

Restriction of p -modular representations of $U(2, 1)$ to a Borel subgroup

Peng Xu

Abstract

Let G be the unramified unitary group $U(2, 1)(E/F)$ over a non-archimedean local field F of odd residue characteristic p , and let B be the standard Borel subgroup of G . In this paper, we study the problem of the restriction of irreducible smooth $\overline{\mathbf{F}}_p$ -representations of G to B , and we prove results which are analogous to that of Paškūnas on $GL_2(F)$ ([Paš07]).

1 Introduction

Let G be the unitary group $U(2, 1)(E/F)$ defined over a non-archimedean local field F of odd residue characteristic p , and let B be the standard Borel subgroup of G . In this paper, we investigate the restriction of irreducible smooth $\overline{\mathbf{F}}_p$ -representations of G to B .

Our first main result is concerning principal series of G :

Theorem 1.1. (Corollary 4.5, 4.6) *Let π be a smooth representation of G . We have:*

(1). *Let ε be a character of B such that $\varepsilon \neq \eta \circ \det$ for any character η of $E^\times$. Then,*

$$\mathrm{Hom}_B(\mathrm{ind}_B^G \varepsilon, \pi) \cong \mathrm{Hom}_G(\mathrm{ind}_B^G \varepsilon, \pi).$$

(2). *For the trivial character of B , we have*

$$\mathrm{Hom}_G(\mathrm{ind}_B^G 1, \pi) \cong \mathrm{Hom}_B(St, \pi).$$

Here St is the Steinberg representation of G .

For a p -adic split connected reductive group, general results on restriction of principal series to a Borel subgroup have been obtained by Vignéras ([Vig08]). Her approach can be modified to work for certain non-split groups of small ranks ([Abd11], [Ly15]). Our result above considers another aspect of this problem.

The work of Abe–Henniart–Herzig–Vignéras([AHHV17]) gives a classification of irreducible *admissible* mod- p representations of a p -adic reductive group

in terms of *admissible* supersingular representations. Roughly speaking, supersingular representations are the mod- p analogue of supercuspidal representations. However, besides the group $GL_2(\mathbf{Q}_p)$ ([Bre03]) and a few closely related cases, supersingular representations remain mysterious largely; indeed, such representations might not even be *admissible* in general, as is shown in the work of Le ([Le19]). Note that Herzig–Koziol–Vignéras have proved the existence of admissible supersingular representation for any p -adic connected reductive group over F of characteristic 0 ([HKV20]).

The following is our main result on supersingular representations of G :

Theorem 1.2. *We have:*

(1). (Theorem 5.8) *Let π be a supersingular representation of G . Then*

$$\pi|_B \text{ is irreducible.}$$

(2). (Theorem 5.10) *Give two smooth representations π and π' of G . Suppose π is supersingular. Then, we have*

$$\mathrm{Hom}_G(\pi, \pi') \cong \mathrm{Hom}_B(\pi, \pi').$$

An immediate application of (1) above gives that the usual Jacquet module of a supersingular representation vanishes (Corollary 5.9).

Our results are analogous to results of Paškūnas on $GL_2(F)$ ([Paš07]), and we follow his strategy closely. To complete the proof of (2) of Theorem 1.2, we came to a new phenomenon which does not exist for GL_2 . The operator S_- is an analogue of the element $\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$ in $GL_2(F)$, but in our case it can always happen that $S_- \cdot v = 0$ for some $v \in \pi^{I_{1,K}}$. This causes essential troubles when we study mod- p representations of the group G . For the problem considered in this paper, we conquer such difficulty, see the argument of Theorem 5.8. (For unexplained notations, see section 2 and section 3).

When $F = \mathbf{Q}_p$, Paškūnas' results were firstly discovered by Berger ([Ber10]), where he uses the theory of (φ, Γ) -modules and classification of supersingular representations. In the work of Colmez on p -adic local Langlands correspondence of $GL_2(\mathbf{Q}_p)$ ([Col10]), the restriction to a Borel subgroup plays a prominent role. We expect our results would also have some interesting arithmetic applications in the future.

Remark 1.3. *Most part of this work (except for (2) of Theorem 1.2) was done when the author was a postdoc at Einstein Institute of Mathematics (2017-2018), and versions of that have been put on Arxiv in early of 2019 (see [Xu19b]). Sometime after that, we were aware of that Abdellatif and Hauseux have announced their results on the same problem in which they work for groups of semi-simple rank one ([Abd21]). As far as we know (as of April/2024), their work has not appeared yet, and indeed some of their argument is close to ours.*

This paper is organized as follows. In section 2, after setting up general notations, we recall some preliminaries on weights and the Hecke operator T . In section 3, we define certain $I_{1,K}$ -invariant maps S_K and S_- , and verify their basic properties. In section 4, we prove Theorem 1.1. In section 5, we prove Theorem 1.2.

2 Notations and Preliminaries

2.1 General notations

Let E/F be a unramified quadratic extension of non-archimedean local fields of odd residue characteristic p . Let $\mathfrak{o}_E$ be the ring of integers of E , $\mathfrak{p}_E$ be the maximal ideal of $\mathfrak{o}_E$, and $k_E = \mathfrak{o}_E/\mathfrak{p}_E$ be the residue field. Fix a uniformizer ϖ_E in E . Equip E^3 with the Hermitian form h :

$$h : E^3 \times E^3 \rightarrow E, (v_1, v_2) \mapsto v_1^T \beta \overline{v_2}, v_1, v_2 \in E^3.$$

Here, $-$ is a generator of $\text{Gal}(E/F)$, and β is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The unitary group G is defined as:

$$G = \{g \in \text{GL}(3, E) \mid h(gv_1, gv_2) = h(v_1, v_2), \forall v_1, v_2 \in E^3\}.$$

Let $B = HN$ (resp, $B' = HN'$) be the subgroup of upper (resp, lower) triangular matrices of G , with N (resp, N') the unipotent radical of B (resp, B') and H the diagonal subgroup of G . A typical element in H is of the following form and is denoted by $h(x, y)$:

$$h(x, y) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix}$$

for $x \in E^\times, y \in E^1$. We will write $h(x, -\bar{x}x^{-1})$ as $h(x)$ for short. An element in N and N' is of the following form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -\bar{x} & 1 \end{pmatrix}$$

and are denoted by $n(x, y)$ and $n'(x, y)$. Here we recall that $(x, y) \in E^2$ satisfies the relation $x\bar{x} + y + \bar{y} = 0$.

