Log Calabi-Yau fibrations

Caucher Birkar

Abstract. In this paper we study boundedness properties and singularities of log Calabi-Yau fibrations, particularly those admitting Fano type structures. A log Calabi-Yau fibration roughly consists of a pair (X, B) with good singularities and a projective morphism $X \to Z$ such that $K_X + B$ is numerically trivial over Z. This class includes many central ingredients of birational geometry such as Calabi-Yau and Fano varieties and also fibre spaces of such varieties, flipping and divisorial contractions, crepant models, germs of singularities, etc.

Contents

Date: January 25, 2022.
2010 MSC: 14J45, 14J32, 14J10, 14E30, 14J17, 14C20, 14E05.

1. **Introduction**

We work over a fixed algebraically closed field k of characteristic zero unless stated otherwise. According to the minimal model program (including the abundance conjecture) every variety W is birational to a projective variety X with good singularities such that either

- X is canonically polarised (i.e. K_X is ample), or
- X admits a Mori-Fano fibration $X \to Z$ (i.e. K_X is anti-ample over Z), or
- X admits a Calabi-Yau fibration $X \to Z$ (i.e. K_X is numerically trivial over Z).

This reduces the birational classification of algebraic varieties to classifying such X. From the point of view of moduli theory it makes perfect sense to focus on such X as they have a better chance of having a reasonable moduli theory due to the special geometric structures they carry. For this and other reasons Fano and Calabi-Yau varieties and their fibrations are central to birational geometry. They are also of great importance in many other parts of mathematics such as arithmetic geometry, differential geometry, mirror symmetry, and mathematical physics.

Boundedness properties of canonically polarised varieties and Fano varieties have been extensively studied in the literature leading to recent advances [16][5][4] but much less is known about Calabi-Yau varieties. With the above philosophy of the minimal model program in mind, there is a natural urge to extend such studies to the more general framework of Fano and Calabi-Yau fibrations. It is also more fruitful and more flexible to discuss this in the context of pairs.

Now we introduce the notion which unifies many central ingredients of birational geometry. A *log Calabi-Yau fibration* consists of a pair (X, B) with log canonical singularities and a contraction $f\colon X \to Z$ (i.e. a surjective projective morphism with connected fibres) such that $K_X + B \sim_{\mathbb{R}} 0$ relatively over Z. We usually denote the fibration by $(X, B) \to Z$. Note that we allow the two extreme cases: when f is birational and when f is constant. When f is birational such a fibration is a crepant model of (Z, f_*B) (see below). When f is constant, that is, when Z is a point, we just say (X, B) is a *log Calabi-Yau pair*. In general, if F is a general fibre of f and if we let $K_F + B_F = (K_X + B)|_F$, then $K_F + B_F \sim_{\mathbb{R}} 0$, hence (F, B_F) is a log Calabi-Yau pair justifying the terminology.

The class of log Calabi-Yau fibrations includes all log Fano and log Calabi-Yau varieties and much more. For example, if X is a variety which is Fano over a base Z, then we can easily find B so that $(X,B) \to Z$ is a log Calabi-Yau fibration. This includes all Mori fibre spaces. Since we allow birational contractions, it also includes all divisorial and flipping contractions. Another interesting example of log Calabi-Yau fibrations $(X,B) \to Z$ is when $X \to Z$ is the identity morphism; the set of such fibrations simply coincides with the set of pairs with log canonical singularities. On the other hand, a surface with a minimal elliptic fibration over a curve is another instance of a log Calabi-Yau fibration.

Besides the classification problem mentioned above, there are other motivations for considering log Calabi-Yau fibrations. Indeed, they are very useful for inductive treatment of various problems in algebraic geometry. For example they are used to treat the minimal model and abundance and Iitaka conjectures, to construct complements on Fano varieties, etc. They appear in the literature with other names, e.g. lc-trivial fibrations [1].

The following are general guiding questions which are the focus of this paper:

Questions. *Under what conditions do log Calabi-Yau fibrations form bounded families?*

How do singularities behave on the total space and base of log Calabi-Yau fibrations?

When do bounded (klt or lc) complements exist for log Calabi-Yau fibrations?

These questions are naturally related to many problems in birational geometry. In this paper we investigate these questions giving particular attention to those log Calabi-Yau fibrations which in some sense carry full or partial Fano type structures.

A log Calabi-Yau fibration $(X,B) \to Z$ is of *Fano type* if X is of Fano type over Z, that is, if $-(K_X + C)$ is ample over Z and (X,C) is klt for some boundary C. When (X,B) is klt, this is equivalent to saying that $-K_X$ is big over Z. We introduce some notation, somewhat similar to [20], to simplify the statements of our results below.

Definition 1.1. *Let d,r be natural numbers and ϵ be a positive real number. A (d,r,ϵ)-*Fano type (log Calabi-Yau) fibration *consists of a pair (X,B) and a contraction $f\colon X \to Z$ such that we have the following:*

- *(X,B) is a projective ϵ-lc pair of dimension d,*
- *$K_X + B \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-divisor L,*
- *$-K_X$ is big over Z, i.e. X is of Fano type over Z,*
- *A is a very ample divisor on Z with $A^{\dim Z} \le r$, and*
- *$A - L$ is ample.*

That is, a (d,r,ϵ)-Fano type fibration is a log Calabi-Yau fibration which is of Fano type and with certain geometric and numerical data bounded by the numbers d,r,ϵ. The condition $A^{\dim Z} \le r$ means that Z belongs to a bounded family of varieties. Ampleness of $A-L$ means that the "degree" of K_X+B is in some sense bounded (this degree is measured with respect to A). When Z is a point the last two conditions in the definition are vacuous: in this case the fibration is simply a Fano type ϵ-lc Calabi-Yau pair of dimension d.

In the rest of this introduction we will state some of the main results of this paper. To keep the introduction as simple as possible we have moved further results and remarks to Section 2.

Boundedness of log Calabi-Yau fibrations with Fano type structure. Our first result concerns the boundedness of Fano type fibrations as defined above. This maybe considered as a relative version of the so-called BAB conjecture [4, Theorem 1.1] which is about boundedness of Fano varieties in the global setting.

Theorem 1.2. *Let d,r be natural numbers and ϵ be a positive real number. Consider the set of all (d,r,ϵ)-Fano type fibrations $(X,B)\to Z$ as in 1.1. Then the X form a bounded family.*

The theorem also holds in the more general setting of generalised pairs, see 2.2. A key ingredient of the proof is the theory of complements. Indeed in the process of proving the theorem we show that there is $\Lambda \ge 0$ such that (X,Λ) is klt, $K_X+\Lambda \sim_{\mathbb{Q}} 0/Z$ having bounded Cartier index, and (X,Λ) is log bounded.

Jiang [20, Theorem 1.4] considers the setting of the theorem and proves birational boundedness of X modulo several conjectures. We use some of his arguments to get the birational boundedness but we need to do a lot more work to get boundedness.

The boundedness statement of Theorem 1.2 does not say anything about boundedness of $\operatorname{Supp} B$. This is because in general we have no control over $\operatorname{Supp} B$, e.g. when $X=\mathbb{P}^2$ and Z is a point, $\operatorname{Supp} B$ can contain arbitrary hypersufaces. However, if the coefficients of B are bounded away from zero, then indeed $\operatorname{Supp} B$ would also be bounded. More generally we have:

Theorem 1.3. *Let d,r be natural numbers and ϵ,δ be positive real numbers. Consider the set of all (d,r,ϵ)-Fano type fibrations $(X,B)\to Z$ as in 1.1 and $\mathbb{R}$-divisors $0\le \Delta\le B$ whose non-zero coefficients are $\ge \delta$. Then the set of such (X,Δ) is log bounded.*

For applications it is important to consider variants of the above results by replacing the Fano type assumption with a more flexible notion. We say that a contraction $X\to Z$ of normal varieties *factors as a tower of Fano fibrations of length l* if $X\to Z$ factors as a sequence of contractions

$$X=X_1\to X_2\to\cdots\to X_l=Z$$

where $-K_{X_i}$ is ample over X_{i+1}, for each $1\le i\le l-1$. For practical convenience we allow $X_i\to X_{i+1}$ to be an isomorphism and allow $\dim X_i=0$, so l is not uniquely determined by $X\to Z$. The next result replaces Fano type with existence of a tower of Fano fibrations. It will be a crucial ingredient of the proof of 1.5 below.

Theorem 1.4. *Let d,r,l be natural numbers and ϵ,τ be positive real numbers. Consider pairs (X,B) and contractions $f\colon X\to Z$ such that*

- *(X,B) is projective ϵ-lc of dimension d,*
- *the non-zero coefficients of B are $\ge\tau$,*
- *$K_X+B\sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-divisor L,*
- *$X\to Z$ factors as a tower of Fano fibrations of length l,*
- *there is a very ample divisor A on Z with $A^{\dim Z}\le r$, and*
- *$A-L$ is ample.*

Then the set of such (X,B) forms a log bounded family.

Special cases of this are proved in [11, 1.9 and 1.10].

Boundedness of crepant models. It is interesting to look at the special cases of Theorem 1.2 when f is birational and when it is constant. In the latter case, the theorem is equivalent to the BAB conjecture [4, Theorem 1.1 and Corollary 1.2] but in the former case, which says something about *crepant models*, a lot work is needed to derive it from the BAB.

Given a pair (Z,B_Z) and a birational contraction $\phi\colon X\to Z$, we can write $K_X+B=\phi^*(K_Z+B_Z)$ for some uniquely determined B. We say (X,B) is a crepant model of (Z,B_Z) if $B\ge 0$. The birational case of Theorem 1.2 then essentially says that if Z belongs to a bounded family, if the "degree" of B_Z is bounded with respect to some very ample divisor, and if (Z, B_Z) is ϵ-lc, then the underlying varieties of all the crepant models of such pairs form a bounded family; this is quite non-trivial even in the case $Z = \mathbb{P}^3$ (actually it is already challenging for $Z = \mathbb{P}^2$ if we do not use BAB). Special cases of boundedness of crepant models have appeared in the literature assuming that $\operatorname{Supp} B_Z$ is bounded, see [29, Lemma 10.5][15, Propositions 2.5, 2.9][10, Proposition 4.8]. The key point here is that we remove such assumptions on the support of B_Z.

Note that the ϵ-lc condition and boundedness of "degree" of B_Z are both necessary. Indeed if we replace ϵ-lc by lc, then the crepant models will not be bounded, e.g. considering $(Z = \mathbb{P}^2, B_Z)$ where B_Z is the union of three lines intersecting transversally and successively blowing up intersection points in the boundary gives an infinite sequence of crepant models with no bound on their Picard number. On the other hand, if B_Z can have arbitrary degree, then we can easily choose it so that $(Z = \mathbb{P}^2, B_Z)$ is $\frac{1}{2}$-lc having a crepant model of arbitrarily large Picard number.

Boundedness of log Calabi-Yau pairs. Without appropriate restrictions, the set of log Calabi-Yau pairs of a fixed dimension is far from being bounded. For example, it is well-known that the set of K3 surfaces is not bounded although they are topologically bounded. In this respect log Calabi-Yau pairs with non-zero boundary behave better as the next result illustrates.

Theorem 1.5. *Let d be a natural number and ϵ, τ be positive real numbers. Consider pairs (X, B) with the following properties:*

- *(X, B) is projective ϵ-lc of dimension d,*
- *$K_X + B \sim_{\mathbb{R}} 0$,*
- *$B \neq 0$ and its coefficients are $\geq \tau$, and*
- *(X, B) is not of product type.*

Then the set of such (X, B) is log bounded up to isomorphism in codimension one.

Boundedness up to isomorphism in codimension one means that there is a bounded family of projective varieties Y such that for each X in the theorem we can find some Y together with a birational map $X \dashrightarrow Y$ which is an isomorphism in codimension one; a similar definition applies to the case of pairs.

Di Cerbo and Svaldi [11, Theorem 1.3] studied this statement and proved it in dimension ≤ 4 when the coefficients of B belong to a fixed DCC set; in this case we can replace ϵ-lc with klt as ϵ-lc would then follow from the other assumptions. They use MMP to obtain a tower of Mori fibre spaces on a birational model of X (see 3.28) and use this tower to prove the claimed boundedness. We will use this strategy and 1.3 and 1.4 to prove the theorem.

The condition of (X, B) not being of product type means that if there is a birational map $\phi\colon X \dashrightarrow Y$ to a normal projective variety whose inverse does not contract divisors and if $g\colon Y \to Z$ is a contraction with $\dim Y > \dim Z > 0$, then $K_Z \not\equiv 0$. This is a technical condition related to moduli and Hodge theory but without it the statement fails. We will see that $K_Y + B_Y$ is numerically equivalent to $g^*(K_Z + B_Z + M_Z)$ where B_Z, M_Z are the discriminant and moduli divisors of adjunction, and the condition says that $B_Z + M_Z$ is not numerically trivial.

When X is of Fano type the theorem was already known [5, Theorem 1.4] (or [15] when the coefficients of B are in a fixed DCC set) in which case we can remove the non-product type assumption as it follows from the Fano type property. But the main point of the theorem is that we have replaced Fano type with the weaker property of not being of product type. The property allows one to reduce the theorem to boundedness of certain towers of

Mori fibre spaces which turns out to be a special case of Theorem 1.4. It it worth mentioning that the theorem also holds in the relative setting similar to 1.4 but for simplicity we prove the above version.

Boundedness of singularities on log Calabi-Yau fibrations. Understanding singularities on log Calabi-Yau fibrations is very important as it naturally appears in inductive arguments. The next statement gives a lower bound for lc thresholds on Fano type fibrations. In particular, it implies [20, Conjecture 1.13] as a special case.

Theorem 1.6. *Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number t depending only on d, r, ϵ satisfying the following. Let $(X, B) \to Z$ be any (d, r, ϵ)-Fano type fibration as in 1.1. If $P \geq 0$ is any $\mathbb{R}$-Cartier divisor on X such that either*

- *$f^*A + B - P$ is pseudo-effective, or*
- *$f^*A - K_X - P$ is pseudo-effective,*

then $(X, B + tP)$ is klt.

In particular, the theorem can be applied to any $0 \leq P \sim_{\mathbb{R}} f^*A + B$ or any $0 \leq P \sim_{\mathbb{R}} f^*A - K_X$ assuming P is $\mathbb{R}$-Cartier. To get a feeling for what the theorem says consider the case when Z is a curve; in this case the theorem implies that the multiplicities of each fibre of f are bounded from above (compare with the main result of [30] for del Pezzo fibrations over curves): indeed, for any closed point $z \in Z$ we can find $0 \leq Q \sim A$ so that z is a component of Q; then applying the theorem to $P := f^*Q$ implies that the multiplicities of the fibre of f over z are bounded.

One can derive the theorem from 1.3 and the results of [4]. However, in practice the theorem is proved together along with 1.3 in an intertwining inductive process.

On the other hand, a fundamental problem on singularities is a conjecture of Shokurov [6, Conjecture 1.2] (a special case of which is due to M$^{\rm c}$Kernan) which roughly says that the singularities on the base of a Fano type fibration are controlled by those on the total space. We will prove Shokurov's conjecture under some boundedness assumptions on the base (1.8) and prove a generalisation of the conjecture under some boundedness assumptions on the general fibres (1.9). These results are of independent interest but also closely related to the other results of this paper.

First we recall adjuction for fibrations also known as canonical bundle formula. If (X, B) is an lc pair and $f: X \to Z$ is a contraction with $K_X + B \sim_{\mathbb{R}} 0/Z$, then by [23][2] we can define a discriminant divisor B_Z and a moduli divisor M_Z so that we have

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).$$

This is a generalisation of the Kodaira canonical bundle formula. Let D be a prime divisor on Z. Let t be the lc threshold of f^*D with respect to (X, B) over the generic point of D. We then put the coefficient of D in B_Z to be $1 - t$. Having defined B_Z, we can find M_Z giving

$$K_X + B + M \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where M_Z is determined up to $\mathbb{R}$-linear equivalence. We call B_Z the *discriminant divisor of adjunction* for (X, B) over Z.

For any birational contraction $Z' \to Z$ from a normal variety there is a birational contraction $X' \to X$ from a normal variety so that the induced map $X' \dashrightarrow Z'$ is a morphism. Let $K_{X'} + B'$ be the pullback of $K_X + B$. We can similarly define $B_{Z'}, M_{Z'}$ for (X', B') over Z'. In this way we get the *discriminant b-divisor* $\mathbf{B}_Z$ *of adjunction* for (X, B) over Z.

The conjecture of Shokurov then can be stated as:

Conjecture 1.7. *Let d be a natural number and ϵ be a positive real number. Then there is a positive real number δ depending only on d, ϵ satisfying the following. Assume that (X, B) is a pair and $f: X \to Z$ is a contraction such that*

- *(X, B) is ϵ-lc of dimension d,*
- *$K_X + B \sim_{\mathbb{R}} 0/Z$, and*
- *$-K_X$ is big over Z.*

Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.

The next result says that Shokurov conjecture holds in the setting of Fano type fibrations. The strength of this result is in the fact that (similar to some of the other results above, e.g. 1.2) we are not assuming any boundedness condition on support of B along the general fibres of f.

Theorem 1.8. *Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number δ depending only on d, r, ϵ satisfying the following. Let $(X, B) \to Z$ be any (d, r, ϵ)-Fano type fibration as in 1.1. Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.*

This can be viewed as a relative version of Ambro's conjecture [4, Theorem 1.4] which is closely related to the BAB conjecture.

Next we prove a variant of Conjecture 1.7 which is weaker in the sense that we assume certain boundedness along the general fibres but it is stronger in the sense that we replace bigness of $-K_X$ over Z with a less restrictive condition. This is important for applications, eg proofs of 1.4 and 1.5.

Theorem 1.9. *Let d, v be natural numbers and ϵ be a positive real number. Then there is a positive real number δ depending only on d, v, ϵ satisfying the following. Assume that (X, B) is a pair and $f: X \to Z$ is a contraction such that*

- *(X, B) is ϵ-lc of dimension d,*
- *$K_X + B \sim_{\mathbb{R}} 0/Z$, and*
- *there is an integral divisor $G \geq 0$ with*
$$0 < \operatorname{vol}((\operatorname{Supp} B + G)|_F) < v$$
for the general fibres F of f.

Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.

Consider a minimal elliptic surface, that is, a smooth projective surface X endowed with an elliptic fibration $f: X \to Z$, that is, a contraction with $K_X \sim_{\mathbb{Q}} 0/Z$ (the general fibres are then elliptic curves). In general, f can have singular fibres of arbitrarily large multiplicity (cf. [31, Example 7.17]), hence the discriminant divisor B_Z can have coefficients arbitrarily close to 1. Here $(X, B) \to Z$ satisfies all the assumptions of the theorem with $B = 0$ and $\epsilon = 1$ except the last condition involving G. This illustrates the role of G in the theorem. Indeed, if we additionally assume that f has a multi-section of fixed degree l (that is, if there is a horizontal/Z curve G with degree of $G \to Z$ being l), then the discriminant divisor B_Z would have coefficients bounded away from 1.

Boundedness of relative-global complements. One of the key tools used in this paper is the theory of complements. We need to prove a more general version of the bounded "complements" statement of [4, Theorem 1.7]. Most likely this will be useful elsewhere. These complements are relative but are controlled globally.

Theorem 1.10. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $d, \mathfrak{R}$ satisfying the following. Assume*

- *(X,B) is a projective lc pair of dimension d,*
- *the coefficients of B are in $\mathfrak{R}$,*
- *M is a semi-ample Cartier divisor on X defining a contraction $f\colon X \to Z$,*
- *X is of Fano type over Z,*
- *$M - (K_X + B)$ is nef and big, and*
- *S is a non-klt centre of (X,B) with $M|_S \equiv 0$.*

Then there is a $\mathbb{Q}$-divisor $\Lambda \geq B$ such that

- *(X,Λ) is lc over a neighbourhood of $z := f(S)$, and*
- *$n(K_X + \Lambda) \sim (n+2)M$.*

We prove a stronger statement when singularities are milder which is proved along with the other results above.

Theorem 1.11. *Let d, r be natural numbers, ϵ be a positive real number, and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists natural numbers n, m depending only on $d, r, \epsilon, \mathfrak{R}$ satisfying the following. Assume that $(X,B) \to Z$ is a (d,r,ϵ)-Fano type fibration and that*

- *we have $0 \leq \Delta \leq B$ with coefficients in $\mathfrak{R}$, and*
- *$-(K_X + \Delta)$ is big over Z.*

Then there is a $\mathbb{Q}$-divisor $\Lambda \geq \Delta$ such that

- *(X,Λ) is klt, and*
- *$n(K_X + \Lambda) \sim mf^*A$.*

For example we can apply the theorem under the assumptions of 1.2 by taking $\Delta = 0$.

Announcement of results of a sequel paper. Here we state several results whose proofs will appear in a sequel paper joint with Di Cerbo and Svaldi. For simplicity we do not state them in their most general form.

Theorem 1.12. *Let d, p be natural numbers. Consider pairs (X,B) with the following properties:*

- *(X,B) is projective klt of dimension d,*
- *$p(K_X + B) \sim 0$, and*
- *X is rationally connected.*

Then the set of such (X,B) is log bounded up to isomorphism in codimension one.

In low dimension we have a stronger statement.

Theorem 1.13. *Let ϵ be a positive real number. Consider pairs (X,B) with the following properties:*

- *(X,B) is projective ϵ-lc of dimension 3,*
- *$-(K_X + B)$ is nef,*
- *$K_X \not\equiv 0$, and*
- *X is rationally connected.*

Then the set of such X is bounded up to isomorphism in codimension one.

The above results have applications to Calabi-Yau varieties with elliptic fibrations.

Theorem 1.14. *Let d, p be natural numbers. Consider varieties X with the following properties:*

- *X is projective klt of dimension d,*
- *$pK_X \sim 0$,*
- *there is an elliptic fibration $X \to Y$ admitting a rational section, and*
- *Y is rationally connected.*

Then the set of such X is bounded up to isomorphism in codimension one.

This generalises [11, Theorem 1.1] to every dimension.

Acknowledgements. This work was supported by a grant of the Leverhulme Trust. Part of it was done while visiting the University of Tokyo in March 2018 which was arranged by Yujiro Kawamata and Yoshinori Gongyo, and I would like to thank them for their hospitality. Thanks to Christopher Hacon for answering our questions regarding moduli part of adjunction. Thanks to Yifei Chen for numerous valuable comments. Thanks to Jingjun Han, Roberto Svaldi, and Yanning Xu for useful comments on an earlier version of this paper.

2. **Further results and remarks**

In this section we state further results and remarks working mostly in the context of generalised pairs.

A framework for classification of Fano fibrations. Here we illustrate how the results above can be used towards classification of Fano fibrations such as Mori fibre spaces in the context of birational classification of algebraic varieties. Suppose that we are given a normal projective variety Z with a very ample divisor A on it. The aim is to somehow classify a given set of Fano fibrations over Z. We naturally want to fix or bound certain invariants. Let d be a natural number and ϵ be a positive real number. Assume $\mathcal{P}$ is a set of contractions $f\colon X \to Z$ such that

- X is projective of dimension d with ϵ-lc singularities, and
- $-K_X$ is ample over Z.

For each non-negative integer l, let $\mathcal{P}_l$ be the set of all $X \to Z$ in $\mathcal{P}$ such that

$$l = \min\{a \in \mathbb{Z}^{\geq 0} \mid af^*A - K_X \text{ is ample}\}.$$

For example, $X \to Z \in \mathcal{P}_0$ means that $-K_X$ is ample, hence X is globally a Fano variety; $X \to Z \in \mathcal{P}_1$ means that $-K_X$ is not ample but $f^*A - K_X$ is ample. Thus we have a disjoint union

$$\mathcal{P} = \bigcup_{l \in \mathbb{Z}^{\geq 0}} \mathcal{P}_l.$$

For each $X \to Z$ in $\mathcal{P}_l$, we can choose a general $0 \leq B \sim_{\mathbb{Q}} lf^*A - K_X$, so that $(X, B) \to Z$ is a

$$(d, ((l+1)A)^{\dim Z}, \epsilon)\text{-Fano type fibration}$$

perhaps after a slight decrease of ϵ (we need to decrease only if $\epsilon = 1$). Thus the set of such X forms a bounded family, by Theorem 1.2. That is, we can write $\mathcal{P}$ as a disjoint union of bounded sets. The next step is to study each set $\mathcal{P}_l$ more closely to get a finer classification.

Lets look at the simplest non-trivial case of surfaces, that is, consider the set $\mathcal{P}$ of Mori fibre spaces $X \to Z = \mathbb{P}^1$ where $d = \dim X = 2$ and X is smooth. In this case it is well-known that $\mathcal{P}$ coincides with the sequence of Hirzebruch surfaces, that is, $\mathbb{P}^1$-bundles $f_i\colon X_i \to Z$ having a section E_i satisfying $E_i^2 = -i$, for $i = 0, 1, \dots$. Applying the divisorial adjunction formula gives $K_{X_i} \cdot E_i = i - 2$. Letting A be a point on Z and using the fact that the Picard group of X_i is generated by E_i and a fibre of f_i, it is easy to check that X_0 and X_1 are Fano, and $(i-1)f_i^*A - K_{X_i}$ is ample but $(i-2)f_i^*A - K_{X_i}$ is not ample for $i \geq 2$. Therefore, under the notation introduced above, we have

$$\mathcal{P}_0 = \{X_0 \to Z, X_1 \to Z\}, \text{ and } \mathcal{P}_l = \{X_{l+1} \to Z\} \text{ for } l \geq 1.$$

Of course we have used the classification of ruled surfaces over $\mathbb{P}^1$ but the point we want to make is that conversely studying $\mathcal{P}$ and each subset $\mathcal{P}_l$ closely will naturally lead us to the classification of ruled surfaces over $\mathbb{P}^1$.

Boundedness of generalised Fano type fibrations. We will prove some of the results stated above in the context of generalised pairs. For the basics of generalised pairs see [9] and [5] and the preliminaries below. A *generalised log Calabi-Yau fibration* consists of a generalised pair $(X, B+M)$ with generalised lc singularities and a contraction $X \to Z$ such that $K_X + B + M \sim_{\mathbb{R}} 0/Z$. We define generalised Fano type fibrations similar to 1.1.

Definition 2.1. *Let d, r be natural numbers and ϵ be a positive real number. A generalised (d, r, ϵ)-Fano type (log Calabi-Yau) fibration consists of a projective generalised pair $(X, B+M)$ with data $X' \to X$ and M', and a contraction $f\colon X \to Z$ such that we have:*

- *$(X, B+M)$ is generalised ϵ-lc of dimension d,*
- *$K_X + B + M \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-divisor L,*
- *$-K_X$ is big over Z, i.e. X is of Fano type over Z,*
- *A is a very ample divisor on Z with $A^{\dim Z} \leq r$, and*
- *$A - L$ is ample.*

Note that M' is assumed to be nef globally. We usually write $(X, B+M) \to Z$ to denote the fibration. Theorem 1.3 can be extended to the case of generalised pairs, that is:

Theorem 2.2. *Let d, r be natural numbers and ϵ, τ be positive real numbers. Consider the set of all generalised (d, r, ϵ)-Fano type fibrations $(X, B+M) \to Z$ such that*

- *we have $0 \leq \Delta \leq B$ whose non-zero coefficients are $\geq \tau$, and*
- *$-(K_X + \Delta)$ is big over Z.*

Then the set of such (X, Δ) is log bounded.

Singularities on generalised log Calabi-Yau fibrations. Theorem 1.6 also holds for generalised pairs, that is:

Theorem 2.3. *Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number t depending only on d, r, ϵ satisfying the following. Let $(X, B+M) \to Z$ be any generalised (d, r, ϵ)-Fano type fibration as in 2.1. If $P \geq 0$ is any $\mathbb{R}$-Cartier divisor on X such that either*

- *$f^*A + B + M - P$ is pseudo-effective, or*
- *$f^*A - K_X - P$ is pseudo-effective,*

then $(X, B + tP + M)$ is generalised klt.

In particular, the theorem can be applied to any $0 \leq P \sim_{\mathbb{R}} f^*A + B + M$ or any $0 \leq P \sim_{\mathbb{R}} f^*A - K_X$ assuming P is $\mathbb{R}$-Cartier.

Adjunction for fibrations also makes sense for generalised pairs. That is, if $(X, B+M) \to Z$ is a generalised log Calabi-Yau pair, then we can define a discriminant divisor B_Z and a moduli divisor M_Z giving

$$K_X + B + M \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z).$$

Moreover, for any birational contraction $Z' \to Z$ from a normal variety we can define the discriminant divisor $B_{Z'}$ whose pushdown to Z is just B_Z. In this way we get the discriminant b-divisor $\mathbf{B}_Z$. See 6.1 for more details.

Now we state a generalised version of Shokurov's conjecture 1.7.

Conjecture 2.4. *Let d be a natural number and ϵ be a positive real number. Then there is a positive real number δ depending only on d, ϵ satisfying the following. Assume that $(X, B+M)$ is a generalised pair with data $X' \to X \xrightarrow{f} Z$ and M' where f is a contraction such that*

- *$(X, B+M)$ is generalised ϵ-lc of dimension d,*
- *$K_X + B + M \sim_{\mathbb{R}} 0/Z$, and*
- *$-K_X$ is big over Z.*

Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.

Note that in particular we are assuming that M' is nef over Z as this is part of the definition of a generalised pair. The next result says that 2.4 holds in the setting of generalised Fano type fibrations.

Theorem 2.5. *Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number δ depending only on d, r, ϵ satisfying the following. Let $(X, B+M) \to Z$ be any generalised (d, r, ϵ)-Fano type fibration as in 2.1. Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.*

Now we propose a conjecture which is stronger than 2.4 in the sense that we replace the bigness of $-K_X$ over Z with a weaker condition.

Conjecture 2.6. *Let d, v be natural numbers and ϵ be a positive real number. Then there is a positive real number δ depending only on d, v, ϵ satisfying the following. Assume that $(X, B+M)$ is a generalised pair with data $X' \to X \xrightarrow{f} Z$ and M' where f is a contraction such that*

- *$(X, B+M)$ is generalised ϵ-lc of dimension d,*
- *$K_X + B + M \sim_{\mathbb{R}} 0/Z$, and*
- *there is an integral divisor $G \geq 0$ on X with*

$$0 < \mathrm{vol}((B+M+G)|_F) < v$$

  for the general fibres F of f.

Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.

When $-K_X$ is big over Z, the general fibres F belong to a bounded family by [4], so $\mathrm{vol}((B+M)|_F) = \mathrm{vol}(-K_X|_F)$ is positive and bounded from above, hence in this case we can take $G = 0$. That is, 2.4 is a special case of 2.6.

The next statement says that Conjecture 2.6 holds if we put some boundedness assumptions on the general fibres.

Theorem 2.7. *Let d, v, p be natural numbers and ϵ be a positive real number. Then there is a positive real number δ depending only on d, v, p, ϵ satisfying the following. Assume that*

$(X, B+M)$ is a generalised pair with data $X' \to X \xrightarrow{f} Z$ and M' where f is a contraction such that

- *$(X, B+M)$ is generalised ϵ-lc of dimension d,*
- *$K_X + B + M \sim_{\mathbb{R}} 0/Z$,*
- *there is an integral divisor $G \geq 0$ on X with*

$$0 < \mathrm{vol}(((\operatorname{Supp} B) + M + G)|_F) < v$$

  for the general fibres F of f, and
- *pM' is b-Cartier.*

Then the discriminant b-divisor $\mathbf{B}_Z$ has coefficients in $(-\infty, 1-\delta]$.

The b-Cartier condition of pM' means that the pullback of pM' to some resolution of X is a Cartier divisor.

Corollary 2.8. *Let p be a natural number and τ be a positive real number. Then Conjectures 2.4 and 2.6 hold for those $(X, B+M)$ which in addition satisfy:*

- *any horizontal$/Z$ component of B has coefficient $\geq \tau$, and*
- *pM' is b-Cartier.*

Note that we allow the case when B has no horizontal$/Z$ components.

Plan of the paper. We will prove Theorem 1.10 in Section 4, Theorems 1.2, 1.3, 2.2, 1.6, 1.11 in Section 5, and Theorems 1.8, 1.9, 2.3, 2.5, 2.7 and Corollary 2.8 in Section 6, and Theorems 1.4 and 1.5 in Section 7.

3. **Preliminaries**

All the varieties in this paper are quasi-projective over a fixed algebraically closed field k of characteristic zero unless stated otherwise.

3.1. **Numbers.** Let $\mathfrak{R}$ be a subset of $[0, 1]$. Following [31, 3.2] we define

$$\Phi(\mathfrak{R}) = \Big\{1 - \frac{r}{m} \mid r \in \mathfrak{R},\ m \in \mathbb{N}\Big\}$$

to be the set of *hyperstandard multiplicities* associated to $\mathfrak{R}$.

3.2. **Contractions.** By a *contraction* we mean a projective morphism $f \colon X \to Y$ of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ (f is not necessarily birational). In particular, f is surjective and has connected fibres.

3.3. **Divisors.** Let X be a variety. If D is a prime divisor on birational models of X whose centre on X is non-empty, then we say D is a prime divisor *over* X. If X is normal and M is an $\mathbb{R}$-divisor on X, we let

$$|M|_{\mathbb{R}} = \{N \geq 0 \mid N \sim_{\mathbb{R}} M\}.$$

Recall that $N \sim_{\mathbb{R}} M$ means that $N - M = \sum r_i \operatorname{Div}(\alpha_i)$ for certain real numbers r_i and rational functions α_i. When all the r_i can be chosen to be rational numbers, then we write $N \sim_{\mathbb{Q}} M$. We define $|M|_{\mathbb{Q}}$ similarly by replacing $\sim_{\mathbb{R}}$ with $\sim_{\mathbb{Q}}$.

Assume $\rho \colon X \dashrightarrow Y/Z$ is a rational map of normal varieties projective over a base variety Z. For an $\mathbb{R}$-Cartier divisor L on Y we define the pullback $\rho^* L$ as follows. Take a common resolution $\phi \colon W \to X$ and $\psi \colon W \to Y$. Then let $\rho^* L := \phi_* \psi^* L$. It is easy to see that this does not depend on the choice of the common resolution as any two such resolutions are dominated by a third one.

Lemma 3.4. *Assume $Y \to X$ is a contraction of normal projective varieties, C is a nef $\mathbb{R}$-divisor on Y and A is the pullback of an ample $\mathbb{R}$-divisor on X. If C is semi-ample over X, then $C + aA$ is semi-ample (globally) for any real number $a > 0$.*

Proof. Since C is semi-ample over X, it defines a contraction $\phi\colon Y \to Z/X$ to a normal projective variety. Replacing Y with Z and replacing C, A with ϕ_*C, ϕ_*A, respectively, we can assume C is ample over X. Pick $a > 0$. Now $C + bA$ is ample for some $b \gg a$ because A is the pullback of an ample divisor on X. Since C is globally nef,

$$C + tbA = (1 - t)C + t(C + bA)$$

is ample for any $t \in (0, 1]$. In particular, taking $t = \frac{a}{b}$ we see that $C + aA$ is ample. □

3.5. **Linear systems.** Let X be a normal projective variety and M be an integral Weil divisor on X. A *sub-linear system* $L \subseteq |M|$ is given by some linear subspace $V \subseteq \mathbb{P}(H^0(M))$, that is,

$$L = \{\mathrm{Div}(\alpha) + M \mid \alpha \in V\}.$$

The general members of L, by definition, are those $\mathrm{Div}(\alpha) + M$ where α is in some given non-empty open subset $W \subseteq V$ (open in the Zariski topology). Being a general member then depends on the choice of W but we usually shrink it if necessary without notice.

Lemma 3.6. *Let X be a normal projective variety, M be an integral Weil divisor on X, and $L \subseteq |M|$ be a sub-linear system. Assume that $x \in X$ is a smooth closed point and that some member of L is smooth at x. Then a general member of L is smooth at x.*

Proof. We can assume that every member of L passes through x otherwise the general members do not pass through x, hence the statement holds trivially. By assumption, there is $D \in L$ such that D is smooth at x. Let H be a general hypersurface section of X passing through x. Then H and $H|_D$ are both smooth at x. In particular, $D|_H$ is also smooth at x because, in a neighbourhood of x, both $H|_D$ and $D|_H$ considered as schemes coincide with the scheme-theoretic intersection $H \cap D$.

We can choose H so that it is not a component of any member of L (e.g. enough to choose H so that $H^d > H^{d-1} \cdot M$). Let $N := L|_H$ be the restriction of L to H, that is, N consists of divisors $E|_H$ where $E \in L$. Although E may not be $\mathbb{Q}$-Cartier but $E|_H$ is well-defined as a Weil divisor by our choice of H. Moreover, again by our choice of H, the map $H^0(M) \to H^0(M|_H)$ is injective, hence induces a map $\mathbb{P}(H^0(M)) \to \mathbb{P}(H^0(M|_H))$, and N is given by the image of V under this map.

By construction, $D|_H \in N$ is smooth at x. Then by induction a general member of N is smooth at x. This is possible only if a general member of L is smooth at x which can be seen as follows. Let E be a general member of L and let h, e be the defining equations of H, E near x. Since H and $E|_H$ are smooth at x, we can choose a system of local parameters $e, t_2, \dots, t_r$ for H at x where $r = \dim H$. But then $h, e, t_2, \dots, t_r$ is a system of local parameters for X at x, hence E is smooth at x. □

Lemma 3.7. *Let X be a normal projective variety and A be a very ample divisor on X. For each pair of closed points $x, y \in X$, let $L_{x,y}$ be the sub-linear system of $|2A|$ consisting of the members that pass through both x, y. Then there is a non-empty open subset $U \subseteq X$ such that for any pair of closed points $x, y \in U$, a general member of $L_{x,y}$ is smooth at both x, y.*

Proof. By definition, a general member of $|A|$ is the element given by a section in some non-empty open subset W of $\mathbb{P}(H^0(A))$. Perhaps after shrinking W, we can assume that the restriction of these general members to the smooth locus of X are smooth.

We claim that there is a finite set Π of closed points of X (depending on W) such that for each closed point $x \in X \setminus \Pi$ we can find a general member of $|A|$ passing through x: indeed since A is very ample, the set H_z of elements of $\mathbb{P}(H^0(A))$ vanishing at a given closed point z is a hyperplane, and for distinct points z, z' we have $H_z \neq H_{z'}$; but there are at most finitely many z with $W \cap H_z = \emptyset$ because the complement of W in $\mathbb{P}(H^0(A))$ is a proper closed set; hence for any closed point x other than those finite set we can find an element of W vanishing at x which proves the claim. Thus there is a non-empty open subset U of the smooth locus of X such that for each closed point $x \in U$ we can find a member of $|A|$ passing through x which is smooth at x.

Now pick a closed point $x \in U$ and let L_x be the sub-linear system of $|A|$ consisting of members passing through x. By the above arguments some member of L_x is smooth at x, hence a general member of L_x is also smooth at x, by Lemma 3.6. In particular, for any other closed point $y \in U$, we can pick a member of L_x smooth at x but not containing y.

Now let $x, y \in U$ be a pair of distinct closed points and let $L_{x,y}$ be the sub-linear system of $|2A|$ consisting of members passing through x, y. By the previous paragraph, there exists a member D (resp. E) of $|A|$ which passes through and smooth at x (resp. y) but not containing y (resp. x). Then $D + E$ is a member of $L_{x,y}$ passing through x, y and smooth at both x, y. Therefore, a general member of $L_{x,y}$ is smooth at both x, y, by Lemma 3.6. □

3.8. **Pairs and singularities.** A *pair* (X, B) consists of a normal variety X and an $\mathbb{R}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $\phi \colon W \to X$ be a log resolution of (X, B) and let

$$K_W + B_W = \phi^*(K_X + B).$$

The *log discrepancy* of a prime divisor D on W with respect to (X, B) is $1 - \mu_D B_W$ and it is denoted by $a(D, X, B)$. We say (X, B) is *lc* (resp. *klt*)(resp. *ϵ-lc*) if $a(D, X, B)$ is ≥ 0 (resp. > 0)(resp. $\geq \epsilon$) for every D. Note that if (X, B) is ϵ-lc, then automatically $\epsilon \leq 1$ because $a(D, X, B) = 1$ for almost all D.

A *non-klt place* of (X, B) is a prime divisor D over X, that is, on birational models of X, such that $a(D, X, B) \leq 0$. A *non-klt centre* is the image on X of a non-klt place.

Sub-pairs and their singularities are defined similarly by letting the coefficients of B to be any real number. In this case instead of lc, klt, etc, we say sub-lc, sub-klt, etc.

Lemma 3.9. *Let (X, B) be a projective ϵ-lc pair for some $\epsilon > 0$ and $f \colon X \to Z$ be a contraction. Let A be a very ample divisor on Z. For each pair of closed points $z, z' \in Z$, let $L_{z,z'}$ be the sub-linear system of $|2A|$ consisting of the members that pass through z, z'. Then there is a non-empty open subset $U \subseteq Z$ such that if $z, z' \in U$ are closed points and if H is a general member of $L_{z,z'}$, then $(X, B + G)$ is a plt pair where $G := f^*H$. In particular, G is normal and (G, B_G) is an ϵ-lc pair where*

$$K_G + B_G := (K_X + B + G)|_G.$$

Proof. Let U be as in Lemma 3.7 chosen for $|2A|$ on Z. We can assume that U is contained in the smooth locus of Z. Let $\phi \colon W \to X$ be a log resolution of (X, B) and let Σ be the union of the exceptional divisors of ϕ and the birational transform of $\operatorname{Supp} B$. Shrinking U we can assume that for any stratum S of (W, Σ) the morphism $S \to Z$ is smooth over U. A stratum of (W, Σ) is either W itself or an irreducible component of $\bigcap_{i \in I} D_i$ for some

$I \subseteq \{1, \ldots, r\}$ where $D_1, \ldots, D_r$ are the irreducible components of Σ. For each stratum S we can assume that either $S \to Z$ is surjective or that its image is contained in $Z \setminus U$.

Pick closed points $z, z' \in U$ and a general member H of $L_{z,z'}$, and let $G = f^*H$ and $E = \phi^*G$. We claim that $(W, \Sigma + E)$ is log smooth. Let S be a stratum of (W, Σ). Since $L_{z,z'}$ is base point free outside z, z' by definition of $L_{z,z'}$, the pullback of $L_{z,z'}$ to S is base point free outside the fibres of $S \to Z$ over z, z'. Thus any singular point of $E|_S$ (if there is any) is mapped to z or z'. In particular, if $S \to Z$ is not surjective, then $E|_S$ is smooth because in this case $E|_S$ has no point mapping to either z or z'. Assume that $S \to Z$ is surjective. Then $E|_S \to H$ is surjective. Moreover, by our choice of U, the fibres of $E|_S \to H$ over z, z' are both smooth as they coincide with the fibres of $S \to Z$ over z, z'. Therefore, $E|_S$ is smooth because H is smooth at z, z' by the choice of U and by Lemma 3.7. To summarise we have shown that $E|_S$ is smooth for each stratum S of (W, Σ). This implies that $(W, \Sigma + E)$ is log smooth.

Let $K_W + B_W$ be the pullback of $K_X + B$. Then each coefficient of B_W is $\leq 1 - \epsilon$. Since $K_W + B_W + E$ is the pullback of $K_X + B + G$, we deduce that $(X, B + G)$ is plt, hence G is normal [27, Proposition 5.51]. On the other hand,

$$K_E + B_E := K_E + B_W|_E = (K_W + B_W + E)|_E$$

is the pullback of

$$K_G + B_G := (K_X + B + G)|_G.$$

Moreover, since $(W, B_W + E)$ is log smooth and since E is not a component of B_W, the coefficients of B_E are each at most $1 - \epsilon$. Therefore, (G, B_G) is an ϵ-lc pair.

□

3.10. **Rational approximation of boundary divisors.**

Lemma 3.11. *Let (X, B) be an ϵ-lc pair and $X \to Z$ be a contraction such that $K_X + B \sim_{\mathbb{R}} 0/Z$. Then for each positive real number δ we can find $\mathbb{Q}$-boundaries B_i and real numbers $r_i > 0$ such that*

- $\sum r_i = 1$,
- $K_X + B = \sum r_i(K_X + B_i)$,
- (X, B_i) *is* $\frac{\epsilon}{2}$*-lc,*
- $K_X + B_i \sim_{\mathbb{Q}} 0/Z$,
- $\operatorname{Supp} B_i = \operatorname{Supp} B$*, and*
- *the coefficients of* $B - B_i$ *are in* $[-\delta, \delta]$.

Proof. If B is a $\mathbb{Q}$-boundary, then the statement holds trivially by taking $r_1 = 1$ and $B_1 = B$. We can then assume that B is not a $\mathbb{Q}$-boundary, in particular, $B \neq 0$. Write $B = \sum_1^s b_l D_l$ where D_l are the irreducible components of B. Since $K_X + B \sim_{\mathbb{R}} 0/Z$, we can write

$$K_X + B = K_X + \sum_1^s b_l D_l = \sum_1^p \alpha_j \operatorname{Div}(f_j) + \sum_1^q \beta_k P_k$$

where α_j, β_k are real numbers, f_j are rational functions, and P_k are pullbacks of Cartier divisors on Z. Consider the affine space $\mathbb{A}^{s+p+q}$ with coordinates

$$u_1, \cdots, u_s, v_1, \cdots, v_p, w_1, \cdots, w_q.$$

Let $H \subset \mathbb{A}^{s+p+q}$ be the set of points satisfying

$$K_X + \sum_1^s u_l D_l = \sum_1^p v_j \operatorname{Div}(f_j) + \sum_1^q w_k P_k.$$

Then H is non-empty as it contains the point

$$\mathbf{x} := (b_1, \cdots, b_s, \alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q).$$

Moreover, H is affine, that is, if $\mathbf{x}_i \in H$ and if $\sum e_i = 1$ where e_i are real numbers, then $\sum e_i \mathbf{x}_i \in H$. Furthermore, since $K_X, D_l, \mathrm{Div}(f_j), P_k$ are all integral divisors, H is a rational affine subspace, that is, it is generated by finitely many points with rational coordinates. In particular, we can choose $\mathbf{x}_i \in H$ with rational coordinates and real numbers $r_i > 0$ with $\sum r_i = 1$ such that $\mathbf{x} = \sum r_i \mathbf{x}_i$, and we can assume that the coordinates of $\mathbf{x}_i$ are arbitrarily close to those of $\mathbf{x}$. The first s coordinates of $\mathbf{x}_i$ define B_i which satisfy all the properties of the lemma.

□

3.12. **Fano type varieties.** Assume X is a variety and $X \to Z$ is a contraction. We say X is *of Fano type over Z* if there is a boundary C such that (X, C) is klt and $-(K_X + C)$ is ample over Z (or equivalently, nef and big over Z). This is equivalent to having a boundary B such that (X, B) is klt, $K_X + B \sim_{\mathbb{R}} 0/Z$ and B is big over Z. By [8], we can run MMP over Z on any $\mathbb{R}$-Cartier divisor M on X and the MMP ends with a minimal model or a Mori fibre space for M.

Lemma 3.13. *Assume $f \colon X \to Z$ is a contraction of normal projective varieties, L is an $\mathbb{R}$-divisor on X and A is an ample $\mathbb{R}$-divisor on Z. If L is pseudo-effective and X is of Fano type over Z, then $|L + af^*A|_{\mathbb{R}} \neq \emptyset$ for any real number $a > 0$.*

Proof. Since L is pseudo-effective and X is of Fano type over Z, L has a minimal model over Z. Replacing X with the minimal model we can assume L is semi-ample over Z. Thus L defines a contraction $X \to Y/Z$ and L is the pullback of an ample/Z $\mathbb{R}$-divisor N on Y. Then $N + bg^*A$ is ample for any $b \gg 0$ where g denotes $Y \to Z$. Since L is pseudo-effective, N is pseudo-effective, hence

$$|N + tbg^*A|_{\mathbb{R}} = |(1-t)N + t(N + bg^*A)|_{\mathbb{R}} \neq \emptyset$$

for any $t \in (0, 1]$. In particular, if $b > a > 0$, then letting $t = \frac{a}{b}$ we see that $|N + ag^*A|_{\mathbb{R}} \neq \emptyset$ which in turn implies $|L + af^*A|_{\mathbb{R}} \neq \emptyset$.

□

3.14. **Complements.** Let (X, B) be a pair and let $X \to Z$ be a contraction. A *strong n-complement* of $K_X + B$ over a point $z \in Z$ is of the form $K_X + B^+$ such that over some neighbourhood of z we have the following properties:

- (X, B^+) is lc,
- $n(K_X + B^+) \sim 0$, and
- $B^+ \geq B$.

When Z is a point, we just say that $K_X + B^+$ is a strong n-complement of $K_X + B$. We recall one of the main results of [5].

Theorem 3.15 ([5, Theroem 1.7])**.** *Let d be a natural number and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension d,*
- *the coefficients of B are in $\Phi(\mathfrak{R})$,*
- *X is of Fano type, and*
- *$-(K_X + B)$ is nef.*

Then there is a strong n-complement K_X+B^+ of K_X+B. Moreover, the complement is also a strong mn-complement for any $m\in\mathbb{N}$.

3.16. **Bounded families of pairs.** A *couple* (X,D) consists of a normal projective variety X and a divisor D on X whose non-zero coefficients are all equal to 1, i.e. D is a reduced divisor. The reason we call (X,D) a couple rather than a pair is that we are concerned with D rather than K_X+D and we do not want to assume K_X+D to be $\mathbb{Q}$-Cartier or with nice singularities. Two couples (X,D) and (X',D') are isomorphic (resp. isomorphic in codimension one) if there is an isomorphism $X\to X'$ (resp. birational map $X\dashrightarrow X'$ which is an isomorphism in codimension one) mapping D onto D' (resp. such that D is the birational transform of D').

We say that a set $\mathcal{P}$ of couples is *birationally bounded* if there exist finitely many projective morphisms $V^i\to T^i$ of varieties and reduced divisors C^i on V^i such that for each $(X,D)\in\mathcal{P}$ there exist an i, a closed point $t\in T^i$, and a birational isomorphism $\phi\colon V^i_t\dashrightarrow X$ such that (V^i_t,C^i_t) is a couple and $E\le C^i_t$ where V^i_t and C^i_t are the fibres over t of the morphisms $V^i\to T^i$ and $C^i\to T^i$ respectively, and E is the sum of the birational transform of D and the reduced exceptional divisor of ϕ. We say $\mathcal{P}$ is *bounded* if we can choose ϕ to be an isomorphism.

We say that a set $\mathcal{P}$ of couples is *bounded up to isomorphism in codimension one* if there is a bounded set $\mathcal{P}'$ of couples such that each $(X,D)\in\mathcal{P}$ is isomorphic in codimension one with some $(X',D')\in\mathcal{P}'$.

A set $\mathcal{R}$ of projective pairs (X,B) is said to be *log birationally bounded* (resp. *log bounded*, etc) if the set of the corresponding couples $(X,\operatorname{Supp}B)$ is birationally bounded (resp. bounded, etc). Note that this does not put any condition on the coefficients of B, e.g. we are not requiring the coefficients of B to be in a finite set. If $B=0$ for all the $(X,B)\in\mathcal{R}$ we usually remove the log and just say the set is birationally bounded (resp. bounded, etc).

Lemma 3.17. *Let $\mathcal{P}$ be a bounded set of couples and $\mathfrak{R}\subset[0,1]$ be a finite set of rational numbers. Then there is a natural number I depending only on $\mathcal{P},\mathfrak{R}$ such that if*

- *(X,B) is a projective klt pair,*
- *the coefficients of B are in $\mathfrak{R}$, and*
- *$(X,\operatorname{Supp}B)\in\mathcal{P}$,*

then $I(K_X+B)$ is Cartier.

Proof. When K_X is $\mathbb{Q}$-Cartier, the lemma follows from [5, Lemma 2.24]. The proof of the general case is actually quite similar to the proof of [5, Lemma 2.24]. We write the details for convenience.

Assume there is a sequence (X_i,B_i) of pairs as in the lemma such that if I_i is the smallest natural number so that $I_i(K_{X_i}+B_i)$ is Cartier, then the I_i form a strictly increasing sequence of numbers. Perhaps after replacing the sequence with a subsequence, by [5, Lemma 2.21], we can assume there is a projective morphism $V\to T$ of varieties, a reduced divisor C on V, and a dense set of closed points $t_i\in T$ such that X_i is the fibre of $V\to T$ over t_i and $\operatorname{Supp}B_i$ is the fibre of $C\to T$ over t_i. Since X_i are normal, replacing V with its normalisation and replacing C with its inverse image with reduced structure, we can assume V is normal.

Let $\phi\colon W\to V$ be a resolution of V and let Δ be the reduced exceptional divisor of ϕ. Running an MMP$/V$ on $K_W+\Delta$ with scaling of an ample divisor, we reach a model V' on which $K_{V'}+\Delta'$ is a limit of movable$/V$ divisors. Let $V'\to V$ be the induced morphism and

X_i', Δ_i' be the fibres of $V' \to T$ and $\Delta' \to T$ over t_i, respectively (note that $\Delta_i' = \Delta'|_{X_i'}$ and since we work in characteristic zero, we can assume Δ_i' is reduced). Now we can assume X_i' are general fibres of $V' \to T$, hence Δ_i' is the reduced exceptional divisor of $X_i' \to X_i$. Since (X_i, B_i) is klt, we can write the pullback of $K_{X_i} + B_i$ to X_i' as $K_{X_i'} + B_i'$ where B_i' has coefficients strictly less than 1. But then since X_i' are general fibres,

$$\Delta_i' - B_i' = K_{X_i'} + \Delta_i' - (K_{X_i'} + B_i') \sim_{\mathbb{Q}} K_{X_i'} + \Delta_i'/X_i$$

is a limit of movable/X_i divisors, hence $\Delta_i' - B_i' \leq 0$ by the general negativity lemma [7, Lemma 3.3] which in turn implies $\Delta_i' = 0$ as Δ_i' is reduced. Thus $X_i' \to X_i$ is a small contraction.

There is a $\mathbb{Q}$-divisor $\Gamma_i' \geq 0$ on X_i' which is anti-ample over X_i. Rescaling it we can assume $(X_i', B_i' + \Gamma_i')$ is klt. In particular, $X_i' \to X_i$ is a $K_{X_i'} + B_i' + \Gamma_i'$-negative contraction of an extremal face of the Mori-Kleiman cone of X_i'. Thus by the cone theorem [27, Theorem 3.7], the Cartier index of $K_{X_i'} + B_i'$ and $K_{X_i} + B_i$ are the same.

If $C' \subset V'$ denotes the birational transform of C, then $\operatorname{Supp} B_i'$ is the fibre of $C' \to T$ over t_i. Thus replacing V, C with V', C' we can replace (X_i, B_i) with (X_i', B_i'), hence assume V is $\mathbb{Q}$-factorial. Moreover, since X_i is a general fibre, $K_{X_i} = K_V|_{X_i}$ which shows that the Cartier index of K_{X_i} is bounded, so we need to bound the Cartier index of B_i.

Pick I so that IC is Cartier. Let $D_i = \operatorname{Supp} B_i$. Then $D_i = C|_{X_i}$, hence ID_i is Cartier. This gives a contradiction if B_i are all irreducible. In general, let $h_i \in \mathbb{Q}$ be the largest number such that $B_i - h_i D_i \geq 0$. Then $B_i - h_i D_i$ has at least one component less than B_i, and the coefficients of $B_i - h_i D_i$ belong to some finite set which is independent of i. Thus we can apply induction on the number of components of B_i (which is a bounded number) to derive a contradiction.

$\square$

3.18. **BAB and lower bound on lc thresholds.** We recall some of the main results of [4] regarding boundedness of Fano's and lc thresholds in families.

Theorem 3.19 ([4, Theorem 1.1])**.** *Let d be a natural number and ϵ a positive real number. Then the projective varieties X such that*
- *(X, B) is ϵ-lc of dimension d for some boundary B, and*
- *$-(K_X + B)$ is nef and big,*

form a bounded family.

On the other hand, lc thresholds are bounded from below under suitable assumptions:

Theorem 3.20 ([4, Theorem 1.6])**.** *Let d, r be natural numbers and ϵ be a positive real number. Then there is a positive real number t depending only on d, r, ϵ satisfying the following. Assume*
- *(X, B) is a projective ϵ-lc pair of dimension d,*
- *A is a very ample divisor on X with $A^d \leq r$,*
- *$A - B$ is ample, and*
- *$M \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor with $|A - M|_{\mathbb{R}} \neq \emptyset$.*

Then

$$\operatorname{lct}(X, B, |M|_{\mathbb{R}}) \geq \operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

Note that the conditions on A, B, M essentially say that X belongs to a bounded family and that the "degrees" of B, M with respect A are bounded.

3.21. **b-divisors.** We recall some definitions regarding b-divisors but not in full generality. Let X be a variety. A *b-$\mathbb{R}$-Cartier b-divisor over X* is the choice of a projective birational morphism $Y \to X$ from a normal variety and an $\mathbb{R}$-Cartier divisor M on Y up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal variety and an $\mathbb{R}$-Cartier divisor M' defines the same b-$\mathbb{R}$-Cartier b-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullbacks of M and M' coincide.

A b-$\mathbb{R}$-Cartier b-divisor represented by some $Y \to X$ and M is *b-Cartier* if M is b-Cartier, i.e. its pullback to some resolution is Cartier.

3.22. **Generalised pairs.** A *generalised pair* consists of

- a normal variety X equipped with a projective morphism $X \to Z$,
- an $\mathbb{R}$-divisor $B \geq 0$ on X, and
- a b-$\mathbb{R}$-Cartier b-divisor over X represented by some projective birational morphism $X' \overset{\phi}{\to} X$ and $\mathbb{R}$-Cartier divisor M' on X'

such that M' is nef$/Z$ and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* M'$.

We usually refer to the pair by saying $(X, B+M)$ is a generalised pair with data $X' \to X \to Z$ and M', and call M' the nef part. Note that our notation here, which seems to be preferred by others, is slightly different from those in [9][5].

We now define generalised singularities. Replacing X' we can assume ϕ is a log resolution of (X, B). We can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

for some uniquely determined B'. For a prime divisor D on X' the *generalised log discrepancy* $a(D, X, B+M)$ is defined to be $1 - \mu_D B'$. We say $(X, B+M)$ is *generalised lc* (resp. *generalised klt*)(resp. *generalised ϵ-lc*) if for each D the generalised log discrepancy $a(D, X, B+M)$ is ≥ 0 (resp. > 0)(resp. $\geq \epsilon$).

A *generalised non-klt centre* of a generalised pair $(X, B+M)$ is the image on X of a prime divisor D over X with $a(D, X, B+M) \leq 0$, and the *generalised non-klt locus* of the generalised pair is the union of all the generalised non-klt centres.

Generalised sub-pairs and their singularities are similarly defined by allowing the coefficients of B to be any real number.

To state the next lemma we need the notion of *very exceptional divisors*. Given a contraction $f\colon X \to Z$ of normal varieties and an $\mathbb{R}$-divisor N on X, we say that N is very exceptional over Z if $\operatorname{Supp} N$ is vertical over Z and that for any prime divisor D on Z there is a prime divisor S on X mapping onto D but such that S is not a component of N.

