PROPER HOLOMORPHIC MAPPPINGS BETWEEN REINHARDT DOMAINS IN $\mathbb{C}^2$

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Abstract. We describe all possibilities of existence of non-elementary proper holomorphic maps between non-hyperbolic Reinhardt domains in $\mathbb{C}^2$ and the corresponding pairs of domains.

1. Introduction

Given $\alpha \in \mathbb{R}^n$, and $z \in \mathbb{C}_*^n$ we put $|z^\alpha| := |z|^{\alpha_1} \dots |z|^{\alpha_n}$ whenever it makes sense. Let $\mathbb{A}_{r^-,r^+} = \{z \in \mathbb{C} : \ r^- < |z| < r^+\}$ for $-\infty < r^- < r^+ < \infty,\ r^+ > 0$. By $\mathbb{D}$ we always denote the unit disc in $\mathbb{C}$. For a domain $D \subset \mathbb{C}^n$, $D \setminus \{0\}$ is denoted by D_*.

Following [Zwo2], for $A = (A_k^j)_{j=1,\dots,m,\ k=1,\dots,n} \in \mathbb{Z}^{m\times n}$ and $b = (b_1, \dots, b_m) \in \mathbb{C}_*^m$ we define:

$$\varphi_A(z) := z^A := (z^{A^1}, \dots, z^{A^m}), \quad z \in \mathbb{C}_*^n,$$
$$\varphi_{A,b}(z) := (b_1 z^{A^1}, \dots, b_m z^{A^m}), \quad z \in \mathbb{C}_*^n.$$

Such maps are called *elementary algebraic* (or briefly *elementary* maps).

The aim of this paper is to describe non-elementary proper holomorphic maps between non-hyperbolic Reinhardt domains in $\mathbb{C}^2$ as well as the corresponding pairs of domains. Additionally, we obtain some partial results for proper maps between domains of the form $\mathbb{C}_*^2$, $\mathbb{C}^2$ and $\mathbb{C}\times\mathbb{C}_*$ and we give some more general results related to proper holomorphic mappings.

Recall that if D, G are Reinhardt domains and $f : D \to G$ is a biholomorphic mapping, then f can be represented as composition of automorphism of D and G and an elementary mapping between these domains (see [Kru] and [Shi2]). Thus, the description of non-elementary biholomorphic mappings between Reinhardt domains reduces to the investigation of their group of automorphisms. It is a general problem of complex geometry of Reinhardt domains considered in many papers. In [Shi1] the

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author using group-theoretic methods investigated the holomorphic equivalence of bounded Reinhardt domains in $\mathbb{C}^n$ not containing the origin and determined automorphisms of a certain class of Reinhardt domains. Similar results were obtained by Barrett in [Bar], however his approach was analytic. The groups of automorphisms of all bounded Reinhardt domains containing the origin were determined in [Sun]. This work has been extended in [Kru] by dropping the assumption that the origin belongs to domain. The situation when domains D and G may be unbounded were considered for example in [Shi3] and [Edi-Zwo].

Obviously, the problem of description of proper holomorphic mappings is harder to deal with. Proper maps between non-hyperbolic, pseudoconvex Reinhardt domains have been considered in [Edi-Zwo] and [Kos]. In the bounded case partial results were obtained in [Ber-Pin], [Lan-Spi] and [Din-Sel]. The final result for bounded domains in $\mathbb{C}^2$, which may be viewed as the completion of this research, has been lastly obtained in the paper [Isa-Kru] of A.V. Isaev and N.G. Kruzhilin. The authors explicitly described all possibilities of existence of non-elementary proper holomorphic mappings between bounded Reinhardt domains in $\mathbb{C}^2$.

Our results finish the problem of characterization of proper holomorphic mappings between Reinhardt domains in $\mathbb{C}^2$.

2. Preliminaries and statement of results

It is well known that for any pseudoconvex Reinhardt domain D in $\mathbb{C}^n$ its logarithmic image $\log D$ is convex. Moreover, any proper holomorphic mapping between domains D_1, D_2 in $\mathbb{C}^n$ can be extended to a proper map between the envelopes of holomorphy $\widehat{D}_1, \widehat{D}_2$ of D_1, D_2 respectively (see e.g. [Ker]).

Let us introduce some notation. First we define

$$V_\iota := \mathbb{C}^{\iota-1} \times \{0\} \times \mathbb{C}^{n-\iota} \subset \mathbb{C}^n, \quad \iota = 1, \dots, n, \tag{1}$$

$$\text{and} \quad M := \bigcup_{\iota=1}^{n} V_\iota.$$

With a given Reinhardt domain D we associate the following constants:

$d(D)$:=the maximal possible dimension of the linear subspace contained in the logarithmic image of the envelope of holomorphy of D;

$t(D)$:=the number of j such that $\widehat{D} \cap V_j \neq \emptyset$.

