Stabilizer Circuits, Quadratic Forms, and Computing Matrix Rank

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Abstract

We show that a form of strong simulation for *n*-qubit quantum stabilizer circuits C is computable in $O(s + n^{\omega})$ time, where ω is the exponent of matrix multiplication. Solution counting for quadratic forms over \mathbb{F}_2 is also placed into $O(n^{\omega})$ time. This improves previous $O(n^3)$ bounds. Our methods in fact show an $O(n^2)$ -time reduction from matrix rank over \mathbb{F}_2 to computing $p = |\langle 0^n | C | 0^n \rangle|^2$ (hence also to solution counting) and a converse reduction that is $O(s + n^2)$ except for matrix multiplications used to decide whether p > 0. The current best-known worst-case time for matrix rank is $O(n^{\omega})$ over \mathbb{F}_2 , indeed over any field, while ω is currently upper-bounded by 2.3728... Our methods draw on properties of classical quadratic forms over \mathbb{Z}_4 . We study possible distributions of Feynman paths in the circuits and prove that the differences in +1 vs. -1 counts and +*i* vs. -*i* counts are always 0 or a power of 2. Further properties of quantum graph states and connections to graph and matroid theory are discussed.

1 Introduction

Consider the following algorithm for computing the rank r of an $n \times n$ matrix \mathbf{A}_0 over the field \mathbb{F}_2 :

- 1. Form the symmetric block matrix $\mathbf{A} = \begin{bmatrix} 0 & \mathbf{A}_0 \\ \mathbf{A}_0^\top & 0 \end{bmatrix}$.
- 2. Form the quantum graph state circuit $C_{\mathbf{A}}$ for the bipartite graph with adjacency matrix \mathbf{A} .
- 3. Calculate p = the quantum probability that $C_{\mathbf{A}}(0^{2n}) = 0^{2n}$. The bipartite case assures p > 0.
- 4. Output $r = \log_2(1/\sqrt{p})$.

All steps except 3 take $O(n^2)$ time. Hence, for dense matrices, this is a linear-time reduction from r to p. In the converse direction, we will show the following algorithm for computing the amplitude $\langle 0^n | C | 0^n \rangle$ for any *n*-qubit quantum stabilizer circuit C:

- 1. Convert C to a classical quadratic form q_C over \mathbb{Z}_4 that retains all quantum properties of C.
- 2. Take the matrix **A** of q_C over \mathbb{Z}_4 and associate a canonical $n \times n$ matrix **B** over \mathbb{F}_2 to it.
- 3. Compute the decomposition $\mathbf{B} = \mathbf{P}\mathbf{L}\mathbf{D}\mathbf{L}^{\top}\mathbf{P}^{\top}$ over \mathbb{F}_2 where \mathbf{P} is a permutation matrix, \mathbf{L} is lower-triangular, and \mathbf{D} is block-diagonal with blocks that are either 1×1 or 2×2 .
- 4. Take \mathbf{L}^{-1} over \mathbb{F}_2 but compute $\mathbf{D}' = \mathbf{L}^{-1} \mathbf{P}^\top \mathbf{A} \mathbf{P} (\mathbf{L}^{-1})^\top$ over \mathbb{Z}_4 . (Note $\mathbf{P}^\top = \mathbf{P}^{-1}$.) If any diagonal 1×1 block of \mathbf{D} has become 2 in \mathbf{D}' , output $\langle 0^n | C | 0^n \rangle = 0$. Else, $\langle 0^n | C | 0^n \rangle$ is nonzero and is obtained by a simple O(n)-time recursion.

Here step 1 from [RCG18] takes time linear in the number s of quantum gates in C, which for standard-basis inputs can be bounded above by $O(n^2/\log n)$ with O(n) quantum Hadamard gates [AG04]. Step 3 is computable in $O(n^{\omega})$ time by [DP18], where ω is the exponent of matrix multiplication and is at most $n^{2.372865}$ [Sto10, Wil12, Gal14]. This is also the best known time for computing $n \times n$ matrix rank over any field and for the particular inverses and products in step 4 as well (see [CKL13]). However, when $\langle 0^n | C | 0^n \rangle \neq 0$ we show that its absolute value is computable quickly from r alone after step 2.

Graph-state circuits and the larger but equivalent class of stabilizer (aka. Clifford) circuits are commonly quoted as simulatable in $O(n^2)$ time but this applies only with a bounded number of single-qubit measurements [AG04, AB06] (see also [GM13, GMC14, GM15]). Computing the probability $p = |\langle 0^n | C | 0^n \rangle|^2$ is classed as a form of *strong simulation* by [JvdN14] and is representative of the tasks designated STR(n) in [Koh17] for standard-basis inputs. The best times stated for computing p in the above-cited papers are $O(n^3)$ for the general n-qubit case. Thus our results improve the asymptotic running time as well as show a near-tight relationship to the task of computing matrix rank that seems not to be noticed in these papers. They also improve the $O(n^3)$ -time algorithm for solution counting of quadratic forms over \mathbb{Z}_2 , as given in [EK90], to $O(n^{\omega})$. (As is common in the literature, we slur the distinction between ω as an infimum and as an upper bound—the latter usage should properly say "in time $n^{\omega+o(1)}$ " or similar. Our machine model is implicitly a RAM that can handle log n-sized words in unit time; for other models we can say the time bounds ignore log factors.)

- **Theorem 1.1** (main). (a) Strong simulation of n-qubit stabilizer circuits of size s with h Hadamard gates (or other nondeterministic single-qubit gates) on standard-basis inputs is in time $O(s+n+h^{\omega})$ where $2 \leq \omega < 2.3729$. This works for amplitude as well as probability.
 - (b) Computing $n \times n$ matrix rank is linear-time equivalent to computing the probability p (for circuits where h = Theta(n) and $s = O(n^2)$) on the promise that p is positive, and equivalent to computing p on the narrower promise that the graphs underlying the circuits are bipartite.

In view of the normal form of [AG04] and in practice, the restriction on h and s in (b) is highly reasonable. The "promise" formulation of (b) is ignorable in the direction from the rank r to p, but not from p to r. The sense of the latter direction is that if rank for dense matrices comes to have a lesser time t(n) with $n^2 \leq t(n) < n^{\omega}$ than matrix multiplication, then computing p correctly in cases where p > 0 will have exactly the same time t(n), whereas computing p in all cases might remain in n^{ω} time. We do not have a reduction from matrix multiplication itself (over \mathbb{F}_2) to strong simulation, hence our results do not imply an asymptotic equivalence between those. To be sure, we note as a practical caveat that among the known sub-cubic algorithms for matrix multiplication, only Strassen's original one [Str69], which runs in time $O(n^{2.81})$, is considered competitive for problem sizes in the range of thousands of qubits that are addressed concretely in the above-cited papers.

The connections used in our proof run through the real-time conversion of quantum circuits Cto "phase polynomials" q_C over \mathbb{Z}_K for $K = 2^k$, $k \ge 1$ in [RC09, RCG18], which extended results by [DHH⁺04] for k = 1, and the analysis of quadratic forms over \mathbb{Z}_4 by Schmidt [Sch09] drawing on [Alb38, Bro72]. In the case of graph-state circuits and *stabilizer circuits* more generally, q_C becomes a classical quadratic form over \mathbb{Z}_4 , as treated also in [CGW18]. Our approach is related to ones involving Gauss sums [BvDR08, CCLL10, CGW18, Bk18] but exploits the availability of normal forms. For bipartite **A** as above, it further devolves into a quadratic form q'_C over \mathbb{F}_2 that is *alternating* (as defined below) plus an ancillary vector v. A linear change in basis—which also sends v to a vector w but leaves the probability computation unaffected—gives over \mathbb{Z}_4 the normal form

$$q'_{C} = y_{1}y_{2} + y_{3}y_{4} + \dots + y_{2g-1}y_{2g} + \sum_{j=1}^{n} 2y_{j}w_{k}.$$
 (1)

Here the rank r must be even and g = r/2. This corresponds to block-diagonal matrices **D** with g-many 2×2 blocks as produced by [DP18], together with 1×1 blocks coming from w. The 1×1 blocks matter most for j > r. The matrix **D**' over \mathbb{Z}_4 may no longer be block-diagonal but its diagonal reveals w.

Provided the terms in w do not cause global cancellation, equation (1) will yield p from r in an invertible manner, without needing to compute the change in basis. Let $N_c(q)$ stand for the number of arguments $x \in \{0, 1\}^n$ giving $q(x) = c \pmod{4}$ for c = 0, 1, 2, 3. Along the way to our main theorem, we prove that for any classical quadratic form q over \mathbb{Z}_4 , the differences $|N_0(q) - N_2(q)|$ and $|N_1(q) - N_3(q)|$ are either zero or a power of 2. This resolves the effects of the "w" part of the normal form (1) for the alternating case in particular. Sections 2 and 3 cover stabilizer circuits and quadratic forms before sections 4 proves part (a) of Theorem 1.1 and the rank-to-strong-simulation direction of part (b).

The other direction—which was the original goal—requires computing r plus information about w. The datum needed is whether a \mathbb{Z}_4 vector corresponding to w has an entry of value 2, or equivalently, whether the graph underlying C belongs to a family we call "net-zero" graphs. Section 5 analyzes a concept of "self-dual" quadratic forms yielded by the probability computation and reduces from the general to the alternating case—which effectively strips phase gates from the circuit and self-loops from the graph—and finishes the proof of part (b) of Theorem 1.1.

A concluding section Section 6 discusses the "net-zero" graphs and observes that the amplitude function $a(G) = \langle 0^n | C_G | 0^n \rangle$ is a generalized Tutte invariant per [OW93, Nob06]. It then contrasts the integral but non-classical quadratic forms which arise from adding the controlled-phase gate to the stabilizer gates to form a universal gate set. Finally we raise possible implications of this work for solution counting and for graph theory.

2 Quantum Stabilizer Circuits

A fundamental problem in quantum computing is whether all quantum circuits of s gates acting on n qubits can be simulated in time polynomial in s and n. A quantum circuit C effects a unitary linear transformation on \mathbb{C}^N where $N = 2^n$. A fixed basis of \mathbb{C}^N is identified with $\{0,1\}^n$. The circuit is a sequence of gates g, each of which effects a transformation U_g that acts on some k of the qubits. The gate g can be represented by a $2^k \times 2^k$ unitary gate matrix \mathbf{M}_g and the subset S_g of qubits acted on.

