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# First-Passage Duality

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**Abstract.** We show that the distribution of times for a diffusing particle to first hit an absorber is *independent* of the direction of an external flow field, when we condition on the event that the particle reaches the target for flow away from the target. Thus, in one dimension, the average time for a particle to travel to an absorber a distance $\ell$ away is $\ell/|v|$, *independent* of the sign of $v$. This duality extends to all moments of the hitting time. In two dimensions, the distribution of first-passage times to an absorbing circle in the radial velocity field $v(r) = Q/(2\pi r)$ again exhibits duality. Our approach also gives a new perspective on how varying the radial velocity is equivalent to changing the spatial dimension, as well as the transition between transience and strong transience in diffusion.

## 1. Introduction

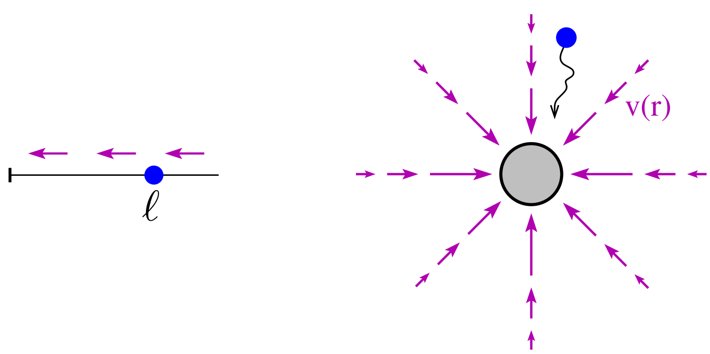
How long does it take a diffusing particle to travel from a starting point to a target? This first-passage, or hitting time, is a fundamental characteristic of diffusion and has a myriad of applications—to chemical kinetics [1, 2], options pricing [3, 4], and neuronal dynamics [5–7] to name a few examples. An intriguing property of diffusion in spatial dimensions $d \leq 2$ is that a diffusing particle is certain to reach any finite-size target, but the average time for this event is infinite. This property of eventually reaching any target is known as *recurrence*. The dichotomy between reaching a target with certainty, but taking an infinite time for this event to occur for $d \leq 2$, helps make diffusion such a compelling and vibrant problem even after a century of intensive study [8–12].

In this article, we investigate basic first-passage characteristics of a simple convection-diffusion system in which a velocity field is either directed toward or away from a target. There has been considerable work on elucidating the characteristics convection-diffusion systems in a variety of geometries (see, e.g., [13–22]). Nevertheless, their first-passage properties, even in the simplest settings, still exhibit surprises, as we present below.

Consider first a uniform velocity field of magnitude $|v|$ that drives a diffusing particle, initially at $\ell > 0$, toward a target at $x = 0$ in one dimension (Fig. 1(a)).

Since the particle is recurrent in the absence of convection, the particle will certainly reach $x = 0$ when the velocity is directed toward the target. The corresponding average hitting time is finite and equals $\ell/|v|$, as one might naively expect. Diffusion plays no role in this average hitting time, but does affect higher moments, as we will derive below.

Conversely, if the velocity is directed away from the target, the probability for the particle to eventually reach the target is less than 1. However, for the fraction of particle trajectories that do reach the target, their average *conditional* hitting time is again $\ell/v$. Here, the conditional hitting time is defined as the average time for those trajectories that actually reach the target. For this subset of "return" trajectories, they must reach the target quickly for $v \gg 1$, or else they will be convected away and never reach the target. We call the equality between these two hitting times as *duality*. As we shall demonstrate, this duality is quite general and applies for the *distribution* of hitting times (unconditional for inward flow and conditional for outward flow).




(a) (b)

**Figure 1.** Illustration of the system in (a) one dimension and (b) two dimensions.

In two dimensions, the natural analog of a constant flow field is radial potential flow, $\mathbf{v} = Q\,\hat{\mathbf{r}}/2\pi r$ (Fig. 1(b)). In keeping with this radial symmetry, we define the absorber to be a circle of radius $a > 0$; a non-zero radius is needed so that a diffusing point particle can actually hit the absorber. When the flow field is inward, $Q < 0$, a diffusing particle hits the absorber with certainty, but the average hitting time is finite only for $Q < -4\pi D$, where $D$ is the diffusion coefficient. Conversely, for sufficiently strong outward flow, namely, $Q > 4\pi D$, the conditional average hitting time is identical to the unconditional average hitting time for inward flow when $Q < -4\pi D$. More generally, the distributions of hitting times (unconditional for inward flow and conditional for outward flow) are identical.