Moreover, in the case $D \subset \mathbb{C}^2$ we put

$s(D)$:=the number of $j = 1, 2$ such that $V_j \cap \widehat{D}$ is equal to $\mathbb{C}$;
$s_*(D)$:=the number of $j = 1, 2$ such that $V_j \cap \widehat{D}$ is equal to $\mathbb{C}_*$;

It turns out that the objects introduced above are invariant under proper holomorphic mappings $f : D \to G$, where D, G are Reinhardt domains in $\mathbb{C}^2$, except for the case when $\alpha\mathbb{R} + \beta \subset \log D$ for some $\alpha \in \mathbb{Q}^2$, $\beta \in \mathbb{R}^2$. In particular, we shall obtain the following

Theorem 1. *Let D, G be Reinhardt domains in $\mathbb{C}^2$ such that the set of proper holomorphic mappings from D onto G is non-empty. Then*

$$d(D) = d(G). \tag{2}$$

If moreover $d(D) = d(G) = 0$, then

$$(s(D), s_*(D), t(D)) = (s(G), s_*(G), t(G)). \tag{3}$$

Recall here that pseudoconvex Reinhardt domains which are algebraically biholomorphic to bounded Reinhardt domains have been described by W. Zwonek in [Zwo1]. This result is of key importance for our considerations so we quote it below:

Theorem 2. *Assume that D is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(i) D is Brody hyperbolic, i.e. any holomorphic mapping from $\mathbb{C}$ to D is constant,
(ii) (a) $\log D$ *contains no affine lines and*
  (b) $D \cap V_j$ is either empty or c-hyperbolic, $j = 1, \dots, n$, (viewed as domain in $\mathbb{C}^{n-1}$);
(iii) D is algebraically biholomorphic to a bounded Reinhardt domain, i.e. there is $A \in \mathbb{Z}^{n\times n}$, $|\det A| = 1$, such that $\varphi_A(D)$ is bounded and $(\varphi_A)_{|D}$ is a biholomorphism onto the image.

Note that a Reinhardt domain D in $\mathbb{C}^2$ satisfies the condition (ii) of Theorem 2 if and only if $s(D) = s_*(D) = d(D) = 0$.

Let D_1, D_2 be Reinhardt domains in $\mathbb{C}^2$ and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Assume that f is non-elementary. Our aim is to get the explicit formulas for the mapping f as well for the domains D_1, D_2.

In view of Theorem 2 we see that case $d(D_i) = s(D_i) = s_*(D_i) = 0$, $i = 1, 2$, has been described in [Isa-Kru]. Moreover, in [Edi-Zwo] and [Kos] the authors gave the explicit formulas for all proper holomorphic mappings $f : D_1 \to D_2$ between pseudoconvex Reinhardt domains D_1 and D_2 such that $d(D_1) = d(D_2) = 1$, that is domains whose logarithmic image is equal to a strip or a half plane. One may apply direct and tedious calculations which allow to determine all possibilities of the form of Reinhardt domains D_1', D_2' whose envelopes of holomorphy are equal to D_1 and D_2 respectively and such that the restriction $f|D_1' : D_1' \to D_2'$ is proper.

On the other hand there is no proper holomorphic mapping between hyperbolic and non-hyperbolic domains (see Lemma 6) so we shall focus our considerations on proper holomorphic mappings between non-hyperbolic domains.

Summing up, to obtain a desired description of the set of non-elementary proper holomorphic mappings between non-hyperbolic Reinhardt domains D_1, D_2 in $\mathbb{C}^2$ whose envelopes of holomorphy do not contain $\mathbb{C}_*^2$ it suffices to confine ourselves to the cases when $d(D_1) = d(D_2) = 0$ and $s(D_1) = s(D_2) \neq 0$ or $s_*(D_1) = s_*(D_2) \neq 0$.

Now we are in position to formulate the main result of this paper:

Theorem 3. *Let D_1, D_2 be non-hyperbolic Reinhardt domains in $\mathbb{C}^2$ such that $d(D_1) = d(D_2) = 0$ and $s(D_i) \neq 0$ or $s_*(D_i) \neq 0$, $i = 1, 2$. Assume that there is a proper, non-elementary holomorphic mapping $f : D_1 \to D_2$.*

Then the following two scenarios obtain:

(i) Up to a permutation of the components of f and the variables, the map f has the form

$$f(z, w) = (\mu_1 z^k B(C_1 z^{p_1} w^{q_1}), \mu_2 w^l), \tag{4}$$

where $k, l \in \mathbb{N}$, $p_1, q_1 > 0$ are relatively prime integers, B is a non-constant finite Blaschke product non-vanishing at 0, $C_1 > 0$ and $\mu_1, \mu_2 \in \mathbb{C}_*$. In this case, the domains D_1 and D_2 have the form:

$$D_i = \{(z, w) \in \mathbb{C}^2 : \ C_i |z|^{p_i} |w|^{q_i} < 1, \ |w| < E_i\} \setminus (P_i \times \{0\}), \ i = 1, 2, \tag{5}$$

where $E_1, E_2 > 0$, $p_2, q_2 > 0$ are relatively prime integers satisfying the equation $\frac{q_2}{p_2} = \frac{kq_1}{lp_1}$ and P_1 is any closed proper Reinhardt subset of $\mathbb{C}$ (then, obviously, P_2 is of the form $\{\mu_1 \zeta^k B(0) : \ \zeta \in P_1\}$).

