ON INSTABILITY OF RADIAL STANDING WAVES FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

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Abstract. We show the strong instability of radial ground state standing waves for the focusing L^2-supercritical nonlinear Schrödinger equation with inverse-square potential

$$i\partial_t u + \Delta u + c|x|^{-2}u = -|u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d,$$

where $d \geq 3$, $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$, $c \neq 0$ satisfies $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$ and $\frac{4}{d} < \alpha < \frac{4}{d-2}$. This result extends a recent result of Bensouilah-Dinh-Zhu [*On stability and instability of standing waves for the nonlinear Schrödinger equation with inverse-square potential*, arXiv:1805.01245] where the stability and instability of standing waves were shown in the L^2-subcritical and L^2-critical cases.

1. Introduction

In the last decade, there has been a great deal of interest in studying the nonlinear Schrödinger equation with inverse-square potential, namely

$$i\partial_t u + \Delta u + c|x|^{-2}u = \mu|u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{1.1}$$

where $d \geq 3$, $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$, $c \neq 0$ satisfies $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$, $\mu \in \mathbb{R}$ and $\alpha > 0$. The nonlinear Schrödinger equation (1.1) appears in a variety of physical settings, such as quantum field equations or black hole solutions of the Einstein's equations (see e.g. [8, 9, 19]) and quantum gas theory (see e.g. [1, 26, 27]). The mathematical interest in the nonlinear Schrödinger equation with inverse-square potential comes from the fact that the potential is homogeneous of degree -2 and thus scales exactly the same as the Laplacian. Recently, the equation (1.1) has been intensively studied (see e.g. [2, 3, 4, 7, 12, 13, 20, 21, 24, 28, 32] and references therein).

In this paper, we consider the L^2-supercritical nonlinear Schrödinger equation with inverse-square potential, namely

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u + c|x|^{-2}u &=& -|u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0) &=& u_0 \in H^1, \end{array} \right. \tag{1.2}$$

where $d \geq 3$, $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$, $u_0 : \mathbb{R}^d \to \mathbb{C}$, $c \neq 0$ satisfies $c < \lambda(d)$ and $\frac{4}{d} < \alpha < \frac{4}{d-2}$.

The main purpose of this paper is to study the instability of radial ground state standing waves for (1.2). Before stating our result, let us recall known results related

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to the stability and instability of standing waves for the nonlinear Schrödinger-like equations. The stability of standing waves for the classical nonlinear Schrödinger equation (i.e. $c = 0$ in (1.2)) is widely pursued by physicists and mathematicians (see e.g. [14] for reviews). To our knowledge, the first work addressed the orbital stability of standing waves for the classical NLS belongs to Cazenave-Lions [10] via the concentration-compactness principle. Later, Weinstein in [29, 30] gave another approach to prove the orbital stability of standing waves for the classical NLS. Afterwards, Grillakis-Shatah-Strauss in [17, 18] gave a criterion based on a form of coercivity for the action functional (see (1.4)) to prove the stability of standing waves for a Hamiltonian system which is invariant under a one-parameter group of operators. Since then, a lot of results on the orbital stability of standing waves for nonlinear dispersive equations were obtained. For the nonlinear Schrödinger equation with a harmonic potential, Zhang [31] succeeded in obtaining the orbital stability of standing waves by the weighted compactness lemma. Recently, the orbital stability phenomenon was proved for the fractional nonlinear Schrödinger equation by establishing the profile decomposition for bounded sequences in H^s (see e.g. [25, 33]). The instability of standing waves for the classical NLS was first studied by Berestycki-Cazenave [5] (see also [11]). Later, Le Coz in [22] gave an alternative, simple proof of the classical result of Berestycki-Cazenave. The key point is to establish the finite time blow-up by using the variational characterization of the ground states as minimizers of the action functional and the virial identity. For the Schrödinger equations with more general nonlinearities, this method does not work due the the lack of virial identities. In such cases, one may use a powerful tool of Grillakis-Shatah-Strauss [17, 18] to derive the instability of standing waves.

Recently, the authors in [4] succeeded, using a profile decomposition theorem proved by the first author [2], to establish the stability of standing waves for (1.2) in the L^2-subcritical regime and the instability by blow-up in the L^2-critical regime. The main goal here is to extend these results to the L^2-supercritical case but only for radial ground state standing waves.

Throughout this paper, we call a standing wave a solution of (1.2) of the form $e^{i\omega t}\phi_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $\phi_\omega \in H^1$ is a nontrivial solution to the elliptic equation

$$-\Delta\phi_\omega + \omega\phi_\omega - c|x|^{-2}\phi_\omega - |\phi_\omega|^\alpha\phi_\omega = 0. \tag{1.3}$$

Note that the existence of positive radial solutions to the elliptic equation

$$-\Delta\phi + \phi - c|x|^{-2}\phi - |\phi|^\alpha\phi = 0$$

was shown in [21, Theorem 3.1] and [13, Theorem 4.1]. By setting $\phi_\omega(x) := (\sqrt{\omega})^{\frac{2}{\alpha}}\,\phi(\sqrt{\omega}x)$, it is easy to see that ϕ_ω is a solution of (1.3). This shows the existence of positive radial solutions to (1.3).

Note also that (1.3) can be written as $S'_\omega(\phi_\omega) = 0$, where

$$\begin{aligned} S_\omega(v) &:= E(v) + \frac{\omega}{2}\|v\|^2_{L^2} \\ &= \frac{1}{2}\|v\|^2_{\dot{H}^1_c} + \frac{\omega}{2}\|v\|^2_{L^2} - \frac{1}{\alpha+2}\|v\|^{\alpha+2}_{L^{\alpha+2}} \end{aligned} \tag{1.4}$$

is the action functional. Here

$$\|v\|^2_{\dot{H}^1_c} := \|\nabla v\|^2_{L^2} - c\||x|^{-1}v\|^2_{L^2} \tag{1.5}$$

is the Hardy functional.

We denote the set of non-trivial radial solutions of (1.3) by

$$\mathcal{A}_{\mathrm{rad},\omega} := \left\{ v \in H^1_{\mathrm{rad}} \backslash \{0\} \ : \ S'_\omega(v) = 0 \right\},$$

where H^1_{rad} is the space of radial H^1 functions.