Lemma 3.23. *Let $(X, B+M)$ be a $\mathbb{Q}$-factorial generalised klt generalised pair with data $X' \to X \to Z$ and M'. Assume $K_X + B + M \sim_{\mathbb{R}} N/Z$ where $N \geq 0$ is very exceptional over Z. Then any MMP on $K_X + B + M$ over Z with scaling of an ample divisor terminates with a model Y on which $K_Y + B_Y + M_Y \sim_{\mathbb{R}} 0/Z$.*

Proof. This proof is similar to that of [9, Theorem 1.8]. Let $C \geq 0$ be an ample $\mathbb{R}$-divisor such that $(X, B+C+M)$ is generalised klt (with the same nef part M') and $K_X + B + C + M$ is ample over Z. Run the MMP on $K_X + B + M$ over Z with scaling of C (as defined before [9, Lemma 4.4]). This consists of a sequence $X_i \dashrightarrow X_{i+1}$ of divisorial contractions and flips where $X = X_1$. Let λ_i be the numbers obtained in the process so that $K_{X_i} + B_i + \lambda_i C_i + M_i$ is nef over Z. If $\lambda = \lim \lambda_i > 0$, then the MMP terminates by [9, Lemma 4.4] because the MMP is also an MMP on $K_X + B + \frac{\lambda}{2} C + M$, so in this case replacing X with the minimal model we can assume $K_X + B + M$ is nef over Z. If $\lambda = \lim \lambda_i = 0$, then replacing X with X_i for some $i \gg 0$, we can assume that $K_X + B + M$ is a limit of movable$/Z$ $\mathbb{R}$-divisors.

In either case (that is, $\lambda > 0$ or $\lambda = 0$), for any prime divisor S on X and for the general curves Γ of S contracted over Z, we have $N \cdot \Gamma \geq 0$. Therefore, $N = 0$ by the general negativity lemma [7, Lemma 3.3] as N is very exceptional over Z. In other words the MMP contracts N.

□

The next lemma is useful for reducing problems about generalised log Calabi-Yau fibrations to usual log Calabi-Yau fibrations.

Lemma 3.24. *Let d, r be natural numbers and ϵ be a positive real number. Let $(X, B + M) \to Z$ be a generalised (d, r, ϵ)-Fano type fibration (as in 2.1) such that $-(K_X + \Delta)$ is big over Z for some $0 \leq \Delta \leq B$. Then we can find a boundary $\Theta \geq \Delta$ such that $(X, \Theta) \to Z$ is a $(d, r, \frac{\epsilon}{2})$-Fano type fibration.*

Proof. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Since $B - \Delta$ is effective and M is pseudo-effective (as it is the pushdown of a nef divisor), $B - \Delta + M$ is pseudo-effective. Moreover, since $-(K_X + \Delta)$ is big over Z,
$$B - \Delta + M \sim_{\mathbb{R}} -(K_X + \Delta)/Z$$
is big over Z. Thus
$$t(B - \Delta + M) + f^*A$$
is globally big for any sufficiently small $t > 0$, hence $B - \Delta + M + f^*A$ is globally big as $B - \Delta + M$ is pseudo-effective.

Let $\phi\colon X' \to X$ be a log resolution of (X, B) on which the nef part M' of $(X, B + M)$ resides. Write
$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$
and
$$K_{X'} + \Delta' = \phi^*(K_X + \Delta).$$
Then
$$B' - \Delta' + M' = \phi^*(B - \Delta + M).$$
Since $(X, B + M)$ is generalised ϵ-lc, the coefficients of B' do not exceed $1 - \epsilon$. In addition, since M' is nef and $B \geq \Delta$, we have $B' \geq \Delta'$ by the negativity lemma applied to $B' - \Delta'$ and the morphism ϕ.

Since $B - \Delta + M + f^*A$ is big, we can write
$$B' - \Delta' + M' + \phi^* f^*A = \phi^*(B - \Delta + M + f^*A) \sim_{\mathbb{R}} G' + H'$$
where $G' \geq 0$ and H' is ample. Replacing ϕ we can assume ϕ is a log resolution of $(X, B + \phi_* G')$. Pick a small real number $\alpha > 0$ and pick a general
$$0 \leq R' \sim_{\mathbb{R}} \alpha H' + (1 - \alpha) M'.$$
Let
$$\Theta' := \Delta' + (1 - \alpha)(B' - \Delta') + \alpha G' + R'.$$
We can make the above choices so that (X', Θ') is log smooth. Since
$$\Delta' \leq \Delta' + (1 - \alpha)(B' - \Delta') \leq \Delta' + (B' - \Delta') = B',$$
we have
$$\Delta' \leq \Theta' \leq B' + \alpha G' + R'.$$
In particular, we can choose α and R' so that the coefficients of Θ' do not exceed $1 - \frac{\epsilon}{2}$.

By construction, we have
$$K_{X'} + \Theta' = K_{X'} + \Delta' + (1 - \alpha)(B' - \Delta') + \alpha G' + R'$$

$$\sim_{\mathbb{R}} K_{X'} + \Delta' + (1-\alpha)(B' - \Delta') + \alpha G' + \alpha H' + (1-\alpha)M'$$

$$\sim_{\mathbb{R}} K_{X'} + \Delta' + (1-\alpha)(B' - \Delta') + \alpha(B' - \Delta' + M' + \phi^* f^* A) + (1-\alpha)M'$$

$$= K_{X'} + \Delta' + B' - \Delta' + \alpha(M' + \phi^* f^* A) + (1-\alpha)M'$$

$$= K_{X'} + B' + M' + \alpha\phi^* f^* A \sim_{\mathbb{R}} \phi^* f^*(L + \alpha A).$$

Therefore, letting $\Theta = \phi_*\Theta'$, we have

$$K_X + \Theta \sim_{\mathbb{R}} f^*(L + \alpha A).$$

Choosing α small enough we can ensure $A-(L+\alpha A)$ is ample. Moreover, since $K_{X'}+\Theta' \sim_{\mathbb{R}} 0/X$ and since the coefficients of Θ' do not exceed $1-\frac{\epsilon}{2}$, the pair (X, Θ) is $\frac{\epsilon}{2}$-lc. Thus $(X, \Theta) \to Z$ is a $(d, r, \frac{\epsilon}{2})$-Fano type fibration. Finally, it is obvious that $\Theta \geq \Delta$.

□

3.25. **Bound on singularities.**

Lemma 3.26. *Let $d, p \in \mathbb{N}$ and $\Phi \subset [0, 1]$ be a DCC set. Then there is a real number $\epsilon > 0$ depending only on d, p and Φ such that if $(X, B + M)$ is a projective generalised pair with data $X' \to X$ and M' satisfying:*

- *$(X, B + M)$ is generalised klt of dimension d,*
- *the coefficients of B are in Φ,*
- *pM' is b-Cartier, and*
- *$K_X + B + M \sim_{\mathbb{R}} 0$,*

then $(X, B + M)$ is generalised ϵ-lc.

Proof. If the lemma does not hold, then there exist a decreasing sequence ϵ_i of numbers approaching 0 and a sequence $(X_i, B_i + M_i)$ of pairs as in the statement such that $(X_i, B_i + M_i)$ is not generalised ϵ_i-lc. There is a prime divisor D_i over X_i with generalised log discrepancy

$$a(D_i, X_i, B_i + M_i) < \epsilon_i.$$

If D_i is a divisor on X_i, we let $X''_i \to X_i$ be the identity morphism. If not, then since $(X_i, B_i + M_i)$ is generalised klt, there is a birational morphism $X''_i \to X_i$ extracting D_i but no other divisors. We can assume that the induced map $X'_i \dashrightarrow X''_i$ is a morphism.

Let $K_{X''_i} + B''_i + M''_i$ be the pullback of $K_{X_i} + B_i + M_i$ where M''_i is the pushdown of M_i. We consider $(X''_i, B''_i + M''_i)$ as a generalised pair with data $X'_i \to X''_i$ and M'_i. Let

$$b_i = 1 - a(D_i, X_i, B_i + M_i)$$

which is the coefficient of D_i in B''_i. Then $b_i \geq 1 - \epsilon_i$ and B''_i has coefficients in $\Phi'' := \Phi \cup \{b_i \mid i \in \mathbb{N}\}$. Replacing the sequence, we can assume Φ'' is a DCC set. Now we get a contradiction, by [9, Theorem 1.6], because

$$K_{X''_i} + B''_i + M''_i \sim_{\mathbb{R}} 0$$

and because $\{b_i \mid i \in \mathbb{N}\}$ is not finite as the b_i form an infinite sequence approaching 1.

□

3.27. **Towers of Mori fibre spaces.** We will use the following result of [11] in the proof of Theorem 1.5.

Theorem 3.28 ([11, Theorem 3.2])**.** *Let (X,B) be a projective klt pair such that $K_X+B \sim_{\mathbb{R}} 0$, $B \neq 0$, and (X,B) is not of product type. Then there exist a birational map $\phi\colon X \dashrightarrow X_1$ and a sequence of contractions*

$$X_1 \to X_2 \to \cdots \to X_l$$

such that ϕ^{-1} does not contract divisors, each $X_i \to X_{i+1}$ is a K_{X_i}-Mori fibre space, and X_l is a point.

The main point in the proof of the theorem is similar to the following:

Lemma 3.29. *Suppose that*

- *(X,B) is a $\mathbb{Q}$-factorial projective klt pair,*
- *$h\colon X \to Y$ is a Mori fibre space structure, i.e. a non-birational extremal contraction,*
- *$Y \to Z$ is a contraction,*
- *$K_X + B \sim_{\mathbb{R}} 0/Z$, and*
- *$Y \dashrightarrow Y'/Z$ is a birational map to a $\mathbb{Q}$-factorial normal projective variety which is an isomorphism in codimension one.*

Then there exist a birational map $X \dashrightarrow X'$ to a $\mathbb{Q}$-factorial normal projective variety which is an isomorphism in codimension one and such that the induced map $X' \dashrightarrow Y'$ is an extremal contraction (hence a morphism).

Proof. Since $K_X + B \sim_{\mathbb{R}} 0/Y$, by adjunction, we can write

$$K_X + B \sim_{\mathbb{R}} h^*(K_Y + B_Y + M_Y)$$

where we consider $(Y, B_Y + M_Y)$ as a generalised pair (see 6.2 below) which is generalised klt. Since X is $\mathbb{Q}$-factorial and since $X \to Y$ is extremal and non-birational, Y is also $\mathbb{Q}$-factorial [27, Corollary 3.18].

Since $K_Y + B_Y + M_Y \sim_{\mathbb{R}} 0/Z$, by the same arguments as in [22] we can decompose $Y \dashrightarrow Y'$ into a sequence of flops: indeed if $H_{Y'}$ is an ample $\mathbb{Q}$-divisor on Y', then after rescaling $H_{Y'}$, $(Y, B_Y + H_Y + M_Y)$ is generalised klt where H_Y is the birational transform of $H_{Y'}$; now running an MMP on $K_Y + B_Y + H_Y + M_Y$ over Z ends with Y' as $K_{Y'} + B_{Y'} + H_{Y'} + M_{Y'}$ is ample; in particular only flips can occur in the MMP which are flops with respect to $K_Y + B_Y + M_Y$.

By the previous paragraph, to obtain X' it is enough to consider the case when $Y \dashrightarrow Y'/Z$ is one single flop (in particular, we can assume $Y \to Z$ is an extremal flopping contraction). Let $H_{Y'}, H_Y$ be as before, and let G be the pullback of H_Y to X. Since B is big over Y and since $Y \to Z$ is birational, B is also big over Z. Thus X is of Fano type over Z as (X,B) is klt and $\mathrm{K}_X + B \sim_{\mathbb{R}} 0/Z$. Therefore, there is a minimal model X' for G over Z. The birational transform of G on X', say G', is semi-ample over Z, hence it defines a contraction $X' \to V'/Z$ and G' is the pullback of an ample/Z divisor $H_{V'}$. By construction, V' is the ample model of G over Z. Since G is the pullback of H_Y, V' is also the ample model of H_Y over Z. But the ample model of H_Y is Y', hence $V' = Y'$. In particular, $X' \dashrightarrow Y'$ is a morphism.

Finally, since $X \to Z$ has relative Picard number two, $X' \to Z$ also has relative Picard number two as $X \dashrightarrow X'$ is an isomorphism in codimension one. On the other hand, $Y' \to Z$ has relative Picard number one, so $X' \to Y'$ is an extremal contraction.

□

4. **Boundedness of complements**

In this section we construct certain kinds of complements as in Theorem 1.10 which are not usual complements but rather similar to those of [4, Theorem 1.7]. The main differences with [4, Theorem 1.7] are that $M-(K_X+B)$ is no longer assumed to be ample, X is not necessarily $\mathbb{Q}$-factorial, $M|_S\sim 0$ is replaced with $M|_S\equiv 0$, and $n+1$ is replaced with $n+2$. We start with treating a special case of the theorem.

Proposition 4.1. *Theorem 1.10 holds under the additional assumption that there is a boundary Γ such that (X,Γ) is plt with $S=\lfloor\Gamma\rfloor$ and such that $\alpha M-(K_X+\Gamma)$ is ample for some real number $\alpha>0$.*

Proof. Step 1. In this step we consider bounded complements on S. Since $M-(K_X+B)$ is nef and big and $\alpha M-(K_X+\Gamma)$ is ample,

$$(1-t+t\alpha)M-(K_X+(1-t)B+t\Gamma)=(1-t)(M-(K_X+B))+t(\alpha M-(K_X+\Gamma))$$

is ample for any $t\in(0,1)$. Thus replacing Γ with $(1-t)B+t\Gamma$ for some sufficiently small number t, we can replace α by some rational number in $(0,2)$. Note that since (X,B) is lc and $S\le\lfloor B\rfloor$, this change preserves the plt property of (X,Γ) and the condition $S=\lfloor\Gamma\rfloor$.

Define $K_S+B_S=(K_X+B)|_S$ by adjunction. Then (S,B_S) is lc and the coefficients of B_S belong to $\Phi(\mathfrak{S})$ for some finite set $\mathfrak{S}\subset[0,1]$ of rational numbers depending only on $\mathfrak{R}$ [31, Proposition 3.8][5, Lemma 3.3]. By assumption, $M|_S\equiv 0$, hence $M|_S\sim_{\mathbb{Q}}0$ as M is semi-ample. In particular,

$$-(K_X+\Gamma)|_S\sim_{\mathbb{R}}(\alpha M-(K_X+\Gamma))|_S$$

is ample, and since (X,Γ) is plt, we deduce that S is of Fano type. Therefore, as

$$-(K_S+B_S)\sim_{\mathbb{Q}}(M-(K_X+B))|_S$$

is nef, there is a natural number n depending only on $d,\mathfrak{S}$ such that K_S+B_S has a strong n-complement $K_S+B_S^+$ which by definition satisfies $B_S^+\ge B_S$ [5, Theorem 1.7] (=Theorem 3.15). In addition, we can assume that nB is an integral divisor after replacing n with a bounded multiple depending on $\mathfrak{R}$.

Note that since S is of Fano type and $M|_S\sim_{\mathbb{Q}}0$, in fact we have $M|_S\sim 0$ as $\mathrm{Pic}(S)$ is torsion-free (cf. [19, Proposition 2.1.2] the main point being the vanishing $h^i(\mathcal{O}_S)=0$ for $i>0$ which is a consequence of Kawamata-Viehweg vanishing theorem).

Step 2. In this step we take a resolution and define appropriate divisors on it. Let $\phi\colon X'\to X$ be a log resolution of $(X,B+\Gamma)$, S' be the birational transform of S, and $\psi\colon S'\to S$ be the induced morphism. Put

$$N:=M-(K_X+B)$$

and let $K_{X'}+B',M',N'$ be the pullbacks of K_X+B,M,N, respectively. Let E' be the sum of the components of B' which have coefficient 1, and let $\Delta'=B'-E'$. Define

$$L':=(n+2)M'-nK_{X'}-nE'-\lfloor(n+1)\Delta'\rfloor$$

which is an integral divisor. Note that

$$\begin{aligned}L'&=(n+2)M'-nK_{X'}-nB'+n\Delta'-\lfloor(n+1)\Delta'\rfloor\\&=2M'+n(M'-K_{X'}-B')+n\Delta'-\lfloor(n+1)\Delta'\rfloor\\&=2M'+nN'+n\Delta'-\lfloor(n+1)\Delta'\rfloor.\end{aligned}$$

Now write $K_{X'}+\Gamma'=\phi^*(K_X+\Gamma)$. We can assume $B'-\Gamma'$ has sufficiently small (positive or negative) coefficients by taking t in the beginning of Step 1 to be sufficiently small.

Step 3. In this step we introduce a boundary Λ' and study related divisors. Let P' be the unique integral divisor so that

$$\Lambda' := \Gamma' + n\Delta' - \lfloor (n+1)\Delta' \rfloor + P'$$

is a boundary, (X',Λ') is plt, and $\lfloor \Lambda' \rfloor = S'$ (in particular, we are assuming $\Lambda' \geq 0$). More precisely, we let $\mu_{S'}P'=0$ and for each prime divisor $D'\neq S'$, we let

$$\mu_{D'}P' := -\mu_{D'} \lfloor \Gamma' + n\Delta' - \lfloor (n+1)\Delta' \rfloor \rfloor$$

which satisfies

$$\mu_{D'}P' = -\mu_{D'} \lfloor \Gamma' - \Delta' + \langle (n+1)\Delta' \rangle \rfloor$$

where $\langle (n+1)\Delta' \rangle$ is the fractional part of $(n+1)\Delta'$. This implies $0 \leq \mu_{D'}P' \leq 1$ for any prime divisor D': this is obvious if $D'=S'$, so assume $D'\neq S'$; if D' is a component of E', then D' is not a component of Δ' and $\mu_{D'}\Gamma' \in (0,1)$ as $B'-\Gamma'$ has small coefficients, hence $\mu_{D'}P'=0$; on the other hand, if D' is not a component of E', then the absolute value of $\mu_{D'}(\Gamma'-\Delta')=\mu_{D'}(\Gamma'-B')$ is sufficiently small, hence $0\leq \mu_{D'}P' \leq 1$.

We show P' is exceptional$/X$. Assume D' is a component of P' which is not exceptional$/X$ and let D be its pushdown. Then $D'\neq S'$ and D' is a component of Δ' as $\mu_{D'}\Gamma'=\mu_D\Gamma \in [0,1)$, hence

$$1 > \mu_{D'}\Delta' = \mu_{D'}B' = \mu_D B \geq 0.$$

Moreover, since nB is integral, $\mu_{D'}n\Delta'$ is integral, hence $\mu_{D'}\lfloor (n+1)\Delta' \rfloor = \mu_{D'}n\Delta'$ which implies

$$\mu_{D'}P' = -\mu_{D'} \lfloor \Gamma' \rfloor = -\mu_D \lfloor \Gamma \rfloor = 0,$$

a contradiction.

Step 4. In this step we show that sections of $(L'+P')|_{S'}$ can be lifted to X'. Let $A := \alpha M - (K_X+\Gamma)$. Letting $A'=\phi^*A$ we have

$$K_{X'}+\Gamma'+A'-\alpha M' = 0.$$

Then

$$\begin{aligned} L'+P' &= 2M' + nN' + n\Delta' - \lfloor (n+1)\Delta' \rfloor + P' \\ &= K_{X'}+\Gamma'+A'-\alpha M' + 2M' + nN' + n\Delta' - \lfloor (n+1)\Delta' \rfloor + P' \\ &= K_{X'}+\Lambda'+A'+nN'+(2-\alpha)M'. \end{aligned}$$

Since $A'+nN'+(2-\alpha)M'$ is nef and big and (X',Λ') is plt with $\lfloor \Lambda' \rfloor = S'$, we have $h^1(L'+P'-S')=0$ by the Kawamata-Viehweg vanishing theorem. Thus

$$H^0(L'+P') \to H^0((L'+P')|_{S'})$$

is surjective.

Step 5. In this step we introduce an effective divisor $G_{S'} \sim (L'+P')|_{S'}$. Recall the n-complement $K_S+B_S^+$ from step 1. Let $R_S := B_S^+ - B_S$ which satisfies

$$-n(K_S+B_S) = -n(K_S+B_S^+ + B_S - B_S^+) \sim -n(B_S - B_S^+) = nR_S \geq 0.$$

Letting $R_{S'}$ be the pullback of R_S, we get

$$\begin{aligned} nN'|_{S'} &= n(M'-(K_{X'}+B'))|_{S'} \sim -n(K_{X'}+B')|_{S'} \\ &= -n\psi^*(K_S+B_S) \sim n\psi^*R_S = nR_{S'} \geq 0. \end{aligned}$$

Then

$$(L'+P')|_{S'} = (2M'+nN'+n\Delta'-\lfloor(n+1)\Delta'\rfloor+P')|_{S'}$$
$$\sim G_{S'} := nR_{S'}+n\Delta_{S'}-\lfloor(n+1)\Delta_{S'}\rfloor+P_{S'}$$

where $\Delta_{S'}=\Delta'|_{S'}$ and $P_{S'}=P'|_{S'}$. Note that $\lfloor(n+1)\Delta'\rfloor|_{S'}=\lfloor(n+1)\Delta'|_{S'}\rfloor$ since Δ' and S' intersect transversally.

We show $G_{S'}\geq 0$. Assume C' is a component of $G_{S'}$ with negative coefficient. Then there is a component D' of Δ' such that C' is a component of $D'|_{S'}$. But

$$\mu_{C'}(n\Delta_{S'}-\lfloor(n+1)\Delta_{S'}\rfloor)=\mu_{C'}(-\Delta_{S'}+\langle(n+1)\Delta_{S'}\rangle)\geq-\mu_{C'}\Delta_{S'}=-\mu_{D'}\Delta'>-1$$

which gives $\mu_{C'}G_{S'}>-1$ and this in turn implies $\mu_{C'}G_{S'}\geq 0$ because $G_{S'}$ is integral, a contradiction. Therefore $G_{S'}\geq 0$, and by Step 4, $L'+P'\sim G'$ for some effective divisor G' whose support does not contain S' and $G'|_{S'}=G_{S'}$.

Step 6. In this step we introduce Λ and show that it satisfies the properties listed in the theorem. Let L,P,G,E,Δ be the pushdowns to X of L',P',G',E',Δ'. By the definition of L', by the previous step, and by the exceptionality of P', we have

$$(n+2)M-nK_X-nE-\lfloor(n+1)\Delta\rfloor=L=L+P\sim G\geq 0.$$

Since nB is integral, $\lfloor(n+1)\Delta\rfloor=n\Delta$, so

$$(n+2)M-n(K_X+B)$$
$$=(n+2)M-nK_X-nE-n\Delta=L\sim nR:=G\geq 0.$$

Let $\Lambda:=B^+:=B+R$. By construction, $n(K_X+B^+)\sim(n+2)M$. It remains to show that (X,B^+) is lc over $z=f(S)$. First we show that (X,B^+) is lc near S: this follows from inversion of adjunction [21], if we show

$$K_S+B_S^+=(K_X+B^+)|_S$$

which is equivalent to showing $R|_S=R_S$. Since

$$nR':=G'-P'+\lfloor(n+1)\Delta'\rfloor-n\Delta'\sim L'+\lfloor(n+1)\Delta'\rfloor-n\Delta'=2M'+nN'\sim_{\mathbb{Q}}0/X$$

and since $\lfloor(n+1)\Delta\rfloor-n\Delta=0$, we get $\phi_*nR'=G=nR$ and that R' is the pullback of R. Now

$$nR_{S'}=G_{S'}-P_{S'}+\lfloor(n+1)\Delta_{S'}\rfloor-n\Delta_{S'}$$
$$=(G'-P'+\lfloor(n+1)\Delta'\rfloor-n\Delta')|_{S'}=nR'|_{S'}$$

which means $R_{S'}=R'|_{S'}$, hence R_S and $R|_S$ both pull back to $R_{S'}$ which implies $R_S=R|_S$.

Finally, (X,B^+) is lc over $z=f(S)$ otherwise by the plt property of (X,Γ) we can take $u>0$ to be sufficiently small so that the non-klt locus of

$$(X,(1-u)B^++u\Gamma)$$

has at least two connected components (one of which is S) near the fibre $f^{-1}\{z\}$. This contradicts the connectedness principle [26, Theorem 17.4] as

$$-(K_X+(1-u)B^++u\Gamma)=-(1-u)(K_X+B^+)-u(K_X+\Gamma)$$
$$\sim_{\mathbb{R}}-u(K_X+\Gamma)\sim_{\mathbb{R}}u\alpha M-u(K_X+\Gamma)/Z$$

is ample over Z.

$\square$

Proof. (of Theorem 1.10) *Step 1. In this step we make some modifications of the setting of the theorem and introduce a boundary* Δ. Adding 1 to $\mathfrak{R}$ and replacing (X,B) with a $\mathbb{Q}$-factorial dlt model, we can assume X is $\mathbb{Q}$-factorial and that S is a component of $\lfloor B \rfloor$. All the assumptions of the theorem are preserved. By assumption, M is the pullback of an ample divisor on Z. Moreover, $M-(K_X+B)$ is nef and big, hence in particular it is semi-ample over Z since X is of Fano type over Z. Thus by Lemma 3.4, $aM-(K_X+B)$ is semi-ample for any rational number $a>1$, so it defines a birational contraction $X \to Y$ which is simply the contraction over Z defined by $M-(K_X+B)$: indeed, for any curve C on Y,

$$(aM-(K_X+B))\cdot C = 0 \text{ iff } (M-(K_X+B))\cdot C=0 \text{ and } M\cdot C=0.$$

In particular, the induced map $Y \dashrightarrow Z$ is a morphism and $K_X+B \sim_{\mathbb{Q}} 0/Y$.

After running an MMP over Y on $-(K_X+S)$ and replacing X with the resulting model we can assume $-(K_X+S)$ is semi-ample over Y. Note that S is not contracted by the MMP since the MMP is also an MMP on $(B-S)$ whose support does not contain S. Letting $\Delta=(1-b)B+bS$ for a sufficiently small rational number $b>0$ (depending on a). Then (X,Δ) is lc, and since $aM-(K_X+B)$ is the pullback of an ample divisor on Y and $-(K_X+S)$ is semi-ample over Y, we see that

$$aM-(K_X+\Delta)=(1-b)(aM-(K_X+B))+b(aM-(K_X+S))$$

is semi-ample and nef and big. Moreover, every non-klt centre of (X,Δ) is also a non-klt centre of (X,S), hence such centres are contained in S because X is $\mathbb{Q}$-factorial and of Fano type over Z which ensures $(X,0)$ is klt.

Replacing (X,B) once again with a $\mathbb{Q}$-factorial dlt model and replacing $K_X+\Delta$ with its pullback, we can assume that

- we have a boundary $\Delta \le B$,
- (X,Δ) is $\mathbb{Q}$-factorial dlt,
- S is a component of $\lfloor \Delta \rfloor$,
- $aM-(K_X+B)$ and $aM-(K_X+\Delta)$ are semi-ample and nef and big for some rational number $a>1$,
- if $X \to Y$ and $X \to V$ are the contractions defined by $aM-(K_X+B)$ and $aM-(K_X+\Delta)$, respectively, then $V \dashrightarrow Y$ and $Y \dashrightarrow Z$ are morphisms, and
- all the non-klt centres of (X,Δ) map to $z=f(S)$.

In particular, $M|_{\lfloor \Delta \rfloor} \sim_{\mathbb{Q}} 0$.

Step 2. In this step we introduce another boundary $\tilde{\Delta}$. By Step 1, $aM-(K_X+\Delta)$ is semi-ample defining a contraction $X\to V$ so that $V \dashrightarrow Y$ is a morphism. After running an MMP on $-K_X$ over V we can assume that $-K_X$ is semi-ample over V. This preserves all the properties listed in Step 1 except that the dlt property of (X,Δ) maybe lost and S maybe contracted. We will recover these properties in a bit. Let $\tilde{\Delta}=(1-c)\Delta$ for some sufficiently small $c>0$. Then $(X,\tilde{\Delta})$ is klt as $(X,0)$ is klt. Moreover, we can assume that

$$aM-(K_X+\tilde{\Delta})=(1-c)(aM-(K_X+\Delta))+c(aM-K_X)$$

is semi-ample and nef and big because $aM-(K_X+\Delta)$ is the pullback of an ample divisor on V and $aM-K_X$ is semi-ample over V. Now replacing (X,Δ) with a $\mathbb{Q}$-factorial dlt model on which S of Step 1 is a divisor, and replacing K_X+B and $K_X+\tilde{\Delta}$ with their pullbacks to the dlt model we can assume that in addition to the properties listed in Step 1 we have

- a boundary $\tilde{\Delta} \le \Delta$,

- $(X, \tilde{\Delta})$ is klt,
- the coefficients of $\Delta - \tilde{\Delta}$ are sufficiently small, and
- that $aM - (K_X + \tilde{\Delta})$ is nef and big.

Note that the coefficients of B are still in $\mathfrak{R}$ because $\Delta \le B$ implies that taking the above $\mathbb{Q}$-factorial dlt model only extracts divisors whose log discrepancy with respect to (X, B) is zero.

Step 3. In this step we introduce divisors H, G and deal with the case when $\operatorname{Supp} G$ *does not contain non-klt centres of* (X, Δ). Write

$$aM - (K_X + \Delta) \sim_{\mathbb{Q}} H + G$$

where $H, G \ge 0$ are $\mathbb{Q}$-divisors and H is ample. First assume that $\operatorname{Supp} G$ does not contain any non-klt centre of (X, Δ). Then, for some small $\delta > 0$,

$$aM - (K_X + \Delta + \delta G) \sim_{\mathbb{Q}} H + G - \delta G = \delta H + (1 - \delta)(H + G)$$

is ample and $(X, \Delta + \delta G)$ is lc. Perturbing the coefficients of $\Delta + \delta G$ we can then produce a boundary Γ such that (X, Γ) is plt, $S = \lfloor \Gamma \rfloor \subseteq \lfloor B \rfloor$ and such that $aM - (K_X + \Gamma)$ is ample. Then apply Proposition 4.1. From now on we can assume that $\operatorname{Supp} G$ contains some non-klt centre of (X, Δ).

