A priori bounds for geodesic diameter. Part III. A Sobolev-Poincaré inequality and applications to a variety of geometric variational problems

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Abstract

Based on a novel type of Sobolev-Poincaré inequality (for generalised weakly differentiable functions on varifolds), we establish a finite upper bound of the geodesic diameter of generalised compact connected surfaces-with-boundary of arbitrary dimension in Euclidean space in terms of the mean curvatures of the surface and its boundary. Our varifold setting includes smooth immersions, surfaces with finite Willmore energy, two-convex hypersurfaces in level-set mean curvature flow, integral currents with prescribed mean curvature vector, area minimising integral chains with coefficients in a complete normed commutative group, varifold solutions to Plateau's problem furnished by min-max methods or by Brakke flow, and compact sets solving Plateau problems based on Čech homology. Due to the generally inevitable presence of singularities, path-connectedness was previously known neither for the class of varifolds (even in the absence of boundary) nor for the solutions to the Plateau problems considered.

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Contents

1 Introduction

Throughout this introduction, we suppose *m and n are integers, $2 \leq m \leq n$, and B is a nonempty compact $m-1$ dimensional submanifold of class 2 of $\mathbf{R}^n$.*

Definition (see 7.1)**.** For closed subsets A of $\mathbf{R}^n$, their *geodesic diameter* is the supremum of all numbers $\sigma(a,x)$ corresponding to $a, x \in A$, where $\sigma(a,x)$ is the infimum of the set of lengths of continuous paths in A connecting a and x.

We derive upper bounds for the geodesic diameter of sets A associated with solutions to a variety of geometric variational problems. They imply finiteness and are given solely in terms of the boundary data and the dimensions involved.

1.1 Plateau problems

We illustrate our varifold-theoretic results by their implication on the two most prominent formulations of Plateau's problem in geometric measure theory.

Theorem A (see 8.4 and 8.5)**.** *Suppose $S \in \mathbf{I}_m(\mathbf{R}^n)$, ∂S is indecomposable, $\| \partial S \| = \mathscr{H}^{m-1} \llcorner B$,*

$$\|S\|(\mathbf{R}^n) \leq \|T\|(\mathbf{R}^n) \quad \textit{whenever } T \in \mathbf{I}_m(\mathbf{R}^n) \textit{ and } \partial T = \partial S,$$

and d denotes the geodesic diameter of $\operatorname{spt} \|S\|$.

Then, for some positive finite number Γ determined by m, there holds

$$d \leq \Gamma \textstyle\int_B |\mathbf{h}(B,b)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1} \, b;$$

here, by convention, we stipulate $\int_B |\mathbf{h}(B,b)|^0 \, \mathrm{d}\mathscr{H}^1 \, b = \mathscr{H}^1(B)$ regarding $m = 2$.

By the fundamental results of H. Federer and W. Fleming, there exists an absolutely area minimising integral current S satisfying the hypotheses of Theorem A whenever B is connected and orientable. Even in case $A = \operatorname{spt} \|S\|$ is an m dimensional submanifold-with-boundary of class 2 and $\partial A = B$ no a priori estimate for the geodesic diameter was known prior to this work. If $m < n-1$, even finiteness of d is new; in fact, it was only known that $A \sim B$ is connected by [DLDPHM23, Corollary 1.10 (b)] by C. De Lellis, G. De Philippis, J. Hirsch, and A. Massaccesi provided B is class 4 and that in this case the geodesic distance on $A \sim B$ (in the sense of [Men16b, Definition 6.6]) is real valued and continuous with respect to the Euclidean metric by [Men16b, Theorem 6.8 (1)] by the first author. If $m = n - 1$, the properties of the geodesic distance on $A \sim B$ may be combined with the studies of R. Hardt and L. Simon in [HS79] to yield finiteness of d without providing any a priori estimate on d, see 8.7. Theorem A transfers unchanged to the context of integral flat chains modulo ν; in fact, in 8.4, we provide a formulation of Theorem A in terms of the integral chains with coefficients in a complete normed commutative group constructed in the first paper of our series [MS22], see 8.6.

The role of the indecomposability hypothesis on ∂S is merely to guarantee the indecomposability of S, see 8.4 and 8.5. Thus, we obtain a nonexistence criterion for indecomposable solutions to Plateau's problem, see 8.8.

Next, we describe the theorem which results from combining our result on area-stationary sets in 8.21 with the studies of the Reifenberg-type Plateau problem by Y. Fang, S. Kolasiński, H. Pugh, and C. Labourie in [FK18, Pug19, Lab22].

Theorem B (see 8.22)**.** *Suppose B is connected, G is a commutative group, L is a subgroup of the $(m-1)$-th Čech homology group of B, $\check{\mathscr{C}}(B, L, G)$ denotes the family of closed subsets of $\mathbf{R}^n$ spanning L, $E \in \check{\mathscr{C}}(B, L, G)$,*

$$\mathscr{H}^m(E) = \inf \left\{ \mathscr{H}^m(F) : F \in \check{\mathscr{C}}(B, L, G) \right\}, \quad A = \operatorname{spt}(\mathscr{H}^m \llcorner E),$$

and d is the geodesic diameter of A.

Then, for some positive finite number Γ determined by n, there holds

$$d \leq \Gamma \operatorname{reach}(B)^{-m} \mathscr{H}^{m-1}(B)^{m/(m-1)} \textstyle\int_B |\mathbf{h}(B, b)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1} \, b;$$

here, by convention, we stipulate $\int_B |\mathbf{h}(B, b)|^0 \, \mathrm{d}\mathscr{H}^1 \, b = \mathscr{H}^1(B)$ regarding $m = 2$.

This yields an a priori bound on d determined by the boundary B and the dimension n. As before, no bound was known even in the smooth case and, due to inevitable singularities, only points in $A \sim B$ in the same connected component of A were known to admit a connecting path in A of finite length.

1.2 The applications to geometric variational problems

The scope of our present results is much wider than the relatively regular setting of minimisers of Plateau problems: It encompasses the Willmore energy with clamped boundary condition studied by M. Novaga and M. Pozzetta, see 7.6, level-set mean curvature flow of two-convex hypersurfaces as studied by P. Gianniotis and R. Haslhofer in [GH20], see 7.8, λ-minimising currents (which in turn include integral currents with prescribed mean curvature vector and codimension-one area minimising integral currents with prescribed volume, as studied by F. Duzaar, K. Steffen, and M. Fuchs in [DS93a, DF90, DF92, Duz93, DS92, DS93b], see 7.21), integral varifolds stationary in $\mathbf{R}^n \sim B$ as furnished either by Brakke flow with fixed boundary studied by S. Stuvard and Y. Tonegawa in [ST21], see 8.16 and 8.20, or by min-max methods studied by C. De Lellis, J. Ramic, and R. Montezuma in [DLR18, Mon20], see 8.18.

1.3 The general results in the varifold-setting

Here, we discuss our key results—Theorems C, D, and E—in the varifold-setting. The following subsection will then include corresponding statements—Corollaries 1, 3, and 2—formulated in a purely differential-geometric setting as well as two further theorems in the varifold setting—Theorems F and G. The latter are tailored for applications to geometric variational problems.

Hypotheses 1 (First variation)**.** *Suppose V is an m dimensional varifold in an open subset U of $\mathbf{R}^n$ and the first variation,*

$$\delta V \in \mathscr{D}'(U, \mathbf{R}^n),$$

of V is representable by integration (equivalently, the variation measure, $\|\delta V\|$, of δV is a Radon measure).

We recall that an m dimensional varifold V in U is a Radon measure over the Cartesian product of U with the Grassmann manifold $\mathbf{G}(n, m)$ consisting of all (unoriented) m dimensional vector subspaces of $\mathbf{R}^n$, that the projection of V onto the first factor is termed the weight of V and is denoted by $\|V\|$, and that $\mathbf{T}(V)$ is the class consisting of all real valued generalised V weakly differentiable functions, see [MS18, Definition 4.2] by the authors, which includes all locally Lipschitzian functions $f : U \to \mathbf{R}$, see [MS18, Lemma 4.6 (1)].

Definition (see [Men16a, Definition 5.1] by the first author)**.** Suppose V satisfies the Hypotheses 1 and E is measurable with respect to $\|V\|$ and $\|\delta V\|$.

Then, the *distributional V boundary* of E is defined by

$$V \, \partial \, E = (\delta V) \llcorner E - \delta(V \llcorner E \times \mathbf{G}(n, m)) \in \mathscr{D}'(U, \mathbf{R}^n).$$

If V and E are suitably regular (see H. Federer's characterisation of sets of locally finite perimeter in [Fed69, 4.5.11]), then $V \, \partial \, E$ may be expressed in terms of the exterior normal of E (with respect to V), see [Men16a, Theorem 5.9]. The notion of boundary allows us to formulate indecomposability of a varifold by considering, for a given class of functions, how many of its superlevel sets split the varifold in a nontrivial way and yet have no distributional boundary.

Definition (see [MS23, 7.1])**.** Suppose that V satisfies the Hypotheses 1 and that $\Psi \subset \mathbf{T}(V)$.

Then, V is called *indecomposable of type* Ψ if and only if, whenever $f \in \Psi$, the set of $y \in \mathbf{R}$, such that $E(y) = \{x : f(x) > y\}$ satisfies

$$\|V\|(E(y)) > 0, \quad \|V\|(U \sim E(y)) > 0, \quad V \, \partial \, E(y) = 0$$

has $\mathscr{L}^1$ measure zero.

It is crucial for the present final paper of our series, that indecomposability of type $\mathscr{D}(U, \mathbf{R})$ is strictly weaker than indecomposability as defined in [Men16a, Definition 6.2]. An in-depth comparison of notions of indecomposability has been carried out in the second paper [MS23]; for instance, two touching spheres yield a varifold which is decomposable but indecomposable of type $\mathscr{D}(U, \mathbf{R})$, see Example 2 and Corollary 2 therein. The formulation of our main results involves two further sets of hypotheses whose meaning and significance we shall discuss next.

Hypotheses 2 (Density and mean curvature)**.** *Suppose V is an m dimensional varifold in an open subset U of $\mathbf{R}^n$, $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, $1 \leq p \leq \infty$, and the generalised mean curvature vector $\mathbf{h}(V, \cdot)$ of V belongs to $\mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$.*

Unlike in the differential-geometric case, $\mathbf{h}(V, \cdot)$ may have a nontrivial tangential component related to variations of $\boldsymbol{\Theta}^m(\|V\|, \cdot)$. In this regard, considering the example of a weighted properly embedded smooth submanifold (see [Men16a, Remark 7.6, Lemma 15.2]), the following question seems natural; if V is integral, an affirmative answer follows from [Men13, Theorem 4.8] of the first author.

Question 1. Suppose V satisfies the Hypotheses 1, $\tau = \operatorname{Tan}^m(\|V\|, \cdot)_\natural$ is the tangent plane[1] function, and $\Theta(x) \geq 1$ for $\|V\|$ almost all x, where $\Theta = \boldsymbol{\Theta}^m(\|V\|, \cdot)$. Does it follow that both functions τ and Θ are $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all x, and, if so, does there hold—denoting $(\|V\|, m)$ approximate derivatives by the prefix "ap"—the equation

$$\mathbf{h}(V, x) \bullet u = T(\operatorname{ap} \mathrm{D}\, \tau(x) \circ \tau(x)) \bullet u + (\operatorname{ap} \mathrm{D}(\log \circ \Theta)(x) \circ \tau(x))(u) \quad \text{for } u \in \mathbf{R}^n$$

for $\|V\|$ almost all x, where the trace operator T is as in [Men16a, 15.1]?

Concerning the significance of the Hypotheses 2, we recall that, if $p \geq m$, then $\operatorname{spt} \|V\|$ is in many ways well-behaved: For instance, there holds

$$\boldsymbol{\Theta}_*^m(\|V\|, x) \geq 1 \quad \text{for } x \in \operatorname{spt} \|V\|$$

by [Men09, Remark 2.7] of the first author—in particular, $\operatorname{spt} \|V\|$ has locally finite $\mathscr{H}^m$ measure—, the set $\operatorname{spt} \|V\|$ is locally connected (see [Men16a, Corollary 6.14 (3)]), decompositions of V are locally finite (see [Men16a, Remark 6.11]) and non-uniquely refine the decomposition of $\operatorname{spt} \|V\|$ into connected components (see [Men16a, Remark 6.13, Corollary 6.14 (1)]), connected components of $\operatorname{spt} \|V\|$ are locally connected by paths of finite length (see [Men16a, Theorem 14.2]), and the resulting geodesic distance thereon is a continuous Sobolev function with bounded generalised weak derivative, see [Men16b, Theorem 6.8 (1)]. A substantial challenge for the present development arises from the fact that, if $p < m$, then $\operatorname{spt} \|V\|$ has substantially less geometric significance: Whenever X is an open subset of U, there exists a varifold V such that $\operatorname{spt} \|V\|$ equals the closure of X relative to U, see [Men16a, Example 14.1]. However, one is at least assured that $\mathscr{H}^{m-p}$ almost all $x \in \operatorname{spt} \|V\|$ satisfy the dichotomy

$$\text{either } \boldsymbol{\Theta}_*^m(\|V\|, x) \geq 1 \quad \text{or } \boldsymbol{\Theta}^m(\|V\|, x) = 0$$

by [Men09, Remark 2.11].

The next set of hypotheses concerns the formulation of a boundary condition for varifolds. This is complicated by the absence of a boundary operator as is available for currents. In this regard, the distribution $B \in \mathscr{D}'(U, \mathbf{R}^n)$ defined by

$$B(\theta) = (\delta V)(\theta) + \textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\|\, x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n)$$

may act as a replacement whenever V satisfies the Hypotheses 1. According to W. Allard in [All72, 4.3], $\|B\| = \|\delta V\|_{\mathrm{sing}}$ *in some sense* is the boundary of V;

[1] For $\|V\|$ almost all x, the closed cone $\operatorname{Tan}^m(\|V\|, x)$ is an m dimensional plane and $\tau(x)$ is the orthogonal projection retracting $\mathbf{R}^n$ onto $\operatorname{Tan}^m(\|V\|, x)$.

here we have employed $\|\delta V\|_{\mathrm{sing}}$ to denote the unique Radon measure over U such that

$$\|\delta V\| = \|\delta V\|_{\|V\|} + \|\delta V\|_{\mathrm{sing}}.$$

Yet, it appears more accurate to consider $\|B\|$ as stemming from two ingredients: firstly indeed, the *geometric boundary* of V but, on top of that, the singular part of the distributional derivative of the tangent plane function of V; in [Men17, Example 15] by the first author, this is illustrated by a varifold V associated with a properly embedded submanifold-with-boundary M of class 1 of $\mathbf{R}^n$ such that the support of $\|B\|$ is not contained in ∂M. (A more basic example with M of class 0 but not of class 1 is given by the varifold associated with the boundary of an $m+1$ dimensional cube, see Footnote 2 on page 6.)

Hypotheses 3 (Density and boundary)**.** *Suppose V and W are m and $m-1$ dimensional varifolds in an open subset U of $\mathbf{R}^n$, respectively, $\|\delta V\|$ and $\|\delta W\|$ are Radon measures, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, $\boldsymbol{\Theta}^{m-1}(\|W\|, x) \geq 1$ for $\|W\|$ almost all x,*

$$W = 0 \text{ if } m = 2, \quad \|\delta V\| \leq \|V\| \llcorner |\mathbf{h}(V, \cdot)| + \|W\| \text{ if } m > 2,$$
$$\|\delta W\| \text{ is absolutely continuous with respect to } \|W\| \text{ if } m > 3.$$

The displayed inequality is equivalent to requiring $\|B\| \leq \|W\|$. It implies that the geometric boundary of V is contained in W but not necessarily equal to W. There are good geometric reasons to consider the stronger condition

$$|B(\theta)| \leq \textstyle\int |S_\natural^\perp(\theta(x))| \, \mathrm{d}W\,(x, S) \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n)$$

which may be seen as the boundary part of F. Almgren's concept of *regular pair* (V, W) described in [Alm66, Subsection 4-3]. This stronger condition is also employed by T. Ekholm, B. White, and D. Wienholtz in [EWW02, Section 7]. For our present purposes, the weaker condition will be sufficient. Finally, the last condition in the Hypotheses 3 excludes the presence of boundary for W.[2]

To formulate the next theorem, we recall that $V\,\mathbf{D}\,f$ denotes the generalised V weak derivative of f whenever $f \in \mathbf{T}(V)$, see [MS18, Definition 4.2], and that $\mathbf{T}_{\mathrm{Bdry}\,U}(V)$ denotes the subclass of those members of $\mathbf{T}(V)$ which are nonnegative and have zero boundary values on $\mathrm{Bdry}\,U$, see [MS18, Definition 4.16]. It constitutes the varifold formulation of the novel type of Sobolev-Poincaré inequality formulated in the context of submanifolds-with-boundary of class 2 in Corollary 1 below.

Theorem C (see 4.5)**.** *Suppose V and W satisfy the Hypotheses 3,*

$$\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n), \quad \text{if } m > 3 \text{ then } \mathbf{h}(W, \cdot) \in \mathbf{L}_{m-2}^{\mathrm{loc}}(\|W\|, \mathbf{R}^n),$$
$$f \in \mathbf{T}_{\mathrm{Bdry}\,U}(V) \cap \mathbf{T}_{\mathrm{Bdry}\,U}(W),$$
$$|V\,\mathbf{D}\,f(x)| \leq 1 \text{ for } \|V\| \text{ almost all } x, \quad |W\,\mathbf{D}\,f(x)| \leq 1 \text{ for } \|W\| \text{ almost all } x,$$

and $E = \{x : f(x) > 0\}$.

[2]This condition is natural from the differential-geometric point of view. However, following F. Almgren's original approach to compactness (see [Alm65, Theorem 10.8]), one might also study tuples $(V_0, \ldots, V_m)$ consisting of i dimensional varifolds V_i such that V_{i-1} controls the boundary behaviour of V_i for $i > 0$. This would include m dimensional cubes, for instance.

Then, there exists a Borel subset Y of $\mathbf{R}$ such that

$$f(x) \in Y \quad \text{for } \|V\| \text{ almost all } x$$

and such that, for some positive finite number Γ determined by m,

(1) *if $m = 2$, then $\mathscr{L}^1(Y) \le \Gamma\big(\|V\|(E)^{1/2} + \|\delta V\|(E)\big)$;*

(2) *if $m = 3$, then*

$$\mathscr{L}^1(Y) \le \Gamma\big(\|V\|(E)^{1/3} + \textstyle\int_E |\mathbf{h}(V,\cdot)|^2 \,\mathrm{d}\|V\| + \|W\|(E)^{1/2} + \|\delta W\|(E)\big);$$

(3) *if $m > 3$, then*

$$\begin{aligned}\mathscr{L}^1(Y) \le \Gamma\big(\|V\|(E)^{1/m} &+ \textstyle\int_E |\mathbf{h}(V,\cdot)|^{m-1} \,\mathrm{d}\|V\| \\ &+ \|W\|(E)^{1/(m-1)} + \textstyle\int_E |\mathbf{h}(W,\cdot)|^{m-2} \,\mathrm{d}\|W\|\big).\end{aligned}$$

The utility of these estimates stems from [MS23, Theorem B]: namely, if V is indecomposable of type $\{f\}$, then $\operatorname{spt} f_\# \|V\|$ is an interval and satisfies the bound

$$\operatorname{diam} \operatorname{spt} f_\# \|V\| \le \mathscr{L}^1(Y).$$

Moreover, if f is continuous, then $f[\operatorname{spt} \|V\|] \subset \operatorname{spt} f_\# \|V\|$, see [MS23, 7.13 (2)]. The resulting oscillation estimate, is the key to establish the next two theorems.

Theorem D (see 6.7 (1))**.** *Suppose V and W satisfy the Hypotheses 3, if $m = 2$ then $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$,*[3] *$\|\delta W\|$ is absolutely continuous with respect to $\|W\|$, $m - 1 \le p < m$, $\mathbf{h}(V,\cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, if $m > 2$ then $\mathbf{h}(W,\cdot) \in \mathbf{L}_{p-1}^{\mathrm{loc}}(\|W\|, \mathbf{R}^n)$, and V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$.*

Then, there holds

$$\text{either } \boldsymbol{\Theta}_*^m(\|V\|, x) \ge 1 \quad \text{or } \boldsymbol{\Theta}_*^{m-1}(\|W\|, x) \ge 1$$

for $\mathscr{H}^{m-p}$ almost all $x \in \operatorname{spt} \|V\|$; in particular, $\mathscr{H}^m \llcorner \operatorname{spt} \|V\| \le \|V\|$.

The result is already significant in case $W = 0$ because it implies that indecomposability of type $\mathscr{D}(U, \mathbf{R})$ allows to discard the alternative $\boldsymbol{\Theta}^m(\|V\|, x) = 0$ from the afore-mentioned dichotomy. In general, this alternative may not be omitted as is shown by suitable decomposable varifolds, see [Men09, 2.11].

