UNITIFICATION OF WEAKLY P. Q.-BAER $*$-RINGS

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Abstract. In this paper, we introduce a concept of weakly principally quasi-Baer $*$-rings in terms of central cover. We prove that a $*$-ring is a principally quasi-Baer $*$-ring if and only if it is weakly principally quasi-Baer $*$-ring with unity. A partial solution to the problem similar to unitification problem raised by S. K. Berberian is obtained.

Keywords: weakly p.q.-Baer $*$-rings, p.q.-Baer $*$-rings, Rickart $*$-rings, Baer $*$-rings.

1. Introduction

Throughout this paper, R denotes an associative ring. A $*$-ring (ring with involution) R is a ring equipped with an involution $x \to x^*$, that is an additive anti-automorphism of period at most two. An element e in a $*$-ring R is called a *projection* if it is self adjoint (i.e., $e = e^*$) and *idempotent* (i.e., $e^2 = e$). Let S be a nonempty subset of R. We write $r_R(S) = \{a \in R \mid sa = 0, \ \forall\ s \in S\}$, called the *right annihilator* of S in R, and $l_R(S) = \{a \in R \mid as = 0, \ \forall\ s \in S\}$, called the *left annihilator* of S in R. For projections e, f, we write $e \leq f$ in case $e = ef$. By [2, Proposition 1, page 4], the relation $\leq$ is a partial order relation on the set of projections in a $*$-ring.

In [9], Kaplansky introduced Baer rings and Baer $*$-rings to abstract various properties of AW^*-algebras, von Neumann algebras and complete $*$-regular rings. A $*$-ring R is said to be a *Baer $*$-ring*, if for every nonempty subset S of R, $r_R(S) = eR$, where e is a projection in R. Herstien was convinced that the simplicity of the simple Lie algebra should follow solely from the fact that $R = M_n(F)$ (F-a field) is a simple ring. This led him in the 1950's to develop his Lie theory of arbitrary simple rings with involutions. Early motivation for studying $*$-rings came from rings of operators. If $\mathscr{B}(H)$ is the set of all bounded linear operators on a (real or complex) Hilbert space H, then each $\phi \in \mathscr{B}(H)$ has an adjoint, $adj(\phi) \in \mathscr{B}(H)$, and $\phi \to adj(\phi)$ is an involution on the ring $\mathscr{B}(H)$.

According to Birkenmeier et al. [6], a $*$-ring R is said to be a *quasi-Baer $*$-ring* if the right annihilator of every ideal of R is generated, as a right ideal, by a projection in R. In the same paper [6], they provide examples of Baer rings which are quasi-Baer $*$-rings but not Baer $*$-rings. In [2], Berberian developed the theory of Baer $*$-rings and Rickart (p.p.) $*$-rings.

The following definitions are required in sequel, can be found in [2]. A $*$-ring R is said to be a *Rickart $*$-ring*, if for each $x \in R$, $r_R(\{x\}) = eR$, where e is a projection in R. For each element a in a Rickart $*$-ring, there is unique projection e such that $ae = a$ and

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$ax = 0$ if and only if $ex = 0$, called the *right projection* of a denoted by $RP(a)$. Similarly, the left projection $LP(a)$ is defined for each element a in a Rickart $*$-ring A. A $*$-ring R is said to be a *weakly Rickart $*$-ring*, if for any $x \in R$, there exists a projection e such that (1) $xe = x$, and (2) if $xy = 0$ then $ey = 0$. Let R be a $*$-ring and $x \in R$, we say that x possesses a *central cover* if there exists a smallest central projection h such that $hx = x$. If such projection h exists, then it is unique, and is called the central cover of x, denoted by $h = C(x)$ or $h = C_R(x)$.

In [3], Birkenmeier introduced principally quasi-Baer (p.q.-Baer) rings as a generalization of quasi-Baer rings. Birkenmeier et al. [7] introduced principally quasi-Baer (p.q.-Baer) $*$-rings. A $*$-ring R is said to be *a p.q.-Baer $*$-ring* if, for every principal right ideal aR of R, $r_R(aR) = eR$, where e is a projection in R. From the above definition, it follows that $l_R(aR) = Rf$ for a suitable projection f. Note that, R is a quasi-Baer $*$-ring if and only if $M_n(R)$ ($n \times n$ matrix ring with $*$ transpose involution) is a quasi-Baer (hence p.q.-Baer) $*$-ring for all $n \geq 1$ [5, Proposition 2.6]. In general this statement is not true for Baer $*$-rings [10, Theorem 6]. Thus the class of p.q.-Baer $*$-rings is very much larger than that of Baer $*$-rings.

