Spectral theory of a class of Block Jacobi matrices and applications

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Abstract

We develop a spectral analysis of a class of block Jacobi operators based on the conjugate operator method of Mourre. We give several applications including scalar Jacobi operators with periodic coefficients, a class of difference operators on cylindrical domains such as discrete wave propagators, and certain fourth-order difference operators.

Keyword: Essential spectrum, absolutely continuous spectrum, Mourre estimate

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1 Introduction

Let $\mathcal{H} = l^2(\mathbb{Z}, \mathbb{C}^N)$, for some integer $N \geq 1$, be the Hilbert space of square summable vector-valued sequences $(\psi_n)_{n \in \mathbb{Z}}$ endowed with the scalar product

$$\langle \phi, \psi \rangle = \sum_{n \in \mathbb{Z}} \langle \phi_n, \psi_n \rangle_{\mathbb{C}^N}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ is the usual scalar product of $\mathbb{C}^N$. Let A_n and B_n be two sequences of N by N matrices such that $B_n = B_n^*$ for all $n \in \mathbb{Z}$. Here we denote by T^* the adjoint matrix of a given matrix T. Let us consider the difference operator $H = H(\{A_n\}, \{B_n\})$ acting in $\mathcal{H}$ by

$$(H\psi)_n = A_{n-1}^* \psi_{n-1} + B_n \psi_n + A_n \psi_{n+1}, \quad \text{for all } n \in \mathbb{Z}. \tag{1.1}$$

In the literature, H is usually called the block Jacobi operator realized by the sequences A_n and B_n. Operators of such a form are naturally related to matrix orthogonal polynomials theory, see for example [2, 9, 10, 12, 14, 21, 22] and references therein. Of course, the case where $N = 1$ corresponds to the usual scalar Jacobi operators that are well studied by different approaches, see e.g. [2, 6, 7, 8, 10, 11, 13, 15, 16, 17, 18, 20, 23, 25, 27, 28] and their references. (This list is very restrictive and contains only some papers related directly to the present one). Applications given in the second part of this paper represent additional motivations of our interest to block Jacobi operators.

Our main goal here is to study spectral properties of the operator H in terms of the asymptotic behavior of the entries A_n and B_n. The key step of our analysis is the construction of a conjugate operator for H in the sense of the Mourre estimate, see [24]. Recall that such an estimate has many important consequences on the spectral measure of H as well as on the asymptotic behavior of the resolvent of $(H - z)^{-1}$ when the complex parameter z approaches the real axis, see [1, 4, 5, 24, 26].

We need the following standard notations. For a self-adjoint operator T we denote by $E_T(\cdot)$ its spectral measure, $\sigma(T)$ its spectrum, $\sigma_{ess}(T)$ its essential spectrum, $\sigma_p(T)$ the set of its eigenvalues, $\sigma_{sc}(T)$ its singular continuous spectrum, and $\sigma_{ac}(T)$ its absolutely continuous spectrum.

In this paper we will focus on the so-called generalized Nevai class. More specifically, we assume that there exist two matrices A and B such that,

$$\lim_{|n|\to\infty} (\|A_n - A\| + \|B_n - B\|) = 0. \tag{1.2}$$

In particular, H is a bounded self-adjoint operator in $\mathcal{H}$. Notice that in the literature, it is customary to assume that the matrices A_n are positive definite or at least non singular. Here we do not demand such an assumption, which turns out to be quite useful in many applications, see Section 10.

It is more convenient to decompose H into the sum $H = H_0 + V$ where the operator $H_0 = H_0(A, B)$ is defined on $\mathcal{H}$ by

$$(H_0\psi)_n = A^*\psi_{n-1} + B\psi_n + A\psi_{n+1}, \quad \text{for all } n \in \mathbb{Z} \tag{1.3}$$

and $V = H(\{A_n - A\}, \{B_n - B\})$ is the Block Jacobi operator realized by the sequences $A_n - A, B_n - B$. The operator H_0 is clearly bounded and self-adjoint in $\mathcal{H}$ while, according to (1.2), the perturbation V is compact. Hence, by Weyl theorem, H and H_0 have the same essential spectra,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0).$$

The point here is that H_0 is a block Toeplitz matrix whose spectrum can be determined explicitly. Indeed, by using Fourier transform we easily show that H_0 is unitarily equivalent to the direct integral

$$\hat{H}_0 = \int_{[-\pi,\pi]}^{\oplus} h(p)dp \quad \text{acting in} \quad \hat{\mathcal{H}} = \int_{[-\pi,\pi]}^{\oplus} \mathbb{C}^N dp \tag{1.4}$$

where the reduced operators $h(p)$ are given by,

$$h(p) = e^{-ip}A + e^{ip}A^* + B. \tag{1.5}$$

The family $h(p), p \in [-\pi, \pi]$, is an analytic family of self-adjoint matrices. So, see [19], there exist two analytic functions λ_j and W_j such that, for all $p \in [-\pi, \pi]$, $\{\lambda_j(p); 1 \leq j \leq N\}$ are the repeated eigenvalues of $h(p)$ and $\{W_j(p); 1 \leq j \leq N\}$ is a corresponding orthonormal basis of eigenvectors. In particular, for any $j = 1, \cdots, N$, the critical set of λ_j defined by

$$\kappa(\lambda_j) = \{\lambda_j(p) \mid \lambda_j'(p) = 0\}$$

is clearly finite. Therefore, the critical set of H_0 defined by

$$\kappa(H_0) = \cup_{i=1}^{i=N} \kappa(\lambda_j)$$

is finite too. We have

Proposition 1.1 *The spectrum of H_0 is purely absolutely continuous outside the critical set $\kappa(H_0)$. More precisely, $\sigma(H_0) = \cup_{i=1}^{i=N} \lambda_i([-\pi, \pi])$, $\sigma_p(H_0) \subset \kappa(H_0)$ and $\sigma_{sc}(H_0) = \emptyset$.*

Let $\alpha_j = \min_{p\in[-\pi,\pi]} \lambda_j(p)$ and $\beta_j = \max_{p\in[-\pi,\pi]} \lambda_j(p)$. Assume that the λ_j's are arranged so that $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_N$. So each λ_j gives arise to a spectral band $\Sigma_j = \lambda_j([-\pi, \pi]) = [\alpha_j, \beta_j]$, $j = 1, \cdots, N$. Two successive bands Σ_j, Σ_{j+1} may overlap if $\alpha_{j+1} \leq \beta_j$; or may be disjoint if $\beta_j < \alpha_{j+1}$. In the second case $[\beta_j, \alpha_{j+1}]$ is called a non degenerate spectral gap. Notice also that, in some situations, the band Σ_j can degenerate into a single point, i.e. $\alpha_j = \beta_j$. This happens if, and only if, $\lambda_j(p)$ is constant on $[-\pi, \pi]$. In which case, this constant value is an infinitely degenerate eigenvalue of H_0. To illustrate these considerations we will give in Section 2 some explicit examples and show necessary and/or sufficient conditions to a spectral band to be non degenerate.

Using the eigenvalues $\lambda_j(p)$'s of the reduced operators $h(p)$, we will construct a conjugate operator for H_0. That is to say a self-adjoint operator $\mathbb{A}$ such that, for each compact real interval J included in $\mathbb{R} \setminus \kappa(H_0)$, we have

$$E_{H_0}(J)[H_0, i\mathbb{A}]E_{H_0}(J) \geq c\, E_{H_0}(J) \tag{1.6}$$

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for some $c > 0$. For the definition of the commutator $[H_0, i\mathbb{A}]$ see Section 3. In particular, we obtain several important resolvent estimates for H_0, see Theorem 5.1. We used a similar strategy in [3] to construct conjugate operators from dispersion curves for perturbations of fibered systems in cylinders.

Next we prove that if $A_n - A$ and $B_n - B$ tend to zero fast enough as n tends to infinity, then the Mourre estimate (1.6) is preserved for H up to a compact operator. More precisely, we will show that, if (1.8) below holds, then for the same $\mathbb{A}, J$ and $c > 0$ as in (1.6),

$$E_H(J)[H, i\mathbb{A}]E_H(J) \geq c\, E_H(J) + K \tag{1.7}$$

for some compact operator K. In particular, we get

Theorem 1.1 *Suppose that*

$$\lim_{|n| \to \infty} |n|(\|A_n - A\| + \|B_n - B\|) = 0. \tag{1.8}$$

Then outside $\kappa(H_0)$ the eigenvalues of H are all finitely degenerate and their possible accumulation points are contained in $\kappa(H_0)$.

Put $\langle x \rangle = \sqrt{1 + x^2}$ for a real number x. We denote by $\mathbf{N}$ the multiplication operator defined by $(\mathbf{N}\psi)_n = n\psi_n$ for all $\psi \in \mathcal{H}$, and by $B(\mathcal{H})$ the Banach algebra of bounded operators in the Hilbert space $\mathcal{H}$. We also need the interpolation space $\mathcal{K} := (D(\mathbf{N}), \mathcal{H})_{1/2,1}$ which can be described, according to Theorem 3.6.2 of [1], by the norm,

$$\|\psi\|_{1/2,1} = \sum_{j=0}^{\infty} 2^{j/2}\|\theta(2^{-j}|\mathbf{N}|)\psi\|$$

where $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(x) > 0$ if $2^{-1} < x < 2$ and $\theta(x) = 0$ otherwise. Clearly, $\langle \mathbf{N} \rangle^{-s}\mathcal{H} \subset \mathcal{K} \subset \mathcal{H}$ for any $s > 1/2$.

Theorem 1.2 *Suppose that the perturbation V satisfies*

$$\int_1^\infty \|\theta(|\boldsymbol{N}|/r)V\|dr < \infty. \tag{1.9}$$

Then the limits $(H - x \mp i0)^{-1} := \lim_{\mu \to 0+}(H - x \mp i\mu)^{-1}$ exist locally uniformly on $\mathbb{R} \setminus [\kappa(H_0) \cup \sigma_p(H)]$ for the weak* topology of $B(\mathcal{K}, \mathcal{K}^*)$. In particular, the singular continuous spectrum $\sigma_{sc}(H)$ is empty.

Mention that one may show that condition (1.9) is equivalent to

$$\int_1^\infty \sup_{r < |n| < 2r} (\|A_n - A\| + \|B_n - B\|)dr < \infty. \tag{1.10}$$

Hence these conditions are clearly satisfied if, for some $\theta > 0$, one has

$$\sup_{n \in \mathbb{Z}} |n|^{1+\theta}(\|A_n - A\| + \|B_n - B\|) < \infty. \tag{1.11}$$

Remark also that Theroem 1.2 implies that $\langle \mathbf{N} \rangle^{-s}(H - x \mp i0)^{-1}\langle \mathbf{N} \rangle^{-s} \in B(\mathcal{H})$, for any $s > \frac{1}{2}$. In our next theorem we describe continuity properties of these boundary values of the resolvent of H as functions of x, as well as some of their propagation consequences. For we need the following definition. Let $(E, \|\cdot\|)$ be a Banach space and $f : \mathbb{R} \to E$ be a bounded continuous function. For an integer $m \geq 1$ let w_m be the modulus of continuity of order m of f defined on $(0, 1)$ by

$$w_m(f, \varepsilon) = \sup_{x \in \mathbb{R}} \left\| \sum_{j=0}^m (-1)^j \binom{m}{j} f(x + j\varepsilon) \right\|.$$

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We say that $f \in \Lambda^\alpha$, $\alpha > 0$, if there is an integer $m > \alpha$ such that

$$\sup_{0 < \varepsilon < 1} \varepsilon^{-\alpha} w_m(\varepsilon) < \infty.$$

Notice that if $\alpha \in (0, 1)$ then Λ^α is nothing but the space of Hőlder continuous functions of order α. In contrast, if $\alpha = 1$ then Λ^1 consists of smooth functions in Zygmund's sense (they are not Lipschitz in general, see [29]). Finally, if $\alpha > 1$ and n_α is the greatest integer strictly less than α then f belongs to Λ^α if, and only if, f is $n_\alpha-$times continuously differentiable with bounded derivatives and its derivative $f^{(n_\alpha)}$ of order n_α is of class $\Lambda^{\alpha-n_\alpha}$. For example, f belongs to Λ^2 means that f is continuously differentiable with a bounded derivative and f' is of class Λ^1 (f' is not Lipschitz in general).

Theorem 1.3 *Suppose that (1.11) holds. Then, for any $\frac{1}{2} < s \le \frac{1}{2} + \theta$, the maps*

$$x \longmapsto \langle \boldsymbol{N} \rangle^{-s} (H - x \mp i0)^{-1} \langle \boldsymbol{N} \rangle^{-s} \in B(\mathcal{H})$$

are locally of class $\Lambda^{s-\frac{1}{2}}$ on $\mathbb{R} \setminus [\kappa(H_0) \cup \sigma_p(H)]$. Moreover,

$$||\langle \boldsymbol{N} \rangle^{-s} e^{-iHt} \varphi(H) \langle \boldsymbol{N} \rangle^{-s}|| \le C(1 + |t|)^{-(s-\frac{1}{2})}.$$

for all $\varphi \in C_0^\infty(\mathbb{R} \setminus [\kappa(H_0) \cup \sigma_p(H)])$ and some $C > 0$.