Theorem E (see 7.4)**.** *Suppose V and W satisfy the Hypotheses 3 with $U = \mathbf{R}^n$, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, we have $(\|V\| + \|W\|)(\mathbf{R}^n) < \infty$, and d denotes the geodesic diameter of $\operatorname{spt} \|V\|$.*

Then, for some positive finite number Γ determined by m, there holds

(1) *if $m = 2$, then $d \le \Gamma \|\delta V\|(\mathbf{R}^n)$;*

(2) *if $m = 3$, then $d \le \Gamma\big(\int |\mathbf{h}(V,\cdot)|^2 \,\mathrm{d}\|V\| + \|\delta W\|(\mathbf{R}^n)\big)$; and,*

(3) *if $m > 3$, then $d \le \Gamma\big(\int |\mathbf{h}(V,\cdot)|^{m-1} \,\mathrm{d}\|V\| + \int |\mathbf{h}(W,\cdot)|^{m-2} \,\mathrm{d}\|W\|\big)$.*

[3] More generally, absolute continuity may be relaxed to $\|\delta V\| \le \|V\| \llcorner |\mathbf{h}(V,\cdot)| + \|W\|$ and "$W = 0$ if $m = 2$" may be omitted from the Hypotheses 3 for the present theorem.

In particular, if the sum on the right hand side of the inequality is finite, then $\operatorname{spt} \|V\|$ is a compact subset of $\mathbf{R}^n$ and any two points of $\operatorname{spt} \|V\|$ may be connected by a path of finite length in $\operatorname{spt} \|V\|$. In analogy with the properties described for the case $p = m$ of the Hypotheses 2, an array of further questions arises. In the absence of boundary, the most immediate ones read as follows.

Question 2. Suppose V satisfies the Hypotheses 3 with $W = 0$, V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$, and $\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$.[4]

(1) Is $\operatorname{spt} \|V\|$ locally connected?

(2) If so, is $\operatorname{spt} \|V\|$ locally connected by paths of finite length?

(3) If so, is the geodesic distance induced on connected components of $\operatorname{spt} \|V\|$ a Sobolev function with bounded generalised weak derivative and what are the continuity properties of this particular (or, any such) function?

The last item thereof relates to the possible study of *intermediate conditions on the mean curvature*, that is, to $1 < p < m$ in the Hypotheses 2 (see [Men16a, p. 990]); a special case of that item was already raised as fifth question in the MSc thesis of the second author supervised by the first author, see [Sch16, Section A].

1.4 The challenges and their resolution

The starting point of our line of research (see [MS18] and the previous two parts [MS22] and [MS23] of our series) culminating in the present final paper was the following a priori bound for geodesic diameter by P. Topping.

Theorem (see [Top08, Theorem 1.1])**.** *Suppose that M is a compact connected m dimensional manifold (without boundary) of class 2, that $F : M \to \mathbf{R}^n$ is an immersion of class 2, that g is the Riemannian metric on M induced by F, and that σ is the Riemannian distance associated with (M, g).*

Then, for some positive finite number Γ determined by m, there holds

$$\operatorname{diam}_\sigma M \leq \Gamma \textstyle\int_M |\mathbf{h}(F, x)|^{m-1} \, \mathrm{d}\mathscr{H}_\sigma^m \, x,$$

where the vector $\mathbf{h}(F, x)$ in $\mathbf{R}^n$ denotes the mean curvature of F at x in M.

1.4.1 Aim 1: the generalisation to a varifold-setting

The varifold-setting is the natural one to model generalised surfaces with mean curvature. The corresponding generalisation involved two challenges:

(i) how to rephrase the connectedness hypothesis for varifolds; and,

(ii) how to handle the low summability of the mean curvature.

Regarding (i), the first notion of connectedness developed for varifolds—termed *indecomposability*—was available from the theory of generalised weakly differentiable functions on varifolds, see [Men16a, Definition 6.2]. Proceeding to (ii), we recall that the Hypotheses 2 with $U = \mathbf{R}^n$ and $p = m - 1$ do not guarantee the required geometric significance of $\operatorname{spt} \|V\|$.

[4] If $m > 2$, the first and last condition are equivalent to the Hypotheses 2 with $p = m - 1$; if $m = 2$, they do not require $\|\delta V\|$ to be absolutely continuous with respect to $\|V\|$.

The key to resolve these two challenges is to treat them simultaneously by means of the insight that *indecomposability has a strong regularising effect*; in particular, the known examples exhibiting undesirable behaviour of $\operatorname{spt} \|V\|$, such as [Men16a, Example 14.1] or the earlier one of a similar structure in [Men09, Example 1.2], are decomposable. The analysis carried out in [Sch16], not only yields a complete generalisation of the above estimate of the geodesic diameter to the varifold setting (i.e., Theorem E with $W = 0$) but it also yields a lower density bound for the varifolds involved (i.e., Theorem D with $W = 0$ and $p = m - 1$) ensuring the geometric significance of $\operatorname{spt} \|V\|$ via

$$\boldsymbol{\Theta}_*^m(\|V\|, x) \geq 1 \quad \text{for } \mathscr{H}^1 \text{ almost all } x \in \operatorname{spt} \|V\|.$$

We note that both theorems describe a one-dimensional property of $\operatorname{spt} \|V\|$: the length of geodesics therein and a lower density bound $\mathscr{H}^1$ almost everywhere.

1.4.2 Aim 2: the treatment of surfaces with boundary

To allow for boundary is evidently a prerequisite for applications to geometric variational problems such as the Plateau problem. An initial step had been made by S.-H. Paeng: If $m = 2$ and M is a manifold-with-boundary, then the estimate in the preceding theorem may be replaced by

$$\operatorname{diam}_\sigma M \leq \Gamma\big(\textstyle\int_M |\mathbf{h}(F, x)| \, \mathrm{d}\mathscr{H}^2_\sigma \, x + \mathscr{H}^1_\sigma(\partial M)\big),$$

provided (M, g) *is convex*, see [Pae14, Theorem 2 (a)]. Therefore, to achieve the second aim, we had to resolve the following three additional challenges:

(iii) for $m = 2$, how to remove the convexity hypothesis;

(iv) for $m > 2$, how to take the geometry of the boundary into account; and,

(v) how to phrase the boundary condition in a varifold-setting.

In [Pae14], the convexity hypothesis is mainly used to ensure that interior points of length-minimising geodesics cannot meet ∂M. Viewing this as regularity consideration, a method sufficiently robust for the varifold-setting is likely to accommodate (iii) as well. For (iv), the $m - 1$ dimensional Hausdorff measure of ∂M (raised to the appropriate power) does not yield a valid estimate, see 7.14. Instead, the mean curvature of $F|\partial M$ turns out to be an adequate choice; in particular, the boundary is naturally represented by an $m - 1$ dimensional varifold. Regarding (v), we recall that the difficulty stems from the lack of a boundary operator for varifolds and is resolved by employing the conditions described in the Hypotheses 3 above as a substitute.

The second aim was then achieved in the first version of the present publication (see `arXiv:1709.05504v1`) which contained two additional insights with respect to [Sch16]: Firstly, it introduced the notion of indecomposability of type $\mathscr{D}(U, \mathbf{R})$. This connectedness property is strictly weaker than indecomposability but yet strong enough for the deduction of the intended geometric consequences; in fact, this weakening of our hypotheses has later turned out to be crucial for the applicability of our theory to geometric variational problems (see Aim 3 below). The second insight was the novel type of Sobolev-Poincaré inequality formulated in Theorem C which is foundational for both the varifold-results on density and those on geodesic diameter in Theorems D and E. To discuss the nature of Theorem C, we shall now describe the corollary resulting from it in the special case of properly embedded *connected* submanifolds.

Corollary 1 (Novel type of Sobolev-Poincaré inequality, see 4.8)**.** *Suppose U is an open subset of $\mathbf{R}^n$, M is a properly embedded, connected m dimensional submanifold-with-boundary of class 2 of U, $f : M \to \mathbf{R}$ is a function of class 1 relative to M,* $\operatorname{spt} f$ *is compact, and*

$$E = M \cap \{x : f(x) \neq 0\}, \quad \kappa = \sup\{|\operatorname{D} f(x)| : x \in M \sim \partial M\}.$$

Then, for some positive finite number Γ determined by m, there holds

$$\begin{aligned}\operatorname{diam} f[M] \leq \Gamma\big(&\mathscr{H}^m(E \cap M)^{1/m} + \textstyle\int_{E\cap M} |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x \\ &+ \mathscr{H}^{m-1}(E \cap \partial M)^{1/(m-1)} + \textstyle\int_{E\cap\partial M} |\mathbf{h}(\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} x\big)\kappa;\end{aligned}$$

here, the summand $\int_{E\cap\partial M} |\mathbf{h}(\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} x$ shall be omitted if $m = 2$.

In case M is compact, $\operatorname{spt} f$ is automatically compact and, applying the isoperimetric inequality to M and ∂M yields the following *Poincaré inequality* (see 7.15) with a positive finite number Δ determined by m:

$$\operatorname{diam} f[M] \leq \Delta\big(\textstyle\int_M |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x + \int_{\partial M} |\mathbf{h}(\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} x\big)\kappa;$$

regarding $m = 2$, we stipulate $\int_{\partial M} |\mathbf{h}(\partial M,x)|^0 \,\mathrm{d}\mathscr{H}^1 x = \mathscr{H}^1(\partial M)$ by convention. Earlier estimates of the essential oscillation of f, see [Men16a, Theorems 10.1 (1d), 10.7 (4), and 10.9 (4)], differ in several aspects: Firstly, they are limited to the case $\partial M = \varnothing$ but do not require connectedness of M. Secondly, they are local in nature rather than global. Thirdly, they involve the m-th power of the mean curvature (in fact, an integral smallness condition thereon) instead of the power $m - 1$. Finally, they allow for q-th power integrals of $\operatorname{D} f$ for some $m < q < \infty$ which in fact turns out to be impossible in our setting, see 7.17. The preceding Poincaré inequality is readily seen to be equivalent to an a priori bound for the geodesic diameter of M. Applying differential-topologic density results (to remove the embeddedness hypothesis) then yields the following extension of P. Topping's result to immersions of manifolds-with-boundary which corresponds to Theorem E in the varifold-setting.

Corollary 2 (Geodesic diameter bound for immersions, see 7.12)**.** *Suppose M is a compact connected m dimensional manifold-with-boundary of class 2, $F : M \to \mathbf{R}^n$ is an immersion of class 2, g is the Riemannian metric on M induced by F, and σ is the Riemannian distance associated with (M, g).*

Then, for some positive finite number Γ determined by m, there holds

$$\operatorname{diam}_\sigma M \leq \Gamma\big(\textstyle\int_M |\mathbf{h}(F,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m_\sigma x + \int_{\partial M} |\mathbf{h}(F|\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}_\sigma x\big);$$

regarding $m = 2$, we stipulate $\int_{\partial M} |\mathbf{h}(\partial M,x)|^0 \,\mathrm{d}\mathscr{H}^1_\sigma x = \mathscr{H}^1_\sigma(\partial M)$.

For $m \geq 3$, this leads to the following question which is open even if $\mathbf{h}(F, \cdot) = 0$ and F is an embedding. If the answer were in the affirmative, then Corollary 2 would be a consequence of the resulting statement applied to both F and $F|\partial M$.

Question 3. May the summand $\int_{\partial M} |\mathbf{h}(F,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}_\sigma x$ in Corollary 2 be replaced by the sum of the geodesic diameters of the connected components of ∂M computed with respect to the induced Riemannian distance on ∂M?

1.4.3 Interlude: lower density ratio bounds

The insights discussed so far allow us to deduce lower bounds on the density based on the connectedness hypothesis. To illustrate this, we state here the underlying conditional lower density ratio bound in the case of properly embedded submanifolds regarding the varifold-result on the density in Theorem D.

Corollary 3 (Conditional lower density ratio bound, see 6.3)**.** *Suppose U is an open subset of $\mathbf{R}^n$, M is a properly embedded, connected m dimensional submanifold-with-boundary of U of class 2, $a \in M$, $0 < r < \infty$, $\mathbf{B}(a,r) \subset U$, and $M \sim \mathbf{U}(a,r) \neq \varnothing$.*

Then, for some positive finite number Γ determined by m, there holds

$$\begin{aligned}\Gamma^{-1} r \leq{}& \mathscr{H}^m(\mathbf{U}(a,r) \cap M)^{1/m} + \textstyle\int_{\mathbf{U}(a,r)\cap M} |\mathbf{h}(M,\cdot)|^{m-1} \,\mathrm{d}\mathscr{H}^m \\ &+ \mathscr{H}^{m-1}(\mathbf{U}(a,r) \cap \partial M)^{1/(m-1)} + \textstyle\int_{\mathbf{U}(a,r)\cap \partial M} |\mathbf{h}(\partial M,\cdot)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1};\end{aligned}$$

here, the summand $\int_{\mathbf{U}(a,r)\cap\partial M} |\mathbf{h}(\partial M,\cdot)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}$ shall be omitted if $m = 2$.

This is a consequence of Corollary 1 generalised to Lipschitzian functions and applied with $f(x) = \sup\{r - |x-a|, 0\}$ for $x \in U$. Since $M \sim \mathbf{U}(a,r) \neq \varnothing$, our novel type of Sobolev-Poincaré inequality for this f takes the flavour of a *Sobolev inequality* because $\sup \operatorname{im} f = \operatorname{diam} f[M]$. Small spheres show that the hypothesis $M \sim \mathbf{U}(a,r) \neq \varnothing$ cannot be omitted. Therefore, whereas $p = m$ is the critical exponent in general, *the critical exponent for the summability of the mean curvature vector in the connected case is $p = m-1$.* This is the key to the regularising effects of indecomposability. For further illustration, we consider the following class of submanifolds (not necessarily properly embedded).

Hypotheses 4. *Suppose $1 \leq p < \infty$, M is a connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$ which meets every compact subset of $\mathbf{R}^n$ in a set of finite $\mathscr{H}^m$ measure such that its second fundamental form $\mathbf{b}(M,\cdot)$ satisfies $\int_M \|\mathbf{b}(M,x)\|^p \,\mathrm{d}\mathscr{H}^m x < \infty$ and the divergence theorem holds in the sense that*

$$\textstyle\int_M \operatorname{Tan}(M,x)_\natural \bullet \mathrm{D}\,\theta(x) \,\mathrm{d}\mathscr{H}^m x = - \int_M \mathbf{h}(M,x) \bullet \theta(x) \,\mathrm{d}\mathscr{H}^m x$$

for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$, and $A = (\operatorname{Clos} M) \cap \{a : \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0\}$.

In view of [MS23, 3.3, 6.1], Theorem D guarantees that

$$\mathscr{H}^{m-p}(A) = 0 \quad \text{in case } m-1 \leq p < m.$$

The next example shows not only that this bound is sharp but also that there is no corresponding result in the range $1 \leq p < m-1$. *This exhibits a discontinuity in p regarding the optimal upper bound on the Hausdorff dimension of A.*

Example (Sharpness of the lower density bounds, see 5.5 and 5.7)**.** *Suppose $m < n$. Then, the following two statements hold.*

(1) If $m - 1 < q < m$, then there exists M satisfying the Hypotheses 4 for $1 \leq p < q$ such that $\mathscr{H}^{m-q}(A) > 0$ for the associated set A.

(2) If $m \geq 3$, then there exists M satisfying the Hypotheses 4 for $1 \leq p < m-1$ such that $\mathscr{H}^m(A) > 0$ for the associated set A.

1.4.4 Aim 3: applicability to geometric variational problems

For this purpose, we were guided by the following two model cases:

(A) The Plateau problem for integral currents (more generally, G chains).

(B) The Plateau problem for sets using Čech homology.

The solution to each of these problems gives rise to an associated varifold V which is stationary away from the boundary. This entails the challenge:

(vi) Does the natural connectedness property of the solution (G chain or set) entail a suitable indecomposability property of the associated varifold?

Introducing indecomposability of type $\mathscr{D}(U, \mathbf{R})$, (vi) has been resolved in [MS23], see Theorem G therein for Case (A) and Corollary 2 therein for Case (B); in particular, for Case (B), it suffices to verify connectedness of $\operatorname{spt} \|V\|$. On the other hand, for Case (B), two considerations which do not pose a particular difficulty in Case (A), see 8.1 and 8.5, add two final challenges to our list:

(vii) Is there an a priori estimate for $\|\delta V\|$ in terms of the boundary data?

(viii) Does connectedness of the boundary imply connectedness of $\operatorname{spt} \|V\|$?

For these challenges in the present Case (B), a key ingredient is provided by C. Labourie with [Lab22, Lemmata 1.2.2 and 2.2.1] which ensure

$$B \subset \operatorname{spt} \|V\|,$$

see 8.22. The treatment of (vii) then uses the following theorem which rests on a refinement of W. Allard's estimates regarding boundary behaviour in [All75]. It employs H. Federer's concept of reach from [Fed59, 4.1].

Theorem F (see 8.12)**.** *Suppose $R = \operatorname{reach}(B)$, V is an m dimensional varifold in $\mathbf{R}^n$, $\|V\|(B) = 0$, $\operatorname{spt} \delta V \subset B \subset \operatorname{spt} \|V\|$, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, and*

$$M = R^{-m} \sup\{\|V\| \mathbf{U}(b, R/2) : b \in B\}.$$

Then, for some positive finite number Γ determined by m, there holds

$$\|\delta V\| \leq \Gamma M \mathscr{H}^{m-1} \llcorner B.$$

For Case (B), the $\|V\|$ measure of $\mathbf{R}^n$, and thus the number M, can readily be estimated in terms of B and m, see 8.22. Regarding (viii), we can rely on the study of connected components of $\operatorname{spt} \|V\|$ from [Men16a, Corollary 6.14] in combination with the isoperimetric inequality to deduce connectedness of $\operatorname{spt} \|V\|$ from that of its subset B, see 8.13. In combination with Theorem E, this yields the following theorem which is also applicable to varifolds constructed by min-max methods or Brakke flows, see 8.18 and 8.20.

Theorem G (see 8.17)**.** *Suppose B is connected, V is an m dimensional varifold in $\mathbf{R}^n$, $\|V\|(\mathbf{R}^n) < \infty$, $1 \leq \lambda < \infty$,*

$$B \subset \operatorname{spt} \|V\|, \quad \|\delta V\| \leq \lambda \mathscr{H}^{m-1} \llcorner B,$$

$\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, and d is the geodesic diameter of $\operatorname{spt} \|V\|$*.*

Then, there holds

$$d \leq \Gamma_{7.4}(m) \lambda \textstyle\int_B |\mathbf{h}(B, b)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1} \, b;$$

here, by convention, we stipulate $\int_B |\mathbf{h}(B, b)|^0 \, \mathrm{d}\mathscr{H}^1 \, b = \mathscr{H}^1(B)$ regarding $m = 2$.

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2 Notation

Basic sources As in Parts I and II (see [MS22, MS23]), our notation follows [Men16a] and is thus largely consistent with H. Federer's terminology for geometric measure theory listed in [Fed69, pp. 669–676] and W. Allard's notation for varifolds introduced in [All72]. This includes, for distributions S, the variation measure $\|S\|$, see [Men16a, 2.18]; for certain varifolds V and sets E, the notion of *distributional boundary*, $V \,\partial\, E$, of E with respect to V, see [Men16a, 5.1]; for certain varifolds V, concepts relating to generalised V weak differentiability—that is, the space $\mathbf{T}(V)$ of generalised V weakly differentiable real valued functions f, the *generalised V weak derivative*, $V \,\mathbf{D}\, f$, for such f, and the subspace $\mathbf{T}_G(V)$ of *nonnegative* members of $\mathbf{T}(V)$ realising the concept of *zero boundary values* on an open subset G of $\operatorname{Bdry} U$—, see [Men16a, 8.3, 9.1].

Review Here, we list symbols not already reviewed in [Men16a, Introduction, Section 1]: the Grassmann manifold, $\mathbf{G}(n, m)$, of all m dimensional subspaces of $\mathbf{R}^n$, see [Fed69, 1.6.2]; the norms associated with inner products, $|\cdot|$, see [Fed69, 1.7.1]; the seminorm, $\|\cdot\|$, on $\operatorname{Hom}(V, W)$ associated with normed spaces V and W, see [Fed69, 1.10.5]; the infimum and supremum, $\inf S$ and $\sup S$, of a subset S of the extended real numbers, see [Fed69, 2.1.1]; the class $\mathbf{2}^X$ of all subsets of X, see [Fed69, 2.1.2]; the restriction, $\phi \llcorner A$, of a measure ϕ to a set A, see [Fed69, 2.1.2]; the support, abbreviated $\operatorname{spt} \phi$, of a measure ϕ, see [Fed69, 2.2.1]; the least Lipschitz constant, $\operatorname{Lip} f$, of a map f between metric spaces, see [Fed69, 2.2.7]; the Lebesgue spaces, $\mathbf{L}_p(\phi, Y)$, see [Fed69, 2.4.12]; the support, denoted $\operatorname{spt} f$, of a member f of $\mathscr{K}(X)$, see [Fed69, 2.5.13]; the variation, $\mathbf{V}_a^b g$, of g from a to b for maps of the real line into a complete metric space, see [Fed69, 2.5.16]; the n dimensional Lebesgue measure, $\mathscr{L}^n$, see [Fed69, 2.6.5]; the diameter of S, $\operatorname{diam} S$, see [Fed69, 2.8.8]; the absolutely continuous part ψ_ϕ of a measure ψ with respect to ϕ, see [Fed69, 2.9.1];[5] the derivative, g', of a function on the real line, see [Fed69, 2.9.19]; the number $\boldsymbol{\alpha}(m)$ and the m dimensional Hausdorff measure, $\mathscr{H}^m$, for $0 \leq m < \infty$, see [Fed69, 2.10.2]; the m dimensional densities, $\boldsymbol{\Theta}^{*m}(\phi, a)$, $\boldsymbol{\Theta}^m_*(\phi, a)$, and $\boldsymbol{\Theta}^m(\phi, a)$, see [Fed69, 2.10.19]; the differential, $\mathrm{D} f$, see [Fed69, 3.1.1]; the closed cones of tangent and normal vectors, $\operatorname{Tan}(S, a)$ and $\operatorname{Nor}(S, a)$, see [Fed69, 3.1.21]; the unit sphere, $\mathbf{S}^{n-1}$, in $\mathbf{R}^n$, see [Fed69, 3.2.13]; the vector space, $\mathscr{E}(U, Y)$, of functions of class ∞ and the support, $\operatorname{spt} T$, of a distribution T in U of type Y, see [Fed69, 4.1.1]; the chain complex of *integral currents* in $\mathbf{R}^n$ with m-th chain group $\mathbf{I}_m(\mathbf{R}^n)$ and boundary operator ∂, see [Fed69, 4.1.7, 4.1.24]; the member $[u, v]$ of $\mathbf{I}_1(\mathbf{R}^n)$ associated with the line segment from u to v, see [Fed69, 4.1.8]; the weight, $\|V\|$, of a varifold V, see [All72, 3.1]; the image, $f_\# V$, of a varifold V under a map f of class ∞, see [All72, 3.2]; and, the first variation, δV, of a varifold V, see [All72, 4.2].

