Quantum Annealed Criticality

Premala Chandra,[1] Piers Coleman,[1,2] Mucio A. Continentino,[3] and Gilbert G. Lonzarich[4]

[1] *Center for Materials Theory, Rutgers University, Piscataway, New Jersey, 08854, USA*
[2] *Department of Physics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK*
[3] *Centro Brasileiro de Pesquisas Fiscicas, 22290-180, Rio de Janeiro, RJ, Brazil*
[4] *Cavendish Laboratory, Cambridge University, Cambridge CB3 0HE, United Kingdom*

(Dated: February 20, 2022)

Experimentally there exist many materials with first-order phase transitions at finite temperature that display quantum criticality. Classically, a strain-energy density coupling is known to drive first-order transitions in compressible systems, and here we generalize this Larkin-Pikin[1] mechanism to the quantum case. We show that if the $T = 0$ system lies above its upper critical dimension, the line of first-order transitions can end in a quantum annealed critical point where zero-point fluctuations restore the underlying criticality of the order parameter.

The interplay of first-order phase transitions with quantum fluctuations is an active area [2–9] in the study of exotic quantum states near zero-temperature phase transitions [10–14]. In many metallic quantum ferromagnets, coupling of the magnetization to low energy particle-hole excitations transforms a high temperature continuous phase transition into a low temperature discontinuous one, and the resulting classical tricritical points have been observed in many systems [2–9]. Experimentally there also exist insulating materials that have classical first-order transitions that display quantum criticality [15–19], and here we provide a theoretical basis for this observed behavior.

At a first-order transition the quartic mode-mode coupling of the effective action becomes negative. One mechanism for this phenomenon, studied by Larkin and Pikin [1] (LP), involves the interaction of strain with a fluctuating critical order parameter. LP found that a diverging specific heat in the clamped system of fixed dimensions leads to a first-order transition in the unclamped system at constant pressure. Specifically, the Larkin-Pikin criterion [1, 20] for a first order phase transition is

$$\kappa \; < \; \frac{\Delta C_V}{T_c}\left(\frac{dT_c}{d\ln V}\right)^2 \tag{1}$$

where V is the volume, ΔC_V is the singular part of the specific heat capacity in the clamped system, T_c is the transition temperature and $\frac{dT_c}{d\ln V}$ is its strain derivative. The effective bulk modulus κ is defined as $\kappa^{-1} = K^{-1} - (K + \frac{4}{3}\mu)^{-1}$ where K and μ are the bare bulk and the shear moduli in the absence of coupling to the order parameter fields; more physically $\kappa \sim K\frac{c_L^2}{c_T^2}$ where c_L and c_T are the longitudinal and the transverse sound velocities [21]. We note that shear strain plays a crucial role in this approach that requires $\mu > 0$. Short-range fluctuations in the atomic displacements renormalize the quartic coupling of the critical modes, but it is the coupling of the uniform ($q = 0$) strain to the energy density, the modulus squared of the critical order parameter, that results in a *macroscopic* instability of the critical point leading to a discontinuous transition.

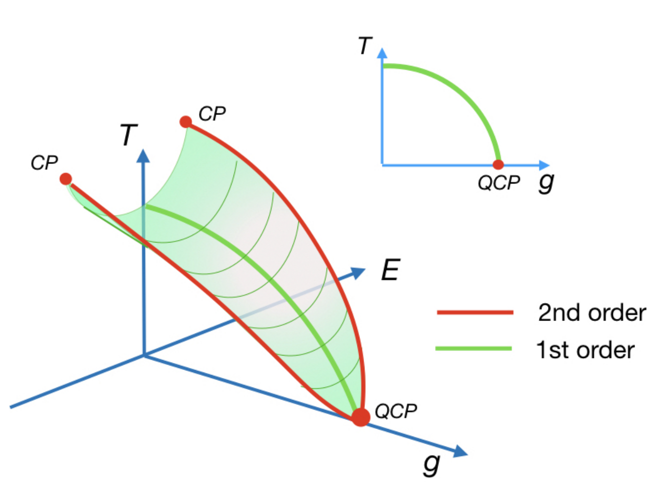

FIG. 1. Proposed Temperature-Field-Pressure Phase Diagram with a sheet of first-order transitions bounded by second-order phase lines linking the three critical points, two classical and one quantum; Inset: Temperature-Pressure "slice" indicating a line of classical phase transitions ending in a "quantum annealed critical point" with the standard temperature fan where the underlying order parameter criticality is restored by zero-point fluctuations.

Here we rewrite the Larkin-Pikin criterion in terms of correlation functions so that it can be generalized to the quantum case. We show that if the $T = 0$ quantum system lies above its upper critical dimension, the corrections to the renormalized bulk modulus are non-universal; the line of classical first-order transitions can end in a "quantum annealed critical point" where zero-point fluctuations restore the underlying criticality of the order parameter. We end with a discussion of the temperature-field-pressure phase diagram and specific measurements to probe it (cf. Fig. 1).

Low-temperature measurements on ferroelectric insulators provide a key motivation for our study [15–19]. At finite temperatures and ambient pressure these materials often display first-order transitions due to strong electromechanical coupling [22]; yet in many cases [15–

19] their dielectric susceptibilities suggest the presence of pressure-induced quantum criticality associated with zero-temperature continuous transitions [15–19]. It is thus natural to explore whether a quantum generalization of the Larkin-Pikin approach [1], involving the coupling of critical order parameter fluctuations to long wavelength elastic degrees of freedom, can be developed to describe this phenomenon.

In the simplest case of a scalar order parameter ψ and isotropic elasticity, the Larkin-Pikin (LP) mechanism [23] refers to a system where the order parameter $\psi(\vec{x})$ is coupled to the volumetric strain with interaction energy

$$H_I = \lambda \int d^3x \; e_{ll}(\vec{x}) \; \psi^2(\vec{x}) \tag{2}$$

where $e_{ab}(\vec{x}) = \frac{1}{2}\left(\frac{\partial u_a}{\partial x_b} + \frac{\partial u_b}{\partial x_a}\right)$ is the strain tensor, $u_a(\vec{x})$ is the atomic displacement, $e_{ll}(x) = \mathrm{Tr}[e(\vec{x})]$ is the volumetric strain and λ is a coupling constant associated with the strain-dependence of T_c, $\lambda = \left(\frac{dT_c}{d\ln V}\right)$. Though the elastic degrees of freedom are assumed to be Gaussian, and thus can be formally integrated out exactly, this must be done with some care. This is because the strain field separates into a uniform ($\vec{q} = 0$) term defined by boundary conditions and a finite-momentum ($\vec{q} \neq 0$) contribution determined by fluctuating atomic displacements

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{V}\sum_{\vec{q}\neq 0} \frac{i}{2}[q_a u_b(\vec{q}) + q_b u_a(\vec{q})]e^{i\vec{q}\cdot\vec{x}}, \tag{3}$$

where $\{a, b\} \in [1, 3]$ and $u_a(q)$ is the Fourier transform of $u_a(x)$. Here we employ periodic boundary conditions to a finite size system with volume $V = L^3$ and discrete momenta $\vec{q} = \frac{2\pi}{L}(l, m, n)$, where l, m, n are integers.