Notice that for H_0 we have more than that, see Theorem 5.1. Moreover, Theorems 1.1-1.3 are actually valid for operators of the form $H = H_0 + V$ where V is a symmetric compact perturbation such that $\langle \mathbf{N} \rangle^{1+\theta} V$ is bounded with $\theta > 0$. See section 6 where this remark is discussed more precisely.

Finally, in the second part of the paper, we apply these abstract results to some concrete models. The first application concerns the case where A is positive definite. It is one of the most common assumptions in the studies of block Jacobi matrices and their applications to matrix orthogonal polynomial theory. In this case, all the eigenvalues of $h(p)$ are monotonic functions of p. This applies directly to a class of difference operators on cylindrical domains such as discrete wave propagators. We devote the next applications to cases where the matrix A is singular. Then we show how Jacobi operators with periodic coefficients fit in this situation. Notice that while the spectral results of Jacobi operators with periodic coefficients are well known our Mourre estimate is a new ingredient. Finally, we study a model where A is a lower triangular matrix that we apply to symmetric difference operators of fourth-order.

The paper is organized as follows.

- In Section 2 we study in details the unperturbed operator H_0.

- Section 3 contains a brief review on what we need from Mourre's theory.

- In Sections 4 and 5 we construct a conjugate operator for H_0.

- In section 6 we extend our Mourre estimate to the operator H.

- In section 7 we study the case where A is positive definite.

- Section 8 is devoted to discrete wave propagators in stratified cylinders.

- In Sections 9-11 we discuss cases where A is singular.

- In section 12 we consider a case where A is a lower triangular.

- Finally in Section 13 we explain how our method extends to difference operator of higher order with matrix coefficients.

4

2 Basic properties of H_0

Consider the unitary transform $\mathcal{F} : l^2(\mathbb{Z}, \mathbb{C}^N) \to L^2([-\pi, \pi], \mathbb{C}^N)$ defined by

$$\mathcal{F}(\psi)(p) = \widehat{\psi}(p) = \sum_{n \in \mathbb{Z}} e^{inp} \psi_n \ , \ \text{for each} \quad \psi \in l^2(\mathbb{Z}, \mathbb{C}^N).$$

Recall that the unperturbed operator H_0 is defined in $\mathcal{H}$ by

$$(H_0 \psi)_n = A^* \psi_{n-1} + B \psi_n + A \psi_{n+1}, \quad \text{for all } n \in \mathbb{Z}.$$

Direct computation shows that, for each $\psi \in l^2(\mathbb{Z}, \mathbb{C}^N)$,

$$\widehat{H_0 \psi}(p) = (e^{-ip} A + e^{ip} A^* + B) \widehat{\psi}(p) = h(p) \widehat{\psi}(p) \ , \ \text{for all } \ p \in [-\pi, \pi] \ .$$

Hence (1.4)-(1.5) are proved. Recall that, for all $p \in \mathbb{R}$, $\{\lambda_j(p); 1 \leq j \leq N\}$ are the repeated eigenvalues of $h(p)$ and $\{W_j(p); 1 \leq j \leq N\}$ is a corresponding orthonormal basis of eigenvectors. The Proposition 1.1 follows from:

Proposition 2.1 *The operator H_0 is unitarily equivalent to $M = \oplus_{j=1}^{j=N} \lambda_j(p)$ acting in the direct sum $\oplus_{j=1}^{j=N} L^2([-\pi, \pi])$.*

Proof Consider the operator $\mathcal{U} : l^2(\mathbb{Z}, \mathbb{C}^N) \to \oplus_{j=1}^{j=N} L^2([-\pi, \pi])$ defined by $\mathcal{U}\psi = (f_j)_{1 \leq j \leq N}$, where

$$f_j(p) = \langle W_j(p), \widehat{\psi}(p) \rangle_{\mathbb{C}^N} = \sum_{n \in \mathbb{Z}} \langle W_j(p), \psi_n \rangle_{\mathbb{C}^N} e^{inp}. \tag{2.1}$$

Clearly, $\mathcal{U}$ is a unitary operator with the following inversion formula:

$$(\mathcal{U}^{-1} f)_n = \sum_{j=1}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(p) W_j(p) e^{-inp} dp,$$

for all $f = (f_1, \cdots, f_N)$ in $\oplus_{j=1}^{j=N} L^2([-\pi, \pi])$. Moreover, direct computation shows that, for each $\psi \in l^2(\mathbb{Z}, \mathbb{C}^N)$,

$$(\mathcal{U} H_0 \psi)_j(p) = \langle W_j(p), \widehat{H_0 \psi}(p) \rangle_{\mathbb{C}^N} = \langle W_j(p), h(p) \widehat{\psi}(p) \rangle_{\mathbb{C}^N} = \lambda_j(p) f_j(p).$$

The proof is complete.

In the sequel we denote the imaginary part of a complex number z by $\Im z$.

Proposition 2.2 *The eigenvalue $\lambda_j(p)$ is identically constant on $[-\pi, \pi]$ if, and only if,*

$$\lambda_j'(p) = \Im(e^{ip} \langle A W_j(p), W_j(p) \rangle_{\mathbb{C}^N}) = 0, \quad \text{for all} \ \ p \in [-\pi, \pi].$$

In this case, the spectral band $\Sigma_j = \lambda_j([-\pi, \pi])$ degenerates into a single point which is an infinitely degenerate eigenvalue of H_0.

Proof: We know that, for all $p \in [-\pi, \pi]$,

$$\lambda_j(p) = \langle h(p) W_j(p), W_j(p) \rangle_{\mathbb{C}^N}.$$

Since $\|W_j(p)\|^2 = 1$ then $\langle W_j'(p), W_j(p) \rangle_{\mathbb{C}^N} + \langle W_j(p), W_j'(p) \rangle_{\mathbb{C}^N} = 0$. Thus

$$\begin{aligned}
\lambda_j'(p) = \langle h(p)' W_j(p), W_j(p) \rangle_{\mathbb{C}^N} &= \langle i(-e^{-ip} A + e^{ip} A^*) W_j(p), W_j(p) \rangle_{\mathbb{C}^N} \\
&= -2\Im(e^{ip} \langle A W_j(p), W_j(p) \rangle_{\mathbb{C}^N}).
\end{aligned}$$

The proof is complete.

Remark In particular, the jth spectral band $\Sigma_j = \lambda_j([-\pi, \pi])$ is not degenerate if, and only if, there is $p \in [-\pi, \pi]$ such that $\lambda_j'(p) = -2\Im(e^{ip} \langle A W_j(p), W_j(p) \rangle_{\mathbb{C}^N}) \neq 0$. In this case, $\sigma_p(H_0) \cap \Sigma_j$ is finite and $\sigma_{sc}(H_0) \cap \Sigma_j = \emptyset$. Moreover, Σ_j contains an eigenvalues $\mathbf{e}$ of H_0 if and only if there exists $i \neq j$ such that $\lambda_i(q) = \mathbf{e}$ for all $q \in [-\pi, \pi]$. Finally, if for all $j = 1, \cdots, N$ there is $p \in [-\pi, \pi]$ such that $\lambda_j'(p) \neq 0$, then the spectrum of H_0 is purely absolutely continuous, i.e. $\sigma_p(H_0) = \sigma_{sc}(H_0) = \emptyset$. We deduce immediately the following corollary:

Corollary 2.1 *If A is positive definite then $\lambda_1(p), \cdots, \lambda_N(p)$ are even on $[-\pi, \pi]$, decreasing on $[0, \pi]$ and for all $j = 1, \cdots, N$, we have*

$$\lambda_j'(p) = -2\langle AW_j(p), W_j(p)\rangle_{\mathbb{C}^N} \sin p , \quad \text{for all } p \in [-\pi, \pi].$$

In particular, $\Sigma_j = [\lambda_j(\pi), \lambda_j(0)]$, $\kappa(\lambda_j) = \{\lambda_j(\pi), \lambda_j(0)\}$ and the spectrum of H_0 is purely absolutely continuous with $\sigma(H_0) = \cup_{1 \leq j \leq N}[\lambda_j(\pi), \lambda_j(0)]$.

Corollary 2.2 *If A is a symmetric invertible matrix then $\lambda_1(p), \cdots, \lambda_N(p)$ are non constant even functions on $[-\pi, \pi]$. In particular, the spectrum of H_0 is purely absolutely continuous.*

Proof Assume by contradiction that $\lambda_1(p) = c$ for all $p \in [-\pi, \pi]$ for some constant c. Then for any $p \in [-\pi, \pi]$ there exists $W(p)$ an eigenvector of $h(p)$ associated to c so that $h(p)W(p) = cW(p)$. Hence for any $p \in [-\pi, \pi]$, $(2\cos p)AW(p) = [(c - B)A^{-1}]AW(p)$, which means that $[-2, 2] \subset \sigma((c - B)A^{-1})$ which is impossible.

Corollary 2.3 *Assume that A is a symmetric matrix such that for any eigenvector w of B, $\langle Aw, w\rangle \neq 0$. Then the conclusion of Corollary 2.2 holds true.*

Proof Here remark that $h(\pi/2) = B$. So $\lambda_j(\pi/2)$ is an eigenvalue of B and $W_j(\pi/2)$ is an associated eigenvector. So $\lambda_j'(\pi/2) = -2\langle AW_j(\pi/2), W_j(\pi/2)\rangle \neq 0$, by hypothesis. The proof is complete.

Corollary 2.4 *Assume that for any eigenvector w of $i(A - A^*) + B$ one has $\langle (A + A^*)w, w\rangle \neq 0$. Then $\lambda_1(p), \cdots, \lambda_N(p)$ are non constant on $[-\pi, \pi]$. In particular, the spectrum of H_0 is purely absolutely continuous and there exist $\alpha_1 < \beta_1, \cdots, \alpha_n < \beta_n$ such that $\sigma(H_0) = \cup_{1 \leq j \leq N}[\alpha_j, \beta_j]$.*

Proof It is enough to remark that $h(-\pi/2) = i(A - A^*) + B$.

We close this section by few explicit examples to illustrate these consideration and our comment just after Proposition 1.1 on the spectrum of H_0.

Example 2.1 *Assume that A and B are both diagonals with diagonal elements $A_{jj} = a_j \geq 0$ and $B_{jj} = b_j \in \mathbb{R}$. In this case, the eigenvalues of the reduced operators $h(p)$ are given by $\lambda_j(p) = b_j + 2a_j \cos p$. Therefore, we immediately see that by playing with the parameters a_j and b_j one may realize whatever we said with the associated spectral bands $\Sigma_j = [b_j - 2a_j, b_j + 2a_j]$.*

Example 2.2 *Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & a \\ a & -b \end{pmatrix}$ where $a, b > 0$. In this case, the eigenvalues of $h(p)$ are given, for all $p \in [-\pi, \pi]$, by*

$$\lambda_j(p) = (-1)^j \sqrt{b^2 + (a + 2\cos p)^2} \quad \text{with } j = 1, 2.$$

Put $\alpha = \sqrt{b^2 + (a - 2)^2}$ and $\beta = \sqrt{b^2 + (a + 2)^2}$. Hence we have the following.

1. *If $a \geq 2$ then λ_1 is increasing on $[0, \pi]$ while λ_2 is decreasing and $\kappa(H_0) = \{\lambda_j(0), \lambda_j(\pi)\}_{j=1,2}$. Moreover, the spectrum of H_0 is purely absolutely continuous and consists of two spectral bands with a gap between them of length 2α and $\sigma(H_0) = [-\beta, -\alpha] \cup [\alpha, \beta]$.*

2. *If $0 < a < 2$ then the critical points of λ_1 and λ_2 are $0, \pi$ plus an additional point $p_0 \in \left]\pi/2, \pi\right[$ defined by $\cos(p_0) = -a/2$. In such case, $\sigma(H_0) = [-\beta, -b] \cup [b, \beta]$ and $\kappa(H_0) = \{\lambda_j(0), \lambda_j(p_0), \lambda_j(\pi)\}_{j=1,2}$. So we have two spectral bands with a gap between them of length $2b$.*

Example 2.3 *Let $A = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}, B = \begin{pmatrix} b & a_1 \\ a_1 & -b \end{pmatrix}$ where $a_1, a_2 > 0$ and $b \in \mathbb{R}$. In this case, the eigenvalues of reduced operators $h(p)$ are given by*

$$\lambda_j(p) = (-1)^j \sqrt{b^2 + a_1^2 + a_2^2 + 2a_1 a_2 \cos p} , \quad j = 1, 2.$$

Then $\lambda_1(p)$ is an increasing function of $p \in [0, \pi]$ while $\lambda_2(p)$ is decreasing and $\lambda_1(p) < \lambda_2(p)$, for all $p \in]0, \pi[$. Therefore, $\kappa(H_0) = \{\lambda_1(0), \lambda_1(\pi), \lambda_2(\pi), \lambda_2(0)\}$ and $\sigma(H_0) = [\lambda_1(0), \lambda_1(\pi)] \cup [\lambda_2(\pi), \lambda_2(0)]$. This spectrum is purely absolutely continuous and consists of two spectral bands with a gap between them of length $L = \lambda_2(\pi) - \lambda_1(\pi) = 2\sqrt{b^2 + (a_1 - a_2)^2}$. That gap is degenerate (i.e. $L = 0$), if and only if, $b = 0$ and $a_1 = a_2$. In such case, the two spectral bands have one common point $\lambda_1(\pi) = \lambda_2(\pi) = 2a$ which is a double eigenvalue of $h(\pi)$.