Modification For the push forward, $f_\# \phi$, of a measure ϕ by a function f, we use the definition in [MS22, 3.9] which extends [Fed69, 2.1.2].

Amendments Modelled on [Fed69, 4.1.7], we employ the restriction notation, $\phi \llcorner f$, for the measure ϕ weighted by f introduced in [MS22, 3.6]; this weighted measure was discussed—without name—in [Fed69, 2.4.10]. For subsets A of Euclidean space, we adopt the concept of reach and the symbol $\operatorname{reach}(A)$ from [Fed59, 4.1] as well as those of *approximate differentiability of order* 2 with the corresponding *approximate mean curvature*, denoted by $\operatorname{ap} \mathbf{h}(A, \cdot)$, from [San19, 3.8] and [MS23, 6.9]. For certain immersions F into an open subset U of $\mathbf{R}^n$, we use the concepts of *mean curvature vector*, denoted $\mathbf{h}(F, \cdot)$, of *varifold associated with* (F, U), and of *Riemannian distance* associated with F as laid down in [MS23, 6.10, 6.13, 10.1], respectively. The terms *immersion* and *embedding* are employed in accordance with [Hir94, p. 21]. Whenever k is a positive integer or $k = \infty$, we mean by a [*sub*]*manifold-with-boundary of class* k a Hausdorff topological space with a countable base of its topology that is, in the terminology of [Hir94, pp. 29–30], a C^k [sub]manifold. For manifolds-with-boundary M of class k, we similarly adapt the notion of *chart of class* k and *Riemannian metric of class* $k-1$ from [Hir94, p. 29, p. 95] and denote by ∂M its *boundary* as in [Hir94, p. 30]. Whenever G is a complete normed commutative group as defined in [MS22, 3.1], we employ the following notation regarding the group $\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G)$ of m dimensional *locally rectifiable G chains S* in $\mathbf{R}^n$, see [MS22, 4.5]: the notion of weight measure, $\|S\|$, see [MS22, 4.5]; the homomorphisms

$$f_\# : \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G) \to \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^\nu, G)$$

associated with locally Lipschitzian maps $f : \mathbf{R}^n \to \mathbf{R}^\nu$, see [MS22, 4.6]; the Cartesian product

$$\times : \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \times \mathscr{R}_\mu^{\mathrm{loc}}(\mathbf{R}^\nu, G) \to \mathscr{R}_{m+\mu}^{\mathrm{loc}}(\mathbf{R}^n \times \mathbf{R}^\nu, G),$$

[5] As was stipulated in [Men16a] for similar notions, ψ_ϕ will also be employed in the case where one of the measures ψ or ϕ on X fails to be finite on bounded sets but there exist open sets $U_1, U_2, U_3, \ldots$ covering X at which ϕ and ψ are finite.

see [MS22, 4.7]; the isomorphism $\iota_{\mathbf{R}^n,m} : \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n) \to \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z})$, see [MS22, 5.1]; the chain complex of *integral G chains* in $\mathbf{R}^n$ with m-th chain group $\mathbf{I}_m(\mathbf{R}^n, G)$—identified with a subgroup of $\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G)$—and boundary operator ∂_G, see [MS22, 5.11, 5.13, 5.17]; and, *indecomposability* of members of $\mathbf{I}_m(\mathbf{R}^n, G)$, see [MS23, 6.6]. Finally, for certain varifolds, we make use of the concept of *indecomposability of type* Ψ introduced in [MS23, 7.1].

Definitions in the text Following [Men16b, 6.6], the terms *geodesic distance* and *geodesic diameter* are laid down in 7.1. The locally convex space $\mathscr{C}^k(M, Y)$ is defined in 7.10.

3 The special case of extrinsic diameter and no boundary

To highlight some of the principal ideas, we provide the short proof a special case of Theorem E in this section; the general case will be treated in 7.4.

3.1 Theorem. *Suppose m and n are integers, $2 \le m \le n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\operatorname{spt} \|V\|$ is compact, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x, V is indecomposable, and*

(1) if $m = 2$, then $\psi = \|\delta V\|$, and

(2) if $m > 2$, then $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and $\psi = \|V\| \llcorner |\mathbf{h}(V, \cdot)|^{m-1}$.

Then, there holds

$$\operatorname{diam} \operatorname{spt} \|V\| \le \Gamma \psi(\mathbf{R}^n),$$

where Γ is a positive, finite number determined by m.

Proof. By [All72, 3.5 (1b), 5.5 (1)], V is rectifiable and $\|V\| = \mathscr{H}^m \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot)$.

Assume $\psi(\mathbf{R}^n) < \infty$. Let A denote the set of all $a \in \operatorname{spt} \|V\|$ such that

$$\limsup_{s \to 0+} \|V\|(\mathbf{B}(a, s))^{(1/m)-1} \|\delta V\| \mathbf{B}(a, s) < (2\boldsymbol{\gamma}(m))^{-1}.$$

Note $\|V\|(\mathbf{R}^n \sim A) = 0$ by [Fed69, 2.8.18, 2.9.5], hence A is dense in $\operatorname{spt} \|V\|$ and

$$\operatorname{diam} \operatorname{spt} \|V\| = \sup\{\operatorname{diam} p[A] : p \in \mathbf{O}^*(n, 1)\}.$$

Next, suppose $p \in \mathbf{O}^*(n, 1)$ and let $\phi = p_\# \psi$. Then, for each $b \in p[A]$, there exists $0 < r < \infty$ such that

$$r \le \Delta \phi \, \mathbf{B}(b, r),$$

where $\Delta = m(2\boldsymbol{\gamma}(m))^m$; in fact, one may choose $a \in A$ with $p(a) = b$ and take

$$r = \inf\left\{s : \|\delta V\| \mathbf{B}(a, s) > (2\boldsymbol{\gamma}(m))^{-1} \|V\|(\mathbf{B}(a, s))^{1-1/m}\right\},$$

hence $0 < r < \infty$ and $\|V\| \mathbf{B}(a, r) \ge (2m\boldsymbol{\gamma}(m))^{-m} r^m$ by [Men09, 2.5] implying

$$\begin{aligned} (2\boldsymbol{\gamma}(m))^{-1} \|V\|(\mathbf{B}(a, r))^{1-1/m} &\le \|\delta V\| \mathbf{B}(a, r) \\ &\le \|V\|(\mathbf{B}(a, r))^{1-1/(m-1)} \psi(\mathbf{B}(a, r))^{1/(m-1)}, \end{aligned}$$

$$(2m\boldsymbol{\gamma}(m))^{-1} r \le \|V\|(\mathbf{B}(a, r))^{1/m} \le (2\boldsymbol{\gamma}(m))^{m-1} \psi \, \mathbf{B}(a, r).$$

Consequently,

$$\mathscr{L}^1(p[A]) \le 2\Delta\boldsymbol{\beta}(1)\psi(\mathbf{R}^n).$$

The proof will be concluded by showing

$$\operatorname{diam} p[A] = \mathscr{L}^1(p[A]).$$

If this were not the case, there would exist $b \in \mathbf{R}$ such that

$$\inf p[A] < b < \sup p[A], \quad \boldsymbol{\Theta}^1(p_\# \|V\|, b) = 0$$

by [Fed69, 2.2.17, 2.10.19 (4)] since $(p_\# \|V\|)(\mathbf{R} \sim p[A]) = 0$, hence one could use [All72, 4.10 (1)] or [Men16a, 8.7, 8.29] to infer

$$\delta(V \llcorner \{(a,S) : p(a) > b\}) = (\delta V) \llcorner \{a : p(a) > b\}$$

which would be incompatible with the indecomposability hypothesis on V. $\square$

3.2 Remark. Inspection of the final argument shows that the indecomposability hypothesis on V may be weakened to indecomposability of type $\mathbf{O}^*(n,1)$.

3.3 Remark. In case $m = 2$, a related inequality for submanifolds involving the second fundamental form is provided in [Sim93, Lemma 1.2].

4 Sobolev-Poincaré inequality

The purpose of this section is to establish (see 4.5) our new Sobolev-Poincaré inequality, Theorem C; the key ingredient therein is a monotonicity lemma (see 4.1) based on the isoperimetric inequality. Following this, the setting of connected submanifolds, Corollary 1, results as corollary (see 4.8). We also include a simpler version of our Sobolev-Poincaré inequality for varifolds that are suitably indecomposable and have no boundary (see 4.10).

The lemma below will presently be applied only with $q = \infty$. We include the case $q < \infty$ since it yields a new proof of previous Sobolev inequalities (see 4.3).

4.1 Lemma. *Suppose m and n are positive integers, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x, $f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V)$, $0 \le s \le r \le \|V\|_{(\infty)}(f)$,*

$$\text{either,} \quad m = q = 1 \text{ and } \lambda = 1,$$
$$\text{or,} \quad m < q \le \infty \text{ and } \lambda = \big((1/m - 1/q)\big/(1-1/q)\big)^{1-1/q},$$

$V \,\mathbf{D}\, f \in \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))$, $0 < \epsilon < \boldsymbol{\gamma}(m)^{-1}$,

$$\|V\| \{x : f(x) \ge y\} < \infty, \quad \|\delta V\| \{x : f(x) \ge y\} \le \epsilon \, \|V\|(\{x : f(x) \ge y\})^{1-1/m}$$

for $s < y < r$, and $\delta = \boldsymbol{\gamma}(m)^{-1} - \epsilon$.

Then, the quantities

$$\|V\|(\{x : f(x) \ge y\})^{1/m-1/q} (\|V\| \llcorner \{x : f(x) \ge y\})_{(q)}(V \,\mathbf{D}\, f) + \delta\lambda y, \quad \text{if } q < \infty,$$
$$\|V\|(\{x : f(x) \ge y\})^{1/m} \|V\|_{(\infty)}(V \,\mathbf{D}\, f) + \delta\lambda y, \quad \text{if } q = \infty,$$

are nonincreasing in y, for $s < y < r$.

Proof. We treat the case $m < q < \infty$. The cases $m = q = 1$ and $q = \infty$ follow by a similar but simpler argument. Abbreviate $\alpha = 1 - 1/q$ and $\beta = 1 - 1/m$. Let $i : U \to \mathbf{R}^n$ denote the inclusion, define

$$E(v) = \{x : f(x) \geq -v\}, \quad g(v) = \|V\|(E(v)), \quad G(v) = g(v)^{1-\beta/\alpha},$$
$$h(v) = \textstyle\int_{E(v)} |V \,\mathbf{D}\, f|^q \,\mathrm{d}\|V\|, \quad W_v = i_\#(V \llcorner E(v) \times \mathbf{G}(n,m)) \in \mathbf{V}_m(\mathbf{R}^n)$$

for $-r < v < -s$, and notice that

$$\boldsymbol{\Theta}^m(\|W_v\|, x) \geq 1 \quad \text{for } \|W_v\| \text{ almost all } x$$

by [Fed69, 2.8.9, 2.8.18, 2.9.11]. Furthermore, as $(\|V\| + \|\delta V\|)\,\{x : f(x) = v\} = 0$ for all but countably many v, we have (see [Men16a, 9.1])

$$\|\delta W_v\| \leq i_\#\big(\|\delta V\| \llcorner E(v) + \|V \,\partial\, E(v)\|\big) \quad \text{for } \mathscr{L}^1 \text{ almost all } -r < v < -s.$$

For such v, the isoperimetric inequality (see [MS18, 3.5, 3.7]) yields

$$\boldsymbol{\gamma}(m)^{-1} g(v)^{1-1/m} \leq \|\delta V\|(E(v)) + \|V \,\partial\, E(v)\|(U).$$

In view of [Men16a, 8.29] and [Fed69, 2.9.19], we deduce the inequalities

$$0 < (\boldsymbol{\gamma}(m)^{-1} - \epsilon) g(v)^{1-1/m} \leq \|V \,\partial\, E(v)\|(U) \leq g'(v)^{1-1/q} h'(v)^{1/q},$$
$$\delta\lambda = (\boldsymbol{\gamma}(m)^{-1} - \epsilon)(1 - \beta/\alpha)^\alpha \leq (g^{1-\beta/\alpha})'(v)^\alpha h'(v)^{1-\alpha} = G'(v)^\alpha h'(v)^{1-\alpha}$$

for $\mathscr{L}^1$ almost all $-r < v < -s$. Therefore, noting $0 < \int_{-r}^{v} h' \,\mathrm{d}\mathscr{L}^1 \leq h(v)$ for $-r < v < -s$ by [Fed69, 2.9.19], we obtain (using the inequality relating geometric and arithmetic means)

$$\delta\lambda \leq \alpha G'(v) G(v)^{\alpha-1} h(v)^{1-\alpha} + (1-\alpha) h'(v) h(v)^{-\alpha} G(v)^\alpha = (G^\alpha h^{1-\alpha})'(v)$$

for $\mathscr{L}^1$ almost all $-r < v < -s$, whence the conclusion follows by integration with respect to $\mathscr{L}^1$ using [Fed69, 2.9.19]. □

4.2 Remark. For $q = \infty$, the pattern of the preceding proof is that of [All72, 8.3].

4.3 Remark. The preceding lemma in particular entails the estimates [Men16a, 10.1 (2b) (2d)] with a different, somewhat more explicit constant.

We next gather the set of conditions on density and first variation that we assume for both varifolds occurring in the Sobolev-Poincaré estimate in 4.5.

4.4. Suppose U is an open subset of $\mathbf{R}^n$, V is a varifold in U, $m = \dim V$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, and, either $m = 1$ and $\phi = \|V\|$, $m = 2$ and $\phi = \|\delta V\|$, or $m > 2$, $\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, and $\phi = \|V\| \llcorner |\mathbf{h}(V, \cdot)|^{m-1}$.

4.5 Theorem. *Suppose m and n are integers, $2 \leq m \leq n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$ and $V_2 \in \mathbf{V}_{m-1}(U)$ satisfy the conditions of 4.4,*

$$V_2 = 0 \quad \text{if } m = 2, \qquad \|\delta V_1\| \leq \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \text{if } m > 2,$$
$$\|\delta V_2\| \text{ is absolutely continuous with respect to } \|V_2\| \quad \text{if } m > 3,$$
$$f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V_i), \qquad |V_i \,\mathbf{D}\, f(x)| \leq 1 \quad \text{for } \|V_i\| \text{ almost all } x,$$
$$\phi_i \text{ are associated with } V_i \text{ as in 4.4},$$

for $i \in \{1,2\}$, and $E = \{x : f(x) > 0\}$.

Then, there exists a Borel subset Y of $\mathbf{R}$ such that

$$f(x) \in Y \quad \text{for } \|V_1\| \text{ almost all } x,$$
$$\mathscr{L}^1(Y) \le \Gamma\big(\|V_1\|(E)^{1/m} + \phi_1(E) + \|V_2\|(E)^{1/(m-1)} + \phi_2(E)\big),$$

where Γ is a positive finite number determined by m.

Proof. We assume $(\|V_i\| + \phi_i)(E) < \infty$ for $i \in \{1,2\}$; in particular, we have $\|\delta V_i\|(E) < \infty$. Define $I = \mathbf{R} \cap \{y : y > 0\}$,

$$\mu_i = f_\# \|V_i\|, \quad \nu_i = f_\# \|\delta V_i\|, \quad \text{and} \quad \omega_i = f_\# \phi_i$$

for $i \in \{1,2\}$. Let $\alpha = \omega_1$ if $m = 2$ and $\alpha = f_\#(\|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)|)$ if $m > 2$. With

$$\Delta_1 = 2m\boldsymbol{\gamma}(m), \quad \Delta_2 = \sup\big\{2(m-1)\boldsymbol{\gamma}(m-1), 2\Delta_1(2\boldsymbol{\gamma}(m))^{1/(m-1)}\big\},$$

we define

$$\lambda_i = \Delta_i\, \mu_i(I)^{1/\dim V_i} \quad \text{for } i \in \{1,2\}$$

and functions $r_i : \mathbf{R} \to \mathbf{R}$, for $i \in \{1,2\}$, by

$$r_i(b) = \sup\big\{s : 0 \le s < b \text{ and } \Delta_i\, \mu_i(\mathbf{B}(b,s))^{1/\dim V_i} \ge s\big\} \quad \text{whenever } b \in \mathbf{R}.$$

Since the sets $(\mathbf{R} \times \mathbf{R}) \cap \big\{(b,s) : 0 \le s < b \text{ and } \Delta_i\, \mu_i(\mathbf{B}(b,s))^{1/\dim V_i} \ge s\big\}$ are relatively closed in $(\mathbf{R} \times \mathbf{R}) \cap \{(b,s) : s < b\}$, we may deduce that r_i are Borel functions for $i \in \{1,2\}$. We also note $r_i(b) \le \lambda_i$ for $b \in \mathbf{R}$ and $i \in \{1,2\}$. Let $C = \{b : \mu_1\{b\} > 0\}$ and notice that C is countable. Moreover, we define

$$Q_i = \mathbf{R} \cap \{b : r_i(b) > 0\} \quad \text{for } i \in \{1,2\}, \qquad B = \{b : r_1(b) > r_2(b)\}.$$

Our two estimates below rest on the basic fact that

$$\nu_i\, \mathbf{B}(b, r_i(b)) \ge (2\boldsymbol{\gamma}(\dim V_i))^{-1} \mu_i(\mathbf{B}(b, r_i(b)))^{1-1/\dim V_i}$$

whenever $\lambda_i < b \in Q_i$ and $i \in \{1,2\}$; in fact, we note [Men16a, 8.12, 8.13, 9.9] and apply, for small $s > 0$, 4.1 with m, V, s, r, q, $f(x)$, and ϵ replaced by $\dim V_i$, V_i, 0, s, ∞, $\sup\{r_i(b) + s - |f(x) - b|, 0\}$, and $(2\boldsymbol{\gamma}(\dim V_i))^{-1}$.

Next, the following two estimates will be proven

$$\mathscr{L}^1(Q_2 \cap \{b : b > \lambda_2\}) \le \Delta_3\, \omega_2(I), \quad \mathscr{L}^1(B \cap \{b : b > \lambda_1\}) \le \Delta_4\, \omega_1(I),$$

where $\Delta_3 = 2^{m+1}\Delta_2\boldsymbol{\gamma}(m-1)^{m-2}$ and $\Delta_4 = 2^{m(m-1)+3}\boldsymbol{\gamma}(m)^{m-1}\Delta_1$. Whenever $\lambda_2 < b \in Q_2$, the basic fact and Hölder's inequality yield

$$\Delta_2^{-1} r_2(b) \le \mu_2(\mathbf{B}(b, r_2(b)))^{1/(m-1)} \le (2\boldsymbol{\gamma}(m-1))^{m-2} \omega_2\, \mathbf{B}(b, r_2(b)),$$

whence we deduce the first estimate by Vitali's covering theorem (see [Fed69, 2.8.5, 2.8.8] with $\delta = \operatorname{diam}$ and $\tau = 3/2$). To similarly prove the second estimate suppose $\lambda_1 < b \in B$. We first notice that $b > r_1(b) > r_2(b)$ yields

$$\begin{aligned} \mu_2(\mathbf{B}(b, r_1(b)))^{1/(m-1)} &\le \Delta_2^{-1} r_1(b) \\ &\le 2^{-1}(2\boldsymbol{\gamma}(m))^{1/(1-m)} \mu_1(\mathbf{B}(b, r_1(b))^{1/m} < \infty. \end{aligned}$$

Combining this estimate with the following consequence of the basic fact,

$$\begin{aligned}\mu_1(\mathbf{B}(b,r_1(b)))^{1/m} &\le (2\boldsymbol{\gamma}(m))^{1/(m-1)}\nu_1(\mathbf{B}(b,r_1(b)))^{1/(m-1)} \\ &\le (2\boldsymbol{\gamma}(m))^{1/(m-1)}\big(\alpha(\mathbf{B}(b,r_1(b)))^{1/(m-1)} + \mu_2(\mathbf{B}(b,r_1(b)))^{1/(m-1)}\big),\end{aligned}$$

we first obtain

$$2^{-1}\mu_1(\mathbf{B}(b,r_1(b)))^{1/m} \le (2\boldsymbol{\gamma}(m))^{1/(m-1)}\alpha(\mathbf{B}(b,r_1(b)))^{1/(m-1)},$$

and then, using Hölder's inequality,

$$\Delta_1^{-1} r_1(b) \le \mu_1(\mathbf{B}(b,r_1(b)))^{1/m} \le 2^{m(m-1)}\boldsymbol{\gamma}(m)^{m-1}\,\omega_1\,\mathbf{B}(b,r_1(b)).$$

Vitali's covering theorem now yields the second estimate.