Step 4. In this step we introduce yet another boundary Ω and study some of its properties. Let t be the lc threshold of $G + \Delta - \tilde{\Delta}$ with respect to $(X, \tilde{\Delta})$. Let

$$\Omega = \tilde{\Delta} + t(G + \Delta - \tilde{\Delta}).$$

We claim that we can ensure that any non-klt place of (X, Ω) is a non-klt place of (X, Δ). Indeed let $W \to X$ be a log resolution of $(X, \Delta + G)$, and let $K_W + \Delta_W$, $K_W + \tilde{\Delta}_W$, $K_W + \Omega_W$, G_W be the pullbacks of $K_X + \Delta$, $K_X + \tilde{\Delta}$, $K_X + \Omega$, G, respectively. Since $\operatorname{Supp} G$ contains some non-klt centre of (X, Δ), we can assume that some component T of $\lfloor \Delta_W \rfloor$ is a component of G_W. Since $\Delta - \tilde{\Delta}$ has sufficiently small coefficients, we can assume that $\Delta_W - \tilde{\Delta}_W$ also has arbitrarily small coefficients, perhaps after replacing $\tilde{\Delta}$. In particular, the lc threshold of G with respect to $(X, \tilde{\Delta})$ is sufficiently small as $\mu_T \tilde{\Delta}_W$ can be made arbitrarily close to 1. Thus t is also sufficiently small. Moreover, we can assume that $\Delta_W - \Omega_W$ also has arbitrarily small positive or negative coefficients, hence $\lfloor \Omega_W \rfloor \subseteq \lfloor \Delta_W \rfloor$ which implies that any non-klt place of (X, Ω) is a non-klt place of (X, Δ). In particular, each non-klt centre of (X, Ω) is mapped to z, by construction of Δ.

By construction,

$$\begin{aligned}
aM - (K_X + \Omega) &= aM - (K_X + \tilde{\Delta} + t(G + \Delta - \tilde{\Delta})) \\
&= aM - K_X - \tilde{\Delta} - t(G + \Delta - \tilde{\Delta}) \\
&= aM - (K_X + \Delta) + \Delta - \tilde{\Delta} - t(G + \Delta - \tilde{\Delta}) \\
&\sim_{\mathbb{Q}} H + G + \Delta - \tilde{\Delta} - t(G + \Delta - \tilde{\Delta}) \\
&= H + (1 - t)(G + \Delta - \tilde{\Delta}) \\
&= tH + (1 - t)(H + G + \Delta - \tilde{\Delta})
\end{aligned}$$

which implies that $aM - (K_X + \Omega)$ is ample because

$$aM - (K_X + \tilde{\Delta}) = aM - (K_X + \Delta) + \Delta - \tilde{\Delta} \sim_{\mathbb{Q}} H + G + \Delta - \tilde{\Delta}$$

is nef and big by Step 2.

Step 5. In this step we produce a plt pair and apply Proposition 4.1 to finish the proof. Assume that $\lfloor \Omega \rfloor \neq 0$. Replacing S we can assume it is a component of $\lfloor \Omega \rfloor \leq \lfloor \Delta \rfloor$. Since (X, Δ) is $\mathbb{Q}$-factorial dlt by Step 1, (X, S) is plt. Thus we can produce a boundary Γ out of Ω so that (X, Γ) is plt, $S = \lfloor \Gamma \rfloor$ maps to z, and $aM - (K_X + \Gamma)$ is ample. We then apply Proposition 4.1.

Now assume $\lfloor \Omega \rfloor = 0$. Let (X', Ω') be a $\mathbb{Q}$-factorial dlt model of (X, Ω). By Step 4, each non-klt place of (X, Ω) is a non-klt place of (X, Δ) which is in turn a non-klt place of (X, B) as $\Delta \leq B$. Thus if we denote the pullback of $K_X + B$ to X' by $K_{X'} + B'$, then each exceptional/X divisor on X' appears in B' with coefficient 1. Running an MMP on $K_{X'} + \lfloor \Omega' \rfloor$ over X ends with X because $\lfloor \Omega' \rfloor$ is the reduced exceptional divisor of $X' \to X$ and because X is $\mathbb{Q}$-factorial klt. The last step of the MMP is a divisorial contraction $X'' \to X$ contracting one prime divisor S''. Then (X'', S'') is plt and $-(K_{X''} + S'')$ is ample over X.

Define $\Gamma'' = (1 - v)\Omega'' + vS''$ for some sufficiently small $v > 0$. Let M'' be the pullback of M. Assume $\alpha = (1 - v)a$. Since $aM - (K_X + \Omega)$ is ample and since $-(K_{X''} + S'')$ is ample over X,

$$\begin{aligned} \alpha M'' - (K_{X''} + \Gamma'') &= (1 - v)aM'' - (K_{X''} + (1 - v)\Omega'' + vS'') \\ &= (1 - v)aM'' - (1 - v)(K_{X''} + \Omega'') - v(K_{X''} + S'') \\ &= (1 - v)(aM'' - (K_{X''} + \Omega'')) - v(K_{X''} + S'') \end{aligned}$$

is ample. Moreover, (X'', Γ'') is plt and $S'' = \lfloor \Gamma'' \rfloor$ maps to z. If $K_{X''} + B''$ is the pullback of $K_X + B$, then we can replace (X, B) with (X'', B'') and apply Proposition 4.1.

□

5. **Boundedness of Fano type fibrations**

In this section we treat boundedness properties of Fano type log Calabi-Yau fibrations. We will frequently refer to Definition 1.1 and use the notation therein.

5.1. **Numerical boundedness.** We start with bounding numerical properties. The next statement and its proof are similar to [20, Lemma 3.2].

Proposition 5.2. *Let d, r be natural numbers and ϵ be a positive real number. Assume that Theorem 1.6 holds in dimension $d - 1$. Then there is a natural number l depending only on d, r, ϵ satisfying the following. Let $(X, B) \to Z$ be a (d, r, ϵ)-Fano type fibration (as in 1.1) such that*

- *$-(K_X + \Delta)$ is nef over Z for some $\mathbb{R}$-divisor $\Delta \geq 0$, and*
- *$f^*A + B - \Delta$ is pseudo-effective.*

Then $lf^*A - (K_X + \Delta)$ is nef (globally).

Proof. Step 1. In this step we do some basic preparations and introduce some notation. Replacing X with a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. All the assumptions of the lemma are preserved. Put $C := f^*A - (K_X + B)$. Let $m \geq 2$ be a natural number. We can write

$$\begin{aligned} (m + 2)f^*A - (K_X + \Delta) &= 2f^*A + m(K_X + B) + mC - (K_X + \Delta) \\ &= 2f^*A + m(K_X + B) - (K_X + \Delta) + mC \\ &= 2f^*A + (m - 1)(K_X + B) + B - \Delta + mC \\ &= (m - 1)(K_X + B + \tfrac{1}{m-1}(2f^*A + B - \Delta)) + mC. \end{aligned}$$

On the other hand, since $f^*A+B-\Delta$ is pseudo-effective and since X is of Fano type over Z, there is
$$0 \le P \sim_{\mathbb{R}} 2f^*A+B-\Delta$$
by Lemma 3.13.

Step 2. In this step we find a real number $t>0$ depending only on d,r,ϵ such that the non-klt locus of $(X,B+tP)$ is mapped to a finite set of closed points of Z. We can assume $\dim Z>0$ otherwise the lemma is trivial. Let $H\in |A|$ be a general element and let $G=f^*H$, and g be the induced morphism $G\to H$. Let
$$K_G+B_G := (K_X+B+G)|_G.$$
By definition of G, we have $B_G=B|_G$. Then

- (G,B_G) is ϵ-lc as (X,B) is ϵ-lc,
- $-K_G$ is big over H as $-K_G=-(K_X+G)|_G\sim -K_X|_G/H$,
- we have
$$K_G+B_G \sim_{\mathbb{R}} (f^*L+f^*H)|_G \sim_{\mathbb{R}} g^*(L+H)|_H \sim_{\mathbb{R}} g^*(L+A)|_H,$$
and
- $2A|_H-(L+A)|_H$ is ample as $A-L$ is ample.

Therefore, $(G,B_G)\to H$ is a $(d-1,2^{d-1}r,\epsilon)$-Fano type fibration.

Letting $P_G=P|_G$ and $Q_G=\Delta|_G$ we have
$$P_G+Q_G \sim_{\mathbb{R}} (2f^*A+B-\Delta)|_G+\Delta|_G=(2f^*A+B)|_G\sim g^*2A|_H+B_G.$$
Since we are assuming Theorem 1.6 in dimension $d-1$, we deduce that there is a real number $t>0$ depending only on d,r,ϵ such that (G,B_G+tP_G) is klt. Note that here we used the assumption $\Delta\ge 0$ to ensure that $g^*2A|_H+B_G-P_G$ is pseudo-effective.

By inversion of adjunction [27, Theorem 5.50] (which is stated for $\mathbb{Q}$-divisors but also holds for $\mathbb{R}$-divisors) and by the previous paragraph,
$$(X,B+G+tP)$$
is plt near G. Since G is a general member of $|f^*A|$, we deduce that the non-klt locus of $(X,B+tP)$ (possibly empty) is mapped to a finite set of closed points of Z.

Step 3. In this step we consider $(m+2)f^*A-(K_X+\Delta)$-negative extremal rays. Fix a natural number $m\ge 2$ so that $\frac{1}{m-1}<t$. Let $\Theta=B+\frac{1}{m-1}P$. By Step 1 and definition of P, we have
$$(m+2)f^*A-(K_X+\Delta)=(m-1)(K_X+B+\frac{1}{m-1}(2f^*A+B-\Delta))+mC$$
$$\sim_{\mathbb{R}} (m-1)(K_X+B+\frac{1}{m-1}P)+mC=(m-1)(K_X+\Theta)+mC.$$
Assume that R is an extremal ray of X with
$$((m+2)f^*A-(K_X+\Delta))\cdot R<0.$$
Since $-(K_X+\Delta)$ is nef over Z, R is not vertical over Z, that is, $f^*A\cdot R>0$. On the other hand, by the previous paragraph,
$$((m-1)(K_X+\Theta)+mC)\cdot R<0$$
which in turn implies
$$(K_X+\Theta)\cdot R<0$$

because $C \sim_{\mathbb{R}} f^*(A-L)$ is nef.

Step 4. In this step we apply boundedness of length of extremal rays and finish the proof. Let T be the non-klt locus of (X,Θ). By Step 2, T is mapped to a finite set of closed points of Z. Let V be the image of $\overline{NE}(T) \to \overline{NE}(X)$ where by convention we put $\overline{NE}(T)=0$ if T is zero-dimensional or empty. Then $V \cap R = 0$ as f^*A intersects R positively but intersects every class in V trivially. Therefore, by [13, Theorem 1.1(5)], R is generated by a curve Γ with

$$-2d \leq (K_X+\Theta)\cdot\Gamma,$$

hence

$$\begin{aligned}-2d(m-1) \leq (m-1)(K_X+\Theta)\cdot\Gamma &\leq ((m-1)(K_X+\Theta)+mC)\cdot\Gamma\\ &= ((m+2)f^*A-(K_X+\Delta))\cdot\Gamma.\end{aligned}$$

Moreover, $f^*A\cdot\Gamma \geq 1$. Therefore, taking $l=m+2+2d(m-1)$ we have

$$0 \leq -2d(m-1)+2d(m-1)f^*A\cdot\Gamma \leq (lf^*A-(K_X+\Delta))\cdot R$$

ensuring that $lf^*A-(K_X+\Delta)$ is nef.

□

5.3. **Bounded very ampleness.**

Lemma 5.4. *Let d,r be natural numbers, ϵ be a positive real number, and $\mathfrak{R}\subset[0,1]$ be a finite set of rational numbers. Assume that Theorem 1.6 holds in dimension $d-1$ and that Theorem 1.3 holds in dimension d. Then there exist natural numbers l,m depending only on $d,r,\epsilon,\mathfrak{R}$ satisfying the following. If $(X,B)\to Z$ is a (d,r,ϵ)-Fano type fibration (as in 1.1) such that*

- *we have $\Delta\leq B$ with coefficients in $\mathfrak{R}$, and*
- *$-(K_X+\Delta)$ is ample over Z,*

then

$$m(lf^*A-(K_X+\Delta))$$

is very ample.

Proof. Since we are assuming Theorem 1.3, (X,Δ) is log bounded. Thus by Lemma 3.17, there is a bounded natural number m such that $m(K_X+\Delta)$ is Cartier. On the other hand, by Proposition 5.2, there is a bounded natural number l such that $lf^*A-(K_X+\Delta)$ is nef. As $-(K_X+\Delta)$ is ample over Z, replacing l with $2l+1$ we can assume that $lf^*A-2(K_X+\Delta)$ is ample (which also implies that $lf^*A-(K_X+\Delta)$ is ample).

Now

$$m(lf^*A-(K_X+\Delta)) = K_X+\Delta+(m(lf^*A-(K_X+\Delta))-(K_X+\Delta))$$

where the term

$$(m(lf^*A-(K_X+\Delta))-(K_X+\Delta)) = m(lf^*A-(1+\frac{1}{m})(K_X+\Delta))$$

is ample because $1+\frac{1}{m}\leq 2$ and $lf^*A-2(K_X+\Delta)$ is ample. Thus replacing m we can assume

$$|m(lf^*A-(K_X+\Delta))|$$

is base point free, by the effective base point free theorem [25, Theorem 1.1]. Therefore,

$$(d+4)m(lf^*A-(K_X+\Delta))$$

is very ample by [25, Lemma 1.2]. Now replace m with $(d+4)m$.

□

5.5. **Effective birationality.** The next statement and its proof are somewhat similar to [20, Lemma 3.3].

Proposition 5.6. *Let d, r be natural numbers and ϵ be a positive real number. Assume that Theorems 1.3 and 1.6 hold in dimension $d-1$. Then there exist natural numbers l, m depending only on d, r, ϵ satisfying the following. If $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1), then the linear system $|m(lf^*A - K_X)|$ defines a birational map.*

Proof. We first replace X with a $\mathbb{Q}$-factorialisation so that K_X is $\mathbb{Q}$-Cartier. Then after running an MMP on $-K_X$ over Z we can assume $-K_X$ is nef and big over Z. Replacing X with the ample model of $-K_X$ over Z we can assume $-K_X$ is ample over Z (X may no longer be $\mathbb{Q}$-factorial but we do not need it any more). If $\dim Z = 0$, then the proposition holds by [5, Theorem 1.2]. We then assume $\dim Z > 0$. By Proposition 5.2, there is $l \in \mathbb{N}$ depending only on d, r, ϵ such that $lf^*A - K_X$ is nef. Since $-K_X$ is ample over Z, replacing l with $l+1$ we can assume $lf^*A - K_X$ is ample.

By Lemma 3.9, there is a non-empty open subset $U \subseteq Z$ such that for any pair of closed points $z, z' \in U$ and any general member H of the sub-linear system $L_{z,z'}$ of $|2A|$ consisting of elements passing through z, z', the pullback $G = f^*H$ is normal and (G, B_G) is an ϵ-lc pair where

$$K_G + B_G = (K_X + B + G)|_G.$$

Pick distinct closed points $x, x' \in X$ such that $z := f(x)$ and $z' := f(x')$ are in U. Let H, G, B_G be as in the previous paragraph constructed for z, z'. Denoting $G \to H$ by g we then have

- (G, B_G) is ϵ-lc,
- $K_G + B_G \sim_{\mathbb{R}} (f^*L + G)|_G \sim g^*(L + 2A)|_H$,
- $-K_G = -(K_X + G)|_G$ is ample over H,
- the divisor
$$3A|_H - (L + 2A)|_H = (A - L)|_H$$
is ample, and
- the divisor
$$(l+4)g^*A|_H - K_G \sim (l+4)f^*A|_G - (K_X + G)|_G \sim ((l+2)f^*A - K_X)|_G$$
is ample.

In particular, (G, B_G) is a $(d-1, 2(3^{d-1})r, \epsilon)$-Fano type fibration. Applying Lemma 5.4 and perhaps replacing l, we can assume that

$$m((l+4)g^*A|_H - K_G)$$

is very ample, for some bounded natural number $m \geq 2$.

On the other hand, by assumption

$$C := f^*A - (K_X + B) \sim_{\mathbb{R}} f^*(A - L)$$

is nef, and we can write

$$\begin{aligned} m((l+2)f^*A - K_X) - G &= 2mf^*A + m(lf^*A - K_X) - G \\ &\sim_{\mathbb{R}} (2m-2)f^*A + m(lf^*A - K_X) \\ &\sim_{\mathbb{R}} K_X + B + C + (2m-3)f^*A + m(lf^*A - K_X). \end{aligned}$$

Thus by the Kawamata-Viehweg vanishing theorem,

$$h^1(m((l+2)f^*A - K_X) - G) = 0,$$

hence the map

$$H^0(m((l+2)f^*A-K_X)) \to H^0(m((l+2)f^*A-K_X)|_G)$$

is surjective. Recall that by the previous paragraph,

$$H^0(m((l+2)f^*A-K_X)|_G) = H^0(m((l+4)g^*A|_H-K_G)).$$

Now since $m((l+4)g^*A|_H-K_G)$ is very ample, we can find a section in

$$H^0(m((l+4)g^*A|_H-K_G))$$

vanishing at x but not at x' (and vice versa). This in turn gives a section in

$$H^0(m((l+2)f^*A-K_X))$$

vanishing at x but not at x' (and vice versa). Therefore, $|m((l+2)f^*A-K_X)|$ defines a birational map. Now replace l with $l+2$.

$\square$

5.7. **Boundedness of volume.** The next statement and its proof are similar to [20, Theorem 4.1] (also see [11, 4.1]).

Proposition 5.8. *Let d,r,l be natural numbers and ϵ be a positive real number. Then there exists a natural number v depending only on d,r,l,ϵ satisfying the following. If $(X,B)\to Z$ is a (d,r,ϵ)-Fano type fibration (as in 1.1), then* $\operatorname{vol}(lf^*A-K_X)\le v$.

Proof. We can assume that lf^*A-K_X is big otherwise $\operatorname{vol}(lf^*A-K_X)=0$. Moreover, taking a $\mathbb{Q}$-factorialisation, we can assume X is $\mathbb{Q}$-factorial. If $\dim Z=0$, then X belongs to a bounded family [4, Corollary 1.2] so the proposition follows. We then assume $\dim Z>0$.

Let p be the largest integer such that pf^*A-K_X is not big. Then $p<l$. Let $H\in|A|$ be a general element and $G=f^*H$. Then

$$lf^*A-K_X = pf^*A-K_X+(l-p)f^*A \sim pf^*A-K_X+(l-p)G,$$

so by [20, Lemma 2.5],

$$\operatorname{vol}(lf^*A-K_X)\le \operatorname{vol}(pf^*A-K_X)+d(l-p)\operatorname{vol}((lf^*A-K_X)|_G).$$

Since pf^*A-K_X is not big, $\operatorname{vol}(pf^*A-K_X)=0$, so it is then enough to bound

$$d(l-p)\operatorname{vol}(lf^*A-K_X)|_G)$$

from above.

Define $K_G+B_G=(K_X+B+G)|_G$. Then (G,B_G) is ϵ-lc, $-K_G$ is big over H, $K_G+B_G\sim_{\mathbb{R}} g^*(L+A)|_H$ where g denotes $G\to H$, and $2A|_H-(L+A)|_H$ is ample. Thus $(G,B_G)\to H$ is a $(d-1,2^{d-1}r,\epsilon)$-Fano type fibration, hence applying induction on dimension shows that

$$\begin{aligned}\operatorname{vol}((lf^*A-K_X)|_G) &= \operatorname{vol}(((l+1)f^*A-(K_X+G))|_G)\\ &= \operatorname{vol}((l+1)g^*A|_H-K_G)\le \operatorname{vol}((l+1)g^*2A|_H-K_G)\end{aligned}$$

is bounded from above. Therefore, it is enough to show that $l-p$ is bounded from above which is equivalent to showing that p is bounded from below.

By definition of p, $(p+1)f^*A-K_X$ is big. Thus we can find

$$0\le R\sim_{\mathbb{Q}}(p+1)f^*A-K_X.$$

Let F be a general fibre of $f\colon X\to Z$. Then

$$K_F+B_F:=(K_X+B)|_F\sim_{\mathbb{R}} 0$$

and

$$R_F:=R|_F\sim_{\mathbb{Q}} -K_X|_F=-K_F\sim_{\mathbb{R}} B_F.$$

By [4, Corollary 1.2], F belongs to a bounded family of varieties as (F, B_F) is ϵ-lc and $-K_F$ is big. We can then find a very ample divisor J_F on F with bounded degree $J_F^{\dim F}$ such that

$$J_F - R_F \sim_{\mathbb{R}} J_F - B_F \sim_{\mathbb{Q}} J_F + K_F$$

is ample. Therefore, by [4, Theorem 1.6] (=Theorem 3.20), the pair $(F, B_F + tR_F)$ is klt for some real number $t \in (0,1)$ depending only on $d, J_F^{\dim F}, \epsilon$. In particular, letting

$$\Delta := (1-t)B + tR \text{ and } \Delta_F := \Delta|_F,$$

we see that

$$(F, \Delta_F = (1-t)B_F + tR_F)$$

is klt. Therefore, (X, Δ) is klt near the generic fibre of f. By construction,

$$\begin{aligned} K_X + \Delta = K_X + (1-t)B + tR &\sim_{\mathbb{R}} K_X + (1-t)B + t(p+1)f^*A - tK_X \\ = (1-t)(K_X + B) + t(p+1)f^*A &\sim_{\mathbb{R}} f^*((1-t)L + t(p+1)A). \end{aligned}$$

Now by adjunction we can write

$$K_X + \Delta \sim_{\mathbb{R}} f^*(K_Z + \Delta_Z + M_Z)$$

where Δ_Z is the discriminant divisor and M_Z is the moduli divisor (see 6.1 below for more details on adjunction). In particular,

$$(1-t)L + t(p+1)A \sim_{\mathbb{R}} K_Z + \Delta_Z + M_Z.$$

By [5, Theorem 3.6], M_Z is pseudo-effective. On the other hand, if $\psi\colon Z' \to Z$ is a resolution, then $K_{Z'} + 3d\psi^*A$ is big [5, Lemma 2.46], hence $K_Z + 3dA$ is big. This implies that

$$K_Z + \Delta_Z + M_Z + 3dA$$

is big. But then

$$(1-t)L + t(p+1)A + 3dA$$

is big which in turn implies that

$$(1-t)A + t(p+1)A + 3dA = (1-t)(A-L) + (1-t)L + t(p+1)A + 3dA$$

is big as $A - L$ is ample. Therefore, $(1-t) + t(p+1) + 3d > 0$ which implies that p is bounded from below as required.

□

5.9. **Log birational boundedness.**

Proposition 5.10. *Let d, r, l be natural numbers and ϵ, δ be positive real numbers. Assume that Theorems 1.3 and 1.6 hold in dimension $d-1$. Then there exists a bounded set of couples $\mathcal{P}$ depending only on $d, r, l, \epsilon, \delta$ satisfying the following. Assume that $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1) and that $\Lambda \geq 0$ is an $\mathbb{R}$-divisor such that*

- *each non-zero coefficient of Λ is $\geq \delta$, and*
- *$lf^*A - (K_X + \Lambda)$ is pseudo-effective.*

Then there exist a couple $(\overline{X}, \overline{\Sigma})$, a very ample divisor $\overline{D} \geq 0$ on $\overline{X}$ and a birational map $\rho\colon \overline{X} \dashrightarrow X/Z$ such that

1. *$(\overline{X}, \overline{\Sigma} + \overline{D})$ is log smooth and belongs to $\mathcal{P}$,*
2. *$\overline{\Sigma}$ contains the exceptional divisors of ρ union the birational transform of* $\operatorname{Supp}\Lambda$,
3. *and if $N := lf^*A - K_X$ is nef and $\overline{N} := \rho^*N$ (defined as in 3.3), then $\overline{D} - \overline{N}$ is ample.*

Proof. Step 1. In this step we introduce some notation. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. By Proposition 5.6, perhaps after replacing l with a bounded multiple, we can assume that there exists a natural number m depending only on d, r, ϵ such that the linear system $|m(lf^*A - K_X)|$ defines a birational map. Pick

$$0 \le M \sim m(lf^*A - K_X).$$

Take a log resolution $\phi\colon W \to X$ of $(X, \Lambda + M)$ such that $|\phi^*M|$ decomposes as the sum of a free part $|F_W|$ plus fixed part R_W (cf. [5, Lemma 2.6]). Let Σ_W be the sum of the reduced exceptional divisor of ϕ and the birational transform of $\operatorname{Supp}(\Lambda+M)$ plus a general element G_W of $|\phi^*f^*A+F_W|$. Let Σ, F, R, G be the pushdowns of Σ_W, F_W, R_W, G_W to X.

Step 2. In this step we show that $\operatorname{vol}(K_W + \Sigma_W + (4d+2)G_W)$ *is bounded from above.* From the definition of Σ_W and the assumption that the non-zero coefficients of Λ are $\ge \delta$, we can see that

$$\Sigma = \operatorname{Supp}(\Lambda + M) + G \le \frac{1}{\delta}\Lambda + M + G.$$

Moreover,

$$G + R \sim f^*A + F + R \sim f^*A + M,$$

and by assumption $lf^*A - (K_X + \Lambda)$ is pseudo-effective. Taking all these into account and letting $p = 4d+2$ and $q = \frac{1}{\delta}$ we then have

$$\begin{aligned}
\operatorname{vol}(K_W + \Sigma_W + pG_W) &\le \operatorname{vol}(K_X + \Sigma + pG)\\
&\le \operatorname{vol}(K_X + q\Lambda + M + (p+1)G)\\
&\le \operatorname{vol}(K_X + q\Lambda + M + (p+1)f^*A + (p+1)M)\\
&= \operatorname{vol}(K_X + q\Lambda + (p+1)f^*A + (p+2)M)\\
&= \operatorname{vol}((1-q)K_X + q(K_X+\Lambda) + (p+1)f^*A + (p+2)M)\\
&\le \operatorname{vol}((1-q)K_X + qlf^*A + (p+1)f^*A + (p+2)M)\\
&= \operatorname{vol}((1-q)K_X + (ql+p+1)f^*A + (p+2)M)\\
&= \operatorname{vol}((ql+p+1)f^*A + (p+2)mlf^*A - ((p+2)m+q-1)K_X).
\end{aligned}$$

The latter volume is bounded from above by Proposition 5.8 as all the numbers l, m, p, q are fixed. Thus $\operatorname{vol}(K_W + \Sigma_W + pG_W)$ is bounded from above.

Step 3. In this step we show that $(X, \operatorname{Supp}(\Lambda + M))$ *is log birationally bounded.* Since $G_W \sim \phi^*f^*A+F_W$, $|G_W|$ is base point free and it defines a birational contraction $W \to \overline{X}'$. In particular,

$$K_W + \Sigma_W + (p-1)G_W$$

is big (cf. [5, Lemma 2.46]), hence

$$\operatorname{vol}(G_W) \le \operatorname{vol}(K_W + \Sigma_W + pG_W)$$

which implies that the left hand side is also bounded from above. Moreover, by [18, Lemma 3.2], $\Sigma_W \cdot G_W^{d-1}$ is bounded from above. Therefore, if $\overline{\Sigma}'$ is the pushdown of Σ_W, then $(\overline{X}', \overline{\Sigma}')$ is log bounded (this follows from [18, Lemma 2.4.2(4)]). Also the induced map $\overline{X}' \dashrightarrow Z$ is a morphism by the choice of G_W: indeed any curve contracted by $W \to \overline{X}'$ intersects G_W trivially hence it intersects the pullback of A trivially which means the curve is also contracted over Z.

Now we can take a log resolution $\overline{X} \to \overline{X}'$ of $(\overline{X}', \overline{\Sigma}')$ such that if $\overline{\Sigma}$ is the union of the exceptional divisors and the birational transform of $\overline{\Sigma}'$, then $(\overline{X}, \overline{\Sigma})$ is log smooth and log bounded. By construction, $\overline{\Sigma}$ contains the reduced exceptional divisor of the induce map $\rho\colon \overline{X} \dashrightarrow X$ union the birational transform of $\operatorname{Supp}(\Lambda + M)$. This settles (1) and (2) of the proposition except that we need to add $\overline{D}$.

Step 4. In this step we prove the existence of the very ample divisor $\overline{D}$. Denote the induced map $\overline{X} \dashrightarrow W$ by α. By construction, $G_W \sim 0/\overline{X}'$, hence $\overline{G} := \alpha^* G_W \sim 0/\overline{X}'$ (pullback under α is defined as in 3.3). Moreover, $\overline{G} \le \overline{\Sigma}$. In particular, $(\overline{X}, \overline{G})$ is log bounded and $\overline{G}$ is big, hence we can find a bounded natural number b and a very ample divisor $\overline{D}$ such that $b\overline{G} - \overline{D}$ is big. Then $(\overline{X}, \overline{\Sigma} + \overline{D})$ is log bounded, hence it belongs to some fixed bounded set of couples $\mathcal{P}$.

From now on we assume that $N = lf^*A - K_X$ is nef. Then $M \sim mN$ is also nef. We will show that we can choose $\overline{D}$ so that $\overline{D} - \overline{M}$ is ample where $\overline{M} = \rho^* M$. Let $\pi\colon V \to W$ and $\mu\colon V \to \overline{X}$ be a common resolution. Then

$$\begin{aligned}
\overline{D}^{d-1} \cdot \overline{M} = \mu^*\overline{D}^{d-1} \cdot \mu^*\overline{M} &= \mu^*\overline{D}^{d-1} \cdot \pi^*\phi^* M \\
&\le \operatorname{vol}(\mu^*\overline{D} + \pi^*\phi^* M) \\
&\le \operatorname{vol}(b\mu^*\overline{G} + \pi^*\phi^* M) \\
&= \operatorname{vol}(b\pi^* G_W + \pi^*\phi^* M) \\
&= \operatorname{vol}(bG_W + \phi^* M) \\
&\le \operatorname{vol}(bG + M) \\
&\le \operatorname{vol}(bf^*A + bM + M) \\
&= \operatorname{vol}(bf^*A + (b+1)mlf^*A - (b+1)mK_X)
\end{aligned}$$

where to get the second equality we use the fact $\overline{M} = \mu_*(\pi^*\phi^* M)$ and to get the first inequality we have used the fact that $\mu^*\overline{D}, \pi^*\phi^* M$ are both nef. Therefore, $\overline{D}^{d-1} \cdot \overline{M}$ is bounded from above as the latter volume is bounded from above by Proposition 5.8. This implies that the coefficients of $\overline{M}$ are bounded from above.