Mention that λ_1 and λ_2 are identically constant on $[-\pi, \pi]$ if, and only if, $a_1 a_2 = 0$. For example, if $a_2 > 0$ and $a_1 = 0$ then the spectrum of H_0 consists of two infinitely degenerate eigenvalues $\pm\sqrt{b^2 + a_2^2}$.

3 The conjugate operator theory

The following brief review on the conjugate operator theory is based on [1, 4, 5, 26]. Let $\mathbb{A}$ be a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$ and $S \in B(\mathcal{H})$.

Definition 3.1 *(i) Let $k \geq 1$ be an integer and $\sigma > 0$. We say that S is of class $C^k(\mathbb{A})$, respectively of class $\mathcal{C}^\sigma(\mathbb{A})$, if the map*

$$t \longmapsto S(t) = e^{-i\mathbb{A}t} S e^{i\mathbb{A}t} \in B(\mathcal{H})$$

is strongly of class C^k, respectively of class Λ^σ on $\mathbb{R}$.
(ii) We say that S is of class $\mathcal{C}^{1,1}(\mathbb{A})$ if

$$\int_0^1 \|e^{-i\mathbb{A}\varepsilon} S e^{i\mathbb{A}\varepsilon} - 2S + e^{i\mathbb{A}\varepsilon} S e^{-i\mathbb{A}\varepsilon} S\| \frac{d\varepsilon}{\varepsilon^2} < \infty.$$

Remarks (i) We have the following inclusions $\mathcal{C}^s(\mathbb{A}) \subset \mathcal{C}^{1,1}(\mathbb{A}) \subset C^1(\mathbb{A})$, for any $s > 1$.

(ii) One may show that S is of class $C^1(\mathbb{A})$, if and only if, the sequilinear form defined on $D(\mathbb{A})$ by $[S, \mathbb{A}] = S\mathbb{A} - \mathbb{A}S$ has a continuous extension to $\mathcal{H}$, which we identify with the associated bounded operator in $\mathcal{H}$ (from the Riesz Lemma) that we denote by the same symbol. Moreover, $[S, i\mathbb{A}] = \frac{d}{dt}|_{t=0} S(t)$.

(iii) Recall that $\langle \mathbb{A} \rangle = (1 + \mathbb{A}^2)^{1/2}$. To prove that S is of class $\mathcal{C}^\sigma(\mathbb{A})$ it is enough to show that $\langle \mathbb{A} \rangle^\sigma S$ is bounded in $\mathcal{H}$, see for example the appendix of [6]. In particular, in the case where $\sigma = 1 + \theta$ for some $\theta > 0$, the operator S is of class $C^{1+\theta}(\mathbb{A})$ if one of the following conditions is true:
(1) $\langle \mathbb{A} \rangle^\sigma S$ is bounded in $\mathcal{H}$ or
(2) S is of class $C^1(\mathbb{A})$ and $\langle \mathbb{A} \rangle^\theta [S, i\mathbb{A}]$ is bounded in $\mathcal{H}$.

In the sequel of this section, let H be a bounded self-adjoint operator which is at least of class $C^1(\mathbb{A})$. Then $[H, i\mathbb{A}]$ defines a bounded operator in $\mathcal{H}$ that we still denote by the same symbol $[H, i\mathbb{A}]$.

We define the open set $\tilde{\mu}^{\mathbb{A}}(H)$ of real numbers x such that, for some constant $a > 0$, a neighborhood Δ of x and a compact operator K in $\mathcal{H}$, we have

$$E_H(\Delta)[H, i\mathbb{A}]E_H(\Delta) \geq a E_H(\Delta) + K. \tag{3.2}$$

The inequality (3.2) is called *the Mourre estimate* and the set of point $x \in \tilde{\mu}^{\mathbb{A}}(H)$ where it holds with $K = 0$ will be denoted by $\mu^{\mathbb{A}}(H)$.

Theorem 3.1 *The set $\tilde{\mu}^{\mathbb{A}}(H)$ contains at most a discrete set of eigenvalues of H and all these eigenvalues are finitely degenerate. Moreover, $\mu^{\mathbb{A}}(H) = \tilde{\mu}^{\mathbb{A}}(H) \setminus \sigma_p(H)$.*

Recall that $\mathcal{K}_{\mathbb{A}} := \mathcal{H}_{1/2,1}$ is the Besov space associated to $\mathbb{A}$ defined by the norm

$$\|\psi\|_{1/2,1} = \|\tilde{\theta}(\mathbb{A})\psi\| + \sum_{j=0}^{\infty} 2^{j/2} \|\theta(2^{-j}|\mathbb{A}|)\psi\|.$$

where $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(x) > 0$ if $2^{-1} < x < 2$ and $\theta(x) = 0$ otherwise, and $\tilde{\theta} \in C_0^\infty(\mathbb{R})$ with $\tilde{\theta}(x) > 0$ if $|x| < 2$ and $\tilde{\theta}(x) = 0$ otherwise. One has, see [4, 26]:

Theorem 3.2 *Assume that H is of class $\mathcal{C}^{1,1}(\mathbb{A})$. Then H has no singular continuous spectrum in $\mu^{\mathbb{A}}(H)$. Moreover, the limits $(H - x \mp i0)^{-1} := \lim_{\mu \to 0+}(H - x \mp i\mu)^{-1}$ exist locally uniformly on $\mu^{\mathbb{A}}(H)$ for the weak* topology of $B(\mathcal{K}_{\mathbb{A}}, \mathcal{K}_{\mathbb{A}}^*)$.*

We also have the following:

Theorem 3.3 *Assume that H is of class $\mathcal{C}^{s+\frac{1}{2}}(\mathbb{A})$ for some $s > 1/2$. Then the maps $x \mapsto \langle \mathbb{A} \rangle^{-s}(H - x \mp i0)^{-1}\langle \mathbb{A} \rangle^{-s}$ are locally of class $\Lambda^{s-\frac{1}{2}}$ on $\mu^{\mathbb{A}}(H)$. Moreover, for every $\varphi \in C_0^{\infty}(\mu^{\mathbb{A}}(H))$ we have*

$$||\langle \mathbb{A} \rangle^{-s} e^{-iHt} \varphi(H) \langle \mathbb{A} \rangle^{-s}|| \leq C < t >^{-(s-\frac{1}{2})}.$$

Since $\|e^{-iHt}\varphi(H)\| = \|\varphi(H)\|$, then by using the complex interpolation one obtains much more than this, see [4, 5, 26]. For example, if H is of class $C^{\infty}(\mathbb{A})$, then, for any $\varphi \in C_0^{\infty}(\mu^{\mathbb{A}}(H))$ and $\sigma, \varepsilon > 0$, we have

$$||\langle \mathbb{A} \rangle^{-\sigma} e^{-iHt} \varphi(H) \langle \mathbb{A} \rangle^{-\sigma}|| \leq C(1 + |t|)^{-\sigma+\varepsilon}, \quad \text{for all } t \in \mathbb{R}$$

4 Mourre estimate for λ_j

Here we denote the operator of multiplication by a function f in $L^2([-\pi, \pi])$ by the same symbol f or by $f(p)$ when we want to stress the p-dependence.

According to Section 2, H_0 is unitarily equivalent to $\oplus_{j=1}^{N} \lambda_j$. So it is convenient to prove first the Mourre estimate for each multiplication operator λ_j. So, let us fix $j \in \{1, \cdots, N\}$ and put $\lambda = \lambda_j$. Recall that the set $\kappa(\lambda)$ of critical values of λ is finite. Let $\Delta \subset \lambda([-\pi, \pi])$ be a compact set, such that $\Delta \cap \kappa(\lambda) = \emptyset$. Obviously, on the set $\lambda^{-1}(\Delta)$ one has $|\lambda'(p)|^2 \geq c > 0$. One may then construct a function $F \in C^{\infty}([-\pi, \pi])$, such that

$$F(p)\lambda'(p) \geq c > 0 \quad \text{on } \lambda^{-1}(\Delta). \tag{4.1}$$

For example, $F = \lambda'$ perfectly fulfills this role. Nevertheless, in some concrete situations more appropriate choices can be made, see next sections. We are now able to define our conjugate operator $\mathbf{a}$ to λ on $L^2([-\pi, \pi])$ by,

$$\mathbf{a} = \frac{i}{2}\{F(p)\frac{d}{dp} + \frac{d}{dp}F(p)\} = iF(p)\frac{d}{dp} + \frac{i}{2}F'(p). \tag{4.2}$$

It is clear that $\mathbf{a}$ is an essentially self-adjoint operator in $L^2([-\pi, \pi])$. Moreover, direct computation yields to

$$[\lambda, i\mathbf{a}] = F(p)\lambda'(p).$$

So λ is of class $C^1(\mathbf{a})$, and by repeating the same calculation, we show that λ is of class $C^{\infty}(\mathbf{a})$. Moreover, thanks to (4.1), $\mathbf{a}$ is strictly conjugate to λ on Δ:

$$E_{\lambda}(\Delta)[\lambda, i\mathbf{a}]E_{\lambda}(\Delta) = E_{\lambda}(\Delta)\big(F(p)\lambda'(p)\big)E_{\lambda}(\Delta) \geq cE_{\lambda}(\Delta) \tag{4.3}$$

From now on we put $F = \lambda'$. We have:

Proposition 4.1 *The operator λ is of class $C^{\infty}(\mathbf{a})$ and $\mathbf{a}$ is locally strictly conjugate to λ on $\mathbb{R} \setminus \kappa(\lambda)$, i.e. $\mu^{\mathbf{a}}(\lambda) = \mathbb{R} \setminus \kappa(\lambda)$.*

Proof First, recall that $\mathbb{R} \setminus \sigma(\lambda) \subset \mu^{\mathbf{a}}(\lambda)$. So, if λ is constant then the assertion is trivial, since $\sigma(\lambda) = \kappa(\lambda)$. Suppose that λ is non constant and let $x \in \sigma(\lambda) \setminus \kappa(\lambda)$. Then choose a compact interval Δ about x sufficiently small so that $\Delta \cap \kappa(\lambda) = \emptyset$. The last discussion shows that $\mathbf{a}$ is strictly conjugate to λ on Δ, i.e. $x \in \mu^{\mathbf{a}}(\lambda)$. The proof is complete.

5 Mourre estimate for H_0

The direct sum $M = \oplus_{1 \leq j \leq N} \lambda_j$ is a self-adjoint bounded operator in the Hilbert space $\oplus_{1 \leq j \leq N} L^2([-\pi, \pi])$. Recall that $\kappa(H_0) = \cup_{1 \leq j \leq N} \kappa(\lambda_j)$ is finite. For any $j = 1, \cdots, N$, let us denote by $\mathbf{a}_j$ the operator given by (4.2), with $F = F_j$ constructed in the last section with $\lambda = \lambda_j$, that is

$$\mathbf{a}_j = \frac{i}{2}\{F_j(p)\frac{d}{dp} + \frac{d}{dp}F_j(p)\} = iF_j(p)\frac{d}{dp} + \frac{i}{2}F_j'(p). \tag{5.1}$$

and define the operator

$$\widetilde{A} = \oplus_{1 \leq j \leq N} \mathbf{a}_j. \tag{5.2}$$

Proposition 5.1 *The operator M is of class $C^{\infty}(\widetilde{A})$ and $\widetilde{A}$ is locally strictly conjugate to M on $\mathbb{R} \setminus \kappa(H_0)$, i.e. $\mu^{\widetilde{A}}(M) = \mathbb{R} \setminus \kappa(H_0)$.*

Proof: Let $x \in \mathbb{R} \setminus \kappa(H_0)$. By definition $\kappa(H_0) = \cup_{1 \leq j \leq N} \kappa(\lambda_j)$, and according to Proposition 4.1, $\mu^{\mathbf{a}_j}(\lambda_j) = \mathbb{R} \setminus \kappa(\lambda_j)$. Hence, $x \in \cap_{1 \leq j \leq N} \mu^{\mathbf{a}_j}(\lambda_j)$. Then for any $j = 1, \cdots, N$, there exists a constant $c_j > 0$ and a compact interval Δ_j about x such that

$$E_{\lambda_j}(\Delta_j)[\lambda_j, i\mathbf{a}_j]E_{\lambda_j}(\Delta_j) \geq c_j E_{\lambda_j}(\Delta_j).$$

But, for any Borel set J, one has $E_M(J) = \oplus_{1 \leq j \leq N} E_{\lambda_j}(J)$ and $[M, i\tilde{A}] = \oplus_{1 \leq j \leq N}[\lambda_j, i\mathbf{a}_j]$, Hence, we immediately conclude that, for $\Delta = \cap_{j=1}^N \Delta_j$ and $c_j = \min_{1 \leq j \leq N} c_j$ we have,

$$E_M(\Delta)[M, i\tilde{A}]E_M(\Delta) \geq c E_M(\Delta),$$

that is, $x \in \mu^{\tilde{A}}(M)$. The proof is complete.