(ii) Up to a permutation of the components of f and the variables, the map f has the form

$$f(z, w) = ((e^{it_1} z^{a_1} + s)^{a_2}, e^{it_2} \exp(2\bar{s} e^{it_1} z^{a_1} + |s|^2)^{-c_2} w^{c_1 c_2}), \tag{6}$$

where $a_1, a_2, c_1, c_2 \in \mathbb{N}$, $s \in \mathbb{C}_*$ *and* $t_1, t_2 \in \mathbb{R}$. *In this case domains have the forms*

$$D_i = \{(z,w) \in \mathbb{C}^2 : \ |w| < C_i \exp(-E_i |z|^{k_i})\}, \ i = 1, 2, \tag{7}$$

where $k_1 = 2a_1$, $k_2 = 2/a_2$ *and* $C_1, C_2, E_1, E_2 > 0$.

As mentioned before, in Section 4 we shall also obtain some results related to proper mappings $f : D \to G$ in the case when $d(D) = d(G) = 2$. It is clear that for any pseudoconvex domain D in $\mathbb{C}^2$, $d(D) = 2$ if and only if $\log D = \mathbb{R}^2$.

3. Proofs

We start with the following

Lemma 4. *Let* $\varphi : D_1 \to D_2$ *be a proper holomorphic mapping, where* $D_1, D_2 \subset \mathbb{C}^n$ *are pseudoconvex Reinhardt domains.*

- *(a) Assume that* $d(D_2) = 0$ *and suppose that there is a non-constant holomorphic mapping* $\psi : \mathbb{C} \to D_1$. *Then* $\varphi(\psi(\mathbb{C})) \subset M$.
- *(b) If* $\tilde{\psi} : \mathbb{C} \to D_2$ *is a non constant holomorphic mapping and* $d(D_1) = 0$, *then* $\varphi^{-1}(\tilde{\psi}(\mathbb{C})) \subset M$.

Proof. a) By Lemma 6 in [Jar-Pfl] there is a nonempty open set $U \subset \mathbb{R}^n$ and there is a positive R such that for any $v \in U$ the set $\log D_2$ is contained in $\{x \in \mathbb{R}^n : x_1 v_1 + \ldots + x_n v_n < R\}$. Thus, there are linearly independent $\alpha^1, \ldots, \alpha^n \in \mathbb{R}^n$, $\alpha^\iota = (\alpha_1^\iota, \ldots, \alpha_n^\iota)$, such that D_2 is contained in $\{z \in \mathbb{C}^n : \ |z^{\alpha^\iota}| < e^R\}$, $\iota = 1, \ldots, n$. Put

$$u_\iota(z) = |\varphi(\psi(z))^{\alpha^\iota}|, \quad z \in \mathbb{C}, \quad \iota = 1, \ldots, n. \tag{8}$$

Obviously u_ι are bounded and subharmonic functions on $\mathbb{C}$, so they are constant, say $u_\iota = \rho_\iota$, $\iota = 1, \ldots, n$. It suffices to notice that $\rho_\iota = 0$ for some ι. Indeed, if $\rho_\iota \neq 0$ for every $\iota = 1, \ldots, n$, then obviously $\sum_{j=1}^n \alpha_j^\iota \log |\varphi_j(\psi(z))| = \log \rho_\iota$. Applying Cramer rules we would find that the mapping $\varphi \circ \psi$ would be constant (recall that $\alpha^1, \ldots, \alpha^n$ are linearly independent). However, it would be obviously in contradiction with the properness of the mapping φ (as the mapping ψ is unbounded).

b) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_*^n$ and $R > 1$ be such that $\log D_1$ is contained in $\{x \in \mathbb{R}^n : \ x_1\alpha_1 + \ldots + x_n\alpha_n < R\}$ and for any $t \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : \ x_1\alpha_1 +$

$\ldots + x_n\alpha_n = t\} \cap \log D_1$ is bounded. Put $u(z) = |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n}$ for $z \in D_1$. It is well known that the function

$$v(z) = v_\alpha(z) = \max u(\varphi^{-1}(\tilde{\psi}(z))), \quad z \in \mathbb{C}, \tag{9}$$

is subharmonic. As it is bounded we find that v is constant. Let ρ be such that $v = \rho$. Similarly as in the previous part of the proof it is sufficient to show that ρ is equal to 0.

Suppose not. One can see that there is a sequence $\{w_\mu\}_{\mu=1}^\infty \subset D_1$ such that $|w_\mu^\alpha| = \rho$ for any $\mu \in \mathbb{N}$ and $w_\mu \to w_0 \in \partial D_1$, $(\mu \to \infty)$. Moreover, $|w^\alpha| \le \rho$ for every w such that $\varphi(w) \in \tilde{\psi}(\mathbb{C})$. Take the supporting hyperplane H of $\log D_1$ at the point $\log w_0$ and let $\beta \in \mathbb{R}^n$ be such that $H = \{x \in \mathbb{R}^n : \ x_1\beta_1 + \ldots + x_n\beta_n = \hat{\rho}\}$ for some $\hat{\rho} \in \mathbb{R}$. Repeating the above reasoning (here the assumption of the boundedness of $H \cap \log D_1$ is unnecessary) applied to a function $v = v_\beta$ (see (9)), we find that there is $\tilde{\rho} < e^{\hat{\rho}}$ such that $|w^\beta| \le \tilde{\rho}$ for any $w \in \varphi^{-1}(\tilde{\psi}(\mathbb{C}))$. However, $|w_\mu^\beta| \to e^{\hat{\rho}}$, $(\mu \to \infty)$, which immediately gives a desired contradiction. □