A salient subclass of quantum circuits that have a deterministic polynomial-time simulation are stabilizer circuits. They can be generated by the following three gate matrices \mathbf{M}_{g} :

$$\mathsf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathsf{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Their extensions to act on \mathbb{C}^N by tensor product with the identity on the qubits outside S_g generate the *n*-qubit *Clifford group*, so these are also called *Clifford circuits*. The original polynomial-time algorithm by Gottesman and Knill [Got98] involved Gaussian elimination and so ran for all intents and purposes in order-of n^3 time. Aaronson and Gottesman [AG04] improved this to $O(n^2)$ time with a tableau method and also showed that every stabilizer circuit has an equivalent one with $O(n^2/\log n)$ gates. Anders and Briegel [AB06] improved the running time concretely and for circuits of size $s = o(n^2)$ using a graph-state representation, as we will also do. Dehaene and De Moor [DM03] described quantum states produced by stabilizer circuits via linear and quadratic forms over \mathbb{F}_2 in ways simplified and extended by van den Nest [vdN09].

We seek even simpler and faster methods that lend themselves to further algorithmic properties, such as quick update when changes are made to C in the sense of "dynamic algorithms." We employ the theory of classical quadratic forms over \mathbb{Z}_4 as developed by Schmidt [Sch09] and more recently by Cai, Guo, and Williams [CGW18]. The quadratic forms are built using the real-time algorithm of [RC09, RCG18] for computing what we call the *additive partition polynomials* q_C for quantum circuits C that meet a mild "balance" condition. Related works involving low-degree polynomials and counting complexity include [BvDR08, BJS10, Mon17, KPS17].

The polynomial q_C has variables x_1, \ldots, x_n corresponding to binary input values, z_1, \ldots, z_n for the binary output values, and y_1, \ldots, y_h representing nondeterminism from Hadamard (and possibly other) gates. For any $a, b \in \{0, 1\}^n$, letting q_{ab} denote q with those values substituted for the x_i and z_i variables, we have for some R > 0:

$$\langle b | C | a \rangle = \frac{1}{R} \sum_{y \in \{0,1\}^h} \omega^{q_{ab}(y)},$$
(2)

where ω is a K-th root of unity such that all phases produced by the circuit are powers of ω . Stabilizer circuits give K = 4 so that the powers in this exponential sum belong to \mathbb{Z}_4 . Generally $R = 2^{h/2}$ but its value is reduced if some nondeterministic y_j variables are forced to equal outputs.

The rules for calculating q are straightforward. Initially q = 0 and each qubit line i has its current annotation u_i defined by $u_i = x_i$. In general, let u_i stand for the current annotation of line i, and let $y_1, \ldots, y_{\ell-1}$ be the nondeterministic variables allocated thus far.

- Hadamard gate on line i: Allocate a new variable y_{ℓ} , do $q + 2u_i y_{\ell}$, and reassign u_i to be y_{ℓ} .
- Phase gate S on line i: $q \neq u_i$, u_i unchanged.
- CZ gate on lines i and j: $q \neq 2u_iu_j$, no other change.
- At the end of each qubit line *i*, we can identify z_i with the variable last denoted by u_i .

Since we are concerned only with 0, 1 as arguments, we can also do $q += u_i^2$ in the case of S, thus making all terms homogeneously quadratic. The *conjugate polynomial* q^* does $q^* += 3u_i^2$ instead, but does the same as q for H and CZ.

We mention some other Clifford gates and their rules for completeness. The first three (plus the identity I) are the *Pauli gates*:

$$\mathsf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathsf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- X: Change u_i to $1 u_i$; no other change.
- Z: $q \neq 2u_i$; same for q^* ; no other change.

- Y: Treat via Y = iXZ and ignore the global scalar *i*.
- CNOT gate on lines i and j: Change the target u_i to $u_i + u_j 2u_iu_j$; no other action.

Note that X = HZH, and similarly, $CNOT = (I \otimes H)CZ(I \otimes H)$, so we could have composed the previous rules for those gates. Changing the annotations u_i and u_j (in the respective cases), however, avoids introducing new nondeterministic variables.

The annotation u_j becomes quadratic in the case of CNOT, but the degree does not rise any higher: In rules where u_i is multiplied, the multiplier contains a factor 2 which cancels the $2u_iu_j$ modulo 4. The last subtlety is what happens when an annotation that is not a single variable is to be equated with a variable z_j . If it has the form $u_i + u_j - 2u_iu_j$ then we add to q the term $2w(u_i + u_j - 2u_iu_j - z_j) = 2wu_i + 2wu_j + 2wz_j \pmod{4}$ where w is a fresh variable. For binary values from the standard basis, if z_j does not equal the XOR of u_i and u_j then the added term reduces to 2w. Because w appears nowhere else, assignments with w = 0 and those with w = 1will globally cancel in (2). Thus only cases with $z_j = u_i \oplus u_j$ contribute. This proves

Theorem 2.1 ([RCG18]). When C is a stabilizer circuit, the polynomial q in (2) becomes a quadratic form over \mathbb{Z}_4 in which all terms involving two variables have coefficient 2.

Such forms are called *classical*, reflecting the historical definition of a quadratic form as given by $x^{\top} \mathbf{A} x$ for some integer $n \times n$ matrix A that is symmetric—so that all cross terms have even coefficients. For \mathbb{Z}_4 they coincide with those called *affine* in [CLX14, CGW18]. For contrast, we note the effect of using a controlled-phase gate:

$$\mathsf{CS} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

If i, j are the qubit lines involved, the rule is to do $q += u_i u_j$ and $q^* += 3u_i u_j$. The CS gate is not a Clifford gate and its inclusion creates a universal gate set. Nevertheless, (2) still holds, as does the following:

Theorem 2.2 ([RCG18]). For any Clifford+CS circuit C, input x, outcome z, and letting V stand for the set of unfixed variables overall,

$$\Pr[C(x) = z] = \frac{1}{R^2} \left[|\{v, v' \in \{0, 1\}^{|V|} : q_{x,z}(v) + q_{x,z}^*(v') = 0\} |$$
(3)

$$|\{v, v' \in \{0, 1\}^{|V|} : q_{x,z}(v) + q_{x,z}^*(v') = 2\}|],$$
(4)

with R as in (2) and values modulo 4. Moreover, for any set Z of binary outcomes defined by fixing a subset of the variables to a value z', $\Pr[C(x) \in Z]$ is given by an analogous formula over pairs v, v' of assignments that agree on the remaining z_j variables involving $q_{x,z'}$ and $q_{x,z'}^*$.

Using the " w_j " variables as above to equate outputs makes $R = 2^h$. As we will note in Lemma 5.1(d) in section 5, it is unnecessary to put absolute-value bars on the difference in (3). There we will symmetrize the roles of x and z and address the general topic of a "self-dual" (classical) quadratic form. We further note the remarkably fine-cut "dichotomy" that although counting solutions to q(v) = 0 over $v \in \mathbb{Z}_4^r$ is in polynomial time for any quadratic form, counting them over $v \in \{0, 1\}^r$ is #P-complete for a non-classical quadratic form [CLX14]. Classical quadratic forms are indifferent

between 0 and 2 as arguments, likewise 1 versus 3, because $2^2 = 0$, $3^2 = 1$, and $2 \cdot 1 = 2 \cdot 3 = 2$ modulo 4, so counting solutions over \mathbb{Z}_4^r and over $\{0,1\}^r$ is equivalent for them.

Let us bear in mind that since (2) computes all amplitudes, the polynomial $q = q_C$ includes all information about the quantum behavior of the circuit C. Thus nothing is lost by manipulating (only) q_C . As an application, we deduce the known fact that graph-state circuits are entirely representative of stabilizer circuits with O(s + n) overhead. Such circuits consist of:

- An initial *n*-ary Walsh-Hadamard transform $\mathsf{H}^{\otimes n}$, effected by placing one Hadamard gate at the start of each qubit line.
- For every edge (i, j) in the given graph G, place a CZ gate between lines i and j. Order does not matter because these operations commute.
- If G has a self-loop at node i, place an S gate there.
- A final $\mathsf{H}^{\otimes n}$.

Proposition 2.3. There is a real-time procedure that given any n-qubit stabilizer circuit C with h Hadamard gates and $x, z \in \{0, 1\}^n$ constructs a graph state circuit C_G on h qubits such that $\langle z | C | x \rangle = \langle 0^h | C_G | 0^h \rangle$.

Proof. Build q_C in real time as above and substitute for x and z. This leaves h variables y_k from the Hadamard gates plus any w_j variables that were employed. Now define the graph G to have an edge (i, j) for every term $2y_iy_j$ (or $2y_iw_j$) in q_C , and a self-loops at i for every term ay_i^2 , a = 1, 2, 3. Note that the coefficients a of the self-loop terms may arise from the substitutions for particular binary values of x and z. The corresponding graph-state circuit has inputs x', z' of its own, but those are zeroed in forming $\langle 0^h | C_G | 0^h \rangle$. The leftover terms in q_{C_G} are identical to those of q_C after the substitution.

If $h = \Theta(n)$ then the number of variables is linear in n. Our original aim was to use this correspondence to be competitive with the above-cited $O(n^2)$ algorithms—and ones that improve then the graph is sparse—in the concrete sense of better leading constants and simplified cases. For those algorithms previously not known to have time better than $O(n^3)$ or similar, our practical objective in what follows is not so much reducing the exponent to ω but rather to $O(n^2)$ time given knowledge of the rank r, for contexts where r might be foreknown or well approximated.

3 Properties of Classical Quadratic Forms Over \mathbb{Z}_4

A classical quadratic form f in variables $\mathbf{x} = (x_1, \dots, x_n)$ is one induced by a symmetric $n \times n$ integer matrix \mathbf{A} as

$$f(\mathbf{x}) = \mathbf{x}^{\top} \mathbf{A} \mathbf{x}.$$
 (5)

This makes every coefficient of a cross term $x_i x_j$ even, and over \mathbb{Z}_4 all nonzero cross terms have coefficient 2. Such a form over \mathbb{Z}_4 treats arguments 0 and 2 the same, likewise 1 and 3, so we may regard it as a function of $\{0, 1\}^n$ into \mathbb{Z}_4 . Then we want to regard (5) as composed of matrix-vector operations over \mathbb{F}_2 plus some extra calculation to get the answer in \mathbb{Z}_4 where 2, 3 as well as 0, 1 may be values.