In [2], it is proved that every element of a Baer $*$-ring has a central cover. In the second section of this paper, we extend this result for p.q.-Baer $*$-rings. In fact, we generalize results of Baer $*$-rings to p.q. Baer $*$-rings. We have given an example of a $*$-ring R, which is a Rickart $*$-ring as well as a p.q.-Baer $*$-ring, and an element $x \in R$ such that $RP(x)$ is not equal to $C(y)$ for any $y \in R$.

In the third section, we introduce the concept of weakly p.q.-Baer $*$-rings. We provide an example of an abelian p.q.-Baer $*$-ring which is not a weakly Rickart $*$-ring. Efforts are taken to obtain the exact difference between p.q.-Baer $*$-rings and weakly p.q.-Baer $*$-rings.

Berberian [2, page 33], raised the following Problem.

Problem 1: Can every weakly Rickart $*$-ring be embedded in a Rickart $*$-ring? with preservation of RP's?

Berberian has given a partial solution to this problem. In [11], Thakare and Waphare gave more general partial solution to Problem 1. Till date this problem is open. In view of Problem 1, it is natural to raise the following problem.

Problem 2: Can every weakly p.q.-Baer $*$-ring be embedded in a p.q.-Baer $*$-ring? with preservation of central covers?

In the last section we give the partial solution of Problem 2, analogous to the partial solution of Problem 1 given by Thakare and Waphare in [11].

2. Central Cover

In this section, we extend the results of Baer $*$-rings (proved in [2]) to p.q.-Baer $*$-rings. Also, we study the properties of a central cover of an element in a p.q.-Baer $*$-ring.

According to Birkenmeier et al. [4], the involution of a $*$-ring R is *semi-proper* if $aRa^* \neq 0$ for every nonzero element $a \in R$.

Proposition 2.1. *If R is a p.q.-Baer $*$-ring, then R has the unity element and the involution of R is semi-proper.*

Proof. Since R is a p.q.-Baer $*$-ring, there exists a projection $e \in R$ such that $r_R(0R) = R = eR$. As $e \in eR$, we have $e = er$ for some $r \in R$. Let $x \in R = eR$, so $x = er'$ for some $r' \in R$. Clearly $ex = e(er') = e^2r' = er' = x$. Hence e is the left unity in R. Since R is a

$*$-ring, e is also the right unity in R. Thus e is an unity in R. By [1, Proposition 2], $*$ is semi-proper. □

Remark 2.2. Note that in a p.q.-Baer $*$-ring, the projection which generates the right annihilator of a principal right ideal is central. For this, let k be a projection in a p.q.-Baer $*$-ring R such that $r_R(zR) = kR$ for some $z \in R$. Let $r \in R$. Since $rk \in r_R(zR) = kR$, we have $rk = kr'$ for some $r' \in R$. Hence $krk = kr' = rk$. Thus $kr = (r^*k)^* = (kr^*k)^* = krk = rk$.

Theorem 2.3. *Let R be a p.q.-Baer $*$-ring and $x \in R$. Then x has a central cover $e \in R$. Further, $xRy = 0$ if and only if $yRx = 0$ if and only if $ey = 0$.*
That is $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1-e)R = R(1-e)$.

Proof. Let R be a p.q.-Baer $*$-ring and $x \in R$. First we prove that x has a central cover. Since R is a p.q.-Baer $*$-ring, there exists a projection $g \in R$ such that $r_R(xR) = gR$. Let $e = 1 - g$. Consider $xe = x(1-g) = x - xg = x$. By Remark 2.2, e is central. To prove e is the smallest central projection with the property $xe = x$, suppose $e' \in R$ be a central projection such that $xe' = x$. So $x(1-e') = 0$. Thus $xR(1-e') = 0$. Consequently $(1-e') \in r_R(xR) = (1-e)R$. It follows that $1 - e' = (1-e)t$ for some $t \in R$. This gives $(1-e')(1-e) = (1-e)t = (1-e')$, that is $(1-e') \leq (1-e)$. Therefore $e \leq e'$. Thus $e = C(x)$.

If $xRy = 0$, then $y \in r_R(xR) = gR$. Hence $y = gr$ for some $r \in R$. As g is a projection, we have $gy = g^2r = gr = y$. Therefore $ey = (1-g)y = y - gy = 0$. Conversely, suppose $ey = 0$. Then $y = y - ey = (1-e)y = gy$. Since $g \in r_R(xR)$, we have $xry = xrgy = 0$ for any $r \in R$. This yields $xRy = 0$. Similarly, there exists a central projection $f \in R$ such that, $fx = x$; and $yRx = 0$ if and only if $yf = 0$.