The uniform strain vanishes when the crystal is externally clamped. The main effect of integrating out the finite wavevector fluctuations in the strain is to induce a finite correction to the short-range interactions of the critical fluctuations that can be absorbed into the quartic ψ^4 terms in the action. By contrast, fluctuations in the uniform component of the strain induce an infinite-range attractive interaction between the critical modes (see Supplementary Materials), and it is this component of the interactions that is responsible for driving first order behavior. The problem is then reduced to the interaction of critical order parameter modes, mediated by the fluctuations of a uniform strain field ϕ with bulk modulus κ (for details see Supplementary Materials). Conceptually, the Larkin-Pikin approach amounts to a study of critical phenomena in a clamped system, followed by a stability analysis of the critical point once the clamping is removed.

Recently it was proposed to adapt the Larkin-Pikin approach to pressure($\mathcal{P}$)-tuned quantum magnets where it is often found that $\frac{dT_c}{d\mathcal{P}} \to \infty$ as $T_c \to 0$; the authors argued that the associated quantum phase transitions should then be first-order [24–26]. However such a diverging coupling of the critical order parameter fluctuations and the lattice should lead to structural instabilities near the quantum phase transition that have not been observed [9, 27]. Furthermore dynamics must be included when treating thermodynamic quantities at zero temperature [28, 29].

We recast the Larkin-Pikin criterion in the language of correlation functions, generalizing the LP approach to the quantum case summing over all possible space-time configurations. The strain field again separates into two contributions as in equation (3), one associated with static uniform boundary conditions and the other determined by short wavelength displacements fluctuating at all frequencies

$$e_{ab}(\vec{x}, \tau) = e_{ab} + \frac{1}{\beta V}\sum_{i\nu_n}\sum_{\vec{q}\neq 0} \frac{i}{2}[q_a u_b(q) + q_b u_a(q)]e^{i(\vec{q}\cdot\vec{x} - \nu_n \tau)} \tag{4}$$

where $q_\alpha \equiv (\vec{q}, i\nu_n)$ with $\alpha \in [1, 4]$, $u_b(q) \equiv u_b(\vec{q}, i\nu_n)$ and $\nu_n = 2\pi n T$ is a Matsubara frequency ($k_B = 1$). A detailed analysis indicates that when these space-time elastic degrees of freedom are integrated out, they lead to the coupling of the quantum critical order parameter modes to a *classical* strain field ϕ, uniform in both space and time, with the same effective bulk modulus κ as in the finite-temperature case (see Supplementary Material). The resulting effective action takes the form

$$S_{eff}[\psi, \phi] = \int_0^\beta d\tau \int d^3x \left[\mathcal{L}[\psi] + \lambda\phi \; \psi^2(\vec{x}, \tau) + \frac{1}{2}\kappa\phi^2\right], \tag{5}$$

where $(\vec{x}, \tau)$ are the Euclidean space-time co-ordinates and $\mathcal{L}[\psi]$ is the Lagrangian of the order parameter $\psi(\vec{x}, \tau)$ that undergoes a continuous transition in the clamped system; in the simplest case $\mathcal{L}[\psi]$ is a ψ^4 field theory

$$\mathcal{L}[\psi] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4. \tag{6}$$

The partition function of the unclamped system is then

$$Z[\phi] = e^{-\beta F[\phi]} = \int \mathcal{D}[\psi] e^{-S_{eff}[\psi, \phi]}, \tag{7}$$

where the trace is over the internal variable ψ, and $Z[\phi]$ to be evaluated at the stationary point $F'[\phi] = 0$. The renormalized bulk modulus, $\tilde{\kappa} = \kappa - \Delta\kappa$, is

$$\tilde{\kappa} = \frac{1}{V}\frac{\partial^2 F}{\partial \phi^2} = \kappa - \lambda^2 \int d^3x d\tau \langle \delta\psi^2(\vec{x}, \tau)\delta\psi^2(0)\rangle, \tag{8}$$

where $\delta\psi^2(\vec{x}, \tau) = \psi^2(\vec{x}, \tau) - \langle\psi^2\rangle$. In the classical problem there is no time-dependence, and $\int_0^\beta d\tau \to \beta \equiv 1/T$

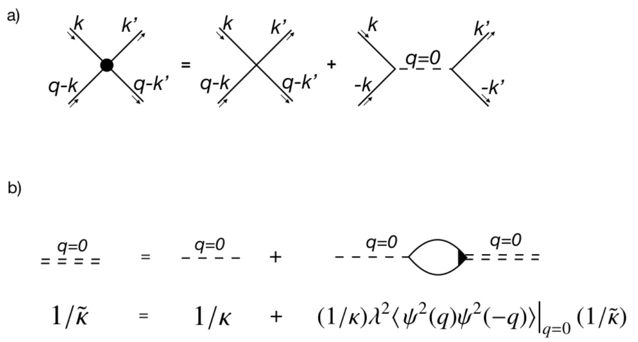

FIG. 2. Diagrammatic approach to the generalized Larkin-Pikin criterion a) Bare interaction is a sum of a local and a nonlocal contribution mediated by fluctuations in the strain; b) Feynman diagram showing renormalization of the strain propagator by coupling to energy fluctuations.

so at the transition

$$\tilde{\kappa} = \frac{1}{V}\frac{\partial^2 F}{\partial \phi^2} = \kappa - \frac{\lambda^2}{T_c}\int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle = \kappa - \Delta\kappa. \tag{9}$$

$\Delta\kappa$ in (9) is proportional to energy fluctuations, and can be re-expressed as $\frac{\lambda^2}{T_c}\Delta C_V$; we thus recover the LP criterion (1) ($\kappa < \Delta\kappa$ or $\tilde{\kappa} < 0$) for a first-order transition. We note that the renormalized quartic mode-mode coupling coefficient associated with (23) changes sign concomitantly with the renormalized bulk modulus; the former has contributions from both the strain coupling and from higher order parameter fluctuations [30].

The renormalized bulk modulus $\tilde{\kappa}$ can also be obtained diagrammatically (cf. Figure 2). In the low-energy effective action, the quartic term now has a contribution from the coupling of the order parameter fluctuations to the effective uniform strain. We then can use a Dyson equation for the strain propagator to determine $\tilde{\kappa}$. More specifically we can write

$$\left(\frac{1}{\tilde{\kappa}}\right) = \left(\frac{1}{\kappa}\right) + \left(\frac{1}{\kappa}\right)\lambda^2 \langle \psi^2(q)\psi^2(-q)\rangle\big|_{q=0}\left(\frac{1}{\tilde{\kappa}}\right) \tag{10}$$

that results in

$$\tilde{\kappa} = \kappa - \Delta\kappa = \kappa - \lambda^2 \chi_{\psi^2} \tag{11}$$

where $\chi_{\psi^2} = \chi_{\psi^2}(\vec{q}, i\nu_n)\big|_{\vec{q}, i\nu_n = 0}$ is the static susceptibility for ψ^2, where

$$\chi_{\psi^2}(\vec{q}, i\nu_n) = \int_0^\beta d\tau \int d^dx \langle \delta\psi^2(\vec{x},\tau)\delta\psi^2(0)\rangle e^{i\nu_n\tau - i\vec{q}\cdot\vec{x}}, \tag{12}$$

is the Fourier transform of the fluctuations in ψ^2 and $d = 3$. The sign of $\tilde{\kappa}$ in (11) is determined by the infrared behavior of $\Delta\kappa$; if it diverges, as it does classically (for a scalar order parameter and isotropic elasticity), then this correction is universal and the transition is first order.