For any $k \in \mathbb{Z}$, denote by N_k and N'_k respectively the following subgroups of N and N'

$$N_k = \{n(x, y) \in N \mid y \in \mathfrak{p}_E^k\}, \\ N'_k = \{n'(x, y) \in N' \mid y \in \mathfrak{p}_E^k\}.$$

We record a useful identity in G : for $y \neq 0$,

$$\beta n(x, y) = n(\bar{y}^{-1}x, y^{-1}) \cdot h(\bar{y}^{-1}) \cdot n'(-\bar{y}^{-1}\bar{x}, y^{-1}). \quad (1)$$

Up to conjugacy, the group G has two maximal compact open subgroups K_0 and K_1 , given by:

$$K_0 = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \end{pmatrix} \cap G, \quad K_1 = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G.$$

The maximal normal pro- p subgroups of K_0 and K_1 are respectively:

$$K_0^1 = 1 + \varpi_E M_3(\mathfrak{o}_E) \cap G, \quad K_1^1 = \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap G.$$

Let α be the following diagonal matrix in G :

$$\begin{pmatrix} \varpi_E^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi_E \end{pmatrix},$$

and put $\beta' = \beta\alpha^{-1}$. Note that $\beta \in K_0$ and $\beta' \in K_1$. We use β_K to denote the unique element in $K \cap \{\beta, \beta'\}$.

Let $K \in \{K_0, K_1\}$, and K^1 be the maximal normal pro- p subgroup of K . We identify the finite group $\Gamma_K = K/K^1$ with the k_F -points of an algebraic group defined over k_F . Let $\mathbb{B}$ (resp, $\mathbb{B}'$) be the upper (resp, lower) triangular subgroup of Γ_K , and $\mathbb{U}$ (resp, $\mathbb{U}'$) be its unipotent radical. The Iwahori subgroup I_K (resp, I'_K) and pro- p Iwahori subgroup $I_{1,K}$ (resp, $I'_{1,K}$) in K are the inverse images of $\mathbb{B}$ (resp, $\mathbb{B}'$) and $\mathbb{U}$ (resp, $\mathbb{U}'$).

Denote by n_K and m_K the unique integers such that $N \cap I_{1,K} = N_{n_K}$ and $N' \cap I_{1,K} = N'_{m_K}$. We have $n_K + m_K = 1$. Note that the coset spaces N_{n_K}/N_{n_K+1} and N'_{m_K}/N'_{m_K+1} are indeed groups of order respectively q^{t_K} and q^{4-t_K} , where $t_K = 3$ or 1 , depending on K is hyperspecial or not.

All representations in this note are smooth over $\overline{\mathbf{F}}_p$.

2.2 Weights

Let σ be an irreducible smooth representation of K . As K_1 is pro- p and normal in K , σ factors through the finite group Γ_K , i.e., σ is the inflation of an irreducible representation of Γ_K . Conversely, any irreducible representation of Γ_K inflates to an irreducible smooth representation of K . We may therefore identify irreducible smooth representations of K with irreducible representations of Γ_K , and we shall call them *weights* of K or Γ_K from now on.

For a weight σ of K , it is known that $\sigma^{I_{1,K}}$ and $\sigma_{I'_{1,K}}$ are one-dimensional, and that the natural composition map $\sigma^{I_{1,K}} \hookrightarrow \sigma \twoheadrightarrow \sigma_{I'_{1,K}}$ is an isomorphism of vector spaces ([CE04, Theorem 6.12]). This implies there exists a unique $\lambda_{\beta_K, \sigma} \in \overline{\mathbf{F}}_p$, such that $\beta_K \cdot v - \lambda_{\beta_K, \sigma} v \in \sigma(I'_{1,K})$, for $v \in \sigma^{I_{1,K}}$. By [HV15, Proposition 3.16] and the fact that $\beta_K \notin I_K \cdot I'_K$, the scalar $\lambda_{\beta_K, \sigma}$ is zero if $\dim \sigma > 1$, and is equal to $\sigma(\beta_K)$ if $\dim \sigma = 1$.

2.3 The Hecke operator T

Let $K \in \{K_0, K_1\}$, and σ be a weight of K . Let $\text{ind}_K^G \sigma$ be the maximal compact induction and $\mathcal{H}(K, \sigma) := \text{End}_G(\text{ind}_K^G \sigma)$ be the associate spherical Hecke algebra. The algebra $\mathcal{H}(K, \sigma)$ is isomorphic to $\overline{\mathbf{F}}_p[T]$, for certain $T \in \mathcal{H}(K, \sigma)$ ([Her11, Corollary 1.3], see also [Xu19a, Proposition 3.3]).

We don't recall the exact definition of T but only its formula on a specific function. For a non-zero vector $v \in \sigma$, denote by $\hat{f}_v$ the function in $\text{ind}_K^G \sigma$ supported on K and having value v at Id .

Proposition 2.1. *Take a non-zero vector v_0 in $\sigma^{I_{1,K}}$. Then, we have*

$$T\hat{f}_{v_0} = \sum_{u \in N_{n_K}/N_{n_K+2}} u\alpha^{-1} \cdot \hat{f}_{v_0} + \lambda_{\beta_K, \sigma} \sum_{u \in N_{n_K+1}/N_{n_K+2}} \beta_K u\alpha^{-1} \cdot \hat{f}_{v_0} \quad (2)$$

Proof. This is [Xu19a, Proposition 3.6]. Note that the above formula determines T uniquely, as the function $\hat{f}_{v_0}$ generates the whole representation $\text{ind}_K^G \sigma$. $\square$

3 The $I_{1,K}$ -invariant maps S_K and S_-

In this section, we recall some partial linear operators on a smooth representation π , and their certain invariant properties.

Definition 3.1. *Let π be a smooth representation of G . We define:*

$$\begin{aligned} S_K : \pi^{N'_{m_K}} &\rightarrow \pi^{N_{n_K}}, \\ v &\mapsto \sum_{u \in N_{n_K}/N_{n_K+1}} u\beta_K v. \\ S_- : \pi^{N_{n_K}} &\rightarrow \pi^{N'_{m_K}}, \\ v &\mapsto \sum_{u \in N_{n_K+1}/N_{n_K+2}} \beta_K u\alpha^{-1} v \end{aligned}$$

It is simple to check both S_K and S_- are well-defined.

Proposition 3.2. *We have:*

- (1). *Let $h \in H_1 = I_{1,K} \cap H$. Then $S_K(hv) = h^s \cdot S_K v$, for $v \in \pi^{N'_{m_K}}$, and $S_-(hv) = h^s \cdot S_- v$, for $v \in \pi^{N_{n_K}}$, where h^s is short for $\beta_K h \beta_K$.*
- (2). *If v is fixed by $I_{1,K}$, the same is true for $S_K \cdot v$ and $S_- \cdot v$.*

Proof. This is firstly proved in [Xu17]. We reproduce the arguemnt in full here.

For (1), we note that the group H_1 acts on $\pi^{N_{n_K}}$ and $\pi^{N'_{m_K}}$, as it normalizes N_{n_K} and N'_{m_K} . The statement then follows from the definitions.

To prove (2), we need the following Lemma.

Lemma 3.3. *Given a $u' \in N'_{m_K}$ and a $u \in N_{n_K}$.*

- (1). *There is a unique $u_1 \in N_{n_K}, h \in H_1, u'_1 \in N'_{m_K}$ so that the following identity*

$$u'u = u_1 h u'_1$$

holds.

(2). For any $l > m \geq 0$, when u goes through N_{n_K+m}/N_{n_K+l} , the element u_1 also goes through N_{n_K+m}/N_{n_K+l} .