We now define $Y = C \cup Q_1$. Since $Q_1 \subset B \cup Q_2 \subset I$, the preceding two estimates imply that the asserted property of Y may be established by proving

$$\mu_1(\mathbf{R} \sim (C \cup Q_1)) = 0.$$

For this purpose, we define

$$\Upsilon = (\operatorname{spt}\mu_1) \cap \left\{ y : y > 0 \text{ and } \limsup_{s\to 0+} \frac{\nu_1\,\mathbf{B}(y,s)}{\mu_1(\mathbf{B}(y,s))^{1-1/m}} \ge (2\boldsymbol{\gamma}(m))^{-1} \right\}$$

and notice that $\mu_1(\Upsilon \sim C) = 0$ by [Fed69, 2.8.9, 2.8.18, 2.9.5]. The assertion then follows verifying $(\operatorname{spt}\mu_1) \sim \Upsilon \subset Q_1 \cup \{0\}$; in fact, for $0 < b \in (\operatorname{spt}\mu_1) \sim \Upsilon$ and small $s > 0$, we note [Men16a, 8.12, 8.13, 9.9] and apply 4.1 with V, s, r, q, $f(x)$, and ϵ replaced by V_1, 0, s, ∞, $\sup\{s - |f(x) - b|, 0\}$, and $(2\boldsymbol{\gamma}(m))^{-1}$, to infer $r_1(b) > 0$. □

4.6 Remark. We recall from [MS23, 7.12, 7.13 (1)] that, under the hypotheses of the preceding theorem, if V_1 is indecomposable of type $\{f\}$, then

$$\operatorname{diam}\operatorname{spt} f_\# \|V_1\| \le \mathscr{L}^1(Y)$$

and if additionally $\|V_1\|\,\{x : f(x) \le y\} > 0$ for $0 < y < \infty$, then

$$\operatorname{diam}\operatorname{spt} f_\# \|V_1\| = \|V_1\|_{(\infty)}(f).$$

4.7 Remark. We notice that $\mathbf{T}_{\operatorname{Bdry} U}(V_i) = \mathbf{T}(V_i) \cap \{f : f \ge 0\}$ in case $U = \mathbf{R}^n$ by [Men16a, 9.2].

In the special case of *connected* properly embedded submanifolds of class 2 and a function of class 1 thereon, the following corollary, Corollary 1, results.

4.8 Corollary. *Suppose m and n are integers, $2 \le m \le n$, U is an open subset of $\mathbf{R}^n$, M is a properly embedded, connected m dimensional submanifold-with-boundary of U of class 2, $f : M \to \mathbf{R}$ is of class 1 relative to M, $\operatorname{spt} f$ is compact, $|\operatorname{D} f(x)| \le 1$ for $x \in M \sim \partial M$, and $E = M \cap \{x : f(x) \ne 0\}$.*

Then, there holds

$$\begin{aligned}\operatorname{diam} f[M] \le 2\Gamma_{4.5}(m)\big(&\mathscr{H}^m(E\cap M)^{1/m} + \textstyle\int_{E\cap M} |\mathbf{h}(M,x)|^{m-1}\,\mathrm{d}\mathscr{H}^m x \\ &+ \mathscr{H}^{m-1}(E\cap\partial M)^{1/(m-1)} + \textstyle\int_{E\cap\partial M}|\mathbf{h}(\partial M,x)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1} x\big);\end{aligned}$$

here, the summand $\int_{E\cap\partial M}|\mathbf{h}(\partial M,x)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1}x$ shall be omitted if $m = 2$.

Proof. We recall [MS23, 6.14]. We define $V_1 \in \mathbf{RV}_m(U)$ and $V_2 \in \mathbf{RV}_{m-1}(U)$ such that

$$\|V_1\| = \mathscr{H}^m \llcorner M, \quad V_2 = 0 \text{ if } m = 2, \quad \|V_2\| = \mathscr{H}^{m-1} \llcorner \partial M \text{ if } m > 2;$$

hence, V_1 is indecomposable of type $\mathscr{D}(U, \mathbf{R})$ by [MS23, 7.9]. Employing [Fed69, 3.1.22], we construct a function $g : U \to \mathbf{R}$ of class 1 with compact support such that $g|M = f$. From [Men16a, 8.7, 8.16, 9.2, 9.4], we infer that $|g| \in \mathbf{T}_{\mathrm{Bdry}\, U}(V_i)$ with

$$\big|V_i \, \mathbf{D}\, |g|(x)\big| \leq 1 \quad \text{for } \|V_i\| \text{ almost all } x$$

for $i \in \{1, 2\}$. In view of [MS23, 7.13 (2)] the conclusion now follows from 4.5 and 4.6 applied with f replaced by $|g|$ because $\operatorname{diam} f[M] \leq 2 \operatorname{diam} g[M]$, if $0 \in \operatorname{im} f$, and $\operatorname{diam} f[M] = \operatorname{diam} g[M]$ as $f[M]$ is an interval, if $0 \notin \operatorname{im} f$. ☐

To prepare for the case without boundary, we collect another set of hypotheses on density and first variation. It will be assumed to hold in 4.10.

4.9. Suppose U is an open subset of $\mathbf{R}^n$, V is a varifold in U, $2 \leq m = \dim V$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, and, either $m = 2$ and $\psi = \|\delta V\|$, or $m > 2$, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, $\mathbf{h}(V, \cdot) \in \mathbf{L}^{\mathrm{loc}}_{m-1}(\|V\|, \mathbf{R}^n)$, and $\psi = \|V\| \llcorner |\mathbf{h}(V, \cdot)|^{m-1}$.

4.10 Corollary. *Suppose U, V, and ψ are as in 4.9, $\Gamma = 2^{m+3} m \boldsymbol{\gamma}(m)$, $f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V)$, V is indecomposable of type $\{f\}$, and*

$$|V \, \mathbf{D}\, f(x)| \leq 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Then, there holds

$$\operatorname{diam} \operatorname{spt} f_\# \|V\| \leq \Gamma\big(\|V\|(\{x : f(x) > 0\})^{1/m} + \boldsymbol{\gamma}(m)^{m-1} \psi\, \{x : f(x) > 0\}\big).$$

Proof. With a possibly larger number Γ, this follows from 4.5 and 4.6 with $V_1 = V$ and $V_2 = 0$. We verify the eligibility of the present number Γ by noting that, for $V_2 = 0$, we can take $\Delta_4 = 2^{m+2} \Delta_1 \boldsymbol{\gamma}(m)^{m-1}$ in the proof of 4.5; in fact,

$$\Delta_1^{-1} r_1(b) \leq \mu_1(\mathbf{B}(b, r_1(b)))^{1/m} \leq (2\boldsymbol{\gamma}(m))^{m-1} \omega_1\, \mathbf{B}(b, r_1(b))$$

whenever $\lambda_1 < b \in B$ by the basic fact and Hölder's inequality. ☐

4.11 Remark. As in 4.6, we note that, in case $\|V\|\, \{x : f(x) \leq y\} > 0$ whenever $0 < y < \infty$, we have $\operatorname{diam} \operatorname{spt} f_\# \|V\| = \|V\|_{(\infty)}(f)$.

5 Examples

In the present section, we construct (see 5.5 and 5.7) the Example mentioned the introduction which shows the sharpness of Theorem D. As preparations, we list an arithmetic formula and terminology for cylinders (see 5.1–5.2) and then indicate a procedure to smooth out corners (see 5.3–5.4).

5.1. If $0 \leq x < 1$ and i is a nonnegative integer, then

$$\sum_{j=i}^{\infty} (j+1) x^j = (1-x)^{-2} \big((i+1)x^i - i x^{i+1}\big).$$

5.2. Whenever $u \in \mathbf{S}^{n-1}$, $a \in \mathbf{R}^n$, $0 < r < \infty$, and $0 \le h \le \infty$, we define

$$Z(a,r,u,h) = \mathbf{R}^n \cap \big\{x : |x-a|^2 = ((x-a) \bullet u)^2 + r^2, 0 \le (x-a) \bullet u \le h\big\}.$$

5.3 Lemma. *Suppose n is an integer, $n \ge 2$, Y is an $n-1$ dimensional submanifold-with-boundary of class ∞ of $\mathbf{R}^{n-1}$, ∂Y is connected and compact, $\epsilon > 0$, and, identifying $\mathbf{R}^n \simeq \mathbf{R}^{n-1} \times \mathbf{R}$, the subsets Q and U of $\mathbf{R}^n$ satisfy*

$$Q \simeq Y \times \{t : 0 \le t < \infty\}, \quad U \simeq (Y \cap \{y : \operatorname{dist}(y, \partial Y) < \epsilon\}) \times \{z : 0 \le z < \epsilon\}.$$

Then, there exists a properly embedded n dimensional submanifold-with-boundary M of class ∞ of $\mathbf{R}^n$ such that ∂M is connected and $M \sim U = Q \sim U$.

Proof. As ∂Y is compact, we employ [MS23, 3.9, 3.12] to construct $\delta > 0$ such that the function $f : G \to \mathbf{R}$, with $G = \mathbf{R}^{n-1} \cap \{y : \operatorname{dist}(y, \partial Y) < \delta\}$ and

$$f(y) = \operatorname{dist}(y, \partial Y) \quad \text{if } y \in Y, \qquad f(y) = -\operatorname{dist}(y, \partial Y) \quad \text{else},$$

for $y \in G$, is of class ∞ and satisfies $|\operatorname{D} f(y)| = 1$ for $y \in G$. Then, defining $g : G \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ by $g(y,z) = (f(y), z)$ for $y \in G$ and $z \in \mathbf{R}$ and noting $\operatorname{im} \operatorname{D} g(y,z) = \mathbf{R} \times \mathbf{R}$ for $y \in G$ and $z \in \mathbf{R}$, the assertion reduces (e.g., by [Fed69, 3.1.18]) to the case $n = 2$ and $Y = \{y : 0 \le y < \infty\}$ which is elementary. □

5.4 Remark. By induction on n, the preceding lemma implies the following proposition: *If $2 \le n \in \mathbf{Z}$, $-\infty < a_k < b_k < \infty$ for $k = 1, \ldots, n$, $\epsilon > 0$, and*

$$Q = \mathbf{R}^n \cap \{x : a_k \le x \bullet e_k \le b_k \text{ for } k = 1, \ldots, n\},$$
$$U = Q \cap \big\{x : \operatorname{dist}(x, Q \cap \{\chi : \operatorname{card}\{k : \chi \bullet e_k = a_k \text{ or } \chi \bullet e_k = b_k\} \ge 2\}) < \epsilon\big\},$$

where $e_1, \ldots, e_n$ form the standard base of $\mathbf{R}^n$, then there exists a properly embedded n dimensional submanifold-with-boundary M of $\mathbf{R}^n$ of class ∞ such that ∂M is connected and $M \sim U = Q \sim U$.

In the first example below related to the sharpness of Theorem D, we are able to control the second fundamental form $\mathbf{b}(M, \cdot)$ instead of merely $\mathbf{h}(M, \cdot)$.

5.5 Theorem. *Whenever m and n are positive integers, $2 \le m < n$, and $m - 1 < q < m$, there exists a bounded, connected m dimensional submanifold M of class ∞ of $\mathbf{R}^n$ such that*

$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \, \mathrm{d}\mathscr{H}^m x < \infty \quad \textit{whenever } 1 \le p < q,$$
$$\textstyle\int_M \operatorname{Tan}(M,x)_\natural \bullet \operatorname{D}\theta(x) \, \mathrm{d}\mathscr{H}^m x = -\int_M \mathbf{h}(M,x) \bullet \theta(x) \, \mathrm{d}\mathscr{H}^m x$$

for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$, and such that $A = (\operatorname{Clos} M) \sim M$ satisfies

$$\mathscr{H}^{m-q}(A) = \boldsymbol{\alpha}(m-q) 2^{1-m+q}, \qquad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \textit{for } a \in A.$$

Proof. We assume $n = m+1$ and let $d = m - q$. Whenever J is a compact subinterval of $\mathbf{R}$, we denote by $\Phi(J)$ the family consisting of the two disjoint subintervals

$$\big\{t : \inf J \le t \le \inf J + 2^{-1/d} \operatorname{diam} J\big\}, \quad \big\{t : \sup J - 2^{-1/d} \operatorname{diam} J \le t \le \sup J\big\}$$

of J. Letting $G_0 = \{\{t : -1/2 \le t \le 1/2\}\}$, we define $G_i = \bigcup\{\Phi(S) : S \in G_{i-1}\}$ for every positive integer i and $C = \bigcap_{i=0}^{\infty} \bigcup G_i$. By [Fed69, 2.10.28], there holds

$$\mathscr{H}^d(C) = \boldsymbol{\alpha}(d) 2^{-d}.$$

For every nonnegative integer i, we let $r_i = 2^{-i/d}$ and $s_i = \sum_{j=i}^{\infty} (j+1) r_j$, hence $\operatorname{diam} J = r_i$ whenever $J \in G_i$ and, using 5.1, we compute

$$s_i = (1 - 2^{-1/d})^{-1} \big(i + (1 - 2^{-1/d})^{-1} \big) r_i.$$

Suppose $e_1, \ldots, e_n$ form the standard base of $\mathbf{R}^n$. We observe that 5.3 may be employed to construct a subset N of the cube

$$\mathbf{R}^n \cap \big\{ x : 0 < x \bullet e_n < 1 \text{ and } |x \bullet e_k| < \tfrac{1}{2} \text{ for } k = 1, \ldots, n-1 \big\}$$

such that its union with (see 5.2)

$$Z\big(\tfrac{r_1 - 1}{2} e_1 + e_n, \tfrac{r_1}{4}, e_n, \infty\big) \cup Z\big(\tfrac{1 - r_1}{2} e_1 + e_n, \tfrac{r_1}{4}, e_n, \infty\big) \cup Z\big(0, \tfrac{1}{4}, -e_n, \infty\big)$$

is a properly embedded, connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$. In particular, there exists $0 \le \kappa < \infty$ satisfying

$$\mathscr{H}^m(N) \le \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(N, \cdot)\| \le \kappa.$$

With $N(a, r) = \mathbf{R}^n \cap \{x : r^{-1}(x - a) \in N\}$ for $a \in \mathbf{R}^n$ and $0 < r < \infty$, we use

$$\begin{aligned} X_i &= \big\{ Z\big(\tfrac{\sup J + \inf J}{2} e_1 + (s_0 - s_i) e_n, \tfrac{r_i}{4}, e_n, i r_i\big) : J \in G_i \big\}, \\ \Psi_i &= \big\{ N\big(\tfrac{\sup J + \inf J}{2} e_1 + (s_0 - s_i + i r_i) e_n, r_i\big) : J \in G_i \big\}, \end{aligned}$$

$H_i = \bigcup_{j=0}^{i} \bigcup (X_j \cup \Psi_j)$, and $M_i = \{x : x \in H_i \text{ or } x - 2(x \bullet e_n) e_n \in H_i\}$, to define

$$M = \bigcup \{M_i : i \text{ is a nonnegative integer}\}.$$

Clearly, M is a connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$ and

$$A = \mathbf{R}^n \cap \{x : x \bullet e_1 \in C,\ |x \bullet e_n| = s_0, \text{ and } x \bullet e_k = 0 \text{ for } k = 2, \ldots, n-1\},$$

where $A = (\operatorname{Clos} M) \sim M$.

Next, we will show the following assertion. *There holds*

$$\mathscr{H}^m(M \cap \mathbf{B}(a, r)) \le 2^{2 + m/d} (1 - 2^{1 - m/d})^{-2} (m \boldsymbol{\alpha}(m) + \kappa) (i+1)^{1 + d - m} r^m$$

whenever $a \in A$, i is a nonnegative integer, and $s_{i+1} \le r \le s_i$. For this purpose, we let $I = \{t : |t - a \bullet e_1| \le r\}$ and firstly estimate

$$\lambda_i = \operatorname{card}\{J : I \cap J \ne \varnothing, J \in G_i\} \le 4(1 - 2^{-1/d})^{-2} (i+1)^d;$$

in fact, G_i is special for $\{t : -1/2 \le t \le 1/2\}$ by [Fed69, 2.10.28 (2) (4)], hence

$$\operatorname{card}\{J : I \supset J \in G_i\} r_i^d \le (2r)^d \le 2(1 - 2^{-1/d})^{-2} (i+1)^d r_i^d.$$

Since $2^{-i} \sum_{j=i}^{\infty} 2^j (j+1) r_j^m \le (1 - 2^{1 - m/d})^{-2} (i+1) r_i^m$ by 5.1, we estimate

$$\mathscr{H}^m(M \cap \mathbf{B}(a, r)) \le \lambda_i (m \boldsymbol{\alpha}(m) + \kappa) (1 - 2^{1 - m/d})^{-2} (i+1) r_i^m.$$

Hence, together with the first estimate and $r_i \le 2^{1/d}(1-2^{-1/d})(i+1)^{-1}s_{i+1}$, the assertion follows.

The assertion of the preceding paragraph implies

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \text{whenever } a \in A$$

and, as $\operatorname{Clos} M$ is compact, also $\mathscr{H}^m(M) < \infty$. Noting

$$\begin{gathered} \textstyle\int_{Z(a,r,u,h)} \|\mathbf{b}(Z(a,r,u,h),x)\|^p \,\mathrm{d}\mathscr{H}^m x \le m\boldsymbol{\alpha}(m) r^{m-p}(h/r), \\ \textstyle\int_{N(a,r)} \|\mathbf{b}(N(a,r),x)\|^p \,\mathrm{d}\mathscr{H}^m x \le \kappa^{1+p} r^{m-p} \end{gathered}$$

for $a \in \mathbf{R}^n$, $0 < r < \infty$, $u \in \mathbf{S}^{n-1}$, and $0 \le h \le \infty$, we estimate

$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \,\mathrm{d}\mathscr{H}^m x \le 2(\kappa^{1+p} + 4m\boldsymbol{\alpha}(m)) \sum_{i=0}^\infty (i+1) 2^{i(1+(p-m)/d)} < \infty$$

whenever $1 \le p < q$. Since $\mathscr{H}^{m-1}(\partial M_i) \le m\boldsymbol{\alpha}(m) 2^{i(1+(1-m)/d)}$ for every positive integer i, the conclusion now readily follows. □

5.6 Remark. The preceding theorem answers the third question posed in [Sch16, Section A].

In the second example below related to the sharpness of Theorem D, we are again able to control the second fundamental form $\mathbf{b}(M,\cdot)$ instead of $\mathbf{h}(M,\cdot)$.