Definition 1.1 (Radial ground states)**.** A function $\phi \in \mathcal{A}_{\mathrm{rad},\omega}$ is called **a radial ground state** for (1.3) if it is a minimizer of S_ω over the set $\mathcal{A}_{\mathrm{rad},\omega}$. The set of radial ground states is denoted by $\mathcal{G}_{\mathrm{rad},\omega}$. In particular,

$$\mathcal{G}_{\mathrm{rad},\omega} = \{ \phi \in \mathcal{A}_{\mathrm{rad},\omega} \ : \ S_\omega(\phi) \leq S_\omega(v), \ \forall v \in \mathcal{A}_{\mathrm{rad},\omega} \}.$$

We have the following result on the existence of radial ground states for (1.3).

Proposition 1.2. *Let $d \geq 3$, $c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$ and $\omega > 0$. Then the set $\mathcal{G}_{\mathrm{rad},\omega}$ is not empty, and it is characterized by*

$$\mathcal{G}_{\mathrm{rad},\omega} = \left\{ v \in H^1_{\mathrm{rad}} \backslash \{0\}, \ : \ S_\omega(v) = d(\mathrm{rad},\omega), \ K_\omega(v) = 0 \right\},$$

where

$$K_\omega(v) := \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \|v\|^2_{\dot{H}^1_c} + \omega \|v\|^2_{L^2} - \|v\|^{\alpha+2}_{L^{\alpha+2}}$$

is the Nehari functional and

$$d(\mathrm{rad},\omega) := \inf \left\{ S_\omega(v) \ : \ v \in H^1_{rad} \backslash \{0\}, \ K_\omega(v) = 0 \right\}. \tag{1.6}$$

We refer the reader to Section 2 for the proof of the above result.

Remark 1.3. Recently, Fukaya-Ohta in [15] studied the instability of standing waves for the nonlinear Schrödinger equation with an attractive inverse power potential, namely

$$i\partial_t u + \Delta u + \gamma |x|^{-\alpha} u = -|u|^{p-1} u,$$

where $\gamma > 0, 0 < \alpha < \min\{2, d\}$ and $\frac{4}{d} < p - 1 < \frac{4}{d-2}$ if $d \geq 3$ and $\frac{4}{d} < p - 1 < \infty$ if $d = 1$ or $d = 2$. The potential $V(x) = \gamma |x|^{-\alpha}$ belongs to $L^r(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for some $r > \min\{1, d/2\}$. This special property allows them to use the weak continuity of the potential energy (see e.g. [23, Theorem 11.4]) to prove the existence of non-radial ground states. In our case, the inverse-square potential $V(x) = c|x|^{-2}$ does not belong to $L^{\frac{d}{2}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, so the weak continuity of potential energy is not applicable to our potential. At the moment, we do not know how to show the existence of non-radial ground states for (1.3). We hope to consider this problem in a future work.

Let us now recall the definition of the strong instability.

Definition 1.4 (Strong instability)**.** We say that the standing wave $e^{i\omega t}\phi_\omega$ is strongly unstable if for any $\epsilon > 0$, there exists $u_0 \in H^1$ such that $\|u_0 - \phi_\omega\|_{H^1} < \epsilon$ and the solution $u(t)$ of (1.2) with initial data u_0 blows up in finite time.

Our main result of this paper is the following:

Theorem 1.5. *Let $d \geq 3$, $c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$, $\omega > 0$ and $\phi_\omega \in \mathcal{G}_{\mathrm{rad},\omega}$. Then the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is strongly unstable.*

To our knowledge, the usual strategy to show the strong instability of standing waves is to use the characterization of ground states combined with the virial identity. However, in the presence of the inverse-square potential, the existence of ground states is well-known. However, the regularity as well as the decay of ground states are not yet known. Therefore, it is not known that the ground states ϕ_ω

belongs to the weighted space $\Sigma := H^1 \cap L^2(|x|^2 dx)$ in order to apply the virial identity. This is a reason why we only consider the instability of radial ground state standing waves in this paper. If one can show that $\phi_\omega \in \Sigma$, then one can study the instability of non-radial ground state standing waves.

The proof of Theorem 1.5 is based on the characterization of the radial ground states and the localized virial estimates. Thanks to the radial symmetry of the ground state, we are able to use the localized virial estimates derived by the second author in [13] to show the finite time blow-up. We refer the reader to Section 3 for more details.

The rest of the paper is organized as follows. In Section 2, we give the proof of the existence of radial ground states for (1.3) given in Proposition 1.2. The proof of our main result-Theorem 1.5 will be given in Section 3.

2. Existence of radial ground states

In this section, we give the proof the existence of radial ground states for (1.3) given in Proposition 1.2. The proof of Proposition 1.2 follows from several lemmas. Let us denote the ω-Hardy functional by

$$H_\omega(v) := \|v\|^2_{\dot{H}^1_c} + \omega \|v\|^2_{L^2}.$$

Using the sharp Hardy inequality

$$\lambda(d) \||x|^{-1} v\|^2_{L^2} \leq \|\nabla v\|^2_{L^2},$$

we see that for $c < \lambda(d)$ and $\omega > 0$ fixed,

$$H_\omega(v) \sim \|v\|^2_{H^1}. \tag{2.1}$$

We note that the action functional can be rewritten as

$$S_\omega(v) := \frac{1}{2} K_\omega(v) + \frac{\alpha}{2(\alpha+2)} \|v\|^{\alpha+2}_{L^{\alpha+2}} = \frac{1}{\alpha+2} K_\omega(v) + \frac{\alpha}{2(\alpha+2)} H_\omega(v). \tag{2.2}$$

Let us start with the following result.

Lemma 2.1. $d(\mathrm{rad}, \omega) > 0$.