Proof. The uniqueness statement is clear, and only the existence needs to be proved. Assume $u = n(x, y) \in N$, $u' \in n'(x', y') \in N'$. Then, if $1 + xx' + \overline{yy'}$ is in $E^\times$, we have

$$u'u = u_1 h u'_1,$$

where $u_1 = n(x_1, y_1) \in N$ in which x_1, y_1 are given by

$$x_1 = \frac{x - \overline{yx'}}{1 + xx' + \overline{yy'}}, y_1 = \frac{y}{1 + xx' + \overline{yy'}},$$

and h is the diagonal matrix

$$\begin{pmatrix} \frac{1}{1 + xx' + \overline{yy'}} & 0 & 0 \\ 0 & \frac{1 + xx' + \overline{yy'}}{1 + xx' + \overline{yy'}} & 0 \\ 0 & 0 & 1 + \overline{xx'} + yy' \end{pmatrix},$$

and $u'_1 = n'(x'_1, y'_1) \in N'$ in which

$$x'_1 = \frac{x' - \overline{xy'}}{1 + xx' + \overline{yy'}}, y'_1 = \frac{y'}{1 + xx' + \overline{yy'}}.$$

Under our assumption that $u' \in N'_{m_K}$ and $u \in N_{n_K}$, the condition $1 + xx' + \overline{yy'} \in E^\times$ holds automatically. The existence is established.

We continue to prove (2). From the formulae of x_1 and y_1 given above, one checks by a direct computation that

$$u_1 = u N_{n_K+m+1}, \text{ if } u \in N_{n_K+m} \text{ for some } m \geq 0.$$

Explicitly,

$$u_1 = u \cdot n(*, yz) \tag{3}$$

holds for some $z \in \mathfrak{p}_E$. Recall that $u = n(x, y) \in N_{n_K}$, $u' = n'(x', y') \in N'_{m_K}$.

We may therefore view u' as a map

$$\begin{aligned} u' : N_{n_K+m}/N_{n_K+l} &\rightarrow N_{n_K+m}/N_{n_K+l} \\ u N_{n_K+l} &\mapsto u_1 N_{n_K+l} \end{aligned}$$

It suffices to show the map is injective. Assume for another $w \in N_{n_K}$, we have a decomposition $u'w = u_2 b''$ for $u_2 \in N_{n_K}$ and $b'' \in B'$. We have to prove:

$$u_2 \in u_1 N_{n_K+l} \text{ implies } w \in u N_{n_K+l}.$$

Write $u_1^{-1} u_2$ as u_3 . A little algebraic transform gives:

$$w = u \cdot b'^{-1} u_3 b''$$

We need to check that the element $b'^{-1} u_3 b'' \in N_{n_K}$, denoted by u_4 , lies in N_{n_K+l} . The element b' can be written as $h \cdot u'_1$, for a diagonal matrix $h \in H_1$ and $u'_1 \in N'_{m_K}$. We therefore get

$$u'_1 u_4 = (h^{-1} u_3 h) \cdot h^{-1} b'',$$

where the right hand side is a decomposition of $u'_1 u_4$ given in (1). The uniqueness of such a decomposition implies

$$u_4 \in N_{n_K+l} \text{ if and only if } h^{-1}u_3h \in N_{n_K+l}$$

for any $l \geq 0$. Our assumption is that $u_3 = u_1^{-1}u_2 \in N_{n_K+l}$, which is the same as $h^{-1}u_3h \in N_{n_K+l}$ ($h \in H_1$). We are done. $\square$

We proceed to complete the argument of (2) of the Proposition. By (1) and the decomposition of $I_{1,K} = N'_{m_K} \times H_1 \times N_{n_K}$, it suffices to check that, for $u' = n'(x, y) \in N'_{m_K}$, the element $u' \cdot S_K v$

$$u' \cdot S_K v = \sum_{u \in N_{n_K}/N_{n_K+1}} u' u \beta_K v$$

is still equal to $S_K v = \sum_{u \in N_{n_K}/N_{n_K+1}} u \beta_K v$. By (1) of Lemma 3.3, the right hand side of above sum is equal to:

$$\sum_{u \in N_{n_K}/N_{n_K+1}} u_1 h u'_1 \beta_K v.$$

We get:

$$u' \cdot S_K v = \sum_{u \in N_{n_K}/N_{n_K+1}} u_1 \beta_K (\beta_K h u'_1 \beta_K) v = \sum_{u \in N_{n_K}/N_{n_K+1}} u_1 \beta_K v,$$

which, by (2) of Lemma 3.3, is just $\sum_{u_1 \in N_{n_K}/N_{n_K+1}} u_1 \beta_K v$. The argument for the statement $S_K v \in \pi^{I_1, \kappa}$ for $v \in \pi^{I_1, \kappa}$ is now complete.

By almost the same argument, one can verify that $S_- \cdot v \in \pi^{I_1, \kappa}$ for $v \in \pi^{I_1, \kappa}$. $\square$

Remark 3.4. A slight variant of (2) holds by the same argument. When u goes through $N_{n_K+m} \setminus N_{n_K+n}/N_{n_K+l}$, the element u_1 also goes through $N_{n_K+m} \setminus N_{n_K+n}/N_{n_K+l}$, for any $l > n \geq m \geq 0$.

We apply the operators S_- and S_K to the $I_{1,K}$ -invariants of a principal series $\text{ind}_B^G \varepsilon$. Recall the space $(\text{ind}_B^G \varepsilon)^{I_1, \kappa}$ is spanned by the functions g_1 and g_2 which are characterized by: $g_1(\text{Id}) = 1, g_1(\beta_K) = 0, g_2(\text{Id}) = 0, g_2(\beta_K) = 1$.

Proposition 3.5. We have:

$$(1). S_K g_1 = g_2, S_K g_2 = d_{q^{t_K}} \cdot g_2.$$

$$(2). S_- g_1 = d_{q^{4-t_K}} \cdot g_1, S_- g_2 = \varepsilon(\alpha) g_1.$$

Here, $d_{q^{t_K}} = \sum_{(x,t) \in L_{q^{t_K}}^\times} \varepsilon_0((h(t))), d_{q^{4-t_K}} = \sum_{(x,t) \in L_{q^{4-t_K}}^\times} \varepsilon_0((h(t))), \varepsilon_0 = \varepsilon|_{B \cap K}$.

Proof. By Proposition 3.2, it suffices to compute the values of the functions in consideration at Id and β_K . We omit the details. Note that the exact values of $d_{q^{t_K}}$ and $d_{q^{4-t_K}}$ depend on the nature of ε_0 , see [KX15, Appendix A]. $\square$

Later on, we will use the composition $S_+ = S_K \circ S_-$, which lies in $\text{End}_{\mathbf{F}_p}(\pi^{N_{n_K}})$. Explicitly for $v \in \pi^{N_{n_K}}$, we have

$$S_+ v = \sum_{u \in N_{n_K}/N_{n_K+2}} u \alpha^{-1} v.$$

By Proposition 3.2, it preserves the $I_{1,K}$ -invariants of a smooth representation.