First note that by the symmetry, every off-diagonal entry of \mathbf{A} may without loss of generality be 0 or 1. Next, define a binary vector \mathbf{v} by $v_j = 1$ if the *j*-th main-diagonal entry of \mathbf{A} is 2 or 3,

else $v_i = 0$. Finally define a binary matrix **B** from **A** by

$$\mathbf{B} = \mathbf{A} - 2diag(\mathbf{v}). \tag{6}$$

Then we have

$$f(\mathbf{x}) = \mathbf{x}^{\top} \mathbf{B} \mathbf{x} + 2\mathbf{x}^{\top} \cdot \mathbf{v}$$
⁽⁷⁾

with calculation in \mathbb{Z}_4 . The $\mathbf{x}^\top \mathbf{B} \mathbf{x}$ calculation is now valid in \mathbb{F}_2 , however. The quadratic form is *alternating* if the main diagonal of \mathbf{B} is all zero, else it is *non-alternating*. When \mathbf{B} comes from or is regarded as the adjacency matrix of a graph, alternating means the graph is simple and undirected (as will hold in our reductions from rank using a simple bipartite graph) and non-alternating means the graph is undirected but with one or more self-loops.

We note the general development of this decomposition and associated concepts by Schmidt [Sch09] in a way not wedded to the standard basis. Since we fix 4 as the modulus throughout this section, we follow [Sch09] in now using K to denote $\{0,1\}$ as a subset of \mathbb{Z}_4 , defining an operation \oplus on K by $a \oplus b := (a + b)^2$, and defining V as an n-dimensional vector space "over K" noting that (K, \oplus, \cdot) is the same as the field \mathbb{F}_2 . Then classical quadratic forms are equivalently defined as follows:

Definition 3.1 (see [Alb38, Sch09]). A symmetric bilinear form on V is a mapping $B : V \times V \to K$ that satisfies

- 1. symmetry: $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x});$
- 2. bilinearity: $B(\alpha \mathbf{x} \oplus \beta \mathbf{y}, \mathbf{z}) = \alpha B(\mathbf{x}, \mathbf{z}) \oplus \beta B(\mathbf{y}, \mathbf{z})$ for $\alpha, \beta \in K$.

B is alternating if $B(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in V$, else it is non-alternating. Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be any basis for *V* over *K*. Then *B* is uniquely determined (relative to this basis) by the $n \times n$ matrix **B** with entries $b_{ij} = B(\lambda_i, \lambda_j)$. The rank of *B* is the rank of its matrix **B**.

Definition 3.2 (see [Bro72, Sch09]). A \mathbb{Z}_4 -valued classical quadratic form is a mapping $f: V \to \mathbb{Z}_4$ that satisfies:

Then f is alternating if the associated bilinear form B is alternating, non-alternating otherwise, and its rank r is the rank of B.

Proposition 3.1 ([Sch09]). There is a vector $\mathbf{v} \in K^n$ such that for all $\mathbf{x} \in K^n$ over the basis Λ ,

$$f(\mathbf{x}) = \mathbf{x}^\top \mathbf{B} \mathbf{x} + 2\mathbf{x}^\top \cdot \mathbf{v}$$

The point of dropping down to \mathbb{F}_2 is to leverage the notions of matrix similarity over \mathbb{F}_2 and the following theorem about changes of basis in V. Over \mathbb{F}_2 the appropriate definition of \mathbf{B} and \mathbf{B}' being *similar* (from [Alb38]) is that there exists an invertible matrix \mathbf{Q} such that $\mathbf{B}' = \mathbf{Q}^{\top} \mathbf{B} \mathbf{Q}$. This preserves the property that similar matrices have the same rank. The notions of *alternating* and *non-alternating* are the same as given for the binary matrix \mathbf{B} above, depending on whether the main diagonal of \mathbf{B} is all zero or not.

Theorem 3.2 ([Alb38]). Let **A** be a K-valued $n \times n$ symmetric matrix of rank r.

- (a) If **A** is alternating, then **A** has even rank and is similar to a matrix that has zeros everywhere except on the subdiagonal and the superdiagonal, which are $1010 \cdots 10100 \cdots 0$ with r/2 ones.
- (b) If **A** is non-alternating, then **A** is similar to a diagonal matrix, whose main diagonal is of *r*-many ones.

With the representation of $f(\mathbf{x}) = \mathbf{x}^{\top} \mathbf{B} + 2\mathbf{x}^{\top} \cdot \mathbf{v}$, the paper [Sch09] uses this to define normal forms with regard to \mathbb{Z}_4 :

Corollary 3.3 ([Sch09]). Given a quadratic form f of rank r as above over the basis Λ , we can find a basis $M = (\mu_1, \ldots, \mu_n)$ for V over K, mapping $\mathbf{x} = (x_1, \ldots, x_n)$ over Λ in V to $\mathbf{y} = (y_1, \ldots, y_n)$ such that:

(a) If f is alternating, then

$$f(\mathbf{y}) = 2\sum_{j=1}^{r/2} y_{2j-1}y_{2j} + 2\sum_{i=1}^{n} w_i y_i,$$

for some $\mathbf{w} = (w_1, \cdots, w_n) \in K^n$.

(b) If f is non-alternating, then there is the equivalent linear form

$$f(\mathbf{y}) = \sum_{j=1}^{r} y_j + 2\sum_{i=1}^{n} w_i y_i,$$

for some $\mathbf{w} = (w_1, \cdots, w_n) \in K^n$.

Schmidt actually retains the symbols \mathbf{x} and \mathbf{v} in his statement but we have used \mathbf{y} and \mathbf{w} to indicate the change of basis. Our analysis in the next section will, however, treat \mathbf{y} as the standard basis, so the generic symbols x_1, \ldots, x_n will re-appear, and w_1, \ldots, w_n will just be ordinary 0-1 values. This switch will be echoed in the next section in that once we substitute for the input qubit values x_i and output values z_j in the quadratic form q_C from Section 2, the actual variables of q_C left over will be named y_1, \ldots, y_h where h = O(n). But to emphasize that the counting lemmas preceding the main results hold apart from the quantum context, we will revert to the standard symbols x_1, \ldots, x_n in their statements and proofs.

Now we reference [DP18] to note some facts about matrix decompositions related to the above normal forms. Note that the inverse of a non-singular lower triangular matrix is lower triangular.

- **Lemma 3.4.** (a) For every symmetric $n \times n$ matrix **B** over \mathbb{F}_2 there is a permutation matrix **P** such that the symmetric matrix $\mathbf{B}' = \mathbf{P}^{\top} \mathbf{B} \mathbf{P}$ has the decomposition $\mathbf{B}' = \mathbf{L} \mathbf{D} \mathbf{L}^{\top}$. Here **L** is an $n \times n$ lower triangular matrix with unit diagonal and **D** is diagonal if **B** is non-alternating, else **D** is block-diagonal as described in Theorem 3.2(a).
 - (b) The matrix **D** in (a) is permutation-equivalent to any matrix **D**' fulfilling the corresponding case of Theorem 3.2 when applied to **B**' or to **B**.
 - (c) The matrix \mathbf{D} in (a) is unique among LDU decompositions applied to \mathbf{B}' .

(d) When $\mathbf{D}' = \mathbf{L}^{-1} \mathbf{P}^{\top} \mathbf{A} \mathbf{P} (\mathbf{L}^{-1})^{\top}$ is computed over \mathbb{Z}_4 rather than \mathbb{F}_2 , it may no longer be diagonal or block-diagonal, but it represents the same quadratic form with arguments in V and values \mathbb{Z}_4 in as in Corollary 3.3 over the new basis. In both the alternating and nonalternating cases, the main diagonal of \mathbf{D}' equals the main diagonal of \mathbf{D} plus 2w where w is the vector in Corollary 3.3.

Proof. (a) This is known and noted in [DP18]. A key point from Gaussian elimination is that if we alternate elementary matrices \mathbf{L}_i that do elimination in the *i*th column of the lower triangle and swaps $\mathbf{P}_{j,k}$ of rows *j* and *k*, then we can rewrite $\mathbf{P}_{j,k}\mathbf{L}_i$ where j, k > i as $\mathbf{L}'_i\mathbf{P}_{j,k}$. The matrix \mathbf{L}'_i is obtained by interchanging the entries in rows *j* and *k* of column *i* and those in positions *j* and *k* on the main diagonal. (The latter is unnecessary when all diagonal entries are 1) and is still lower-triangular. Since each \mathbf{L}'_i is still lower triangular and we can repeat the switch for further row swaps, we obtain the lower-triangular matrix formally designated as \mathbf{L}^{-1} as the product of the \mathbf{L}'_i and the matrix designated as \mathbf{P}^{\top} as the product of all swaps. Since **B** is symmetric, corresponding events on the right give $\mathbf{D} = \mathbf{L}^{-1}\mathbf{P}^{\top}\mathbf{B}\mathbf{P}(\mathbf{L}^{-1})^{\top}$ of the diagonal or block-diagonal forms stated in all of [Alb38, Sch09, DP18].

Part (b) follows simply because \mathbf{D} and \mathbf{D}' have the same rank and the same block-diagonal structure in the alternating case or diagonal structure in the non-alternating case). The proof of (c), which is not strictly needed for our key point (d), is in the Appendix.

The point in (d) is that when computed over \mathbb{Z}_4 , $\mathbf{D}' = \mathbf{L}^{-1} \mathbf{P}^\top \mathbf{A} \mathbf{P} (\mathbf{L}^{-1})^\top$ represents the same quadratic form f originally given by \mathbf{A} in (5) but over the transformed basis that maps \mathbf{x} to \mathbf{y} . Thus

$$f(\mathbf{y}) = \mathbf{y}^{\top} \mathbf{D}' \mathbf{y} = \mathbf{y}^{\top} \mathbf{D} \mathbf{y} + 2 \sum_{i=1}^{n} y_i w_i.$$
 (8)

In the non-alternating case, this means any symmetric pairs $d'_{j,k}$, $d'_{k,j}$ of off-diagonal elements of \mathbf{D}' must sum to 0 modulo 4, and likewise off-diagonal elements in the alternating case apart from the block elements on the super-diagonal and sub-diagonal. The diagonal must satisfy $d'_{j,j} = d_{j,j} + 2w_j$ (mod 4) in either case.

Put more simply, the decomposition in [DP18] is the same as that obtained in [Sch09] following [Alb38, Bro72], so the normal forms for classical quadratic forms over \mathbb{Z}_4 in the latter papers inherit the $O(n^{\omega})$ time computability from [DP18] working over \mathbb{F}_2 . For some examples, consider the alternating form $q(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$. It gives

$$\mathbf{A} = \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which is the adjacency matrix of the triangle graph. Gaussian elimination begins by swapping row 1 and row 2, then no more swaps are needed. So we have:

$$\mathbf{P} = \mathbf{P}_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{P}^{\top}, \quad \mathbf{B}' = \mathbf{P}^{\top} \mathbf{B} \mathbf{P} = \mathbf{B}, \text{ and } \mathbf{L}^{-1} = \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

This gives over \mathbb{F}_2 ,

$$\mathbf{D} = \mathbf{L}\mathbf{B}\mathbf{L}^{\top} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \mathbf{L}^{\top} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But over \mathbb{Z}_4 , we get

$$\mathbf{LA} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \text{ which times } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \mathbf{D}' \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The presence of a 2 in the lower-right corner of \mathbf{D}' , corresponding to a 1×1 block in the diagonal matrix \mathbf{D} , signals a cancellation in the 0-1 assignments $a \in K^n$ giving q(a) = 0 versus those giving q(a) = 2. That is, $N_0(q) - N_2(q) = 0$. In section 6 we will call the simple triangle graph a "net-zero" graph.