## 2. Duality in One Dimension

Let us first treat a diffusing particle with diffusivity $D$ that starts at $x = \ell > 0$ on the infinite line, and is absorbed when $x = 0$ is reached. The particle is also driven by a spatially uniform velocity field of magnitude $v$. We first analyze the situation where the drift velocity $v<0$ drives the particle toward the origin. All moments of the hitting time can be computed from the first-passage probability, namely, the probability for the particle to reach the origin for the first time at time $t$ [11]:

$$F(t)=\frac{\ell}{\sqrt{4\pi Dt^3}}\,e^{-(\ell-|v|t)^2/4Dt}\,. \tag{1}$$

It is straightforward to verify that the eventual hitting probability $H$ equals 1; that is, $H\equiv\int_0^\infty dt\,F(t)=1$. Since an isotropically diffusing particle eventually reaches the origin, the origin is certainly reached when there is a negative drift velocity.

The moments of the hitting time are given by

$$\langle t^n\rangle=\int_0^\infty dt\;t^n\frac{\ell}{\sqrt{4\pi Dt^3}}\,e^{-(\ell-|v|t)^2/4Dt}\,. \tag{2a}$$

This quantity is properly normalized because the zeroth moment, which is the eventual hitting probability, equals 1. To simplify our results for the hitting time moments, we define the convection time $t_c\equiv\ell/|v|$, the diffusion time $t_D\equiv\ell^2/(2D)$, and their ratio $z=t_c/t_D=2D/\ell|v|$; the latter is the inverse of the Péclet number [23]. Using these variables and also introducing the dimensionless time $\tau\equiv t/t_c$, the moments of the hitting time are

$$\begin{aligned}\langle t^n\rangle&=\int_0^\infty dt\,t^n\,\frac{\ell}{\sqrt{4\pi Dt^3}}\;\exp\left[-\frac{\ell^2}{4Dt}+\frac{|v|\ell}{2D}-\frac{v^2t}{4D}\right]\\&=\frac{t_c^n}{\sqrt{2\pi z}}\,e^{1/z}\int_0^\infty d\tau\,\tau^{n-3/2}\,\exp\left[-(\tau+\tau^{-1})/2z\right]\\&=t_c^n\,\sqrt{\frac{2}{\pi z}}\,e^{1/z}\;K_{\frac{1}{2}-n}(1/z)\,,\end{aligned} \tag{2b}$$

where $K_\mu$ is the modified Bessel function of the second kind of order $\mu$ [24].

From the above expression, the average hitting time is $\langle t\rangle=\ell/|v|$, in agreement with naive intuition; also notice that $\langle t\rangle$ is independent of the diffusion coefficient $D$. The next few moments are:

$$\begin{aligned}\langle t^2\rangle&=t_c^2(1+z)\,,\\\langle t^3\rangle&=t_c^3(1+3z+3z^2)\,,\\\langle t^4\rangle&=t_c^4(1+6z+15z^2+15z^3)\,,\end{aligned}$$

etc. From these moments, the cumulants [25] are given by: $\kappa_1=t_c$, $\kappa_2=zt_c^2$, $\kappa_3=3z^2t_c^3$, etc.; the general result is $\kappa_n=(2n-3)!!\;z^{n-1}t_c^n$. The standard deviation in the hitting time is $\sqrt{\kappa_2}/\kappa_1=\sqrt{t_c/t_D}$. Thus as the drift velocity $v\to 0$, fluctuations in the hitting time between different trajectories diverge. As a curious sidenote, the numerical coefficient of the $n^{\text{th}}$ cumulant is the same as the $n^{\text{th}}$ moment of the Gaussian distribution $P(y)=\exp(-y^2/2)/\sqrt{2\pi}$.