5.7 Theorem. *Whenever m and n are positive integers and $3 \le m < n$, there exists a bounded, connected m dimensional submanifold M of class ∞ of $\mathbf{R}^n$ such that*

$$\begin{gathered} \textstyle\int_M \|\mathbf{b}(M,x)\|^p \,\mathrm{d}\mathscr{H}^m x < \infty \quad \textit{whenever } 1 \le p < m-1, \\ \textstyle\int_M \operatorname{Tan}(M,x)_\natural \bullet \mathrm{D}\,\theta(x) \,\mathrm{d}\mathscr{H}^m x = -\int_M \mathbf{h}(M,x) \bullet \theta(x) \,\mathrm{d}\mathscr{H}^m x \end{gathered}$$

for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$, and such that

$$\mathscr{H}^m((\operatorname{Clos} M) \sim M) = 1, \qquad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \textit{for } a \in (\operatorname{Clos} M) \sim M.$$

Proof. We assume $n = m+1$. With each $A \subset \mathbf{R}^n$, we associate sets

$$A(a,r) = \mathbf{R}^n \cap \{x : r^{-1}(x-a) \in A\} \quad \text{for } a \in \mathbf{R}^n \text{ and } 0 < r < \infty.$$

Let $e_1, \ldots, e_n$ denote the standard base vectors of $\mathbf{R}^n$. We define

$$\begin{aligned} D &= \mathbf{R}^n \cap \{x : x \bullet e_n = 0 \text{ and } |x| < 1\}, \\ H &= \mathbf{R}^n \cap \{u : \text{for some } k \in \{1,\ldots,m\},\ u = e_k \text{ or } u = -e_k\}. \end{aligned}$$

We define $\gamma : \mathbf{R}^n \to \mathbf{2}^H$ by

$$\gamma(x) = H \sim \{u : \text{for some } k,\ x \bullet e_k = 1 \text{ and } u = e_k \text{ or } x \bullet e_k = 0 \text{ and } u = -e_k\}$$

for $x \in \mathbf{R}^n$. Notice that 5.3 may be used to construct a subset R of the cylinder

$$\mathbf{R}^n \cap \{x : 0 \le x \bullet e_n < \tfrac{1}{4} \text{ and } |x - (x \bullet e_n)e_n| < \tfrac{1}{4}\}$$

such that $R \cup \big(\mathbf{R}^n \cap \{x : x \bullet e_n = 0\} \sim D(0,\frac{1}{4})\big) \cup Z(\frac{e_n}{4}, \frac{1}{8}, e_n, \infty)$, see 5.2, is a properly embedded, connected m dimensional submanifold of $\mathbf{R}^n$ of class ∞. Let

S denote the reflection $\{x : x - 2(x \bullet e_n)e_n \in R\}$ of R along $\mathbf{R}^n \cap \{x : x \bullet e_n = 0\}$. Considering the submanifold furnished by 5.4 applied with $\epsilon = \frac{1}{16}$ and (a_k, b_k) replaced by $(-\frac{3}{8}, \frac{3}{8})$ if $k < n$ and $(-\frac{1}{4}, \frac{1}{4})$ if $k = n$, we observe that 5.3 may also be employed to construct, for $G \subset H$, a subset N_G of the cuboid

$$\mathbf{R}^n \cap \big\{x : |x \bullet e_n| \leq \tfrac{1}{4} \text{ and } |x \bullet e_k| < \tfrac{1}{2} \text{ for } k = 1, \ldots, m\big\}$$

such that N_G contains $D(\frac{e_n}{4}, \frac{1}{4}) \cup D(-\frac{e_n}{4}, \frac{1}{4})$ and such that

$$N_G \cup \bigcup \big\{Z(\tfrac{u}{2}, \tfrac{1}{8}, u, \infty) : u \in G\big\}$$

is a properly embedded, connected m dimensional submanifold of $\mathbf{R}^n$ of class ∞. Clearly, there exists $0 \leq \kappa < \infty$ satisfying

$$\mathscr{H}^m(R) \leq \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(R, \cdot)\| \leq \kappa, \quad \mathscr{H}^m(N_G) \leq \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(N_G, \cdot)\| \leq \kappa$$

whenever $G \subset H$.

In this paragraph, we define various objects for each positive integer i. Let $r_i = 2^{-i(i+1)}$ and define C_i to consist of those $x \in \mathbf{R}^n$ such that

$$x \bullet e_n = 2^{-i}, \quad 0 \leq x \bullet e_k \leq 1, \quad \text{and} \quad 2^{i-1} x \bullet e_k \in \mathbf{Z}$$

for $k = 1, \ldots, m$. We have $\operatorname{card} C_i = (2^{i-1} + 1)^m$. Then, noting $\frac{r_i}{2} < 2^{-i}$, we define $X_i(u)$, for $u \in H$, to be the family consisting of the sets

$$Z\big(x + \tfrac{r_i}{2} u, \tfrac{r_i}{8}, u, 2^{-i} - \tfrac{r_i}{2}\big)$$

corresponding to $x \in C_i$ with $u \in \gamma(x)$. We have $\operatorname{card} X_i(u) = (2^{i-1}+1)^{m-1} 2^{i-1}$. With $C_0 = \varnothing$, we define Ψ_i to be the family consisting of the sets

$$N_{\gamma(x)}(x, r_i) \sim D(x - \tfrac{r_i}{4} e_n, \tfrac{r_{i+1}}{4})$$

corresponding to $x \in C_i$ with $x + 2^{-i} e_n \notin C_{i-1}$ as well as the sets

$$N_{\gamma(x)}(x, r_i) \sim \big(D(x + \tfrac{r_i}{4} e_n, \tfrac{r_i}{4}) \cup D(x - \tfrac{r_i}{4} e_n, \tfrac{r_{i+1}}{4})\big)$$

corresponding to $x \in C_i$ with $x + 2^{-i} e_n \in C_{i-1}$. Noting $\frac{r_i}{4} + \frac{3 r_{i+1}}{4} < r_i \leq 2^{-i-1}$, we also define Ω_i to be the family consisting of the sets

$$\begin{gathered} S\big(x - \tfrac{r_i}{4} e_n, r_{i+1}\big) \cup Z\big(x - (\tfrac{r_i}{4} + \tfrac{r_{i+1}}{4}) e_n, \tfrac{r_{i+1}}{8}, -e_n, 2^{-i-1} - \tfrac{r_i}{4} - \tfrac{3 r_{i+1}}{4}\big) \\ \cup R\big(x - (2^{-i-1} - \tfrac{r_{i+1}}{4}) e_n, r_{i+1}\big) \end{gathered}$$

corresponding to $x \in C_i$. Clearly, we have $\operatorname{card} \Psi_i = \operatorname{card} C_i = \operatorname{card} \Omega_i$. Finally, we let $M_i = \bigcup_{j=1}^{i} \bigcup_{u \in H} \bigcup (X_j(u) \cup \Psi_j \cup \Omega_j)$.

Now, we define $M = \bigcup_{i=1}^{\infty} M_i$ and notice that M is a bounded, connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$ such that

$$(\operatorname{Clos} M) \sim M = \mathbf{R}^n \cap \{x : x \bullet e_n = 0 \text{ and } 0 \leq x_k \leq 1 \text{ for } k = 1, \ldots, m\}.$$

Since we have $\mathscr{H}^{m-1}(\partial D(a, r)) = m \boldsymbol{\alpha}(m) r^{m-1}$ and

$$\begin{gathered} \mathscr{H}^m(Z(a, r, u, h)) = m \boldsymbol{\alpha}(m) r^{m-1} h, \quad \sup \operatorname{im} \|\mathbf{b}(Z(a, r, u, h), \cdot)\| = r^{-1}, \\ \mathscr{H}^m(R(a, r)) \leq \kappa r^m, \quad \sup \operatorname{im} \|\mathbf{b}(R(a, r), \cdot)\| \leq \kappa r^{-1}, \\ \mathscr{H}^m(N_G(a, r)) \leq \kappa r^m, \quad \sup \operatorname{im} \|\mathbf{b}(N_G(a, r), \cdot)\| \leq \kappa r^{-1} \end{gathered}$$

whenever $a \in \mathbf{R}^n$, $0 < r < \infty$, $u \in \mathbf{S}^{n-1}$, $0 < h < \infty$, and $G \subset H$, one may use the fact that $\sum_{i=1}^{\infty} 2^{i\lambda} r_i^{\epsilon} < \infty$ whenever $\lambda \in \mathbf{R}$ and $\epsilon > 0$ to conclude

$$\lim_{i\to\infty} 2^{im} \mathscr{H}^m (M \sim M_{i-1}) = 0, \quad \lim_{i\to\infty} \mathscr{H}^{m-1}(\partial M_i) = 0,$$
$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \,\mathrm{d}\mathscr{H}^m\, x < \infty \quad \text{for } 1 \leq p < m-1,$$

whence we readily deduce the asserted conclusion. □

5.8 Remark. The construction bears some similarities with [Men09, 1.2].

6 Lower density bounds

In this section, we provide (see 6.7) Theorem D. The key to this are conditional lower density ratio bounds (see 6.3–6.6) which are in turn based on the Sobolev-Poincaré inequality (see 4.5). Moreover, to treat small positive density ratios, a compactness lemma (see 6.1) is employed.

6.1 Lemma. *Suppose* $1 \leq M < \infty$.

Then, there exists a positive, finite number Γ *with the following property.*

If m *and* n *are positive integers,* $m \leq n \leq M$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{V}_m(\mathbf{U}(a,r))$, $\|\delta V\|\, \mathbf{U}(a,r) \leq \Gamma^{-1} r^{m-1}$,

$$\|V\|\, \mathbf{B}(a,s) \geq M^{-1} \boldsymbol{\alpha}(m) s^m \quad \textit{whenever } 0 < s < r,$$

and $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ *for* $\|V\|$ *almost all* x, *then*

$$\|V\|\, \mathbf{U}(a,r) \geq (1 - M^{-1}) \boldsymbol{\alpha}(m) r^m.$$

Proof. If the lemma were false for some M, there would exist sequences Γ_i with $\Gamma_i \to \infty$ as $i \to \infty$ and sequences m_i, n_i, a_i, r_i, and V_i showing that $\Gamma = \Gamma_i$ does not have the asserted property.

We could assume for some positive integers m and n that $m \leq n \leq M$, $m = m_i$, $n = n_i$, $a_i = 0$, and $r_i = 1$ whenever i is a positive integer. Defining $V \in \mathbf{V}_m(\mathbf{R}^n \cap \mathbf{U}(0,1))$ to be the limit of some subsequence of V_i, we would obtain

$$\|V\|\, \mathbf{U}(0,1) \leq (1 - M^{-1}) \boldsymbol{\alpha}(m), \quad 0 \in \operatorname{spt} \|V\|, \quad \delta V = 0.$$

Finally, using [All72, 5.6, 8.6, 5.1 (2)], we would then conclude that

$$\boldsymbol{\Theta}^m(\|V\|, x) \geq 1 \quad \text{for } \|V\| \text{ almost all } x,$$
$$\boldsymbol{\Theta}^m(\|V\|, 0) \geq 1, \quad \|V\|\, \mathbf{U}(0,1) \geq \boldsymbol{\alpha}(m),$$

a contradiction. □

6.2 Remark. The pattern of the preceding proof is that of [Men16a, 7.3].

The conditional lower density bounds follow rather immediately from the Sobolev-Poincaré inequality (see 4.5) and its corollary (see 4.10), respectively.

6.3 Lemma. *Suppose m and n are positive inters, $2 \le m \le n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$ and $V_2 \in \mathbf{V}_{m-1}(U)$ satisfy the conditions of 4.4, V_1 is indecomposable of type $\mathscr{D}(U, \mathbf{R})$,*

$$V_2 = 0 \quad \text{if } m = 2, \qquad \|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \text{if } m > 2,$$
$$\|\delta V_2\| \text{ is absolutely continuous with respect to } \|V_2\| \quad \text{if } m > 3,$$
$$\phi_i \text{ are associated with } V_1 \text{ as in 4.4 for } i \in \{1, 2\},$$

$a \in \operatorname{spt} \|V_1\|$, $0 < r < \infty$, $\mathbf{B}(a, r) \subset U$, and $(\operatorname{spt} \|V_1\|) \sim \mathbf{U}(a, r) \neq \varnothing$.

Then, there holds

$$\Gamma_{4.5}(m)^{-1} r \le \|V_1\|(\mathbf{U}(a, r))^{1/m} + \phi_1 \mathbf{U}(a, r) + \|V_2\|(\mathbf{U}(a, r))^{1/(m-1)} + \phi_2 \mathbf{U}(a, r).$$

Proof. In view of [MS18, 4.6 (1)], [Men16a, 9.2, 9.4], and [MS23, 7.4], one may apply 4.5 and 4.6 with $f(x)$ replaced by $\sup\{r - |x - a|, 0\}$. ☐

We also include a version without boundary with more explicit constants.

6.4 Lemma. *Suppose U, V, and ψ are as in 4.9, $a \in \operatorname{spt} \|V\|$, $0 < r < \infty$, $\mathbf{B}(a, r) \subset U$, $(\operatorname{spt} \|V\|) \sim \mathbf{U}(a, r) \neq \varnothing$, and V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$.*

Then,

$$2^{-m-3} m^{-1} \boldsymbol{\gamma}(m)^{-1} r \le \|V\|(\mathbf{U}(a, r))^{1/m} + \boldsymbol{\gamma}(m)^{m-1} \psi\, \mathbf{U}(a, r).$$

Proof. In view of [MS18, 4.6 (1)], [Men16a, 9.2, 9.4], and [MS23, 7.4], one may apply 4.10 and 4.11 with $f(x)$ replaced by $\sup\{r - |x - a|, 0\}$. ☐

6.5 Remark. If either $m < n$ or $m = n = 2$, considering small spheres or small disks, respectively, shows that neither the nonemptyness hypothesis nor the indecomposability hypothesis may be omitted.

6.6 Remark. If $m = 1$ and V otherwise is as in 4.9, then $r \le \|V\| \, \mathbf{U}(a, r)$; in fact, the indecomposability hypothesis implies $\{x : |x - a| = s\} \cap \operatorname{spt} \|V\| \neq \varnothing$ for $0 < s < r$, whence the inequality follows, since $\mathscr{H}^1 \llcorner \operatorname{spt} \|V\| \le \|V\|$ by [All72, 3.5 (1b)] and [Men16a, 4.8 (4)].

Theorem D is contained in the first item of the next theorem. The remaining items discuss—for special dimensions—slightly more general boundary conditions.

6.7 Theorem. *Suppose m and n are positive inters, $2 \le m \le n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$, $V_2 \in \mathbf{V}_{m-1}(U)$,*

$$\boldsymbol{\Theta}^{\dim V_i}(\|V_i\|, x) \ge 1 \quad \text{for } \|V_i\| \text{ almost all } x \text{ and } i \in \{1, 2\},$$
$$\|\delta V_1\| \text{ is a Radon measure}, \quad \|\delta V_2\| \text{ is a Radon measure},$$

V_1 is indecomposable of type $\mathscr{D}(U, \mathbf{R})$, and $\lambda = 2^{-4} \boldsymbol{\alpha}(2)^{-1} \boldsymbol{\gamma}(2)^{-2}$.

Then, the following three statements hold.

(1) *If $m - 1 \le p < m$, $\|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\|$, $\|\delta V_2\|$ is absolutely continuous with respect to $\|V_2\|$, $\mathbf{h}(V_1, \cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V_1\|, \mathbf{R}^n)$, and, in case $m > 2$, additionally $\mathbf{h}(V_2, \cdot) \in \mathbf{L}_{p-1}^{\mathrm{loc}}(\|V_2\|, \mathbf{R}^n)$, then*

$$\mathscr{H}^{m-p}\big(\operatorname{spt} \|V_1\| \cap \big\{x : \boldsymbol{\Theta}_*^m(\|V_1\|, x) < 1 \text{ and } \boldsymbol{\Theta}_*^{m-1}(\|V_2\|, x) < 1\big\}\big) = 0.$$

(2) *If* $m = 3$, $\|\delta V_1\| \leq \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\|$, $\mathbf{h}(V_1, \cdot) \in \mathbf{L}_2^{\mathrm{loc}}(\|V_1\|, \mathbf{R}^n)$, *and*

$$A = \operatorname{spt} \|V_1\| \cap \big\{x : \boldsymbol{\Theta}_*^3(\|V_1\|, x) < 1 \text{ and } \boldsymbol{\Theta}_*^2(\|V_2\|, x) < \lambda\big\},$$

then

$$\mathscr{H}^1 \llcorner A \leq \sup\big\{2^7 \boldsymbol{\gamma}(2)^2, 2^2 \Gamma_{4.5}(3)\big\} \|\delta V_2\| \llcorner A.$$

(3) *If* $m = 2$, $V_2 = 0$, *and* $A = \operatorname{spt} \|V_1\| \cap \big\{x : \boldsymbol{\Theta}_*^2(\|V_1\|, x) < \lambda\big\}$, *then*

$$\mathscr{H}^1 \llcorner A \leq 2^8 \boldsymbol{\gamma}(2)^2 \|\delta V_1\| \llcorner A.$$

In particular, in all cases, $\mathscr{H}^m \llcorner \operatorname{spt} \|V_1\| \leq \|V_1\|$.

Proof. Firstly, we notice that in case of (1) we may assume $V_2 = 0$ if $m = 2$; in fact, since $\boldsymbol{\Theta}^1(\|V_2\|, x) \geq 1$ for $x \in \operatorname{spt} \|V_2\|$ by [Men16a, 4.8 (4)], we may otherwise replace U by $U \sim \operatorname{spt} \|V_2\|$ by [MS23, 7.5].

Secondly, we notice for $i \in \{1, 2\}$ that $V_i \in \mathbf{RV}_{\dim V_i}(U)$ by [All72, 5.5 (1)] and that

$$\|V_i\| = \mathscr{H}^{\dim V_i} \llcorner \boldsymbol{\Theta}^{\dim V_i}(\|V_i\|, \cdot)$$

by [All72, 3.5 (1b)]. Taking $p = m - 1$ in case of (2) or (3), we define

$$\psi_1 = \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)|^p \text{ in case of (1) or (2)}, \quad \psi_1 = \|\delta V_1\| \text{ in case of (3)},$$
$$\psi_2 = \|V_2\| \llcorner |\mathbf{h}(V_2, \cdot)|^{p-1} \text{ in case of (1)}, \quad \psi_2 = \|\delta V_2\| \text{ in case of (2) or (3)}.$$

Taking $\epsilon = \inf\big\{2^{-7} \boldsymbol{\gamma}(2)^{-2}, 2^{-2} \Gamma_{4.5}(3)^{-1}\big\}$ and

$$\lambda_1 = 1 \text{ in case of (1) or (2)}, \quad \lambda_1 = \lambda \text{ in case of (3)},$$
$$\lambda_2 = 1 \text{ in case of (1) or (3)}, \quad \lambda_2 = \lambda \text{ in case of (2)},$$
$$\delta = 2^{-m} \boldsymbol{\alpha}(m)^{-1} \Gamma_{4.5}(m)^{-m} \text{ in case of (1) or (2)}, \quad \delta = 0 \text{ in case of (3)},$$
$$\epsilon_1 = 0 \text{ in case of (1) or (2)}, \quad \epsilon_1 = 2^{-8} \boldsymbol{\gamma}(2)^{-2} \text{ in case of (3)},$$
$$\epsilon_2 = 0 \text{ in case of (1) or (3)}, \quad \epsilon_2 = \epsilon \text{ in case of (2)},$$

we furthermore define

$$A_1 = \operatorname{spt} \|V_1\| \cap \big\{x : \boldsymbol{\Theta}_*^m(\|V_1\|, x) < \lambda_1\big\}, \quad A_2 = \big\{x : \boldsymbol{\Theta}_*^{m-1}(\|V_2\|, x) < \lambda_2\big\},$$
$$Q_1 = \big\{x : \boldsymbol{\Theta}_*^m(\|V_1\|, x) > \delta\big\}, \quad Q_2 = \big\{x : \boldsymbol{\Theta}^{*m-1}(\|V_2\|, x) > 0\big\},$$
$$X_i = \big\{x : \boldsymbol{\Theta}^{*m-p}(\psi_i, x) > \epsilon_i\big\} \quad \text{for } i \in \{1, 2\}.$$

Clearly, we have $X_2 = \varnothing$ in case of (3). Moreover, we observe that [Fed69, 2.10.19 (3)] may be employed (cf. [FZ73, p. 152, l. 9–16]) to conclude

$$2^{-8} \boldsymbol{\gamma}(2)^{-2} \mathscr{H}^1 \llcorner X_1 \leq \psi_1 \text{ in case of (3)}, \quad \epsilon \mathscr{H}^1 \llcorner X_2 \leq \psi_2 \text{ in case of (2)},$$
$$\mathscr{H}^{m-p}(X_1) = 0 \text{ in case of (1) or (2)}, \quad \mathscr{H}^{m-p}(X_2) = 0 \text{ in case of (1)}.$$

Clearly, we have $Q_2 = \varnothing$ in case of (3). Applying [Men09, 2.10] with ϵ, Γ, and s replaced by $(2\boldsymbol{\gamma}(2))^{-1}$, $2^4 \boldsymbol{\gamma}(2)$, and 1, we obtain

$$\mathscr{H}^1(A_1 \cap Q_1 \sim X_1) = 0 \text{ in case of (3)}, \quad \mathscr{H}^1(A_2 \cap Q_2 \sim X_2) = 0 \text{ in case of (2)}.$$

According to [Men09, 2.11], there holds

$$\mathscr{H}^{m-p}(A_2 \cap Q_2) = 0 \text{ in case of (1).}$$

Moreover, we obtain

$$A_1 \cap Q_1 \subset X_1 \cup Q_2 \text{ in case of (1) and (2);}$$

in fact, whenever $\sup\{n, 1/\delta\} \leq M < \infty$ and $a \in Q_1 \sim (X_1 \cup Q_2)$, all sufficiently small $r > 0$ satisfy

$$\begin{gathered}\mathbf{B}(a,r) \subset U, \qquad \|V_1\| \, \mathbf{B}(a,s) \geq M^{-1} \boldsymbol{\alpha}(m) s^m \quad \text{for } 0 < s < r, \\ (\boldsymbol{\alpha}(m) r^m)^{1-1/p} \psi_1(\mathbf{U}(a,r))^{1/p} + \|V_2\| \, \mathbf{U}(a,r) \leq \Gamma_{6.1}(M)^{-1} r^{m-1},\end{gathered}$$

whence we infer $\|V_1\| \, \mathbf{U}(a,r) \geq (1 - M^{-1}) \boldsymbol{\alpha}(m) r^m$ by 6.1 and Hölder's inequality. Next, we verify

$$\operatorname{spt} \|V_1\| \subset Q_1 \cup X_1 \cup Q_2 \cup X_2;$$

in fact, this follows from 6.3 and Hölder's inequality in case of (1), from 6.3 alone in case of (2), and from 6.4 in case of (3). Therefore, we obtain

$$\begin{gathered}A_1 \cap A_2 \subset (A_2 \cap Q_2) \cup X_1 \cup X_2 \text{ in case of (1) or (2),} \\ A_1 \subset (A_1 \cap Q_1) \cup X_1 \text{ in case of (3),}\end{gathered}$$

whence the main conclusion follows.