Now, however, let us define $q' = q + 2x_1^2$. This corresponds to adding a self-loop at node 1 to the triangle graph. This goes into the vector \mathbf{v} and does not change \mathbf{B} or the decomposition. At the end, however, we first get that over \mathbb{Z}_4 , $\mathbf{A}' = \mathbf{P}^\top \mathbf{A} \mathbf{P}$ is no longer the same as \mathbf{A} : it moves the 2 from the upper left corner to the center. Then we get

$$\mathbf{LA}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \text{ which times } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \mathbf{D}' \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a 2 on the main diagonal but it is tucked within a 2×2 block of **D**. Here in fact we have $N_0(q') = 6$ and $N_2(q') = 2$.

An example of an alternating form q'' with $N_2(q'') > N_0(q'')$ is $q'' = 2x_1^2 + 2x_2^2 + 2x_1x_2$, which corresponds to a single edge with a self-loop at each end. Replacing each self-loop by a triangle yields a 6-node simple undirected graph with $N_0 = 28$ and $N_2 = 36$. We will show that when $N_0 \neq N_2$ in the alternating case, the absolute difference is a simple function of the rank r of **B** over \mathbb{F}_2 .

4 Main Results

Given any *n*-qubit stabilizer circuit *C* of size *s* with *h* nondeterministic gates, we can obtain its associated quadratic form q_C in O(s) time via the process in Section 2. This form has variables $\mathbf{x} = x_1, \ldots, x_n$ for inputs, $\mathbf{z} = z_1, \ldots, z_n$ for outputs, and y_1, \ldots, y_h for nondeterministic variables (*wlog.* all coming from *h* Hadamard gates). It may also have the variables called " w_j " in Section 2, but those are introduced only to equate the final annotation term on a qubit line *j* with the output variable z_j without thereby forcing a value restriction for nondeterministic variable(s) on that line, and so preserve $2^{h/2}$ as the value of the magnitude divisor *R* in (2). We can either treat w_j as forced by z_j without changing *R*, or avoid introducing w_j by reducing *R*. Since the circuits are allowed to have initial X gates on some lines, treating $\mathbf{x} = (0, \cdots, 0)$ loses no generality. For any output $\mathbf{b} = (b_1, \cdots, b_m)$, the quadratic form then becomes

$$q(\mathbf{y}, \mathbf{b}) = (\sum \alpha_i y_i + \sum 2y_i y_j) + \sum 2y_i b_j \mod 4$$
$$= \mathbf{y}^\top \mathbf{A} \mathbf{y} + \mathbf{y}^\top 2 \mathbf{\Delta} \mathbf{y} \mod 4$$

in the **y** variables only. Here Δ is a diagonal matrix with $\Delta_{i,i} = b_j$. Because we will have $h = \Theta(n)$ for the most part, we still refer to "n" to denote the number of variables in quadratic forms.

Finally, we also fix the outputs b_j all to be 0. We denote by $\mathbf{N} = (N_0, N_1, N_2, N_3)$ the resulting distribution of values of q_C over the 2^h assignments to \mathbf{y} . Reviewing the discussion surrounding Equation (2) in Section 2, we can abbreviate the numerator of the amplitude by

$$a_0(\mathbf{N}) = N_0 - N_2 + i(N_1 - N_3).$$
(9)

We use the N_c and a_0 notation generally for linear and quadratic forms f without reference to their coming from a quantum circuit. Then a_0 gives the value of the exponential sum $\sum_x i^{f(x)}$.

Now the present the main lemmas that underlie the main theorems. Their proofs are in the appendix.

Lemma 4.1. For any linear function $f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i$ over \mathbb{Z}_4 , $|N_0 - N_2|$ and $|N_1 - N_3|$ are 0 or a power of 2.

Lemma 4.2. For any \mathbb{Z}_4 -valued alternating quadratic form $f: V \to \mathbb{Z}_4$ of rank r, there is a basis of V over which f can be rewritten as

$$f(\mathbf{x}) = 2\sum_{j=1}^{g} x_{2j-1} x_{2j} + 2\sum_{i=1}^{n} w_i x_i$$

for some $\mathbf{w} = (w_1, \cdots, w_n) \in K^n$, and

$$N_0 - N_2 = 0$$
 or $(-1)^k 2^{n-g}$

where 2g = r and k is the number of (w_{2j-1}, w_{2j}) -pairs in f such that $(w_{2j-1}, w_{2j}) = (1, 1)$ for $j \in \{1, \dots, g\}$. Also $N_1 = N_3 = 0$.

Lemma 4.3. For any \mathbb{Z}_4 -valued non-alternating quadratic form $f: V \to \mathbb{Z}_4$ of rank r, there is a basis of V over which f can be rewritten as

$$f(\mathbf{x}) = \sum_{j=1}^{r} x_j + 2\sum_{i=1}^{n} w_i x_i = \sum_{j=1}^{r} (1+2w_j)x_j + 2\sum_{i=r+1}^{n} w_i x_i$$

for some $\mathbf{w} = (w_1, \dots, w_n)$. Define c to be the number of w_i 's such that $w_i = 0$ with $i \in \{r + 1, \dots, n\}$ and d to be the number of pairs such that $(1+2w_j, 1+2w_{j'}) = (1,3)$ with $j, j' \in \{1, \dots, r\}$. Also let m = n - c - 2d and rewrite m = 4a + b, and define η such that $\eta = 0$ if the rest m-many coefficients are all 1's but $\eta = 1$ if they are all 3's. Then the differences $N_0 - N_2$ and $N_1 - N_3$ take one of the following values:

- if b = 0, then $N_0 N_2 = (-1)^a 2^{(n+c)/2}$, $N_1 N_3 = 0$;
- if b = 1, then $N_0 N_2 = (-1)^a 2^{(n+c-1)/2}$, $N_1 N_3 = (-1)^{a+\eta} 2^{(n+c-1)/2}$;
- if b = 2, then $N_0 N_2 = 0$, $N_1 N_3 = (-1)^{a+\eta} 2^{(n+c)/2}$;
- if b = 3, then $N_0 N_2 = (-1)^{a+1} 2^{(n+c-1)/2}$, $N_0 N_2 = (-1)^{a+\eta} 2^{(n+c-1)/2}$.

The connection between rank and solution counting is expressed by our main theorem about quadratic forms after the normalization process in Lemmas 4.1 to 4.3 is applied:

Theorem 4.4. Given any normalized classical quadratic form f in n variables, we can compute N_0, N_1, N_2, N_3 and hence $a_0(\mathbf{N})$ in time O(n). Furthermore, $|a_0(\mathbf{N})|^2$ is either 0 or 2^{2n-r} where r is the rank of f.

This means that the bulk of the computing time for the whole process goes into the decomposition in Lemma 3.4, which is used to compute the normal forms asserted in Corollary 3.3. After that, the up-to- n^2 denseness of the original form does not matter and the computation needs only O(n)time. *Proof.* We show this separately for the alternating and non-alternating cases. By Corollary 3.3, a normalized alternating quadratic form is of the form

$$f(\mathbf{x}) = 2\sum_{j=1}^{r/2} x_{2j-1} x_{2j} + 2\sum_{i=1}^{n} w_i x_i \pmod{4},$$

for some $\mathbf{w} = (w_1, \dots, w_n) \in K^n$. It is easy to see that $N_1 - N_3$ is always zero since there is no assignment to $\mathbf{x} = (x_1, \dots, x_n)$ that would give $f(\mathbf{x}) = 1$ or 3. Lemma 4.2 gives out

$$N_0 - N_2 = 0$$
 or $(-1)^k 2^{n-g}$,

which can be done in time O(n). Hence if this is non-zero, then we have

$$a_0(\mathbf{N}) = (-1)^k 2^{n-g},$$

and

$$|a_0(\mathbf{N})|^2 = 2^{2n-r}$$

Similarly, a normalized non-alternating quadratic form is written as

$$f(\mathbf{x}) = \sum_{j=1}^{r} x_j + 2\sum_{i=1}^{n} w_i x_i = \sum_{j=1}^{r} (1+2w_j)x_j + 2\sum_{i=r+1}^{n} w_i x_i \pmod{4},$$

for some $\mathbf{w} = (w_1, \dots, w_n) \in \{0, 1\}^n$. Things become trivial if $2w_i = 2$ for some $i \in \{r+1, \dots, n\}$. This makes $N_0 - N_2 = N_1 - N_3 = 0$.

Now assume $N_0 - N_2$ and $N_1 - N_3$ are not both zero at the same time. Then we can derive from Lemma 4.3 that

$$a_0(\mathbf{N}) = N_0 - N_2 + i(N_1 - N_3)$$

takes one of the following values:

- if b = 0, then $a_0(\mathbf{N}) = (-1)^a 2^{(n+c)/2}$;
- if b = 1, then $a_0(\mathbf{N}) = (-1)^a 2^{(n+c-1)/2} + i(-1)^{a+\eta} 2^{(n+c-1)/2}$;
- if b = 2, then $a_0(\mathbf{N}) = i(-1)^{a+\eta} 2^{(n+c)/2};;$
- if b = 3, then $a_0(\mathbf{N}) = (-1)^{a+1} 2^{(n+c-1)/2} + i(-1)^{a+\eta} 2^{(n+c-1)/2}$,

where a, b and c are as defined in Lemma 4.3. Note that c = n - r. Together we have

$$|a_0(\mathbf{N})|^2 = 2^{2n-r}$$

and again this can be computed in O(n) time.

Now we rejoin the process of evaluating the stabilizer circuit C. It will normalize q_C to q' in one of the two forms in Corollary 3.3, which will give a matrix \mathbf{D}' such that $q'(\mathbf{y}) = \mathbf{y}^\top \mathbf{D}' \mathbf{y}$. With such \mathbf{D}' , the acceptance probability can be derived directly by Theorem 4.4. We repeat the statement of our main theorem from section 1 but split (b) into two pieces, proving part (b1) here and part (b2) in the next section.