Another possibility is revealed in the zero-temperature long-wavelength Gaussian approximation of (11). If we make the Gaussian approximation $\langle \delta\psi^2(x)\delta\psi^2(0)\rangle \approx (\langle \delta\psi(x)\delta\psi(0)\rangle)^2$, then

$$\lim_{T\to 0} \Delta\kappa \propto \int dq\ d\nu\ q^{d-1}\ [\chi_\psi(\vec{q}, i\nu)]^2 \tag{13}$$

where $\chi_\psi(\vec{q}, i\nu)$, the order parameter susceptibility, is the Fourier transform of the correlator $\langle \psi(x)\psi(0)\rangle$. Since dimensionally $[\chi] = \left[\frac{1}{q^2}\right]$ and $[\nu] = [q^z]$, we find that in the approach to the quantum phase transition

$$\lim_{T\to 0}\ [\Delta\kappa] = \frac{[q^{d+z}]}{[q^4]} \tag{14}$$

so that the quantum corrections to κ are non-singular for $d + z > 4$. The presence of quantum zero-point fluctuations increases the effective dimensionality of the phase space for order parameter fluctuations. If the effective dimensionality of the quantum system lies above its upper critical dimensionality, this will have the effect of liberating the quantum critical point from the inevitable infrared slavery experienced by its finite-temperature classical counterpart. In particular the correction to the renormalized bulk modulus is then non-universal, allowing for quantum annealed criticality where zero-point fluctuations toughen the system against the macroscopic instability present classically, restoring its underlying continuous phase transition.

We have therefore identified a theoretical scenario where there is a quantum continuous transition even though all transitions at finite temperature are first-order. Application of a field conjugate and parallel/antiparallel to the order parameter in such a system leads to a line of first-order transitions ending in two classical critical points. Therefore by continuity there is a surface of first-order phase transitions in the phase diagram (cf. Figure 1) connecting the three critical points, one quantum and two classical, bounded by second-order phase lines. This phase diagram then presents an alternative scenario of the interplay of discontinuous transitions and fluctuations to that studied in metallic magnets where applied field is needed to observe quantum criticality in addition to the tuning parameter [9].

The specific heat exponent α plays a key role in the universality of the classical Larkin-Pikin criterion (1) since the coupling of the order parameter to the lattice is a strain-energy density. For the scalar ($n = 1$) case considered here, $\alpha > 0$, so that $\Delta\kappa$ is singular and the finite-temperature transition is always first-order; for $d+z > 4$, there is a quantum annealed criticality but no quantum tricritical point since the quartic mode-mode term in the effective action jumps from negative to positive due to the change of effective dimension.

For systems with multi-component order parameters ($n \geq 2$), α is negative so the correction to the renormalized bulk modulus will be nonuniversal even at finite temperatures [20, 31, 32]. In this case, there can be a classical tricritical point at finite pressures with a second-order transition that continues to zero temperature; this situation should be robust to everpresent disorder following the Harris criterion [33]. By contrast everpresent elastic anisotropy is known to destabilize criticality in the classical isotropic elastic scalar ($n = 1$) lattice and to drive it first-order into an inhomogeneous state [20, 31, 32]; here quantum annealed criticality may still be possible due to the increase of effective dimensionality. The coupling of domain dynamics to anistropic strain has been studied classically for ferroelectrics [34], and implications for the quantum case are a topic for future work.

Because of its underlying non-universal nature, the possibility of pressure-tuned quantum annealed criticality must be determined in specific setttings. Ferroelectrics have a dynamical exponent $z = 1$, so such three-dimensional materials are in their marginal dimension; logarithmic corrections to the bulk modulus are certainly present but they are not expected to be singular. Indeed such contributions to the dielectric susceptibility, χ, in the approach to ferroelectric quantum critical points have not been observed to date [18]; furthermore here the temperature-dependence of χ is described well by a self-consistent Gaussian approach appropriate above its upper critical dimension [18, 19]. Therefore there may be a very weak first-order quantum phase transition but experimentally it appears to be indistinguishable from a continuous one. We note that near quantum criticality the main effect of long-range dipolar interactions, not included in this treatment, is to produce a gap in the logitudinal fluctuations, but the transverse fluctuations remain critical [35–37]; the excellent agreement between theory and experiment at ferroelectric quantum criticality confirms that this is the case [18, 19].

Dielectric loss and hysteresis measurements can be used to probe the line of classical first-order transitions, and to determine the nature of the quantum phase transition. The Gruneisen ratio (Γ), the ratio of the thermal expansion and the specific heat, is known to change signs across the quantum phase transition [38, 39]; furthermore it is predicted to diverge at a 3D ferroelectric quantum critical point as $\Gamma \propto \frac{1}{T^2}$ so this would be a good indicator of underlying quantum criticality [19]. Both the bulk modulus and the longitudinal sound velocity should display jumps near quantum annealed criticality, though specifics are material-dependent since the fluctutation contributions to both are non-universal.

In summary, we have developed a theoretical framework to describe compressible insulating systems that have classical first-order transitions and display pressure-induced quantum criticality. We have generalized the Larkin-Pikin criterion [1] in the language of correlation and response functions; from this standpoint it is clear that the correction to the renormalized bulk modulus, singular at finite temperature, is non-universal at $T = 0$ for $d + z > 4$ so then the quantum transition may be continuous. Our analyis has been performed for the case of a scalar order parameter and isotropic elasticity where the phase transition is first-order for all finite temperature; in this extreme instance we argue that it is still possible to have quantum annealed criticality. Naturally the presence of a finite-pressure classical tricritical point ensures a continuous quantum phase transition. The key point is that a compressible material can host a quantum critical phase even if it displays a first-order transition at ambient pressure. More generally the order of the classical phase transition can be different from its quantum counterpart.

We note in ending that there are experiments on metallic systems [40–42] that also suggest quantum annealed criticality, so a quantum generalization of the electronic case [43] with possible links to previous work on metallic magnets should be pursued [9]; implications for doped paraelectric materials and polar metals [19] will also be explored. Extension of this work to quantum transitions between two distinct ordered states separated by first-order classical transitions may be relevant to the iron-based superconductors [44] and to the enigmatic heavy fermion material URu_2Si_2 where quantum critical endpoints have been suggested [45]. Finally the possibility of quantum annealed criticality in compressible materials, magnetic and ferroelectric, provides new settings for the exploration of exotic quantum phases where a broad temperature range can be probed with easily accessible pressures due to the lattice-sensitivity of these systems.