4 Non-supersingular representations

4.1 Some recaps

In this subsection, we recall briefly the restriction of a principal series to a Borel subgroup. For readers' convenience, we reproduce certain details below where we mainly follow the approach in [Vig08].

For a character ε of B , consider the principal series $\text{ind}_B^G \varepsilon$. Recall that $\text{ind}_B^G \varepsilon$ is reducible if and only if $\varepsilon = \eta \circ \det$ for some character η , and in this case it is of length two. Evaluating an $f \in \text{ind}_B^G \varepsilon$ at the identity, we get a B -map from the principal series to the character ε . Denote the kernel by κ_ε . Then we have a short exact sequence of B -representations:

$$0 \rightarrow \kappa_\varepsilon \rightarrow \text{ind}_B^G \varepsilon \rightarrow \varepsilon \rightarrow 0$$

By almost the same argument of [Vig08, Theorem 5], one shows that the B -representation κ_ε is irreducible. Indeed, as explained below, one may even prove κ_ε is irreducible as a representation of $\alpha^\mathbb{Z} N$. But by the same map Φ below, one may verify that $St|_B \cong \kappa_1$. This gives irreducibility of $St|_B$.

Lemma 4.1. *The B -representation κ_ε is irreducible.*

Proof. 1). We firstly identify the underlying space of κ_ε with $C_c^\infty(N)$.

$$\begin{aligned} \Phi : \kappa_\varepsilon &\rightarrow C_c^\infty(N), f \mapsto \Phi(f), \Phi(f)(u) = f(\beta u), \forall u \in N. \\ \Psi : C_c^\infty(N) &\rightarrow \kappa_\varepsilon, f \mapsto \Psi(f), \Psi(f)(b\beta u) = \varepsilon(b)f(u), \forall b \in B, u \in N. \end{aligned}$$

One verifies easily that Ψ and Φ are inverse to each other. This gives $C_c^\infty(N)$ a structure of B -representation.

2). We modify the argument of [Ly15, Proposition 5.2] to our case. Let V be a non-zero B -stable subspace of $C_c^\infty(N)$, and f be a non-zero function in V . As f is compactly supported and N has the decreasing open compact cover $(N_k)_{k \in \mathbb{Z}}$, we may assume the support of f is contained in N_k for some integer k . Write V_k for the subspace of V consisting of functions supported in N_k , we have $V_k \neq 0$. By [BL95, Lemma 1], we know $V_k^{N_k} \neq 0$. This shows that V contains the characteristic function 1_{N_k} of N_k .

Now for any $n \in \mathbb{Z}, u \in N$, we have $u\alpha^n \cdot 1_{N_k} = \varepsilon(\alpha^{-n})1_{N_{k-2n}u^{-1}}$, and as V is B -stable we conclude V contains $1_{N_{k-2n}u^{-1}}$. Note that $1_{N_{k-1}} = \sum_{u \in N_{k-1}/N_k} u^{-1} \cdot 1_{N_k}$, so we have $1_{N_{k-1}} \in V$. Repeating the previous process, we conclude V contains $1_{N_{k-2n-1}u^{-1}}$. In all we have shown V contains all the functions $1_{N_k u}$ for any $k \in \mathbb{Z}$ and $u \in N$. As all the functions $\{1_{N_k u} \mid k \in \mathbb{Z}, u \in N\}$ span the underlying space of $C_c^\infty(N)$, we get $V = C_c^\infty(N)$. $\square$

4.2 Proof of Theorem 1.1

We now come to the main input of this section.

Theorem 4.2. *The restriction map induces an isomorphism between the following spaces:*

$$\mathrm{Hom}_G(\mathrm{ind}_B^G \varepsilon, \pi) \cong \mathrm{Hom}_B(\kappa_\varepsilon, \pi)$$

Proof. We show firstly that the restriction map is injective.

Given ϕ_1 and ϕ_2 in the space $\mathrm{Hom}_G(\mathrm{ind}_B^G \varepsilon, \pi)$, suppose that $\phi = \phi_1 - \phi_2$ vanishes at the subspace κ_ε . By the remark proceeding Lemma 4.1, ϕ induces a B -map from the character ε to π , for which we still denote by ϕ .

Lemma 4.3. *If $\varepsilon \neq \eta \circ \det$ for any character η , then $\mathrm{Hom}_B(\varepsilon, \pi) = 0$.*

Proof. Assume $\phi \neq 0$. As π is smooth, the vector $\phi(1) \in \pi$ is fixed by some N'_{m_K+2k} for large enough k . Using the following identity repeatedly

$$\alpha N'_{m_K+2k-2} \alpha^{-1} = N'_{m_K+2k}$$

and $\phi(\alpha \cdot 1) = \varepsilon(\alpha)\phi(1) = \alpha \cdot \phi(1)$, we see $\phi(1)$ is fixed by N' . As the group G is generated by B and N' , we see ε extends uniquely to a character of G (put $\varepsilon(N') = 1$). In such a situation, $\mathrm{Hom}_B(\varepsilon, \pi) \cong \mathrm{Hom}_G(\varepsilon, \pi)$. $\square$

Remark 4.4. *Under the same assumption on ε , the Lemma implies that any non-zero map in $\mathrm{Hom}_B(\mathrm{ind}_B^G \varepsilon, \pi)$ is an injection. Take $\pi = \mathrm{ind}_B^G \varepsilon$. We conclude that $\mathrm{End}_B(\mathrm{ind}_B^G \varepsilon)$ is one-dimensional. This is because, by the proceeding remark, any non-zero map in the former space will induce a non-zero map in $\mathrm{End}_B(\kappa_\varepsilon)$ which is one-dimensional by Lemma 4.1. We deduce that $\mathrm{End}_B(\mathrm{ind}_B^G \varepsilon) \cong \mathrm{End}_G(\mathrm{ind}_B^G \varepsilon)$.*

We are done if $\varepsilon \neq \eta \circ \det$ for any character η . Assume $\varepsilon = \eta \circ \det$ for some η . After a twist we may assume $\eta = 1$. If $\phi \neq 0$, it induces a non-zero map in $\mathrm{Hom}_B(1, \pi) \cong \mathrm{Hom}_G(1, \pi)$. This in turn implies the map $\phi \in \mathrm{Hom}_G(\mathrm{ind}_B^G 1, \pi)$ realizes the trivial character of G as a quotient of $\mathrm{ind}_B^G 1$ (note that ϕ is not injective by our assumption), which is not true.

We proceed to prove the restriction map is surjective.

Recall again that the space $(\mathrm{ind}_B^G \varepsilon)^{I_1, \kappa}$ is two dimensional with a basis of functions g_1 and g_2 characterized by: $g_1(\mathrm{Id}) = 1, g_1(\beta_K) = 0, g_2(\mathrm{Id}) = 0, g_2(\beta_K) = 1$. By Proposition 3.5, we have

$$S_+ g_2 = \varepsilon(\alpha) g_2 \tag{4}$$

Then, by Lemma 5.3 the K -representation $\langle K \cdot g_2 \rangle = \langle K \cdot S_+ g_2 \rangle$ is a weight, denoted by σ , of dimension greater than one (note that I_K acts on g_2 by the character ε_0^s).