Now we prove that $e = f$. Since $xR(e-f) = 0$, we have $e(e-f) = 0$, that is $e = ef$. Similarly, since $(e-f)Rx = 0$, we have $(e-f)f = 0$, that is, $ef = f$. This yields $e = f$. Therefore $xRy = 0$ if and only if $yRx = 0$ if and only if $ey = 0$.

Observe that, $r_R(xR) = (1-e)R$ and $l_R(Rx) = R(1-e)$. As e is central, $1-e$ is also central. Consequently, $r_R(xR) = l_R(Rx) = R(1-e)$. To prove $r_R(eR) = R(1-e)$, let $z \in r_R(eR)$. Then $eRz = 0$, so $ez = 0 = ze$. Hence $z = z(1-e) \in R(1-e)$. Thus $r_R(eR) \subseteq R(1-e)$. Since $1-e$ is central, we have $R(1-e) \subseteq r_R(eR)$. This gives $r_R(eR) = R(1-e)$. In nutshell, we get $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1-e)R = R(1-e)$. □

The following example shows that RP's in Rickart $*$-rings are not necessarily central covers in p.q.-Baer $*$-rings.

Example 2.4. Let $A = M_2(\mathbb{Z}_3)$, which is a Baer $*$-ring (hence a p.q.-Baer $*$-ring and a Rickart $*$-ring) with transpose as an involution. By [2, Exercise 17, page 10], the set of all projections $P(A)$ in A is, $P(A) = \{0, 1, e, f, 1-e, 1-f\}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Out of these only 0 and 1 are central projections in A. Note that the set of all right projections in A is $P(A)$, whereas the set of all central covers in A (being central) is $\{0, 1\}$. Therefore there is an element $x \in A$ such that $RP(x)$ is not equal to $C(y)$ for any $y \in A$. In particular, $r_A(x) \neq r_A(xA)$ for some $x \in A$.

Now we provide properties of a central cover of an element in p.q.-Baer $*$-rings.

Theorem 2.5. *If R is a p.q.-Baer $*$-ring, then*
(1) $C(x) = C(x^*)$; (2) $xRy = 0$ if and only if $C(x)C(y) = 0$.

Proof. (1): Let $e = C(x^*)$ and $f = C(x)$. By the definition of central cover, $x^*e = x^*$ and hence $ex = x = fx$. Therefore $ex - fx = 0$. This gives $rex - rfx = 0$ for all $r \in R$. Since e and f are central, $erx - frx = 0$. That is $(e - f)rx = 0$ for all $r \in R$. It follows that $(e - f)Rx = 0$. By Theorem 2.3, we have $(e - f)f = 0$. Hence $ef = f$. Similarly, if $fx = x$ then $x^*f = x^* = x^*e$, this yields $ef = e$. Thus $e = f$.

(2): Let $e = C(x)$ and $f = C(y)$. If $xRy = 0$, then $ey = 0$. As e is central, $eRy = 0$. Hence $ef = 0$. Conversely, suppose $ef = 0$. We have $eRy = 0$, that is, $ey = 0$. Thus $xRy = 0$. □

Projections $e,\ f$ in a p.q.-Baer $*$-ring are called *very orthogonal* if $C(e)C(f) = 0$ (equivalently, in view of Theorem 2.5, $eRf = 0$).

Corollary 2.6. *Let R be a p.q.-Baer $*$-ring whose only central projections are 0 and 1. If $x, y \in R$, then $xRy = 0$ if and only if $x = 0$ or $y = 0$.*

Proof. If $x \neq 0$ and $y \neq 0$, then $C(x) = C(y) = 1$. Therefore $C(x)C(y) \neq 0$. By Theorem 2.5, $xRy \neq 0$. If $x = 0$ or $y = 0$ then by the definition of a central cover, we have $C(x) = 0$ or $C(y) = 0$. That is $C(x)C(y) = 0$. So by Theorem 2.5, $xRy = 0$. □

According to [1], a $*$-ring R is said to satisfy the $*$-*Insertion of Factors Property* (simply, $*$-IFP) if $ab = 0$ implies $aRb^* = 0$ for all $a, b \in R$.