Since $\lambda_2 \, \mathscr{H}^{m-1} \llcorner U \sim A_2 \leq \|V_2\|$ by [Fed69, 2.10.19 (3)] and

$$\mathscr{H}^m \, \{x : 0 < \boldsymbol{\Theta}^m(\|V_1\|, x) < 1\} = 0,$$

the main conclusion yields $\boldsymbol{\Theta}^m(\|V_1\|, x) \geq 1$ for $\mathscr{H}^m$ almost all $x \in \operatorname{spt} \|V_1\|$ and the postscript follows. □

6.8 Remark. The exponent of the Hausdorff measure $\mathscr{H}^{m-p}$ in (1) may not be replaced by any smaller number determined by m and p even if $V_2 = 0$ by [MS23, 6.1] and 5.5. Similarly, if $m \geq 3$, the hypothesis $m - 1 \leq p$ in (1) may not be replaced by $m - 1 - \epsilon \leq p$ for any $0 < \epsilon \leq 1$ determined by m and p even if $V_2 = 0$ by [MS23, 6.1] and 5.7.

6.9 Remark. The case $m = 1$, not treated here, was studied in [Men16a, 4.8]; similarly, results on the case $p = m$ and $V_2 = 0$ are summarised in [Men16a, 7.6].

7 Geodesic diameter

In this section, we establish (see 7.4 and 7.12) Theorem E and Corollary 2. For this purpose, we firstly study and characterise the geodesic diameter of closed subsets of Euclidean space (see 7.1–7.3). Then, we deduce (see 7.4–7.9) the bounds on the geodesic diameter in the varifold setting. As corollaries, we treat the cases of immersions (see 7.10–7.14), submanifolds (see 7.15–7.17), and λ-minimising currents (see 7.18–7.21).

7.1 Definition (see [Men16b, 6.6])**.** Whenever X is a boundedly compact metric space, the *geodesic distance* on X is the pseudometric on X whose value at $(a,x) \in X \times X$ equals the infimum of the set of numbers

$$\mathbf{V}_{\inf I}^{\sup I} C$$

corresponding to all continuous maps $C : \mathbf{R} \to X$ such that $C(\inf I) = a$ and $C(\sup I) = x$ for some compact non-empty subinterval I of $\mathbf{R}$. Moreover, the diameter with respect to the geodesic distance is termed *geodesic diameter*.

7.2 Remark (see [Men16b, 6.3]). The same definition results if one considers maps $C : \{y : 0 \le y \le b\} \to X$ with $\operatorname{Lip} C \le 1$ and $b = \mathbf{V}_0^b C$ corresponding to $0 \le b < \infty$. (In fact, if it is finite, the infimum is attained by some such C.)

7.3 Lemma. *Suppose X is a closed subset of $\mathbf{R}^n$ and d denotes the geodesic diameter of X.*

Then, there holds

$$d = \sup\{\operatorname{diam} f[X] : 0 \le f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}) \text{ and } |\operatorname{D} f(x)| \le 1 \text{ for } x \in X\}.$$

Proof. In view of [Fed69, 2.9.20] and 7.2, the supremum does not exceed d.

To prove the converse inequality, we define pseudometrics $\sigma_\delta : X \times X \to \overline{\mathbf{R}}$ by letting $\sigma_\delta(a,x)$, for $(a,x) \in X \times X$ and $0 < \delta \le 1$, denote the infimum of the set of numbers

$$\sum_{i=1}^{j} |x_i - x_{i-1}|$$

corresponding to all finite sequences $x_0, x_1, \ldots, x_j \in X$ with $x_0 = a$, $x_j = x$, and $|x_i - x_{i-1}| \le \delta$ for $i = 1, \ldots, j$. One readily verifies that

$$\sigma_\delta(\chi, a) \le \sigma_\delta(x,a) + |x - \chi| \quad \text{whenever } a, x, \chi \in X \text{ and } |x - \chi| \le \delta;$$

in particular, $\operatorname{Lip}(\sigma_\delta(\cdot, a) | \mathbf{B}(x,\delta)) \le 1$ in case $\sigma_\delta(a,x) < \infty$. Denoting by ϱ the geodesic distance on X, we have

$$|a - x| \le \sigma_\delta(a,x) \le \varrho(a,x) \quad \text{and} \quad \lim_{\delta \to 0+} \sigma_\delta(a,x) = \varrho(a,x) \quad \text{for } a, x \in X$$

by [Men16b, 6.3]. Consequently, one readily verifies[6]

$$d \le \sup\{\operatorname{diam} \operatorname{im} \sup\{s - \sigma_\delta(\cdot, a), 0\} : 0 \le s < \infty,\ 0 < \delta \le 1, \text{ and } a \in X\}.$$

This estimate implies that the conclusion is a consequence of the following assertion: if $\epsilon > 0$, $0 \le s < \infty$, $0 < \delta \le 1$, $a \in X$, and $\zeta = \sup\{s - \sigma_\delta(\cdot, a), 0\}$, then there exists a nonnegative function $Z \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ such that

$$|Z(x) - \zeta(x)| \le \epsilon \quad \text{and} \quad |\operatorname{D} Z(x)| \le 1 \quad \text{whenever } x \in X.$$

To prove this assertion, we first observe that ζ is a real valued function with $\operatorname{Lip}(\zeta | \mathbf{B}(x,\delta)) \le 1$ for $x \in X$. Moreover, since $\sup \operatorname{im} \zeta < \infty$, it is sufficient

[6] In fact, as σ_δ is real valued in case $d < \infty$, we have

$$d = \sup\{\operatorname{diam} \operatorname{im} \sup\{s - \sigma_\delta(\cdot, a), 0\} : 0 \le s < \infty,\ 0 < \delta \le 1, \text{ and } a \in X\}.$$

to prove the assertion with $|\operatorname{D} Z(x)| \leq 1$ replaced by $|\operatorname{D} Z(x)| \leq 1 + \epsilon$. For this purpose, we will employ the partition of unity given in [Fed69, 3.1.13]; in particular, let V_1 be the number constructed there, $\kappa = \sup\{1, V_1\}$, and define

$$\Phi = \{\mathbf{U}(\chi, \delta) : \chi \in X \cap \mathbf{U}(a, s+\delta)\} \cup \{\mathbf{R}^n \sim \mathbf{B}(a,s)\}, \quad U = \bigcup \Phi,$$
$$h(x) = \tfrac{1}{20} \sup \big\{ \inf\{\operatorname{dist}(x, \mathbf{R}^n \sim T), 1\} : T \in \Phi \big\} \quad \text{for } x \in U.$$

Employing a Lipschitzian extension (see [Fed69, 2.10.44]) and convolution, we construct, for each $T \in \Phi$, a nonnegative function $g_T \in \mathscr{E}(\mathbf{R}^n, \mathbf{R})$ satisfying

$$|g_T(x) - \zeta(x)| \leq (129)^{-n}(20\kappa)^{-1}\delta\epsilon \quad \text{for } x \in T, \qquad |\operatorname{D} g_T(x)| \leq 1 \quad \text{for } x \in \mathbf{R}^n,$$

where we may assume that $g_T = 0$ if $T = \mathbf{R}^n \sim \mathbf{B}(a,s)$, since $\operatorname{spt} \zeta \subset \mathbf{B}(a,s)$. Taking S, S_x, and v_s, for $s \in S$, as in [Fed69, 3.1.13] and choosing $\tau : S \to \Phi$ such that $\operatorname{spt} v_s \subset \tau(s)$ for $s \in S$, we define $G = \sum_{s \in S} v_s g_{\tau(s)}$. Clearly, we have $|G(x) - \zeta(x)| \leq \epsilon$ for $x \in X$ and $\operatorname{spt} G \subset \mathbf{U}(a, s + 2\delta)$. Noting $h(x) \geq \frac{\delta}{20}$ for $x \in X$, we furthermore estimate

$$\Big|\sum_{s \in S} \operatorname{D} v_s(x) g_{\tau(s)}(x)\Big| \leq \sum_{s \in S_x} |\operatorname{D} v_s(x)||g_{\tau(s)}(x) - \zeta(x)| \leq \epsilon, \quad |\operatorname{D} G(x)| \leq 1 + \epsilon$$

for $x \in X$. Applying [MS22, 3.16] with U, E_0, and E_1 replaced by $\mathbf{R}^n$, $\mathbf{R}^n \sim U$, and X to obtain f with the properties listed there, we may take $Z \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ defined by $Z(x) = f(x)G(x)$ for $x \in U$ and $Z(x) = 0$ for $x \in \mathbf{R}^n \sim U$. ☐

Next, we turn to Theorem E, the general a priori estimate of the geodesic diameter in the varifold setting with boundary.

7.4 Theorem. *Suppose m and n are integers, $2 \leq m \leq n$, $V_1 \in \mathbf{V}_m(\mathbf{R}^n)$ and $V_2 \in \mathbf{V}_{m-1}(\mathbf{R}^n)$ satisfy the conditions of 4.4 with $U = \mathbf{R}^n$, V_1 is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $(\|V_1\| + \|V_2\|)(\mathbf{R}^n) < \infty$,*

$$V_2 = 0 \quad \text{if } m = 2, \qquad \|\delta V_1\| \leq \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \text{if } m > 2,$$
$$\|\delta V_2\| \text{ is absolutely continuous with respect to } \|V_2\| \quad \text{if } m > 3,$$

ϕ_i are associated with V_i as in 4.4, for $i \in \{1, 2\}$, and d denotes the geodesic diameter of $\operatorname{spt} \|V_1\|$.

Then, there holds, for some positive finite number Γ determined by m,

$$d \leq \Gamma(\phi_1 + \phi_2)(\mathbf{R}^n).$$

Proof. The isoperimetric inequality and Hölder's inequality yield

$$\|V_2\|(\mathbf{R}^n)^{1/(m-1)} \leq \boldsymbol{\gamma}(m-1)^{m-2}\, \phi_2(\mathbf{R}^n).$$

We will show

$$\|V_1\|(\mathbf{R}^n)^{1/m} \leq (2\boldsymbol{\gamma}(m))^{m-1}\, \phi_1(\mathbf{R}^n) + (2\boldsymbol{\gamma}(m))^{1/(m-1)} \boldsymbol{\gamma}(m-1)^{m-2}\, \phi_2(\mathbf{R}^n);$$

in fact, if $m = 2$, then $\|V_1\|(\mathbf{R}^n)^{1/2} \leq \boldsymbol{\gamma}(2)\, \phi_1(\mathbf{R}^n)$ by the isoperimetric inequality, and, if $m > 2$, then we may assume $\|V_1\|(\mathbf{R}^n)^{1-1/m} > 2\boldsymbol{\gamma}(m)\, \|V_2\|(\mathbf{R}^n)$, in which case the isoperimetric inequality may be used to obtain

$$\|V_1\|(\mathbf{R}^n)^{1-1/m} \leq 2\boldsymbol{\gamma}(m) \textstyle\int |\mathbf{h}(V_1, x)| \,\mathrm{d}\|V_1\| x,$$

whence the asserted inequality follows by Hölder's inequality.

Next, suppose $X = \operatorname{spt} \|V_1\|$ and f satisfies the conditions of 7.3. Then, $f \in \mathbf{T}_\varnothing(V_i)$ and $\|V_i\|_{(\infty)}(V_i \mathbf{D} f) \leq 1$ for $i \in \{1, 2\}$ by [MS18, 4.6 (1)] and [Men16a, 9.2]. Hence, 4.5 and 4.6 yield

$$\operatorname{diam} \operatorname{spt} f_\# \|V_1\| \leq \Delta(\phi_1 + \phi_2)(\mathbf{R}^n),$$

where $\Delta = \Gamma_{4.5}(m)\big(1+\boldsymbol{\gamma}(m-1)^{m-2}(1+(2\boldsymbol{\gamma}(m))^{1/(m-1)})+(2\boldsymbol{\gamma}(m))^{m-1}\big)$. Finally, we notice $f[X] \subset \operatorname{spt} f_\# \|V_1\|$ as f is continuous.[7] □

7.5 Remark. The preceding theorem answers the fourth question posed in [Sch16, Section A].

7.6 Remark. By [MS23, 10.21], integral varifolds satisfying the hypotheses with $m = 2$ and $n = 3$ occur in the minimisation of the Willmore energy with *clamped boundary condition* amongst connected surfaces; see [NP20, Theorem 4.1].

In the case without boundary, somewhat more explicit constants may be obtained by using 4.10 instead of 4.5.

7.7 Corollary. *Suppose V and ψ are as in 4.9 with $U = \mathbf{R}^n$, $\|V\|(\mathbf{R}^n) < \infty$, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, d denotes the geodesic diameter of* $\operatorname{spt} \|V\|$*, and $\Gamma = 2^{m+4} m \boldsymbol{\gamma}(m)^m$.*

Then, there holds

$$d \leq \Gamma\, \psi(\mathbf{R}^n).$$

Proof. With a possibly larger number Γ, this follows from 7.4 with $V_1 = V$ and $V_2 = 0$. We verify the eligibility of the present number Γ by noting that

$$\operatorname{diam} \operatorname{spt} f_\# \|V\| \leq 2^{m+3} m \boldsymbol{\gamma}(m) \big(\|V\|(\mathbf{R}^n)^{1/m} + \boldsymbol{\gamma}(m)^{m-1}\, \psi(\mathbf{R}^n) \big) \leq \Gamma\, \psi(\mathbf{R}^n)$$

by 4.10 in conjunction with the isoperimetric inequality and Hölder's inequality, whenever f satisfies the conditions of 7.3 with $X = \operatorname{spt} \|V\|$. □

7.8 Remark. Each component (see [Men16a, 6.12] and [MS23, 5.1, 7.2]) of a varifold occurring in the *level-set mean curvature flow of two-convex submanifolds* of dimension m in $\mathbf{R}^{m+1}$ satisfies the hypotheses, see [GH20, Corollary 1.1].

7.9 Remark. Here, we compare our proof with that of P. Topping's analogous result for immersions (see [Top08, Theorem 1.1]). The principal geometric idea—suitable smallness of mean curvature implies lower density ratio bounds in balls—is the same. Our formulation (see 4.1 and 6.4) may be traced back to [All72, 8.3], whereas his formulation (see [Top08, Lemma 1.2]) was inspired by his local non-collapsing result for Ricci flow (see [Top05, Theorem 4.2]). To implement this geometric idea for varifolds, one faces the difficulty that one cannot—a priori—assume the existence of either geodesics or lower density bounds; the latter are employed to obtain [Top08, Lemma 1.2]. Instead, our proof avoids these tools—though, lower density bounds are independently proven in 6.7—and proceeds through the characterisation of geodesic diameter (see 7.3) and the Sobolev-Poincaré inequality (see 4.5) in conjunction with basic properties from the study of indecomposability (see [MS23, 7.12]).

To prepare for the use of the Whitney-type approximation results in 7.12, we firstly define the appropriate topological function space.

[7] In fact, as f is closed, we have $f[X] = \operatorname{spt} f_\# \|V_1\|$.

7.10 Definition. Suppose k is a positive integer, M is a compact manifold-with-boundary of class k, and Y is a Banach space.

Then, $\mathscr{C}^k(M,Y)$ is defined (see [Men16b, 2.4]) to be the locally convex space of all maps from M into Y of class k topologised by the family of all seminorms, that correspond to charts ϕ of M of class k and compact subsets K of $\operatorname{dmn}\phi$, and have value

$$\sup\big(\{0\}\cup\big\{\|\operatorname{D}^l(F\circ\phi^{-1})(x)\|: x\in K\sim\phi[\partial M], l=0,\ldots,k\big\}\big)$$

at $F\in\mathscr{C}^k(M,Y)$.

7.11 Remark. Choosing a positive integer ι and charts ϕ_i of M of class k with compact subsets K_i of $\operatorname{dmn}\phi_i$, for $i=1,\ldots,\iota$, satisfying $M=\bigcup_{i=1}^{\iota}\operatorname{Int}K_i$, the topology of the locally convex space $\mathscr{C}^k(M,Y)$ is induced by the norm ν on $\mathscr{C}^k(M,Y)$ whose value at $F\in\mathscr{C}^k(M,Y)$ equals

$$\sup\big(\{0\}\cup\big\{\|\operatorname{D}^j(F\circ\phi_i^{-1})(x)\|: x\in K_i\sim\phi_i[\partial M],\ i=0,\ldots,\iota,\ j=0,\ldots,k\big\}\big);$$

in fact, each seminorm occurring in 7.10 is bounded by a finite multiple of ν by the general formula for the differentials of a composition, see [Fed69, 3.1.11]. Similarly, we see that the topology of $\mathscr{C}^k(M,\mathbf{R}^n)$ agrees with that of the space named "$C^k_W(M,\mathbf{R}^n)$" in [Hir94, p. 35]. Consequently, if $n>2\dim M$, the set of embeddings of M into $\mathbf{R}^n$ of class 2 is dense in $\mathscr{C}^2(M,\mathbf{R}^n)$ by [Hir94, 2.1.0].

Next, we present Corollary 2, the a priori estimate of the geodesic diameter of immersions of compact manifolds-with-boundary.

7.12 Corollary. *Suppose m and n are positive integers, $2\le m\le n$, M is a compact connected m dimensional manifold-with-boundary of class 2, the map $F: M\to\mathbf{R}^n$ is an immersion of class 2, g is the Riemannian metric on M induced by F, and σ is the Riemannian distance associated with (M,g).*

Then, there holds

$$\operatorname{diam}_\sigma M\le\Gamma_{7.4}(m)\big(\textstyle\int_M|\mathbf{h}(F,\cdot)|^{m-1}\,\mathrm{d}\mathscr{H}^m_\sigma+\int_{\partial M}|\mathbf{h}(F|\partial M,\cdot)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1}_\sigma\big);$$

here $0^0=1$.

Proof. First, the *special case*, that F is an embedding, will be treated; in this case, F induces an isometry between σ and the geodesic distance on $F[M]$ by [Fed69, 2.10.13, 3.2.3 (1)] and 7.2. We define $V_1\in\mathbf{V}_m(\mathbf{R}^n)$ to be associated with $(F,\mathbf{R}^n)$ and $V_2\in\mathbf{V}_{m-1}(\mathbf{R}^n)$ to be 0 if $m=2$ and to be associated with $(F|\partial M,\mathbf{R}^n)$ if $m>2$. Hence, [MS23, 6.14, 10.3] yield

$$\begin{gathered}\boldsymbol{\Theta}^{\dim V_i}(\|V_i\|,x)\ge 1\quad\text{for }\|V_i\|\text{ almost all }x\text{ and }i\in\{1,2\},\\ \|V_1\|=F_\#\mathscr{H}^m_\sigma,\quad\|\delta V_1\|=\|V_1\|\llcorner|\mathbf{h}(F[M\sim\partial M],\cdot)|+F_\#(\mathscr{H}^{m-1}_\sigma\llcorner\partial M),\\ \|V_2\|=F_\#(\mathscr{H}^{m-1}_\sigma\llcorner\partial M)\text{ if }m>2,\quad\|\delta V_2\|=\|V_2\|\llcorner|\mathbf{h}(F[\partial M],\cdot)|\text{ if }m>2.\end{gathered}$$

Since V_1 is indecomposable of type $\mathscr{D}(U,\mathbf{R})$ by [MS23, 7.9 (1) (4)], the special case now follows from 7.4.

In the general case, we assume $n>2m$ and obtain from 7.11 a sequence of embeddings $F_i: M\to\mathbf{R}^n$ of class 2 converging to F in $\mathscr{C}^2(M,\mathbf{R}^n)$ as $i\to\infty$; in particular, $\mathbf{h}(F_i,x)\to\mathbf{h}(F,x)$, uniformly for $x\in M$, as $i\to\infty$ by [MS23, 6.11] and 7.11. Moreover, denoting by g_i the Riemannian metrics on M induced by F_i and by σ_i the Riemannian distance of (M, g_i), we observe that, given $1 < \lambda < \infty$, all sufficiently large i satisfy

$$\lambda^{-2}\langle (w,w), g(z)\rangle \le \langle (w,w), g_i(z)\rangle \le \lambda^2 \langle (w,w), g(z)\rangle$$

whenever $z \in M$ and w belongs to the tangent space of M at z, whence we infer $\lambda^{-1}\sigma \le \sigma_i \le \lambda\sigma$ and $\lambda^{-k}\mathscr{H}^k_\sigma \le \mathscr{H}^k_{\sigma_i} \le \lambda^k \mathscr{H}^k_\sigma$ for $0 \le k < \infty$. Therefore, the conclusion follows from the special case applied with F replaced by F_i. □

7.13 Remark. In the case $m = 2$ with $\operatorname{diam}_\sigma f[M]$ replaced by $\operatorname{diam} f[M]$ a better constant is obtained in [Miu22, Theorem 1.1].

7.14 Remark. The integral $\int_{\partial M} |\mathbf{h}(F|\partial M, \cdot)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}_\sigma$ equals $\mathscr{H}^1_\sigma(\partial M)$ if $m = 2$ but it may not be replaced by $\mathscr{H}^{m-1}_\sigma(\partial M)^{1/(m-1)}$ if $m > 2$; it suffices to take $m = n$.