Let ϕ be a non-zero B -map from κ_ε to π . The function g_2 by definition is supported on $B\beta_K I_K$ so it lies in κ_ε . As κ_ε is irreducible (Lemma 4.1), we have $\phi(g_2)$ is non-zero. Since ϕ respects the action of B , the vector $\phi(g_2)$ is fixed by $B \cap I_{1, K}$. Now we compute $\phi(S_+ g_2)$:

$$\phi(S_+ g_2) = \varepsilon(\alpha) \phi(g_2) = S_+ \phi(g_2)$$

that is

$$\phi(g_2) = \varepsilon(\alpha)^{-1} S_+ \phi(g_2) \quad (5)$$

As π is smooth, the vector $\phi(g_2)$ is fixed by some N'_{m_K+2k} for k large enough. Now by applying the argument of Lemma 3.3 to the above equality, we see $\phi(g_2)$ is fixed by N'_{m_K+2k-2} . Repeating such a process enough times, we prove that the vector $\phi(g_2)$ is fixed by N'_{m_K} . By Iwahori decomposition $I_{1,K} = N'_{m_K} \cdot (B \cap I_{1,K})$, we conclude that $\phi(g_2)$ is fixed by $I_{1,K}$.

By Lemma 5.3 again the representation $\langle K \cdot \phi(g_2) \rangle$ is a weight σ' of dimension greater than one. We claim that $\sigma \cong \sigma'$. The Iwahori group I_K acts on the vector $\phi(g_2)$ by ε_0^s . If $\varepsilon_0 \neq \eta \circ \det$ for $K = K_0$ (resp, $\varepsilon_0 \neq \varepsilon_0^s$ for $K = K_1$), the claim follows as in this case a weight is determined by the character of I_K acting on its $I_{1,K}$ -invariants. If it is in the other case, we are also done: neither σ or σ' is a one-dimensional character, and as quotients of the principal series $\text{Ind}_{I_K}^K \varepsilon_0^s$ they are both isomorphic to $st \otimes \varepsilon_0$.

By [Xu18b, Proposition 4.14, 4.15], we have an isomorphism, unique up to a scalar,

$$\text{ind}_K^G \sigma / (T_\sigma - \varepsilon(\alpha)) \cong \text{ind}_B^G \varepsilon.$$

As the representation $\langle G \cdot \phi(g_2) \rangle$ contains the weight σ , it is therefore a quotient of the above representation (by (5) and that σ is not a character). We have shown ϕ extends to a G -map as required. $\square$

With the last Theorem proved, we conclude with the following corollaries.

Corollary 4.5. *Suppose $\varepsilon \neq \eta \circ \det$ for any character η . Then*

$$\text{Hom}_G(\text{ind}_B^G \varepsilon, \pi) \cong \text{Hom}_B(\text{ind}_B^G \varepsilon, \pi)$$

Proof. Let $\phi \in \text{Hom}_B(\text{ind}_B^G \varepsilon, \pi)$ be non-zero. By Remark 4.4, we know ϕ is injective. We have:

- 1). Using Remark 4.4 again, the image of ϕ is contained in $\langle G \cdot \phi(\kappa_\varepsilon) \rangle$.
- 2). By Theorem 4.2 and irreducibility of $\text{ind}_B^G \varepsilon$, there is an isomorphism $\text{ind}_B^G \varepsilon \cong \langle G \cdot \phi(\kappa_\varepsilon) \rangle$.

With these ingredients available, the proof is then identical to that of [Paš07, Corollary 5.5]. $\square$

Corollary 4.6. *We have*

$$\text{Hom}_B(St, \pi) \cong \text{Hom}_G(\text{ind}_B^G 1, \pi)$$

Proof. As $St|_B \cong \kappa_1$ (remarks before Lemma 4.1), the assertion is a special case of Theorem 4.2. Note that the result can not be improved by replacing $\text{ind}_B^G 1$ in the statement by St : the space $\text{Hom}_G(\text{ind}_B^G 1, \text{ind}_B^G 1) \neq 0$ but $\text{Hom}_G(St, \text{ind}_B^G 1) = 0$. $\square$

5 Supersingular representations

5.1 Definition

Recall we have defined the Hecke operator T in subsection 2.3. To define the supersingular representations, we modify it in the following way. If $\dim \sigma = 1$, we put $T_\sigma = T + 1$; otherwise, we put $T_\sigma = T$. Note that this modification reflects the fact the group G has two maximal compact open subgroups, up to conjugacy.

Definition 5.1. *An irreducible smooth $\overline{\mathbf{F}}_p$ -representation π of G is called supersingular if it is a quotient of $\text{ind}_K^G \sigma / (T_\sigma)$, for some weight σ of K .*

5.2 A key property

Let π be an irreducible smooth representation of G , and σ be a weight of K contained in π . By [Xu18a, Theorem 1.1], the representation π admits Hecke eigenvalues for the spherical Hecke algebra $\mathcal{H}(K, \sigma) \cong \overline{\mathbf{F}}_p[T_\sigma]$. This implies that the representation π is a quotient of $\text{ind}_K^G \sigma / (T_\sigma - \lambda)$, for some scalar λ . By [Xu18b, Theorem 1.1], the representation π is supersingular if $\lambda = 0$.

Lemma 5.2. *Let π be a supersingular representation of G , and assume ϕ is a non-zero G -map from $\text{ind}_K^G \sigma$ to π . Then, for large enough $k \geq 1$, we have*

$$\phi \circ T_\sigma^k = 0.$$

Proof. By [Xu18a, Corollary 4.2], there is a non-constant polynomial $P(X)$ such that $\phi \circ P(T_\sigma) = 0$. Assume $P(X)$ is such a polynomial of minimal degree. Take a root λ of $P(X)$, and write $P(X) = (X - \lambda)P_1(X)$. Put $\phi' = \phi \circ P_1(T_\sigma)$. Note that ϕ' is still a G -map from $\text{ind}_K^G \sigma$ to π . By our assumption, the map ϕ' is non-zero and factors through $\text{ind}_K^G \sigma / (T_\sigma - \lambda)$. As π is supersingular, we have $\lambda = 0$ ([Xu18b, Theorem 1.1]). We conclude $P(X) = X^n$ for some $n \geq 1$. $\square$

Lemma 5.3. *Let π be a smooth representation of G . Assume v is a non-zero vector in $\pi^{I_1, \kappa}$, such that I_K acts on v as a character. Then, either $S_+ v = 0$, or $S_+ v$ generates a weight of K of dimension greater than one.*

Proof. Assume $S_+ v \neq 0$. We put $w = S_- v$. By definition,

$$S_+ v = S_K w,$$

and we see w must be non-zero. Consider the K -representation $\kappa = \langle K \cdot w \rangle$. As I_K acts on v by a character χ , I_K acts on w by χ^s (Proposition 3.2). By Frobenius reciprocity, there is a surjective K -map from $\text{Ind}_{I_K}^K \chi^s$ to κ , sending φ_{χ^s} to w . Here, φ_{χ^s} is the function in $\text{Ind}_{I_K}^K \chi^s$ supported on I_K and having value 1 at Id .