According to the Proposition 2.1, $H_0 = \mathcal{U}^{-1}M\mathcal{U}$. Define the operator $\mathbb{A}$ by

$$\mathbb{A} = \mathcal{U}^{-1}\widetilde{A}\mathcal{U}. \tag{5.3}$$

By a direct application of the previous proposition, we get

Corollary 5.1 *The operator H_0 is of class $C^{\infty}(\mathbb{A})$ and $\mathbb{A}$ is locally strictly conjugate to H_0 on $\mathbb{R} \setminus \kappa(H_0)$, i.e. $\mu^{\mathbb{A}}(H_0) = \mathbb{R} \setminus \kappa(H_0)$.*

In particular, the spectrum of H_0 is purely absolutely continuous on $\mathbb{R} \setminus \kappa(H_0)$. In fact, combining this corollary, Lemma 6.1, and the results of Section 3, we immediately get,

Theorem 5.1 *(i) The limits $(H_0 - x \mp i0)^{-1} := \lim_{\mu \to 0+}(H_0 - x \mp i\mu)^{-1}$ exist locally uniformly on $\mathbb{R} \setminus \kappa(H_0)$ for the weak* topology of $B(\mathcal{K}, \mathcal{K}^*)$.*
(ii) For any $s > 1/2$, the maps $x \mapsto \langle \mathbf{N} \rangle^{-s}(H_0 - x \mp i0)^{-1}\langle \mathbf{N} \rangle^{-s} \in B(\mathcal{H})$ are locally of class $\Lambda^{s-\frac{1}{2}}$.
(iii) For any $\varphi \in C_0^{\infty}(\mathbb{R} \setminus \kappa(H_0))$ and $\sigma, \varepsilon > 0$.

$$||\langle \mathbf{N} \rangle^{-\sigma} e^{-iH_0 t}\varphi(H_0)\langle \mathbf{N} \rangle^{-\sigma}|| \leq C(1 + |t|)^{-\sigma+\varepsilon}.$$

6 Mourre estimate for H

In this section we extend our analysis to the operator H. For we start by recalling that $H = H_0 + V$ where V is the difference operator acting in $\mathcal{H}$ by

$$(V\psi)_n = (A_{n-1} - A)^*\psi_{n-1} + (B_n - B)\psi_n + (A_n - A)\psi_{n+1}, \quad \text{for all } n \in \mathbb{Z}. \tag{6.1}$$

According to (1.2), V is a compact operator in $\mathcal{H}$. In particular,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \cup_{j=1}^{j=N}\lambda_j([-\pi, \pi])$$

The main result of this section is the extension, up to a compact operator, of the Mourre estimate obtained in the last section for H_0 to H. In fact we will prove the following more general result,

Theorem 6.1 *Let V be a self-adjoint compact operator on $\mathcal{H}$ (not necessarily of the form (6.1)) such that $V < \boldsymbol{N} >$ is also compact. Then the self-adjoint operator $H = H_0 + V$ is of class $C^1(\mathbb{A})$ and $\mathbb{A}$ is locally conjugate to H on $\mathbb{R} \setminus \kappa(H_0)$, i.e. $\tilde{\mu}^{\mathbb{A}}(H) = \mu^{\mathbb{A}}(H_0) = \mathbb{R} \setminus \kappa(H_0)$. In particular, outside the finite set $\kappa(H_0)$ the eigenvalues of $H_0 + V$ are all finitely degenerate and their possible accumulation points are included in $\kappa(H_0)$.*

Proof of Theorem 6.1 According to the last section, the operator H_0 is of class $C^{\infty}(\mathbb{A})$ with $\mathbb{A} = \mathcal{U}^{-1} \widetilde{A} \mathcal{U}$, and $\widetilde{A}$ is defined by (5.1) and (5.2). So to prove Theorem 6.1 we first show that V is of class $C^1(\mathbb{A})$ and that $[V, i\mathbb{A}]$ is a compact operator in $\mathcal{H}$.

According to Lemma 6.1 below, the operator $\langle \boldsymbol{N} \rangle^{-1} \mathbb{A}$ is bounded in $\mathcal{H}$. Hence, $V\mathbb{A} = V\langle \boldsymbol{N} \rangle (\langle \boldsymbol{N} \rangle^{-1} \mathbb{A})$ is a compact operator in $\mathcal{H}$. By adjonction, $\mathbb{A}V$ is also a compact operator. Thus, the commutator $[V, i\mathbb{A}]$ is a compact operator in $\mathcal{H}$. Moreover, for any Borel set J, $E_H(J) - E_{H_0}(J)$ is a compact operator. Hence, for any $J \subset \mu^{\mathbb{A}}(H_0)$,

$$
\begin{aligned}
E_H(J)[H, i\mathbb{A}]E_H(J) &= E_H(J)[H_0, i\mathbb{A}]E_H(J) + E_H(J)[V, i\mathbb{A}]E_H(J) \\
&= E_{H_0}(J)[H_0, i\mathbb{A}]E_{H_0}(J) + K \\
&\geq cE_{H_0}(J) + K \\
&\geq cE_H(J) + K'
\end{aligned}
$$

for some $c > 0$ and compact operators K, K'. The proof is finished.

Lemma 6.1 *For every integer $m > 0$ the operators $\langle \boldsymbol{N} \rangle^{-m} \mathbb{A}^m$ and $\mathbb{A}^m \langle \boldsymbol{N} \rangle^{-m}$ are bounded in $\mathcal{H}$.*

Proof: The proof can be done by induction. We only illustrate the idea for $m = 2$ and deal with $\langle \boldsymbol{N} \rangle^{-2} \mathbb{A}^2$ (the remaining cases are similar). Let us show that this operator is bounded in $\mathcal{H}$.

We check that $\mathbb{A}^2 = \mathcal{U}^{-1} \widetilde{A}^2 \mathcal{U}$ is given by

$$
(\mathbb{A}^2 \psi)_n = \sum_{j=1}^{N} \int_{-\pi}^{\pi} \left[G_j^0(p) \frac{d^2 f_j}{dp^2}(p) + +G_j^1(p)\frac{df_j}{dp}(p) + G_j^2(p)f_j(p) \right] W_j(p) e^{-ipn} \, dp,
$$

where f_j is defined by (2.1), and G_j^0, G_j^1 and G_j^2 are smooth functions on $[-\pi, \pi]$. Integrations by parts show that $(\langle \boldsymbol{N} \rangle^{-2} \mathbb{A}^2 \psi)_n$ is a finite sum of terms of the form

$$
\langle n \rangle^{-2}(an^2 + bn + c) \int_{-\pi}^{\pi} f_j(p)W(p) e^{-ipn} \, dp,
$$

where a, b and c are constants, and W is a smooth function. Since, on the one hand, $\langle n \rangle^{-2}(an^2 + bn + c)$ is bounded, and using, on the other hand, Plancherel's theorem:

$$
\begin{aligned}
\left\| \left(\int_{-\pi}^{\pi} f_j(p)W(p)e^{-ipn} \, dp \right)_n \right\|_{l^2(\mathbb{Z})}^2 &= \|f_j(p)W(p)\|_{L^2([-\pi,\pi])}^2 \\
&\leq C\|f_j(p)\|_{L^2([-\pi,\pi])}^2 \\
&\leq C\|\psi\|_{\mathcal{H}}^2,
\end{aligned}
$$

we obtain $\|\langle \boldsymbol{N} \rangle^{-2} \mathbb{A}^2 \psi\|_{\mathcal{H}}^2 \leq C'\|\psi\|_{\mathcal{H}}^2$, for some constants $C, C' > 0$.

Proof of Theorem 1.1 First, the assumption (1.8) implies that $V\langle \boldsymbol{N} \rangle$ is a compact operator in $\mathcal{H}$. Hence Theorem 1.1 is an immediate consequence of Theorem 6.1 and Theorem 3.1.

Proof of Theorem 1.2 Since condition (1.9) implies (1.8), $\tilde{\mu}^{\mathbb{A}}(H) = \mathbb{R} \setminus \kappa(H_0)$. So, according to Theorem 3.2 and Lemma 6.1, we only have to prove that H is of class $\mathcal{C}^{1,1}(\mathbb{A})$. As in the last proof, it is enough to show that V is of class $\mathcal{C}^{1,1}(\mathbb{A})$. But this is true according to the appendix of [6]. The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3 As in the precedent proof, according to Theorem 3.3, it is enough to show that the condition (1.11) implies that V is of class $\mathcal{C}^{1+\theta}(\mathbb{A})$. But (1.11) ensures that the operator

$\langle \mathbf{N} \rangle^{1+\theta} V$ is bounded in $B(\mathcal{H})$. The preceding lemma shows that $\langle \mathbb{A} \rangle^{1+\theta} V = (\langle \mathbb{A} \rangle^{1+\theta} \langle \mathbf{N} \rangle^{-(1+\theta)}) \langle \mathbf{N} \rangle^{1+\theta} V$ is bounded too. The proof of Theorem 1.3 is finished by the point (ii) of the remark given just after Definition 3.1.

Remark Here also it should be clear that Theorem 1.2 and Theorem 1.3 are valid for $H = H_0 + V$ where the perturbation V is any bounded operator in $\mathcal{H}$ that satisfies one of the following:
(i) $\langle \mathbf{N} \rangle^{1+\theta} V \in B(\mathcal{H})$ for some $\theta > 0$, or
(ii) V is a compact operator on $\mathcal{H}$ such that $\langle \mathbf{N} \rangle^{\theta} [V, \mathbb{A}] \in B(\mathcal{H})$.

7 The case where A is positive definite

In this section we focus on the case where H is the block Jacobi operator defined by (1.1)-(1.2) such that A is positive definite. Then, thanks to the Corollary 2.1, we have:

1. for all $j = 1, \cdots, N$, we have

$$\lambda'_j(p) = -2 < AW_j(p), W_j(p) >_{\mathbb{C}^N} \sin p, \text{ for all } p \in [-\pi, \pi].$$

2. the eigenvalues $\lambda_1(p), \cdots, \lambda_N(p)$ are even on $[-\pi, \pi]$ and decreasing on $[0, \pi]$. In particular, $\Sigma_j = [\lambda_j(\pi), \lambda_j(0)]$ and $\kappa(\lambda_j) = \{\lambda_j(\pi), \lambda_j(0)\}$.

3. the spectrum of H_0 is purely absolutely continuous and

$$\sigma(H_0) = \cup_{1 \le j \le N} [\lambda_j(\pi), \lambda_j(0)].$$

Moreover, according to the last section, the operator H_0 is of class $C^{\infty}(\mathbb{A})$ and $\mu^{\mathbb{A}}(H_0) = \mathbb{R} \setminus \kappa(H_0)$, with $\mathbb{A} = \mathcal{U}^{-1} \widetilde{A} \mathcal{U}$, and $\widetilde{A}$ is defined by (5.1) and (5.2). The advantage now is that, one may choose

$$F_j(p) = -\sin p, j = 1, \cdots, N,$$

in the definition of the operator $\mathbb{A}$. Indeed,

$$F_j(p) \lambda'_j(p) = (-\sin p) \lambda'_j(p) = 2 < AW_j(p), W_j(p) >_{\mathbb{C}^N} (\sin p)^2,$$

which is strictly positive on $]0, \pi[$. But then, with this choice, one may easily show that

$$\mathbb{A} = \mathbb{D} + L \tag{7.2}$$

for some bounded operator L in $\mathcal{H}$ and where $\mathbb{D}$ is defined in $\mathcal{H}$ by,

$$(\mathbb{D}\psi)_n = i(n + \frac{1}{2})\psi_{n+1} - i(n - \frac{1}{2})\psi_{n-1} \tag{7.3}$$

This allows us to handle the larger class of perturbations V satisfying

$$\lim_{|n| \to \infty} |n|(\|A_{n+1} - A_n\| + \|B_{n+1} - B_n\|) = 0. \tag{7.4}$$

Remark that the condition (1.8) implies (7.4) but not the inverse, think to the example $A_n = A + \frac{1}{\ln(1+|n|)} A'$, with any matrix A'. So we get the following more general result in comparison with Theorem 1.1:

Theorem 7.1 *If (7.4) holds then H is of class $C^1(\mathbb{A})$ and $\tilde{\mu}^{\mathbb{A}}(H) = \mathbb{R} \setminus \{\alpha_j, \beta_j\}_{1 \le j \le N}$. In particular, the possible eigenvalues of H in $\mathbb{R} \setminus \{\alpha_j, \beta_j\}_{1 \le j \le N}$ are all finitely degenerate and cannot accumulate outside $\{\alpha_j, \beta_j\}_{1 \le j \le N}$.*

Proof The only point we have to verify is the compactness of $[V, i\mathbb{A}]$ in $\mathcal{H}$ under the conditions (1.2) and (7.4). Indeed,

$$[V, i\mathbb{A}] = [V, i\mathbb{D}] + [V, iL].$$

But $[V, iL] = i(VL - LV)$ is clearly a compact operator since each term is (V being compact and L bounded). Moreover, a direct computation shows that the only nonzero matrix coefficients of the commutator $K = [V, i\mathbb{D}]$ are:

$K_{n,n+2} = K^*_{n+2,n} = (n + \frac{1}{2})(A_{n+1} - A_n) - (A_{n+1} - A);$

$K_{n,n+1} = K^*_{n+1,n} = (n + \frac{1}{2})(B_{n+1} - B_n);$

$K_{n,n} = (n + \frac{1}{2})[(A_n - A_{n-1}) + (A_n - A_{n-1})^*] + (A_{n-1} - A) + (A_{n-1} - A)^*.$

All these coefficients tend to zero at infinity, thanks to (1.2) and (7.4). Hence, $[V, i\mathbb{D}]$ is a compact operator and the proof is complete.