Corollary 2.7. *Let R be a p.q.-Baer $*$-ring. If $C(xy) = C(x)C(y)$ for all $x, y \in R$, then R is a Rickart $*$-ring. Moreover $RP(x) = C(x)$ for all $x \in R$.*

Proof. Let $x, y \in R$ be such that $xy = 0$. This gives $C(xy) = 0$. By assumption, we have $C(x)C(y) = 0$. Clearly $C(y) = C(y^*)$, therefore $C(x)C(y^*) = 0$. By Theorem 2.5, $xRy^* = 0$. Therefore R satisfies $*$-IFP. By [1, Proposition 9], R is reduced and hence a Rickart $*$-ring. Also, observe that $r_R(x) = r_R(xR)$ and hence $RP(x) = C(x)$ for all $x \in R$. □

Proposition 2.8. *Let R be a p.q.-Baer $*$-ring and suppose (e_i) is a family of projections that has a supremum, say e. If $x \in R$, then $xRe = 0$ if and only if $xRe_i = 0$ for all i.*

Proof. Let $x \in R$ and $C(x) = e'$. Suppose $xRe = 0$. As $C(x) = e'$, $e'e = 0$, which yields $e(1 - e') = e$. Therefore $e \leq (1 - e')$. Since $e_i \leq e$ for all i, we have $e_i \leq (1 - e')$ for all i. Hence $e_i(1 - e') = e_i$. This gives $e_ie' = 0$, that is, $e'e_i = 0$ for all i. As $C(x) = e'$, $xRe_i = 0$ for all i. By reversing the steps, we get the converse part. □

Let R be a ring and $S \subseteq R$ be nonempty, then $S' = \{x \in R \mid xs = sx,\ \forall\ s \in S\}$. A subring B of a $*$-ring R is said to be $*$-subring, if $x \in B$ imply $x^* \in B$.

Theorem 2.9. *Let R be a p.q.-Baer $*$-ring and B be a $*$-subring of R such that $B = (B')'$. If (e_i) is a family of central projections in B that possesses a supremum, say $e \in R$, then $e \in B$.*

Proof. Since $e_i \in B = (B')'$, we have $e_iy = ye_i$ for all $y \in B'$. Clearly $e_i(y - ye) = e_iy - e_iye = e_iy - ye_ie = e_iy - ye_i$ (as $e_i \leq e$). Since $e_iy = ye_i$, we get $e_i(y - ye) = 0$. Moreover, each e_i is central, $e_iR(y - ye) = 0$. By Proposition 2.8, $eR(y - ye) = 0$. Hence $e(y - ye) = 0$, that is $ey = eye$ for all $y \in B'$. If $y \in B'$, then $y^* \in B'$. Therefore $ey^* = ey^*e$, which gives $ye = eye$. Consequently, $ey = ye$ for all $y \in B'$. Thus $e \in (B')' = B$. □

3. Weakly $p.q.$-Baer $*$-rings

In this section, we introduce weakly p.q.-Baer $*$-rings. By Proposition 2.1, p.q.-Baer $*$-rings has unity. What about the unit-less case? The answer, in principle, is to modify the arguments or attempt to adjoin the unity element. First, let us review the unit case.

Proposition 3.1. *A $*$-ring R is a p.q.-Baer $*$-ring if and only if*
(a) R has the unity element.
(b) For each $x \in R$, there exists a central projection $e \in R$ such that $r_R(xR) = r_R(eR)$.

Proof. If R is a p.q.-Baer $*$-ring, then by Proposition 2.1, R has the unity element and by Theorem 2.3, the condition (b) is satisfied. Conversely suppose that the conditions (a) and (b) are satisfied. Since e is a central projection, $1-e$ is a central projection. So $(1-e)R \subseteq r_R(eR)$. Let $y \in r_R(eR)$. Then $ey = 0$. Hence $y = y - ey = (1-e)y$, so $y \in (1-e)R$. Therefore $r_R(eR) \subseteq (1-e)R$. This yields $r_R(xR) = r_R(eR) = (1-e)R$. Thus R is a p.q.-Baer $*$-ring. □

The following example shows that the condition (a) in Proposition 3.1 is necessary.

Example 3.2 (Exercise 1, page 32, [2])**.** Let R be a $*$-ring with $R^2 = \{0\}$. Then 0 is the only projection in R and $r_R(xR) = r_R(0R) = R$ for every element $x \in R$, but $x0 \neq x$, when $x \neq 0$.

To study the unit-less case, we introduce the concept of weakly p.q.-Baer $*$-rings as follows.

Definition 3.3. A $*$-ring R is said to be a weakly p.q.-Baer $*$-ring, if every $x \in R$ has a central cover $e \in R$ such that, $xRy = 0$ if and only if $ey = 0$.

The following is an example of an abelian p.q.-Baer $*$-ring which is not a Rickart $*$-ring, see [8].

Example 3.4. Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d,\ b \equiv 0,\ \text{and } c \equiv 0\ (mod\ 2) \right\}$. Consider involution $*$ on R as the transpose of the matrix. In [8, Example 2(1)], it is shown that R is neither right p.p. nor left p.p. (hence not a Rickart $*$-ring) but $r_R(uR) = \{0\} = 0R$ for any nonzero element $u \in R$. Therefore R is a p.q.-Baer $*$-ring.