Proof. Let $v \in H^1_{\mathrm{rad}} \backslash \{0\}$ be such that $K_\omega(v) = 0$. By the Sobolev embedding, (2.1) and the fact $H_\omega(v) = \|v\|^{\alpha+2}_{L^{\alpha+2}}$, we have

$$\|v\|^2_{L^{\alpha+2}} \leq C_1 \|v\|^2_{H^1} \leq C_2 H_\omega(v) = C_2 \|v\|^{\alpha+2}_{L^{\alpha+2}},$$

for some $C_1, C_2 > 0$. This implies that

$$\frac{\alpha}{2(\alpha+2)} \|v\|^{\alpha+2}_{L^{\alpha+2}} \geq \frac{\alpha}{2(\alpha+2)} \left(\frac{1}{C_2}\right)^{\frac{\alpha+2}{\alpha}}.$$

Taking the infimun over $v \in H^1_{\mathrm{rad}} \backslash \{0\}$, we obtain $d(\mathrm{rad}, \omega) > 0$. □

We now denote the set of all minimizers of (1.6) by

$$\mathcal{M}_{\mathrm{rad},\omega} := \left\{ v \in H^1_{\mathrm{rad}} \backslash \{0\} \ : \ K_\omega(v) = 0, \ S_\omega(v) = d(\mathrm{rad}, \omega) \right\}.$$

Lemma 2.2. *The set $\mathcal{M}_{\mathrm{rad},\omega}$ is non-empty.*

Proof. Let $(v_n)_{n\geq 1}$ be a minimizing sequence of $d(\text{rad},\omega)$, i.e. $v_n \in H^1_{\text{rad}}\backslash\{0\}$, $K_\omega(v_n) = 0$ and $S_\omega(v_n) \to d(\text{rad},\omega)$ as $n \to \infty$. Since $K_\omega(v_n) = 0$, we have $H_\omega(v_n) = \|v_n\|^{\alpha+2}_{L^{\alpha+2}}$ for any $n \geq 1$. Using (2.2), the fact $S_\omega(v_n) \to d(\text{rad},\omega)$ as $n \to \infty$ implies that

$$\frac{\alpha}{2(\alpha+2)} H_\omega(v_n) = \frac{\alpha}{2(\alpha+2)} \|v_n\|^{\alpha+2}_{L^{\alpha+2}} \to d(\text{rad},\omega),$$

as $n \to \infty$. We infer that there exists $C > 0$ such that

$$H_\omega(v_n) \leq \frac{2(\alpha+2)}{\alpha} d(\text{rad},\omega) + C,$$

for all $n \geq 1$. It follows from (2.1) that $(v_n)_{n\geq 1}$ is a bounded sequence in H^1_{rad}. Using the compact embedding $H^1_{\text{rad}} \hookrightarrow L^{\alpha+2}$, there exists $v_0 \in H^1_{\text{rad}}$ such that

$$v_n \rightharpoonup v_0 \text{ weakly in } H^1 \text{ and strongly in } L^{\alpha+2} \text{ as } n \to \infty.$$

Writting $v_n = v_0 + r_n$, where $r_n \rightharpoonup 0$ weakly in H^1 as $n \to \infty$. We have

$$K_\omega(v_n) = H_\omega(v_n) - \|v_n\|^{\alpha+2}_{L^{\alpha+2}} = H_\omega(v_0) + H_\omega(r_n) - \|v_n\|^{\alpha+2}_{L^{\alpha+2}} + o_n(1),$$

as $n \to \infty$. Here $o_n(1)$ means that $o_n(1) \to 0$ as $n \to \infty$. Since $K_\omega(v_n) = 0$ and $H_\omega(r_n) \geq 0$ for all $n \geq 1$, we get

$$H_\omega(v_0) \leq \|v_n\|^{\alpha+2}_{L^{\alpha+2}} + o_n(1),$$

as $n \to \infty$. Taking the limit $n \to \infty$, we obtain

$$H_\omega(v_0) \leq \frac{2(\alpha+2)}{\alpha} d(\text{rad},\omega).$$

Since $v_n \to v_0$ strongly in $L^{\alpha+2}$, it follows that

$$\|v_0\|^{\alpha+2}_{L^{\alpha+2}} = \lim_{n\to\infty} \|v_n\|^{\alpha+2}_{L^{\alpha+2}} = \frac{2(\alpha+2)}{\alpha} d(\text{rad},\omega).$$

We thus get $K_\omega(v_0) \leq 0$. Now suppose that $K_\omega(v_0) < 0$. We have for $\mu > 0$,

$$K_\omega(\mu v_0) = \mu^2 H_\omega(v_0) - \mu^{\alpha+2}\|v_0\|^{\alpha+2}_{L^{\alpha+2}}.$$

It is easy to see that the equation $K_\omega(\mu v_0) = 0$ admits a unique non-zero solution

$$\mu_0 = \left(\frac{H_\omega(v_0)}{\|v_0\|^{\alpha+2}_{L^{\alpha+2}}}\right)^{\frac{1}{\alpha}}.$$

Since $K_\omega(v_0) < 0$, we have $\mu_0 \in (0,1)$. By the definition of $d(\text{rad},\omega)$ and (2.2), we get

$$\begin{aligned} d(\text{rad},\omega) \leq S_\omega(\mu_0 v_0) = \frac{\alpha}{2(\alpha+2)} H_\omega(\mu_0 v_0) &= \mu_0^2 \frac{\alpha}{2(\alpha+2)} H_\omega(v_0) \\ &< \frac{\alpha}{2(\alpha+2)} H_\omega(v_0) \leq d(\text{rad},\omega), \end{aligned}$$

which is a contradiction. Therefore, $K_\omega(v_0) = 0$. Moreover,

$$S_\omega(v_0) = \frac{\alpha}{2(\alpha+2)} \|v_0\|^{\alpha+2}_{L^{\alpha+2}} = d(\text{rad},\omega).$$

This shows that v_0 is a minimizer of $d(\text{rad},\omega)$. The proof is complete. □

Lemma 2.3. $\mathcal{M}_{\text{rad},\omega} \subset \mathcal{G}_{\text{rad},\omega}$.