Turning to the case of submanifolds, we state the immediate corollary and then discuss the impossiblity of a seemingly natural sharpening of the result.

7.15 Corollary. *Suppose m and n are integers, $2 \le m \le n$, M is a compact connected m dimensional submanifold-with-boundary of class 2 of $\mathbf{R}^n$, $f : M \to \mathbf{R}$ is of class 1 relative to M, and $\kappa = \sup\{|\operatorname{D} f(x)| : x \in M \sim \partial M\}$.*

Then, there holds (here, $0^0 = 0$)

$$\operatorname{diam} f[M] \le \Gamma_{7.4}(m)\big(\textstyle\int_M |\mathbf{h}(M,\cdot)|^{m-1} \,\mathrm{d}\mathscr{H}^m + \int_{\partial M} |\mathbf{h}(\partial M, \cdot)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}\big)\kappa.$$

Proof. We combine [Fed69, 2.9.20], [MS23, 10.3], and 7.12. □

7.16 Remark. In contrast to other Poincaré inequalities, the derivative $\operatorname{D} f$ may not be measured with respect to $(\mathscr{H}^m \llcorner M)_{(q)}$ for any $m < q < \infty$, see 7.17.

7.17 Example. Whenever m is an integer and $2 \le m < q < \infty$, the infimum of the set of numbers

$$\gamma(M)^{1-m/q} \cdot \big(\textstyle\int_M |\operatorname{D} f|^q \,\mathrm{d}\mathscr{H}^m\big)^{1/q},$$

where $\gamma(M) = \int_M |\mathbf{h}(M,z)|^{m-1} \,\mathrm{d}\mathscr{H}^m z + \int_{\partial M} |\mathbf{h}(\partial M, z)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} z$, corresponding to all compact connected m dimensional submanifolds-with-boundary of class ∞ of $\mathbf{R}^{m+1}$ and functions $f : M \to \mathbf{R}$ of class 1 relative to M with $\operatorname{diam} f[M] = 1$, equals 0; in fact, the same holds if we require $\partial M = \varnothing$.

Proof. Whenever $1 \le r < \infty$, we consider

$$M = \mathbf{S}^{m-1} \times \{y : 0 \le y \le r\} \subset \mathbf{R}^m \times \mathbf{R} \simeq \mathbf{R}^{m+1},$$

and $f : M \to \mathbf{R}$ such that $f(x,y) = y/r$ for $(x,y) \in M$; hence, $\operatorname{diam} f[M] = 1$. Noting $|\operatorname{D} f(x,y)| = 1/r$ for $(x,y) \in M \sim \partial M$, we use [Fed69, 3.2.23] to compute

$$\begin{aligned} \textstyle\int_M |\mathbf{h}(M,\cdot)|^{m-1} \,\mathrm{d}\mathscr{H}^m &= \mathscr{H}^{m-1}(\mathbf{S}^{m-1})(m-1)^{m-1} r, \\ \textstyle\int_{\partial M} |\mathbf{h}(\partial M,\cdot)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} &= 2\mathscr{H}^{m-1}(\mathbf{S}^{m-1})(m-1)^{m-2}, \\ \textstyle\int_M |\operatorname{D} f|^q \,\mathrm{d}\mathscr{H}^m &= \mathscr{H}^{m-1}(\mathbf{S}^{m-1}) r^{1-q}. \end{aligned}$$

The principal assertion follows and the postscript may be obtained by adding half-spheres with boundary ∂M to the cylinder M and approximation. □

Finally, we shall introduce the setting of λ-minimising currents, apply our general diameter estimate to it, and discuss its significance.

7.18 Example. Suppose m and n are integers, $2 \leq m \leq n$, $0 \leq \lambda < \infty$, and, following [DS93a], the integral current $Q \in \mathbf{I}_m(\mathbf{R}^n)$ satisfies

$$\|Q\|(\mathbf{R}^n) \leq \|Q + \partial R\|(\mathbf{R}^n) + \lambda \|R\|(\mathbf{R}^n) \quad \text{whenever } R \in \mathbf{I}_{m+1}(\mathbf{R}^n).$$

Noting [Fed69, 4.1.28] and [All72, 3.5 (1c)], we take $V \in \mathbf{IV}_m(\mathbf{R}^n)$ characterised by $\|V\| = \|Q\|$. Then,

$$\|\delta V\| \leq \lambda \|V\| + \|\partial Q\|$$

by [DS93a, (2.3)], hence, noting $\|\partial Q\|_{\|Q\|} = 0$ by [Fed69, 4.1.28], we obtain

$$\|\delta V\|_{\|V\|} \leq \lambda \|V\|, \quad \|\delta V\| - \|\delta V\|_{\|V\|} \leq \|\partial Q\|;$$

in fact, [Fed69, 2.9.2] applied with $\psi = \|\partial Q\|$ and $\phi = \|V\|$ yields a Borel set B such that

$$\|\partial Q\|(B) = 0 \quad \text{and} \quad \|V\|(\mathbf{R}^n \sim B) = 0,$$

whence we infer $\|\delta V\|_{\|V\|} \leq \|\delta V\| \llcorner B \leq \lambda \|V\|$ and thus $\|\delta V\| \llcorner B \leq \|\delta V\|_{\|V\|}$. Recalling $\|\delta V\|_{\|V\|} = \|V\| \llcorner |\mathbf{h}(V, \cdot)|$ from [MS23, 3.21], we deduce

$$\|V\|_{(\infty)}(\mathbf{h}(V, \cdot)) \leq \lambda \quad \text{and} \quad \|\delta V\| \leq \|V\| \llcorner |\mathbf{h}(V, \cdot)| + \|\partial Q\|.$$

7.19 Corollary. *Suppose m and n are integers, $2 \leq m \leq n$, $0 \leq \lambda < \infty$, $Q \in \mathbf{I}_m(\mathbf{R}^n)$ is indecomposable,*

$$\|Q\|(\mathbf{R}^n) \leq \|Q + \partial R\|(\mathbf{R}^n) + \lambda \|R\|(\mathbf{R}^n) \quad \text{for } R \in \mathbf{I}_{m+1}(\mathbf{R}^n),$$

d denotes the geodesic diameter of the set $\operatorname{spt} Q$, and $\gamma = \Gamma_{7.4}(m)$; if $m > 2$, then suppose $W \in \mathbf{V}_{m-1}(\mathbf{R}^n)$, $\|\delta W\|$ is a Radon measure, $\boldsymbol{\Theta}^{m-1}(\|W\|, x) \geq 1$ for $\|W\|$ almost all x, and $\|\partial Q\| \leq \|W\|$; if $m > 3$, then suppose $\|\delta W\|$ is absolutely continuous with respect to $\|W\|$ and $\mathbf{h}(W, \cdot) \in \mathbf{L}^{\mathrm{loc}}_{m-2}(\|W\|, \mathbf{R}^n)$.

Then, $\mathscr{H}^m \llcorner \operatorname{spt} Q \leq \|Q\|$, the set $\operatorname{spt} Q$ is approximately differentiable of order 2 at $\|Q\|$ almost all a, and the following three statements hold.

(1) If $m = 2$, then $d \leq \gamma\big(\int |\operatorname{ap} \mathbf{h}(\operatorname{spt} Q, \cdot)| \,\mathrm{d}\|Q\| + \|\partial Q\|(\mathbf{R}^n)\big)$.

(2) If $m = 3$, then $d \leq \gamma\big(\int |\operatorname{ap} \mathbf{h}(\operatorname{spt} Q, \cdot)|^2 \,\mathrm{d}\|Q\| + \|\delta W\|(\mathbf{R}^n)\big)$.

(3) If $m > 3$, then $d \leq \gamma\big(\int |\operatorname{ap} \mathbf{h}(\operatorname{spt} Q, \cdot)|^{m-1} \,\mathrm{d}\|Q\| + \int |\mathbf{h}(W, \cdot)|^{m-2} \,\mathrm{d}\|W\|\big)$.

Proof. We associate $V \in \mathbf{IV}_m(\mathbf{R}^n)$ with Q as in 7.18. Noting [MS22, 5.1] and [MS23, 6.7], the varifold V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$ by [MS23, 10.9]. Therefore, applying 6.7 and 7.4 with (V_1, V_2) replaced by $(V, 0)$ if $m = 2$ and (V, W) if $m > 2$ yields $\mathscr{H}^m \llcorner \operatorname{spt} Q \leq \|Q\|$ and that the last three conclusions hold with $\operatorname{ap} \mathbf{h}(\operatorname{spt} Q, \cdot)$ replaced by $\mathbf{h}(V, \cdot)$. Finally, [Men13, 4.8], in conjunction with [Fed69, 2.10.19 (4)] and [San19, 3.22], shows that the set $\operatorname{spt} Q$ is approximately differentiable of order 2 with $\operatorname{ap} \mathbf{h}(\operatorname{spt} Q, a) = \mathbf{h}(V, a)$ at $\|Q\|$ almost all a. ☐

7.20 Remark. For any $Q \in \mathbf{I}_m(\mathbf{R}^n)$ satisfying the condition

> "There exists no $R \in \mathbf{I}_m(\mathbf{R}^n)$ with $\partial R = 0$ such that $R \neq 0 \neq Q - R$ and $\|R\| + \|Q - R\| = \|Q\|$."

indecomposability of ∂Q implies that of Q. For Q as in 7.18, this condition is guaranteed by

$$\lambda^m \|Q\|(\mathbf{R}^n) \leq \boldsymbol{\alpha}(m+1)(m+1)^{m+1}$$

via the optimal isoperimetric inequality for integral currents of [Alm86, § 10].

7.21 Remark. Both [DS93a] and the preceding theorem are tailored to apply to the integral currents with prescribed mean curvature vector studied in [DF90, DF92, Duz93] and to the codimension-one area minimising integral currents with prescribed volume and boundary constructed in [DS92]. The condition discussed in 7.20 is particularly natural in this context as exemplified by [DS92, 1.2, 1.3] and [Duz93, 2.3]; the same holds for the mass bound guaranteeing it in view of [DF90, 6.1]. For $n - m = 1$, its usage to obtain connectedness of the regular part of $\operatorname{spt} Q$ appears in [DS93b, p. 358].

8 Plateau problems

In this section, we demonstrate how to apply our geodesic diameter estimate (see 7.4) to solutions of Plateau problems. Firstly, we consider the setting of integral chains with coefficients in a complete normed commutative group (see 8.1–8.8); in particular, we obtain Theorem A. Then, we study (see 8.9–8.21) natural classes of varifolds with conditions on their first variation away from the boundary; this results (see 8.12 and 8.17) in Theorems F and G. Drawing from the literature, our theory finally becomes applicable to the Plateau problem for sets based on Čech homology yielding (see 8.22) Theorem B.

8.1 Lemma (classical)**.** *Suppose m and n are integers, $1 \leq m \leq n$, G is a complete normed commutative group, $S \in \mathbf{I}_m(\mathbf{R}^n, G)$,*

$$\|S\|(\mathbf{R}^n) \leq \|T\|(\mathbf{R}^n) \quad \textit{whenever } T \in \mathbf{I}_m(\mathbf{R}^n, G) \textit{ and } \partial_G T = \partial_G S,$$

and $V \in \mathbf{RV}_m(\mathbf{R}^n)$ is characterised by $\|V\| = \|S\|$.

Then, there holds

$$|(\delta V)(g)| \leq \textstyle\int |\operatorname{Nor}^{m-1}(\|\partial_G S\|, x)_\natural(g(x))| \,\mathrm{d}\|\partial_G S\| x \quad \textit{for } g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n).$$

Proof. Suppose $g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ and $\lambda = \operatorname{Lip} g$. We define $h : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ and $h_t : \mathbf{R}^n \to \mathbf{R}^n$ such that $h(t, x) = h_t(x) = x + tg(x)$ whenever $(t, x) \in \mathbf{R} \times \mathbf{R}^n$; we abbreviate $I_t = \operatorname{spt}[0, t]$ for $t \in \mathbf{R}$ and $\phi = \|\partial_G S\|$. Identifying $[0, t] \in \mathbf{I}_1(\mathbf{R})$ with $\iota_{\mathbf{R},1}([0, t]) \in \mathbf{I}_1(\mathbf{R}, \mathbf{Z})$, see [MS22, 5.1], we notice

$$[0, t] \times (\partial_G S) \in \mathscr{R}_m(\mathbf{R} \times \mathbf{R}^n, G) \quad \text{with} \quad \|[0, t] \times \partial_G S\| = (\mathscr{L}^1 \llcorner I_t) \times \phi$$

whenever $t \in \mathbf{R}$ by [MS22, 4.7]. Employing [MS22, 5.13 (6), 5.16, 5.13 (4), 4.5, 4.6], we then obtain

$$\begin{gathered}
(h_t)_\# S \in \mathbf{I}_m(\mathbf{R}^n, G), \quad h_\#([0, t] \times S) \in \mathbf{I}_{m+1}(\mathbf{R}^n, G), \\
h_\#([0, t] \times \partial_G S) \in \mathbf{I}_m(\mathbf{R}^n, G), \\
(h_t)_\# S - S = \partial_G h_\#([0, t] \times S) + h_\#([0, t] \times \partial_G S), \\
\partial_G \big((h_t)_\# S - h_\#([0, t] \times \partial_G S)\big) = \partial_G S, \\
\|S\|(\mathbf{R}^n) - \|h_\#([0, t] \times \partial_G S)\|(\mathbf{R}^n) \leq \|(h_t)_\# S\|(\mathbf{R}^n), \\
\|h_\#([0, t] \times \partial_G S)\|(\mathbf{R}^n) \leq \textstyle\int_{I_t \times \mathbf{R}^n} \big\| \textstyle\bigwedge_m (\mathscr{L}^1 \times \phi, m) \operatorname{ap} \mathrm{D}\, h \big\| \,\mathrm{d}\mathscr{L}^1 \times \phi
\end{gathered}$$

whenever $t \in \mathbf{R}$; in case $|t|\lambda < 1$, we also observe that h_t is a diffeomorphism with $\operatorname{im} h_t = \mathbf{R}^n$, hence $\|(h_t)_\# V\|(\mathbf{R}^n) = \|(h_t)_\# S\|(\mathbf{R}^n)$ by [MS22, 3.27, 3.28, 4.6]. For $\mathscr{L}^1 \times \phi$ almost all $(s,x) \in I_t \times \mathbf{R}^n$, recalling

$$\operatorname{Tan}^m(\mathscr{L}^1 \times \phi, (s,x)) = \mathbf{R} \times \operatorname{Tan}^{m-1}(\phi, x)$$

from [MS22, 4.4, 4.7], we finally estimate

$$\begin{aligned}\big\|{\textstyle\bigwedge_m}(\mathscr{L}^1 \times \phi, m) \operatorname{ap} \mathrm{D}\, h(s,x)\big\| \leq (1+|t|\lambda)^{m-1}\big|\operatorname{Nor}^{m-1}(\phi,x)_\natural(g(x))\big| \\ +(m-1)|t|\lambda(1+|t|\lambda)^{m-2}\big|\operatorname{Tan}^{m-1}(\phi,x)_\natural(g(x))\big|;\end{aligned}$$

thus, [All72, 4.1] yields the conclusion. □

8.2 Remark. The case $G = \mathbf{Z}$ appears in [All72, 4.8 (4)] whose method we employ.

8.3 Remark. It follows that if such S is indecomposable, then $\operatorname{spt}\|S\|$ is connected by [MS23, 7.7, 10.9].

Theorem A corresponds to the case $G = \mathbf{Z}$ in the next theorem, see [MS22, 5.1].

8.4 Theorem. *Suppose m and n are integers, $2 \leq m \leq n$, G is a complete normed commutative group, $S \in \mathbf{I}_m(\mathbf{R}^n, G)$ is indecomposable,*

$$\|S\|(\mathbf{R}^n) \leq \|T\|(\mathbf{R}^n) \quad \textit{whenever } T \in \mathbf{I}_m(\mathbf{R}^n, G) \textit{ and } \partial_G T = \partial_G S,$$

$\boldsymbol{\Theta}^m(\|S\|, x) \geq 1$ for $\|S\|$ almost all x, d denotes the geodesic diameter of $\operatorname{spt}\|S\|$, and $\gamma = \Gamma_{7.4}(m)$; if $m > 2$, then suppose $W \in \mathbf{V}_{m-1}(\mathbf{R}^n)$, $\|\delta W\|$ is a Radon measure, $\boldsymbol{\Theta}^{m-1}(\|W\|, x) \geq 1$ for $\|W\|$ almost all x, and $\|\partial_G S\| \leq \|W\|$; if $m > 3$, then suppose $\|\delta W\|$ is absolutely continuous with respect to $\|W\|$.

Then, the following three statements hold.

(1) If $m = 2$, then $d \leq \gamma \|\partial_G S\|(\mathbf{R}^n)$.

(2) If $m = 3$, then $d \leq \gamma \|\delta W\|(\mathbf{R}^n)$.

(3) If $m > 3$, then $d \leq \gamma \int |\mathbf{h}(W,x)|^{m-2} \,\mathrm{d}\|W\|\, x$.

Proof. Let $V \in \mathbf{RV}_m(\mathbf{R}^n)$ be characterised by $\|V\| = \|S\|$. From 8.1, we obtain $\|\delta V\| \leq \|\partial_G S\|$; in particular, $\mathbf{h}(V, \cdot) = 0$ by [All72, 3.5 (1b)]. In view of [MS23, 10.9], the conclusion then follows by applying 7.4 with (V_1, V_2) replaced by $(V, 0)$ if $m = 2$ and (V, W) if $m > 2$. □

8.5 Remark. For any $S \in \mathbf{I}_m(\mathbf{R}^n, G)$ satisfying

$$\|S\|(\mathbf{R}^n) \leq \|T\|(\mathbf{R}^n) \quad \text{whenever } T \in \mathbf{I}_m(\mathbf{R}^n, G) \text{ and } \partial_G T = \partial_G S,$$

indecomposability of $\partial_G S$ implies indecomposability of S.

8.6 Remark. We recall [MS22, 5.1, 6.2] and suppose $G = \mathbf{Z}$ or $G = \mathbf{Z}/d\mathbf{Z}$ for some positive integer d. Then, whenever $B \in \mathbf{I}_{m-1}(\mathbf{R}^n, G)$ with $\partial_G B = 0$, there exists $S \in \mathbf{I}_m(\mathbf{R}^n, G)$ with $\partial_G S = B$ satisfying

$$\|S\|(\mathbf{R}^n) \leq \|T\|(\mathbf{R}^n) \quad \text{whenever } T \in \mathbf{I}_m(\mathbf{R}^n, G) \text{ and } \partial_G T = \partial_G S;$$

in fact, in view of [MS22, 4.6, 5.13 (6)], we combine [Fed69, 4.1.11, 4.1.16, 4.2.17 (2), 4.2.26]. For these G, the density hypotheses are redundant. For general

G, the validity of a rectifiability theorem analogous to [Fed69, 4.2.16 (3)]—which is central to this minimisation process—has been characterised in [Whi99, 7.1] in the context of the flat m-chains over G introduced in [Fle66]; see [MS22, p. 7] regarding the pending comparison to the present concept of G chains.

8.7 Remark. Here, we discuss the novelty of our results in the case that $G = \mathbf{Z}$, B is a connected orientable compact $m-1$ dimensional submanifold of $\mathbf{R}^n$, and $\| \partial_{\mathbf{Z}} S\| = \mathscr{H}^{m-1} \llcorner B$: If B is of class 4, then connectedness of $\operatorname{spt} \|S\|$ was first noted—as a consequence of profound regularity results—in [DLDPHM23, p. 6]; yet, the implication itself is elementary, see 8.3. That $\operatorname{spt} \|S\|$ is necessarily path-connected—in fact, must have finite geodesic diameter—is new. If B is of class 2 and $m = n-1$, then finiteness of the geodesic diameter of $\operatorname{spt} \|S\|$ can be deduced from earlier results—recalling [Fed69, 4.1.20, 4.1.31 (2)], it suffices to combine [Men16b, Theorem 6.8 (1)] with [HS79, 11.2 (1) (3)]—whereas the a priori bound on the geodesic diameter in terms of B and m is new.

8.8 Remark. As in [Miu22, Section 3], the preceding theorem implies nonexistence of indecomposable solutions to the Plateau problem in the case that the diameter of $\operatorname{spt} \| \partial_G S\|$ strictly exceeds the upper bound for d in the conclusion by [MS22, 5.13 (5)].

Now, we return to the setting of varifolds.

8.9 Lemma. *Suppose m and n are positive integers, $m \le n$, B is an $m-1$ dimensional submanifold of $\mathbf{R}^n$ of class 2, $0 < s < R$, $b \in B$, $V \in \mathbf{V}_m(\mathbf{U}(b,s))$, $b \in \operatorname{spt} \|V\|$, $\|V\|(B \cap \mathbf{U}(b,s)) = 0$,*

$$\big| \operatorname{Nor}(B,z)_\natural (z-y) \big| \le (2R)^{-1} |z-y|^2 \quad \textit{for } y,z \in B,$$
$$\mathbf{U}(b, s/(1-s/R)) \cap (\operatorname{Clos} B) \sim B = \varnothing,$$

$\operatorname{spt} \delta V \subset B$, and $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x.

Then, the following two statements hold.