We have benefitted from discussions with colleagues including A.V. Balatsky, L.B. Ioffe, D. Khmelnitskii and P.B. Littlewood. PC and PC gratefully acknowledge the Centro Brasileiro de Pesquisas Fisicas (CBPF), Trinity College (Cambridge) and the Cavendish Laboratory where this project was initiated. Early stages of this work were supported by CAPES and FAPERJ grants CAPES-AUXPE-EAE-705/2013 (PC and PC), FAPERJ-E-26/110.030/2103 (PC and PC), and NSF grants DMR-1309929 (P. Coleman) and DMR-1334428 (P. Chandra). MAC acknowledges the Brazilian agencies CNPq and FAPERJ for partial financial support. GGL acknowledges support from grant no. EP/K012984/1 of the ERPSRC and the CNPq/Science without Borders Program. PC, PC and GGL thank the Aspen Center for Physics and NSF grant PHYS-1066293 for hospitality where this work was further developed and discussed. PC and PC thank S. Nakatsuji and the Institute for Solid State Physics (U. Tokyo) for hospitality where this work was completed.

[1] A.I. Larkin and S.A. Pikin, "Phase Transitions of the First Order but Nearly of the Second," Sov. Phys. JETP **29**, 891 (1969).

[2] D. Belitz, T. R. Kirkpatrick, and Thomas Vojta, "First Order Transitions and Multicritical Points in Weak Itinerant Ferromagnets," Phys. Rev. Lett. **82**, 4707–4710 (1999).

[3] S.A. Grigera, R.S. Perry, A.J. Schofield, M. Chiao, S.R. Julian, G.G. Lonzarich, S.I. Kieda, Y. Meano, A.J. Millis, and A.P. Mackenzie, "Magnetic Field-Tuned Quantum Criticality in the Metallic Ruthenate $Sr_3Ru_2O_7$," Science **294**, 329–332 (2001).

[4] Andrey V. Chubukov, Catherine Pépin, and Jerome Rech, "Instability of the Quantum-Critical Point of Itinerant Ferromagnets," Phys. Rev. Lett. **92**, 147003 (2004).

[5] D. Belitz, T. R. Kirkpatrick, and Thomas Vojta, "How Generic Scale Invariance Influences Quantum and Classical Phase Transitions," Rev. Mod. Phys. **77**, 579–632 (2005).

[6] Dmitrii L. Maslov, Andrey V. Chubukov, and Ronojoy Saha, "Nonanalytic Magnetic Response of Fermi and Non-Fermi Liquids," Phys. Rev. B **74**, 220402 (2006).

[7] Jérôme Rech, Catherine Pépin, and Andrey V. Chubukov, "Quantum Critical Behavior in Itinerant Electron Systems: Eliashberg Theory and Instability of a Ferromagnetic Quantum Critical Point," Phys. Rev. B **74**, 195126 (2006).

[8] T. R. Kirkpatrick and D. Belitz, "Universal Low-Temperature Tricritical Point in Metallic Ferromagnets and Ferrimagnets," Phys. Rev. B **85**, 134451 (2012).

[9] M. Brando, D. Belitz, F. M. Grosche, and T. R. Kirkpatrick, "Metallic Quantum Ferromagnets," Rev. Mod. Phys. **88**, 025006 (2016).

[10] P. Chandra, P. Coleman, and A.I. Larkin, "Ising Phase Transition in Frustrated Heisenberg Models," Phys. Rev. Lett. **64**, 88 (1990).

[11] P. Chandra and P. Coleman, "New Outlooks and Old Dreams in Quantum Antiferromagnets," in Les Houches Lecture Notes (Session LVI), edited by B. Doucot and J. Zinn-Justin (Elsevier, Amsterdam, 1995) pp. 495–594.

[12] L. Balents, "Spin Liquids in Frustrated Magnets," Nature **464**, 199–208 (2010).

[13] M. R. Norman, "Colloquium: Herbertsmithite and the search for the quantum spin liquid," Rev. Mod. Phys. **88**, 041002 (2016).

[14] Rafael M. Fernandes, Peter P. Orth, and Jörg Schmalian, "Intertwined Vestigial Order in Quantum Materials: Nematicity and Beyond," arXiv:1804.00818 (2018).

[15] T. Ishidate, S. Abe, H. Takahashi, and N. Môri, "Phase Diagram of $BaTiO_3$," Phys. Rev. Lett. **78**, 2397–2400 (1997).

[16] T. Suski, S. Takaoka, K. Murase, and S. Porowski, "New Phenomena of Low Temperature Resistivity Enhancement in Quantum Ferroelectric Semiconductors," Solid State Communications **45**, 259–262 (1983).

[17] S. Horiuchi, K. Kobayashi, R. Kumai, N. Minami, F. Kagawa, and Y. Tokura, "Quantum Ferroelectricity in Charge-Transfer Complex Crystals," Nature Communications **6**, 7469 (2015).

[18] S.E. Rowley, L.J. Spalek, R.P. Smith, M.P.M. Dean, M. Itoh, J.F. Scott, G.G. Lonzarich, and S.S. Saxena, "Ferroelectric Quantum Criticality," Nature Physics **10**, 367–372 (2014).

[19] P. Chandra, G.G. Lonzarich, S.E. Rowley, and J.F. Scott, "Prospects and Applications near Ferroelectric Quantum Phase Transitions," Rep. Prog. Phys. **80**, 112502 (2017).

[20] D.J. Bergman and B.I. Halperin, "Critical Behavior of an Ising Model on a Cubic Compressible Lattice," Phys. Rev. B **13**, 2145 (1976).

[21] L.D. Landau and E.M. Lifshitz, Theory of Elasticity, 3rd Edition (Pergamon Press, 1986).

[22] M.E. Lines and A.M. Glass, "Principles and Applications of Ferroelectrics and Related Materials," Clarendon Press, Oxford (1977).

[23] A.I. Larkin and D.E. Khmelnitskii, "Phase Transitions in Uniaxial Ferroelectrics," Sov. Phys. JETP **29**, 11231128 (1969).

[24] G. Gehring, "Pressure-Induced Quantum Phase Transitions," Europhys. Lett. **82**, 60004 (2008).

[25] G. Gehring and A. Ahmed, "Landau Theory of Compressible Magnets Near a Quantum Critical Point," J. Appl. Phys. **107**, 09E125 (2010).

[26] V.P. Mineev, "On the Phase Diagram of UGe_2," Comptes Rendus Physique **12**, 567–572 (2011).

[27] C.P. Bean and D.S. Rodbell, "Magnetic Disorder as a First-Order Phase Transition," Phys. Rev. **126**, 104 (1962).

[28] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, 1999).

[29] M. A. Continentino, "Quantum Scaling in Many-Body Systems: An Approach to Quantum Phase Transitions," Cambridge Unversity Press (2017).

[30] N. Das, "Effects of Strain Coupling and Marginal Dimensionality in the Nature of Phase Transitions in Quantum Paralelectrics," Int. J. Mod. Phys. B. **27**, 1350028 (2013).

[31] J. Bruno and J. Sak, "Renormalization Group for First-Order Phase Transitions: Equation of State of the Compressible Ising Magnet," Phys. Rev. B **22**, 3302 (1980).

[32] M.A. de Moura, T.C. Lubensky, Y. Imry, and A. Aharony, "Coupling to Anisotropic Elastic Media: Magnetic and Liquid-Crystal Phase Transitions," Phys. Rev. B **13**, 2176–2185 (1976).

[33] A.B. Harris, "Effects of Random Defects on the Critical Behaviour of Ising Models," J. Phys. C **7**, 1671 (1974).