Corollary 5. *Let $D, G \subset \mathbb{C}^n$ be pseudoconvex Reinhardt domains such that $d(D) = 0$ and $d(G) \ge 1$. Then the sets* $\operatorname{Prop}(D, G)$ *and* $\operatorname{Prop}(G, D)$ *are empty.*

Proof. Take $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ such that $\alpha\mathbb{R} + \beta \subset \log G$. Note that for any $z \in G$ the set $\psi_z(\mathbb{C})$ is contained in G, where ψ_z is given by a formula

$$\psi_z(\zeta) = (z_1 e^{\alpha_1 \zeta}, \ldots, z_n e^{\alpha_n \zeta}), \quad \zeta \in \mathbb{C}. \tag{10}$$

Thus, if $f : D \to G$ (or $g : G \to D$) would be a proper holomorphic mapping then, by Lemma 4 $f^{-1}(G) \subset M$ (resp. $g(G) \subset M$). This immediately gives a contradiction. □

Lemma 6. *Let $D, G \subset \mathbb{C}^n$ be domains. Assume that D is bounded and G is not Brody-hyperbolic. Then there is no proper holomorphic mapping from D onto G.*

Proof. Suppose that $\varphi : D \to G$ is a proper holomorphic mapping. Put $A = \{z \in D : \det \varphi'(z) = 0\}$. The set A is a variety in D and, by the properness of φ, $A \neq D$. Moreover, there is an integer m such that $\#\varphi^{-1}(w) = m$ for any $w \in G \setminus \varphi(A)$.

Put

$$\pi_k(\lambda) = \sum_{1 \le i_1 < \ldots < i_k \le m} \lambda_{i_1} \ldots \lambda_{i_k}, \quad \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m, \ k = 1, \ldots, m$$

and $\pi=(\pi_1,\dots,\pi_m)$. Moreover, for $z^j=(z^j_1,\dots,z^j_n)\in\mathbb{C}^n,\ j=1,\dots,m$, define

$$\sigma(z^1,\dots,z^m):=(\pi(z^1_1,\dots,z^m_1),\dots,\pi(z^1_n,\dots,z^m_n)). \tag{11}$$

Obviously $\sigma:(\mathbb{C}^n)^m\to\mathbb{C}^{nm}$ is a proper holomorphic mapping with multiplicity equal to $(m!)^n$.

Let $\varphi^{-1}(w)=\{\zeta_1(w),\dots,\zeta_m(w)\},\ w\in G\setminus\varphi(A)$. Since φ is locally biholomorphic near any $\zeta_i(w)$, $i=1,\dots,m$, and the mapping σ given by the formula (11) is symmetric, we find that the mapping $\psi=\sigma\circ(\zeta_1,\dots,\zeta_m)$ is holomorphic in $G\setminus\varphi(A)$. Because $\varphi(A)$ is analytic in G and ψ is bounded, we may extend ψ to bounded holomorphic mapping on the whole G. Let $\tilde\psi$ be such an extension. Take any $\gamma:\mathbb{C}\to G$ non-constant and holomorphic. Then $\tilde\psi\circ\gamma$ is bounded and holomorphic on $\mathbb{C}$; in particular, $\tilde\psi\circ\gamma$ is constant.

Let us take any $z'\in\mathbb{C}$. If $\gamma(z')$ belongs to $G\setminus\varphi(A)$, then obviously $\tilde\psi(\gamma(z'))=\sigma(\zeta_1(\gamma(z)),\dots,\zeta_m(\gamma(z)))$. Suppose now that $\gamma(z')$ is a critical value of φ. Let $x=(x_1,\dots,x_m)\in(\mathbb{C}^n)^m$ be such that $\tilde\psi(\gamma(z))=\sigma(x),\ z\in\mathbb{C}$.

Take any ζ such that $\varphi(\zeta)=\gamma(z')$, and let $(\zeta_n)\subset D\setminus A$ be such that $\zeta_n\to\zeta$. Observe that $\sigma(\varphi^{-1}(\varphi(\zeta_n)))=\tilde\psi(\varphi(\zeta_n))\to\sigma(x)$. In particular, using properness of σ, we find that $\zeta\in\sigma^{-1}(\sigma(x))$, so we have shown that $\varphi^{-1}(\gamma(z'))\subset\sigma^{-1}(\sigma(x_1,\dots,x_1))$.

It follows that for any $w\in\gamma(\mathbb{C})$ $\varphi^{-1}(w)$ is contained in the finite set $\sigma^{-1}(\sigma(x))$. Since the mapping γ is unbounded, we immediately get a contradiction with the properness of φ. $\square$

Remark 7. Since the mapping $\mathbb{C}\setminus\{0,1\}\ni z\to\frac{1}{z(z-1)}\in\mathbb{C}$ is proper, the above theorem is not true if we only assume that the domain D is Brody-hyperbolic (instead of bounded). On the other hand, since in the class of pseudoconvex Reinhardt domains the property of Brody-hyperbolicity means, up to algebraic mappings, the boundedness, we easily see that there is no proper holomorphic mapping between hyperbolic and non-hyperbolic pseudoconvex Reinhardt domains.