Remark: In addition of (1.2) suppose that

$$\int_1^\infty \sup_{r < |n| < 2r} \left(|n|(\|A_n - A_{n-1}\| + \|B_n - B_{n-1}\|) + \|A_n - A\| \right) \frac{dr}{r} < \infty. \tag{7.5}$$

Then according to the appendix of [6] we get H is of class $\mathcal{C}^{1,1}(\mathbb{A})$ (in fact we have slightly more than that but we will not go in these details here). Remark that (7.5) follows if, for some $\theta > 0$, we have

$$\sup_{n \in \mathbb{Z}} |n|^\theta \left(|n|(\|A_n - A_{n-1}\| + \|B_n - B_{n-1}\|) + \|A_n - A\| \right) < \infty. \tag{7.6}$$

In this case it is clear that $\langle \mathbf{N} \rangle^\theta [V, i\mathbb{A}]$ is bounded. Hence $\langle \mathbb{A} \rangle^\theta [V, i\mathbb{A}] = (\langle \mathbb{A} \rangle^\theta \langle \mathbf{N} \rangle^{-\theta}) \langle \mathbf{N} \rangle^\theta [V, i\mathbb{A}]$ is bounded too. Consequently, H is of class $\mathcal{C}^{1+\theta}(\mathbb{A})$ so that, we have

Theorem 7.2 *Assume (1.2). Theorems 1.2, respectively Theorem 1.3, remain valid for this model if we replace the condition (1.9), respectively (1.11), by (7.5), respectively (7.6).*

Example 7.1 *Let us consider the Jacobi operator J acting in $l^2(\mathbb{Z}, \mathbb{C})$ by*

$$(J\psi)_n = a_{n-1}\psi_{n-1} + b_n\psi_n + a_n\psi_{n+1}, \quad \text{for all } n \in \mathbb{Z}.$$

Here-above $a_n > 0$ and $b_n \in \mathbb{R}$. Assume that there exist $a^\pm > 0, b^\pm \in \mathbb{R}$ such that,

$$\lim_{n \to \pm\infty} a_n = a^\pm \quad and \quad \lim_{n \to \pm\infty} b_n = b^\pm. \tag{7.7}$$

This model has been studied in [7] where the authors used some ideas of three body problem to establish their Mourre estimate. Here our present approach applies in straightforward way. Indeed, Let $U : l^2(\mathbb{Z}) \to l^2(\mathbb{N}, \mathbb{C}^2)$ defined by $(U\psi)_n = (\psi_n, \psi_{-n+1})$. It is clear, see [2], that U is unitary operator and $UJU^{-1} = H$ where H is the block Jacobi operator realized by

$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & a_{-n} \end{pmatrix}, \; n \geq 1; B_n = \begin{pmatrix} b_n & 0 \\ 0 & b_{-n+1} \end{pmatrix}, \; n \geq 2 \quad and \quad B_1 := \begin{pmatrix} b_1 & a_0 \\ a_0 & b_0 \end{pmatrix}.$$

In particular, one may deduce directly by applying the last discussion that $\sigma_{ess}(J) = [b^- - 2a^-, b^- + 2a^-] \cup [b^+ - 2a^+, b^+ + 2a^+]$. Moreover, if

$$\lim_{n \to \pm\infty} |n|^\theta \left(|n|(|a_n - a_{n+1}| + |b_n - b_{n+1}|) + |a_n - a^\pm| \right) = 0, \quad \text{for some } \theta \geq 0$$

holds then $\sigma_p(J)$ has no accumulation points outside $\kappa(J) = \{b^- \pm 2a^-, b^+ \pm 2a^+\}$. Moreover, if $\theta > 0$ then $\sigma_{sc}(J) = \emptyset$.

8 Difference operators on cylindrical domains

Consider the configuration space $X = \mathbb{Z} \times \{1, \cdots, N\}$, for some integer N, and the difference operator J acting in $l^2(X)$ by

$$(J\psi)_{n,k} = a_{n-1,k}\psi_{n-1,k} + b_{n,k}\psi_{n,k} + a_{n,k}\psi_{n+1,k} + c_{n,k-1}\psi_{n,k-1} + c_{n,k}\psi_{n,k+1} \tag{8.8}$$

initialized by $\psi_{n,0} = \psi_{n,N+1} = 0$ and where $a_{n,k}, c_{n,k}$ are positive while $b_{n,k}$ are real numbers.

Theorem 8.1 *Assume that, there is some positive numbers $a_1, \cdots, a_N, c_1, \cdots, c_{N-1}$ and real numbers $b_1, \cdots, b_N$ such that for all $j = 1, \cdots, N$,*

$$\lim_{n \to \pm\infty} |a_{n,j} - a_j| + |b_{n,j} - b_j| + |c_{n,j} - c_j| = 0 , \quad (\text{with the convention } c_{n,N} = c_N = 0).$$

Then J is a bounded symmetric operator whose essential spectrum is of the form $\cup_{j=1}^N [\alpha_j, \beta_j]$ for some numbers $\alpha_j < \beta_j$. If in addition, we have

$$\lim_{n \to \pm\infty} |n|(|a_{n+1,j} - a_{n,j}| + |b_{n+1,j} - b_{n,j}| + |c_{n+1,j} - c_{n,j}|) = 0,$$

then there is a self-adjoint operator $\mathbb{A}'$ such that $\tilde{\mu}^{\mathbb{A}'}(J) = \mathbb{R} \setminus \kappa(J)$ with $\kappa(J) = \cup_{1 \leq j \leq N} \{\alpha_j, \beta_j\}$. In particular, the possible eigenvalues of J are all finitely degenerate and cannot accumulate outside $\kappa(J)$. Finally, $\sigma_{sc}(J) = \emptyset$ and conclusions of Theorems 1.2 and 1.3 hold for J if, for some $\theta > 0$, we have

$$\sup_n |n|^\theta \Big(|n| \big(|a_{n+1,j} - a_{n,j}| + |b_{n+1,j} - b_{n,j}| + |c_{n+1,j} - c_{n,j}| \big) + |a_{n,j} - a_j| \Big) < \infty.$$

Before to show this result let us fix some notations that we will use in the sequel of this paper. Let E_{ij} be the elementary matrix whose only non zero entry is 1 placed at the intersection of the ith ligne and the jth column and

$$\text{diag}(d_1, \cdots, d_N) = \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & d_N \end{pmatrix} \tag{8.9}$$

$$\text{tridiag}(\{\alpha_i\}_{1 \leq i \leq N-1}, \{\beta_i\}_{1 \leq i \leq N}) = \begin{pmatrix} \beta_1 & \alpha_1 & 0 & \ldots & 0 \\ \alpha_1 & \beta_2 & \alpha_2 & 0 & \ldots \\ 0 & \alpha_2 & \beta_3 & \alpha_2 & \ldots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & \alpha_{N-1} & \beta_N \end{pmatrix} \tag{8.10}$$

Proof of Theorem 8.1 Let $U : l^2(\mathbb{Z} \times \{1, \cdots, N\}) \to l^2(\mathbb{Z}, \mathbb{C}^N)$ defined by $(U\psi)_n = (\psi_{n,1}, \cdots, \psi_{n,N})$. It is clear that U is unitary operator and $UJU^{-1} = H(\{A_n\}, \{B_n\})$ where

$$A_n = \text{diag}(a_{n,1}, \cdots, a_{n,N}) \quad , \quad B_n = \text{tridiag}(\{c_{n,i}\}_{1 \leq i \leq N-1}, \{b_{n,i}\}_{1 \leq i \leq N}) \tag{8.11}$$

forall $n \in \mathbb{Z}$. By hypothesis the matrices A_n and B_n satisfy (1.2) with

$$A = \text{diag}(a_1, \cdots, a_N) \quad \text{and} \quad B = \text{tridiag}(\{c_i\}_{1 \leq i \leq N-1}, \{b_i\}_{1 \leq i \leq N}) \tag{8.12}$$

Since A is positive definite, the results of the last section apply directly to the operator $H(\{A_n\}, \{B_n\})$. In addition, the conjugate operator of J is given by $\mathbb{A}' = U^{-1}\mathbb{A}U$ where $\mathbb{A}$ is defined by (7.2) and (7.3). The proof is finished.

Example 8.1 *To have explicit formulae, assume in (8.12) that*

$$a_i = a > 0, \quad b_i = 0 \quad and \quad c_i = c > 0.$$

In this case, $h(p)$ is N by N Toeplitz matrix whose spectrum is given by

$$\lambda_j(p) = 2a \cos p + 2c\epsilon_j \quad with \quad \epsilon_j = \cos \frac{(N - j + 1)\pi}{N + 1} \ , \ j = 1, \cdots, N.$$

Here again, we observe explicitly $\sigma(H_0) = \cup_{j=1}^N [\lambda_j(\pi), \lambda_j(0)]$. Remark that j^{th} gap is non trivial, if and only if,

$$L_j = \lambda_{j+1}(\pi) - \lambda_j(0) = 2c\Delta_j - 4a > 0 \quad where \quad \Delta_j = \epsilon_{j+1} - \epsilon_j > 0.$$

Then j^{th} gap is non trivial, if and only if, $c > 2a/\Delta_j$.

Example 8.2 *Discrete wave propagator.* *Consider on $X = \mathbb{Z} \times \{1, \cdots, N\}$ the discrete Laplacian Δ given by*

$$(\Delta\psi)_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1} \ , \ for \ all \ x = (n,m) \in X$$

with the initial boundary condition $\psi_{0,m} = \psi_{n,0} = \psi_{n,N+1} = 0$. Let $\rho : X \to]0, \infty[$ be the local propagation speed. The discrete wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = \rho^2 \Delta u.$$

Clearly, the change of variable $u = \rho v$ transforms the last equation to

$$\frac{\partial^2 v}{\partial t^2} = \rho \Delta \rho v,$$

Hence the spectral analysis of the operator $H = \rho \Delta \rho$ on $\mathcal{H} = l^2(X)$ is of interest for discrete wave equation. Assume that the cylinder X is stratified, i.e. for some constant $\rho_1, \cdots, \rho_N > 0$, one has $\rho(n,m) - \rho_m \to 0$ as $n \to \infty$. Hence $H = H(\{A_n\}, \{B_n\})$ given in (8.11) with

$$a_{n,m} = \rho(n,m)\rho(n+1,m) \quad , \quad b_{n,m} = 0 \quad and \quad c_{n,m} = \rho(n,m)\rho(n,m+1).$$

Of course we assume that $\rho_0 = \rho_{N+1} = 0$. This case can be clearly studied by Theorem 8.1 since

$$a_{n,m} \to a_m = \rho_m^2 \quad and \ c_{n,m} \to c_m = \rho_m \rho_{m+1} \quad as \ |n| \to \infty.$$

But we will not state the corresponding result separately.

For example, if the stratification is simple in the sense that ρ is constant, say $\rho_m = 1$ for all m. Then the essential spectrum of H has no gaps since the spectral bands Σ_j and Σ_{j+1} overlap. Indeed, with the notations of example 8.1 we have:

$$L_j = \lambda_{j+1}(\pi) - \lambda_j(0) = 2\Delta_j - 4 \le 0 \quad here \ \Delta_j = \epsilon_{j+1} - \epsilon_j > 0$$

9 A first case where A is singular

The simplest situation one may consider is a naive modification of the example considered in the previous section, namely where B is a tridiagonal matrix and A has only one nonzero coefficient on the diagonal. More specifically, let $a_1, a_2, \cdots, a_N > 0$ and $b_1, \cdots, b_N \in \mathbb{R}$ be given and H be the block Jacobi operator defined by (1.1) where A_n and B_n satisfy (1.2) with

$$A = a_N E_{N,N} \quad and \quad B = \text{tridiag}(\{a_i\}_{1 \le i \le N-1}, \{b_i\}_{1 \le i \le N}) \tag{9.1}$$

Theorem 9.1 *1. There exist $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 \cdots < \alpha_N < \beta_N$ such that the spectrum of H_0 is purely absolutely continuous and $\sigma(H_0) = \cup_{j=1}^N [\alpha_j, \beta_j]$ and $\kappa(H_0) = \{\alpha_1, \beta_1, \alpha_2, \beta_2 \cdots, \alpha_N, \beta_N\}$.*

  2. If (1.8) holds then we have a Mourre estimate for H_0 and H on $\mathbb{R} \setminus \kappa(H_0)$, i.e. $\mu^{\mathbb{A}}(H_0) = \tilde{\mu}^{\mathbb{A}}(H) = \mathbb{R} \setminus \kappa(H_0)$. In particular, the eigenvalues of H are all finitely degenerate and cannot accumulate outside $\kappa(H_0)$.