- **Theorem 4.5** (Main Theorem). (a) Strong simulation of n-qubit stabilizer circuits C with h nondeterministic single-qubit gates on standard-basis inputs (amplitude as well as the probability) is in time $O(s + n + h^{\omega})$ where $2 \le \omega < 2.3729$.
- (b1) Computing $n \times n$ matrix rank over \mathbb{F}_2 reduces in linear time to computing one instance of the strong simulation probability $|\langle 0^n | C | 0^n \rangle|^2$.
- (b2) Computing the strong simulation probability $p = |\langle 0^n | C | 0^n \rangle|^2$ reduces in linear time to computing one instance of $n \times n$ matrix rank over \mathbb{F}_2 on the promise that p > 0.

Proof of (a) and (b1). (a) Let C be given, take **A** to be the matrix over \mathbb{Z}_4 of its classical quadratic form q_C , and take **B** be the associated symmetric matrix over \mathbb{F}_2 . By Lemma 3.4 and the algorithm of [DP18] there is a decomposition $\mathbf{B} = \mathbf{PLDL}^{\top}\mathbf{P}^{\top}$ over \mathbb{F}_2 that is computable in $O(n^{\omega})$ time such that **D** is diagonal (in the non-alternating case) or 2×2 block-diagonal (in the alternating case) and equals the matrix **D** in Theorem 3.2. This also computes the rank r of **B**. Then compute $\mathbf{D}' = \mathbf{L}^{-1}\mathbf{P}^{\top}\mathbf{AP}(\mathbf{L}^{-1})^{\top}$ over \mathbb{Z}_4 which again takes $O(n^{\omega})$ time. By Lemma 3.4(d), **D**' and **D** yield the vector **w** in the normal form of Corollary 3.3 for q_C . Then Theorem 4.4 yields not only the probability $p = |\langle 0^n | C | 0^n \rangle|^2$ but also the entire distribution of phases as powers of *i*, and hence yield the amplitude $\langle 0^n | C | 0^n \rangle$.

(b1) To compute the rank r of an $n \times n$ matrix over \mathbb{F}_2 , make an equivalent symmetric matrix \mathbf{A} by the block-transpose trick in the introduction. Not only is \mathbf{A} alternating but it is the adjacency matrix of a bipartite graph G = (V, V', E). To see that the corresponding graph state circuit C gives $p = |\langle 0^n | C | 0^n \rangle|^2 > 0$, consider any assignment a to the variables in V. This reduces q_C to a linear form $2\ell(x')$ of the variables x' corresponding to nodes of the other partition. If $\ell(x')$ vanishes modulo 2, then all extensions of a to a' on x' contribute 0 modulo 4. Otherwise, $2\ell(x')$ has a nonzero term $2x'_i$ for some i. Assignments a' to x' pair off with canceling contributions 0 and 2 according to the value a'_i of x'_i . Thus there are never more values of 2 than 0. Finally, the all-zero assignment to x makes $\ell(x')$ vanish, so the difference between the numbers of 0 values and 2 values is positive. Thus the normal form for q_C with input and output 0^n cannot have global cancellation, so r is a simple function of p.

To get the converse simulation in (b2) we must consider the non-alternating case, which arises when the stabilizer circuit C has an odd number of S or S^* gates on some qubit line(s), and allow for the possibility $\langle 0^n | C | 0^n \rangle = 0$. The algorithm for amplitude in the non-alternating case needs knowledge of individual entries in the normal form over \mathbb{Z}_4 besides the rank r of q_C . We show that for the probability computation $p = |\langle 0^n | C | 0^n \rangle|^2$ one can reduce to the alternating case—that is, produce in $O(s + h^2)$ time an alternating quadratic form q'_C of rank 2r such that p is a simple function of r provided p > 0. This development leads us to a concept of "self-dual" quadratic forms in order to complete the proof of (b2) in the next section.

5 Self-Dual Forms and Probability Reduction to Rank

To take stock, we have shown that strong simulation of stabilizer circuits (on input 0^n) is in matrixmultiplication time, which is currently the best-known time for computing matrix rank (over \mathbb{F}_2). We have also reduced the rank computation to the simulation of the restricted class of circuits that come from bipartite graphs. The issue now becomes whether there is a more-general equivalence of strong simulation of stabilizer circuits C to computing the rank. We show a *yes* answer provided the decision problem of telling whether the probability $p = |\langle 0^n | C | 0^n \rangle|^2$ is nonzero, for C having no phase gates, is in $O(n^2)$ time.

To do so, we begin by abstracting the quadratic forms $q + q^*$ in Theorem 2.2 into a notion of "self-duality." We do not need to legislate the separate presence of input variables x_i and output variables z_j . Our definition is at the level of the long literature of quadratic forms but we have not found a reference for it.

Definition 5.1. Let π be a permutation of the variable set V such that $\pi^2 = 1$ (that is, an involution). Let f(v) be a quadratic form in reduced form as a sum of terms over \mathbb{Z}_K . Then f is self-dual provided:

- Whenever f has the term av_i^2 , v_i is not fixed by π and f also has the term $(K-a)\pi(v_i)^2$.
- Whenever f has the term bv_iv_j , at least one of the variables is not fixed by Π and f also has the term $(K b)\pi(v_i)\pi(v_j)$.

If f had av_i^2 with v_i fixed by π then f would also have $(K-a)v_i^2$ so the terms would cancel, and similarly if bv_iv_j were fixed by π . The polynomials $q(v) + q^*(v')$ in Theorem 3 meet this definition even before the substitution of values for x_i and z_j variables since those are considered fixed by the permutation sending v to v'. The definition allows b to be odd in bv_iv_j but classical forms with K = 4 only allow b = 2. They can have terms of the form $v_i^2 + 3(v_i')^2$ or $3v_i^2 + (v_i')^2$, but our first results will eliminate them.

We can fix any partition of the set V of non-fixed variables into U and U', writing $u' = \pi(u)$ for $u \in U$. Also, given a Boolean assignment a, let $a' = \pi(a)$.

Lemma 5.1. Given any self-dual quadratic form f over \mathbb{Z}_4 :

- (a) The form obtained by identifying any u, u' pair is self-dual.
- (b) For all Boolean assignments $a, f(a) + f(a') = 0 \pmod{4}$. Hence it follows that the numbers of 1 values and 3 values are equal, so their contributions cancel.
- (c) Values of 1 and 3 exist only when f has terms 3u + u' or u + 3u' where u is in U, and when they happen, exactly half the values are 1 and 3. Hence it suffices to count the number of zeroes of f.

Proof. Part (a) is immediate. Note that if f has terms $3u^2 + u'^2$ or 2xu + 2xu' with x fixed this makes them cancel. But 2uv + 2u'v' becomes 2uv + 2uv' which is not yet trivial. This also compares to point 3 of Lemma 3.1 in [CGW18]. Part (b) follows individually for each term t plus $t' = \pi(t)$. For (c), let T be the set of $u \in U$ such that f has $3u^2 + u'^2$ or $u^2 + 3u'^2$. Then an assignment a makes f(a) odd iff it sets an odd number of pairs (u, u') with $u \in T$ to different values, and so exactly half the assignments do so. On the other assignments, those terms contribute 0 (mod 4) so those terms do not affect the calculation.

Now we derive a combinatorial lemma that effectively eliminates the terms with odd coefficients. Let $T_0 \subseteq U$ collect the variables u such that $u^2 + 3u'^2$ is a term, and $T_1 \subseteq U$ those for which $3u^2 + u'^2$ is a term. Now define

$$S = \{a : V \to \{0, 1\} : a(u) \neq a(u') \text{ for an even number of } u \in T_0 \cup T_1, a(u) = 1\}$$

Then S is a linear subspace of $\mathbb{F}_2^{[V]}$. Assignments a outside S make f(a) have value 1 or 3 and hence contribute to a global cancellation by Theorem 5.1(c). We will now replace f by an alternating form f' that preserves the cancellation outside S and gives the same distribution of 0 and 2 values inside S. Let t collect the terms in f with odd coefficients and define a quadratic form f' as follows:

- 1. For every pair (u, v) with u, v both in T_0 or both in T_1 , r has the term-pair 2uv + 2u'v'.
- 2. For every pair (u, v) with $u \in T_0$ and $v \in T_1$ or vice-versa, r has the term-pair 2uv' + 2u'v.
- 3. Finally choose any variable $u \in T$ and substitute 2u by twice the sum of all the other variables in T into f - t + r. The result is f'.

We remark that in the second step, r introduces terms that "cross the partition" from U to U'. The self-dual forms $q_C + q_C^*$ arising from Theorem 2.2 have a canonical partition with no crossing edges. The import here is that the presence of terms that cross the partition does not affect the analysis—they will just become edges in a larger graph. Except for the element crossing the partition, r represents exclusive-or-ing by a clique of edges between pairs of nodes in $T_0 \cup T_1$. This also accounts for our current statement of the reduction time having a $+n^2$ term that is not reduced when the size s is $o(n^2)$.

The final form f' in the third step need not even be self-dual. For an example, consider f = 3u + u' + 3v + v', which has no cross-terms. Then $T = T_1 = \{u, v\}$. So r = 2uv + 2u'v'. The subspace S is defined by $u + u' + v + v' = 0 \pmod{2}$. Because f - t + r = r only has even terms, we can substitute $2u + 2u' + 2v + 2v' = 0 \pmod{4}$. Choosing 2v as the term to substitute for makes

$$f' = u(2u + 2u' + 2v') + 2u'v' = 2u^{2} + 2uu' + 2uv' + 2u'v'.$$

This is not self-dual because of the absence of terms $2u'^2$, 2u'v, and 2uv, plus it has a term of the form 2uu', which even if it were allowed in Definition 5.1 would be canceled in the process of forming $q + q^*$. It is, however, alternating and equivalent to f restricted to S pointwise, not just in distribution. That is, for any assignment a to (u, u', v'), letting a'(v) = a(u) + a(u') + a(v') (and a'(u) = a(u), a'(u') = a(u'), a'(v') = a(v')), we get f'(a) = f(a'). Now we prove this in general.

Lemma 5.2. The classical alternating quadratic form f' equals f restricted to S, hence $N_0(f') - N_2(f') = N_0(f) - N_2(f)$.

Proof. The substitution step 3 is valid because f - t + r only has even coefficients. Hence we need only show that f - t + r agrees with f on S. For any assignment $a : V \to \{0, 1\}$, define Dto be the set of variables in u T such that a(u') = 1 - a(u). Pick any $u \in D$, for instance the lexicographically first variable in D. Then define a' to be the assignment with a'(u) = 1 - a(u)and a'(u') = 1 - a(u') = 1 - a'(u). Going from a to a' changes the form t by +2 modulo 4. The only term-pairs in r that change are (2uv + 2u'v') or (2uv' + 2u'v) for $v \in D$ and those change by 2. Hence if d = |D| is even, there is a matching change by 2, so f'(a) - f'(a') = f(a) - f(a'). Define the closure of a under such "flip" operations by F(a).