[34] R. T. Brierley and P. B. Littlewood, "Domain wall fluctuations in ferroelectrics coupled to strain," Phys. Rev. B **89**, 184104 (2014).

[35] A.B. Rechester, "Contribution to the Theory of Second-Order Phase Transitions at Low Temperatures," Sov. Phys. JETP **33**, 423–430 (1971).

[36] D.E. Khmelnitskii and V.L. Shneerson, "Phase Transition of the Displacement Type in Crystals at Very Low Temperature," Sov. Phys. JETP **37**, 164–170 (1973).

[37] R. Roussev and A.J. Millis, "Theory of the Quantum Paraelectric-Ferroelectric Transition," Physical Review B **67**, 014105 (2003).

[38] L. Zhu, M. Garst, A. Rosch, and Q. Si, "Universal Diverging Gruneisen Parameter and the Magnetocaloric Effect Close to Quantum Critical Points," Physical Review Letters **91**, 066404 (2003).

[39] M. Garst and A. Rosch, "Sign Change of the Grüneisen Parameter and Magnetocaloric Effect near Quantum

Critical Points," Physical Review B **72**, 205129 (2005).

[40] G.M. Schmiedeshoff, E.D. Mun, A.W. Lounsbury, S.J. Tracy, E.C. Palm, S.T. Hannahs, J.-H. Park, T.P. Murphy, S.L. Bud/ko, and P.C. Canfield, "Multiple Regions of Quantum Criticality in $YbAgGe$," Phys. Rev. B **83**, 180408 (2011).

[41] Y. Tokiwa, M. Garst, P. Gegenwart, S.L. Bud/ko, and P.C. Canfield, "Quantum Bicriticality in the Heavy Fermion Metamagnet $YbAgGe$," Phys. Rev. Lett. **111**, 116401 (2013).

[42] A. Steppke, R. Kuchler, S. Lausberg, Edit Lengyel, Lucia Steinke, R. Borth, T. Luhmann, C. Krellner, M. Nicklas, C. Geibel, F. Steglich, and M. Brando, "Ferromagnetic Quantum Critical Point in the Heavy-Fermion Metal $YbNi_4(P_{1-x}As_x)_2$," Science **339**, 933 (2013).

[43] S. A. Pikin, "The Nature of Magnetic Transitions in Metals," Sov. Phys. JETP **31**, 753 (1970).

[44] K. Quader and M. Widom, "Lifshitz and other transitions in alkaline-earth 122 pnictides under pressure," Phys. Rev. B **90**, 144512 (2014).

[45] P. Chandra, P. Coleman, and R Flint, "Hastatic Order in URu_2Si_2," Nature **493**, 621 (2013).

SUPPLEMENTARY MATERIAL FOR QUANTUM ANNEALED CRITICALITY

Overview

The key idea of the Larkin-Pikin approach is that we integrate out the Gaussian strain degrees of freedom from the action to derive an effective action for the order parameter field so that

$$Z = \int \mathcal{D}[u] \int \mathcal{D}[\psi] e^{-S[\psi,u]} \qquad \longrightarrow \qquad Z = \int \mathcal{D}[\phi] \int \mathcal{D}[\psi] e^{-S_{eff}[\psi,\phi]}. \tag{15}$$

The key element in this procedure is a separation of the strain field into uniform and fluctuating components. When we integrate out the uniform component of the strain, it induces an infinite-range attractive interaction between the order parameter modes mediated by a *classical* field ϕ that is uniform in both space and time. The main effect of the integration of the fluctuating strain component is to renormalize the short-range interactions between the order parameter modes; however completion of the Gaussian integral also leads to an infinite range repulsive order parameter interaction. The overall infinite range interaction is attractive, but this subtlety needs to be checked carefully in both the classical and quantum cases, as is performed explicitly in this Supplementary Material; here we summarize its main results. The relevant quantum generalization of the effective action in (15) is

$$S_{eff}[\psi, \phi] = \int d^4x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi, b^*] + \lambda \left(\phi + \frac{P}{K} \right) \psi^2[x] \right] \tag{16}$$

where

$$\mathcal{L}[\psi, b^*] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu} \tag{17}$$

and an effective bulk modulus

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}. \tag{18}$$

In the classical case

$$\int d^4x \longrightarrow \frac{1}{T} \int d^3x \tag{19}$$

and so we recover the classical effective action

$$S_{eff}[\psi, \phi] = \frac{1}{T} \int d^3x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi, b^*] + \lambda(\phi + \frac{P}{K})\psi^2[\vec{x}] \right] \tag{20}$$

with definitions as above. We note that in the main text we have replaced the renormalized b^* by b and we have set $P = 0$ for presentational simplicity.

Preliminaries

The partition function can be written as an integral over the order parameter and strain fields

$$Z = \int \mathcal{D}[\psi] \int \mathcal{D}[u] e^{-S[\psi,u]} \tag{21}$$

where ψ is the order parameter field and $u(x)$ is the local displacement of the lattice that determines the strain fields according to the relation

$$e_{ab}(x) = \frac{1}{2} \left(\frac{\partial u_a}{\partial x_b} + \frac{\partial u_b}{\partial x_a} \right). \tag{22}$$

Here the action is is determined by an integral over the Lagrangian L, $S = \int d^4xL$. In the quantum case, $\int d^4x \equiv \int d\tau \int d^3x$ is a space-time integral over configurations that are periodic in imaginary time $\tau \in [0, \beta]$, where the β inverse temperature ($k_B = 1$). In the classical case the time-dependence disappears and the integral over τ is replaced by $1/T$ so that $S = \frac{1}{T}\int d^3xL$.

The action divides up into three parts

$$S = S_A + S_I + S_B = \int d^4x(L_A[u] + L_I[\psi, e] + L_B[\psi]), \tag{23}$$

where the contributions to the Lagrangian are: (i) a Gaussian term describing the elastic degrees of freedom in an isotropic system

$$L_A[u] = \frac{1}{2}\left[\rho \dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right) e_{ll}^2 + 2\mu e_{ab}^2\right] - \sigma_{ab}e_{ab} \tag{24}$$

where σ_{ab} is the external stress and we have assumed a summation convention in which repeated indices are summed over, so that for instance, $e_{ll} \equiv \sum_{l=1,3} e_{ll}$ and $e_{ab}^2 \equiv e_{ab}e_{ab} = \sum_{a,b=1,3} e_{ab}^2$; (ii) an interaction term

$$L_I[\psi, e] = \lambda e_{ll}\psi^2 \tag{25}$$

describing the coupling between the volumetric strain $e_{ll} = \mathrm{Tr}[e]$ and the "energy density" ψ^2 of the order parameter ψ; (iii) the Lagrangian $L_B[\psi]$ of the order parameter that, in the simplest case, is a ψ^4 field theory

$$L_B[\psi, b] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4, \tag{26}$$

where we have explicitly noted its dependence on the interaction strength b. At a finite temperature critical point, all time-derivative terms are dropped from these expressions.