Now since $\operatorname{Supp} \overline{M} \le \overline{\Sigma}$, $(\overline{X}, \operatorname{Supp} \overline{M})$ is log bounded. Thus replacing $\overline{D}$ we can assume that $\overline{D} - \overline{M}$ is ample. Finally, note that $\overline{N} \sim_{\mathbb{Q}} \frac{1}{m}\overline{M}$, hence

$$\overline{D} - \overline{N} \sim_{\mathbb{Q}} \overline{D} - \frac{1}{m}\overline{M} = \frac{1}{m}(m\overline{D} - \overline{M})$$

is ample.

$\square$

5.11. **Lower bound on lc thresholds: special case.** We prove a special case of Theorem 1.6 which is crucial for the rest of this section.

Proposition 5.12. *Let d, r, l be natural numbers and ϵ be a positive real number. Assume that Theorems 1.3 and 1.6 hold in dimension $d-1$. Then there exists a positive real number t depending only on d, r, l, ϵ satisfying the following. Assume that $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1) and that $0 \le P \sim_{\mathbb{R}} lf^*A$. Then $(X, B + tP)$ is klt.*

We prove a lemma before proving the proposition.

Lemma 5.13. *Let d, r, l, n be natural numbers and ϵ be a positive real number. Assume that Theorems 1.3 and 1.6 hold in dimension $d-1$. Then there exists a natural number v depending only on d, r, l, n, ϵ satisfying the following. Assume that $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1) and that*

- *$0 \le P \sim_{\mathbb{R}} lf^*A$,*
- *T is a prime divisor over X,*
- *(X, Λ) is lc over a neighbourhood of z where $\Lambda \ge 0$ and z is the generic point of the image of T on Z,*
- *$a(T, X, B) \le 1$ and $a(T, X, \Lambda) = 0$, and*

- $n(K_X + \Lambda) \sim (n+2)lf^*A$.

Then $\mu_T P \le v$.

Proof. Step 1. In this step we discuss log birational boundedness of $(X, \operatorname{Supp}\Lambda)$. Taking a $\mathbb{Q}$-factorialisation we can assume that X is $\mathbb{Q}$-factorial. After running an MMP on $-K_X$ over Z, we can assume that $-K_X$ is nef over Z. By Lemma 5.2, $qf^*A - K_X$ is globally nef for some bounded natural number q. By Lemma 5.10, $(X, \operatorname{Supp}\Lambda)$ is log birationally bounded, that is, there exist a couple $(\overline{X}, \overline{\Sigma})$, a very ample divisor $\overline{D}$, and a birational map $\rho\colon \overline{X} \dashrightarrow X/Z$ such that

- $(\overline{X}, \overline{\Sigma} + \overline{D})$ is log smooth and belongs to a bounded set of couples $\mathcal{P}$,
- $\overline{\Sigma}$ contains the exceptional divisors of ρ union the birational transform of $\operatorname{Supp}\Lambda$,
- and if $N := qf^*A - K_X$ and $\overline{N} = \rho^*N$, then $\overline{D} - \overline{N}$ is ample.

Let $K_{\overline{X}} + \overline{B}$ and $K_{\overline{X}} + \overline{\Lambda}$ be the pullbacks of $K_X + B$ and $K_X + \Lambda$, respectively. Since (X, Λ) is lc over z and $K_X + \Lambda \sim_{\mathbb{Q}} 0/Z$, $(\overline{X}, \overline{\Lambda})$ is sub-lc over z. Moreover, from $a(T, X, \Lambda) = 0$ we get $a(T, \overline{X}, \overline{\Lambda}) = 0$. But then since $\operatorname{Supp}\overline{\Lambda} \subseteq \overline{\Sigma}$, we have $\overline{\Lambda} \le \overline{\Sigma}$ over z, hence T is also an lc place of $(\overline{X}, \overline{\Sigma})$, that is,

$$a(T, \overline{X}, \overline{\Sigma}) = 0.$$

Step 2. In this step we study numerical properties of $\overline{D}$. Since $(\overline{X}, \overline{\Sigma})$ is log bounded, we can assume that $\overline{D} - \overline{\Sigma}$ is ample. Moreover, by adding a general element of $|(n+2)f^*A|$ to Λ we can assume that some element of $|(n+2)\overline{f}^*A|$ is a component of $\Sigma_{\overline{X}}$ where $\overline{f}$ denotes $\overline{X} \to Z$; this requires replacing l with $l+n$ to preserve the condition $n(K_X + \Lambda) \sim (n+2)lf^*A$, and replacing P accordingly. Thus we can also assume that $\overline{D} - (n+2)\overline{f}^*A$ is ample and that $\overline{D} - \overline{P}$ is ample where $\overline{P} = \rho^*P$.

Now

$$\begin{aligned} \overline{D} - (K_{\overline{X}} + \overline{B}) &\sim_{\mathbb{R}} \overline{D} - \overline{f}^*A + \overline{f}^*A - (K_{\overline{X}} + \overline{B}) \\ &\sim_{\mathbb{R}} \overline{D} - \overline{f}^*A + \overline{f}^*(A - L) \end{aligned}$$

is ample. In addition we can assume $\overline{D} + K_{\overline{X}}$ is ample as well, hence replacing $\overline{D}$ with $2\overline{D}$ we can assume that $\overline{D} - \overline{B}$ is ample.

Step 3. In this step we prove the lemma assuming that the coefficients of $\overline{B}$ *are bounded from below.* That is, assume that the coefficients of $\overline{B}$ are $\ge p$ for some fixed integer p. Under this assumption, there is $c \in (0,1)$ depending only on p such that

$$\overline{\Delta} := c\overline{B} + (1-c)\overline{\Sigma} \ge 0$$

because the components of $\overline{B}$ with negative coefficients are exceptional over X, hence are components of $\overline{\Sigma}$. In particular, since $(\overline{X}, \overline{B})$ is sub-ϵ-lc, $(\overline{X}, \overline{\Delta})$ is an $c\epsilon$-lc pair. Moreover,

$$a(T, \overline{X}, \overline{\Delta}) = ca(T, \overline{X}, \overline{B}) + (1-c)a(T, \overline{X}, \overline{\Sigma}) = ca(T, \overline{X}, \overline{B}) < 1.$$

In addition,

$$\overline{D} - \overline{\Delta} = \overline{D} - c\overline{B} - (1-c)\overline{\Sigma} = c(\overline{D} - \overline{B}) + (1-c)(\overline{D} - \overline{\Sigma})$$

is ample.

Now applying [4, Theorem 1.6] (=Theorem 3.20), we deduce that $(\overline{X}, \overline{\Delta} + t\overline{P})$ is klt for some real number $t > 0$ bounded away from zero. Therefore, $\mu_T P = \mu_T \overline{P}$ is bounded from above because

$$0 \le a(T, \overline{X}, \overline{\Delta} + t\overline{P}) = a(T, \overline{X}, \overline{\Delta}) - t\mu_T\overline{P} < 1 - t\mu_T\overline{P}.$$

Step 4. Finally it is enough to show that the coefficients of $\overline{B}$ are bounded from below. Define $K_{\overline{X}}+\overline{E}=\rho^*K_X$. It is enough to show that the coefficients of $\overline{E}$ are bounded from below because $\overline{E}\le\overline{B}$. Write $\overline{E}$ as the difference $\overline{E}^+-\overline{E}^-$ where $\overline{E}^+,\overline{E}^-\ge 0$ have no common components. Observe that

$$\overline{N}=\rho^*(qf^*A-K_X)=q\overline{f}^*A-(K_{\overline{X}}+\overline{E})=q\overline{f}^*A-(K_{\overline{X}}+\overline{E}^+)+\overline{E}^-,$$

hence

$$2\overline{D}-\overline{E}^-\sim_{\mathbb{R}}\overline{D}-\overline{N}+\overline{D}-(K_{\overline{X}}+\overline{E}^+)+q\overline{f}^*A.$$

By Step 1, $\overline{D}-\overline{N}$ is ample and $\overline{E}^+\le\overline{\Sigma}$. Replacing $\overline{D}$ with a multiple we can assume that $\overline{D}-(K_{\overline{X}}+\overline{E}^+)$ is ample. Thus $2\overline{D}-\overline{E}^-$ is ample which implies that $\overline{D}^{d-1}\cdot\overline{E}^-$ is bounded from above, hence the coefficients of $\overline{E}^-$ are bounded from above which in turn implies that the coefficients of $\overline{E}$ are bounded from below as required.

□

Proof. (of Proposition 5.12) *Step 1. In this step we will translate the problem into showing that the multiplicity of P along certain divisors is bounded from above.* First we can assume $P\ne 0$ otherwise the statement is trivial. In particular, $\dim Z>0$. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Pick a small $\epsilon'\in(0,\epsilon)$. Let s be the ϵ'-lc threshold of P with respect to (X,B), that is, s is the largest number such that $(X,B+sP)$ is ϵ'-lc. It is enough to show that s is bounded from below away from zero. In particular, we can assume $s<1$.

There is a prime divisor T over X with log discrepancy

$$a(T,X,B+sP)=\epsilon'<1.$$

Since P is vertical over Z, T is vertical over Z. It is enough to show that μ_TP, the coefficient of T in the pullback of P on any resolution, is bounded from above because

$$s\mu_TP=a(T,X,B)-a(T,X,B+sP)\ge\epsilon-\epsilon'.$$

We devote the rest of the proof to showing that μ_TP is bounded from above.

Step 2. In this step we apply induction and reduce to the case when T maps to a closed point on Z. By the choice of P,

$$K_X+B+sP\sim_{\mathbb{R}}f^*(L+slA),$$

and since $s<1$, $(l+1)A-(L+slA)$ is ample. Thus replacing B with $B+sP$, replacing A with $(l+1)A$ (and replacing r accordingly), and replacing ϵ with ϵ', we can assume that ϵ is sufficiently small and that $a(T,X,B)=\epsilon$ (we will not use s any more). Extracting T we can also assume T is a divisor on X. Our goal still is to show that μ_TP is bounded from above.

Take a hyperplane section $H\sim A$ of Z and let $G=f^*H$. Consider

$$K_G+B_G:=(K_X+B+G)|_G$$

and $P_G:=P|_G$. Then (G,B_G) is ϵ-lc, $-K_G$ is big over H, $K_G+B_G\sim_{\mathbb{R}}g^*(L+A)|_H$ where g denotes $G\to H$, and $2A|_H-(L+A)|_H$ is ample. Thus $(G,B_G)\to H$ is a $(d-1,2^{d-1}r,\epsilon)$-Fano type fibration. Moreover, $2P_G\sim_{\mathbb{R}}lg^*2A|_H$. Applying induction on dimension we find a real number $u>0$ depending only on d,r,l,ϵ such that (G,B_G+uP_G) is klt. Then by inversion of adjuction [27, Theorem 5.50] (which is stated for $\mathbb{Q}$-divisors but also holds for $\mathbb{R}$-divisors), the pair $(X,B+G+uP)$ is plt near G. In particular, if the image of T on Z

is positive-dimensional, then T intersects G, so $\mu_T P$ is bounded from above. Therefore, we can assume that the image of T on Z is a closed point.

Step 3. In this step we finish the proof by applying Lemma 5.13. Let $\Theta = T$. Since Θ is vertical over Z, $-(K_X + \Theta)$ is big over Z. Run an MMP on $-(K_X + \Theta)$ over Z and let X' be the resulting model. We denote the pushdown of each divisor D to X' by D'. Then $-(K_{X'} + \Theta')$ is nef and big over Z. By construction, $-\epsilon T' \le B' - \Theta'$, hence since T' is mapped to a closed point on Z, $(f^*A)' + B' - \Theta'$ is pseudo-effective. Thus by Proposition 5.2, we can assume that $(lf^*A)' - (K_{X'} + \Theta')$ is nef for some bounded natural number l. Increasing l by 1 we can assume $(lf^*A)' - (K_{X'} + \Theta')$ is nef and big.

On the other hand, $(X', \Theta' - \epsilon T')$ is klt as $\Theta' - \epsilon T' \le B'$. Since ϵ is assumed to be sufficiently small, by the ACC for lc thresholds [17, Theorem 1.1], (X', Θ') is lc. Then applying Theorem 1.10 (by taking $B = \Theta'$, $M = (lf^*A)'$ and S to be the centre of T on X'), there exist a bounded natural numbers n and $\Lambda' \ge \Theta'$ such that (X', Λ') is lc over z and $n(K_{X'} + \Lambda') \sim (n+2)(lf^*A)'$. Since $X \dashrightarrow X'$ is an MMP on $-(K_X + \Theta)$ over Z, taking $K_X + \Lambda$ to be the crepant pullback of $K_{X'} + \Lambda'$ to X we get $\Lambda \ge \Theta \ge \Delta$ such that (X, Λ) is lc over z and $n(K_X + \Lambda) \sim (n+2)lf^*A$. Finally, apply Lemma 5.13 to deduce that $\mu_T P$ is bounded.

□

5.14. **Bounded klt complements.** In this subsection we treat Theorem 1.11 inductively. We first consider a weak version.

Proposition 5.15. *Let d, r be natural numbers, ϵ be a positive real number, and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Assume that Theorems 1.3 and 1.6 hold in dimension $d-1$. Then there exist natural numbers n, m depending only on $d, r, \epsilon, \mathfrak{R}$ satisfying the following. Assume that $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1) and that*

- *we have $0 \le \Delta \le B$ with coefficients in $\mathfrak{R}$, and*
- *$-(K_X + \Delta)$ is big over Z.*

Then for each point $z \in Z$ there is a $\mathbb{Q}$-divisor $\Lambda \ge \Delta$ such that

- *(X, Λ) is lc over z, and*
- *$n(K_X + \Lambda) \sim mf^*A$.*

Proof. Step 1. In this step we create singularities over z. We can assume that $\dim Z > 0$ otherwise we apply [5, Theorem 1.7]. It is enough to prove the proposition with z replaced by any closed point z' in the closure $\bar{z}$ because any open neighbourhood of z' contains z. Thus from now on we assume that z is a closed point. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Consider the sub-linear system V_z of $|f^*A|$ consisting of elements containing the fibre $f^{-1}\{z\}$, and pick P in V_z. Since A is very ample, V_z is base point free outside $f^{-1}\{z\}$. Replacing A with $2A$ we can assume that $\dim V_z > 0$.

Let p be a natural number such that $\frac{1}{p} < 1 - \epsilon$. Pick distinct general elements $M_1, \dots, M_{p(d+1)}$ in V_z and let

$$M = \frac{1}{p}(M_1 + \dots + M_{p(d+1)}).$$

Then $(X, B+M)$ is ϵ-lc outside $f^{-1}\{z\}$ by generality of the M_i and the assumption $\frac{1}{p} < 1 - \epsilon$. On the other hand, $(X, B+M)$ is not lc at any point of $f^{-1}\{z\}$ by [26, Theorem 18.22].

Step 2. In this step we reduce the problem to the situation when there is a prime divisor T on X mapping to z with $a(T,X,B)=\epsilon$ sufficiently small. Now pick a sufficiently small rational number $\epsilon'\in(0,\epsilon)$ and let u be the largest number such that $(X,B+uM)$ is ϵ'-lc. There is a prime divisor T over X such that

$$a(T,X,B+uM)=\epsilon'.$$

As $(X,B+M)$ is not lc near $f^{-1}\{z\}$, $u<1$. Since $(X,B+uM)$ is ϵ-lc outside $f^{-1}\{z\}$ and since $\epsilon'<\epsilon$, the centre of T on X is contained in $f^{-1}\{z\}$. On the other hand, it is clear that

$$K_X+B+uM\sim_{\mathbb{R}} f^*(L+u(d+1)A).$$

Replacing ϵ with ϵ' and replacing B with $B+uM$ (and replacing A,r accordingly) we can assume that ϵ is sufficiently small and that there is a prime divisor T over X mapping to z with $a(T,X,B)=\epsilon$. Extracting T we can assume it is a divisor on X; if T is not exceptional over the original X, we increase the coefficient of T in Δ to $1-\epsilon$; but if T is exceptional over the original X, then we let Δ be the birational transform of the original Δ plus $(1-\epsilon)T$. The bigness of $-(K_X+\Delta)$ over Z is preserved as T is vertical over Z.

Step 3. In this step we find a bounded complement of $K_X+\Delta$ using Theorem 1.10. Let Θ be the same as Δ except that we increase the coefficient of T to 1. Adding 1 to $\mathfrak{R}$ we can assume that the coefficients of Θ are in $\mathfrak{R}$. Since T is vertical over Z, $-(K_X+\Theta)$ is big over Z. Run an MMP on $-(K_X+\Theta)$ over Z and let X' be the resulting model. We denote the pushdown of each divisor D to X' by D'. Then $-(K_{X'}+\Theta')$ is nef and big over Z. By construction, $-\epsilon T'\le B'-\Theta'$, hence since T is mapped to a closed point on Z, $(f^*A)'+B'-\Theta'$ is pseudo-effective. Thus by Proposition 5.2, we can assume that $(lf^*A)'-(K_{X'}+\Theta')$ is nef for some bounded natural number l. Increasing l by 1 we can assume $(lf^*A)'-(K_{X'}+\Theta')$ is nef and big.

On the other hand, $(X',\Theta'-\epsilon T')$ is klt as $\Theta'-\epsilon T'\le B'$. Since ϵ is assumed to be sufficiently small, by the ACC for lc thresholds [17, Theorem 1.1], (X',Θ') is lc. Then applying Theorem 1.10 (by taking $B=\Theta'$, $M=(lf^*A)'$ and S to be the centre of T on X'), there exist a bounded natural number n and $\Lambda'\ge\Theta'$ such that (X',Λ') is lc over z and $n(K_{X'}+\Lambda')\sim(mf^*A)'$ where $m:=l(n+2)$. Since $X\dashrightarrow X'$ is an MMP on $-(K_X+\Theta)$ over Z, taking $K_X+\Lambda$ to be the crepant pullback of $K_{X'}+\Lambda'$ to X we get $\Lambda\ge\Theta\ge\Delta$ such that (X,Λ) is lc over z and $n(K_X+\Lambda)\sim mf^*A$.

□

Now we strengthen the previous statement by replacing lc over z with klt over z.

Proposition 5.16. *Let d,r be natural numbers, ϵ be a positive real number, and $\mathfrak{R}\subset[0,1]$ be a finite set of rational numbers. Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then there exist natural numbers n,m depending only on $d,r,\epsilon,\mathfrak{R}$ satisfying the following. Assume that $(X,B)\to Z$ is a (d,r,ϵ)-Fano type fibration (as in 1.1) and that*

- *we have $0\le\Delta\le B$ with coefficients in $\mathfrak{R}$, and*
- *$-(K_X+\Delta)$ is big over Z.*

Then for each point $z\in Z$ there is a $\mathbb{Q}$-divisor $\Lambda\ge\Delta$ such that

- *(X,Λ) is klt over z, and*
- *$n(K_X+\Lambda)\sim mf^*A$.*

Proof. Step 1. In this step we modify B and introduce a divisor $\tilde{\Delta}$. We can assume that $\dim Z>0$ otherwise we apply [5, Corollary 1.2] which shows that (X,Δ) is log bounded.

Moreover, it is enough to prove the lemma by replacing z with any closed point z' in $\bar{z}$ because (X,Λ) being klt over z' implies that it is klt over z. Thus from now on we assume that z is a closed point. Then as A is very ample we can find $P \in |f^*A|$ containing $f^{-1}\{z\}$.

By Proposition 5.12, there is a rational number $t>0$ depending only on d,r,ϵ such that $(X, B+2tP)$ is lc. Since

$$B+tP = \frac{1}{2}B + \frac{1}{2}(B+2tP),$$

the pair $(X,B+tP)$ is $\frac{\epsilon}{2}$-lc. Moreover,

$$K_X+B+tP \sim_{\mathbb{R}} f^*(L+tA).$$

Thus replacing B with $B+tP$ and replacing ϵ with $\frac{\epsilon}{2}$, we can assume that $B \ge \tilde{\Delta} := \Delta+tP$ for some fixed rational number $t\in(0,1)$ (here we can replace A with $2A$ to ensure that $f^*A-(K_X+B)$ is still nef, and then replace r accordingly). Since P is integral and t is fixed, the coefficients of $\tilde{\Delta}$ belong to a fixed finite set, so expanding $\mathfrak{R}$ we can assume they belong to $\mathfrak{R}$.

Step 2. In this step we reduce the proposition to existence of a special lc complement. Assume that there exist bounded natural numbers n,m and a $\mathbb{Q}$-divisor $\Lambda \ge \tilde{\Delta}$ such that

(1) (X,Λ) is lc over z,
(2) the non-klt locus of (X,Λ) is mapped to a finite set of closed points on Z, and
(3) that $n(K_X+\Lambda)\sim mf^*A$.

Assume that $Q\in|f^*A|$ is general and let $\Lambda' := \Lambda - tP + tQ$. By (2), any non-klt centre of (X,Λ) intersecting $f^{-1}\{z\}$ is actually contained in $f^{-1}\{z\}$. Thus since P contains $f^{-1}\{z\}$, (X,Λ') is klt over z. Moreover, $\Lambda'\ge\Delta$, and perhaps after replacing n,m with a bounded multiple we have

$$n(K_X+\Lambda') = n(K_X+\Lambda-tP+tQ) \sim n(K_X+\Lambda)\sim mf^*A.$$

Therefore, it is enough to find n,m,Λ as in (1)-(3). At this point we replace Δ with $\tilde{\Delta}$. The bigness of $-(K_X+\Delta)$ over Z is preserved as P is vertical.

Step 3. In this step we find a bounded lc complement of $K_X+\Delta$ and study it. After taking a $\mathbb{Q}$-factoriallisation of X and running an MMP on $-(K_X+\Delta)$ over Z we can assume that $-(K_X+\Delta)$ is nef over Z. Applying Proposition 5.2, there is a bounded natural number l such that $lf^*A-(K_X+\Delta)$ is nef globally. Replacing l with $l+1$ we can assume $lf^*A-(K_X+\Delta)$ is nef and big.

By Proposition 5.15, there exist bounded natural numbers n,m and a $\mathbb{Q}$-divisor $\Lambda\ge\Delta$ such that (X,Λ) is lc over z and $n(K_X+\Lambda)\sim mf^*A$. Then

$$n(\Lambda-\Delta) = n(K_X+\Lambda)-n(K_X+\Delta)\sim mf^*A - n(K_X+\Delta),$$

hence

$$n(\Lambda-\Delta)\in|mf^*A-n(K_X+\Delta)|.$$

Multiplying n,m by a bounded number we can assume that $n\Delta$ is integral.

Now by adding a general member of $|2lf^*A|$ to Λ and replacing m with $m+2nl$ to preserve $n(K_X+\Lambda)\sim mf^*A$, we can assume that $m-1\ge l(n+1)$, hence

$$(m-1)f^*A-(n+1)(K_X+\Delta)$$

is nef and big.

Step 4. In this step we consider the restriction of $|mf^*A - n(K_X + \Delta)|$ *to a general member of* $|f^*A|$. Let H be a general member of $|A|$ and let $G = f^*H$. Then

$$mf^*A - n(K_X + \Delta) - G \sim K_X + \Delta + (m-1)f^*A - (n+1)(K_X + \Delta).$$

Thus

$$H^1(mf^*A - n(K_X + \Delta) - G) = 0$$

by the Kawamata-Viehweg vanishing theorem, hence the restriction map

$$H^0(mf^*A - n(K_X + \Delta)) \to H^0((mf^*A - n(K_X + \Delta))|_G)$$

is surjective. Note that for any Weil divisor D on X, we have

$$\mathcal{O}_X(D) \otimes \mathcal{O}_G \simeq \mathcal{O}_G(D|_G)$$

by the choice of G (see [5, 2.41]). This is used to get the above surjectivity.

Step 5. In this step we consider complements on G. Define

$$K_G + B_G = (K_X + B + G)|_G$$

and

$$K_G + \Delta_G = (K_X + \Delta + G)|_G.$$

Then as we have seen several times in this section, $(G, B_G) \to H$ is a $(d-1, r', \epsilon)$-Fano type fibration for some fixed r'. Moreover, $\Delta_G \le B_G$, the coefficients of Δ_G are in $\mathfrak{R}$, and $-(K_G + \Delta_G)$ is big over H.

Since we are assuming Theorem 1.11 in dimension $d-1$, there exist bounded natural numbers p, q and there is a $\mathbb{Q}$-divisor $\Lambda'_G \ge \Delta_G$ such that (G, Λ'_G) is klt and $p(K_G + \Lambda'_G) \sim qg^*A|_H$ where g denotes the morphism $G \to H$. Replacing both n and p with np and then replacing m and q with mp and nq, respectively, we can assume that $n = p$. Next if $q < m + n$, then we increase q to $m + n$ by adding $\frac{1}{n}D_G$ to Λ'_G where D_G is a general element of $(m + n - q)g^*A|_H$. If $q \ge m + n$, we similarly increase m to $q - n$ by modifying Λ so that we can again assume $q = m + n$. Thus we now have

$$n(K_G + \Lambda'_G) \sim (m+n)g^*A|_H.$$

Note that in the process, the inequality $m - 1 \ge l(n+1)$ of Step 3 is preserved, so the surjectivity of Step 4 still holds.

Step 6. In this step we finish the proof. By construction,

$$nR_G := n(\Lambda'_G - \Delta_G) \in |(m+n)g^*A|_H - n(K_G + \Delta_G)|,$$

and $(G, \Lambda'_G = \Delta_G + R_G)$ is klt. Thus if we replace nR_G with any general element of

$$|(m+n)g^*A|_H - n(K_G + \Delta_G)|,$$

then the pair $(G, \Delta_G + R_G)$ is still klt. On the other hand,

$$\begin{aligned}(m+n)g^*A|_H - n(K_G + \Delta_G) &= (m+n)g^*A|_H - n(K_X + \Delta + G)|_G \\ &= ((m+n)f^*A - nG - n(K_X + \Delta))|_G \\ &\sim (mf^*A - n(K_X + \Delta))|_G.\end{aligned}$$

Thus, by the surjectivity in Step 4, a general element

$$nR \in |mf^*A - n(K_X + \Delta)|$$

restricts to a general element

$$nR_G \in |(m+n)g^*A|_H - n(K_G + \Delta_G)|.$$

Now in view of

$$K_G + \Delta_G + R_G = (K_X + \Delta + R + G)|_G$$

and inversion of adjunction [27, Theorem 5.50], the pair $(X, \Delta+R+G)$ is plt near G, hence $(X, \Delta + R)$ is klt near G. Therefore, replacing Λ with $\Delta + R$ we can assume that (X, Λ) is klt near G. In other words, the non-klt locus of (X, Λ) is mapped to a finite set of closed points of Z. Note that (X, Λ) is still lc over z. Thus we have satisfied the conditions (1)-(3) of Step 2.

□

Lemma 5.17. *Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then Theorem 1.11 holds in dimension d.*

Proof. By Proposition 5.16, there exist natural numbers n, m depending only on $d, r, \epsilon, \mathfrak{R}$ such that for each point $z \in Z$ there is a $\mathbb{Q}$-divisor $\Gamma \geq \Delta$ such that

- (X, Γ) is klt over some neighbourhood U_z of z, and
- $n(K_X + \Gamma) \sim mf^*A$.

We can find finitely many closed points $z_1, \dots, z_p$ in Z such that the corresponding open sets U_{z_i} cover Z. For each z_i let Γ_i be the corresponding boundary as above.

From

$$n(\Gamma_i - \Delta) = n(K_X + \Gamma_i) - n(K_X + \Delta) \sim mf^*A - n(K_X + \Delta)$$

we get

$$n(\Gamma_i - \Delta) \in |mf^*A - n(K_X + \Delta)|.$$

Therefore, if nR is a general member of $|mf^*A - n(K_X + \Delta)|$ and if we let $\Lambda := \Delta + R$, then

- (X, Λ) is klt over U_{z_i}, and
- $n(K_X + \Lambda) \sim mf^*A$.

Finally, since we have only finitely many open sets U_{z_i} involved, (X, Λ) is klt everywhere.

□

5.18. **A special case of boundedness of Fano type fibrations.** We treat a special case of Theorem 1.3 inductively.

Lemma 5.19. *Let d, r be natural numbers, ϵ be a positive real number, and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Consider the set of all (d, r, ϵ)-Fano type fibrations $(X, B) \to Z$ (as in 1.1) and $\mathbb{R}$-divisors $0 \leq \Delta \leq B$ such that*

- *the coefficients of Δ are in $\mathfrak{R}$, and*
- *$-(K_X + \Delta)$ is ample over Z.*

Then the set of such (X, Δ) is log bounded.

Proof. By Lemma 5.17, our assumptions imply Theorem 1.11 in dimension d, hence there exist natural numbers n, m depending only on $d, r, \epsilon, \mathfrak{R}$ and a $\mathbb{Q}$-divisor $\Lambda \geq \Delta$ such that

- (X, Λ) is klt, and
- $n(K_X + \Lambda) \sim mf^*A$.

We have

$$n(\Lambda - \Delta) \in |mf^*A - n(K_X + \Delta)|.$$

Increasing m (by adding to Λ appropriately) and applying Proposition 5.2, we can assume that $mf^*A - n(K_X + \Delta)$ is nef and that $l := \frac{m}{n}$ is a natural number. Since $-(K_X + \Delta)$ is ample over Z, replacing m with a bounded multiple (which then replaces l with a bounded multiple), we can assume that $mf^*A - n(K_X + \Delta)$ is ample. In particular, $\Lambda - \Delta$ is ample.

Since (X, Λ) is klt and $n(K_X + \Lambda)$ is Cartier, (X, Λ) is $\frac{1}{n}$-lc. Pick a small $t > 0$ such that

$$(X, \Theta := \Lambda + t(\Lambda - \Delta))$$

is $\frac{1}{2n}$-lc. Here t depends on (X, Λ). Then

$$K_X + \Theta = K_X + \Lambda + t(\Lambda - \Delta) \sim_{\mathbb{Q}} lf^*A + t(\Lambda - \Delta)$$

is ample. In addition, since $\operatorname{Supp}(\Lambda - \Delta) \subseteq \Lambda$ and since $n\Lambda$ is integral, each non-zero coefficient of Θ is at least $\frac{1}{n}$.