Note that the ring R in above example is weakly p.q.-Baer $*$-ring but not weakly Rickart $*$-ring (as R has unity).

By [2, Exercise 6, page 32], a weakly Rickart $*$-ring with finitely many elements is a Baer $*$-ring. We give an example of a p.q.-Baer $*$-ring (hence a weakly p.q.-Baer $*$-ring) with finitely many elements which is not a Baer $*$-ring. First, we recall the following corollary.

Corollary 3.5 ([10], Corollary 7)**.** *(i) $M_n(\mathbb{Z}_m)$ is a Baer $*$-ring for $n \geq 2$ if and only if $n = 2$ and m is a square free integer whose every prime factor is of the form $4k+3$.*
(ii) $\mathbb{Z}_m$ is a Baer $*$-ring if and only if m is a square free integer.

Example 3.6. Let $R = M_2(\mathbb{Z}_6)$, which is a $*$-ring with transpose as an involution. By Corollary 3.5(ii), $\mathbb{Z}_6$ is a Baer $*$-ring and hence $\mathbb{Z}_6$ is a quasi-Baer $*$-ring. By [6, Proposition 2.6], R is a quasi-Baer $*$-ring and hence R is a p.q.-Baer $*$-ring. Note that R contains finitely many elements. By Corollary 3.5(i), R is not a Baer $*$-ring.

The following theorem leads to a characterization of weakly p.q.-Baer $*$-rings.

Theorem 3.7. *Let R be a $*$-ring (not necessarily with the unity), $x \in R$ and $e \in R$ be a central projection. Then the following statements are equivalent.*
(a) $C(x) = e$; and $xRy = 0$ if and only if $ey = 0$.
(b) $xe = x$; and $xRy = 0$ if and only if $ey = 0$.

Proof. Observe that, if the statement (a) holds then the statement (b) holds. Conversely suppose statement (b) holds. To prove statement (a), it is sufficient to prove e is the smallest central projection with $xe = x$. Let $e' \in R$ be the central projection such that $xe' = x$. Therefore $x(e - e') = 0$. As $e - e'$ is central, we have $xR(e - e') = 0$. By the assumption, we get $e(e - e') = 0$. Thus $e \leq e'$. □

Corollary 3.8. *R is a weakly p.q.-Baer $*$-ring if and only if for every $x \in R$ there exists a central projection $e \in R$ such that (1) $xe = x$; (2) $xRy = 0$ if and only if $ey = 0$.*

We give one more characterization of a weakly p.q.-Baer $*$-rings as follows.

Theorem 3.9. *The following conditions on a $*$-ring R are equivalent:*
(a) R is a weakly p.q.-Baer $*$-ring.
(b) R has a semi-proper involution, and for each $x \in R$, there exists a central projection e such that $r_R(xR) = r_R(eR)$.

Proof. $(a) \Rightarrow (b)$: Let $a \in R$ be such that $aRa^* = 0$. There exists a projection $e \in R$ such that (1) $e = C(a)$; (2) $aRy = 0$ if and only if $ey = 0$. As $aRa^* = 0$, we have $ea^* = 0$, that is $ae = 0$. This gives $a = 0$. Hence $*$ is semi-proper. Let $y \in r_R(xR)$. Consequently, $xRy = 0$. So $ey = 0$. It follows that $eRy = 0$. This yields $y \in r_R(eR)$. Therefore $r_R(xR) \subseteq r_R(eR)$. Let $z \in r_R(eR)$. Then $eRz = 0$. Consider $xRz = xeRz = 0$. This gives $z \in r_R(xR)$. Hence $r_R(eR) \subseteq r_R(xR)$. Thus $r_R(xR) = r_R(eR)$.

$(b) \Rightarrow (a)$: Let $x \in R$ and e be the central projection such that $r_R(xR) = r_R(eR)$. Since for all $y \in R$, $eR(y - ey) = 0$, we have $xR(y - ey) = 0$. Put $y = x^*$, we get $xR(x^* - ex^*) = 0$. Let $(x - xe)r(x^* - ex^*) \in (x - xe)R(x^* - ex^*) = (x - xe)R(x - xe)^*$, where $r \in R$. Clearly $(x - xe)r(x^* - ex^*) = (x - xe)rx^* - (x - xe)rex^* = xr(x^* - ex^*) = 0$. Therefore $(x - xe)R(x - xe)^* = 0$. As $*$ is semi-proper, we have $x - xe = 0$. Hence $x = xe$. If $xRy = 0$ then $y \in r_R(xR) = r_R(eR)$, so $eRy = 0$. Consequently $eey = 0$, that is, $ey = 0$. Thus, by Corollary 3.8, R is a weakly p.q.-Baer $*$-ring. □

Theorem 3.10. *The following conditions on a $*$-ring R are equivalent:*
(a) R is a p.q.-Baer $*$-ring.
(b) R is a weakly p.q.-Baer $*$-ring with the unity.