Since the integral over the strain fields is Gaussian, the latter can be integrated out of the partition function leading to an effective action of the ψ fields $S_{eff}[\psi] = S_B[\psi] + \Delta S[\psi]$ where

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[u]e^{-(S_A+S_I)}. \tag{27}$$

If we write the elastic action in the schematic, discretized form

$$S_A + S_I = \frac{1}{2}\sum_{i,j} u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \tag{28}$$

then the effective action becomes simply

$$\Delta S = \frac{1}{2}\ln\det[M] - \frac{\lambda^2}{2}\sum_{i,j}\psi_i^2 M_{i,j}^{-1}\psi_j^2 \tag{29}$$

where the second term is recognized as an induced attractive interaction between the order-parameter fields. The subtlety in this procedure derives from the division of the strain field into two parts: a uniform contribution determined by boundary conditions and a fluctuating component in the bulk. For the classical case

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{\sqrt{V}}\sum_{\vec{q}\neq 0}\frac{i}{2}\left(q_a u_b(\vec{q}) + q_b u_a(\vec{q})\right)e^{i\vec{q}\cdot\vec{x}} \tag{30}$$

where the $u_b(\vec{q})$ are the Fourier transform of the atomic displacements, while in the quantum problem

$$e_{ab}(\vec{x}, \tau) = e_{ab} + \frac{1}{\sqrt{V\beta}}\sum_{i\nu_n}\sum_{\vec{q}\neq 0}\frac{i}{2}\left(q_a u_b(q) + q_b u_a(q)\right)e^{i(\vec{q}\cdot\vec{x} - \nu_n\tau)}, \tag{31}$$

where $\nu_n = 2\pi nT$ is the bosonic Matsubara frequency. Note that the exclusion of all terms where $\vec{q} = 0$ from the summation also excludes the special point where both $i\nu_n$ and $\vec{q}$ are zero. As we now demonstrate, the overall attractive interaction ($\propto -\psi_i^2 M_{ij}^{-1}\psi_j^2$) contains both short-range and infinite range components.

The Gaussian Strain integral: Classical Case

Our task is to calculate the Gaussian integral,

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e_{ab}, u_q] e^{-(S_A+S_I)} \tag{32}$$

where the classical action

$$S_A + S_I = \frac{1}{T}\int d^3x \left[\frac{1}{2}\left(K - \frac{2}{3}\mu\right) e_{ll}^2(\vec{x}) + \mu e_{ab}(\vec{x})^2 + (\lambda\psi^2(\vec{x}) + P)e_{ll}(\vec{x})\right], \tag{33}$$

where we have denoted $\sigma_{ab} = -P\delta_{ab}$ in terms of the pressure P. We begin by splitting the strain field into the $q = 0$ and finite q components,

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{\sqrt{V}}\sum_{\vec{q}\neq 0} \frac{i}{2}\left(q_a u_b(\vec{q}) + q_b u_a(\vec{q})\right) e^{i\vec{q}\cdot\vec{x}}. \tag{30}$$

This separation enables us to use periodic boundary conditions, putting the system onto a spatial torus with discrete momenta $\vec{q} = \frac{2\pi}{L}(l, m, n)$. After this transformation, the action divides up into two terms, $S = S[e_{ab}, \psi] + S[u, \psi]$. We shall define the integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]},$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]}. \tag{34}$$

The uniform part of the action is

$$\begin{aligned} S[e_{ab}, \psi] &= \frac{V}{T}\left[\frac{1}{2}\left(K - \frac{2}{3}\mu\right) e_{ll}^2 + \mu e_{ab}^2\right] + \frac{V}{T}(\lambda\psi^2_{q=0} + P)e_{ll} \\ &= \frac{1}{2} e_{ab}\mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \end{aligned} \tag{35}$$

where $\psi^2_{\vec{q}} = \frac{1}{V}\int d^3x \psi^2(\vec{x}) e^{i\vec{q}\cdot\vec{x}}$ is the Fourier transform of the fluctuations in “energy density” and

$$\mathcal{M}_{abcd} = K\overbrace{(\delta_{ab}\delta_{cd})}^{\mathcal{P}^L_{abcd}} + 2\mu\overbrace{\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)}^{\mathcal{P}^T_{abcd}}, \tag{36}$$

$$v_{ab} = \frac{V}{T}(\lambda\psi^2_{q=0} + P)\delta_{ab}. \tag{37}$$

The nonuniform part of the action is

$$S[u, \psi] = \frac{1}{T}\sum_{\vec{q}\neq 0}\left(\frac{1}{2}u_a^*(\vec{q}) M_{ab} u_b(\vec{q}) + \vec{a}(\vec{q})\cdot\vec{u}(\vec{q})\right) \tag{38}$$

where

$$\begin{aligned} M_{ab} &= \left[\left(K - \frac{2}{3}\mu\right) q_a q_b + \mu\left(q^2\delta_{ab} + q_a q_b\right)\right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V}\ \psi^2_{-q}\right)\vec{q}. \end{aligned} \tag{39}$$

When we integrate over the uniform part of the strain field,

$$\frac{1}{2}e_{ab}\mathcal{M}_{abcd}e_{cd} + v_{ab}e_{ab} \rightarrow S_1[\psi] = -\frac{1}{2}v_{ab}\mathcal{M}^{-1}_{abcd}v_{cd} \tag{40}$$

Now the two terms P^L_{abcd} and P^T_{abcd} in $\mathcal{M}$ (36) are independent projection operators ($P^\Gamma_{abef}P^\Gamma_{efcd} = P^\Gamma_{abcd}$, $\Gamma \in L, T$), projecting the longitudinal and transverse components of the strain. The inverse of $\mathcal{M}$ is then given by

$$\mathcal{M}^{-1}_{abcd} = \frac{T}{V}\left[\frac{1}{K}(\delta_{ab}\delta_{cd}) + \frac{1}{2\mu}\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\right], \tag{41}$$

so the Gaussian integral over the uniform part of the strain field gives

$$S_1[\psi] = -\frac{1}{2}v_{ab}\mathcal{M}^{-1}_{abcd}v_{cd} = -\frac{V}{2T}\frac{1}{K}(\lambda\psi^2_{q=0} + P)^2. \tag{42}$$

Now the matrix entering the fluctuating part of the action $S[u,\psi]$, can be projected into the longitudinal and transverse components of the strain

$$M_{ab}(\vec{q}) = q^2\left[\left(K + \frac{4}{3}\mu\right)\hat{q}_a\hat{q}_b + \mu(\delta_{ab} - \hat{q}_a\hat{q}_b)\right] \tag{43}$$

where $\hat{q}_a = q_a/q$ are the direction cosines of $\vec{q}$. The inversion of this matrix is then

$$M^{-1}_{ab}(\vec{q}) = q^{-2}\left[\left(K + \frac{4}{3}\mu\right)^{-1}\hat{q}_a\hat{q}_b + \mu^{-1}(\delta_{ab} - \hat{q}_a\hat{q}_b)\right], \tag{44}$$

so the Gaussian integral over fluctuating part of the strain field leads to

$$\begin{aligned}
&\frac{1}{T}\sum_{\vec{q}\neq 0}\frac{1}{2}u^*_a(\vec{q})M_{ab}(\vec{q})u_b(\vec{q}) + \vec{a}(\vec{q})\cdot\vec{u}(\vec{q}) \rightarrow \\
&\qquad S_2[\psi] = -\frac{1}{2T}\sum_{\vec{q}\neq 0}a_a(-\vec{q})M^{-1}_{ab}(\vec{q})a_b(\vec{q}) \\
&\qquad\qquad = -\frac{V}{2T}\sum_{\vec{q}\neq 0}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu}
\end{aligned} \tag{45}$$

We can rewrite this as a sum over *all* $\vec{q}$, plus a remainder at $\vec{q} = 0$:

$$\begin{aligned}
S_2[\psi] &= -\frac{V}{2T}\sum_{\vec{q}}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu} + \frac{V}{2T}(\psi^2_{q=0})^2\frac{\lambda^2}{K + \frac{4}{3}\mu} \\
&= -\frac{1}{2T}\frac{\lambda^2}{K + \frac{4}{3}\mu}\int d^3x\psi^4(\vec{x}) + \frac{V}{2T}(\psi^2_{q=0})^2\frac{\lambda^2}{K + \frac{4}{3}\mu}.
\end{aligned} \tag{46}$$

The first term is a local attraction while the second term, involving only the $\vec{q} = 0$ Fourier component, corresponds to to a repulsive long range interaction.