For a Reinhardt domain D in $\mathbb{C}^n$ let $I(D)$ denote the set of $i=1,\dots,n$ for which the intersection $V_i\cap D$ is not $c-$hyperbolic (viewed as a domain in $\mathbb{C}^{n-1}$). Put

$$D^{hyp}=D\setminus(\bigcup_{i\in I(D)}V_i). \tag{12}$$

It is clear that $D^{hyp}=D$ if D is c-hyperbolic or $D\subset\mathbb{C}^n_*$. In the sequel by $\widehat{D}^{hyp}$ we shall denote the set $(\widehat{D})^{hyp}$.

Now we are in position to formulate the following

Theorem 8. *Let D_1, D_2 be pseudoconvex Reinhardt domains in $\mathbb{C}^2$ such that $\log D_i$ contains no affine lines, $i = 1, 2$ (i.e. $d(D_1) = d(D_2) = 0$). If $\varphi : D_1 \to D_2$ is a proper holomorphic mapping, then $\varphi(D_1^{hyp}) \subset D_2^{hyp}$ and the restriction $\varphi|_{D_1^{hyp}} : D_1^{hyp} \to D_2^{hyp}$ is proper.*

Proof. Obviously it suffices to prove the following statement:

1) If $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$ then $\varphi(D_1 \cap V_1)$ is contained either in V_1 or in V_2. In particular, if moreover $D_2 \cap V_2$ is c-hyperbolic or empty, then $\varphi(D_1 \cap V_1) \subset V_1$.
2) If $D_2 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$ and $D_1 \cap V_2$ is neither $\mathbb{C}$ nor $\mathbb{C}_*$, then $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$ and $\varphi^{-1}(D_2 \cap V_1) \subset V_1$.

1) From Lemma 4(a) applied to the mapping $\psi(z) = (0, e^z)$, $z \in \mathbb{C}$, we get that $\varphi(D_1 \cap V_1) \subset M$. It follows that $\varphi_1(0, z)\varphi_2(0, z) = 0$ for any $z \in \mathbb{C}_*$, so $\varphi_1(0, \cdot) \equiv 0$ or $\varphi_2(0, \cdot) \equiv 0$. The second statement is clear.

2) If $D_1 \cap V_1$ were neither $\mathbb{C}$ nor $\mathbb{C}_*$, then D_1 would be biholomorphic to a bounded domain, which obviously contradicts Lemma 6. Thus $D_1 \cap V_1 \in \{\mathbb{C}, \mathbb{C}_*\}$.

Suppose that $D_1 \cap V_2$ is non-empty (in the other case Lemma 4 (b) finishes the proof). Pseudoconvexity implies that $\pi_1(D_1)$ is a bounded subset of $\mathbb{C}$, where $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ denotes a projection onto the first variable. Thus, a function given by the formula

$$v(z) := \max |\pi_1(\varphi^{-1}(0, z))|, \quad z \in \mathbb{C}_* \tag{13}$$

is constant (as bounded and subharmonic). Moreover, from Lemma 4 we get that $\varphi^{-1}(D_2 \cap V_1) \subset M$.

Now, one may easily verify that $v = 0$. □

Corollary 9. *Let $D_1, D_2 \subset \mathbb{C}^2$ be Reinhardt domains such that $d(D_1) = d(D_2) = 0$. Assume that* $\operatorname{Prop}(D_1, D_2)$ *is non-empty. Then*

$$(s(D_1), s_*(D_1), t(D_1)) = (s(D_2), s_*(D_2), t(D_2)). \tag{14}$$

Proof. Any proper holomorphic map between domains D_1, D_2 may be extended to the proper map between their envelopes of holomorphy $\widehat{D}_1, \widehat{D}_2$, respectively. Moreover, it is well known (see Corollary 0.3 [Isa-Kru]) that if there exists a proper holomorphic mapping between two given bounded domains, then there also exists an elementary algebraic proper holomorphic mapping between these domains.

Thus, our result is a direct consequence of Theorem 8 and properties of algebraic mappings. □

Note, that applying the methods used in previous theorems we may easily show

Proposition 10. *There are no proper holomorphic mappings between domains D, G and G, D in the following cases:*

1. $D \subset \{z \in D' : \ u(z) < 0\}$, where u is some plurisubharmonic function on the domain $D' \subset \mathbb{C}^n$, non constant on D and $G = \mathbb{C}^n \setminus E$ for some pluripolar set E in $\mathbb{C}^n$.

2. D is hyperconvex (i.e. there is a negative plurisubharmonic exhaustion function for D) and G is not Brody-hyperbolic.

Proof. 1. Obviously $\text{Prop}(\mathbb{C}^n \setminus E, D) = \emptyset$.