  3. If (1.9) holds then the conclusions of Theorems 1.2 and 1.3 are valid.

Mention here that the spectrum of H_0 has a band/gap structure, that is between each two successive bands $\Sigma_j = [\alpha_j, \beta_j]$ and $\Sigma_{j+1} = [\alpha_{j+1}, \beta_{j+1}]$ there is a non trivial gap of length $L_j = \alpha_{j+1} - \beta_j > 0$. To show this result we will study the eigenvalues of $h(p)$ given here by

$$h(p) = \mathrm{tridiag}(\{a_i\}_{1 \le i \le N-1}, \{b_1, \cdots, b_{N-1}, b_N + 2a_N \cos p\})$$

Since $h(-p) = h(p)$, it is enough to restrict ourselves to $p \in [0, \pi]$. Theorem 9.1 is an immediate consequence of the following:

Theorem 9.2 *Let $B_{N-1} = \mathrm{tridiag}(\{a_i\}_{1 \le i \le N-2}, \{b_i\}_{1 \le i \le N-1})$ and $\mu_1 < \mu_1 \cdots < \mu_{N-1}$ its eigenvalues. The eigenvalues $\lambda_1(p), \cdots, \lambda_N(p)$ are simple, decreasing on $[0, \pi]$ and can be chosen so that, for all $p \in [0, \pi]$,*

$$\lambda_1(p) < \mu_1 < \lambda_2(p) \cdots < \mu_{N-1} < \lambda_N(p). \tag{9.2}$$

In particular, $\lambda_j([0, \pi]) = [\alpha_j, \beta_j]$ with $\alpha_j = \lambda_j(\pi)$ and $\beta_j = \lambda_j(0)$. Moreover, $\kappa(\lambda_j) = \{\alpha_j, \beta_i\}$.

Proof Let $p \in [0, \pi]$. The matrix $h(p)$ is symmetric tridiagonal with positive off-diagonal elements and its characteristic polynomial satisfies

$$D(x, p) = \det(h(p) - x) = (b_N + 2a_N \cos p - x)D_{N-1}(x) - a_{N-1}^2 D_{N-2}(x),$$

where $D_{N-j}(x) = \det(B_{N-j} - x)$ and B_{N-j} is obtained from $h(p)$, and so from B, by eliminating the last j rows and columns. Moreover, it is known that $D(x, p)$ has N distincts real roots that are separated by the $N - 1$ distinct roots of $D_{N-1}(x)$. So (9.2) is proved. Moreover, direct calculation shows that

$$\frac{\partial D}{\partial p}(\lambda_j(p), p) = -2a_N(\sin p)D_{N-1}(\lambda_j(p)) \ne 0 , \quad \text{for all } p \in]0, \pi[.$$

Thus by differentiating the identity $D(\lambda_j(p), p) = 0$ we get

$$\lambda_j'(p)\frac{\partial D}{\partial x}(\lambda_j(p), p) + \frac{\partial D}{\partial p}(\lambda_j(p), p) = 0$$

and so $\lambda_j'(p)$ is non zero on $]0, \pi[$. Finally, because the matrix A is non negative we have, for all $j = 1, \cdots, N$,

$$\lambda_j'(p) = -2 < AW_j(p), W_j(p) >_{\mathbb{C}^N} \sin p \le 0, \quad \text{for all } p \in [0, \pi].$$

Hence $\lambda_j'(p) < 0$ for all $p \in]0, \pi[$. We see that $\kappa(\lambda_i) = \{\lambda_j(0), \lambda_j(\pi)\}$. The proof is finished.

  Remark: Mention that if we put the nonzero coefficient at the first place of the diagonal instead of the last one then exactly the same phenomenon happens. In contrast, if we put it in a different place of the diagonal then some drastic changes may arise. For example, assume that B is the same and $A = a_N E_{N-1, N-1}$. Hence

$$h(p) = \mathrm{tridiag}(\{a_i\}_{1 \le i \le N-1}, \{b_1, \cdots, b_{N-2}, b_{N-1} + 2a_N \cos p, b_N\})$$

In this case, on one hand $D(x, p) = \det(B - x) + 2a_N \cos p(b_N - x)D_{N-2}(x)$. On the other hand $\det(B - x) = (b_N - x)D_{N-1}(x) - a_{N-1}^2 D_{N-2}(x)$. Hence,

$$D(x, p) = (b_N - x)\Big(D_{N-1}(x) + 2a_N \cos p(b_N - x)D_{N-2}(x)\Big) - a_{N-1}^2 D_{N-2}(x).$$

Now let us choose first $a_1, \cdots a_{N-3}$ and $b_1, \cdots, b_{N-2}$ then pick b_N among the eigenvalues of B_{N-2}. So $b_N - x$ divide $D_{N-2}(x)$. Hence, b_N is a root of $D(x, p)$ for all p and therefore H_0 has a degenerate spectral band and b_N is an infinitely degenerate eigenvalue of H_0.

Example 9.1 *Let $N = 2$ so that $A = a_2 E_{2,2}$, $B = \text{tridiag}(\{a_1\}, \{b_1, b_2\})$ where $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$. It is easy to check that $\sigma(H_0) = [\lambda_1(\pi), \lambda_1(0)] \cup [\lambda_2(\pi), \lambda_2(0)]$ and that $\lambda_1(0) < b_1 < \lambda_2(\pi)$.*

Let us consider now the inverse problem of this model. More specifically, let $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ be given and find $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$ such that the corresponding H_0 satisfies $\sigma(H_0) = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$. First, if there is a solution then, according to Theorem 9.1 and Theorem 9.2, α_1, α_2 are the eigenvalues of $h(\pi)$ and β_1, β_2 are the eigenvalues of $h(0)$. Thus

$$\alpha_1 + \alpha_2 = b_1 + b_2 - 2a_2 \quad , \quad \alpha_1 \alpha_2 = b_1(b_2 - 2a_2) - a_1^2$$
$$\beta_1 + \beta_2 = b_1 + b_2 + 2a_2 \quad , \quad \beta_1 \beta_2 = b_1(b_2 + 2a_2) - a_1^2$$

Hence $4a_2 = (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2)$ and $4a_2 b_1 = \beta_1 \beta_2 - \alpha_1 \alpha_2$ gives b_1 while $2(b_2 + b_1) = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$ leads to b_2. Finally we get $2a_1^2 = 2b_1 b_2 - \alpha_1 \alpha_2 - \beta_1 \beta_2$ which gives a_1. Of course we have to make sur that $2b_1 b_2 - \alpha_1 \alpha_2 - \beta_1 \beta_2 > 0$ which holds if and only if $\beta_1 < \alpha_2$.

10   A second case where A is singular: Periodic Jacobi operators

In this part we look at the case where the only nonzero entry of A is put at the lower left corner and B is a tridiagonal matrix. It turns out that scalar Jacobi operators with periodic coefficients are covered by this model. More precisely, let $H_0 = H_0(A, B)$ defined by (1.3) with

$$A = a_N E_{N,1} \ , \ B = \text{tridiag}(\{a_i\}_{1 \le i \le N-1}, \{b_i\}_{1 \le i \le N}). \tag{10.1}$$

Then we have

Theorem 10.1   *1. There exist $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \cdots \le \alpha_N < \beta_N$ such that the spectrum of H_0 is purely absolutely continuous and $\sigma(H_0) = \cup_{j=1}^N [\alpha_j, \beta_j]$.*

2. Define the critical set $\kappa(H_0) = \{\alpha_1, \beta_1, \alpha_2, \beta_2 \cdots, \alpha_N, \beta_N\}$. There exists a conjugate operator $\mathbb{A}$ for H_0 i.e. $\mu^{\mathbb{A}}(H_0) = \mathbb{R} \setminus \kappa(H_0)$. In particular, conclusions of Theorem 5.1 hold for H_0.

3. Let V be a symmetric compact operator. Then the sum $H = H_0 + V$ defines a bounded self-adjoint operator in $\mathcal{H}$ whose essential spectrum coincides with $\sigma(H_0)$. Moreover, if $\langle N \rangle V$ is compact then $\tilde{\mu}^{\mathbb{A}}(H) = \mathbb{R} \setminus \kappa(H_0)$. In particular, the eigenvalues of H are all finitely degenerate and cannot accumulate outside $\kappa(H_0)$.

4. If $\langle N \rangle^{1+\theta} V$ is bounded for some $\theta > 0$ then the conclusions of Theorems 1.2 and 1.3 are valid for H.

A consequence of this theorem is the following result on Jacobi operators with periodic coefficients,

Corollary 10.1 *Let J_0 be the periodic Jacobi operator given in $l^2(\mathbb{Z})$ by*

$$(J_0 \psi)_n = a_n \psi_{n+1} + b_n \psi_n + a_{n-1} \psi_{n-1}$$

such that $a_n > 0$ and $b_n \in \mathbb{R}$ with $a_{j+N} = a_j$ and $b_{j+N} = b_j$. Then assertions of Theorem 10.1 hold true when we replace H_0 by J_0.

Proof Consider the unitary operator $U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}, \mathbb{C}^N)$ defined by $(U\psi)_n = (\psi_{(n-1)N+1}, \cdots, \psi_{nN})$. One may show that $U J_0 U^{-1} = H_0(A, B)$ where A, B are given by (10.1). The proof is complete.

Of course spectral properties of J_0 are well known, see [28] for example. The point here is to obtain the Mourre estimate for these operators and also to show how our general approach works in this context.

Here again spectral analysis of H_0 is based on the study of the eigenvalues of the reduced operators $h(p)$ given here by

$$h(p) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N e^{ip} \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{N-2} & b_{N-1} & a_{N-1} \\ a_N e^{-ip} & \cdots & 0 & a_{N-1} & b_N \end{pmatrix} \ , \quad \forall p \in [-\pi, \pi]. \tag{10.2}$$

Since $h(-p)$ is the transpose of $h(p)$, it is enough to restrict ourselves to $p \in [0, \pi]$. Theorem 10.1 is an immediate consequence of the following:

Theorem 10.2 *The eigenvalues $\lambda_1(p), \cdots, \lambda_N(p)$, of $h(p)$ are monotonic on $[0, \pi]$ and can be chosen so that $\lambda_1(p) < \lambda_2(p) \cdots < \lambda_N(p)$, for all $p \in]0, \pi[$. In fact, if λ_j is increasing (respectively decreasing) then λ_{j+1} is decreasing (respectively increasing). Moreover, for $p = 0$ or π the eigenvalues $\lambda_j(p)$ may be double and*

$$\lambda_1(\pi) < \lambda_1(0) \le \lambda_2(0) < \lambda_2(\pi) \le \cdots \le \lambda_N(\pi) < \lambda_N(0) \text{ if } N \text{ is odd };\tag{10.3}$$

$$\lambda_1(0) < \lambda_1(\pi) \le \lambda_2(\pi) < \lambda_2(0) \le \cdots \le \lambda_N(\pi) < \lambda_N(0) \text{ if } N \text{ is even }.\tag{10.4}$$

In particular, there exist $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \cdots \alpha_{N-1} < \beta_{N-1} \le \alpha_N < \beta_N$ such that $\lambda_j([0, \pi]) = [\alpha_j, \beta_j]$ and $\kappa(\lambda_j) = \{\alpha_j, \beta_j\}$. If for some $1 \le j \le N$, $\beta_j = \alpha_{j+1}$ then such a value is a double eigenvalue of $h(p)$ for $p = 0$ or π.

Notice that $h(p)$ for $p = 0$ and π are exactly the periodic and anti-periodic Jacobi matrices studied in [20, 23] where the assertion (10.3)-(10.4) are proved. More generally, eigenvectors of $h(p), p \in [0, \pi]$ are nothing but the so-called Bloch solutions studied in the Floquet theory developed for periodic Jacobi operators. In this sense, Theorem 10.2 is well known with different proofs, see [11, 20, 23, 28]. We will give however a proof, mainly for completeness, but also because it is elementary and based on standard linear algebra. It will be given in few lemmas.

Lemma 10.1 *For every $p \in]0, \pi[$, the eigenvalues $\lambda_1(p), \cdots, \lambda_N(p)$ of $h(p)$ are all simple, while for $p = 0, \pi$ they are at most of multiplicity two. Moreover, they are all monotonic on $(0, \pi)$.*

Proof Let $u = (x_1, \cdots, x_N)$ be an eigenvector of $h(p)$ corresponding to an eigenvalues λ. This is equivalent to,

$$\begin{cases} (b_1 - \lambda)x_1 & + & a_1 x_2 & + & a_N e^{ip} x_N & = 0 \\ a_{i-1}x_{i-1} & + & (b_i - \lambda)x_i & + & a_i x_{i+1} & = 0 \\ a_N e^{-ip} x_1 & + & a_{N-1} x_{N-1} & + & (b_N - \lambda)x_N & = 0 \end{cases}, \quad 2 \le i \le N-1 \tag{10.5}$$

We see that for all $2 \le i \le N - 1$, x_i is a linear combination of x_1 and x_N so that λ is at most of multiplicity two. Moreover, if $x_1 = 0$ then $p = 0$ or $p = \pi$. Indeed, if $x_1 = 0$ then $x_2 = -(a_N/a_1)e^{ip}x_N$. Moreover, using simultanousely the equations $j = 2, 3, \cdots, N - 1$, we get that $x_N = ce^{ip}x_N$ for some $c \in \mathbb{R}$. This means that $x_N = 0$ or $e^{ip} \in \mathbb{R}$. Since u is an eigenvector of $h(p)$, $u \ne 0$ so that $x_N \ne 0$ and therefore $e^{ip} \in \mathbb{R}$, or equivalently $p = 0$ or $p = \pi$. Similarly one may prove that if $x_N = 0$ then $p = 0$ or $p = \pi$.

Now let $p \in (0, \pi)$ and assume that $e^{ip}\overline{x_1}x_N$ is real. Then multiplying our system by $\overline{x_1}$ we get from the first equation that $\overline{x_1}x_2$ is real too. Similarly, using simultanousely the equations $j = 2, 3, \cdots, N - 1$, we get $\overline{x_1}x_N$ is real which is impossible. Thus $\Im(e^{ip}\overline{x_1}x_N) \ne 0$ and so $\lambda'(p) \ne 0$ according to the proposition 2.2. This completes the proof.