When we combine the results of the two integrals (42) and (46) we obtain

$$\Delta S[\psi] = -\frac{V}{2T}\frac{\lambda^2}{\kappa}(\psi^2_{q=0})^2 - \frac{1}{2T}\frac{\lambda^2}{K + \frac{4}{3}\mu}\int d^3x\psi^4(x) - \frac{V}{2T}\frac{1}{K}(2\lambda\psi^2_{q=0}P + P^2) \tag{47}$$

where

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu} \tag{48}$$

is the effective Bulk modulus.

The final step in the procedure, is to carry out a Hubbard Stratonovich transformation, factorizing the long-range attraction in terms of a stochastic uniform field ϕ,

$$-\frac{V}{2T}\frac{\lambda^2}{\kappa}(\psi^2_{q=0})^2 \rightarrow \frac{1}{T}\int d^3x\left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x)\right]. \tag{49}$$

Combining (47) and (49) we obtain the following expression for

$$\Delta S[\psi,\phi] = \frac{1}{T}\int d^3x\left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \lambda\left(\phi + \frac{P}{K}\right)\psi^2(x) - \frac{\lambda^2}{2(K + \frac{4}{3}\mu)}\psi^4(x)\right]. \tag{50}$$

Finally, adding this term to the original order parameter action $S_B[\psi] = \frac{1}{T}\int d^3x L_B[\psi, b]$, our final partition function can be written

$$Z = \int d\phi \int \mathcal{D}[\psi] e^{-S_{eff}[\psi,\phi]} \tag{51}$$

where $S_{eff}[\psi,\phi] = S_B[\psi] + \Delta S[\psi,\phi]$ is given by

$$S_{eff}[\psi,\phi] = \frac{1}{T}\int d^3x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi, b^*] + \lambda(\phi + \frac{P}{K})\psi^2[x]\right] \tag{52}$$

where

$$\mathcal{L}[\psi, b^*] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4.$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu}. \tag{53}$$

Note that in the main text we have dropped the "$*$" on b for presentational simplicity; there b refers to this renormalized interaction (53).

Thus the main effects of integrating out the strain field are a renormalization of the short-range interaction of the order parameter field and the development of an infinite-range interaction mediated by an effective strain field ϕ. If we differentiate the action with respect to the pressure, we obtain the volumetric strain

$$\frac{\delta S}{\delta P(\vec{x})} = e_{ll}(\vec{x}) = \frac{1}{K}\left(P + \lambda\psi^2(\vec{x})\right), \tag{54}$$

which, as a result of integrating out the strain fluctuations, now contains a contribution from the order parameter. Again in the main text we set $P = 0$ for presentational simplicity.

The Gaussian Strain integral: Quantum Case

In the quantum case, the action in the Gaussian strain integral

$$e^{-\Delta S[\Psi]} = \int \mathcal{D}[e_{ab}, u_q] e^{-(S_A + S_I)} \tag{55}$$

now involves an integral over space time, with $S = \int d^4x L \equiv \int_0^\beta d\tau \int d^3x L$. We now restore the kinetic energy terms in Lagrangian (24) and (26), so that now the quantum action takes the form

$$S_A + S_I = \int d\tau d^3x \left[\frac{\rho}{2}\dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right) e_{ll}^2(x) + \frac{1}{2}2\mu e_{ab}(x)^2 + (\lambda\psi^2(x) + P)e_{ll}(x)\right]. \tag{56}$$

Again our task is to cast this into matrix form

$$S_A + S_I = \frac{1}{2}\sum_q u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \to \frac{\lambda^2}{2}\sum_{i,j}\psi_i^2 M_{i,j}^{-1}\psi_j^2. \tag{57}$$

where now the summations run over the discrete wavevector and Matsubara frequencies $q \equiv (i\nu_n, \vec{q})$, where $\nu_n = \frac{2\pi}{\beta}n$, $\vec{q} = \frac{2\pi}{L}(j, l, k)$. As before, we must separate out the static, $\vec{q} = 0$ component of the strain tensor, writing

$$e_{ab}(x,\tau) = e_{ab} + \frac{1}{\sqrt{V\beta}}\sum_{i\nu_n}\sum_{\vec{q}\neq 0}\frac{i}{2}\left(q_a u_b(q) + q_b u_b(q)\right) e^{i(\vec{q}\cdot\vec{x} - \nu_n\tau)}, \tag{58}$$

Note that there is no time-dependence to the uniform part of the strain, since the boundary conditions are static. However the fluctuating component excludes $\vec{q} = 0$, but includes all Matsubara frequencies; with these caveats, the quantum integration of the strain fields closely follows that of the classical case.

Again the action divides up into two terms, $S = S[e_{ab}, \psi] + S[u, \psi]$, corresponding to the distinct unifom and finite $\vec{q}$ contributions to the strain. We shall again define the integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]},$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]}. \tag{59}$$

The uniform part of the action

$$\begin{aligned} S[e_{ab}, \psi] &= \int d\tau \left[\frac{1}{2} \left(K - \frac{2}{3}\mu \right) e_{ll}^2 + \frac{1}{2} 2\mu e_{ab}^2 \right] + \frac{V}{T} (\lambda \psi_{q=0}^2 + P) e_{ll} \\ &= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \mathcal{M}_{abcd} &= \left[K(\delta_{ab}\delta_{cd}) + 2\mu \left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd} \right) \right], \\ v_{ab} &= V\beta(\lambda\psi_{q=0}^2 + P)\delta_{ab}, \end{aligned} \tag{61}$$

is unchanged, but now

$$\psi_q^2 = \frac{1}{V\beta} \int d^4x \psi^2(x) e^{-i(\vec{q}\cdot\vec{x} - \nu_n \tau)} \tag{62}$$

is the space-time Fourier transform of the order parameter intensity. The non-uniform part is now

$$S[u, \psi] = \sum_{i\nu_n} \sum_{\vec{q}\neq 0} \left(\frac{1}{2} u_a^*(q) M_{ab} u_b(q) + \vec{a}(q) \cdot \vec{u}(q) \right), \tag{63}$$

where

$$\begin{aligned} M_{ab} &= \left[\rho\nu_n^2 \left(K - \frac{2}{3}\mu \right) q_a q_b + \mu \left(q^2 \delta_{ab} + q_a q_b \right) \right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V\beta}\ \psi_{-q}^2 \right) \vec{q}. \end{aligned} \tag{64}$$

When we integrate over the uniform part of the strain field, we obtain

$$\begin{aligned} &\frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab} \rightarrow \\ &\qquad S_1[\psi] = -\frac{1}{2} v_{ab} \mathcal{M}_{abcd}^{-1} v_{cd} \\ &\qquad\qquad = -\frac{V\beta}{2K} (\lambda\psi_{q=0}^2 + P)^2, \end{aligned} \tag{65}$$

or

$$S_1[\psi] = -\frac{1}{2K} \int d^4x (\lambda\psi_{q=0}^2 + P)^2. \tag{66}$$

For presentational simplicity, we will now set $P = 0$ since the role of pressure here follows that in the classical treatment already described.