For positive drift velocity, $v > 0$, the probability that the particle eventually hits the origin is

$$H = \int_0^\infty dt\, \frac{\ell}{\sqrt{4\pi D t^3}}\, e^{-(\ell+vt)^2/4Dt} = e^{-v\ell/D}\,. \tag{3}$$

The moments of the hitting time, conditioned on the particle actually reaching the origin, are given by

$$\begin{aligned}
\langle t^n\rangle &= \frac{1}{H}\int_0^\infty dt\; t^n\, \frac{\ell}{\sqrt{4\pi D t^3}}\; e^{-(\ell+vt)^2/4Dt} \\
&= e^{v\ell/D}\int_0^\infty dt\; t^n \frac{\ell}{\sqrt{4\pi D t^3}}\; e^{-(\ell+vt)^2/4Dt} \\
&= \int_0^\infty dt\; t^n \frac{\ell}{\sqrt{4\pi D t^3}}\; e^{-(\ell-vt)^2/4Dt}\,.
\end{aligned} \tag{4}$$

When the prefactor $\frac{1}{H} = e^{v\ell/D}$ is combined with the integrand, its effect is merely to change the sign between $\ell$ and $vt$ in the exponential in the numerator. As a result, the last line of (4) is identical to Eq. (2a). Thus all moments of the *conditional* hitting time for convection-diffusion, with drift *away* from the origin, coincide with the corresponding moments of the *unconditional* hitting time for convection-diffusion, with drift *toward* the origin. This is the statement of first-passage duality in one dimension.

## 3. Duality in Two Dimensions

We now extend duality to two dimensions. In two dimensions, the target must have a non-zero size so there is a positive probability that the target will be hit by a diffusing particle. We take the target to be a disk of radius $a$ and study the hitting time to this disk when the particle starts at an arbitrary point in the exterior region $r > a$.

The natural two-dimensional analog of a constant velocity field in one dimension is radial potential flow, with velocity

$$\mathbf{v}(\mathbf{r}) = \frac{Q}{2\pi r}\,\hat{\mathbf{r}}\,. \tag{5}$$

Such a flow is generated by a point sink (or source) of strength $Q$ at the origin in an incompressible fluid. This flow field has the useful property that in rotationally-symmetric situations the convection term can be absorbed into the diffusion operator by an appropriate shift of the spatial dimension [16,26]. We will exploit this equivalence between the flow field and the spatial dimension in what follows. To simplify matters and without loss of generality, we take the initial condition to be a probability density that is symmetrically concentrated on a ring of radius $r$. By this construction, the system is always rotationally symmetric.

*Hitting probability and average hitting time*

In principle, the hitting probability and average hitting time can be determined by solving the convection-diffusion equation in the flow field (5), then computing the diffusive flux to the absorbing circle, and finally extracting the eventual hitting probability and the average hitting time from moments of this flux [11]. A simpler approach relies on the backward Kolmogorov equation [27], which exploits the Markov nature of the particle motion to write a time-independent equation for the hitting probability.

For example, for a particle that starts at $r$, the hitting probability to a target set can be generically written as (for a discrete hopping process for concreteness)

$$H(r) = \sum_{r'} p(r \to r')\, H(r')\,, \tag{6}$$

where $p(r \to r')$ is the probability to hop from $r$ to $r'$ in a single step. That is, the hitting probability starting at $r$ may be decomposed into the sum of hitting probabilities after one step, multiplied by the probability for the particle to take a single step from $r$ to $r'$. Expanding (6) in a Taylor series leads to the backward Kolmogorov equation for $H(r)$, in which the initial point is the dependent variable. The nature of this equation depends on the geometry of the system and the single-step hopping probabilities.

For radial potential flow, the probability $H(r)$ that a particle, which starts at radius $r$, eventually hits the disk satisfies [11]

$$D\Big(H'' + \frac{1}{r}\, H'\Big) + \frac{Q}{2\pi r}\, H' = 0\,,$$

where prime denotes radial derivative. We rewrite this equation as

$$H'' + \frac{1+q}{r}\, H' = 0\,, \tag{7a}$$

with $q \equiv Q/2\pi D$ is the dimensionless source strength. Let us compare (7a) with the corresponding equation

$$H'' + \frac{d-1}{r}\, H' = 0 \tag{7b}$$

for the hitting probability in $d$ dimensions for pure diffusion (without convection) exterior to the ball. Equations (7a) and (7b) are *identical* when $d = 2+q$. We may thus interpret the hitting probability for diffusion with potential flow in two dimensions as equivalent to the hitting probability for isotropic diffusion in spatial dimension $d = 2+q$. By this correspondence, an increase in $q$ is equivalent to an increase in the spatial dimension. This correspondence accords with naive expectations: by increasing the outward velocity, it becomes less likely that that the absorber will be reached. Such a decrease in the hitting probability also occurs in isotropic diffusion when the spatial dimension is increased. To obtain the hitting probability, we solve (7a) subject to $H(r\!=\!a) = 1$, and obtain

$$H(r) = \begin{cases} 1 & q \le 0\,, \\ r^{-q} & q > 0\,, \end{cases} \tag{8}$$

where we henceforth set $a = 1$ and $D = 1$ to simplify the formulae that follow.