For the *d*-even case, it remains to give some $a_0 \in F(a)$ such that $t(a_0) = r(a_0)$. Take $a_0(u) = 1$, $a_0(u') = 0$ for $u \in T_0 \cap D$ and $a_0(u) = 0$, $a_0(u') = 1$ for $u \in T_1 \cap D$. Then $t(a_0) \equiv d \pmod{4}$. To get the contribution from $r(a_0)$, set $k = |T_0 \cap D|$ and consider:

• For every $v, w \in T \setminus D$, r has the term-pair 2vw + 2v'w' or 2vw' + 2v'w where $a_0(v) = a_0(v')$ and $a_0(w) = a_0(w')$. Each of these pairs contributes a multiple of 4, so the net from these terms is 0.

- For every pair (v, w) with $v \in D$ and $w \in T \setminus D$, the term-pair equals 0 if $a_0(w) = 0$ and 2(v + v') = 2 otherwise. The net for any w with $a_0(w) = 1$ is 2d, which is 0 if d is even.
- For every pair with $v, w \in D$, the contribution is always 2: Either v and w are both in T_0 or both in T_1 and the term pair is 2vw + 2v'w' or they have one in each and the pair is 2vw' + 2v'w. Hence the total contribution is $2\binom{d}{2} = d^2 d$.

The final point is that when d is even, d^2 vanishes so the net from $r(a_0)$ is -d modulo 4, which equals d modulo 4 from t. This finishes the proof that for all assignments $a \in S$, r(a) = t(a).

Proof of Theorem 4.5(b2). Given an *n*-qubit stabilizer circuit *C* of size *s* with *h* nondeterministic gates, we can obtain the self-dual classical quadratic form $f = q_C + q_C^*$ in O(s) time by the process in Theorem 2.2. Then Lemma 5.2 converts *f* to the equivalent alternating quadratic form f' in time $O(s + h^2)$. The alternating case of Theorem 4.4 then gives a simple relation from $p = |\langle 0^n | C | 0^n \rangle|^2 = \frac{1}{R} |N_0(f') - N_2(f')|$ to rank unless p = 0.

If C has at most k qubit lines with odd numbers of phase gates, where $k = O(\sqrt{s})$, then the time is O(s). The most immediate scope for improvement is to reduce the overhead from the quadratic size of r—perhaps sacrificing the exact agreement between f' and f for distributional equivalence.

6 Conclusions

We have improved the asymptotic running time for strong simulation of *n*-qubit stabilizer circuits (with typical size and nondeterminism) from $O(n^3)$ to $O(n^{\omega})$. We have also shown a linear time reduction from matrix rank over \mathbb{F}_2 to strong simulation. One interpretation of the latter is:

The time gap between weak and strong simulation for stabilizer circuits cannot be closed unless $n \times n$ matrix rank over \mathbb{F}_2 is computable in $O(n^2)$ time.

The direction from the quantum simulation to matrix rank comes close to establishing a complete equivalence of them, especially for the simulation probability p. Via analysis of "self-dual" forms we have reduced the probability computation to the alternating case, in which by Lemma 4.2 and Theorem 4.4 we get a simple expression whose absolute value depends only on the rank and whether p = 0. That puts focus on the complexity of deciding whether p, or equivalently the amplitude $a = \langle 0^n | C | 0^n \rangle$, is zero, specifically in the alternating case where a is always real.

The alternating case comes down to graph-state circuits C_G and can be framed in terms apart from quantum computing. Consider black/white two-colorings (not necessarily proper) of the *n* vertices of *G*, and count the number of edges whose two nodes are both colored black. Call those B-B edges. Define c_0 to be the count of colorings that make an even number of B-B edges and $c_1 = 2^n - c_0$ to be the count of colorings that make an odd number of B-B edges. The following is called a(G) for "amplitude" and divides by 2^n not $2^{n/2}$ because C_G has 2n Hadamard gates.

$$a(G) = \frac{c_0 - c_1}{2^n}$$

Definition 6.1. Call an undirected graph G net-zero if a = 0, net-positive if a > 0, and net-negative if a < 0.

The following proposition collects some basic facts:

Proposition 6.1. (a) Every odd cycle graph is net-zero.

- (b) Every bipartite graph is net-positive.
- (c) A graph is net-zero if and only if one of its connected components is net-zero.
- (d) If G is net-zero, then the graph G' obtained by attaching a new node v only to one existing node u, then attaching a second new node w only to v, is also net-zero.

Proof. Part (a) follows because every coloring has an even number of B-W edges. Hence the number of monochrome edges is odd, and so complementing the coloring flips the parity between B-B and W-W edges. Part (b) was part of the proof of Theorem 4.5(b1). Part (c) is intuitive from how the quantum state is a tensor product over the connected components, so the events of all-0 output on each component are independent. The proof of (d) is that whether u is colored black or white, exactly one of the four colorings of v and w creates one more B-B edge. Thus $|\langle 0^{n+2}| C_{G'} |0^{n+2}\rangle|^2$ is directly proportional to $|\langle 0^n | C_G |0^n \rangle|^2$.

The smallest net-zero graph is the triangle graph. The graph made by attaching a second triangle is net-zero, as is the graph made by attaching a triangle to any of the latter's four outer edges. As observed at the end of section 3, the six-node graph consisting of two triangles connected by an edge is net-negative. Here are the connected net-zero graphs of 3, 4, and 5 nodes:



The concept extends to graphs with multiple edges and self-loops. An isolated self-loop is net-zero, while an edge with two self-loops is net-negative. This includes the quadratic forms produced by Lemma 5.2 and we conclude:

Corollary 6.2. If net-zero graphs of n nodes with self-loops allowed are recognizable in $O(n^2)$ time then computing $|\langle 0^n | C | 0^n \rangle|$ for stabilizer circuits C (of $O(n^2)$ size with O(n) nondeterminism) is O(N)-time equivalent to computing $n \times n$ matrix rank, where $N = n^2$.

The concept further extends to graphs with *circles*, which are isolated loops without a vertex and contribute a multiplicative -1, and more generally to graphical 2-polymatroids with rank function $f_G(A)$ defined for any set $A \subseteq E$ to be the total number of vertices touched by edges in A. Then $a(f_G)$ becomes a generalized Tutte invariant (see [OW93, Nob06]) with parameters

$$(r, s, t; a, b, c, d; m, n) = (1/2, -1, 0; 1, -1, 1, -1/2; 1, -1/2).$$

This gives

$$a(G) = \left(-\frac{1}{2}\right)^{n/2} S(f_G; -\sqrt{2}i, \sqrt{2}i), \quad \text{where} \quad S(f; x, y) = \sum_{A \subseteq E} x^{f(E) - f(A)} y^{2|A| - f(A)},$$

by the main theorem of [OW93]. This in turn further simplifies to

$$a(G) = \sum_{A \subseteq E} \frac{(-2)^{|A|}}{2^{f_G(A)}}.$$

Noble [Nob06] shows that computing $S(f_G; x, y)$ is $\#\mathsf{P}$ -hard for any constant rational x, y whenever $xy \neq 1$. The complex irrational point $(-\sqrt{2}i, \sqrt{2}i)$ has xy = 2 but evades his proof because having $y^2 = -2$ makes a denominator vanish. Other connections between quantum graph states and matroids have been shown by Sarvepalli [Sar14], and there is scope for further development along these lines.

We have shown tight connections to the fundamental problems of counting solutions to quadratic forms f over \mathbb{F}_2 and \mathbb{Z}_4 . For \mathbb{F}_2 we get that 2f is an alternating form over \mathbb{Z}_4 with the same solution count over $\{0,1\}^n$, so the near-equivalence to matrix rank applies. In any event we have reduced the \mathbb{Z}_2 case to matrix multiplication in a way that improves the $O(n^3)$ running time stated in [EK90] to $O(n^{\omega})$. For binary solution counting of non-alternating classical quadratic forms over \mathbb{Z}_4 , we obtain $O(n^{\omega})$ runtime via methods that multiply matrices as well as compute rank.

When the non-Clifford gate CS is added to create a universal set, the quadratic forms over \mathbb{Z}_4 have terms xy or 3xy. They are no longer classical and the connection to \mathbb{F}_2 exploited by [Sch09] no longer applies. No such connection can apply, nor any extension of the algorithm in [DP18] *a-fortiori*, unless BQP = P. There is also the sharp dichotomy theorem of [CLX14] that solution counting for these forms over all of \mathbb{Z}_4^n is in polynomial time, but over $\{0,1\}^n$ it is #P-complete. This extends to affine versus non-affine forms over \mathbb{Z}_K^n , $K = 2^k$. Deeper understanding of *why* the dichotomy operates may illuminate exactly which elements of quantum computations create hardness for classical emulation (for this, see [Bac17, Bac18]).

Nevertheless, perhaps these techniques can apply to heuristic or approximative methods on general quantum circuits. The polynomial translation in [RCG18] applies to quantum circuits of all common gate types. There are questions about analyzing circuits that are "mostly Clifford" or those from the Clifford plus T libraries that try to minimize the latter gates, of which we mention[BG16, MFIB18, BBC⁺19]. For example, are there reasonably-tight bounds for the numbers of the non-Clifford gates required to compute certain functions that can be obtained efficiently by algebraic means, without resort to exhaustive search?

A direction for improving the present results is to sharpen the times for sparse cases—reflecting for instance the analysis for bounded degree in section VI of [AB06] and the results for rank in [CKL13]. Our Lemma 5.2 introduces a dense clique of edges even for sparse graphs; perhaps there is a more economical reduction to the alternating case. We have left unused one further manipulation of a classical quadratic form f that is most simply described in terms of the associated graphs G. Subdivide each edge e = (u, v) by a new node s_e and add a second new node r_e connected only to s_e . Finally replace 2uv in the form by $2us_e + 2r_es_e + 2vs_e$, and for each u of odd degree (on non-self edges), add $2u^2$. The resulting form f' is equivalent to f on the linear subspace S of assignments that make $r_e = u + v \pmod{2}$ for each edge e and is equivalent to a *linear* form on that subspace. The drawback is adding upwards of n^2 -many nodes, but it preserves sparseness of the edge set.

A closer look into Lemma 4.2 and Lemma 4.3 suggests that the probability of a specific output or the distribution over the entire output set can serve as a metric to test whether two given quantum stabilizer circuits are (not) equivalent. Let h_0^i, h_1^i be two corresponding h_0, h_1 differences for circuit C_i /quadratic form $f_i(\mathbf{x})$. Then we can define the following two concepts accordingly.