Via aforementioned map, we see $\langle K \cdot S_+ v \rangle$ is the image of $\langle K \cdot S_K \varphi_{\chi^s} \rangle$. But the latter, by [KX15, Proposition 5.7], is an irreducible smooth representation of K of dimension greater than one. The assertion follows. $\square$

Proposition 5.4. *Assume π is a supersingular representation of G , and v is a non-zero vector in $\pi^{I_{1,K}}$. Then, for $k \gg 0$, we have $S_+^k v = 0$.*

Proof. Assume firstly I_K acts on v as a character χ .

Assume $S_+ v \neq 0$. By Lemma 5.3 the K -subrepresentation generated by $S_+ v$ is a weight of dimension greater than one, and denote it by σ . By Frobenius reciprocity, we have a G -map ϕ from $\text{ind}_K^G \sigma$ to π , sending the function $\hat{f}_{S_+ v}$ to $S_+ v$. From Lemma 5.2, there is some $k \geq 1$ such that

$$S_+^k v = 0,$$

and we are done in this special case.

Note that $I_K/I_{1,K}$ is an abelian group of finite order prime to p . For any non-zero $v \in \pi^{I_{1,K}}$, the I_K -representation $\langle I_K \cdot v \rangle$ generated by v is a finite sum of characters, and we may write v as $\sum v_i$ so that I_K acts on v_i by a character χ_i of $I_K/I_{1,K}$. We then apply the previous process to each v_i , and take the largest k_i . We are done. $\square$

5.3 A criteria of Paškūnas

In this part, for an irreducible smooth representation we prove a sufficient condition under which its restriction to the Borel subgroup remains irreducible. We will verify it for supersingular ones in the next part.

Proposition 5.5. *Let π be an irreducible smooth representation of G . If, for any non-zero vector $w \in \pi$, there is a non-zero vector $v \in \pi^{I_{1,K}} \cap \langle B \cdot w \rangle$ such that*

$$S_+ v = 0,$$

then $\pi|_B$ is irreducible.

Proof. Let w be a non-zero vector in π . As π is smooth, there exists a $k \geq 0$ such that w is fixed by N'_{2k+m_K} . Hence, the vector $w_1 = \alpha^{-k} w$ is fixed by N'_{m_K} . Since $I_{1,K} = (I_{1,K} \cap B) \cdot N'_{m_K}$, we see

$$\langle I_{1,K} \cdot w_1 \rangle = \langle (I_{1,K} \cap B) \cdot w_1 \rangle.$$

As $I_{1,K}$ is pro- p , the space $\langle (I_{1,K} \cap B) \cdot w_1 \rangle$ has non-zero $I_{1,K}$ -invariant ([BL95, Lemma 1]). We conclude that $\pi^{I_{1,K}} \cap \langle B \cdot w \rangle \neq 0$.

Lemma 5.6. *If $S_+ v = 0$, then $\beta_K v \in \langle B \cdot v \rangle$.*

Proof. By the assumption $S_+ v = 0$, we get

$$v = -\alpha \cdot \sum_{u \in (N_{n_K} \setminus N_{n_K+2})/N_{n_K+2}} u \alpha^{-1} v$$

or equivalently

$$\beta_K v = - \sum_{u \in (N_{n_K} \setminus N_{n_K+2})/N_{n_K+2}} \beta_K \alpha u \alpha^{-1} v. \quad (6)$$

Applying (1), we see $\beta_K \alpha u \alpha^{-1} \in BN'_{m_K}$, for any $u \in (N_{n_K} \setminus N_{n_K+2})/N_{n_K+2}$. More precisely, for $u = n(*, \varpi_E^{n_K} t) \in N_{n_K} \setminus N_{n_K+1}$, we have

$$\beta_K \alpha u \alpha^{-1} = n(*, \varpi_E^{n_K+2} t^{-1}) h(\bar{t}^{-1}) \alpha^{-2} n'(*, \varpi_E^{m_K+1} t^{-1});$$

for $u = n(*, \varpi_E^{n_K+1} t) \in N_{n_K+1} \setminus N_{n_K+2}$, we have

$$\beta_K \alpha u \alpha^{-1} = n(*, \varpi_E^{n_K+1} t^{-1}) h(\bar{t}^{-1}) \alpha^{-1} n'(*, \varpi_E^{m_K} t^{-1}).$$

That gives $\beta_K \alpha u \alpha^{-1} v \in \langle B \cdot v \rangle$ for all $u \in (N_{n_K} \setminus N_{n_K+2})/N_{n_K+2}$. We conclude $\beta_K v \in \langle B \cdot v \rangle$ from the above equality (6). For our later purpose, we record (6) in a more explicit form:

$$\beta_K v = - \sum_u u \alpha^{-2} h(\bar{t}^{-1}) v - \sum_u u \alpha^{-1} h(\bar{t}^{-1}) v, \quad (7)$$

where, in the first sum $u = n(*, \varpi_E^{n_K+2} t)$ goes through $(N_{n_K+2} \setminus N_{n_K+3})/N_{n_K+4}$; in the second sum $u = n(*, \varpi_E^{n_K+1} t)$ goes through $(N_{n_K+1} \setminus N_{n_K+2})/N_{n_K+2}$. $\square$

We proceed to complete the proof of Proposition 5.5. Choose $0 \neq v \in \pi^{I_{1,K}} \cap \langle B \cdot w \rangle$ such that $S_+ v = 0$. The above Lemma says $\beta_K v \in \langle B \cdot v \rangle$. As π is irreducible, we have $\pi = \langle G \cdot v \rangle$. By the Bruhat decomposition $G = BI_{1,K} \cup B\beta_K I_{1,K}$, we see

$$\pi \subseteq \langle B \cdot v \rangle \subseteq \langle B \cdot w \rangle.$$

Hence, we have proved $\pi = \langle B \cdot w \rangle$ for any $w \in \pi$, and the proposition follows. $\square$

Remark 5.7. *The condition in Proposition 5.5 is merely sufficient: for the Steinberg representation St , its restriction to B is irreducible, but S_+ does not annihilate the line $St^{I_{1,K}}$. More precisely, the representation St is defined as $\text{ind}_B^G 1/(1)$, and its $I_{1,K}$ -invariants is of one dimension. In the notation of section 3, the space $St^{I_{1,K}}$ is spanned by the image of g_1 . By Proposition 3.5, we check that*

$$S_+ g_1 = -g_2$$

which is just $S_+ \overline{g_1} = \overline{g_1}$.