Proof. Let $\phi \in \mathcal{M}_{\mathrm{rad},\omega}$. Since $K_\omega(\phi) = 0$, we have $H_\omega(\phi) = \|\phi\|^{\alpha+2}_{L^{\alpha+2}}$. Since ϕ is a minimizer of $d(\mathrm{rad},\omega)$, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $S'_\omega(\phi) = \mu K'_\omega(\phi)$. We thus have

$$0 = K_\omega(\phi) = \langle S'_\omega(\phi), \phi\rangle = \mu \langle K'_\omega(\phi), \phi\rangle .$$

It is easy to see that

$$K'_\omega(\phi) = -2\Delta\phi + 2\omega\phi - 2c|x|^{-2}\phi - (\alpha+2)|\phi|^\alpha\phi.$$

Therefore,

$$\langle K'_\omega(\phi), \phi\rangle = 2H_\omega(\phi) - (\alpha+2)\|\phi\|^{\alpha+2}_{L^{\alpha+2}} = -\alpha\|\phi\|^{\alpha+2}_{L^{\alpha+2}} < 0.$$

This implies that $\mu = 0$, hence $S'_\omega(\phi) = 0$. In particular, we have $\phi \in \mathcal{A}_{\mathrm{rad},\omega}$. To prove $\phi \in \mathcal{G}_{\mathrm{rad},\omega}$, it remains to show that $S_\omega(\phi) \leq S_\omega(v)$ for all $v \in \mathcal{A}_{\mathrm{rad},\omega}$. To see this, let $v \in \mathcal{A}_{\mathrm{rad},\omega}$. We have

$$K_\omega(v) = \langle S'_\omega(v), v\rangle = 0.$$

By definition of $\mathcal{M}_{\mathrm{rad},\omega}$, we have $S_\omega(\phi) \leq S_\omega(v)$. The proof is complete. ☐

Lemma 2.4. $\mathcal{G}_{\mathrm{rad},\omega} \subset \mathcal{M}_{\mathrm{rad},\omega}$.

Proof. Let $\phi \in \mathcal{G}_{\mathrm{rad},\omega}$. Since $\mathcal{M}_{\mathrm{rad},\omega}$ is not empty, we take $\psi \in \mathcal{M}_{\mathrm{rad},\omega}$. By Lemma 2.3, $\psi \in \mathcal{G}_{\mathrm{rad},\omega}$. In particular, $S_\omega(\phi) = S_\omega(\psi)$. Since $\psi \in \mathcal{M}_{\mathrm{rad},\omega}$, we get

$$S_\omega(\phi) = S_\omega(\psi) = d(\mathrm{rad},\omega).$$

It remains to show that $K_\omega(\phi) = 0$. Since $\phi \in \mathcal{A}_{\mathrm{rad},\omega}$, we have $S'_\omega(\phi) = 0$, hence $K_\omega(\phi) = \langle S'_\omega(\phi), \phi\rangle = 0$. Therefore, $\phi \in \mathcal{M}_{\mathrm{rad},\omega}$ and the proof is complete. ☐

Proof of Proposition 1.2. Proposition 1.2 follows immediately from Lemmas 2.2, 2.3 and 2.4. ☐

3. Instability of radial standing waves

In this section, we give the proof of the instability of radial ground state standing waves given in Theorem 1.5. Let us start by recalling the local well-posedness in the energy space H^1 for (1.2) proved by Okazawa-Suzuki-Yokota [24].

Theorem 3.1 (Local well-posedness [24])**.** *Let $d \geq 3$, $c \neq 0$ be such that $c < \lambda(d)$ and $\frac{4}{d} < \alpha < \frac{4}{d-2}$. Then for any $u_0 \in H^1$, there exists $T \in (0, +\infty]$ and a maximal solution $u \in C([0,T), H^1)$ of (1.2). The maximal time of existence satisfies either $T = +\infty$ or $T < +\infty$ and*

$$\lim_{t\uparrow T} \|\nabla u(t)\|_{L^2} = \infty.$$

Moreover, the local solution enjoys the conservation of mass and energy

$$\begin{aligned} M(u(t)) &= \int |u(t,x)|^2 dx = M(u_0), \\ E(u(t)) &= \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{c}{2}\int |x|^{-2}|u(t,x)|^2 dx - \frac{1}{\alpha+2}\int |u(t,x)|^{\alpha+2} dx \\ &= E(u_0), \end{aligned}$$

for any $t \in [0,T)$.

We refer the reader to [24, Proposition 5.1] for the proof of the above result. Note that the existence of local solution is based on a refined energy method of the well-known energy method proposed by Cazenave [11, Chapter 3]. The uniqueness of local solutions follows from Strichartz estimates proved by Burq-Planchon-Stalker-Zadel [7].

We next recall the so-called Pohozaev's identities for (1.3). We give the proof for the reader's convenience.

Lemma 3.2. *Let $\omega > 0$. If $\phi_\omega \in H^1$ is a solution to (1.3), then*

$$\|\phi_\omega\|^2_{\dot{H}^1_c} + \omega\|\phi_\omega\|^2_{L^2} - \|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}} = 0,$$

and

$$\left(1 - \frac{d}{2}\right)\|\phi_\omega\|^2_{\dot{H}^1_c} - \frac{d\omega}{2}\|\phi_\omega\|^2_{L^2} + \frac{d}{\alpha+2}\|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}}.$$

Proof. Multiplying both sides of (1.3) with ϕ_ω and integrating over $\mathbb{R}^d$, we obtain easily the first identity. Let us prove the second identity. Due to the singularity of the inverse-square potential at zero, we multiply both sides of (1.3) with $x \cdot \nabla\phi_\omega$ and integrate on $P(r,R) := \{x \in \mathbb{R}^d \ : \ r \leq |x| \leq R\}$ for some $R > r > 0$. We have

$$\begin{aligned}
-\int_{P(r,R)} \Delta\phi_\omega(x \cdot \nabla\phi_\omega)dx = \int_{P(r,R)} \nabla\phi_\omega \cdot \nabla(x \cdot \nabla\phi_\omega)dx - \int_{\partial B_r} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_1)dS \\
- \int_{\partial B_R} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_2)dS,
\end{aligned}$$

where $\boldsymbol{n}_1 = -\frac{x}{r}$ is the unit inward normal at $x \in \partial B_r$ and $\boldsymbol{n}_2 = \frac{x}{R}$ is the unit outward normal at $x \in \partial B_R$. We also have

$$\begin{aligned}
\int_{P(r,R)} \nabla\phi_\omega \cdot \nabla(x \cdot \nabla\phi_\omega)dx = \left(1 - \frac{d}{2}\right)\int_{P(r,R)} |\nabla\phi_\omega|^2 dx + \frac{1}{2}\int_{\partial B_r} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_1)dS \\
+ \frac{1}{2}\int_{\partial B_R} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_2)dS.
\end{aligned}$$