Proof. $(a) \Rightarrow (b)$: Follows from Proposition 2.1 and Theorem 2.3.

$(b) \Rightarrow (a)$: By the definition of a weakly p.q.-Baer $*$-ring, for each $x \in R$, there exists a projection e such that (1) $C(x) = e$; (2) $xRy = 0$ if and only if $ey = 0$. By similar steps as in Theorem 3.9, we have $r_R(xR) = r_R(eR)$. By Proposition 3.1, R is a p.q.-Baer $*$-ring. □

4. Unitification

Theorem 3.10 distinguishes p.q.-Baer $*$-rings and weakly p.q.-Baer $*$-rings. With understanding of this difference, we try to adjoin the unity to weakly p.q.-Baer $*$-rings (that is an unitification of weakly p.q.-Baer $*$-rings). This gives a partial solution of **Problem 2**.

First, we define an unitification of a $*$-ring as follows.

Definition 4.1. Let R be a $*$-ring. We say that R_1 is an unitification of R, if there exists an auxiliary ring K, called the ring of scalars (denoted by $\lambda, \mu, ...$) such that,
1) K is an integral domain with involution (necessarily semi-proper), that is, K is a commutative $*$-ring with unity and without divisors of zero (the identity involution is permitted),
2) R is a $*$-algebra over K (that is, R is a left K-module such that, identically $1a = a,\ \lambda(ab) = (\lambda a)b = a(\lambda b),\ and\ (\lambda a)^* = \lambda^* a^*$).
Define $R_1 = R \oplus K$ (the additive group direct sum), thus $(a, \lambda) = (b, \mu)$ means, by the definition that $a = b$ and $\lambda = \mu$, and addition in R_1, is defined by the formula $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$. Define $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$, $(a, \lambda)^* = (a^*, \lambda^*)$. Evidently R_1 is also a $*$-algebra over K, has unity element $(0, 1)$ and R is a $*$-ideal in R_1.

The following lemmas are elementary facts about unitification R_1 of a $*$-ring R.

Lemma 4.2. *With notations as in Definition 4.1, if an involution on R is semi-proper, then so is the involution of R_1*

Proof. Let $(a, \lambda) \in R_1$ be such that $(a, \lambda)R_1(a, \lambda)^* = (0, 0)$. So $(a, \lambda)(0, 1)(a^*, \lambda^*) = (aa^* + \lambda^* a + \lambda a^*, \lambda\lambda^*) = (0, 0)$, This gives $\lambda\lambda^* = 0$. As an involution of K is semi-proper, we have $\lambda = 0$. Therefore, for any $r \in R$, $(a, 0)(r, 0)(a^*, 0) = (0, 0)$. That is $(ara^*, 0) = (0, 0)$. So $ara^* = 0$ for all $r \in R$. Hence $aRa^* = 0$. Also an involution on R is semi-proper, we have $a = 0$. So $(a, \lambda) = (0, 0)$. Thus involution of R_1 is semi-proper. □

Lemma 4.3. *With notations as in Definition 4.1, let $x \in R$ and let e be a projection in R. Then $C(x) = e$ in R if and only if $C((x, 0)) = (e, 0)$ in R_1.*

Proof. Let $x \in R$. Suppose $e = C(x)$. So (1) $xe = x$; (2) $xRy = 0$ if and only if $ey = 0$. Consider $(x, 0)(e, 0) = (xe, 0) = (x, 0)$. Suppose $(x, 0)R_1(b, \lambda) = (0, 0)$. Hence $(0, 0) = (x, 0)R_1(b, \lambda) = (xe, 0)R_1(b, \lambda) = (x, 0)(e, 0)R_1(b, \lambda) = (x, 0)R_1(e, 0)(b, \lambda) = (x, 0)R_1(eb + \lambda e, 0)$. This yields $xR(eb+\lambda e) = 0$. Thus $e(eb+\lambda e) = 0$. Consequently $eb+\lambda e = 0$, that is, $(e, 0)(b, \lambda) = (0, 0)$. Therefore, by Theorem 3.7, $(e, 0)$ is a central cover of $(x, 0)$ in R_1. □

Proposition 4.4. *In a weakly p.q.-Baer $*$-ring R, $xRy = 0$ if and only if $C(x)C(y) = 0$.*