In a similar spirit to the hitting probability, the average hitting time can also be obtained by solving an appropriate backward Kolmogorov equation. In this case, the underlying equation is

$$T(r) = \sum_{r'} p(r \to r') \left[T(r') + \delta t\right] , \tag{9}$$

where we write $T(r) \equiv \langle t(r) \rangle$ for the average hitting time to the circle of radius 1 when the particle starts at $r$. Here, we again decompose the average hitting time starting from $r$ as the time for a single step (the factor $\delta t$), plus the average hitting time starting from neighboring points $r'$, multiplied by the probability of this single step. Performing the same Taylor expansion as that used to recast the general equation (6) for the hitting probability as (7), $T(r)$ satisfies [11]

$$T'' + \frac{1+q}{r}\, T' = -1 \tag{10}$$

when $q \le 0$. We must solve this equation subject to the boundary condition $T(1) = 0$; that is, starting at the disk boundary, the hitting time is zero.

There exists a one-parameter family of solutions to (10):

$$T(r) = \begin{cases} \dfrac{r^2-1}{-2(2+q)} + A\big(r^{-q} - 1\big) & q \neq -2\,, \\[2ex] -\dfrac{r^2}{2}\, \ln r + A\big(r^2 - 1\big) & q = -2\,. \end{cases} \tag{11}$$

In the range $-2 < q \le 0$ the solution for $q \neq -2$ is unacceptable, as it becomes negative for any finite choice of $A$; the solution for $q = -2$ has the same deficiency. The resolution of this apparent paradox is simple: $A = \infty$, so that $T(r) = \infty$ in the range $-2 \le q \le 0$. When $q < -2$, the expression in the first line of (11) with $A = 0$ is the proper solution. The vanishing of this amplitude follows from the observation that with an inward convection field $T(r)$ cannot grow faster than $r^2$ for $r \gg 1$.

When $q > 0$, we want to solve for the average *conditional* hitting time. By generalizing the backward Kolmogorov approach in the appropriate way [11], the governing equation for this average conditional time is

$$(HT)'' + \frac{1+q}{r}\,(HT)' = -H\,, \tag{12}$$

with $H = r^{-q}$ from (8). Solving (12), subject $T(1) = 0$, gives

$$T(r) = \begin{cases} \dfrac{r^2-1}{2(q-2)} + A\big(r^{q} - 1\big) & q > 2\,, \\[2ex] -\dfrac{r^2}{2}\, \ln r + A\big(r^2 - 1\big) & q = 2\,. \end{cases} \tag{13}$$

The same argument as that used for the inward flow field shows that there are no acceptable solutions in the range $0 \leq q \leq 2$; instead $T = \infty$. When $q > 2$, the average hitting time is given by the first line of (13) with $A$ set to 0.

To summarize, the average hitting time to a disk of radius $a$ is

$$T(r) = \begin{cases} \dfrac{r^2 - a^2}{2(|q| - 2)D} & |q| > 2 \\ \infty & |q| \leq 2\,. \end{cases} \tag{14}$$

where the physical variables $a$ and $D$ have been restored. Crucially, $T(r)$ depends only on $|q|$. This is the statement of duality in two dimensions; namely the average hitting time is *independent* of the sign of the velocity. Another important feature is that the average hitting time (either conditional or unconditional) is finite only for $|q| > 2$.

Equation (14) can be given an additional meaning by exploiting the aforementioned correspondence $d = 2 + q$ between magnitude of the flow field in convection diffusion and the spatial dimension in isotropic diffusion. For isotropic diffusion, it is known that spatial dimension $d = 4$ demarcates the transition between transience and strong transience [28, 29]. Transience is the familiar property that a diffusing particle may not necessarily hit a finite absorbing set, a property that occurs when the spatial dimension $d > 2$. However, for the subset of diffusing trajectories that do reach the absorber, the average conditional hitting time is infinite for $2 < d \leq 4$. However, for spatial dimension $d > 4$, this average conditional hitting time becomes finite, a features that is known as strong transience. Thus by varying the strength of the flow, one can vary the effective dimensionality and drive the system from recurrent, to transient, to strongly transient.