Thus,

$$\begin{aligned}
-\int_{P(r,R)} \Delta\phi_\omega(x \cdot \nabla\phi_\omega)dx = \left(1 - \frac{d}{2}\right)\int_{P(r,R)} |\nabla\phi_\omega|^2 dx - \frac{1}{2}\int_{\partial B_r} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_1)dS \\
- \frac{1}{2}\int_{\partial B_R} |\nabla\phi_\omega|^2(x \cdot \boldsymbol{n}_2)dS.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega\int_{P(r,R)} \phi_\omega(x \cdot \nabla\phi_\omega)dx = -\frac{d\omega}{2}\int_{P(r,R)} |\phi_\omega|^2 dx + \frac{\omega}{2}\int_{\partial B_r} |\phi_\omega|^2(x \cdot \boldsymbol{n}_1)dS \\
+ \frac{\omega}{2}\int_{\partial B_R} |\phi_\omega|^2(x \cdot \boldsymbol{n}_2)dS,
\end{aligned}$$

and

$$\begin{aligned}-c\int_{P(r,R)}|x|^{-2}\phi_\omega(x\cdot\nabla\phi_\omega)dx = -c\left(1-\frac{d}{2}\right)&\int_{P(r,R)}|x|^{-2}|\phi_\omega|^2dx\\ -\frac{c}{2}&\int_{\partial B_r}|x|^{-2}|\phi_\omega|^2(x\cdot\boldsymbol{n}_1)dS\\ -\frac{c}{2}&\int_{\partial B_R}|x|^{-2}|\phi_\omega|^2(x\cdot\boldsymbol{n}_2)dS,\end{aligned}$$

and finally

$$\begin{aligned}-\int_{P(r,R)}|\phi_\omega|^\alpha\phi_\omega(x\cdot\nabla\phi_\omega)dx &= \frac{d}{\alpha+2}\int_{P(r,R)}|\phi_\omega|^{\alpha+2}dx\\ &-\frac{1}{\alpha+2}\int_{\partial B_r}|\phi_\omega|^{\alpha+2}(x\cdot\boldsymbol{n}_1)dS\\ &-\frac{1}{\alpha+2}\int_{\partial B_R}|\phi_\omega|^{\alpha+2}(x\cdot\boldsymbol{n}_2)dS.\end{aligned}$$

Adding the above identities, we get

$$\begin{aligned}\left(1-\frac{d}{2}\right)\left[\int_{P(r,R)}|\nabla\phi_\omega|^2dx - c\int_{P(r,R)}|x|^{-2}|\phi_\omega|^2dx\right] - \frac{d\omega}{2}\int_{P(r,R)}|\phi_\omega|^2dx\\ +\frac{d}{\alpha+2}\int_{P(r,R)}|\phi_\omega|^{\alpha+2}dx = I_1(r)+I_2(R), \quad (3.1)\end{aligned}$$

where

$$\begin{aligned}I_1(r) &= \frac{1}{2}\int_{\partial B_r}|\nabla\phi_\omega|^2(x\cdot\boldsymbol{n}_1)dS - \frac{\omega}{2}\int_{\partial B_r}|\phi_\omega|^2(x\cdot\boldsymbol{n}_1)dS\\ &\quad+\frac{c}{2}\int_{\partial B_r}|x|^{-2}|\phi_\omega|^2(x\cdot\boldsymbol{n}_1)dS + \frac{1}{\alpha+2}\int_{\partial B_r}|\phi_\omega|^{\alpha+2}(x\cdot\boldsymbol{n}_1)dS\\ &= -r\left(\int_{\partial B_r}\frac{1}{2}|\nabla\phi_\omega|^2 - \frac{\omega}{2}|\phi_\omega|^2 + \frac{c}{2}|x|^{-2}|\phi_\omega|^2 + \frac{1}{\alpha+2}|\phi_\omega|^{\alpha+2}dS\right),\end{aligned}$$

and

$$\begin{aligned}I_2(R) &= \frac{1}{2}\int_{\partial B_R}|\nabla\phi_\omega|^2(x\cdot\boldsymbol{n}_2)dS - \frac{\omega}{2}\int_{\partial B_R}|\phi_\omega|^2(x\cdot\boldsymbol{n}_2)dS\\ &\quad+\frac{c}{2}\int_{\partial B_R}|x|^{-2}|\phi_\omega|^2(x\cdot\boldsymbol{n}_2)dS + \frac{1}{\alpha+2}\int_{\partial B_R}|\phi_\omega|^{\alpha+2}(x\cdot\boldsymbol{n}_2)dS\\ &= R\left(\int_{\partial B_R}\frac{1}{2}|\nabla\phi_\omega|^2 - \frac{\omega}{2}|\phi_\omega|^2 + \frac{c}{2}|x|^{-2}|\phi_\omega|^2 + \frac{1}{\alpha+2}|\phi_\omega|^{\alpha+2}dS\right).\end{aligned}$$

Denote

$$A(\phi_\omega) = \frac{1}{2}|\nabla\phi_\omega|^2 - \frac{\omega}{2}|\phi_\omega|^2 + \frac{c}{2}|x|^{-2}|\phi_\omega|^2 + \frac{1}{\alpha+2}|\phi_\omega|^{\alpha+2}.$$

We have

$$\int_B A(\phi_\omega)dx = \int_0^1\int_{\partial B_r}A(\phi_\omega)dSdr < \infty, \quad (3.2)$$

where B is the unit ball in $\mathbb{R}^d$. Hence, there exists a sequence $r_n \to 0$ such that

$$r_n \int_{\partial B_{r_n}} A(\phi_\omega) dS \to 0 \quad \text{as } n \to \infty.$$

Indeed, if

$$\liminf_{r\to 0} r \int_{\partial B_r} A(\phi_\omega) dS = c > 0,$$

then

$$\int_{\partial B_r} A(\phi_\omega) dS$$

would not be in $L^1(0,1)$, which contradicts to (3.2). On the other hand, since

$$\int_{\mathbb{R}^d} A(\phi_\omega) dx = \int_0^{+\infty} \int_{\partial B_R} A(\phi_\omega) dS dR < \infty,$$

there exists a sequence $R_n \to +\infty$ such that

$$R_n \int_{\partial B_R} A(\phi_\omega) dS \to 0 \quad \text{as } n \to \infty.$$

This implies that $I_1(r_n) \to 0$ and $I_2(R_n) \to 0$ as $n \to \infty$. Now substituting r by r_n and R by R_n in (3.1) and taking $n \to \infty$, we obtain the second identity. The proof is complete. □