5.4 Proof of (1) of Theorem 1.2

Theorem 5.8. *Let π be a supersingular representation of G . Then $\pi|_B$ is irreducible.*

Proof. Let w be any non-zero vector in π . We already know that (from the argument of Proposition 5.5)

$$\pi^{I_{1,K}} \cap \langle B \cdot w \rangle \neq 0.$$

Take any non-zero vector v in the above space. As π is supersingular, by Proposition 5.4 v will be annihilated by S_+^k for k large enough. Let m be the least positive integer satisfying that. Now the vector $v' = S_+^{m-1} v$ is non-zero. By Proposition 3.2, v' is still $I_{1,K}$ -invariant, and lies in $\langle B \cdot w \rangle$ by the form of S_+ . It satisfies

$$S_+v' = 0.$$

We are done by Proposition 5.5. $\square$

An immediate but interesting application of Theorem 5.8 is the following. Recall the Levi decomposition $B = H \ltimes N$, where H is the diagonal subgroup and N is the upper unipotent radical.

Corollary 5.9. *For a supersingular representation π of G , we have*

$$\pi_N = 0,$$

i.e., the usual Jacquet module of π (with respect to B) vanishes.

Proof. Recall that $\pi_N = \pi/\pi(N)$, and the space $\pi(N) := \langle \pi(u) \cdot v - v \mid u \in N, v \in \pi \rangle$ is B -stable and non-zero. The assertion follows by Theorem 5.8. $\square$

5.5 Proof of (2) of Theorem 1.2

Theorem 5.10. *Give two smooth representations π and π' of G . Suppose π is supersingular. Then, we have*

$$\mathrm{Hom}_G(\pi, \pi') \cong \mathrm{Hom}_B(\pi, \pi').$$

Proof. As we have one side inclusion $\mathrm{Hom}_G(\pi, \pi') \hookrightarrow \mathrm{Hom}_B(\pi, \pi')$, it suffices to prove any non-zero map ϕ in $\mathrm{Hom}_B(\pi, \pi')$ is also a G -map.

We begin with a non-zero vector $v \in \pi^{I_1, K}$, and by replacing v by a proper vector in the space $\langle I_K \cdot v \rangle$, we may assume I_K acts on v by a character χ . As $\pi|_B$ is irreducible (Theorem 5.5), the map ϕ is injective whence $\phi(v) \neq 0$. As π' is smooth, $\phi(v)$ is fixed by N'_{m_K+2m} for some $m \geq 0$.

We assume $m \geq 1$, and proceed to find a non-zero vector $w \in \pi^{I_1, K} \cap \langle B \cdot v \rangle$ such that $\phi(w)$ is fixed by N'_{m_K+2m-2} .

Case I: $S_+v \neq 0$. We take w as S_+v , so it lies in $\pi^{I_1, K} \cap \langle B \cdot v \rangle$. Note that I_K acts on w still by the character χ . Using Lemma 3.3, we see the vector $\phi(w) = S_+\phi(v)$ is fixed by N'_{m_K+2m-2} .

Case II: $S_+v = 0$ and $S_-v \neq 0$. We take S_-v as w , which lies in $\pi^{I_1, K}$ (Lemma 3.2). Our assumption $S_+v = 0$ means

$$S_K w = 0,$$

which is equivalent to

$$w = -\sum_{u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+2}} \beta_K u \alpha^{-1} v$$

An application of (1) gives that for $u = n(*, \varpi_E^{n_K} t) \in N_{n_K} \setminus N_{n_K+1}$

$$\beta_K u \alpha^{-1} = n(*, \varpi_E^{n_K} t^{-1}) \alpha^{-1} h(\bar{t}^{-1}) n'(*, \varpi_E^{m_K+1} t^{-1}),$$

hence we have $\beta_K u \alpha^{-1} \cdot v = \chi(h(\bar{t}^{-1})) n(*, \varpi_E^{n_K} t^{-1}) \alpha^{-1} \cdot v$. In all we get

$$w = -\sum_{u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+2}} \chi(h(\bar{t}^{-1})) u \alpha^{-1} \cdot v \in \langle B \cdot v \rangle,$$

whence that

$$\phi(w) = - \sum_{u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+2}} \chi(h(\bar{t}^{-1})) u \alpha^{-1} \cdot \phi(v)$$

This, combined with Lemma 3.3 (especially (3)) and Remark 3.4, shows $\phi(w)$ is fixed by N'_{m_K+2m-2} .

Case III: $S_- v = 0$ (whence $S_+ v = 0$). By the definition of S_- , this assumption is same as $\sum_{u \in N_{n_K+1}/N_{n_K+2}} u \alpha^{-1} \cdot v = 0$. Then by almost the same argument we have

$$\beta_K v = - \sum_{u \in N_{n_K+1} \setminus N_{n_K+1}/N_{n_K+2}} \chi(h(\bar{t}^{-1})) u \alpha^{-1} \cdot v.$$

Thus

$$\phi(\beta_K v) = - \sum_{u \in N_{n_K+1} \setminus N_{n_K+1}/N_{n_K+2}} \chi(h(\bar{t}^{-1})) u \alpha^{-1} \cdot \phi(v)$$

We conclude that $\phi(\beta_K v)$ is fixed by N'_{m_K+2m-2} , using Lemma 3.3 (especially (3)) and Remark 3.4. Based on this, one verifies further that, for any $u \in N_{n_K}/N_{n_K+1}$, the vector $\phi(u\beta_K v)$ is still fixed by N'_{m_K+2m-2} , using Lemma 3.3 again.

- $S_K v = \sum_{u \in N_{n_K}/N_{n_K+1}} u \beta_K v \neq 0$. In this case, we take $S_K v$ as w . As

$$\phi(w) = \sum_{u \in N_{n_K}/N_{n_K+1}} \phi(u\beta_K v),$$

we conclude that $\phi(w)$ is fixed by N'_{m_K+2m-2} by the proceeding remarks.

- $S_K v = \sum_{u \in N_{n_K}/N_{n_K+1}} u \beta_K v = 0$. This assumption gives us that

$$v \in \langle u\beta_K v \mid u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+1} \rangle.$$

Explicitly, under this assumption we have

$$v = - \sum_{u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+1}} \chi(h(\bar{t}^{-1})) u \beta_K \cdot v$$

Consider the K -representation $\tau = \langle K \cdot v \rangle$. It is spanned by the set $\{v, u\beta_K v \mid u \in N_{n_K}/N_{n_K+1}\}$. Then we see it can be spanned by the set

$$\{u\beta_K v \mid u \in N_{n_K} \setminus N_{n_K+1}/N_{n_K+1}\}$$

Now take a weight σ contained in τ , and the unique line $\sigma^{I_1, K}$ is given by a non-zero vector which we take as our w . Then w is a linear combination of the elements from the above set. This implies, as before that $\phi(w)$ is fixed by N'_{m_K+2m-2} .

Note that in each case the group I_K acts as a character on the vector w we find. By repeating the process, we find a non-zero $w \in \pi^{I_1, K}$ on which I_K acts as a character, such that $\phi(w)$ is fixed by N'_{m_K} . Note that $\phi(w)$ is automatically fixed by $B \cap I_{1, K}$, whence the vector $\phi(w)$ lies in $(\pi')^{I_1, K}$.