1. *If $0 < r < s/2$, then*

$$\|\delta V\| \, \mathbf{B}(b,r) \le \big(r^{-1} + m/(R-s)\big) \|V\| \, \mathbf{B}(b, 2r);$$

   in particular, $\boldsymbol{\Theta}^{*m-1}(\|\delta V\|, b) \le 2^m \boldsymbol{\alpha}(m-1)^{-1} \boldsymbol{\alpha}(m) \boldsymbol{\Theta}^m(\|V\|, b)$.

2. *The function mapping $0 < r \le s$ onto*

$$\frac{\|V\| \, \mathbf{U}(b,r)}{r^m} \exp\left(\frac{3mR}{2(R-s)^2} r \right)$$

   is nondecreasing. Moreover, there holds $1/2 \le \boldsymbol{\Theta}^m(\|V\|, b) < \infty$.

Proof. We notice that [All75, 2.2 (3) (4), 3.1, 3.5 (1) (2)] remain valid if the submanifold B therein is required to be of class 2, instead of class ∞. From [All75, 2.2 (4b)], we thus infer that for $a \in \mathbf{U}(b,s)$ there exists a unique $y \in B$ with $|y-a| = \operatorname{dist}(a, B) < R$. Therefore, [All75, 2.2 (3), 3.1 (1) (2)] yield

$$\|\delta V\|(\psi) \le \textstyle\int |\operatorname{D} \psi(x) \circ S_\natural| \, \mathrm{d} V\,(x,S) + m/(R-s) \textstyle\int \psi \, \mathrm{d} \|V\|$$

for $0 \le \psi \in \mathscr{D}(\mathbf{U}(b,s), \mathbf{R})$; in particular, $\|\delta V\|$ is a Radon measure and hence $V \in \mathbf{RV}_m(\mathbf{U}(b,s))$ by [All72, 5.5 (1)]. Moreover, we have

$$1/2 \le \boldsymbol{\Theta}^m(\|V\|, b) < \infty$$

by [Men16a, 4.8 (2) (4)] if $m = 1$ and by [All75, 3.5 (1) (2)] if $m \geq 2$.

To verify (1), we define $f : \mathbf{U}(b,s) \to \mathbf{R}$ by

$$f(x) = \sup\left\{0, 1 - r^{-1}\operatorname{dist}(x, \mathbf{B}(b,r))\right\} \quad \text{for } x \in \mathbf{U}(b,s),$$

hence $0 \leq f \leq 1$, $f(x) = 1$ for $x \in \mathbf{B}(b,r)$, $\operatorname{Lip} f = r^{-1}$, and $\operatorname{spt} f = \mathbf{B}(b,2r)$. Recalling [All72, 3.5 (1b)] and [Men16a, 8.7], and approximating f by nonnegative members of $\mathscr{D}(\mathbf{U}(b,s),\mathbf{R})$ for instance by means of [Men16b, 3.7], we conclude

$$\|\delta V\|(f) \leq \textstyle\int |V\,\mathbf{D}\,f|\,\mathrm{d}\|V\| + m/(R-s)\int f\,\mathrm{d}\|V\|$$

and infer (1), as $|V\,\mathbf{D}\,f(x)| \leq \operatorname{Lip} f$ for $x \in \operatorname{dmn} V\,\mathbf{D}\,f$.

Finally, the first conclusion of (2) may be obtained by adapting the case $\alpha = 0$ of [All75, 3.4 (2)] as follows:[8] Recalling that [All75, 2.2 (3) (4), 3.1 (3)] remain valid for B therein required to be of class 2 instead of class ∞, in the statement of [All75, 3.4 (2)], omit the definition of μ and modify the definition of $\Phi(r)$ to $\Phi(r) = \frac{3kR}{2(R-s)^2}r$, and, in its proof, omit the first paragraph, replace α by 0 throughout, replace "$g \in \mathscr{X}(\mathbf{U}(0,s))$" by "$g : \mathbf{U}(0,s) \to \mathbf{R}^n$ is of class 1 with compact support", and omit the summand involving μ in the last equation. □

8.10 Remark. The method of proof of (1) is adapted from [All75, 3.4 (1)].

8.11 Remark. If $m = 1$, then the sharp estimate $\boldsymbol{\Theta}^0(\|\delta V\|, b) \leq 2\boldsymbol{\Theta}^1(\|V\|, b)$ holds as we could apply [All75, 3.1 (2)] after the first paragraph of the proof.

The preceding lemma readily yields Theorem F.

8.12 Theorem. *Suppose m and n are positive integers with $m \leq n$, B is a nonempty compact $m-1$ dimensional submanifold of class 2 of $\mathbf{R}^n$, $R = \operatorname{reach}(B)$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\|V\|(B) = 0$, $\operatorname{spt}\delta V \subset B \subset \operatorname{spt}\|V\|$, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x,*

$$M = R^{-m}\sup\{\|V\|\,\mathbf{U}(b, R/2) : b \in B\},$$

and $\Gamma = 4^m\boldsymbol{\alpha}(m-1)^{-1}\exp(3m)$.

Then, there holds $0 < R < \infty$, $\Gamma M \geq 2^{m-1}\boldsymbol{\alpha}(m)\boldsymbol{\alpha}(m-1)^{-1}$, and

$$\|\delta V\| \leq \Gamma M \mathscr{H}^{m-1} \llcorner B.$$

Proof. Since $\operatorname{reach}(B,b) > 0$ for $b \in B$ by [Fed59, 4.12], there holds $R > 0$ by [Fed59, 4.2]. Moreover, observing that B cannot be convex, we obtain $R < \infty$ from [Fed59, 4.2]; in particular, the first conclusion holds. Next, [Fed59, 4.18] yields

$$|\operatorname{Nor}(B,b)_\natural(y-b)| \leq (2R)^{-1}|y-b|^2 \quad \text{whenever } y, b \in B.$$

Applying 8.9 (1) with $s = R/2$ and 8.9 (2) with $r = s = R/2$, we obtain

$$\boldsymbol{\Theta}^{*m-1}(\|\delta V\|, b) \leq 2^m\boldsymbol{\alpha}(m-1)^{-1}\boldsymbol{\alpha}(m)\boldsymbol{\Theta}^m(\|V\|, b) \leq \Gamma M \quad \text{for } b \in B;$$

in particular, the second conclusion holds because $\boldsymbol{\Theta}^m(\|V\|, b) \geq 1/2$ for $b \in B$ by 8.9 (2). As $\boldsymbol{\Theta}^{m-1}(\mathscr{H}^{m-1} \llcorner B, b) = 1$ by [Fed69, 3.1.23, 3.2.17], we infer

$$\limsup_{r\to 0+} \frac{\|\delta V\|\,\mathbf{B}(b,r)}{(\mathscr{H}^{m-1} \llcorner B)\,\mathbf{B}(b,r)} \leq \Gamma M \quad \text{for } b \in B.$$

[8]This presupposes the following typographical corrections to [All75, 3.4 (2)]: in the statement, replace "$m(s)^{1/k}/(R-s)$" by "$km(s)^{1/k}/(R-s) < 1$" on line 3 thereof; on page 427, replace "(6)" by "(b)" on line 7, "$\mathscr{D}(\mathbf{R})$" by "$\mathscr{E}^0(\mathbf{R})$", "$\Phi$" by "$\varphi$", and "near 0" by "near 0 with $\sup\operatorname{spt}\varphi < s$" all on line 19, and "$\xi$" by "$\zeta$" on line 27.

Since $\|\delta V\|(\mathbf{R}^n \sim B) = 0$, this implies firstly that $\|\delta V\|$ is absolutely continuous with respect to $\mathscr{H}^{m-1} \llcorner B$ by [Fed69, 2.9.2, 2.9.15] and then the last conclusion by [Fed69, 2.8.18, 2.9.7]. □

Boundary connectedness may be exploited with the following lemma.

8.13 Lemma. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $X \in \mathbf{V}_m(U)$, $\|X\|(U) < \infty$, $\beta = \infty$ if $m = 1$, $\beta = m/(m-1)$ if $m > 1$,*

$$\sup\{(\delta X)(\theta) : \theta \in \mathscr{D}(U, \mathbf{R}^n), \|X\|_{(\beta)}(\theta) \leq 1\} < \boldsymbol{\gamma}(m)^{-1},$$

and $\boldsymbol{\Theta}^m(\|X\|, x) \geq 1$ for $\|X\|$ almost all x.

Then, the closure of every connected component of $\operatorname{spt}\|X\|$ *meets* $\operatorname{Bdry} U$*; in particular, if* $\operatorname{Bdry} U \cap \operatorname{Clos}\operatorname{spt}\|X\|$ *is connected, then so is* $\operatorname{Clos}\operatorname{spt}\|X\|$.

Proof. Let $i : U \to \mathbf{R}^n$ be the inclusion map and denote by Φ the family of connected components of $\operatorname{spt}\|X\|$. Suppose $C \in \Phi$, hence $\operatorname{spt}(\|X\| \llcorner C) = C$ and $\|\delta(X \llcorner C \times \mathbf{G}(n,m))\| = \|\delta X\| \llcorner C$ by [Men16a, 6.14 (4)] and [MS23, 5.1]. Defining

$$V = i_{\#}(X \llcorner C \times \mathbf{G}(n,m)) \in \mathbf{V}_m(\mathbf{R}^n),$$

we note $0 < \|V\|(\mathbf{R}^n) < \infty$ and infer $\operatorname{spt}\|V\| = \operatorname{Clos} C$ and

$$\operatorname{spt}\delta V \subset \operatorname{spt}\|V\| \subset \operatorname{Clos} U, \quad \|\delta V\| \llcorner U = i_{\#}(\|\delta X\| \llcorner C).$$

We conclude $\|\delta V\|(\operatorname{Bdry} U) > 0$ because, using [MS23, 3.3, 3.4], we may estimate

$$\begin{aligned}\boldsymbol{\gamma}(m)^{-1}\|V\|(\mathbf{R}^n)^{1-1/m} &\leq \|\delta V\|(\mathbf{R}^n) \leq \|\delta X\|(U) + \|\delta V\|(\operatorname{Bdry} U) \\ &< \boldsymbol{\gamma}(m)^{-1}\|V\|(\mathbf{R}^n)^{1-1/m} + \|\delta V\|(\operatorname{Bdry} U);\end{aligned}$$

hence, $\operatorname{Bdry} U \cap \operatorname{spt}\|V\| \neq \varnothing$ and the principal conclusion follows. Accordingly, if $B = \operatorname{Bdry} U \cap \operatorname{Clos}\operatorname{spt}\|X\|$ is connected, then so are $B \cup \bigcup\{\operatorname{Clos} C : C \in \Phi\}$ and its closure $\operatorname{Clos}\operatorname{spt}\|X\|$. □

8.14 Remark. If $m = 1$, then $\boldsymbol{\gamma}(1) = 2^{-1}$ by [MS18, 3.8, 3.9] and $\boldsymbol{\gamma}(1)^{-1}$ may not be replaced by any larger value in the preceding lemma.

8.15 Remark. If $m = 2$ and V is integral, then $\boldsymbol{\gamma}(m)^{-1}$ may be replaced by the larger value $\left(\int_{\mathbf{S}^m} |\mathbf{h}(\mathbf{S}^m, \cdot)|^m \,\mathrm{d}\mathscr{H}^m\right)^{1/m}$ by [KS04, A.18]; the latter value is evidently sharp. (Based on [Men23, 24.1], the same replacement is feasible in case $2 \leq m = n - 1$ and V is integral.)

8.16 Remark. To illustrate the preceding lemma, let $B = \operatorname{Bdry} U \cap \operatorname{Clos}\operatorname{spt}\|W\|$ be the *fixed boundary* of either a *Brakke flow* V_t, $0 \leq t < \infty$, or a stationary integral varifold V_∞, resulting from it as subsequential limit as $t \to \infty$; one is assured of the existence of such objects under broad conditions by [ST21, Theorem 2.2, Corollary 2.4]. Thus, if this boundary B is connected, so must be $\operatorname{Clos}\operatorname{spt}\|V_t\|$, whenever V_t satisfies the first variation condition, and $\operatorname{Clos}\operatorname{spt}\|V_\infty\|$.

In combination with results from [MS23], Theorem G now follows.

8.17 Theorem. *Suppose m and n are integers, $2 \leq m \leq n$, B is a compact connected $m-1$ dimensional submanifold of class 2 of $\mathbf{R}^n$, $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\|V\|(\mathbf{R}^n) < \infty$, $1 \leq \lambda < \infty$,*

$$B \subset \operatorname{spt}\|V\|, \quad \|\delta V\| \leq \lambda \mathscr{H}^{m-1} \llcorner B,$$

$\mathbf{\Theta}^m(\|V\|, x) \geq 1$ *for* $\|V\|$ *almost all* x*, and* d *is the geodesic diameter of* $\operatorname{spt}\|V\|$*.*

Then, there holds

$$d \leq \Gamma_{7.4}(m)\lambda \textstyle\int_B |\mathbf{h}(B,b)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}\, b;$$

here, $0^0 = 1$*.*

Proof. We note $V \in \mathbf{RV}_m(\mathbf{R}^n)$ by [All72, 5.5 (1)]; in particular, $\|V\|(B) = 0$ by [All72, 3.5 (1b)]. Thus, taking $X = V|\mathbf{2}^{(\mathbf{R}^n \sim B)\times \mathbf{G}(n,m)}$, the set

$$\operatorname{spt}\|V\| = \operatorname{Clos}\operatorname{spt}\|X\|$$

is connected by 8.13. We then apply [MS23, 10.16] in conjunction with [MS23, 9.2] to conclude that V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$. We let $W \in \mathbf{V}_{m-1}(\mathbf{R}^n)$ be defined by

$$W(k) = \lambda \textstyle\int_B k(b, \operatorname{Tan}(B,b)) \,\mathrm{d}\mathscr{H}^{m-1}\, b \quad \text{for } k \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, m-1))$$

so that

$$\|\delta V\| \leq \|W\|.$$

The conclusion now follows from 7.4 applied with (V_1, V_2) replaced by $(V, 0)$ if $m = 2$ and (V, W) if $m > 2$, respectively. □

8.18 Remark. Our hypotheses have been arranged so as to include with $\lambda = 1$ the varifolds furnished by the min-max methods of [DLR18, Theorem 2.6 (b)] and [Mon20, Theorem 1.3] when, in these theorems, the prescribed boundary is connected and the ambient Riemannian manifold is a convex body in $\mathbf{R}^n$.

The preceding two theorems combine into two corollaries.

8.19 Corollary. *Suppose* m *and* n *are integers,* $2 \leq m \leq n$*,* B *is a nonempty compact connected* $m-1$ *dimensional submanifold of class* 2 *of* $\mathbf{R}^n$*,* $R = \operatorname{reach}(B)$*,* $V \in \mathbf{V}_m(\mathbf{R}^n)$*,* $\|V\|(\mathbf{R}^n) < \infty$*,* $\|V\|(B) = 0$*,*

$$\operatorname{spt}\delta V \subset B \subset \operatorname{spt}\|V\|,$$

$\mathbf{\Theta}^m(\|V\|, x) \geq 1$ *for* $\|V\|$ *almost all* x*,* $M = R^{-m}\sup\{\|V\|\,\mathbf{U}(b, R/2) : b \in B\}$*, and* d *is the geodesic diameter of* $\operatorname{spt}\|V\|$*.*

Then, there holds

$$d \leq \Gamma M \textstyle\int_B |\mathbf{h}(B,b)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1}\, b,$$

where Γ *is a positive, finite number determined by* m*; here,* $0^0 = 1$*.*

Proof. Let $\kappa = 2^{m-1}\boldsymbol{\alpha}(m)\boldsymbol{\alpha}(m-1)^{-1}$. We will show that one may take

$$\Gamma = \sup\left\{\kappa^{-1}, 1\right\}\Gamma_{7.4}(m)\Gamma_{8.12}(m).$$

From 8.12, we obtain $\Gamma_{8.12}(m)M \geq \kappa$ and

$$\|\delta V\| \leq \Gamma_{8.12}(m) M \mathscr{H}^{m-1} \llcorner B;$$

hence, we may apply 8.17 with $\lambda = \sup\{1, \Gamma_{8.12}(m)M\}$. □

8.20 Remark. The preceding corollary is applicable to the image in $\mathbf{R}^n$ of the varifolds furnished by subsequential limits as time approaches ∞ of the Brakke flow with fixed boundary in [ST21, Corollary 2.4] when the boundary in the cited source is a connected $m-1$ dimensional submanifold of class 2 of $\mathbf{R}^n$; in this case, M is bounded a priori by the initial conditions of the flow.

8.21 Corollary. *Suppose m and n are integers, $2 \le m \le n$, B is a nonempty compact connected $m-1$ dimensional submanifold of class 2 of $\mathbf{R}^n$, $R = \operatorname{reach}(B)$, S is a $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^n$,*

$$\begin{gathered} B \subset S, \quad S = \operatorname{spt}(\mathscr{H}^m \llcorner S), \\ \textstyle\int_S \operatorname{Tan}^m(\mathscr{H}^m \llcorner S, x) \bullet \mathrm{D}\,\theta(x)\,\mathrm{d}\mathscr{H}^m x = 0 \quad \textit{for } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n), \\ M = R^{-m} \sup\{\mathscr{H}^m(S \cap \mathbf{U}(b, R/2)) : b \in B\}, \end{gathered}$$

and d is the geodesic diameter of S.

Then, there holds

$$d \le \Gamma_{8.19}(m) M \textstyle\int_B |\mathbf{h}(B,b)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1} b;$$

here, $0^0 = 1$.

Proof. We take $V \in \mathbf{RV}_m(\mathbf{R}^n)$ such that $\|V\| = \mathscr{H}^m \llcorner S$ in 8.19. ☐

Finally, we include the derivation of Theorem B.

8.22 Remark. Adopting the terminology of [FK18, § 12] and drawing from the literature,[9] we will deduce the following corollary: *If G is a commutative group, L is a subgroup of the $(m-1)$-th Čech homology group of B, $\check{\mathscr{C}}(B, L, G)$ denotes the family of closed subsets of $\mathbf{R}^n$ spanning L, $E \in \check{\mathscr{C}}(B, L, G)$,*

$$\mathscr{H}^m(E) = \inf\big\{\mathscr{H}^m(F) : F \in \check{\mathscr{C}}(B, L, G)\big\}, \quad S = \operatorname{spt}(\mathscr{H}^m \llcorner E),$$

then $\mathscr{H}^m(E) \le \mathscr{H}^{m-1}(B)\operatorname{diam}(B)/m$ and the geodesic diameter of S is bounded by $\Gamma_{8.19}(m)\operatorname{reach}(B)^{-m}\mathscr{H}^m(E)\int_B |\mathbf{h}(B,b)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1} b$. Let C be the convex hull of B. We assume $L \ne \{0\}$ because $L = \{0\}$ implies $\mathscr{H}^m(E) = 0$. Verifying

$$Y_c = \{tb + (1-t)c : b \in B, 0 \le t \le 1\} \in \check{\mathscr{C}}(B, L, G) \quad \text{for } c \in C$$

by means of [ES52, Chapter 9, Theorems 3.4, 4.4, and 5.1], the estimate of $\mathscr{H}^m(E)$ follows from [Fed69, 3.2.20]; thus, noting [FK18, 12.3, 12.4], the set E is $(\mathscr{H}^m, m)$ rectifiable by [FK18, 10.1] applied with U, $\mathscr{C}$, S_i, S, and F replaced by $\mathbf{R}^n \sim B$, $\check{\mathscr{C}}(B, L, G)$, E, E, and $(\mathbf{R}^n \times \mathbf{G}(n,m)) \times \{1\}$ and

$$\textstyle\int_E \operatorname{Tan}^m(\mathscr{H}^m \llcorner E, x) \bullet \mathrm{D}\,\theta(x)\,\mathrm{d}\mathscr{H}^m x = 0 \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$$

by [All72, 4.1]; hence, $\mathscr{H}^m \llcorner S = \mathscr{H}^m \llcorner E$ by [All72, 2.8 (4a), 8.3], $S \subset C$ by [Sim83, 19.2], and, noting [Fed69, 4.1.16], we may assume $E \subset C$ by [FK18, 12.3]. Inferring $S \in \check{\mathscr{C}}(B, L, G)$ from [Lab22, 2.2.1] applied with E_k and E replaced by E and S and noting $B \subset S$ by [Lab22, 1.2.2], the preceding corollary becomes applicable. We also record that, noting [Fed69, 2.10.11, 4.1.16] and [Lab22, 2.1.3], the existence of such E, additionally contained in the convex hull

[9] The authors did not verify the results of [ES52, FK18, Pug19, Lab22] employed.

of B, is guaranteed for instance by [Lab22, 3.2.2] applied with $\Gamma = B$, $d = m$, and $\mathscr{I} = \mathscr{H}^m$. Finally, the isoperimetric inequality [Pug19, Theorem 2] yields

$$\mathscr{H}^m(E) \leq \Delta\, \mathscr{H}^{m-1}(B)^{m/(m-1)},$$

where Δ is a finite number determined by n; in fact, in view of [FK18, 12.3, 12.4], the estimate is entailed by applying the inequality with A, Y, and L, replaced by B, Y_c, and $\mathscr{H}^{m-1}(B)^{1/(m-1)}$ for some $c \in C$ because $Y_c \in \check{\mathscr{C}}(B, L, G)$.

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