Proof. Follows in the same way as the proof of the Part (2) of the Theorem 2.5. □

Theorem 4.5. *Let R be a weakly p.q.-Baer $*$-ring. If e and f are projections in R, then there exists a central projection g such that $e \leq g$ and $f \leq g$ (i.e. g is an upper-bound of the set $\{e, f\}$).*

Proof. Suppose that f is a central projection in R. Let $g = f+C(e-ef)$ and $h = C(e-ef)$, that is, $g = f+h$. As f is a central projection, $C(f) = f$ and $fR(e-ef) = R(e-ef)f = \{0\}$. By Proposition 4.4, $C(f)C(e - ef) = 0$, consequently $fh = 0$. So $g = f + h$ is a central projection in R. Consider $fg = f(f + h) = f + fh = f$. Hence $f \leq g$. Since $h = C(e - ef)$, $(e-ef)h = e-ef$. Therefore $0 = (e-ef)h-(e-ef) = eh-e(fh)-e+ef = eh-e+ef$. Thus $e = ef + eh = e(f + h) = eg$. This yields $e \leq g$. Consequently g a central projection which is an upper-bound of $\{e, f\}$. Moreover g is the least upper bound as a central projection in the set of all upper bounds which are central projections. Now suppose e and f are any two projections (not necessarily any of them to be central) in R. Let g' be a central projection in R (we can take $g' = 0$). Let e' be a central projection which is an upper bound of $\{e, g'\}$ and f' be a central projection which is an upper bound of $\{f, g'\}$. Let g'' be a central projection which is an upper bound of $\{e', f'\}$. Therefore g'' is an upper-bound of $\{e, f\}$. □

Now, we provide a partial solution to the unitification problem of weakly p.q.-Baer $*$ rings.

Theorem 4.6. *A weakly p.q.-Baer $*$-ring R can be embedded in a p.q.-Baer $*$-ring, provided there exists, a ring K such that,*
(i) K is an integral domain with involution,
(ii) R is a $*$-algebra over K,
(iii) For any $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in R$ that is an upper bound for the central covers of the right annihilators of λ, that is, for $t \in R$, if $\lambda\, t = 0$ then $C(t) \leq e_\lambda$.

Proof. Let $R_1 = R \oplus K$ (the additive group direct sum), with operations in Definition 4.1. First we prove that for any $a \in R$ and a nonzero scalar $\gamma \in K$, there exists a greatest central projection g in R such that $ag = \gamma g$. By the condition (iii), there exists a projection $e_\gamma \in R$ that is an upper bound for the central covers of the right annihilators of γ. Let $e_0 = C(a)$. By Theorem 4.5, there exists a central projection e which is an upper bound of $\{e_0, e_\gamma\}$. Therefore e is a projection in R with $e_0 \leq e$, that is, $e_0 = e_0 e$. As $e_0 = C(a)$, we have $ae_0 = a$. It follows that $ae_0 e = ae$. Hence $a = ae$. Since e is central, $a = eae \in eRe$. Therefore $a - \gamma e = eae - e\gamma e = e(a - \gamma e)e \in eRe$. Let $h = C(a - \gamma e)$. So $(a - \gamma e)h = a - \gamma e$. As $(a - \gamma e)e = a - \gamma e$, we have $h \leq e$, i.e., $h = he$. Observe that $g = e - h$ is a central projection. Consider $ag - \gamma eg = a(e - h) - \gamma e(e - h) = ae - ah - \gamma e + \gamma eh = (a - \gamma e) - (a - \gamma e)h = (a - \gamma e) - (a - \gamma e) = 0$. Also $\gamma eg = \gamma e(e - h) = \gamma e - \gamma eh = \gamma e - \gamma he = \gamma e - \gamma h = \gamma(e - h) = \gamma g$. This yields $0 = ag - \gamma eg = ag - \gamma g$. Thus $ag = \gamma g$.

Now we prove maximality of g. Let k be a central projection in R such that $ak = \gamma k$, that is, $ak - \gamma k = 0$. Since $a = ea$, $eak - \gamma k = 0$. Hence $e\gamma k - \gamma k = 0$. Consequently $\gamma(ek - k) = 0$. Let $f = C(ek - k)$. So f is a central projection in R. Therefore $(\gamma f)(ek - k) = f\gamma(ek - k) = 0$. It follows that $(ek - k)(\gamma f) = 0$, which further yields $(ek - k)R(\gamma f) = 0$. Hence $f(\gamma f) = 0$, and consequently $\gamma f = 0$. This gives $C(f) \leq e_\gamma \leq e$. So $f \leq e$, that is, $f = fe$. As e is a central projection and $f = C(ek - k)$, $(ek - k)Re = 0$. Therefore $fe = 0$, thus $f = 0$. Hence $ek - k = f(ek - k) = 0$. So $ek = k$, hence $k \leq e$. Since $ak = \gamma k$ and k is central, we have $(a - \gamma e)Rk = 0$. Further $h = C(a - \gamma e)$, gives $hk = 0$. Consider $kg = k(e - h) = ke - kh = ke - 0 = k$, that is, $k \leq g$. Thus g is the greatest central projection such that $ag = \gamma g$.