We proceed to complete the proof. Recall our condition that π is supersingular. We apply Proposition 5.4 to the vector w . We find some $k \geq 1$ such that $S_+^k w = 0$ but $S_+^{k-1} w \neq 0$. We write the vector $S_+^{k-1} w$ as w' . Then w' is a non-zero $I_{1, K}$ -invariant such that $S_+ w' = 0$. This, as we already recorded in (7), gives that

$$\beta_K w' = - \sum_u u \alpha^{-2} h(\bar{t}^{-1}) w' - \sum_u u \alpha^{-1} h(\bar{t}^{-1}) w'$$

where, in the first sum $u = n(*, \varpi_E^{n_K+2}t)$ goes through $(N_{n_K+2} \setminus N_{n_K+3})/N_{n_K+4}$; in the second sum $u = n(*, \varpi_E^{n_K+1}t)$ goes through $(N_{n_K+1} \setminus N_{n_K+2})/N_{n_K+2}$. As ϕ is a B -map, we firstly see that

$$\phi(\beta_K w') = -\sum_u u \alpha^{-2} h(\bar{t}^{-1}) \phi(w') - \sum_u u \alpha^{-1} h(\bar{t}^{-1}) \phi(w')$$

But we also have $\phi(S_+ w') = S_+ \phi(w') = 0$, which gives us similarly that

$$\beta_K \phi(w') = -\sum_u u \alpha^{-2} h(\bar{t}^{-1}) \phi(w') - \sum_u u \alpha^{-1} h(\bar{t}^{-1}) \phi(w')$$

We conclude

$$\phi(\beta_K w') = \beta_K \phi(w').$$

Recall that the vector w' and its image $\phi(w')$ are both $I_{1,K}$ -invariant. As $G = BI_{1,K} \cup B\beta_K I_{1,K}$ and the non-zero vector w' generates the representation π , we conclude that from the above equality that the B -map ϕ is indeed a G -map. $\square$

Remark 5.11. *In early versions of this paper, we didn't find a full proof of Theorem 5.10. We came to the above proof recently. The operator S_- is an analogue of the element $\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$ in $GL_2(F)$, but the situation that $S_- v = 0$ for some $v \in \pi^{I_{1,K}}$ can always happen, which is not the case for $GL_2(F)$. This is dealt with in our above **Case III** argument, and we consider it as a novelty of our work. It would be interesting to see whether it can be adapted to other problems.*

Acknowledgements

Most part of this work was done when the author was a postdoc at Einstein Institute of Mathematics (HUI), supported by European Research Council project 669655, and he would like to thank the Institute for the hospitality. Our debt owed to the works of Paškūnas ([Paš07]) and Hu ([Hu12]) should be clear to the reader.

References

- [Abd11] Ramla Abdellatif, *Autour des représentations modulo p des groupes réductifs p -adiques de rang 1*, PhD Thesis, University of Paris-Sad, 2011.
- [Abd21] ———, *Restriction of p -modular representations of p -adic groups to minimal parabolic subgroups*, RIMS Kokyuroku2097, 6pp, 2021.
- [AHHV17] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, *A classification of irreducible admissible mod p representations of p -adic reductive groups*, J. Amer. Math. Soc. **30** (2017), no. 2, 495–559. MR 3600042

- [Ber10] Laurent Berger, *Représentations modulaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et représentations galoisiennes de dimension 2*, Astérisque (2010), no. 330, 263–279. MR 2642408
- [BL95] Laure Barthel and Ron Livné, *Modular representations of GL_2 of a local field: the ordinary, unramified case*, J. Number Theory **55** (1995), no. 1, 1–27. MR 1361556 (96m:22036)
- [Bre03] Christophe Breuil, *Sur quelques représentations modulaires et p -adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I*, Compositio Math. **138** (2003), no. 2, 165–188. MR 2018825 (2004k:11062)
- [CE04] Marc Cabanes and Michel Enguehard, *Representation theory of finite reductive groups*, New Mathematical Monographs, vol. 1, Cambridge University Press, Cambridge, 2004. MR 2057756 (2005g:20067)
- [Col10] Pierre Colmez, *Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ) -modules*, Astérisque (2010), no. 330, 281–509. MR 2642409
- [Her11] Florian Herzig, *A Satake isomorphism in characteristic p* , Compos. Math. **147** (2011), no. 1, 263–283. MR 2771132
- [HKV20] Florian Herzig, Karol Koziol, and Marie-France Vignéras, *On the existence of admissible supersingular representations of p -adic reductive groups*, Forum Math. Sigma **8** (2020), 73 (English), Id/No e2.
- [Hu12] Yongquan Hu, *Diagrammes canoniques et représentations modulo p de $\mathrm{GL}_2(F)$* , J. Inst. Math. Jussieu **11** (2012), no. 1, 67–118. MR 2862375
- [HV15] Guy Henniart and Marie-France Vignéras, *A Satake isomorphism for representations modulo p of reductive groups over local fields*, J. Reine Angew. Math. **701** (2015), 33–75. MR 3331726
- [KX15] Karol Koziol and Peng Xu, *Hecke modules and supersingular representations of $\mathrm{U}(2, 1)$* , Representation Theory **19** (2015), 56–93. MR 3321473
- [Le19] Daniel Le, *On some nonadmissible smooth irreducible representations for GL_2* , Math.Research.Letters **26** (2019), no. 6, 1747–1758.
- [Ly15] Tony Ly, *Des représentations modulo p de $\mathrm{GL}(2, D)$, D algèbre à division sur un corps local*, J. Number Theory **151** (2015), 54–106. MR 3314202
- [Paš07] Vytautas Paškūnas, *On the restriction of representations of $\mathrm{GL}_2(F)$ to a Borel subgroup*, Compos. Math. **143** (2007), no. 6, 1533–1544. MR 2371380 (2009a:22013)

- [Vig08] Marie-France Vignéras, *Série principale modulo p de groupes réductifs p -adiques*, Geom. Funct. Anal. **17** (2008), no. 6, 2090–2112. MR 2399093
- [Xu17] Peng Xu, *On certain Iwahori representation of unramified $U(2, 1)$ in characteristic p* , <https://arxiv.org/abs/1706.02674>, Preprint, 2017.
- [Xu18a] ———, *Hecke eigenvalues in p -modular representations of unramified $U(2, 1)$* , <https://sites.google.com/view/xupeng2012/research>, 2018.
- [Xu18b] ———, *Notes on p -modular representations of unramified $U(2, 1)$* , <https://sites.google.com/view/xupeng2012/research>, 2018.
- [Xu19a] ———, *Freeness of spherical Hecke modules of unramified $U(2, 1)$ in characteristic p* , J. Number Theory **195** (2019), 293–311. MR 3867443
- [Xu19b] ———, *Restriction of p -modular representations of $U(2, 1)$ to a Borel subgroup*, <http://arxiv.org/abs/1902.02018>, 2019.

E-mail address: xupeng2012@gmail.com