Lemma 10.2 *Let $a = (-1)^{N-1}a_1 a_2 \cdots a_N$. The characteristic polynomial of $h(p)$ is given by,*

$$D(x, p) := \det(h(p) - x) = \Delta(x) + 2a \cos p,\tag{10.6}$$

$$\Delta(x) = \det(B - x) - a_N^2 \det(\dot{B} - x)\tag{10.7}$$

where $\dot{B}$ is the matrix obtained from B by removing the first and last rows and columns with the convention $\det(\dot{B} - x) = 1$ in the case where $N = 2$.

Proof Direct calculation, see however [17] where the formula is already used.

Remark We emphasize that, according to (10.6), if one of the a_j's is zero then $D(x, p) = \Delta(x)$ which is independent of p. In this case, the spectrum of $h(p)$ is independent of p so that the spectrum of H_0 consists of N infinitely degenerate eigenvalues. This provides one with a general class of block Jacobi matrices with this kind of spectra.

Lemma 10.3 *The derivative Δ' of Δ has $N-1$ roots $\mu_1 < \mu_2 < \cdots < \mu_{N-1}$ and the λ_j's may be chosen so that, for all $j = 1, \cdots, N$,*
(i) $(-1)^j \Delta'(\lambda_j(p)) > 0$ for all $p \in]0, \pi[$;
(ii) $\lambda_j(]0, \pi[) \subset]\mu_{j-1}, \mu_j[$ with $\mu_0 = -\infty$ and $\mu_N = +\infty$.
(iii) For any $j = 1, \cdots, N$, $(-1)^{N+j-1}\lambda_j(p)$ is an increasing function of $p \in [0, \pi]$ and $\kappa(\lambda_j) \subset \{\lambda_j(0), \lambda_j(\pi)\}$.

Proof (i) Let us denote by $\lambda_1(\pi/2) < \lambda_2(\pi/2) < \cdots < \lambda_N(\pi/2)$ the distinct eigenvalues of $h(\pi/2)$ (thanks to the Lemma 10.1). They are the roots of $\Delta(x) = D(x, \pi/2)$. So for any $j = 1, \cdots, N-1$ there is $\mu_j \in]\lambda_j(\pi/2), \lambda_{j+1}(\pi/2)[$ such that $\Delta'(\mu_j) = 0$, Moreover, since Δ is a polynomial of the form $\Delta(x) = (-1)^N x^N + $ lower terms, it is clear that $(-1)^j \Delta'(\lambda_j(\pi/2)) > 0$ for any $j = 1, \cdots N$ which gives us the sign of Δ' on each $]\mu_{j-1}, \mu_j[$.

(ii) Now assume that (ii) is not true for some j, say $j = 1$ (the other cases are similar). Then, by continuity of λ_1 and the fact that $\lambda_1(\pi/2) < \mu_1$, μ_1 would belong to $\lambda_1(]0, \pi[)$. Therefore $\mu_1 = \lambda_1(p_0)$ for some $p_0 \in]0, \pi[$. But then for such p_0 one has:

$$D(\mu_1, p_0) = D(\lambda_1(p_0), p_0) = \frac{\partial D}{\partial x}(\lambda_1(p_0), p_0) = 0.$$

Here-above we used the fact that $\frac{\partial D}{\partial x}(x, p) = \Delta'(x)$. This means that $\mu_1 = \lambda_1(p_0)$ is a multiple eigenvalue of $h(p_0)$ which, according to Lemma 10.1, implies that $p_0 = 0$ or $p_0 = \pi$. But this is impossible since $p_0 \in]0, \pi[$.

(iii) Again using the fundamental equation (10.6) we get,

$$D(\lambda_j(p), p) = 0 \quad \text{for all } p \in (0, \pi) \text{ and } j = 1, \cdots, N.$$

By differentiating this identity with respect to p and using $\frac{\partial D}{\partial p}(x, p) = -2a \sin p$, we get

$$\lambda_j'(p) = \frac{2a \sin p}{\Delta'(\lambda_j(p))} \quad \text{for all } p \in (0, \pi) \text{ and } j = 1, \cdots, N.$$

So the sign of λ_j' on $]0, \pi[$ is that of $a\Delta'$ on $]\mu_{j-1}, \mu_j[$. As $a = (-1)^{N-1} a_1 a_2 \cdots a_N$ and according to the Lemma 10.3 where the sign of Δ' has been found, we see that $(-1)^{N+j-1}\lambda_j'(p) > 0$ for any $p \in]0, \pi[$.

Example 10.1 *Let $N = 2$ so that $A = a_2 E_{2,1}$, $B = tridiag(\{a_1\}, \{b_1, b_2\})$ where $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$. In this case, it is easy to see on one hand that $h(0)$ has two simple eigenvalues. On the other hand, $h(\pi)$ has a double eigenvalues if, and only if, $b_1 = b_2$ and $a_1 = a_2$, in such case the double eigenvalue is $\mu = (b_1 + b_2)/2 = b_1/2$.*

In the second part of this example we consider the inverse problem of this model. More specifically, let $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2$ and find $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$ such that the corresponding H_0 satisfies $\sigma(H_0) = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$. First, if there is a solution then, according to Theorem 10.1, α_1, β_2 are the eigenvalues of $h(0)$ and β_1, α_2 are the eigenvalues of $h(\pi)$. Thus

$$\alpha_1 + \beta_2 = b_1 + b_2 = \alpha_2 + \beta_1 \;, b_1 b_2 = (a_1 - a_2)^2 + \alpha_2 \beta_1 \quad and \quad \alpha_1 \beta_2 = \alpha_2 \beta_1 - 4a_1 a_2$$

In particular, the numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ are not completely arbitrary. In fact, once the identity $\alpha_1 + \beta_2 = \alpha_2 + \beta_1$ holds we get $16 a_1 a_2 = (\beta_2 - \alpha_1)^2 - (\alpha_2 - \beta_1)^2$; and b_1 and b_2 exist if, and only if, the discriminant $(\alpha_2 - \beta_1)^2 - 4(a_1 - a_2)^2 \ge 0$. Here again we see that the gap is degenerate, i.e. $\alpha_2 - \beta_1 = 0$ if, and only if, $a_1 = a_2 = (\beta_2 - \alpha_1)/4$ and $b_1 = b_2 = \alpha_2 + \beta_1$. All that is known for general periodic Jacobi matrices, see [8, 17, 23]. The goal here is to show the usefulness of the identity (10.6) and to contrast with the model considered in the example 9.1 where the inverse problem has a unique solution.

11 A case where A is a singular indefinite matrix

Remark first that the assertions of the previous Theorem 10.2 remain valid if we replace A by $a_N E_{1N}$. Here we study the combination of the two cases: we modify the last model by putting non zero coefficients in the two corners of A. While the band/gap structure of the spectrum of H_0 may break down our Mourre estimate still holds. More precisely, let $N \geq 2$ and $a_1, \cdots, a_{N+1} > 0$, $b_1, \cdots, b_N \in \mathbb{R}$. Let $H_0 = H_0(A, B)$ defined by (1.1) where

$$A = a_N E_{N1} + a_{N+1} E_{1N} \quad \text{and} \quad B = \text{trid}(\{a_i\}_{1 \leq i \leq N-1}, \{b_i\}_{1 \leq i \leq N}) \tag{11.1}$$

Notice that if $N \geq 3$ then A is a singular indefinite matrix, since its spectrum is nothing but $\{0, \pm\sqrt{a_N a_{N+1}}\}$. We have,

Theorem 11.1 1. There exist $\alpha_1 < \beta_1, \alpha_2 < \beta_2, \cdots, \alpha_N < \beta_N$ such that the spectrum of H_0 is purely absolutely continuous and $\sigma(H_0) = \cup_{j=1}^{N}[\alpha_j, \beta_j]$,

2. If (1.8) holds then we have a Mourre estimate for H_0 on $\mathbb{R} \setminus \kappa(H_0)$, i.e. $\mu^{\mathbb{A}}(H_0) = \mathbb{R} \setminus \kappa(H_0)$ ($\kappa(H_0)$ is finite and contains $\{\alpha_1, \beta_1, \cdots, \alpha_N, \beta_N\}$). In particular, conclusions of Theorem 5.1 hold for H_0

It should be clear that one may deduce results for $H = H_0 + V$ for a class of compact perturbations V, but we will not state them separately.

Again we will look at the eigenvalues of $h(p)$ given in this case by

$$h(p) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & z_p \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{N-2} & b_{N-1} & a_{N-1} \\ \overline{z_p} & \cdots & 0 & a_{N-1} & b_N \end{pmatrix} \quad \text{with} \quad z_p = a_N e^{ip} + a_{N+1} e^{-ip}.$$

Since $h(-p)$ is the transpose of $h(p)$, it is enough to study the operators $h(p)$ for $p \in [0, \pi]$. Theorem 11.1 is a consequence of the following:

Theorem 11.2 The eigenvalues $\lambda_1(p), \cdots, \lambda_N(p)$, of $h(p)$ are non constant $[0, \pi]$. They are all simple for all $p \in [0, \pi]$ except for finite set of p's for which some of them may be double. Such set is reduced to $\{0, \pi\}$ if $a_N \neq a_{N+1}$.

The proof of Theorem 11.2 will be done in few steps that we state in lemmas.

Lemma 11.1 (i) The eigenvalues of $h(\pi/2)$ are all simple.
(ii) If $a_N \neq a_{N+1}$ then for all $p \in]0, \pi[$, the eigenvalues of $h(p)$ are all simple, while for $p = 0, \pi$ some of them may be double.

Proof Simple adaptation of the proof of Lemma 10.1.

Lemma 11.2 Let $a' = (-1)^{N-1} a_1 a_2 \cdots a_{N-1}(a_N + a_{N+1})$. The characteristic polynomial of $h(p)$ is given by,
$$D(x, p) = \Delta(x) - 2a_N a_{N+1} \det(\dot{B} - x) \cos^2 p + 2a' \cos p, \tag{11.2}$$

where, with $\dot{B}$ as in Lemma 10.2,

$$\Delta(x) = \det(B - x) - (a_N - a_{N+1})^2 \det(\dot{B} - x) \tag{11.3}$$

Proof Direct calculations.

Lemma 11.3 The derivative Δ' of Δ has $N - 1$ roots $\mu_1 < \mu_2 < \cdots < \mu_{N-1}$ and the λ_j's may be chosen so that, for all $j = 1, \cdots, N$, $(-1)^j \Delta'(\lambda_j(\pi/2)) > 0$.

Proof Again remark that $\Delta(x) = D(x, \pi/2)$ and denote by $\lambda_1(\pi/2) < \lambda_2(\pi/2) < \cdots < \lambda_N(\pi/2)$ the distinct eigenvalues of $h(\pi/2)$. They are roots of $\Delta(x) = D(x, \pi/2)$. So for any $j = 1, \cdots, N-1$ there is $\mu_j \in]\lambda_j(\pi/2), \lambda_{j+1}(\pi/2)[$ such that $\Delta'(\mu_j) = 0$, Moreover, since $\Delta(x) = (-1)^N x^n + $ lower order, it is clear that $(-1)^j \Delta'(\lambda_j(\pi/2)) > 0$ for any $j = 1, \cdots N$. The proof is finished.

Lemma 11.4 *There is a neighborhood of $\pi/2$ on which the eigenvalues $\lambda_j(p)$ of $h(p)$ are monotonic.*

Proof Let us fix $j = 1, \cdots, N$ and put $\mu_0 = -\infty, \mu_N = +\infty$. According to the Lemma 11.3, $\lambda_j(\pi/2) \in]\mu_{j-1}, \mu_j[$ and $(-1)^j \Delta'(\lambda_j(p)) > 0$. We also have $\frac{\partial D}{\partial p}(\lambda_j(\pi/2), \pi/2) = -2a'(a_N^2 + a_{N+1})^2 \neq 0$, Now after differentiation of the identity $D(\lambda_j(p), p) = 0$ at $p = \pi/2$ we find,

$$\lambda_j'(\pi/2) = \frac{2a'(a_N^2 + a_{N+1}^2)}{\Delta'(\lambda_j(\pi/2))} \neq 0.$$

The proof is complete.

Example 11.1 *Let $A = a_3 E_{3,1} + a_4 E_{1,3}$, $B = tridiag(\{a_1, a_2\}, \{b_i = 0\})$, with $a_1, a_2, a_3 + a_4 > 0$ and $a_3, a_4 \geq 0$.*

(i) If $a_3 \neq a_4$ then $h(p)$ has only simple eigenvalues for all $p \in (0, \pi)$. Moreover, $h(0)$ has a double eigenvalue if, and only if, $a_1 = a_2 = a_4 + a_3$, which is equivalent to $h(\pi)$ has a double eigenvalue. In such case, $-(a_4 + a_3)$ (resp. $a_4 + a_3$) is a double eigenvalues of $h(0)$ (resp. $h(\pi)$).