The matrix entering the fluctuating part of the action can be projected into the longitudinal and transverse components

$$M_{ab} = \left[\left(\rho\nu_n^2 + (K + \frac{4}{3}\mu) \right) \hat{q}_a \hat{q}_b + \left(\rho\nu_n^2 + \mu \right) (\delta_{ab} - \hat{q}_a \hat{q}_b) \right], \tag{67}$$

where $\hat{q}_a = q_a/q$ is the unit vector. The inversion of this matrix is then

$$M_{ab}^{-1} = \left[\frac{1}{\rho(\nu_n^2 + c_L^2 q^2)} \hat{q}_a \hat{q}_b + \frac{1}{\rho(\nu_n^2 + c_T^2 q^2)} (\delta_{ab} - \hat{q}_a \hat{q}_b) \right], \tag{68}$$

where

$$c_L^2 = \frac{K + \frac{4}{3}\mu}{\rho}, \qquad c_T^2 = \frac{2\mu}{\rho} \tag{69}$$

are the longitudinal and transverse sound velocities. The two terms appearing in M^{-1} are recognized as the propagators for longitudinal and tranverse phonons.

When we integrate over the fluctuating component of the strain field, only the longitudinal phonons couple to the order parameter:

$$\begin{aligned} \frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0} u_a^*(q)M_{ab}(q)u_b(q) + \vec{a}(q)\cdot\vec{u}(q) \rightarrow \\ S_2[\psi] &= -\frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0} a_a(-q)M_{ab}^{-1}(q)a_b(q) \\ &= -\frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}\neq 0}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right). \end{aligned} \tag{70}$$

Now in this last term,

$$\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) \tag{71}$$

the $\vec{q} = 0$ term vanishes for any finite ν_n, but in the case where $\nu_n = 0$, the limiting $\vec{q} \rightarrow 0$ form of this term is finite:

$$\left.\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right)\right|_{\vec{q}\rightarrow 0} = \begin{cases} 0 & \nu_n \neq 0 \\ \frac{1}{K+\frac{4}{3}\mu} & \nu_n = 0. \end{cases} \tag{72}$$

We can thus replace

$$\sum_{i\nu_n,\vec{q}\neq 0}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) \rightarrow \sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) - \frac{(\psi_{q=0}^2)^2}{K+\frac{4}{3}\mu}. \tag{73}$$

so that

$$S_2[\psi] = \frac{V\beta\lambda^2}{2(K+\frac{4}{3}\mu)}(\psi_{q=0}^2)^2 - \frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right). \tag{74}$$

If we now combine S_1 and S_2, we obtain

$$S_1 + S_2 = \frac{V\beta\lambda^2}{2\kappa}(\psi_{q=0}^2)^2 - \frac{V\beta\lambda^2}{2}\sum_{q}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right). \tag{75}$$

where

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K+\frac{4}{3}\mu} \tag{76}$$

is the effective Bulk modulus, as in the classical case.

Next we carry out a Hubbard-Stratonovich transformation, rewriting the the long-range attraction in terms of a stochastic static and uniform scalar field ϕ as follows

$$-\frac{V\beta}{2}\frac{\lambda^2}{\kappa}(\psi_{q=0}^2)^2 \rightarrow \int d^4x\left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x)\right]. \tag{77}$$

The remaining interaction term can be divided up into two parts as follows

$$\sum_q \psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) = \frac{1}{K+\frac{4}{3}\mu}\sum_q \psi_{-q}^2\psi_q^2\left[1 - \left(\frac{\nu_n^2/c_L^2}{q^2+\nu_n^2/c_L^2}\right)\right]. \tag{78}$$

The first term inside the brackets is independent of momentum and frequency, leading to a finite local attraction term that will act to renormalize the b term in the Lagrangian $\mathcal{L}_{\psi,b}$ as in the classical case. The second term is a non-local and retarded interaction. Due to Lorentz invariance, simple power-counting shows that this term has the same scaling dimensionality as a local repulsive term, and thus it will not modify the critical behavior of the second-order phase transition.

If we transform back into into space-time co-ordinates, then we obtain

$$S_1 + S_2 = \int d^4x \left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x) - \frac{\lambda^2}{2(K+\frac{4}{3}\mu)}\psi^4(x)\right] + S_{NL} = S_{eff}[\psi,\phi] + S_{NL} \tag{79}$$

where

$$S_{NL} = \frac{\lambda^2}{2V\beta(K+\frac{4}{3}\mu)}\int d^4x d^4x' \partial_\tau(\psi^2)(x)V(x-x')\partial_\tau(\psi^2)(x') \tag{80}$$

and

$$V(x) = \int \frac{d^4q}{(2\pi)^4}\left(\frac{c_L^{-2}}{|\vec{q}|^2 + \nu_n^2/c_L^2}\right)e^{i(\vec{q}\cdot\vec{x} - i\nu_n\tau)} = \frac{1}{2\pi c_L^2}\frac{1}{(|\vec{x}|^2 + c_L^2\tau^2)} \tag{81}$$

is the non-local interaction mediated by the acoustic phonons. Then the final quantum partition function resulting from integrating out the strain fields in the quantum case can be written ($P \neq 0$)

$$Z = \int d\phi \int \mathcal{D}[\psi]e^{-S_{eff}[\psi,\phi]-S_{NL}} \tag{82}$$

where

$$S_{eff}[\psi,\phi] = \int d^4x\left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi,b^*] + \lambda\left(\phi + \frac{P}{K}\right)\psi^2[x]\right] \tag{83}$$

and

$$\mathcal{L}[\psi,b^*] = \frac{1}{2}(\partial_\mu\psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4.$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K+\frac{4}{3}\mu} \tag{84}$$

and an effective bulk modulus

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K+\frac{4}{3}\mu}. \tag{85}$$

Thus, as in the classical case, the main effect of integrating out the strain field, is a renormalization of the short-range interaction of the order parameter field, and the development of an infinite range interaction, mediated by an effective strain field ϕ. The introduction of a nonlocal contribution with the same scaling dimensions as the ψ^4 term will not affect the properties of the fixed point, and thus it will not change the universality class of the fixed point, as in the classical Larkin-Pikin case. However we emphasize that in its quantum generalization the effective dimension of the theory is $d_{eff} = d + z$. Again we note that in the main text we have replaced the coefficient of the renormalized interaction b^* in (84) by b for presentational simplicity.