Definition 6.2. Given two quantum circuits C_1 and C_2 , we call C_1 and C_2 are weakly equivalent, denoted by $C_1 \stackrel{w}{\approx} C_2$ if

$$|\langle \mathbf{z} | C_1 | \mathbf{a} \rangle|^2 = |\langle \mathbf{z} | C_2 | \mathbf{x} \rangle|^2$$

for a fixed input \mathbf{a} and all possible output \mathbf{z} .

Definition 6.3. Given two quantum circuits C_1 and C_2 , we call C_1 and C_2 are strongly equivalent, denoted by $C_1 \stackrel{s}{\approx} C_2$ if for all possible output \mathbf{z} , the amplitudes of C_1 and C_2 are the same, that is,

$$|N_j(Q_1(\mathbf{a}, \mathbf{y}, \mathbf{b}))| = |N_j(Q_2(\mathbf{a}, \mathbf{y}, \mathbf{b}))|$$

for a fixed input \mathbf{a} and all possible output \mathbf{b} .

Now consider $C_1 \stackrel{s}{\approx} C_2$ for two given stabilizer circuits. The corresponding $Q_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $Q_2(\mathbf{x}, \mathbf{y}, \mathbf{z})$ will be of the forms as stated in Section 4. Without loss of generality, assume **0**. Note that the resulting $Q_1(\mathbf{y}, \mathbf{z})$ and $Q_2(\mathbf{y}, \mathbf{z})$ can be associated with two graphs. Each graph has two sets of nodes \mathbf{y} and \mathbf{z} . The nodes in \mathbf{y} can be connected by edges in any way, while there is no edge between nodes among \mathbf{z} and each node is connected by exactly one node from \mathbf{y} without overlapping node. Hence, this should be a strict class among graphs and this gives out another interesting question, does $C_1 \stackrel{s}{\approx} C_2$ implies that their associated graphs are isomorphic? If this is true, we will have, $C_1 \stackrel{s}{\approx} C_2$ if and only if their associated graphs are isomorphic. Note that $C_1 \stackrel{s}{\approx} C_2$ says $|N_i(Q_1)| = |N_j(Q_2)|$ for all possible outputs, which are exponentially many. We ask:

- If $|N_j(Q_1)| = |N_j(Q_2)|$ for all possible outputs, does $Q_1(\mathbf{y}, \mathbf{b}) \stackrel{\mathbb{F}_2}{\sim} Q_2(\mathbf{y}, \mathbf{b})$ for all possible \mathbf{b} ?
- For the case where $C_1 \stackrel{w}{\approx} C_2$, can we pose a similar question, but in terms of rank?

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References

- [AB06] S. Anders and H. Briegel. Fast simulation of stabilizer circuits using a graph state representation. *Phys. Rev. A*, 73(022334), 2006.
- [AG04] S. Aaronson and D. Gottesman. Improved simulation of stabilizer circuits. *Phys. Rev.* A, 70(052328), 2004.
- [Alb38] A. Albert. Symmetric and alternate matrices in an arbitrary field, I. Trans. Amer. Math. Soc., 43:386–436, 1938.
- [Bac17] M. Backens. A new holant dichotomy inspired by quantum computation. In Proc. 44th Annual International Conference on Automata, Languages, and Programming, Leibniz International Proceedings in Informatics (LIPIcs), pages 16:1–16:14, 2017.
- [Bac18] M. Backens. A complete dichotomy for complex-valued Holant^c. In Proc. 45th Annual International Conference on Automata, Languages, and Programming, volume 107 of Leibniz International Proceedings in Informatics (LIPIcs), pages 12:1–12:14, 2018.
- [BBC⁺19] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard. Simulation of quantum circuits by low-rank stabilizer decompositions. In *Proceedings of QIP'19*, also https://arxiv.org/abs/1808.00128, 2019.
- [BG16] S. Bravyi and D. Gosset. Improved classical simulation of quantum circuits dominated by Clifford gates. *Physical Review Letters*, 116, 2016.

- [BJS10] M. Bremner, R. Jozsa, and D. Shepherd. Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy. http://arxiv.org/abs/1005.1407, May 2010.
- [Bk18] K. Bu and D. koh. Classical simulation of quantum circuits by half Gauss sums. https://arxiv.org/abs/1812.00224, December 2018.
- [Bro72] Edgar H Brown. Generalizations of the Kervaire invariant. Annals of Mathematics, 95:368–383, 1972.
- [BvDR08] D. Bacon, W. van Dam, and A. Russell. Analyzing algebraic quantum circuits using exponential sums. http://www.cs.ucsb.edu/vandam/LeastAction.pdf, November 2008.
- [CCLL10] J.-Y. Cai, X. Chen, R. Lipton, and P. Lu. On tractable exponential sums. In Proceedings of the 2010 Frontiers in Algorithms Workshop, volume 6213 of Lect. Notes in Comp. Sci., pages 48–59. Springer Verlag, 2010.
- [CGW18] Jin-Yi Cai, Heng Guo, and Tyson Williams. Clifford gates in the holant frameqwork. Theoretical Computer Science, 75:163–171, 2018.
- [CKL13] H.Y. Cheung, T.C. Kwok, and L.C. Lau. Fast matrix rank algorithms and applications. J. Assn. Comp. Mach., 60:1–25, 2013.
- [CLX14] J.-Y. Cai, P. Lu, and M. Xia. The complexity of complex weighted Boolean #CSP. J. Comp. Sys. Sci., 80:217–236, 2014.
- [DHH⁺04] C. Dawson, H. Haselgrove, A. Hines, D. Mortimer, M. Nielsen, and T. Osborne. Quantum computing and polynomial equations over the finite field Z_2 . Quantum Information and Computation, 5:102–112, 2004.
- [DM03] J. Dehaene and B.L.R. De Moor. The Clifford group, stabilizer states, and linear and quadratic operations over GF(2). *Phys. Rev. A*, 68:042318, 2003.
- [DP18] J.-G. Dumas and C. Pernet. Symmetric indefinite triangular factorization revealing the rank profile matrix. In Proc. 43rd International Symposium on Symbolic and Algebraic Computation, pages 151–158, 2018. Also https://arxiv.org/abs/1802.10453.
- [EK90] A. Ehrenfeucht and M. Karpinski. The computational complexity of (XOR, AND)counting problems. Technical Report TR-90-032, Mathematical Sciences Research Institute, University of California at Berkeley, 1990.
- [Gal14] F. Le Gall. Powers of tensors and fast matrix multiplication. In Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation (ISSAC 2014), 2014.
- [GM13] H. Garca and I.L. Markov. Quipu: High-performance simulation of quantum circuits using stabilizer frames. In *Proceedings of ICCD 2013*, pages 404–410, 2013.
- [GM15] H. Garca and I.L. Markov. Simulation of quantum circuits via stabilizer frames. *IEEE Trans. Computers*, 64:2323–2336, 2015. Updated 12/2017 at https://arxiv.org/abs/1712.03554.
- [GMC14] H. Garca, I.L. Markov, and A.W. Cross. On the geometry of stabilizer states. *Quantum Information and Computation*, 14:683–720, 2014.

- [Got98] D. Gottesman. The Heisenberg representation of quantum computers. http://arxiv.org/abs/quant-ph/9807006, 1998.
- [JvdN14] R. Jozsa and M. van den Nest. Classical simulation complexity of extended Clifford circuits. *Quantum Information and Computation*, 14:633–648, 2014.
- [Koh17] D. Koh. Further extensions of Clifford circuits and their classical simulation complexities. *Quantum information and computation*, 17:262–282, 2017.
- [KPS17] D.E. Koh, M.D. Penney, and R.W. Spekkens. Computing quopit Clifford circuit amplitudes by the sum-over-paths technique. https://arxiv.org/pdf/1702.03316, 2017.
- [MFIB18] I. Markov, A. Fatima, S. Isakov, and S. Boixo. Quantum supremacy is both closer and farther than it appears. https://arxiv.org/pdf/1807.10749, 2018.
- [Mon17] A. Montanaro. Quantum circuits and low-degree polynomials over F_2 . Journal of Physics A, 50, 2017.
- [Nob06] S. Noble. Evaluating the rank generating function of a graphic 2-polymatroid. Combinatorics, Probability and Computing, 15:449–461, 2006.
- [OW93] J. Oxley and G. Whittle. A characterization of Tutte invariants of 2-polymatroids. J. Comb. Thy. Ser. B, 59:210–244, 1993.
- [RC09] K. Regan and A. Chakrabarti. Quantum circuits, polynomials, and entanglement measures, 2009. Draft.
- [RCG18] K. Regan, A. Chakrabarti, and C. Guan. Algebraic and logical emulations of quantum circuits. *Transactions on Computational Science*, 10,730:41–76, 2018.
- [Sar14] P. Sarvepalli. Quantum codes and symplectic matroids. In *Proceedings of the 2014 IEEE International Symposium on Information Theory; also https://arxiv.org/abs/1104.1171*, 2014.
- [Sch09] Kai-Uwe Schmidt. Z_4 -valued quadratic forms and quaternary sequence families. *IEEE Transactions on Information Theory*, 55:5803–5810, 2009.
- [Sto10] A. Stothers. On the Complexity of Matrix Multiplication. PhD thesis, University of Edinburgh, 2010.
- [Str69] V. Strassen. Gaussian elimination is not optimal. Numer. Math., 13:354–356, 1969.
- [vdN09] M. van den Nest. Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond, 2009. arXiv:0811.0898.
- [Wil12] V.V. Williams. Multiplying matrices faster than Coppersmith-Winograd. In Proc. 44th Annual ACM Symposium on the Theory of Computing, pages 887–898, 2012.

A Appendix: Other Results and Proofs

Proof for Lemma 3.4(c). Suppose $\mathbf{B}' = \mathbf{L}\mathbf{D}\mathbf{U} = \mathbf{M}\mathbf{E}\mathbf{V}$ where $\mathbf{L}\mathbf{M}$ are lower triangular and \mathbf{U}, \mathbf{V} are upper triangular, not even caring that $\mathbf{U} = \mathbf{L}^{\top}$ and $\mathbf{V} = \mathbf{M}^{\top}$ but just that they are invertible.

First consider the non-alternating case where **D** and **E** are diagonal but not necessarily of full rank. They must have the same rank r. Then \mathbf{M}^{-1} is also lower triangular, so that $\mathbf{C} = \mathbf{M}^{-1}\mathbf{L}\mathbf{D}$ is lower triangular, and \mathbf{U}^{-1} is upper triangular, so that \mathbf{EVU}^{-1} is upper triangular. $\mathbf{C} = \mathbf{MLD} = \mathbf{EVU}^{-1}$, and the only way a lower-triangular matrix can equal an upper-triangular matrix is when both are diagonal. So **C** is diagonal, and we need only argue that $\mathbf{C} = \mathbf{D}$ ($= \mathbf{E}$). This follows because they have the same rank and for any i such that $\mathbf{D}[i, i] = 0$, also $\mathbf{C}[i, i] = 0$.