Throughout this section, we denote the functional

$$Q(v) := \|v\|^2_{\dot{H}^1_c} - \frac{d\alpha}{2(\alpha+2)} \|v\|^{\alpha+2}_{L^{\alpha+2}}.$$

Note that if we take

$$v^\lambda(x) := \lambda^{\frac{d}{2}} v(\lambda x), \tag{3.3}$$

then we have

$$\begin{aligned} \|v^\lambda\|_{L^2} &= \|v\|_{L^2}, & \|\nabla v^\lambda\|_{L^2} &= \lambda \|\nabla v\|_{L^2}, \\ \||x|^{-1} v^\lambda\|_{L^2} &= \lambda \||x|^{-1} v\|_{L^2}, & \|v^\lambda\|_{L^{\alpha+2}} &= \lambda^{\frac{d\alpha}{2(\alpha+2)}} \|v\|_{L^{\alpha+2}}. \end{aligned} \tag{3.4}$$

Thus,

$$S_\omega(v^\lambda) = \frac{\lambda^2}{2} \|v\|^2_{\dot{H}^1_c} + \frac{\omega}{2} \|v\|^2_{L^2} - \frac{\lambda^{\frac{d\alpha}{2}}}{\alpha+2} \|v\|^{\alpha+2}_{L^{\alpha+2}},$$

and

$$Q(v) = \partial_\lambda S_\omega(v^\lambda)\big|_{\lambda=1}.$$

Lemma 3.3. *Let $d \geq 3, c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$ and $\omega > 0$. Let $\phi_\omega \in \mathcal{G}_{\mathrm{rad},\omega}$. Then*

$$S_\omega(\phi_\omega) = \inf\{S_\omega(v) \ : \ v \in H^1_{rad} \backslash \{0\}, Q(v) = 0\}.$$

Proof. Let $d_n := \inf\{S_\omega(v) \ : \ v \in H^1_{\mathrm{rad}} \backslash \{0\}, Q(v) = 0\}$. Thanks to the Pohozaev's identities, it is easy to check that $S_\omega(\phi_\omega) = Q(\phi_\omega) = 0$. By the definition of d_n,

$$S_\omega(\phi_\omega) \geq d_n. \tag{3.5}$$

We now consider $v \in H^1_{\mathrm{rad}} \backslash \{0\}$ be such that $Q(v) = 0$. If $K_\omega(v) = 0$, then by Proposition 1.2, $S_\omega(v) \geq S_\omega(\phi_\omega)$. Assume that $K_\omega(v) \neq 0$. Let v^λ be as in (3.3). We have

$$K_\omega(v^\lambda) = \lambda^2 \|v\|^2_{\dot{H}^1_c} + \omega \|v\|^2_{L^2} - \lambda^{\frac{d\alpha}{2}} \|v\|^{\alpha+2}_{L^{\alpha+2}}.$$

We see that $\lim_{\lambda\to 0} K_\omega(v^\lambda) = \omega\|v\|^2_{L^2} > 0$. Since $\frac{d\alpha}{2} > 2$, we have $\lim_{\lambda\to+\infty} K_\omega(v^\lambda) = -\infty$. Thus, there exists $\lambda_0 > 0$ such that $K_\omega(v^{\lambda_0}) = 0$. By Proposition 1.2, we get $S_\omega(v^{\lambda_0}) \geq S_\omega(\phi_\omega)$. On the other hand, a direct computation shows that

$$\begin{aligned}\partial_\lambda S_\omega(v^\lambda) &= \lambda\|v\|^2_{\dot H^1_c} - \frac{d\alpha}{2(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}}\\ &= \lambda\left(\|v\|^2_{\dot H^1_c} - \frac{d\alpha}{2(\alpha+2)}\lambda^{\frac{d\alpha}{2}-2}\|v\|^{\alpha+2}_{L^{\alpha+2}}\right).\end{aligned}$$

The equation $\partial_\lambda S_\omega(v^\lambda) = 0$ admits a unique non-zero solution

$$\lambda_1 = \left(\frac{\|u\|^2_{\dot H^1_c}}{\frac{d\alpha}{2(\alpha+2)}\|v\|^{\alpha+2}_{L^{\alpha+2}}}\right)^{\frac{2}{d\alpha-4}}$$

which is equal to 1 since $Q(v) = 0$. It follows that $\partial_\lambda S_\omega(v^\lambda) > 0$ if $\lambda \in (0,1)$ and $\partial_\lambda S_\omega(v^\lambda) < 0$ if $\lambda \in (1,\infty)$. In particular, we get $S_\omega(v^\lambda) < S_\omega(v)$ for any $\lambda > 0$ and $\lambda \neq 1$. Since $\lambda_0 > 0$, it follows that $S_\omega(v^{\lambda_0}) \leq S_\omega(v)$. This implies that $S_\omega(v) \geq S_\omega(\phi_\omega)$ for any $v \in H^1_{\mathrm{rad}}\backslash\{0\}$, $Q(v) = 0$. Taking the infimum, we obtain

$$S_\omega(\phi_\omega) \leq d_n. \tag{3.6}$$

Combining (3.5) and (3.6), we prove the result. □

Let $\phi_\omega \in \mathcal{G}_{\mathrm{rad},\omega}$. We denote

$$\mathcal{B}_{\mathrm{rad},\omega} := \{v \in H^1_{\mathrm{rad}}\backslash\{0\} \ : \ S_\omega(v) < S_\omega(\phi_\omega), Q(v) < 0\}.$$

Lemma 3.4. *Let $d \geq 3, c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$ and $\omega > 0$. Let $\phi_\omega \in \mathcal{G}_{rad,\omega}$. Then $\mathcal{B}_{rad,\omega}$ is invariant under the flow of (1.2), that is, if $u_0 \in \mathcal{B}_{rad,\omega}$, then the corresponding solution $u(t)$ to (1.2) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{B}_{rad,\omega}$ for any $t \in [0,T)$.*

Proof. Let $u_0 \in \mathcal{B}_{\mathrm{rad},\omega}$. By the conservation of mass and energy,

$$S_\omega(u(t)) = S_\omega(u_0) < S_\omega(\phi_\omega), \quad \forall t \in [0,T). \tag{3.7}$$