(ii) Assume that $a_3 = a_4$. In this case, $h(p) = (2a \cos p)A + B$. Moreover,

$$\lambda_j'(p) = -2a \sin p \langle AW_j(p), W_j(p) \rangle = -4a_3 x_j z_j \sin p$$

where (x_j, y_j, z_j) are the coordinate of $W_j(p)$. This derivative vanishes if, and only if, $p = 0, \pi$ or $x_j z_j = 0$. By writing the eigenvalue problem for $h(p)$ we find out that, $x_j z_j = 0$ if, and only if,
(i) $x_j = 0$ leading to $\lambda_j = \pm a_2$, $\pm \cos p_\pm = \mp a_1/2a_3$, provided $a_1 \leq 2a_3$ or;
(ii) $z_j = 0$ leading to $\lambda_j = \pm a_1$, $\pm \cos p_\pm' = \mp a_2/2a_3$, provided $a_2 \leq 2a_3$.
Moreover, $h(p)$ has a double eigenvalue if, and only if, $a_1 = a_2 < 2a_3$ and $p = p_\pm = p_\pm'$.

Let us arrange the eigenvalues of $h(p)$ so that $\lambda_1(\pi/2) = -\sqrt{a_1^2 + a_2^2}, \lambda_2(\pi/2) = 0, \lambda_3(\pi/2) = \sqrt{a_1^2 + a_2^2}$. Using the preceding formula of λ_j' one may prove that $\lambda_1'(\pi/2) < 0$, $\lambda_2'(\pi/2) > 0$ and $\lambda_3'(\pi/2) < 0$. In particular,

1. *if $a_1 > 2a_3$ and $a_2 > 2a_3$ then, for all j, $\lambda_j'(p)$ never vanishes on $(0, \pi)$ so they have the sign of $\lambda_j'(\pi/2)$. Hence, the spectrum of H_0 is absolutely continuous and $\sigma(H_0) = [\lambda_1(\pi), \lambda_1(0)] \cup [\lambda_2(0), \lambda_2(\pi)] \cup [\lambda_3(\pi), \lambda_3(0)]$. Moreover, $\lambda_1(0) < \lambda_2(0)$ otherwise one may find $p_0 \in [0, \pi/2]$ such that $\lambda_1(p_0) = \lambda_2(p_0)$ which contradicts the simplicity of the eigenvalues. Similarly, $\lambda_2(\pi) < \lambda_3(\pi)$ so that we have two gaps.*

2. *If $a_1 \leq 2a_3 < a_2$ then there is two points $p_\pm$ given by $\cos p_\pm = \mp a_1/2a_3$ for which $h(p_\pm)$ has $\pm a_2$ as a simple eigenvalue whose corresponding eigenvectors satisfy $x_j z_j = 0$. Hence $\kappa(H_0) = \{\pm a_2, \lambda_j(0), \lambda_j(\pi), j = 1, 2, 3\}$. For example, if $a_1 = a_3 = 1$ and $a_2 = 2$ one may deduce that the spectrum of H_0 is purely absolutely continuous with two gaps and, for some $\alpha > 1$, $\sigma(H_0) = [-\alpha, -1] \cup [-1/2, 1/2] \cup [1, \alpha]$.*

3. *Finally if $a_1 = a_2 < 2a_3$ then, there is two points $p_\pm$ given by $\cos p_\pm = \mp a/2a_3$ for which $h(p_\pm)$ has $\pm a$ as a double eigenvalue whose corresponding eigenvectors satisfy $x_j z_j = 0$. Hence one may deduce that we only have overlapping spectral bands.*

12 An example where A is a lower triangular matrix: Symmetric fourth-order difference operators

Let $a_n, b_n > 0$ and $v_n \in \mathbb{R}$. Consider the difference operator $\mathcal{J} = \mathcal{J}(a_n, b_n, v_n)$ acting in $l^2(\mathbb{Z})$ by

$$(\mathcal{J}x)_n = a_n x_{n+2} + b_n x_{n+1} + v_n x_n + b_{n-1} x_{n-1} + a_{n-2} x_{n-2}.$$

Assume that, for some $a, b, c, d > 0$ and $\sigma \geq 0$, one has

$$\lim_{|n| \to \infty} |n|^\sigma (|a_{2n} - a| + |a_{2n+1} - b| + |b_{2n+1} - c|) + |b_{2n} - d| + |v_n|) = 0,$$

Then $\mathcal{J}$ is a bounded symmetric operator. Moreover, We have,

Theorem 12.1 *(i) There exist $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ such that $\sigma_{ess}(\mathcal{J}) = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$.*
(ii) If $\sigma \geq 1$, then there is a self-adjoint operator $\mathbb{A}'$ such that $\tilde{\mu}^{\mathbb{A}'}(\mathcal{J}) = \mathbb{R} \setminus \kappa(\mathcal{J})$ for some finite set $\kappa(\mathcal{J})$. In particular, all possible eigenvalues of $\mathcal{J}$ are finitely degenerate and cannot accumulate outside $\kappa(\mathcal{J})$.
(iii) Finally, if $\sigma > 1$ then $\mathcal{J}$ has no singular continuous spectrum and conclusions of Theorems 1.2 and 1.3 hold for $\mathcal{J}$.

Proof Consider the unitary operator $U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}, \mathbb{C}^2)$ defined by $(Ux)_n = (x_{2n}, x_{2n+1})$. One may verify that $U\mathcal{J}U^{-1} = H(A_n, B_n)$ with $A_n = \begin{pmatrix} a_{2n} & 0 \\ b_{2n+1} & a_{2n+1} \end{pmatrix}, B = \begin{pmatrix} v_{2n} & b_{2n} \\ b_{2n} & v_{2n+1} \end{pmatrix}$. Our hypothesis means that $H = H(A_n, B_n)$ is a compact perturbation of $H_0 = H_0(A, B)$ defined by (1.3) with $A = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, B = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$. In this case, $h(p) = \begin{pmatrix} 2a \cos p & z_p \\ \overline{z_p} & 2b \cos p \end{pmatrix}$, with $z_p = d + ce^{ip}$. One may easily prove that, $h(p)$ has a double eigenvalue if, and only if, $p = \pi, a = b$ and $c = d$. Moreover, for any $p \in (0, \pi)$, $h(p)$ has two simple eigenvalues given by

$$\lambda_j(p) = (a + b) \cos p + (-1)^j \sqrt{(a - b)^2 \cos^2 p + 4dc \cos^2 p/2 + (d - c)^2}, j = 1, 2.$$

Hence our general method applies directly here also. The proof is complete.

Example 12.1 *Suppose that $a = b$ and $c = d$. Then λ_2 is decreasing and generates the spectral band $\Sigma_2 = [-2a, 2(a + d)]$. In contrast, we have:*

1. *if $K := \frac{d}{4a} > 1$ then λ_1 is increasing and $\Sigma_1 = [2(a - d), -2a]$. In this case, $\kappa(J) = \partial\Sigma_1 \cup \partial\Sigma_1$ where $\partial\Sigma_i$ consists of the edges of Σ_i; and $\sigma(J_0) = [2(a - d), -2a] \cup [-2a, 2(a + d)] = [2(a - d), 2(a + d)]$ (the spectral bands touch in one point $-2a$ which is a double eigenvalue of $h(\pi)$).*

2. *if $K \leq 1$ then λ_1 has a critical point at p_0 given by $\cos p_0/2 = K$. In this case, $\lambda_1(p_0) = -2a(1 + 2K^2)$ and there is two possibilities. If $2a \leq d \leq 4a$ then $\Sigma_1 = [-2a(1 + 2K^2), -2a]$ so that the two bands Σ_1 and Σ_2 touch in one point. In contrary, if $0 < d \leq 2a$ then $\Sigma_1 = [-2a(1 + 2K^2), 2(a - d)]$ so that $\Sigma_1 \cap \Sigma_2 = [-2a, 2(a - d)]$. In both cases $\kappa(J) = \partial\Sigma_1 \cup \partial\Sigma_1 \cup \{\lambda_1(p_0)\}$ and $\sigma(J_0) = [-2a(1 + 2K^2), 2(a + d)]$.*

Of course in this case $\mathcal{J}$ is a compact perturbation of $\mathcal{J}_0 = \mathcal{J}(a_n = a, b_n = d, v_n = 0)$ that one may study directly. Indeed, by Fourier transform, $\mathcal{J}_0$ is unitarily equivalent to the multiplication operator by the function $f(p) = 2d \cos p + 2a \cos 2p$ acting in $L^2(-\pi, \pi)$.

Example 12.2 *Suppose that $a = b$ and $d \neq c$. Then here again λ_2 is decreasing and $\Sigma_2 = [-2a + |d - c|, 2a + d + c]$ and*

1. *if $a < \frac{dc}{2(d+c)} := K_1$ then λ_1 is increasing and $\Sigma_1 = [2a - (d + c), -2a - |d - c|]$. In this case, $\kappa(J) = \partial\Sigma_1 \cup \partial\Sigma_1$ and $\sigma(H_0) = [2a - (d + c), -2a - |d - c|] \cup [-2a + |d - c|, 2a + d + c]$ (a spectral gap of length $2|d - c|$).*

2. *if $a > \frac{dc}{2|d-c|} := K_2$ then λ_1 is decreasing and $\Sigma_1 = [-2a - |d - c|, 2a - (d + c)]$. In this case, $\kappa(J) = \partial\Sigma_1 \cup \partial\Sigma_1$ and $\sigma(H_0) = [-2a - |d - c|, 2a - (d + c)] \cup [-2a + |d - c|, 2a + d + c]$. One may have a non trivial gap in some cases (e.g. $d = 10, c = 1$ and $a = 1$) and overlapping bands in others (e.g. $d = 2, c = 1$ and $a = 3/2$).*

3. If $K_1 \le a \le K_2$ then λ_1 has a critical point at p_0 given by $\cos p_0 = \frac{dc}{8a^2} - \frac{d^2+c^2}{2dc}$. It generates the spectral band $\Sigma_1 = [\alpha, \beta]$ where $\alpha = \min(\lambda_1(0), \lambda_1(p_0), \lambda_1(\pi))$ and $\beta = \max(\lambda_1(0), \lambda_1(p_0), \lambda_1(\pi))$. In this case, $\kappa(J) = \partial\Sigma_1 \cup \partial\Sigma_1 \cup \{-2a - 4K^2\}$ and $\sigma(H_0) = [\alpha, \beta] \cup [-2a, 2(a+d)]$. For example, let $c = 2, d = 1$. Then $K_1 \le a \le K_2$ means that $1/3 \le a \le 1$. $\Sigma_1 = [-9/4, -2]$ and $\Sigma_2 = [-1, 3]$ and we have a spectral gap. $\Sigma_1 = [-3, -1]$ and $\Sigma_2 = [-2, 4]$ and we have overlapping spectral bands.

Example 12.3 *It remains to study the case where $a \ne b$. To be simple we only discuss the following two situations.*

1. *If $c = d = 0$ then $\lambda_1(p) = 2a \cos p$ and $\lambda_2(p) = 2b \cos p$ so that we have two spectral bands one contained in the another.*

2. *In contrast if $d = 0$ and $c > 0$ then both $\lambda'_j s$ are monotonic and one may choose a, b, c to get overlapping bands or bands with a spectral gap between them. Here $\kappa(J)$ is the set of the edges of the spectral bands.*

13   Further developments

In this section we explain how our analysis extends to compact perturbation of difference operators of higher order with matrix coefficients. More specifically, let $H = H_0 + V$ where V is a compact operator in $\mathcal{H} = l^2(\mathbb{Z}, \mathbb{C}^N)$ and H_0 is defined on $\mathcal{H}$ by

$$(H_0\psi)_n = B\psi_n + \sum_{k=1}^{m} A_k \psi_{n+k} + A_k^* \psi_{n-k}, \quad \text{for all } n \in \mathbb{Z}. \tag{13.4}$$

where $A_1, \cdots, A_k$ and $B = B^*$ are $N \times N$ matrices. By using Fourier transform we easily show that H_0 is unitarily equivalent to the direct integral

$$\hat{H}_0 = \int_{[-\pi,\pi]}^{\oplus} h(p)dp \quad \text{acting in} \quad \hat{\mathcal{H}} = \int_{[-\pi,\pi]}^{\oplus} \mathbb{C}^N dp \tag{13.5}$$

where the reduced operators $h(p)$ are given by,

$$h(p) = B + \sum_{k=1}^{m} e^{-ikp} A_k + e^{ikp} A_k^*.$$

So by repeating verbatim the arguments of Sections 2 and 5 we prove that Corollary 5.1 and it consequence Theorem 5.1 are valid for H_0. Similarly, the perturbative argument developed in Section 6 extends to $H = H_0 + V$ with similar assumptions on the perturbation V. In particular, analogous results to Theorem 6.1 and those stated in the introduction can be obtained.

We finish by given the simplest but non trivial example one may consider here. Assume that $A_k = A$ for all $k = 1, \cdots, m$ where A is a positive definite matrix. Then $h(p) = B + (D_m(p) - 1)A$ where $D_m(p) = 1 + 2\sum_{k=1}^{m} \cos(kp)$ is the Dirichlet Kernel. So by repeating the argument of Lemma 2.2 we get that

$$\lambda'_j(p) = D'_m(p) < AW_j(p), W_j(p) >$$

The Dirichlet kernel is known to strongly oscillates, especially when m is large, so is for $\lambda_j(p)$. Hence $\kappa(H_0)$ will be little bit complicate but finite and our general framework applies directly, compare however with section 7.

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