When $q$ passes through $-2$, the average unconditional hitting time changes from infinite to finite, while the eventual hitting probability always equals 1. We term this change as a transition between recurrence (hitting probability equals 1 and infinite average hitting time) and *strong recurrence* (hitting probability equals 1 and finite average hitting time). By the connection $d = 2 + q$ between convection-diffusion in two dimensions and isotropic diffusion in spatial dimension $d$, this transition between recurrence and strong recurrence for pure diffusion occurs when the spatial dimension $d = 0$.

### *Distribution of the hitting time*

Duality can be extended to the full distribution of hitting times. Let $F(r,t)$ denote the first-passage probability, namely, the probability that the particle first hits the disk at time $t$ when starting at $r$. While the moments of hitting time can be computed by the backward Kolmogorov equation approach, a more efficient strategy is to write the backward equation for the Laplace transform of the distribution of hitting times

$$\Pi(r,s) = \int_0^\infty dt\, e^{-st}\, F(r,t) = \left\langle e^{-st} \right\rangle\,, \tag{15}$$

from which all moments (and cumulants) can be extracted.

Following this approach, the function $\Pi(r,s)$ satisfies [30]

$$\Pi'' + \frac{1+q}{r}\,\Pi' = s\Pi \qquad q \le 0\,, \tag{16}$$

subject to the boundary condition $\Pi(1,s)=1$, as the hitting time vanishes at $r=1$. The general solution to (16) can be expressed in terms of the modified Bessel functions [24]:

$$\Pi(r,s) = r^{-\lambda}\left[A_1 I_{-\lambda}\big(r\sqrt{s}\big) + A_2 K_{-\lambda}\big(r\sqrt{s}\big)\right]\,,$$

where $\lambda = q/2$. We fix the constants by the boundary conditions. The Laplace transform obeys the obvious bounds $0<\Pi(r,s)<1$, with the upper bound arising from $\Pi(r,s)<\Pi(r,0)=1$. However, the above general solution diverges when $r\to\infty$. To ensure that the solution remains finite in this limit, we must choose $A_1=0$. The amplitude $A_2$ is fixed by the boundary condition $\Pi(1,s)=1$, from which the Laplace transform of the hitting time distribution is

$$\Pi(r,s) = r^{-\lambda}\,\frac{K_{-\lambda}\big(r\sqrt{s}\big)}{K_{-\lambda}\big(\sqrt{s}\big)} \qquad q \le 0\,. \tag{17a}$$

The same result was derived in [11] by directly solving the convection-diffusion equation for pure diffusion in $d$ dimensions and then computing the diffusive flux to the absorbing ball. To make the correspondence with (17a), we need to replace $d$ in [11] by $2+q$, as discussed above.

When $q>0$, we need to work with the conditional hitting probability. Thus the relevant quantity is the function $\Xi(r,s) \equiv H(r)\Pi(r,s)$, which satisfies the backward equation

$$\Xi'' + \frac{1+q}{r}\,\Xi' = s\,\Xi\,,$$

which is mathematically identical to (16). Hence its solution, subject to the boundary condition $\Pi(1,s)=1$, is identical to (17a). We now use $H=r^{-q}=r^{-2\lambda}$ and the identity $K_\lambda(z)=K_{-\lambda}(z)$ to ultimately find

$$\Pi(r,s) = \frac{\Xi(r,s)}{H(r)} = r^{\lambda}\,\frac{K_{\lambda}\big(r\sqrt{s}\big)}{K_{\lambda}\big(\sqrt{s}\big)} \qquad q > 0\,. \tag{17b}$$