It remains to show that $Q(u(t)) < 0$ for any $t \in [0,T)$. Suppose that there exists $t_0 \in [0,T)$ such that $Q(u(t_0)) \geq 0$. By the continuity of $t \mapsto Q(u(t))$, there exists $t_1 \in (0,t_0]$ such that $Q(u(t_1)) = 0$. By Lemma 3.3, $S_\omega(u(t_1)) \geq S_\omega(\phi_\omega)$ which contradicts to (3.7). □

Lemma 3.5. *Let $d \geq 3, c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$ and $\omega > 0$. Let $\phi_\omega \in \mathcal{G}_{rad,\omega}$. If $v \in \mathcal{B}_{rad,\omega}$, then*

$$Q(v) \leq 2(S_\omega(v) - S_\omega(\phi_\omega)).$$

Proof. Let v^λ be as in (3.3). Set $g(\lambda) := S_\omega(v^\lambda)$. We have

$$\begin{aligned}g(\lambda) &= \frac{\lambda^2}{2}\|v\|^2_{\dot H^1_c} + \frac{\omega}{2}\|v\|^2_{L^2} - \frac{\lambda^{\frac{d\alpha}{2}}}{\alpha+2}\|v\|^{\alpha+2}_{L^{\alpha+2}},\\ g'(\lambda) &= \lambda\|v\|^2_{\dot H^1_c} - \frac{d\alpha}{2(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}} = \frac{Q(v^\lambda)}{\lambda},\end{aligned}$$

and

$$\begin{aligned}(\lambda g'(\lambda))' &= 2\lambda\|v\|^2_{\dot H^1_c} - \frac{d^2\alpha^2}{4(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}} \\ &= 2\left(\lambda\|v\|^2_{\dot H^1_c} - \frac{d\alpha}{2(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}}\right) - \frac{d\alpha(d\alpha-4)}{4(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}} \\ &= 2g'(\lambda) - \frac{d\alpha(d\alpha-4)}{4(\alpha+2)}\lambda^{\frac{d\alpha}{2}-1}\|v\|^{\alpha+2}_{L^{\alpha+2}}.\end{aligned}$$

Since $d\alpha > 4$, we see that

$$(\lambda g'(\lambda))' \leq 2g'(\lambda), \quad \forall \lambda > 0. \tag{3.8}$$

Since $Q(v) < 0$, the equation $\partial_\lambda S_\omega(v^\lambda) = 0$ admits a unique non-zero solution $\lambda_0 \in (0,1)$. Taking the integration over λ_0 and 1 and note that $Q(v^{\lambda_0}) = \lambda_0 \left(\partial_\lambda S_\omega(v^\lambda)\right)\big|_{\lambda=\lambda_0} = 0$, we get

$$Q(v) - Q(v^{\lambda_0}) \leq 2(S_\omega(v) - S_\omega(v^{\lambda_0})) \leq 2(S_\omega(v) - S_\omega(\phi_\omega)).$$

Here, the last inequality comes from the fact $Q(v^{\lambda_0}) = 0$. The proof is complete. □

The key ingredient in showing the strong instability of radial standing waves is to use localized virial estimates to establish the finite time blowup. Let us recall localized virial estimates related to (1.2). Let $\theta : [0,\infty) \to [0,\infty)$ be such that

$$\theta(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq 1, \\ \text{const.} & \text{if } r \geq 2, \end{cases} \quad \text{and} \quad \theta''(r) \leq 2 \text{ for } r \geq 0.$$

The precise constant here is not important. For $R > 1$, we define the radial function

$$\varphi_R(x) = \varphi_R(r) := R^2\theta(r/R), \quad r = |x|. \tag{3.9}$$

We define the virial potential by

$$V_{\varphi_R}(t) := \int \varphi_R(x)|u(t,x)|^2 dx. \tag{3.10}$$

Lemma 3.6 (Radial virial estimate [13])**.** *Let $d \geq 3$, $c \neq 0$ be such that $c < \lambda(d)$, $\frac{4}{d} < \alpha < \frac{4}{d-2}$, $R > 1$ and φ_R be as in (3.9). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to (1.2). Then for any $t \in I$,*

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 8\|u(t)\|^2_{\dot H^1_c} - \frac{4d\alpha}{\alpha+2}\|u(t)\|^{\alpha+2}_{L^{\alpha+2}} + O\left(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot H^1_c}\right) \tag{3.11}$$

$$= 8Q(u(t)) + O\left(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot H^1_c}\right) \tag{3.12}$$

$$= 4d\alpha E(u(t)) - 2(d\alpha-4)\|u(t)\|^2_{\dot H^1_c} + O\left(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot H^1_c}\right). \tag{3.13}$$

The implicit constant depends only on $\|u_0\|_{L^2}, d$ and α. Here $A = O(B)$ means there exists a constant $C > 0$ such that $A = CB$.

We refer the reader to [13, Lemma 5.4] for the proof of the above result.

We are now able to prove our main result.

Proof of Theorem 1.5. Let $\epsilon > 0$, $\omega > 0$ and $\phi_\omega \in \mathcal{G}_{\text{rad},\omega}$. Since $\phi^\lambda_\omega \to \phi_\omega$ in H^1 as $\lambda \to 1$, there exists $\lambda_0 > 1$ such that $\|\phi_\omega - \phi^{\lambda_0}_\omega\|_{H^1} < \epsilon$. By decreasing λ_0 if necessary, we claim that $\phi^{\lambda_0}_\omega \in \mathcal{B}_{\text{rad},\omega}$. To see this, we first notice that $Q(\phi_\omega) = 0$.