Suppose that $\varphi : D \to \mathbb{C}^n \setminus E$ is proper and holomorphic. Put $v(z) = \max u(\varphi^{-1}(z))$, $z \in \mathbb{C}^n \setminus E$. It is seen that the function v is constant. In particular, there is $\rho < 0$ such that $u \leq \rho$ and $u(w_0) = \rho$ for some $w_0 \in D$; a contradiction.

2. It is clear that the set $\text{Prop}(G, D)$ is empty. Suppose that $\varphi : D \to G$ is proper and holomorphic. Let u be a negative plurisubharmonic exhaustion function for D and let $\psi : \mathbb{C} \to G$ be a non-constant holomorphic mapping. Put $v(\zeta) = \max u(\varphi^{-1}(\psi(\zeta)))$, $\zeta \in \mathbb{C}$. The function v is subharmonic on $\mathbb{C}$. Since $v < 0$, it is constant. So we find that $\varphi^{-1}(\psi(\mathbb{C}))$ is a relatively-compact subset of D; a contradiction. □

Proposition 11. *Let $D, G \subset \mathbb{C}^2$ be pseudoconvex Reinhardt domains. If $d(D) \neq d(G)$, then there is no proper holomorphic mapping between D and G.*

Proof. In the view of Corollary 5 it suffices to consider the case when $d(D) = 2$ or $d(G) = 2$. However, this immediately follows from Proposition 10. □

Proof of Theorem 1. It is a direct consequence of Corollary 9 and Proposition 11. □

Proof of Theorem 3. Take any $f : D_1 \to D_2$ proper and holomorphic and suppose that it is non-elementary. Let $f : \widehat{D}_1 \to \widehat{D}_2$ also denotes its extension to the proper mapping between the envelopes of holomorphy of D_1 and D_2. By Proposition 11 $d(D_1) = d(D_2)$.

First we consider the case $d(D_1) = d(D_2) = 0$. Then, by Theorem 8 the restriction $f|_{\widehat{D}_1^{hyp}} : \widehat{D}_1^{hyp} \to \widehat{D}_2^{hyp}$ is also proper.

If $s(D_1) = 2$ or $s_*(D_1) = t(D_1) = 1$, then $\widehat{D}_1^{hyp}$ would be contained in $\mathbb{C}_*^2$ and from the description obtained in [Isa-Kru] we find that $f|_{\widehat{D}_1^{hyp}}$ would be elementary algebraic. It is clear that the identity principle gives a contradiction.

Therefore, we may assume, that $t(D_1) = t(D_2) = 2,\ s(D_1) = s(D_2) = 1$ and $s_*(D_1) = s_*(D_2) = 0$. Up to a permutation of components we may suppose that $\widehat{D}_i \cap V_2 = V_2$ and $\widehat{D}_i \cap V_1$ is bounded, $i = 1, 2$. Therefore, there are $k_1, k_2 \in \mathbb{N}$ such that $\widehat{2}D_i$ is contained in $\{(z,w) \in \mathbb{C}^2 : \ |z||w|^k < c_i\}$ for some positive constants $c_i,\ i = 1, 2$. It follows that Φ_{A_i}, where $A_i = \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix}$, is a biholomorphic mapping from $\widehat{D}_i^{hyp}$ onto the bounded set $\Phi_{A_i}(\widehat{D}_i^{hyp}),\ i = 1, 2$. In particular,

$$g := \Phi_{A_2} \circ f \circ \Phi_{A_1^{-1}} : \Phi_{A_1}(\widehat{D}_1^{hyp}) \to \Phi_{A_2}(\widehat{D}_2^{hyp}) \tag{15}$$

is a proper holomorphic mapping between two bounded domains in $\mathbb{C}^2$. Now, using description obtained in [Isa-Kru] it is straightforward to observe that two possibilities may hold:

(i) $\widehat{D}_i^{hyp} = \{(z,w) \in \mathbb{C}^2 : \ C_i|z|^{p_i}|w|^{p_i k_i + q_i} < 1,\ 0 < |w| < C_i'\}$, where p_i, q_i are relatively prime integers such that $p_i k_i + q_i > 0,\ p_i > 0,\ q_i \leq 0$ and $C_i, C_i' > 0,\ i = 1, 2$.

(ii) $\widehat{D}_1^{hyp} = \{(z,w) \in \mathbb{C}^2 : \ 0 < |w| < C_1 \exp(-E_1|z|^{2a_1}|w|^{2k_1 a_1 - 2b_1})$, and $\widehat{D}_2^c = \{(z,w) \in \mathbb{C}^2 : \ 0 < |w| < C_2 \exp(-E_2|z|^{2/a_2}|w|^{2k_2/a_2 - 2b_2/a_2 c_2})$, where $a_i, b_i, c_i \in \mathbb{N},\ C_i, E_i > 0,\ i = 1, 2$.