Equations (17a) and (17b) exhibit the fundamental duality in the Laplace transform of the first-passage probability with respect to the transformation $q \longleftrightarrow -q$:

$$\Pi(r,s;-q) = \Pi(r,s;q)\,. \tag{18}$$

From the series representation of $\Pi(r,s)$, we can extract all moments of the hitting time and show that there is a transition in the average hitting time when $q=\pm 2$, as already found in (14), as well as a series of transitions for progressively higher moments for $q = \pm 4, \pm 6, \ldots$. Consider the asymptotic behavior of $\Pi(r,s)$ in (17b) as $s \to 0^+$. For $\lambda > 0$ and non integer, we use the identity

$$K_\lambda(z) = \pi\, \frac{I_{-\lambda}(z) - I_\lambda(z)}{2\sin(\pi\lambda)}\,, \tag{19}$$

and the Frobenius series for $I_\lambda$ [24]

$$I_\lambda(z) = \sum_{n\geq 0} \frac{(z/2)^{2n+\lambda}}{\Gamma(n+1)\,\Gamma(n+\lambda+1)} \tag{20}$$

to find, as $s \to 0^+$,

$$\Pi(r,s) = 1 + \left(\frac{s}{4}\right)^{\lambda} \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)} \left(1 - r^{2\lambda}\right) + O(s)\,, \tag{21}$$

when $0 < \lambda < 1$. If the average hitting time was finite, the Laplace transform would have the Taylor-series expansion

$$\Pi(r,s) = 1 - s\langle t(r)\rangle + \ldots\,.$$

Comparing with (21) we see that $\langle t\rangle = \infty$ when $0 < \lambda < 1$. When $\lambda = 1$ we use the known asymptotic behavior of the Bessel function $K_1(z)$ [24] to obtain $\Pi(r,s) - 1 \sim s \ln s$. The absence of a linear term in the expansion again shows that $\langle t\rangle = \infty$ when $\lambda = 1$. When $1 < \lambda < 2$ we can use again (19)–(20) to obtain

$$\Pi(r,s) = 1 - s\langle t(r)\rangle + O\big(s^\lambda\big)\,, \tag{22}$$

with $\langle t\rangle$ given by (14). However, the second moment $\langle t^2\rangle$ still diverges when $1 < \lambda \leq 2$. By using (19)–(20) and focusing on the range $2 < \lambda < 3$, we find

$$\Pi(r,s) = 1 - s\langle t(r)\rangle + \tfrac{1}{2}s^2\langle t(r)^2\rangle + O\big(s^\lambda\big) \tag{23}$$

Thus the first two moments are finite, while the third moment diverges when $2 < \lambda \leq 3$. The second moment is, explicitly,

$$\langle t(r)^2\rangle = \begin{cases} \dfrac{r^4 - a^4}{4(|q|-2)(|q|-4)D^2} - \dfrac{(r^2-a^2)a^2}{2(|q|-2)^2 D^2} & |q| > 4\,, \\ \infty & |q| \leq 4\,, \end{cases} \tag{24}$$

while the variance is described more compactly as

$$\langle t^2\rangle - \langle t\rangle^2 = \frac{r^4 - a^4}{2(|q|-2)^2(|q|-4)D^2} \tag{25}$$

when $|q| > 4$. The transition from the second moment being infinite to being finite when $q$ passes through 4 corresponds to a transience/strong transience transition for the second moment for isotropic diffusion in spatial dimension $d = 6$, while the transition when $q$ passes through $-4$ corresponds to a recurrence/strong recurrence transition in the second moment for isotropic diffusion when $d = -2$. Generally, when $n < \lambda \leq n+1$, the moments $\langle t^j\rangle$ exist when $j = 0, 1, \ldots, n$ and diverge when $j \geq n+1$.

## 4. Discussion

We discovered a simple duality for the distribution of hitting times to an absorber for a diffusing particle in a constant ($1d$) or radial potential ($2d$) velocity field.

In one dimension with flow toward the absorber, all particle trajectories eventually hit the target in a finite time and the basic quantity is the *unconditional* hitting time and all its moments. When the flow is away from the absorber, it is necessary to restrict to the *conditional* hitting time, defined as hitting time of the subset of trajectories that eventually reach the target (and all its moments). We showed that all moments of the unconditional hitting time for flow toward an absorber coincide with the corresponding moments of the conditional hitting time for flow away from the absorber. As a consequence, the conditional hitting time *decreases* when the outward flow velocity *increases*.