Now since $n\Lambda$ is integral and since $lf^*A - (K_X + \Lambda) \sim_{\mathbb{Q}} 0$, (X, Λ) is log birationally bounded, by Proposition 5.10. Thus (X, Θ) is also log birationally bounded as $\operatorname{Supp}\Theta = \operatorname{Supp}\Lambda$. Therefore, (X, Θ) is log bounded by [17, Theorem 1.6] which implies that (X, Δ) is log bounded as $\Delta \leq \Theta$.

□

5.20. **Boundedness of generators of Néron-Severi groups.** To treat the Theorem 1.3 in full generality we need to discuss generators of relative Néron-Severi groups. We start with bounding global Picard numbers.

Lemma 5.21. *Let d, r be natural numbers and ϵ be a positive real number. Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then there is a natural number p depending only on d, r, ϵ satisfying the following. If $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration (as in 1.1), then the Picard number $\rho(X) \leq p$.*

Proof. Replacing X with a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Running an MMP on $-K_X$ over Z we find Y so that $-K_Y$ is nef and big over Z. Replace Y with the ample model of $-K_Y$ over Z so that $-K_Y$ becomes ample over Z. Let $K_Y + B_Y$ be the pushdown of $K_X + B$. Then $(Y, B_Y) \to Z$ is a (d, r, ϵ)-Fano type fibration. Now applying Lemma 5.19 to $(Y, B_Y) \to Z$ we deduce that Y is bounded.

By construction, if D is a prime divisor on X contracted over Y, then

$$a(D, Y, 0) \leq a(D, X, 0) = 1.$$

Thus, by [15, Proposition 2.5], there is a birational morphism $X' \to Y$ from a bounded normal projective variety such that the induced map $X \dashrightarrow X'$ is an isomorphism in codimension one.

We can take a resolution $W \to X'$ such that W is bounded. Then there exist finitely many surjective smooth projective morphisms $V_i \to T_i$ between smooth varieties, depending only on d, r, ϵ, such that W is a fibre of $V_i \to T_i$ over some closed point for some i. Since smooth morphisms are locally products in the complex topology (here we can assume that the ground field is $\mathbb{C}$), $\dim_{\mathbb{R}} H^2(W, \mathbb{R})$ is bounded by some number p depending only on d, r, ϵ. In particular, since the Néron-Severi group $N^1(W)$ is embedded in $H^2(W, \mathbb{R})$ as a vector space, we get

$$\rho(W) \leq \dim_{\mathbb{R}} H^2(W, \mathbb{R}) \leq p.$$

Since $X \dashrightarrow X'$ is an isomorphism in codimension one and since $W \to X'$ is a morphism, the induced map $X \dashrightarrow W$ does not contract divisors, hence $\rho(X) \leq \rho(W) \leq p$.

□

Proposition 5.22. *Let d, r be natural numbers, ϵ be a positive real number, and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then there is a bounded set $\mathcal{P}$ of couples depending only on $d, r, \epsilon, \mathfrak{R}$ satisfying the following. Suppose that*

- *$(X,B)\to Z$ is a (d,r,ϵ)-Fano type fibration (as in 1.1), and that*
- *the coefficients of B are in $\mathfrak{R}$.*

Then there exist a birational map $X \dashrightarrow X'$ and a reduced divisor Σ' on X' such that

- *X' is a $\mathbb{Q}$-factorial normal projective variety,*
- *$X \dashrightarrow X'$ is an isomorphism in codimension one,*
- *(X',Σ') belongs to $\mathcal{P}$,*
- *$\operatorname{Supp} B' \subseteq \Sigma'$ where B' is the birational transform of B, and*
- *the irreducible components of Σ' generate $N^1(X'/Z)$.*

By $N^1(X'/Z)$ we mean $\operatorname{Pic}(X')\otimes\mathbb{R}$ modulo numerical equivalence over Z. Note that there is a natural surjective map $N^1(X')\to N^1(X'/Z)$. We prove some lemmas before giving the proof of the proposition.

Lemma 5.23. *Assume that Proposition 5.22 holds in dimension $\le d-1$. Then the proposition holds in dimension d when X is $\mathbb{Q}$-factorial and there is a non-birational extremal contraction $h\colon X\to Y/Z$.*

Proof. First note that if $\dim Y=0$, then X is an ϵ-lc Fano variety with Picard number one and $K_X+B\sim_{\mathbb{Q}}0$, hence X belongs to a bounded family by [5, Theorem 1.4], hence (X,B) is log bounded as the coefficients of B are in $\mathfrak{R}$ which implies the result in this case as $N^1(X/Z)$ is generated by the components of B. We can then assume that $\dim Y>0$.

Let F be a general fibre of h and let $K_F+B_F:=(K_X+B)|_F$. Then (F,B_F) is ϵ-lc, $K_F+B_F\sim_{\mathbb{Q}}0$, and B_F is big with coefficients in $\mathfrak{R}$, hence F belongs to a bounded family by [5, Theorem 1.4] which implies that (F,B_F) is log bounded. Moreover, by adjunction, we can write

$$K_X+B\sim_{\mathbb{Q}} h^*(K_Y+B_Y+M_Y)$$

where we consider (Y,B_Y+M_Y) as a generalised pair as in Remark 6.2 below. By [6, Theorem 1.4], (Y,B_Y+M_Y) is generalised δ-lc for some fixed $\delta>0$ which depends only on $d,\epsilon,\mathfrak{R}$.

By construction,

$$K_Y+B_Y+M_Y\sim_{\mathbb{R}} g^*L$$

where g denotes the morphism $Y\to Z$. Moreover, since X is of Fano type over Z, Y is also of Fano type over Z (cf, the proof of [5, Lemma 2.12] works in the relative setting). Then $(Y,B_Y+M_Y)\to Z$ is a generalised (d',r,δ)-Fano type fibration for some $d'\le d-1$, as in 2.1. By Lemma 3.24, we can find a boundary Δ_Y so that $(Y,\Delta_Y)\to Z$ is a $(d',r,\frac{\delta}{2})$-Fano type fibration. Therefore, applying Lemma 5.17, there exist bounded natural numbers n,m and a boundary Λ_Y such that (Y,Λ_Y) is klt and $n(K_X+\Lambda_Y)\sim mg^*A$. In particular, $(Y,\Lambda_Y)\to Z$ is a $(d',r',\epsilon':=\frac{1}{n})$-Fano type fibration for some fixed r'. Moreover, increasing m by adding to Λ_Y, we can assume that $m>n$. Since we are assuming Proposition 5.22 in dimension $d-1$, applying it to $(Y,\Lambda_Y)\to Z$ we deduce that there exist a birational map $Y\dashrightarrow Y'/Z$ to a normal projective variety and a reduced divisor $\Sigma_{Y'}$ on Y' satisfying the properties listed in 5.22.

By Lemma 3.29, there exists a birational map $X\dashrightarrow X'/Z$ which is an isomorphism in codimension one so that the induced map $X'\dashrightarrow Y'$ is an extremal contraction (hence a morphism) and X' is normal projective and $\mathbb{Q}$-factorial. Let B' on X' be the birational transform of B and let $\Lambda_{Y'}$ on Y' be the birational transform of Λ_Y. To ease notation we can replace (X,B) and (Y,Λ_Y) with (X',B') and (Y',Λ'_Y) and denote $\Sigma_{Y'}$ by Σ_Y. By construction, $\operatorname{Supp}\Lambda_Y\le\Sigma_Y$.

Since $\operatorname{Supp}\Lambda_Y \subseteq \Sigma_Y$ and since (Y, Σ_Y) is log bounded, there is a very ample divisor H on Y with bounded $s := H^{\dim Y}$ such that

$$H - (K_Y + \Lambda_Y) \sim_{\mathbb{Q}} H - \frac{m}{n} g^* A$$

is ample which implies that $H - g^*A$ is ample as $m > n$. Then

$$H - g^*L = H - g^*A + g^*(A - L)$$

is ample as $A - L$ is ample. Thus $(X, B) \to Y$ is a (d, s, ϵ)-Fano type fibration in view of $K_X + B \sim_{\mathbb{R}} h^*g^*L$.

Replacing H we can in addition assume that $H - \Sigma_Y$ is ample. Thus we can find $0 \le P \sim_{\mathbb{R}} h^*H$ such that $P \ge h^*\Sigma_Y$. Now applying Proposition 5.12 to $(X, B) \to Y$, there is a fixed rational number $t \in (0, 1)$ such that $(X, B + 2tP)$ is klt. Thus $(X, B + 2th^*\Sigma_Y)$ is klt, so

$$(X, \Theta := B + th^*\Sigma_Y)$$

is $\frac{\epsilon}{2}$-lc. Therefore, from

$$K_X + \Theta = K_X + B + th^*\Sigma_Y \sim_{\mathbb{R}} h^*(g^*L + t\Sigma_Y)$$

we deduce that $(X, \Theta) \to Y$ is a $(d, s, \frac{\epsilon}{2})$-Fano type fibration, perhaps after replacing H with $2H$ and replacing s accordingly.

On the other hand, the coefficients of $th^*\Sigma_Y$ belong to a fixed finite set because t is fixed, the Cartier index of Σ_Y is bounded [5, Lemma 2.24], and the coefficients of $th^*\Sigma_Y$ are less than 1. Thus the coefficients of Θ belong to a fixed finite set. Moreover, since $X \to Y$ is extremal and Θ is big over Y, Θ is ample over Y, hence $-(K_X + \frac{1}{2}\Theta) \sim_{\mathbb{R}} \frac{1}{2}\Theta/Y$ is ample over Y. Therefore, applying Lemma 5.19 to $(X, \Theta) \to Y$ (by taking $\Delta = \frac{1}{2}\Theta$) we deduce that $(X, \frac{1}{2}\Theta)$ is log bounded. Since B is big over Y, it is ample over Y, hence the components of B and $h^*\Sigma_Y$ together generate $N^1(X/Z)$ as the components of Σ_Y generate $N^1(Y/Z)$. Now let $\Sigma := \operatorname{Supp}\Theta$.

□

Lemma 5.24. *Proposition 5.22 holds when $X \to Z$ is a small $\mathbb{Q}$-factorialisation.*

Proof. We will apply induction on the relative Picard number $\rho(X/Z) := \dim_{\mathbb{R}} N^1(X/Z)$. By Lemma 5.21, $\rho(X)$ is bounded, so $\rho(X/Z)$ is bounded as well because $\rho(X/Z) \le \rho(X)$. The case $\rho(X/Z) = 0$ is trivial in which case $X \to Z$ is an isomorphism and (X, B) is log bounded, so we assume $\rho(X/Z) > 0$.

By Lemma 5.17, our assumptions imply Theorem 1.11 in dimension d. Since $X \to Z$ is birational, $-(K_X+B)$ is big over Z, hence applying the theorem there exist bounded natural numbers n, m and a boundary $\Lambda \ge B$ such that (X, Λ) is klt and $n(K_X + \Lambda) \sim mf^*A$. In particular, $n(K_X + \Lambda)$ is Cartier and (X, Λ) is $\frac{1}{n}$-lc. Replacing B with Λ, ϵ with $\frac{1}{n}$, A with $2mA$, and replacing $r, \mathfrak{R}$ accordingly, we can assume that $n(K_X + B)$ is Cartier for some fixed natural number n. Replacing n with $2n$ we can assume $n \ge 2$.

Let B_Z be the pushdown of B. By boundedness of length of extremal rays [24], $K_Z + B_Z + (2d + 1)A$ is ample. Thus taking a general member $G \in |n(2d + 1)f^*A|$, adding $\frac{1}{n}G$ to B, and then replacing A with $(2d + 2)A$ (to keep the ampleness of $A - L$), we can assume that $K_X + B$ is the pullback of some ample divisor on Z and that $B - \frac{1}{2}f^*A$ is pseudo-effective. We have used the assumption $n \ge 2$ to make sure that the ϵ-lc property of (X, B) is preserved.

By the cone theorem [27, Theorem 3.7], we can decompose $X \to Z$ into a sequence

$$X = X_1 \to X_2 \to \cdots \to X_l = Z$$

of extremal contractions. Let B_i be the pushdown of B. Then

$$(K_X+B)^d=\operatorname{vol}(K_X+B)=\operatorname{vol}(K_{X_i}+B_i)\le\operatorname{vol}(A)=A^d=r,$$

hence there are only finitely many possibilities for $\operatorname{vol}(K_{X_i}+B_i)$ as $n(K_X+B)$ is Cartier. Therefore, by [28, Theorem 6], the set of such (X_i,B_i) is log bounded. In particular, there is a very ample divisor G_{l-1} on X_{l-1} with bounded G_{l-1}^d such that $G_{l-1}-A_{l-1}$ is ample where A_{l-1} is the pullback of A (here we are using the property that $B-\frac{1}{2}f^*A$ is pseudo-effective).

Let G be the pullback of G_{l-1} to X. Let $\Theta=B+\frac{1}{n}P$ for some general element $P\in|nG|$. Then $K_X+\Theta\sim_{\mathbb{Q}}0/X_{l-1}$ and

$$\begin{aligned}2G-(K_X+\Theta)&=2G-(K_X+B)-\tfrac{1}{n}P\\&\sim_{\mathbb{Q}}G-(K_X+B)\\&=G-f^*A+f^*A-f^*L\end{aligned}$$

is the pullback of an ample divisor on X_{l-1} as $G_{l-1}-A_{l-1}$ and $A-L$ are ample. Thus $(X,\Theta)\to X_{l-1}$ is a (d,u,ϵ)-Fano type fibration for some fixed number u.

Now $\rho(X/X_{l-1})<\rho(X/Z)$. Therefore, by induction on the relative Picard number, there is a birational map $X\dashrightarrow X'/X_{l-1}$ and a reduced divisor Σ' on X' satisfying the properties listed in 5.22 with Θ,X_{l-1} instead of B,Z. Now since P', the birational transform of P, is the pullback of some ample$/Z$ divisor on X_{l-1}, since $P'\le\Sigma'$, and since $X_{l-1}\to X_l=Z$ is extremal, the components of Σ' generate $N^1(X'/Z)$. This proves the lemma.

□

Proof. (of Proposition 5.22) Replacing X with a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. By Lemma 5.21, the Picard number $\rho(X)$ is bounded. We will apply induction on dimension and induction on the relative Picard number $\rho(X/Z)$, in the $\mathbb{Q}$-factorial case. We will assume that $X\to Z$ is not an isomorphism otherwise the proposition holds by taking $X'=X$ and $\Sigma'=\operatorname{Supp}B$ as in this case (X,B) would be log bounded by definition of (d,r,ϵ)-Fano type fibrations.

First we prove the proposition assuming that there is a birational map $h\colon X\dashrightarrow Y/Z$ to a normal projective variety such that h^{-1} does not contract any divisor but h contracts some divisor. Since $K_X+B\sim_{\mathbb{R}}0/Z$, (Y,B_Y) is klt where B_Y is the pushdown of B. Replacing Y with a $\mathbb{Q}$-factorialisation we can assume it is $\mathbb{Q}$-factorial. The log discrepancy of any prime divisor D contracted by h satisfies

$$a(D,Y,B_Y)=a(D,X,B)\le 1.$$

Thus modifying Y by extracting all such divisors except one, we can assume that h contracts a single prime divisor D. Moreover, replacing h with the extraction morphism determined by D, we can assume that h is an extremal divisorial contraction.

Now $(Y,B_Y)\to Z$ is a (d,r,ϵ)-Fano type fibration and $\rho(Y/Z)<\rho(X/Z)$, so by the induction hypothesis, there exist a birational map $Y\dashrightarrow Y'/Z$ to a normal projective variety and a reduced divisor $\Sigma_{Y'}$ on Y' satisfying the properties of the proposition. Replacing Y with Y' and replacing X accordingly (as in the previous paragraph) we can assume $Y=Y'$. We change the notation $\Sigma_{Y'}$ to Σ_Y.

Since $\operatorname{Supp}B_Y\subseteq\Sigma_Y$ and since (Y,Σ_Y) is log bounded, by [4, Theorem 1.6] (=Theorem 3.20), there is a fixed rational number $t>0$ such that

$$(Y,\Theta_Y:=B_Y+t\Sigma_Y)$$

is $\frac{\epsilon}{2}$-lc. Thus since $a(D,Y,\Theta_Y)\le 1$, applying [15, Proposition 2.5] we deduce that there is a birational contraction $X'\to Y$ extracting D but no other divisors and such that if $K_{X'}+\Theta'$ is the pullback of $K_Y+\Theta_Y$, then (X',Θ') is log bounded. In addition, from the proof of

[15, Proposition 2.5] we can see that if $\Sigma' = \mathrm{Supp}(D + \Theta')$, then (X', Σ') is log bounded. Now since Y is $\mathbb{Q}$-factorial, $X' = X$. For convenience we change the notation Θ', Σ' to Θ, Σ. Since $K_X + B \sim_{\mathbb{Q}} 0/Y$ and $B_Y \leq \Theta_Y$, we have $B \leq \Theta$, hence $\mathrm{Supp}\, B \subseteq \mathrm{Supp}\, \Theta \subseteq \Sigma$. By construction, Σ generates $N^1(X/Z)$, so we are done in this case.

Now we prove the proposition in general. If $X \to Z$ is not birational, then running an MMP on K_X ends with a Mori fibre space $\tilde{X} \to Y/Z$; applying the above we can assume that $X \dashrightarrow \tilde{X}$ does not contract any divisor, hence replacing X we can assume $X = \tilde{X}$; we can then apply Lemma 5.23. Now assume that $X \to Z$ is birational. Applying the above again reduces the proposition to the case when $X \to Z$ is a small contraction. But then we can apply Lemma 5.24.

□

5.25. **Boundedness of Fano type fibrations.** In this subsection we treat Theorems 1.2 and 1.3 inductively.

Lemma 5.26. *Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then Theorems 1.2 and 1.3 hold in dimension d.*

Proof. It is enough to treat 1.3 as it implies 1.2 by taking $\Delta = 0$. If $\dim Z = 0$, then X belong to a bounded family by [4, Corollary 1.2] from which we can deduce that (X, Δ) is log bounded. We can then assume that $\dim Z > 0$. Let $(X, B) \to Z$ be a (d, r, ϵ)-Fano type fibration and $0 \leq \Delta \leq B$, as in 1.3. Changing the coefficients of Δ we can assume that all its coefficients are equal to a fixed rational number, and that $\mathrm{Supp}(B - \Delta) = \mathrm{Supp}\, B$. This in particular implies that $-(K_X + \Delta) \sim_{\mathbb{R}} B - \Delta/Z$ is big over Z.

Now by Lemma 5.17, our assumptions imply Theorem 1.11 in dimension d, hence applying the theorem we can find bounded natural numbers $n, m \geq 2$ and a boundary $\Lambda \geq \Delta$ such that (X, Λ) is klt and $n(K_X + \Lambda) \sim mf^*A$. Replacing B with Λ, ϵ with $\frac{1}{n}$, and A with $2mA$ we can assume that the coefficients of B belong to some fixed finite set of rational numbers. Furthermore, taking a general element $P \in |n(2d+1)f^*A|$ and adding $\frac{1}{n}P$ to B and replacing A, r accordingly we can assume that B is big and that $K_X + B$ is nef.

By Proposition 5.22, there exist a birational map $X \dashrightarrow X'/Z$ and a reduced divisor Σ' on X' satisfying the properties listed in the proposition. Since (X', Σ') is log bounded, there is a very ample divisor G' on X' with bounded G'^d and bounded $G'^{d-1} \cdot \Sigma'$. Moreover, as $\mathrm{Supp}\, B' \subseteq \Sigma'$, (X', B') is log bounded. Now since B' is big, there is a fixed natural number $l > 0$ such that $lB' \sim \Sigma' + D'$ where $D' \geq 0$. From $B' \leq \Sigma'$ we deduce that

$$(X', \mathrm{Supp}(\Sigma' + D'))$$

is log bounded because

$$G'^{d-1} \cdot (\Sigma' + D') \leq G'^{d-1} \cdot (\Sigma' + lB') \leq G'^{d-1} \cdot (\Sigma' + l\Sigma')$$

is bounded. Replacing G' with a multiple we can then assume that $G' - B'$ and $G' - (\Sigma' + D')$ are ample. Therefore, by [4, Theorem 1.6] (=Theorem 3.20), there is a fixed rational number $t \in (0, 1)$ such that

$$(X', B' + t\Sigma' + tD')$$

is $\frac{\epsilon}{2}$-lc. Replacing B' with

$$(1-t)B' + \frac{t}{l}(\Sigma' + D') \sim_{\mathbb{Q}} B',$$

replacing B accordingly, replacing Σ' with $\mathrm{Supp}(\Sigma' + D')$, and replacing ϵ with $\frac{\epsilon}{2}$, we can assume that $\mathrm{Supp}\, B' = \Sigma'$. In addition, by the previous paragraph, we can assume that

$B \geq \frac{1}{n}P$ for some general member $P \in |n(2d+1)f^*A|$ hence that the birational transform $P' \leq \Sigma'$.

Let H be an ample $\mathbb{Q}$-divisor on X and let H' be its birational transform on X'. Since the components of Σ' generate $N^1(X'/Z)$, there exists an $\mathbb{R}$-divisor $R' \equiv H'/Z$ such that $\operatorname{Supp} R' \subseteq \Sigma'$. In particular, if R is the birational transform of R' on X, then R is ample over Z. Replacing R' with a small multiple and adding a multiple of P' to it, we can assume that R is globally ample. Since $\operatorname{Supp} R \subseteq \operatorname{Supp} B$, rescaling R we can in addition assume that $\Theta := B + R \geq \frac{1}{2}\Delta$, that the coefficients of Θ are $\geq \frac{\delta}{2}$, and that (X, Θ) is $\frac{\epsilon}{2}$-lc.

By construction, $\operatorname{Supp} \Theta' = \operatorname{Supp} B' = \Sigma'$ where Θ' is the birational transform of Θ. Thus (X, Θ) is log birationally bounded. Moreover, $K_X + \Theta$ is ample as $K_X + B$ is nef and R is ample. Therefore, applying [17, Theorem 1.6] we deduce that (X, Θ) is log bounded which in particular means that (X, Δ) is log bounded.

□

5.27. **Lower bound on lc thresholds.**

Lemma 5.28. *Assume that Theorems 1.3, 1.6, and 1.11 hold in dimension $d-1$. Then Theorem 1.6 holds in dimension d.*

Proof. Assume that $(X, B) \to Z$ is a (d, r, ϵ)-Fano type fibration and $P \geq 0$ is $\mathbb{R}$-Cartier such that either $f^*A + B - P$ or $f^*A - K_X - P$ is pseudo-effective. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. First assume that $f^*A + B - P$ is pseudo-effective. Since $A - L$ is nef, $f^*A - (K_X + B)$ is nef, hence $2f^*A - K_X - P$ is pseudo-effective. Thus replacing A with $2A$, it is enough to treat the theorem in the case when $f^*A - K_X - P$ is pseudo-effective.

By Lemma 5.26, Theorem 1.3 holds in dimension d. Let $D \in |f^*A|$ be a general element. Let $\Theta := B + \frac{1}{2}D$. Then

$$K_X + \Theta \sim_{\mathbb{R}} f^*(L + \frac{1}{2}A)$$

and (X, Θ) is ϵ'-lc where $\epsilon' = \min\{\epsilon, \frac{1}{2}\}$. Thus $(X, \Theta) \to Z$ is a $(d, 2^{d-1}r, \epsilon')$-Fano type fibration. Applying 1.3, we deduce that (X, D) is log bounded. Thus there is a very ample divisor H on X such that H^d is bounded and $H + K_X - D$ is ample.

Since $f^*A - (K_X + B)$ is nef,

$$H - B \sim H + K_X - D + f^*A - (K_X + B)$$

is ample. On the other hand, since $Q := f^*A - K_X - P$ is pseudo-effective,

$$H - P = H + K_X - f^*A + Q$$

is big, hence $|H - P|_{\mathbb{R}} \neq \emptyset$. Now by [4, Theorem 1.6] (=Theorem 3.20), there is a real number $t > 0$ depending only on d, H^d, ϵ such that $(X, B + tP)$ is klt. By construction, t depends only on d, r, ϵ.

□

5.29. **Proofs of 1.2, 1.3, 2.2, 1.6, 1.11.** We are now ready to prove several of the main results of this paper. We apply induction so we assume that 1.3, 1.6, and 1.11 hold in dimension $d-1$.

Proof. (of Theorems 1.2, 1.3, and 2.2) Theorems 1.2 and 1.3 follow from Theorems 1.3, 1.6, and 1.11 in dimension $d-1$, and Lemma 5.26. Theorem 2.2 follows from Lemma 3.24 and Theorem 1.3.

□

Proof. (of Theorem 1.6) This follows from Theorems 1.3, 1.6, and 1.11 in dimension $d-1$, and Lemma 5.28.

$\square$

Proof. (of Theorem 1.11) This follows from Theorems 1.3, 1.6, and 1.11 in dimension $d-1$, and Lemma 5.17.

$\square$

6. **Generalised log Calabi-Yau fibrations**

In this section we discuss singularities and boundedness of log Calabi-Yau fibrations in the context of generalised pairs.

6.1. **Adjunction for generalised fibrations.** Consider the following set-up. Assume that

- $(X, B+M)$ is a generalised sub-pair with data $X' \to X$ and M',
- $f\colon X \to Z$ is a contraction with $\dim Z > 0$,
- $(X, B+M)$ is generalised sub-lc over the generic point of Z, and
- $K_X + B + M \sim_{\mathbb{R}} 0/Z$.

We define the discriminant divisor B_Z for the above setting, similar to the definition in the introduction. Let D be a prime divisor on Z. Let t be the generalised lc threshold of f^*D with respect to $(X, B+M)$ over the generic point of D. This makes sense even if D is not $\mathbb{Q}$-Cartier because we only need the pullback f^*D over the generic point of D where Z is smooth. We then put the coefficient of D in B_Z to be $1-t$. Note that since $(X, B+M)$ is generalised sub-lc over the generic point of Z, t is a real number, that is, it is not $-\infty$ or $+\infty$. Having defined B_Z, we can find M_Z giving

$$K_X + B + M \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where M_Z is determined up to $\mathbb{R}$-linear equivalence. We call B_Z the *discriminant divisor of adjunction* for $(X, B+M)$ over Z.

Let $Z' \to Z$ be a birational contraction from a normal variety. There is a birational contraction $X' \to X$ from a normal variety so that the induced map $X' \dashrightarrow Z'$ is a morphism. Let $K_{X'} + B' + M'$ be the pullback of $K_X + B + M$. We can similarly define $B_{Z'}, M_{Z'}$ for $(X', B'+M')$ over Z'. In this way we get the *discriminant b-divisor* $\mathbf{B}_Z$ *of adjunction* for $(X, B+M)$ over Z. Fixing a choice of M_Z we can pick the $M_{Z'}$ consistently so that it also defines a b-divisor $\mathbf{M}_Z$ which we refer to as the *moduli b-divisor of adjunction* for $(X, B+M)$ over Z.

Remark 6.2. Assume that $M = 0$, B is a $\mathbb{Q}$-divisor, (X, B) is projective, and that (X, B) is lc over the generic point of Z. Then $\mathbf{M}_Z$ is b-nef b-$\mathbb{Q}$-Cartier, that is, we can pick Z' so that $M_{Z'}$ is a nef $\mathbb{Q}$-divisor and for any resolution $Z'' \to Z'$, $M_{Z''}$ is the pullback of $M_{Z'}$ [5, Theorem 3.6] (this is derived from [14] which is in turn derived from [1] and this in turn is based on [23]). We can then consider $(Z, B_Z + M_Z)$ as a generalised pair with nef part $M_{Z'}$. When $M \neq 0$, the situation is more complicated, see [12] for recent advances in this direction which we will not use in this paper.

6.3. **Lower bound for lc thresholds: proof of 2.3.**

Proof. (of Theorem 2.3) *Step 1. In this step we make some preparations.* Let $(X, B+M) \to Z$ and P be as in Theorem 2.3 in dimension d. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Assume that $f^*A + B + M - P$ is pseudo-effective. Since

$$f^*A - (K_X + B + M) \sim_{\mathbb{R}} f^*(A - L)$$

is nef, $2f^*A - K_X - P$ is pseudo-effective. Thus replacing A with $2A$, it is enough to treat Theorem 2.3 in the case when $f^*A - K_X - P$ is pseudo-effective.

Step 2. In this step we take a log resolution and introduce some notation. Since B is effective and M is pseudo-effective (as it is the pushdown of a nef divisor), $B+M$ is pseudo-effective. Moreover, since $-K_X$ is big over Z, $B+M$ is big over Z, hence $B+M+f^*A$ is big globally. Let $\phi\colon X' \to X$ be a log resolution of (X, B) on which the nef part M' of $(X, B+M)$ resides. Write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M).$$

Since $(X, B+M)$ is generalised ϵ-lc, the coefficients of B' do not exceed $1-\epsilon$. We can write

$$\phi^*(B + M + f^*A) \sim_{\mathbb{R}} G' + H'$$

where $G' \ge 0$ and H' is general ample. Replacing ϕ we can assume ϕ is a log resolution of $(X, B + P + \phi_*G')$.

Pick a small $\alpha > 0$ and pick a general

$$0 \le R' \sim_{\mathbb{R}} \alpha H' + (1-\alpha)M'.$$

Since M' is nef, $\phi^*M = M' + E'$ where E' is effective and exceptional. Let

$$\Delta' := B' - \alpha\phi^*B - \alpha E' + \alpha G' + R'.$$

We can make the above choices so that the coefficients of Δ' do not exceed $1 - \frac{\epsilon}{2}$ and so that (X', Δ') is log smooth.