First suppose that (i) holds. From [Isa-Kru] it follows that g must be of the form $g(z,w) = (\lambda_1 z^a w^b B(C_1 z^{p_1} w^{q_1}), \lambda_2 w^c),\ (z,w) \in \Phi(\widehat{D}_1^{hyp})$, where $a, b, c \in \mathbb{Z},\ a, c > 0, aq_1 - bp_1 < 0,\ \frac{q_2}{p_2} = \frac{aq_1 - bp_1}{cp_1}$, B is a Blaschke product non-vanishing at 0 and $\lambda_1, \lambda_2 \in \mathbb{C}_*$. Put $\tilde{q}_i = p_i k_i + q_i$. It is obvious that p_i and $\tilde{q}_i$ are relatively prime. Moreover, from the form of $\widehat{D}_i$ we get that $\tilde{q}_i > 0$. An easy computation gives

$$f(z,w) = (\mu_1 z^a w^{ak_1 - ck_2 + b} B(C_1 z^{p_1} w^{\tilde{q}_1}), \mu_2 w^c), \quad (z,w) \in \widehat{D}_1^{hyp},$$

for some constants μ_1, μ_2. Since f may be extended properly on $\widehat{D}_1,\ ak_1 - ck_2 + b = 0$. Moreover, it is clear that $\frac{\tilde{q}_2}{p_2} = \frac{a\tilde{q}_1}{cp_1}$.

It is straightforward to see that any Reinhardt subdomain of $\widehat{D}_1$ mapped properly by f onto a Reinhardt domain and whose envelopes of holomorphy coincides with $\widehat{D}_1$ is equal to $\widehat{D}_1 \setminus P_1 \times \{0\}$, where P_1 is any closed Reinhardt subset of $\mathbb{C}$.

Now suppose that (ii) holds. Denote $m_1 := k_1 a_1 - b_1,\ m_2 := k_2 c_2 - b_2$. Similarly as before, taking into account the form of $\widehat{D}_1$ and $\widehat{D}_2$ one can see that $m_1, m_2 \geq 0$.

For $s \in \mathbb{C}_*$ and $t_1, t_2 \in \mathbb{R}$ put $h_1(z) := e^{it_1}z+s,\ h_2(z) = e^{it_2}\exp(2\bar{s}e^{it_1z}+|s|^2),\ z \in \mathbb{C}$. An easy calculation and formula for the mapping g (see [Isa-Kru]) give

$$f(z,w) = (h_1(z^{a_1}w^{m_1})^{a_2}h_2(z^{a_1}w^{m_1})^{m_2}w^{-m_2c_2}, h_2(z^{a_1}w^{m_2})^{-c_2}w^{c_1c_2}),\qquad (z,w) \in \widehat{D}_1^{hyp}. \tag{16}$$

Since f may be extended through V_1, $m_1 = m_2 = 0$.

Finally, one may easily verify that any Reinhardt subdomain of $\widehat{D}_1$ whose envelope of holomorphy coincides with $\widehat{D}_1$ and which is mapped properly by f onto a Reinhardt domain is equal to $\widehat{D}_1$. □

4. Remarks on the proper holomorphic mappings $f : D \to G$ between Reinhardt domains in case $d(D) = d(G) = 2$

It is well known that the structure of $\mathrm{Aut}(\mathbb{C}^2),\ \mathrm{Aut}(\mathbb{C}^2_*), \mathrm{Aut}(\mathbb{C}\times\mathbb{C}_*)$ is very complicated and the full description of these groups seems to be not known. Proper maps are harder to deal with, so description of the set of proper holomorphic mappings between pseudoconvex Reinhardt domains D_1 and D_2 in the case, when $\log D_i = \mathbb{R}^2,\ i = 1, 2$, is more difficult.

Below we present some partial results related to these problems.

Proposition 12. *The sets* $\mathrm{Prop}(\mathbb{C}\times\mathbb{C}, \mathbb{C}\times\mathbb{C}_*),\ \mathrm{Prop}(\mathbb{C}\times\mathbb{C}, \mathbb{C}_*\times\mathbb{C}_*)$ *and* $\mathrm{Prop}(\mathbb{C}\times\mathbb{C}_*, \mathbb{C}_*\times\mathbb{C}_*)$ *are empty.*

Proof. First suppose that $f : \mathbb{C}^2 \to \mathbb{C}\times\mathbb{C}_*$ is proper and holomorphic. Obviously there exists a holomorphic mapping $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $f = (\psi_1, e^{\psi_2})$. One can easily verify that the mapping ψ is proper; in particular ψ is surjective. Thus there is a discrete sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^2$ such that $\psi(z_n) = (0, 2n\pi i),\ n \in \mathbb{N}$. It follows that $f(z_n) = (0, 1)$ for $n \in \mathbb{N}$. From this we immediately get a contradiction.

To show that $\mathrm{Prop}(\mathbb{C}^2, \mathbb{C}^2_*) = \emptyset$ we proceed similarly.

Now suppose that $g : \mathbb{C}\times\mathbb{C}_* \to \mathbb{C}^2_*$ is holomorphic and proper. It is seen that there exists a holomorphic mapping $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $g(z, e^w) = (e^{\varphi_1(z,w)}, e^{\varphi_2(z,w)})$ for $z, w \in \mathbb{C}$.