Our derivations in one dimension relied on direct calculations, while in two dimensions we employed the backward Kolmogorov equation. This latter approach can be also used in one dimension. Within this approach, there is a subtlety in the determination of the average hitting time for inward flow. Mathematically, one must solve a second-order equation

$$D\frac{d^2T}{d\ell^2} - v\frac{dT}{d\ell} = -1\,, \tag{26}$$

subject to a single boundary condition $\mathrm{T}(0) = 0$. The one-parameter family of solutions to (26) that satisfies this boundary condition is

$$T(\ell) = \frac{\ell}{v} + A\big(e^{v\ell/D} - 1\big)\,. \tag{27}$$

Setting the amplitude $A = 0$, gives the correct (and unique) solution. The simplest way to show that $A = 0$ is by physical reasoning. For $v \to \infty$, the second term in (27) diverges, whereas the hitting time must go to zero. Thus $A = 0$. One can also obtain $T(\ell)$ by solving this problem in the finite interval $[0, L]$ with absorption at 0 and reflection at $L$, and then take the limit $L \to \infty$. By either approach, the average hitting time is simply $T(\ell) = \ell/v$.

The duality between hitting times also holds in the interval $[0, L]$ with both boundaries absorbing. Here, the basic observables are the hitting probabilities and the conditional hitting times to 0 and to $L$. The approach of Sec. 2 now gives the following: for positive drift velocity $v > 0$, the probability to hit the origin is

$$H(\ell) = \frac{e^{-v\ell/D} - e^{-vL/D}}{(1 - e^{-vL/D})}\,, \tag{28a}$$

while the average conditional hitting time to the origin is

$$T(\ell) = \frac{\ell}{v}\,\frac{e^{-vL/D} + e^{-v\ell/D}}{e^{-v\ell/D} - e^{-vL/D}} - \frac{2L}{v}\,\frac{1 - e^{-v\ell/D}}{1 - e^{-vL/D}}\,\frac{e^{-vL/D}}{e^{-v\ell/D} - e^{-vL/D}}\,. \tag{28b}$$

For $L \to \infty$, we recover the result of the semi-infinite system, $H(\ell) = 1$ and $T(\ell) = \ell/v$. When $v < 0$, the hitting probability to the origin is again given by (28a), but with the opposite sign for $v$. This expression is not invariant under the interchange $v \to -v$. However, the average conditional hitting time (28b) *is* invariant under the interchange $v \to -v$. A similar duality for the finite interval was found previously in Refs. [31, 32] and was extended to general potentials (not just constant drift) in Refs. [33, 34].

In two dimensions, we showed that the distribution of unconditional hitting times for inward flow is the same as the distribution of conditional hitting times for outward flow, for the radial potential velocity field $v = Q/(2\pi r)$. Consequently, all moments of the unconditional hitting times for inward flow coincide with the corresponding moments of the conditional hitting times for outward flow. This relationship between hitting times follows from the connection between convection-diffusion in two dimensions in the presence of a radial flow field and pure diffusion in general spatial dimensions that was previously studied in Ref. [26]. In this work, we exploited this connection to derive a duality in the first-passage properties of radial flow.

A basic feature in two dimensions is that the average hitting time (unconditional for inward flow, conditional for outward flow) diverges for sufficiently weak flow, namely, $|Q| \leq 4\pi D$. We also showed that increasing the magnitude of the outward flow is equivalent to increasing the spatial dimension $d$ through the relation $d = 2 + q$, with $q = Q/(2\pi D)$. Thus the transition between finite and infinite conditional hitting time as $q$ passes through 2 also describes the transition between transience and strong transience [28, 29] as the spatial dimension of a system with isotropic diffusion passes through 4.

The duality between $q \leftrightarrow -q$ also has an intriguing consequence for isotropic diffusion. Because of the equivalence between potential flow in two dimensions with scaled velocity $q$ and isotropic diffusion in spatial dimension $d = 2 + q$, the duality $q \leftrightarrow -q$ translates to the duality $d \leftrightarrow 4 - d$ for isotropic diffusion.

Letting the spatial dimension $d$ to be a free parameter and then developing theoretical approaches based on this parameter, such as expansion about a critical dimension, dimensional regularization, etc., has proven insightful in field theory and statistical physics [35–37]. Such non-integer dimensions arise naturally in many theories of critical phenomena as an intermediate step toward understanding physically relevant spatial dimensions, such as $d = 2$ and $d = 3$. We demonstrated a connection between: (a) two-dimensional convection-diffusion with a radial potential flow field of scaled magnitude $q$, and (b) isotropic diffusion in spatial dimension $d = 2 + q$. It would be exciting to realize these convection-diffusion flows experimentally and thereby probe dynamical processes in spaces whose spatial dimension is not necessarily an integer.

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