Step 3. In this step we show that $(X, \Delta) \to Z$ is a $(d, r, \frac{\epsilon}{2})$-Fano type fibration where $\Delta = \phi_*\Delta'$. By construction, we have

$$\begin{aligned} K_{X'} + \Delta' &= K_{X'} + B' - \alpha\phi^*B - \alpha E' + \alpha G' + R' \\ &\sim_{\mathbb{R}} K_{X'} + B' - \alpha\phi^*B - \alpha E' + \alpha G' + \alpha H' + (1-\alpha)M' \\ &\sim_{\mathbb{R}} K_{X'} + B' - \alpha\phi^*B - \alpha E' + \alpha\phi^*(B + M + f^*A) + (1-\alpha)M' \\ &\sim_{\mathbb{R}} K_{X'} + B' - \alpha\phi^*B - \alpha\phi^*M + \alpha\phi^*(B + M + f^*A) + M' \\ &\sim_{\mathbb{R}} K_{X'} + B' + M' + \alpha\phi^*f^*A \sim_{\mathbb{R}} \phi^*f^*(L + \alpha A). \end{aligned}$$

Therefore,

$$K_X + \Delta \sim_{\mathbb{R}} f^*(L + \alpha A).$$

Choosing α small enough we can ensure $A - (\alpha A + L)$ is ample. On the other hand, since $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0/X$, we have $K_{X'} + \Delta' = \phi^*(K_X + \Delta)$, hence (X, Δ) is $\frac{\epsilon}{2}$-lc because the coefficients of Δ' do not exceed $1-\frac{\epsilon}{2}$. Thus $(X, \Delta) \to Z$ is a $(d, r, \frac{\epsilon}{2})$-Fano type fibration.

Step 4. In this step we finish the proof. By Theorem 1.6, there is a real number $t > 0$ depending only on d, r, ϵ such that $(X, \Delta + 2tP)$ is klt. Then letting $P' = \phi^*P$ we see that the coefficients of $\Delta' + 2tP'$ do not exceed 1 as

$$K_{X'} + \Delta' + 2tP' = \phi^*(K_X + \Delta + 2tP).$$

Thus the coefficients of

$$B' - \alpha\phi^* B - \alpha E' + 2tP'$$

do not exceed 1. Now t is independent of the choice of α, so taking the limit as α approaches zero, we see that the coefficients of $B' + 2tP'$ do not exceed 1. Therefore, the coefficients of $B' + tP'$ are strictly less than 1 because the coefficients of B' do not exceed $1 - \epsilon$, hence $(X, B + tP + M)$ is generalised klt as

$$K_{X'} + B' + tP' + M' = \phi^*(K_X + B + tP + M).$$

$\square$

6.4. **Upper bound for the discriminant b-divisor when the base is bounded.**

Proof. (of Theorems 1.8 and 2.5) Since 1.8 is a special case of 2.5 we treat the latter only. By induction we can assume that Theorem 2.5 holds in dimension $d-1$. Let $(X, B+M) \to Z$ be a generalised (d, r, ϵ)-Fano type fibration. Let D be a prime divisor over Z. First assume that the centre of D on Z is positive-dimensional. Take a resolution $Z' \to Z$ so that D is a divisor on Z'. Take a log resolution $\phi\colon X' \to X$ of (X, B) so that the nef part M' of $(X, B+M)$ is on X' and that the induced map $f'\colon X' \dashrightarrow Z'$ is a morphism. Replacing X' we can assume that ϕ is a log resolution of $(X, B + \phi_* f'^* D)$.

Let $K_{X'} + B' + M'$ be the pullback of $K_X + B + M$. Let t be the generalised lc threshold of f'^*D with respect to $(X', B' + M')$ over the generic point of D: this coincides with the lc threshold of f'^*D with respect to (X', B') over the generic point of D because M' is nef. Since $(X', B' + tf'^*D)$ is log smooth and since it is sub-lc but not sub-klt over the generic point of D, there is a prime divisor S on X' mapping onto D such that $\mu_S B' + t\mu_S f'^* D = 1$.

Let $H \in |A|$ be a general member and let H', G, G' be its pullback to Z', X, X', respectively. Since the centre of D on Z is positive-dimensional, H' intersects D and G' intersects S. By divisorial generalised adjunction we can write

$$K_G + B_G + M_G \sim_{\mathbb{R}} (K_X + B + G + M)|_G$$

where $(G, B_G + M_G)$ is generalised ϵ-lc with nef part $M_{G'} = M'|_{G'}$. Moreover, $-K_G$ is big over H, and

$$K_G + B_G + M_G \sim_{\mathbb{R}} g^*(L + A)|_H$$

where g denotes $G \to H$. Thus $(G, B_G + M_G) \to H$ is a generalised $(d, 2^{d-1}r, \epsilon)$-Fano type fibration. We can write

$$K_{G'} + B_{G'} + M_{G'} \sim_{\mathbb{R}} (K_{X'} + B' + G' + M')|_{G'}$$

where $B_{G'} = B'|_{G'}$ and $K_{G'} + B_{G'} + M_{G'}$ is the pullback of $K_G + B_G + M_G$.

Let C be a component of $D \cap H'$ and let s be the generalised lc threshold of g'^*C with respect to $(G', B_{G'} + M_{G'})$ over the generic point of C where g' denotes $G' \to H'$. Then $1-s$ is the coefficient of C in the discriminant b-divisor of adjunction for $(G, B_G + M_G) \to H$. Thus applying Theorem 2.5 in dimension $d-1$, we deduce that $1 - s \le 1 - \delta$ for some $\delta > 0$ depending only on d, r, ϵ. Thus $s \ge \delta$.

By definition of s, for any prime divisor T on G' mapping onto C, we have the inequality $\mu_T B_{G'} + s\mu_T g'^* C \le 1$. In particular, if we take T to be a component of $S \cap G'$ which maps onto C, then we have

$$\begin{aligned}
\mu_S B' + s\mu_S f'^* D &= \mu_T B'|_{G'} + s\mu_T f'^* D|_{G'} \\
&= \mu_T B_{G'} + s\mu_T g'^* D|_{H'} \\
&= \mu_T B_{G'} + s\mu_T g'^* C \\
&\le 1 \\
&= \mu_S B' + t\mu_S f'^* D
\end{aligned}$$

where we use the fact that over the generic point of C the two divisors $g'^*D|_{H'}$ and g'^*C coincide. Therefore, $\delta \le s \le t$, hence $\mu_D B_{Z'} = 1 - t \le 1 - \delta$ where $B_{Z'}$ is the discriminant divisor on Z' defined for $(X, B+M)$ over Z. Thus we have settled the case when the centre of D on Z is positive-dimensional.

From now on we can assume that the centre of D on Z is a closed point, say z. Let Z', X', f', B', M' be as before. Pick $N \in |A|$ passing through z. Then

$$f^*A + B + M - f^*N \sim B + M$$

is obviously pseudo-effective. Thus by Theorem 2.3 in dimension d, the generalised lc threshold u of f^*N with respect to $(X, B + M)$ is bounded from below by some $\delta > 0$ depending only on d, r, ϵ.

Since N passes through z, we have $\psi^*N \ge D$ where ψ denotes $Z' \to Z$. Thus the generalised lc threshold v of $f'^*\psi^*N$ with respect to $(X', B' + M')$ over the generic point of D is at most as large as the generalised lc threshold t of f'^*D with respect to $(X', B' + M')$ over the generic point of D. On the other hand, the generalised lc threshold u of f^*N with respect to $(X, B + M)$ globally coincides with the generalised lc threshold of $f'^*\psi^*N$ with respect to $(X', B'+M')$ globally which is at most as large as the generalised lc threshold v of $f'^*\psi^*N$ with respect to $(X', B'+M')$ over the generic point of D. Therefore, $\delta \le u \le v \le t$, hence $\mu_D B_{Z'} = 1 - t \le 1 - \delta$.

□

6.5. **Upper bound for the discriminant b-divisor when log general fibres are bounded.** In this subsection, we prove 2.7, 2.8, and 1.9. We first prove 2.7 when the base is one-dimensional. We use ideas similar to the proof of [6, Theorem 1.4].

Proposition 6.6. *Theorem 2.7 holds when* $\dim Z = 1$.

Proof. Step 1. In this step we do some preparations and introduce a boundary Δ' *on* X'. Let D be a prime divisor on Z. We want to show that the coefficient $\mu_D B_Z$ is bounded from above away from 1 where B_Z is the discriminant divisor of adjunction of $(X, B + M)$ over Z. Since this is a local problem near D we will shrink Z around D if necessary. Taking a $\mathbb{Q}$-factorialisation, we can assume X is $\mathbb{Q}$-factorial.

Denote the given morphism $X' \to X$ by ϕ. Replacing ϕ we can assume it is a log resolution of $(X, B+G+f^*D)$. Write $K_{X'}+B'+M'$, G' for the pullbacks of K_X+B+M, G, respectively. Let Σ' be the birational transform of the horizontal over Z part of $\operatorname{Supp}(B+G)$ union the horizontal over Z exceptional divisors of ϕ. Denote $X' \to Z$ by f', and let

$$\Delta' = (1 - \frac{\epsilon}{2})\Sigma' + \operatorname{Supp} f'^*D.$$

Shrinking Z we can assume that $\operatorname{Supp} \Delta'$ coincides with the reduced exceptional divisor of ϕ union the birational transform of $\operatorname{Supp}(B+G+f^*D)$ (so we get rid of divisors which are vertical over Z but do not map to D).

Step 2. In this step we study $K_{X'} + \Delta' + 2M'$. By construction, $(X', \Delta' + 2M')$ is generalised lc globally and generalised $\frac{\epsilon}{2}$-lc over $Z \setminus \{D\}$. Moreover, the coefficients of Δ' belong to $\{1 - \frac{\epsilon}{2}, 1\}$ and $\lfloor \Delta' \rfloor = \operatorname{Supp} f'^*D$. Since $(X, B + M)$ is generalised ϵ-lc, the coefficients of B' are at most $1-\epsilon$. Furthermore, $\operatorname{Supp} B' \subseteq \Sigma' + \operatorname{Supp} f'^*D$. Thus we have

$$B' \le (1 - \epsilon)(\Sigma' + \operatorname{Supp} f'^*D)$$

which in turn gives

$$\Delta' - B' \ge \Delta' - (1 - \epsilon)(\Sigma' + \operatorname{Supp} f'^*D) = \frac{\epsilon}{2}\Sigma' + \epsilon \operatorname{Supp} f'^*D.$$

On the other hand, $\phi^*M - M'$ is effective and exceptional over X, so we can write
$$\phi^*((\operatorname{Supp} B) + M + G) = M' + N'$$
for some $N' \geq 0$ with
$$\operatorname{Supp} N' \subseteq \Sigma' + \operatorname{Supp} f'^*D.$$
In particular, $\Delta' - B' \geq \alpha N'$ for some small $\alpha > 0$.

Now since
$$0 < \operatorname{vol}((\operatorname{Supp} B) + M + G)|_F)$$
for the general fibres F of f, $(\operatorname{Supp} B) + M + G$ is big over Z, hence $M' + N'$ is big over Z. Then since M' is nef over Z,
$$M' + \alpha N' = (1-\alpha)M' + \alpha(M' + N')$$
is big over Z. This in turn implies that $M' + \Delta' - B'$ is big over Z because $\Delta' - B' \geq \alpha N'$. Therefore, from
$$K_{X'} + \Delta' + 2M' \sim_{\mathbb{R}} K_{X'} + \Delta' + 2M' - (K_{X'} + B' + M') = M' + \Delta' - B'/Z$$
we deduce that $K_{X'} + \Delta' + 2M'$ is big over Z. Also note that by assumption pM' is Cartier.

Step 3. In this step we show that $(X', \Delta' + 2M')$ has a generalised lc model over Z, that is, an ample model over Z. We have
$$K_{X'} + \Delta' + 2M' \sim_{\mathbb{R}} K_{X'} + \Delta' - \alpha f'^*D + 2M'/Z.$$
Choosing α to be small enough we can ensure that
$$\Theta' := \Delta' - \alpha f'^*D \geq 0.$$
Then $(X', \Theta' + 2M')$ is generalised klt as $\lfloor \Delta' \rfloor = \operatorname{Supp} f'^*D$, and $K_{X'} + \Theta' + 2M'$ is big over Z. Thus we can run an MMP on $K_{X'} + \Theta' + 2M'$ over Z terminating with a minimal model, say $\tilde{X}''$, on which $K_{\tilde{X}''} + \tilde{\Theta}'' + 2\tilde{M}''$ is semi-ample over Z [9, Lemma 4.4], hence defining a contraction $\tilde{X}'' \to X''/Z$. As
$$K_{X'} + \Delta' + 2M' \sim_{\mathbb{R}} K_{X'} + \Theta' + 2M'/Z,$$
X'' is also the generalised lc model of $(X', \Delta' + 2M')$ over Z. In particular, $(X'', \Delta'' + 2M'')$ is generalised lc with nef part being the pullback of M' to some common resolution of X', X'', and $K_{X''} + \Delta'' + 2M''$ is ample over Z.

Step 4. In this step we obtain lower bound for the volume of $K_{X''} + \Delta'' + 2M''$ restricted to components of the fibre of $X'' \to Z$ over D. Let S be the normalisation of a component T of f''^*D where f'' is the morphism $X'' \to Z$. Since T is a component of $\lfloor \Delta'' \rfloor$, by generalised divisorial adjunction [5, Subsection 3.1], we can write
$$K_S + \Delta''_S + 2M''_S \sim_{\mathbb{R}} (K_{X''} + \Delta'' + 2M'')|_S$$
such that $(S, \Delta''_S + 2M''_S)$ is a generalised pair data $\overline{S} \to S$ and $M_{\overline{S}}$ where the nef part $M_{\overline{S}}$ is the restriction of the nef part of $(X'', \Delta'' + 2M'')$. Since the coefficients of Δ'' are in a fixed finite set and since pM' is Cartier, the coefficients of Δ''_S are in a fixed DCC set Ψ and $M_{\overline{S}}$ is b-Cartier [5, Lemma 3.3]. Moreover, $(S, \Delta''_S + 2M''_S)$ is generalised lc and $K_S + \Delta''_S + 2M''_S$ is ample.

Let $\Lambda_{\overline{S}}$ be the sum of the reduced exceptional divisor of $\overline{S} \to S$ and the birational transform of Δ''_S. Applying [9, Theorem 1.3], we find a natural number m depending only on d, p, Ψ such that
$$|\lfloor m(K_{\overline{S}} + \Lambda_{\overline{S}} + 2M_{\overline{S}}) \rfloor|$$

defines a birational map, hence

$$|\lfloor m(K_S+\Delta''_S+2M''_S)\rfloor|$$

also defines a birational map. In particular,

$$(K_S+\Delta''_S+2M''_S)^{d-1}=\operatorname{vol}(K_S+\Delta''_S+2M''_S)\geq \frac{1}{m^{d-1}}.$$

Step 5. In this step we study the intersection number of $K_{X''}+\Delta''+2M''$ *with the fibres of* f''. Write $f''^*D=\sum m_iT_i$ where T_i are irreducible components, and let S_i be the normalisation of T_i. In later steps we will show that the m_i are bounded from above. As in the previous step we write

$$K_{S_i}+\Delta''_{S_i}+2M''_{S_i}\sim_{\mathbb{R}}(K_{X''}+\Delta''+2M'')|_{S_i}.$$

Then

$$(K_{S_i}+\Delta''_{S_i}+2M''_{S_i})^{d-1}=(K_{X''}+\Delta''+2M'')^{d-1}\cdot T_i$$

where to define the latter intersection number we use the fact that $X''\to Z$ is a projective morphism over a curve. In particular,

$$\begin{aligned}\sum m_i(K_{S_i}+\Delta''_{S_i}+2M''_{S_i})^{d-1}&=\sum m_i(K_{X''}+\Delta''+2M'')^{d-1}\cdot T_i\\&=(K_{X''}+\Delta''+2M'')^{d-1}\cdot(\sum m_iT_i)\\&=(K_{X''}+\Delta''+2M'')^{d-1}\cdot f''^*D.\end{aligned}$$

Thus if F'' is a general fibre of f'', then since $f''^*D\sim F''$ we get

$$\begin{aligned}\sum m_i(K_{S_i}+\Delta''_{S_i}+2M''_{S_i})^{d-1}&=(K_{X''}+\Delta''+2M'')^{d-1}\cdot F''\\&=((K_{X''}+\Delta''+2M'')|_{F''})^{d-1}\\&=\operatorname{vol}((K_{X''}+\Delta''+2M'')|_{F''}).\end{aligned}$$

Step 6. In this step we show that $\operatorname{vol}((K_{X''}+\Delta''+2M'')|_{F''})$ *is bounded from above.* Indeed let F,F' be the fibres of f,f' corresponding to F''. Since $(X'',\Delta''+2M'')$ is the generalised lc model of $(X',\Delta'+2M')$ and since F'' is a general fibre, we have

$$\operatorname{vol}((K_{X''}+\Delta''+2M'')|_{F''})\leq\operatorname{vol}((K_{X'}+\Delta'+2M')|_{F'}).$$

Actually equality holds but we do not need it. On the other hand,

$$\operatorname{vol}((K_{X'}+\Delta'+2M')|_{F'})\leq\operatorname{vol}((K_X+\Delta+2M)|_F)$$

where Δ is the pushdown of Δ', because the pushdown of $(K_{X'}+\Delta'+2M')|_{F'}$ to F is $(K_X+\Delta+2M)|_F$ (we are using the assumption that F is a general fibre).

Let Σ be the pushdown of Σ', that is, Σ is the support of the horizontal$/Z$ part of $B+G$. In particular, $\Sigma\leq(\operatorname{Supp}B)+G$, hence

$$\Sigma+M\leq(\operatorname{Supp}B)+M+G$$

which implies that

$$\operatorname{vol}((\Sigma+M)|_F)\leq\operatorname{vol}(((\operatorname{Supp}B)+M+G)|_F)<v.$$

By construction,

$$\Delta=(1-\frac{\epsilon}{2})\Sigma+\operatorname{Supp}f^*D,$$

so

$$\begin{aligned}\mathrm{vol}((K_X+\Delta+2M)|_F) &= \mathrm{vol}((K_X+(1-\tfrac{\epsilon}{2})\Sigma+2M)|_F)\\ &\le \mathrm{vol}((K_X+\Sigma+2M)|_F)\\ &= \mathrm{vol}((K_X+\Sigma+2M-K_X-B-M)|_F)\\ &= \mathrm{vol}((\Sigma+M-B)|_F)\\ &\le \mathrm{vol}((\Sigma+M)|_F)\\ &< v.\end{aligned}$$

Therefore, $\mathrm{vol}((K_{X''}+\Delta''+2M'')|_{F''})<v$.

Step 7. In this step we show that the m_i are bounded from above. Recall from Step 5 that

$$\mathrm{vol}((K_{X''}+\Delta''+2M'')|_{F''})=\sum m_i(K_{S_i}+\Delta''_{S_i}+2M''_{S_i})^{d-1}.$$

By the previous step, the left hand side is bounded from above. On the other hand, by Step 4,

$$\sum m_i(K_{S_i}+\Delta''_{S_i}+2M''_{S_i})^{d-1}\ge\sum\frac{m_i}{m^{d-1}}.$$

Therefore, the right hand side is bounded from above, hence the m_i are all bounded from above.

Step 8. In this final step we finish the proof. We will denote the pushdown of B' to X'' by B'', etc. By Step 2 we get

$$\Delta''-B''\ge\frac{\epsilon}{2}\Sigma''+\epsilon\,\mathrm{Supp}\,f''^*D.$$

Thus since $f''^*D=\sum m_iT_i$ with m_i bounded, there is a positive real number δ bounded from below away from zero such that

$$Q'':=\Delta''-B''-\delta f''^*D\ge 0.$$

Then

$$K_{X''}+\Delta''+2M''-(K_{X''}+B''+\delta f''^*D+M'')=Q''+M''$$

which in particular means that $Q''+M''$ is $\mathbb{R}$-Cartier. Therefore, since $(X'',\Delta''+2M'')$ is generalised lc,

$$(X'',B''+\delta f''^*D+M'')$$

is generalised sub-lc. This implies that $(X,B+\delta f^*D+M)$ is generalised lc because the pullbacks of

$$K_X+B+\delta f^*D+M$$

and of

$$K_{X''}+B''+\delta f''^*D+M''$$

agree on any common resolution of X,X'' as both divisors are $\mathbb{R}$-linearly trivial over Z. Therefore, $t\ge\delta$ where t is the generalised lc threshold of f^*D with respect to $(X,B+M)$, hence $\mu_DB_Z=1-t\le 1-\delta$.

□

Proof. (of Theorem 2.7) *Step 1. In this step we introduce a boundary Δ' on some resolution of X.* We will reduce the theorem to Proposition 6.6. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Assume D is a prime divisor over Z. First we reduce the statement to the case when D is a divisor on Z. Let $Z'\to Z$ be a resolution such that D is a divisor on Z'. Pick a log resolution $\phi\colon X'\to X$ of (X,B) such that $f'\colon X'\dashrightarrow Z'$ is a morphism and such that the nef part M' of $(X,B+M)$ is on X'. Write $K_{X'}+B'+M'$ for the pullback of K_X+B+M.

Let Γ' be obtained from B' by removing all the components with negative coefficients, and let $\Delta' = \Gamma' + \frac{\epsilon}{2}R'$ where R' is the reduced exceptional divisor of $X' \to X$. Then $(X', \Delta' + M')$ is generalised $\frac{\epsilon}{2}$-lc, and we get

$$K_{X'} + \Delta' + M' \sim_{\mathbb{R}} K_{X'} + \Delta' + M' - (K_{X'} + B' + M') = \Delta' - B'/Z$$

where $E' := \Delta' - B' \geq \frac{\epsilon}{2}R'$ and E' is exceptional$/X$.

Step 2. In this step we consider running MMP on $K_{X'} + \Delta' + M'$ over Z. Run an MMP on $K_{X'} + \Delta' + M'$ over the main component of $X \times_Z Z'$ with scaling of some ample divisor. So the MMP is over both X and Z'. We do not claim that the MMP terminates but we claim that it does terminate over the generic point of Z contracting all the horizontal$/Z$ components of E'. Indeed let U be a non-empty open subset of Z over which $Z' \to Z$ is an isomorphism. Then $X \times_Z Z' \to X$ is an isomorphism over $f^{-1}U$, and since E' is exceptional over X, the MMP terminates over $f^{-1}U$, by Lemma 3.23, contracting E' over $f^{-1}U$. Thus we reach a model X'' on which $E'' = 0$ over $f^{-1}U$, in particular, E'' is vertical over Z'. Moreover, since E'' contains all the exceptional divisors of $X'' \to X$ and since X is $\mathbb{Q}$-factorial, we deduce that $X'' \to X$ is an isomorphism over U.

Let $G' = \lfloor \phi^* G \rfloor$ and let G'' be its pushdown on X''. Then by the previous paragraph,

$$0 < \operatorname{vol}(((\operatorname{Supp} \Delta'') + M'' + G'')|_{F''}) = \operatorname{vol}(((\operatorname{Supp} B) + M + G)|_F) < v$$

where F'' is a general fibre of $X'' \to Z$ and F is the corresponding fibre of $X \to Z$.

Step 3. In this step we consider running MMP on $K_{X''} + \Delta'' + M''$ over Z'. Now run another MMP on $K_{X''} + \Delta'' + M''$ over Z' with scaling of some ample divisor. Since $K_{X''} + \Delta'' + M'' \sim_{\mathbb{R}} 0$ holds over U, the MMP does not do anything over U. We claim that the MMP terminates over the generic point of D. We can assume $E'' \neq 0$ otherwise the claim holds trivially. For the rest of this paragraph we shrink Z' around the generic point of D hence assume that every component of $E'' + f''^*D$ maps onto D where f'' is the induced morphism $X'' \to Z'$. Let α be the largest real number such that $E'' - \alpha f''^* D$ is effective ($\alpha \geq 0$ and $\alpha = 0$ is possible). There is a component of $f''^* D$ which is not a component of $E'' - \alpha f''^* D$. Thus $E'' - \alpha f''^* D$ is very exceptional over Z'. Therefore, the MMP terminates by Lemma 3.23 contracting $E'' - \alpha f''^* D$. In particular, the MMP terminates over the generic point of D.

Step 4. In this step we reduce the theorem to the case when D is a divisor on Z. In the course of the MMP of last step we reach a model X''' on which $(X''', \Delta''' + M''')$ is generalised $\frac{\epsilon}{2}$-lc with nef part being the pullback of M' to some common resolution of X', X''', and that

$$K_{X'''} + \Delta''' + M''' \sim_{\mathbb{R}} 0/Z'$$

holds over the generic point of D. Moreover, if $K_{X'''} + B''' + M'''$ is the pushdown of $K_{X'} + B' + M'$, then $B''' \leq \Delta'''$. In particular, the generalised lc threshold of $f'''^* D$ with respect to $(X''', \Delta''' + M''')$ over the generic point of D is smaller or equal to the generalised lc threshold with respect to $(X''', B''' + M''')$ where f''' is the induced morphism $X''' \to Z'$. Furthermore, by Step 2,

$$0 < \operatorname{vol}(((\operatorname{Supp} \Delta''') + M''' + G''')|_{F'''}) < v$$

for the general fibres F''' of f''' because $X'' \dashrightarrow X'''$ is an isomorphism over the generic point of Z'. Therefore, shrinking Z' around the generic point of D and replacing ϵ, $(X, B + M) \to Z$ with $\frac{\epsilon}{2}$, $(X''', \Delta''' + M''') \to Z'$, we can assume that D is a divisor on Z.

Step 5. In this step we take a hyperplane section of Z. In this step assume $\dim Z > 1$. Let H be a general hyperplane section of Z and let $V = f^*H$. Then by divisorial generalised adjunction we can write

$$K_V + B_V + M_V \sim_{\mathbb{R}} (K_X + B + V + M)|_V$$

where $(V, B_V + M_V)$ is generalised ϵ-lc with nef part $M_{V'} = M'|_{V'}$ where $V' \subset X'$ is the pullback of V. Let $G_V := G|_V$. By generality of H, we have $B_V = B|_V$ and $M_V = M|_V$, hence

$$0 < \mathrm{vol}(((\mathrm{Supp}\, B_V) + M_V + G_V)|_F) = \mathrm{vol}(((\mathrm{Supp}\, B) + M + G)|_F) < v$$

for the general fibres F of $V \to H$ because F is among the general fibres of f. Moreover, $pM_{V'}$ is Cartier as pM' is Cartier by assumption, and

$$K_V + B_V + M_V \sim_{\mathbb{R}} 0/H.$$

Step 6. In this step we finish the proof by applying induction on dimension. If $\dim Z = 1$, then we use Proposition 6.6. Otherwise we apply induction on $\dim Z$ as follows. Let V, H etc be as in the previous step. Let C be a component of $D \cap H$ and let s be the generalised lc threshold of g^*C with respect to $(V, B_V + M_V)$ over the generic point of C where g denotes $V \to H$. Applying induction on dimension, s is bounded from below away from zero, hence it is enough to show that $s \le t$ where t is the generalised lc threshold of f^*D with respect to $(X, B + M)$ over the generic point of D.

Shrinking Z we can assume $C = D|_H$, hence $f^*D|_V = g^*C$. By definition of s,

$$(V, B_V + sg^*C + M_V)$$

is generalised lc over the generic point of C. Shrinking Z around the generic point of C we can assume that it is generalised lc everywhere. But then by generalised inversion of adjunction [5, Lemma 3.2],

$$(X, B + V + sf^*D + M)$$

is generalised lc near V, hence it is generalised lc over a neighbourhood of H which then implies that it is generalised lc over the generic point of D as H intersects D. Thus $s \le t$ as required.

□

In the proof just completed we first changed the base Z so that we could assume D is a divisor on Z. It is worth pointing out that this strategy does not work when dealing with Theorem 2.5 because in this case we need to keep Z varying in a bounded family. That is why the proof of 2.5 is different in the sense that we use hyperplane sections of Z only when the centre of D on Z is positive-dimensional.

Proof. (of Corollary 2.8) We want to prove Conjectures 2.4 and 2.6 under the extra assumptions that: any horizontal/Z component of B has coefficient $\ge \tau$ and pM' is b-Cartier. We can assume $\tau < 1$. First consider 2.6. Since $1 < \frac{1}{\tau}$ and since $M + G$ is pseudo-effective over Z,

$$\begin{aligned} 0 < \mathrm{vol}((B + M + G)|_F) &\le \mathrm{vol}(((\mathrm{Supp}\, B) + M + G)|_F) \\ \le \mathrm{vol}((\tfrac{1}{\tau}B + M + G))|_F) &\le \mathrm{vol}((\tfrac{1}{\tau}(B + M + G))|_F) < \tfrac{v}{\tau^d} \end{aligned}$$

for the general fibres F of f. Thus we can apply Theorem 2.7.

Now consider 2.4. Since $-K_X$ is big over Z, $B+M$ is big over Z, so

$$0<\mathrm{vol}((B+M)|_F)=\mathrm{vol}(-K_X|_F)=\mathrm{vol}(-K_F)$$

for the general fibres F of f. Letting $B_F:=B|_F$ and $M_F:=M|_F$, we see that (F,B_F+M_F) is generalised ϵ-lc, $K_F+B_F+M_F\sim_{\mathbb{R}}0$, and B_F+M_F is big. We can then find a big boundary Δ_F such that (F,Δ_F) is $\frac{\epsilon}{2}$-lc and $K_F+\Delta_F\sim_{\mathbb{R}}0$ (this follows from 3.24). Thus F belongs to a bounded family by [4, Corollary 1.2], hence $\mathrm{vol}(-K_F)$ is bounded from above. Then $\mathrm{vol}((B+M)|_F)$ is bounded from above, so taking $G=0$ we are in the situation of 2.6. Thus we are done by the previous paragraph.

□

Proof. (of Theorem 1.9) This is a special case of Theorem 2.7 which was already proved.

□

In the rest of this section we prove few other results which are not essential for this paper in the sense that they will only be used to give alternatives proofs of 1.4. They will likely be useful elsewhere so it is good to write them here for future reference.