To show R_1 is a p.q.-Baer $*$-ring, it is sufficient to prove that for every element $x \in R_1$ there exists a central projection $e \in R_1$ such that (1) $xe = x$; and (2) $xRy = 0$ if and only if $ey = 0$ (because of Corollary 3.8 and Theorem 3.10). Let $x = (a, \lambda) \in R_1$. If $\lambda = 0$, then by Lemma 4.3, x has a central cover. Suppose $\lambda \neq 0$. By the above discussion, there exists a greatest central projection g in R such that $ag = (-\lambda)g$. So, $xg = (a, \lambda)(g, 0) = (ag + \lambda g, 0) = (0, 0)$. Let $e = 1 - g = (0, 1) - (g, 0) = (-g, 1)$. We prove that $C(x) = e$ in R_1. Consider $xe = (a, \lambda)(-g, 1) = (-ag + a - \lambda g, \lambda) = (\lambda g + a - \lambda g, \lambda) = (a, \lambda) = x$. Suppose $(a, \lambda)R_1(b, \mu) = (0, 0)$. So $(a, \lambda)(0, 1)(b, \mu) = (0, 0)$. Hence $(a, \lambda)(b, \mu) = (0, 0)$, that is, $(ab + \mu a + \lambda b, \lambda\mu) = (0, 0)$. This gives, $ab + \mu a + \lambda b = 0$ and $\lambda\mu = 0$. As $\lambda \neq 0$, we have $\mu = 0$. This yields $ab + \lambda b = 0$. Let $f = C(b)$ in R. Hence (1) $fb = b$; (2) $yRb = 0$ if and only if $yf = 0$. Since $(a, \lambda)R_1(b, \mu) = (0, 0)$ for any $r \in R$, we have $(a, \lambda)(r, 0)(b, 0) = (0, 0)$. That is $(arb + \lambda rb, 0) = (0, 0)$. Therefore $arb + \lambda rb = 0$ for any $r \in R$. Consider $0 = arb + \lambda rb = arb + \lambda rfb = (a + \lambda f)rb$ for any $r \in R$. That is $(a + \lambda f)Rb = 0$, and consequently $(a + \lambda f)f = 0$. Therefore $af = -\lambda f$. Since g is the greatest projection with this property, we have $f \leq g$, i.e., $f = fg$. Consider $ey = (-g, 1)(b, \mu) = (-g, 1)(b, 0) = (-gb + b, 0) = (-gfb + fb, 0) = (-fb + fb, 0) = (0, 0)$. Thus R_1 is a p.q.-Baer $*$-ring. □

A partial solution of **Problem 2** analogous to a partial solution of **Problem 1** given by Berberian [2], can be obtained as a corollary.

Corollary 4.7. *Let R be a weakly p.q.-Baer $*$-ring. If there exists an involutary integral domain K such that R is a $*$-algebra over K and it is torsion free K-module, then R can be embedded in a p.q.-Baer $*$-ring with preservation of central covers.*

Proof. By [11, Corollary 4], if R is torsion-free then R satisfies the condition (iii) in Theorem 4.6. □

Remark 2: Let $A = C_\infty(T) \otimes M_2(\mathbb{Z}_3)$ (external direct product of the $*$-rings), where $C_\infty(T)$ denotes the algebra of all continuous, complex valued functions on T that vanish at ∞. In [2], it is proved that, $C_\infty(T)$ is a commutative weakly Rickart C^*-algebra, when T is a non-compact, locally compact, Hausdorff space. Hence $C_\infty(T)$ is a weakly p.q.-Baer $*$-ring. Since $M_2(\mathbb{Z}_3)$ is a Baer $*$-ring and hence it is a weakly p.q.-Baer $*$-ring. Therefore A is a weakly p.q.-Baer $*$-ring. Also A satisfies condition (iii) in the Theorem 4.6. Since there does not exist an integral domain K such that A is torsion free K-module, so A does not satisfies the hypotheses of Corollary 4.7. However, A satisfies the hypotheses of the Theorem 4.6. Moreover, $A_1 = A \oplus \mathbb{Z}$ (operations as in unitification) is a p.q.-Baer $*$-ring containing A that preserves central covers.

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