Fix $z \in \mathbb{C}$ and put $\tilde{g}_i = g_i(z, \cdot),\ \tilde{\varphi}_i = \varphi_i(z, \cdot),\ i = 1, 2$. Since $\tilde{g}_i(e^w) = e^{\tilde{\varphi}_i(w)}$, we find that $\tilde{\varphi}_i'(w) = \zeta_i(e^w),\ w \in \mathbb{C}$, where ζ_i is a holomorphic function given by the formula $\zeta_i(\lambda) = \frac{\lambda\tilde{g}_i'(\lambda)}{\tilde{g}_i(\lambda)},\ \lambda \in \mathbb{C}_*$. Expanding ζ_i to the Laurent series gives $\tilde{\varphi}_i(w) = a_iw + \sum_{n\in\mathbb{Z}_*} a_{in}e^{nw}$ for some $a_i = a_i(z), a_{in} = a_{in}(z) \in \mathbb{C}$.

Thus, there is a holomorphic mapping $\hat{\varphi}_i(\cdot) = \hat{\varphi}_i(z,\cdot)$ on $\mathbb{C}_*$ such that $\tilde{\varphi}_i(w) = a_i w + \hat{\varphi}_i(e^w),\ w \in \mathbb{C}$. Since $e^{a_i w} = \frac{\tilde{g}_i(e^w)}{e^{\hat{\varphi}_i(e^w)}}$ we immediately find that $a_i \in \mathbb{Z},\ i = 1, 2$.

Therefore $\varphi_i(z,w) = a_i(z)w + \hat{\varphi}_i(z, e^w),\ z, w \in \mathbb{C},\ i = 1, 2$. In particular

$$g(z,w) = (w^{a_1(z)} e^{\hat{\varphi}_1(z,w)}, w^{a_2(z)} e^{\hat{\varphi}_2(z,w)}), \quad (z,w) \in \mathbb{C} \times \mathbb{C}_*. \tag{17}$$

It is straightforward to verify that $a_i(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\frac{\partial g_i}{\partial \lambda}(z,\lambda)}{g_i(z,\lambda)} d\lambda,\ z \in \mathbb{C}$, whence a_i is constant (recall that $a_i(z) \in \mathbb{Z}$) and therefore $\hat{\varphi}_i$ is holomorphic on $\mathbb{C} \times \mathbb{C}_*,\ i = 1, 2$.

Note that we may assume that $a_2 = 0$ (if $a_1 a_2 \neq 0$ one may compose g with a proper holomorphic mapping $F : \mathbb{C}_*^2 \to \mathbb{C}_*^2$ given by the formula $F(z,w) = (z^{a_2}, \frac{w^{a_1}}{z^{a_2}})$).

Put

$$h(z,w) = (w^{a_1} e^{\hat{\varphi}_1(z,w)}, \hat{\varphi}_2(z,w)), \quad (z,w) \in \mathbb{C} \times \mathbb{C}_*,$$

and notice that the mapping $h : \mathbb{C} \times \mathbb{C}_* \to \mathbb{C}_* \times \mathbb{C}$ is proper.

Now, in order to get a contradiction one may proceed exactly as in the case of $\mathrm{Prop}(\mathbb{C}^2, \mathbb{C} \times \mathbb{C}_*)$. □

Corollary 13. $\mathrm{Prop}(A \times \mathbb{C}, A \times \mathbb{C}_*)$ *is empty for any domain* $A \subset \mathbb{C}$.

Proof. If $\#(\mathbb{C} \setminus A) \leq 1$ the result follows directly from Proposition 12. Assume that $\#(\mathbb{C} \setminus A) > 1$ and let $f : A \times \mathbb{C} \to A \times \mathbb{C}_*$ be proper and holomorphic. By the uniformization theorem there is an universal covering $\pi : \mathbb{D} \to A$ and there is $\psi \in \mathcal{O}(\mathbb{D} \times \mathbb{C}, \mathbb{D})$ such that

$$f(\pi(\lambda), w) = (\pi(\psi(\lambda, w)), f_2(\pi(\lambda), w)) \quad \text{for any} \quad (\lambda, w) \in \mathbb{D} \times \mathbb{C}.$$

Fix any $\lambda \in \mathbb{D}$ and note that the mapping $\psi(\lambda, \cdot)$ is constant. From properness of f it easily follows that the mapping $f_2(\pi(\lambda), \cdot) : \mathbb{C} \to \mathbb{C}_*$ is proper; a contradiction. □

Remark 14. Since $\phi : \mathbb{C}_* \ni z \to z + 1/z \in \mathbb{C}$ is proper, there exist proper holomorphic maps from $\mathbb{C}_*^2$ onto $\mathbb{C}^2$, from $\mathbb{C}_*^2$ onto $\mathbb{C} \times \mathbb{C}_*$, and from $\mathbb{C} \times \mathbb{C}_*$ onto $\mathbb{C}^2$. Obviously such maps cannot be algebraic.

On the other hand, the above results and the ones obtained in [Isa-Kru] and [Kos] imply that if there exists a proper holomorphic mapping between two Reinhardt domains $D_1, D_2 \subset \mathbb{C}^2$ such that $\alpha\mathbb{R} + \beta$ is not contained in $\log D_1$ for any $\alpha \in \mathbb{Q}^2,\ \beta \in \mathbb{R}^2$ (hence also in $\log D_2$, see [Kos]), then there also exists an elementary algebraic mapping between these domains.

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