6.7. **Comparing singularities on the total space and base.**

Lemma 6.8. *Let (X,B) be a projective sub-pair and $f\colon X\to Z$ be a contraction such that (X,B) is lc over the generic point of Z, $K_X+B\sim_{\mathbb{Q}}0/Z$, and B is a $\mathbb{Q}$-divisor. Let B_Z,M_Z be the discriminant and moduli parts of adjunction and consider (Z,B_Z+M_Z) as a generalised pair (as in 6.2). Then for any open subset $U\subseteq Z$, (X,B) is sub-lc over U iff (Z,B_Z+M_Z) is generalised sub-lc on U.*

Proof. Choose a log resolution $Z'\to Z$ of (Z,B_Z) such that $M_{Z'}$ is nef and $\mathbf{M}_Z$ is the b-divisor determined by $M_{Z'}$, that is, for any higher resolution $Z''\to Z'$ the divisor $M_{Z''}$ is the pullback of $M_{Z'}$. Pick a log resolution $X'\to X$ of (X,B) such that the induced map $f'\colon X'\dashrightarrow Z'$ is a morphism. Let $K_{X'}+B'$ be the pullback of K_X+B. Let $U\subseteq Z$ be an open subset.

Assume that (X,B) is sub-lc over U. Let D be a prime divisor on Z' whose centre on Z intersects U. Since (X',B') is sub-lc over U, the lc threshold of f'^*D with respect to (X',B') over the generic point of D is non-negative, hence the coefficient of D in $B_{Z'}$ is at most 1. Therefore, the coefficients of the components of $B_{Z'}$ whose generic point map to U do not exceed 1, so $(Z',B_{Z'}+M_{Z'})$ is generalised sub-lc over U which means that (Z,B_Z+M_Z) is generalised sub-lc on U.

Conversely assume that (Z,B_Z+M_Z) is generalised sub-lc on U. Assume (X,B) is not sub-lc over U. Then there is a prime divisor S over X with log discrepancy $a(S,X,B)<0$ whose image on Z intersects U. Since (X,B) is an lc pair over the generic point of Z, S is vertical over Z. Thus replacing X',Z' we can assume that S is a divisor on X' and that the image of S on Z' is a divisor, say D. Since by assumption the coefficient of S in B' exceeds 1, the lc threshold of f'^*D with respect to (X',B') over the generic point of D is negative, hence the coefficient of D in $B_{Z'}$ exceeds 1, a contradiction. Therefore, (X,B) is sub-lc over U.

□

6.9. **Composition of contractions.**

Lemma 6.10. *Let (X,B) be a projective sub-pair and $X\xrightarrow{f}Y\xrightarrow{g}Z$ be contractions such that (X,B) is lc over the generic point of Z, $K_X+B\sim_{\mathbb{Q}}0/Z$, and B is a $\mathbb{Q}$-divisor. Let*

- $\mathbf{B}_Y, \mathbf{M}_Y$ *(resp. B_Y, M_Y) be the discriminant and moduli b-divisors (resp. divisors) of adjunction for (X, B) over Y,*
- $\mathbf{B}_Z, \mathbf{M}_Z$ *be the discriminant and moduli b-divisors of adjunction for (X, B) over Z,*
- *and $\mathbf{C}_Z$ be the discriminant b-divisor of adjunction for $(Y, B_Y + M_Y)$ over Z where we consider $(Y, B_Y + M_Y)$ as a generalised pair (as in 6.2).*

Then $\mathbf{C}_Z = \mathbf{B}_Z$.

Proof. Let D be a prime divisor over Z, say on some resolution $Z' \to Z$. Let c, b be the coefficients of D in $\mathbf{C}_Z, \mathbf{B}_Z$, respectively. We want to show $c = b$. Pick birational contractions $\psi\colon Y' \to Y$ and $\phi\colon X' \to X$ from normal varieties so that ψ, ϕ are isomorphisms over the generic point of Z and so that the induced maps $g'\colon Y' \dashrightarrow Z'$ and $f'\colon X' \dashrightarrow Y'$ are morphisms. Let $K_{X'} + B'$ be the pullback of $K_X + B$. Then (X', B') is lc over the generic point of Z. Moreover, the discriminant and moduli divisors $B'_{Y'}, M'_{Y'}$ defined for (X', B') over Y' coincide with the discriminant and moduli divisors $B_{Y'}, M_{Y'}$ on Y' defined for (X, B) over Y. Similarly, the discriminant and moduli b-divisors $\mathbf{B}'_{Z'}, \mathbf{M}'_{Z'}$ of adjunction for (X', B') over Z' coincide with the discriminant and moduli b-divisors $\mathbf{B}_Z, \mathbf{M}_Z$ of adjunction for (X, B) over Z, and the discriminant b-divisor $\mathbf{C}'_{Z'}$ of adjunction for $(Y', B'_{Y'} + M'_{Y'})$ over Z' coincides with the discriminant b-divisor $\mathbf{C}_Z$ of adjunction for $(Y, B_Y + M_Y)$ over Z. Thus replacing $(X, B), f, g$ with $(X', B'), f', g'$ we can assume D is a divisor on Z and that Z is smooth.

Put $h = gf$. Let t be the lc threshold of h^*D with respect to (X, B) over the generic point of D. Similarly let s be the generalised lc threshold of g^*D with respect to $(Y, B_Y + M_Y)$ over the generic point of D. By definition, $b = 1 - t$ and $c = 1 - s$. On the other hand, for any $\mathbb{R}$-Cartier divisor P_Y on Y, $B_Y + P_Y$ is the discriminant divisor of adjunction for $(X, B + f^*P_Y)$ over Y. In particular, for any real number u, $B_Y + ug^*D$ is the discriminant divisor of $(X, B + uh^*D)$ over Y. Now by Lemma 6.8, $(X, B + uh^*D)$ is sub-lc over the generic point of D iff
$$(Y, B_Y + ug^*D + M_Y)$$
is generalised sub-lc over the generic point of D. Applying this to $u = t$ and $u = s$ shows that $t = s$ which in turn shows that $b = c$.

□

6.11. **DCC property of the discriminant divisor.**

Lemma 6.12. *Let d, p be natural numbers and $\Phi \subset [0, 1]$ be a DCC set. Then there is a DCC set $\Psi \subset [0, 1]$ depending only on d, p, Φ satisfying the following. Assume that $(X, B + M)$ and $X \to Z$ are as in 6.1 and that*

- *$(X, B + M)$ is generalised lc of dimension d,*
- *the coefficients of B are in Φ, and*
- *pM' is b-Cartier where M' is the nef part of $(X, B + M)$.*

Then the discriminant divisor B_Z of adjunction for $(X, B + M)$ over Z has coefficients in Ψ.

Proof. Let D be a prime divisor on Z. Let t be the generalised lc threshold of f^*D with respect to $(X, B + M)$ over the generic point of D. Shrinking Z around the generic point of D we can assume D is Cartier and that t is the generalised lc threshold of f^*D with respect to $(X, B + M)$ (that is, globally not just over the generic point of D). In particular, the coefficients of f^*D are natural numbers, hence they belong to $\Phi \cup \mathbb{N}$ which is a DCC set. Moreover, we can assume that $\frac{1}{p}$ is in Φ. Then by [9, Theorem 1.5] the generalised lc thresholds t above satisfy the ACC. Therefore, $\mu_D B_Z = 1 - t$ belongs to some DCC set Ψ depending only on d, p, Φ.

□

Note that in the proof, unlike some other proofs above, we did not need M' to be nef over Z but only used its nefness over X (indeed M' is not assumed to be nef over Z in the lemma).

7. Boundedness of towers of Fano fibrations and of log Calabi-Yau varieties

In this section we treat Theorems 1.4 and 1.5.

Proof. (of Theorem 1.4) *Step 1. In this step we do some easy reductions.* By assumption, $X \to Z$ factors as a sequence

$$X = X_1 \to \cdots \to Z = X_l$$

of Fano fibrations. If $l = 1$, then the statement holds essentially trivially: indeed, $X = Z$ and $A^{d=\dim Z} \le r$ means X is bounded; also $A - L \sim_{\mathbb{R}} A - (K_X + B)$ being ample implies $A^{d-1} \cdot (K_X + B) < r$, hence $A^{d-1} \cdot B$ is bounded from above which then implies that (X, B) is log bounded as the coefficients of B are $\ge \tau$. Thus we can assume $l \ge 2$. Moreover, applying induction on l we can assume that $\dim X_{l-1} > 0$ otherwise we can replace Z with X_{l-1} and decreasing l.

On the other hand, by Lemma 3.11, we can write $K_X + B = \sum r_i(K_X + B_i)$ for certain real numbers $r_i > 0$ with $\sum r_i = 1$ and rational boundaries B_i such that (X, B_i) is $\frac{\epsilon}{2}$-lc, $K_X + B_i \sim_{\mathbb{Q}} 0/Z$, $\operatorname{Supp} B_i = \operatorname{Supp} B$, and the coefficients of B_i are $\ge \frac{\tau}{2}$. Furthermore, we can choose B_i so that the coefficients of $B - B_i$ are arbitrarily small, hence if $K_X + B_i \sim_{\mathbb{Q}} f^*L_i$, then we can make sure that $A - L_i$ is ample. Now replacing $\epsilon, \tau, (X, B), L$ with $\frac{\epsilon}{2}, \frac{\tau}{2}, (X, B_i), L_i$ for some i we can assume that B has rational coefficients.

Step 2. In this step we apply adjunction to (X, B) *over each* X_i. Let B_i and M_i be the discriminant and moduli divisors of adjunction defined for $(X, B) = (X_1, B_1)$ over X_i whenever $\dim X_i > 0$. We consider $(X_i, B_i + M_i)$ as a generalised pair with data consisting of some high resolution $X_i' \to X_i$ and nef part M_i' on X_i' (as in 6.2).

Denote $X \to X_j$ by g_j. We claim that for each i,

(1) there exists a positive real number δ depending only on d, i, ϵ, τ such that $(X_i, B_i + M_i)$ is generalised δ-lc if $\dim X_i > 0$;
(2) there exist natural numbers $n_1, \dots, n_{i-1}, v$ depending only on d, i, ϵ, τ and there exists an integral divisor $J \ge 0$ on X such that for the general fibres F of $X \to X_i$ we have

$$J|_F \sim -\sum_{j=1}^{i-1} n_j g_j^* K_{X_j}|_F \text{ and } 0 < \operatorname{vol}((B + J)|_F) < v.$$

To prove the claim we will apply induction on d, so we assume that the claim holds in lower dimension.

Step 3. In this step we consider the log general fibres of $(X, B) \to X_i$. Let F be a general fibre of $X = X_1 \to X_i$, say over a closed point v. Then the sequence

$$X_1 \to X_2 \to \cdots \to X_i$$

induces a sequence

$$F = G_1 \to G_2 \to \cdots \to G_{i-1} \to G_i = \{v\}$$

of contractions where G_j is the fibre of $X_j \to X_i$ over v. Moreover, $-K_{G_j}$ is ample over G_{j+1} as $K_{G_j} = K_{X_j}|_{G_j}$ and $-K_{X_j}$ is ample over X_{j+1}. In particular, $F \to \{v\}$ factors as a tower of Fano fibrations of length i. Note that G_j may consist of only one point for some $j < i$: this is the case when $X_j \to X_i$ is birational. In addition, $G_j \to G_{j+1}$ may be an isomorphism for some j.

Let
$$K_F + B_F := (K_X + B)|_F.$$
Then (F, B_F) is projective ϵ-lc, $K_F + B_F \sim_{\mathbb{Q}} 0$, and each non-zero coefficient of B_F is $\geq \tau$.

Step 4. In this step we apply induction on dimension. In this step assume $\dim X_i > 0$ and let F, G_j be as in the previous step. Then $\dim F < d$. Therefore, applying induction on dimension for the claim in Step 2 and using the fact that the general fibres of $F \to G_i = \{v\}$ are just F itself, there exist natural numbers $n_1, \dots, n_{i-1}, v$ depending only on $\dim F, i, \epsilon, \tau$ and there is an integral divisor $J_F \geq 0$ such that
$$J_F \sim -\textstyle\sum_{j=1}^{i-1} n_j h_j^* K_{G_j} \text{ and } 0 < \operatorname{vol}(B_F + J_F) < v$$
where h_j denotes $F \to G_j$. In particular, $-\sum_{j=1}^{i-1} n_j h_j^* K_{G_j}$ is an integral divisor. On the other hand, from $h_j^* K_{G_j} = g_j^* K_{X_j}|_F$, we get
$$J_F \sim -\sum_{j=1}^{i-1} n_j h_j^* K_{G_j} = (-\sum_{j=1}^{i-1} n_j g_j^* K_{X_j})|_F.$$
Since F is a general fibre of $X \to X_i$, we deduce that $-\sum_{j=1}^{i-1} n_j g_j^* K_{X_j}$ is an integral divisor over the generic point of X_i (note that since F is a general fibre we have: if $P = \sum p_k D_k$ is an $\mathbb{R}$-divisor on X where D_k are distinct irreducible components, then $P|_F = \sum p_k D_k|_F$ where $D_k|_F$ are reduced divisors and there is no common component for distinct k; so the set of coefficients of $P|_F$ coincides with the set of horizontal/X_i coefficients of P). Let
$$M = \left\lfloor -\sum_{j=1}^{i-1} n_j g_j^* K_{X_j} \right\rfloor.$$
Then $M|_F \sim J_F \geq 0$, hence there is an integral divisor $0 \leq J \sim M/X_i$ (this can be seen by taking a resolution $W \to X$ and applying base change of cohomology to $W \to X_i$). In particular,
$$J|_F \sim M|_F \sim J_F \sim -\sum_{1}^{i-1} n_j g_j^* K_{X_j}|_F$$
and
$$0 < \operatorname{vol}((B + J)|_F) = \operatorname{vol}(B_F + J_F) < v.$$

Step 5. In this step we establish claim (1) of step 2. As mentioned earlier we can assume that the claim holds in lower dimension. If $i = 1$, the claim holds trivially. Moreover, if $\dim X_i = 0$, then claim (1) holds as it is vacuous in this case. But if $\dim X_i > 0$, then claim (1) follows by applying Corollary 2.8 to $(X, B) \to X_i$ using the integral divisor J of the previous step. We can assume that δ of claim (1) depends only on d, l, ϵ, τ.

Step 6. In this step we work towards establishing claim (2) of step 2. We will prove claim (2). If $\dim X_i > 0$, then it follows from the previous step. So assume $\dim X_i = 0$ which means $i = l$ and that X_{l-1} is a Fano variety. By claim (1), $(X_{l-1}, B_{l-1} + M_{l-1})$ is generalised δ-lc, hence X_{l-1} is a δ-lc Fano variety. Thus X_{l-1} is bounded by [4, Theorem 1.1] (=Theorem 3.19), so there are natural numbers n_{l-1}, v depending only on d, δ such that $-n_{l-1}K_{X_{l-1}}$ is very ample with volume less than v. In particular, we are done if $l = 2$, so we can assume $l \geq 3$. We will construct $n_i, \dots, n_{l-2}$ inductively and during the process we modify n_{l-1}, v.

Denote $X_j \to X_k$ by $e_{j,k}$. Assume that for some $2 \leq j \leq l-1$ there exist natural numbers $n_j, \cdots, n_{l-1}, v$ depending only on d, j, δ such that

$$H_j := -\sum_{k=j}^{l-1} n_k e_{j,k}^* K_{X_k}$$

is very ample on X_j with volume less than v. By claim (1), $(X_{j-1}, B_{j-1} + M_{j-1})$ is generalised δ-lc. By assumption, $-K_{X_{j-1}}$ is ample over X_j. Moreover, since $\dim Z = \dim X_l = 0$, $K_X + B \sim_{\mathbb{Q}} 0$ from which we get

$$K_{X_{j-1}} + B_{j-1} + M_{j-1} \sim_{\mathbb{Q}} 0.$$

Then

$$(X_{j-1}, B_{j-1} + M_{j-1}) \to X_j$$

is a generalised $(\dim X_{j-1}, v, \delta)$-Fano type fibration where we use the assumption that $H_j^{\dim X_j} < v$. Then by Lemma 3.24, there is a boundary Δ_{j-1} such that $(X_{j-1}, \Delta_{j-1}) \to X_j$ is a $(\dim X_{j-1}, v, \frac{\delta}{2})$-Fano type fibration.

Step 7. In this step we establish the claim of step 2 from which we derive the theorem. By Theorem 1.3 (or 2.2), X_{j-1} belongs to a bounded family. On the other hand, by Lemma 5.4, there exist bounded natural numbers p, q such that the divisor

$$H_{j-1} := p(qe_{j-1,j}^* H_j - K_{X_{j-1}})$$

is very ample. In particular, letting $n_{j-1} := p$ and replacing $n_j, \cdots, n_{l-1}$ with the numbers $qn_{j-1}n_j, \cdots, qn_{j-1}n_{l-1}$, respectively, we can rewrite

$$H_{j-1} = -\sum_{k=j-1}^{l-1} n_k e_{j-1,k}^* K_{X_k}.$$

Applying Proposition 5.8, the volume of H_{j-1} is bounded from above, so replacing v we can assume $\operatorname{vol}(H_{j-1}) < v$.

Repeating the above process gives bounded natural numbers $n_1, \cdots, n_{l-1}, v$ depending only on d, l, δ (hence depending only on d, l, ϵ, τ) such that

$$H_1 := -\sum_{k=1}^{l-1} n_k e_{1,k}^* K_{X_k}$$

is very ample with volume less than v. In particular, $X = X_1$ belongs to a bounded family which in turn implies that (X, B) is log bounded because $H_1^{d-1} \cdot B = -H_1^{d-1} \cdot K_X$ is bounded from above and because the coefficients of B are $\geq \tau$. Moreover, we can find

$$0 \leq J \sim H_1 = -\sum_{k=1}^{l-1} n_k e_{1,k}^* K_{X_k} = -\sum_{k=1}^{l-1} n_k g_k^* K_{X_k}$$

and perhaps after replacing v we can assume

$$0 < \operatorname{vol}(B + J) = \operatorname{vol}(-K_X + H_1) < v.$$

This proves claim (2) and finishes the proof of the theorem.

□

Proof. (of Theorem 1.5) We follow the strategy in [11] which reduces the theorem to a special case of 1.4. Taking a $\mathbb{Q}$-factorialisation we can assume X is $\mathbb{Q}$-factorial. Since $B \neq 0$ and (X, B) is not of product type, by [11, Theorem 3.2](=Theorem 3.28), there exist a birational map $\phi\colon X \dashrightarrow X_1$ and a sequence of contractions

$$X_1 \to X_2 \to \cdots \to X_l$$

such that ϕ^{-1} does not contract divisors, each $X_i \to X_{i+1}$ is a Mori fibre space, and X_l is a point. In particular, $l \leq d$. Then applying Theorem 1.4, we deduce that (X_1, B_1) is log bounded where $B_1 = \phi_* B$.

On the other hand, since $K_X + B \sim_{\mathbb{Q}} 0$, each exceptional prime divisor D of ϕ satisfies

$$a(D, X_1, B_1) = a(D, X, B) \leq 1,$$

hence there is a birational contraction $\psi\colon X' \to X_1$ from a normal projective variety such that the induced map $X \dashrightarrow X'$ is an isomorphism in codimension one. Let B' on X' be the birational transform of B. Then (X', B') is a crepant model of (X_1, B_1). Thus (X', B') is log bounded by Theorem 1.3.

□

In the rest of this section we give different proofs of 1.4 in certain special cases.

Alternative proof of 1.4 when coefficients of B are in a fixed DCC set Φ.

Step 1. In this step we apply adjunction to (X, B) over each X_i. First, as in the previous proof we can assume $l \geq 2$ and that $\dim X_{l-1} > 0$. Let B_i and M_i (resp. $\mathbf{B}_i$ and $\mathbf{M}_i$) be the discriminant and moduli divisors (resp. b-divisors) of adjunction defined for $(X_1, B_1) = (X, B)$ over X_i when $\dim X_i > 0$. We consider $(X_i, B_i + M_i)$ as a generalised pair with data consisting of some high resolution $X_i' \to X_i$ and nef part M_i' on X_i' (as in 6.2). A crucial point is that, by Lemma 6.10, we have the following property:

(∗) B_i (resp. $\mathbf{B}_i$) coincides with the discriminant divisor (resp. b-divisor) of generalised adjunction for

$$(X_{i-1}, B_{i-1} + M_{i-1}) \text{ over } X_i.$$

Step 2. In this step we investigate the $(X_i, B_i + M_i)$. We claim that there exist a DCC set Ψ, a natural number p, and a positive real number δ depending only on d, l, Φ, ϵ such that for each i we have:

- the coefficients of B_i belong to Ψ,
- we can choose M_i' in its $\mathbb{Q}$-linear equivalence class so that pM_i' is Cartier, and
- $(X_i, B_i + M_i)$ is generalised δ-lc.

For $i = 1$ the claim holds trivially by taking $\Psi = \Phi$, $p = 1$, and $\delta = \epsilon$. Assuming that we have already found Ψ, p, δ which satisfy the claim up to $i - 1 \geq 1$, we prove the claim for i (where we assume $\dim X_i > 0$). By Lemma 6.12, the coefficients of B_i belong to some DCC set $\tilde{\Psi}$ depending only on d, Φ.

Step 3. In this step we show that we can choose M_i' with bounded Cartier index. Let F be a general fibre of $X_1 \to X_i$ over a point v. Let

$$K_F + B_F := (K_X + B)|_F.$$

By induction on dimension, the set of such (F, B_F) form a log bounded family. Thus there is a bounded natural number $\tilde{p}$ such that we can choose M_i' in its $\mathbb{Q}$-linear equivalence class so that $\tilde{p}M_i'$ is Cartier: this follows from the same arguments as in the proof of [15, Claim 3.2 (3)] (note that in [15, Claim 3.2 (3)] it is implicitly assumed that B_F is big but this is not needed in the proof once we know that (F, B_F) is log bounded). Alternatively, as in [5, Proposition 6.3], we can use boundedness of relative complements for $K_{X_{i-1}} + B_{i-1} + M_{i-1}$ over X_i (similar to that of usual relative complements [5, Theorem 1.8]) to show that $\tilde{p}$ exists.

Step 4. In this step we finish the proof of the claim of Step 3. By $(*)$ above and by Corollary 2.8 applied to $(X_{i-1}, B_{i-1} + M_{i-1})$ over X_i, the b-divisor $\mathbf{B}_i$ has coefficients $\le 1 - \tilde{\delta}$ for some positive real number $\tilde{\delta}$ depending only on d, p, Ψ, δ. In particular, $(X_i, B_i + M_i)$ is generalised $\tilde{\delta}$-lc. Now replace Ψ, p, δ with $\Psi \cup \tilde{\Psi}$, $p\tilde{p}$, $\delta\tilde{\delta}$, respectively. Then we can assume that the coefficients of B_i are in Ψ, pM_i' is Cartier, and that $(X_i, B_i + M_i)$ is generalised δ-lc. This proves the above claim inductively.

Step 5. In this step we show that X_{l-1} is bounded. Denote $X_{l-1} \to X_l$ by h. Pick $0 \le \Delta_{l-1} \sim_{\mathbb{Q}} h^*A$ with coefficients in a fixed finite set so that

$$(X_{l-1}, B_{l-1} + \Delta_{l-1} + M_{l-1})$$

is still generalised δ-lc with nef part M_{l-1}'. By assumption, $-(K_{X_{l-1}} + \Delta_{l-1})$ is ample over X_l. By construction,

$$K_{X_{l-1}} + B_{l-1} + \Delta_{l-1} + M_{l-1} \sim_{\mathbb{Q}} h^*(L + A).$$

Then

$$(X_{l-1}, B_{l-1} + \Delta_{l-1} + M_{l-1}) \to X_l = Z$$

is a generalised $(\dim X_{l-1}, r2^{\dim X_l}, \delta)$-Fano type fibration. Thus the pairs (X_{l-1}, Δ_{l-1}) form a log bounded family, by Lemma 3.24 and Theorem 1.3. In particular, we can find a very ample divisor H on X_{l-1} such that $H^{\dim X_{l-1}}$ is bounded from above and $H - h^*A \sim_{\mathbb{Q}} H - \Delta_{l-1}$ is ample.

Step 6. In this step we finish the proof. Denote $X = X_1 \to X_{l-1}$ by g. Then $K_X + B \sim_{\mathbb{Q}} g^*h^*L$ and

$$H - h^*L = H - h^*A + h^*(A - L)$$

is ample. Therefore, if $l > 2$, then we can apply induction on l. If $l = 2$, then $X = X_{l-1}$ and $h = f$, in particular, X belongs to a bounded family. Moreover,

$$H - (K_X + B) \sim_{\mathbb{Q}} H - f^*L$$

is ample, hence $H^{d-1} \cdot B$ is bounded from above. This implies that (X, B) is log bounded.

Alternative proof of 1.4 when $K_X + B \sim_{\mathbb{Q}} 0$ and the coefficients of B are in a fixed DCC set Φ. The previous proof can be simplified in the case $K_X + B \sim_{\mathbb{Q}} 0$ in the sense that we do not need 2.8. The proof goes along the same lines except that we can show that $(X_i, B_i + M_i)$ is generalised δ-lc by a different argument rather than applying 2.8 to the generalised Fano type fibration

$$(X_{i-1}, B_{i-1} + M_{i-1}) \to X_i.$$

Indeed the generalised pair $(X_i, B_i + M_i)$ is generalised klt satisfying the following properties with a fixed DCC set Ψ and natural number p:

- B_i has coefficients in Ψ, and
- pM'_i is Cartier.

When $K_X + B \sim_{\mathbb{Q}} 0$, we also have

- $K_{X_i} + B_i + M_i \sim_{\mathbb{Q}} 0$.

But then $(X_i, B_i + M_i)$ is generalised δ-lc for some fixed $\delta > 0$, by Lemma **3.26**. The rest of the proof is as before.

References

[1] F. Ambro; *The moduli b-divisor of an lc-trivial fibration.* Compos. Math. **141** (2005), no. 2, 385–403.
[2] F. Ambro; *The Adjunction Conjecture and its applications.* arXiv:math/9903060v3.
[3] C. Birkar, *Birational geometry of algebraic varieties.* arXiv:1801.00013. To appear in the proceedings of the ICM 2018.
[4] C. Birkar, *Singularities of linear systems and boundedness of Fano varieties.* arXiv:1609.05543.
[5] C. Birkar; *Anti-pluricanonical systems on Fano varieties*, arXiv:1603.05765v1.
[6] C. Birkar; *Singularities on the base of a Fano type fibration.* J. Reine Angew Math., 715 (2016), 125-142.
[7] C. Birkar, *Existence of log canonical flips and a special LMMP*, Pub. Math. IHES., **115** (2012), 325–368.
[8] C. Birkar, P. Cascini, C. Hacon and J. M^{c}Kernan; *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405-468.
[9] C. Birkar and D-Q. Zhang; *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Pub. Math. IHES.123 (2016), 283-331.
[10] W. Chen, G. Di Cerbo, J. Han, C. Jiang, R. Svaldi; *Birational boundedness of rationally connected Calabi-Yau 3-folds.* arXiv:1804.09127.
[11] G. Di Cerbo, R. Svaldi; *Birational boundedness of low dimensional elliptic Calabi-Yau varieties with a section.* arXiv:1608.02997v2.
[12] S. Filipazzi, *On a generalized canonical bundle formula and generalized adjunction*, arXiv:1807.04847v1.
[13] O. Fujino, *Fundamental theorems for the log minimal model program*, Publ. Res. Inst. Math. Sci. 47 (2011), no. 3, 727-789.
[14] O. Fujino, Y. Gongyo; *On the moduli b-divisors of lc-trivial fibrations.* Ann. Inst. Fourier, Grenoble **64**, 4 (2014) 1721–1735.
[15] C. D. Hacon and C. Xu; *Boundedness of log Calabi-Yau pairs of Fano type.* Math. Res. Lett, **22** (2015), 1699–1716.
[16] C. D. Hacon, J. M^{c}Kernan and C. Xu, *Boundedness of moduli of varieties of general type*, J. Eur. Math. Soc. (JEMS) 20 (2018), no. 4, 865-901.
[17] C. D. Hacon, J. M^{c}Kernan and C. Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523-571.
[18] C. D. Hacon, J. M^{c}Kernan and C. Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. (2) **177** (2013), no. 3, 1077-1111.
[19] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties.* Algebraic geometry. V., Encyclopaedia Math. Sci., vol. 47, Springer, Berlin, 1999.
[20] C. Jiang, *On birational boundedness of Fano fibrations*, to appear in Amer. J. Math, arXiv:1509.08722v1.
[21] M. Kawakita; *Inversion of adjunction on log canonicity*, Invent. Math. **167** (2007), 129–133.
[22] Y. Kawamata; *Flops connect minimal models.* Publ. RIMS, Kyoto Univ. 44 (2008), 419-423.
[23] Y. Kawamata; *Subadjunction of log canonical divisors, II*, Amer. J. Math. **120** (1998), 893–899.
[24] Y. Kawamata, *On the length of an extremal rational curve*, Invent. Math. **105** (1991), no. 3, 609–611.
[25] J. Kollár; *Effective base point freeness.* Math. Annalen (1993), Volume **296**, Issue 1, pp 595–605.
[26] J. Kollár ét al.; *Flips and abundance for algebraic threefolds,* Astérisque No. **211** (1992).
[27] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, 1998.
[28] D. Martinelli, S. Schreieder, L. Tasin; *On the number and boundedness of log minimal models of general type.* Arxiv arXiv:1610.08932v5. To appear in Annales Scientifique de l'ENS.
[29] J. M^{c}Kernan, Yu. Prokhorov; *Threefold thresholds.* Manuscripta Math. 114 (2004), no. 3, 281–304.
[30] S. Mori, Yu. Prokhorov *Multiple fibers of del Pezzo fibrations.* Proc. Steklov Inst. Math., 264(1):131-145, 2009.

[31] Yu. Prokhorov, V.V. Shokurov; *Towards the second main theorem on complements.* J. Algebraic Geometry, **18** (2009) 151–199.

DPMMS, Centre for Mathematical Sciences
University of Cambridge,
Wilberforce Road, Cambridge CB3 0WB, UK
c.birkar@dpmms.cam.ac.uk