This fact follows from the Pohozaev's identities related to (1.3) given in Lemma 3.2:

$$\omega\|\phi_\omega\|^2_{L^2} = \frac{4-(d-2)\alpha}{2(\alpha+2)}\|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}} = \frac{4-(d-2)\alpha}{d\alpha}\|\phi_\omega\|^2_{\dot{H}^1_c}. \tag{3.14}$$

On the other hand, a direct computation shows

$$\begin{aligned} S_\omega(\phi^\lambda_\omega) &:= \frac{\lambda^2}{2}\|\phi_\omega\|^2_{\dot{H}^1_c} + \frac{\omega}{2}\|\phi_\omega\|^2_{L^2} - \frac{\lambda^{\frac{d\alpha}{2}}}{\alpha+2}\|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}}, \\ \partial_\lambda S_\omega(\phi^\lambda_\omega) &:= \lambda\|\phi_\omega\|^2_{\dot{H}^1_c} - \frac{d\alpha}{2(\alpha+2)}\lambda^{\frac{d\alpha-2}{2}}\|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}} = \frac{Q(\phi^\lambda_\omega)}{\lambda}. \end{aligned}$$

It is easy to see that the equation $\partial_\lambda S_\omega(\phi^\lambda_\omega) = 0$ has a unique non-zero solution

$$\left(\frac{\|\phi_\omega\|^2_{\dot{H}^1_c}}{\frac{d\alpha}{2(\alpha+2)}\|\phi_\omega\|^{\alpha+2}_{L^{\alpha+2}}}\right)^{\frac{2}{d\alpha-4}} = 1.$$

The last inequality comes from the fact $Q(\phi_\omega) = 0$. This implies in particular that

$$\left\{ \begin{array}{ll} \partial_\lambda S_\omega(\phi^\lambda_\omega) > 0 & \text{if } \lambda \in (0,1), \\ \partial_\lambda S_\omega(\phi^\lambda_\omega) < 0 & \text{if } \lambda \in (1,\infty), \end{array} \right.$$

from which we get $S_\omega(\phi^\lambda_\omega) < S_\omega(\phi_\omega)$ for any $\lambda > 0, \lambda \neq 1$. Since $Q(\phi^\lambda_\omega) = \lambda\partial_\lambda S_\omega(\phi^\lambda_\omega)$, we also have

$$\left\{ \begin{array}{ll} Q(\phi^\lambda_\omega) > 0 & \text{if } \lambda \in (0,1), \\ Q(\phi^\lambda_\omega) < 0 & \text{if } \lambda \in (1,\infty). \end{array} \right.$$

As an application of the above argument, we have

$$S_\omega(\phi^{\lambda_0}_\omega) < S_\omega(\phi_\omega), \quad Q(\phi^{\lambda_0}_\omega) < 0.$$

This shows that $\phi^{\lambda_0}_\omega \in \mathcal{B}_{\mathrm{rad},\omega}$ and the claim follows.

By Theorem 3.1, there exists a unique solution $u \in C([0,T), H^1)$ to (1.2) with initial data $u(0) = u_0 = \phi^{\lambda_0}_\omega$, where $T > 0$ is the maximal existence time. Since $u_0 = \phi^{\lambda_0}_\omega$ is radial, it is well-known that the corresponding solution is also radial. The rest of this note is to show that u blows up in finite time. It is done by several steps.

Step 1. We claim that there exists $a > 0$ such that $Q(u(t)) \leq -a$ for any $t \in [0,T)$. Indeed, since $\mathcal{B}_{\mathrm{rad},\omega}$ is invariant under the flow of (1.2), we see that $u(t) \in \mathcal{B}_{\mathrm{rad},\omega}$ for any $t \in [0,T)$. By Lemma 3.5, we get

$$Q(u(t)) \leq 2(S_\omega(u(t)) - S_\omega(\phi_\omega)) = 2(S_\omega(\phi^{\lambda_0}_\omega) - S_\omega(\phi_\omega)).$$

This proves the claim with $a = 2(S_\omega(\phi_\omega) - S_\omega(\phi^{\lambda_0}_\omega)) > 0$.

Step 2. We next claim that there exists $b > 0$ such that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -b, \tag{3.15}$$

for any $t \in [0,T)$, where $V_{\varphi_R}(t)$ is as in (3.10). Indeed, since the solution $u(t)$ is radial, we apply Lemma 3.6 to have

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 4d\alpha E(u(t)) - 2(d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c} + O\left(R^{-2} + R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot{H}^1_c}\right),$$

for any $t \in [0,T)$ and any $R > 1$. The Young inequality implies for any $\epsilon > 0$,

$$R^{-\frac{(d-1)\alpha}{2}}\|u(t)\|^{\frac{\alpha}{2}}_{\dot{H}^1_c} \lesssim \epsilon\|u(t)\|^2_{\dot{H}^1_c} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}.$$

Note that in our consideration, we always have $0<\alpha<4$. We thus get

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq 4d\alpha E(u(t)) - 2(d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c} + C\epsilon\|u(t)\|^2_{\dot{H}^1_c} + O\left(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}\right),$$

for any $t\in[0,T)$, any $R>1$, any $\epsilon>0$ and some constant $C>0$.

To see (3.15), we follow the argument of Bonheure-Castéras-Gou-Jeanjean [6]. Fix $t\in[0,T)$ and denote

$$\mu := \frac{4d\alpha|E(u_0)|+2}{d\alpha-4}.$$

We consider two cases.

Case 1.

$$\|u(t)\|^2_{\dot{H}^1_c} \leq \mu.$$

Since $4d\alpha E(u(t)) - 2(d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c} = 8Q(u(t)) \leq -8a$ for any $t\in[0,T)$, we have

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -8a + C\epsilon\mu + O\left(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}\right).$$

By choosing $\epsilon>0$ small enough and $R>1$ large enough depending on ϵ, we see that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -4a.$$

Case 2.

$$\|u(t)\|^2_{\dot{H}^1_c} > \mu.$$

In this case, we have

$$4d\alpha E(u_0) - 2(d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c} < -2 - (d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c}.$$

Thus,

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -2 - (d\alpha-4)\|u(t)\|^2_{\dot{H}^1_c} + C\epsilon\|u(t)\|^2_{\dot{H}^1_c} + O\left(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}\right).$$

Since $d\alpha-4>0$, we choose $\epsilon>0$ small enough so that

$$d\alpha - 4 - C\epsilon \geq 0.$$

This implies that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -2 + O\left(R^{-2} + \epsilon^{-\frac{\alpha}{4-\alpha}}R^{-\frac{2(d-1)\alpha}{4-\alpha}}\right).$$

We next choose $R>1$ large enough depending on ϵ so that

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -1.$$

Note that in both cases, the choices of $\epsilon>0$ and $R>1$ are independent of t. Therefore, the claim follows with $b=\min\{4a,1\}>0$.

Step 3. By Step 2, the solution $u(t)$ satisfies

$$\frac{d^2}{dt^2}V_{\varphi_R}(t) \leq -b < 0,$$

for any $t\in[0,T)$. The convexity argument of Glassey (see e.g. [16]) implies that the solution blows up in finite time. The proof is complete. $\square$

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