In the alternating case, $\mathbf{M}^{-1}\mathbf{L}$ is lower triangular but its product C with \mathbf{D} can also have a non-zero diagonal above the main diagonal. The product \mathbf{EVU}^{-1} is upper-triangular except for the diagonal below the main. Hence C must be tri-diagonal. Every off-diagonal nonzero element of C equals a diagonal element of $\mathbf{M}^{-1}\mathbf{L}$ multiplied by the corresponding off-diagonal entry of D and also equals a diagonal element of \mathbf{VU}^{-1} multiplying the corresponding entry of \mathbf{E} . By invertibility over \mathbb{F}_2 the diagonal entries are all 1, so we have proved that D and E agree on all off-diagonal entries. The proof that they agree with each other (but not necessarily with \mathbf{C}) in their 1×1 blocks on the diagonal is similar to that for the alternating case.

Proof for Lemma 4.1. The given $f(x_1, \dots, x_n)$ will fall into one of the following cases:

- (a) If some $a_j = 0$, it is safe to drop this *j*-th variable x_j since $\sum_{i=1}^n a_i \cdot x_i \mod 4 = \sum_{i=1, i \neq j}^n a_i \cdot x_i \mod 4$. Define N'_0, N'_1, N'_2, N'_3 with respect to $\mathbf{x}' = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$. We can see that $N_i = 2N'_i$ for i = 0, 1, 2, 3;
- (b) If some $a_j = 2$, then for any $\mathbf{x}_0 = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$, it can be paired with \mathbf{x}_1 such that $f(\mathbf{x}_1) = f(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) = f(\mathbf{x}_0) + 2 \mod 4$. That is, if $f(\mathbf{x}_0) = 0$, then $f(\mathbf{x}_1) = 2$, and vice versa. Same analysis goes to N_1 and N_3 . Hence, the two differences are zero in this case;
- (c) If some $a_j = 1$ and some $a_k = 3$ (without loss of generality, assume $j \leq k$), then for any $\mathbf{x}_{10} = (x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n)$ and $f(\mathbf{x}_{10}) = \sum_{i=1, i \neq j, i \neq k}^n a_i \cdot x_i + 1$ mod 4, we have $f(\mathbf{x}_{01}) = \sum_{i=1, i \neq j, i \neq k}^n a_i \cdot x_i + 3 \mod 4$, which will cancel in the differences. While for $\mathbf{x}_{00} = (x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n)$ and $f(\mathbf{x}_{00}) = \sum_{i=1, i \neq j, i \neq k}^n a_i \cdot x_i \mod 4$, $f(\mathbf{x}_{11}) = \sum_{i=1, i \neq j, i \neq k}^n a_i \cdot x_i + 4 \mod 4 = f(\mathbf{x}_{00})$. Hence, by dropping both *j*-th and *k*-th variables (similar to case 1) and defining N'_i with respect to $\mathbf{x}' = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$, $N_i = 2N'_i$ for i = 0, 1, 2, 3;
- (d) If all x_j 's are 1, then for i = 0, 1, 2, 3, we have

$$N_i = \sum_{m \ge 0} \binom{n}{4m+i}$$

Then Lemma A.1 gives that both differences are powers of 2.

(e) If all x_j 's are 3, then

$$N_0 = \sum_{m \ge 0} \binom{n}{4m}, \quad N_2 = \sum_{m \ge 0} \binom{n}{4m+2},$$

$$N_1 = \sum_{m \ge 0} {n \choose 4m+3}, \quad N_3 = \sum_{m \ge 0} {n \choose 4m+1}.$$

Then it can be reduced to case 4 and hence both differences are powers of 2.

Note that the above procedures can be applied to a given $f(\mathbf{x})$ recursively. Overall, the statement holds.

Lemma A.1.

$$\sum_{r\geq 0} \binom{n}{4r} - \sum_{r\geq 0} \binom{n}{4r+2}$$
$$\sum_{r\geq 0} \binom{n}{4r+1} - \sum_{r\geq 0} \binom{n}{4r+3}$$

are either 0 or a power-of-2.

Proof. It is known that with ω be the *d*-th root of unity,

$$\sum_{r\geq 0} \binom{n}{dr+c} = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-jc} (1+\omega^j)^n$$

where $0 \le c < d$. By simple substitutions with d = 4 and a = 0, 1, 2, 3, we get

$$g_0 = \sum_{r \ge 0} \binom{n}{4r} - \sum_{r \ge 0} \binom{n}{4r+2} = \frac{1}{2} (1+\omega^n)(1+\omega^3)^n,$$

$$g_1 = \sum_{r \ge 0} \binom{n}{4r+1} - \sum_{r \ge 0} \binom{n}{4r+3} = \frac{1}{2} \omega^{-1} (\omega^n - 1)(1+\omega^3)^n,$$

Rewrite n = 4a + b with $a, b \in \mathbb{Z}$ and $0 \le b < 4$. Let $g_0 = \sum_{r \ge 0} \binom{n}{4r} - \sum_{r \ge 0} \binom{n}{4r+2}$ and $g_1 = \sum_{r \ge 0} \binom{n}{4r+1} - \sum_{r \ge 0} \binom{n}{4r+3}$. It is easy to verify that $(1 + \omega^3)^4 = -4$ and hence we can rewrite

$$g_0 = \frac{1}{2}(-4)^a (1+\omega^b)(1+\omega^3)^b,$$

$$g_1 = \frac{1}{2}(-4)^a \omega^3 (\omega^b - 1)(1+\omega^3)^b.$$

Now we can analysis them case by case.

1. b = 0: $g_0 = (-4)^a$ and $g_1 = 0$; 2. b = 1: $g_0 = (-4)^a$ and $g_1 = (-4)^a$; 3. b = 2: $g_0 = 0$ and $g_1 = 2 \cdot (-4)^a$; 4. b = 3: $g_0 = (-2)(-4)^a$ and $g_1 = 2 \cdot (-4)^a$.

Proof for Lemma 4.2. Since f is alternating, by Corollary 3.3 and Theorem 3.2, f has even rank r and

$$f(\mathbf{x}) = 2\sum_{j=1}^{r/2} x_{2j-1} x_{2j} + 2\sum_{i=1}^{n} w_i x_i,$$

for some basis for V over K and for some $\mathbf{w} = (w_1, \dots, w_n) \in K^n$. Let r = 2g for some $g \in \mathbb{Z}$ and we can further rewrite it as

$$f(\mathbf{x}) = 2\sum_{j=1}^{g} (x_{2j-1} + w_{2j})(x_{2j} + w_{2j-1}) + 2\sum_{i=r+1}^{n} w_i x_i - 2\sum_{j=1}^{g} w_{2j-1} w_{2j}$$

Without loss of generality, we first look at the variable pair (x_1, x_2) and its coefficient pair (w_1, w_2) . Denote

$$f'(\mathbf{x}) = 2\sum_{j=2}^{g} (x_{2j-1} + w_{2j})(x_{2j} + w_{2j-1}) + 2\sum_{i=r+1}^{n} w_i x_i - 2\sum_{j=2}^{g} w_{2j-1} w_{2j},$$

and write

$$f(\mathbf{x}) = f'(\mathbf{x}) + 2(x_1 + w_2)(x_2 + w_1) + 2w_1w_2.$$

Note that $f'(\mathbf{x})$ only depends on (x_3, \dots, x_n) , that is, $f'(\mathbf{x}) = f(x_3, \dots, x_n)$. There are only four cases to consider for $h(x_1, x_2) = 2(x_1 + w_2)(x_2 + w_1)$:

- $(w_1, w_2) = (0, 0)$: $h(x_1, x_2) = 2x_1x_2$, and h(0, 0) = 0, h(0, 1) = 0, h(1, 0) = 0, h(1, 1) = 2;
- $(w_1, w_2) = (1, 0)$: $h(x_1, x_2) = 2x_1(1 + x_2)$, and h(0, 0) = 0, h(0, 1) = 0, h(1, 0) = 2, h(1, 1) = 0;
- $(w_1, w_2) = (0, 1)$: $h(x_1, x_2) = 2(1 + x_1)x_2$, and h(0, 0) = 0, h(0, 1) = 2, h(1, 0) = 0, h(1, 1) = 0;
- $(w_1, w_2) = (1, 1)$: $h(x_1, x_2) = 2(1 + x_1)(1 + x_2)$, and h(0, 0) = 2, h(0, 1) = 0, h(1, 0) = 0, h(1, 1) = 0.

Define $Q_i^{00} = \{\mathbf{x} \in V | f(\mathbf{x}) = i \mod 4 \text{ and } x_1 = 0, x_2 = 0\}$ and similarly $Q_i^{01}, Q_i^{10}, Q_i^{11}$. Then we have $Q_i = Q_i^{00} \cup Q_i^{01} \cup Q_i^{10} \cup Q_i^{11}$. Also define $S_i^{00} = \{\mathbf{x} \in V | f'(\mathbf{x}) = i \mod 4 \text{ and } x_1 = 0, x_2 = 0\}$ and analogously $S_i^{01}, S_i^{10}, S_i^{11}$.

and analogously $S_i^{01}, S_i^{10}, S_i^{11}$. Note that $f'(\mathbf{x})$ only depends on (x_3, \dots, x_n) . Let $S'_i = \{(x_3, \dots, x_n) | f'(x_3, \dots, x_n) = i\}$, and we have $|S'_i| = |S_i^{00}| = |S_i^{01}| = |S_i^{10}| = |S_i^{11}|$. Now analyze the above four cases separately:

• $(w_1, w_2) = (0, 0)$: we have

$$f(\mathbf{x}) = f'(\mathbf{x}) + 2x_1x_2,$$

If $\mathbf{x}_{00} \in Q_i^{00}$, then $f(\mathbf{x}_{00}) = f'(\mathbf{x}_{00}) = i$, same for Q_i^{01}, Q_i^{10} , while if $\mathbf{x}_{11} \in Q_i^{11}$, then $f(\mathbf{x}_{11}) = f'(\mathbf{x}_{11}) + 2$ and hence $f'(\mathbf{x}_{11}) = i + 2$. Now for some $c \in \{0, 1\}$,

$$N_{c} - N_{c+2} = |Q_{c}^{00}| + |Q_{c}^{01}| + |Q_{c}^{10}| + |Q_{c}^{11}| - (|Q_{c+2}^{00}| + |Q_{c+2}^{01}| + |Q_{c+2}^{10}| + |Q_{c+2}^{11}| + |Q_{c+2}^{$$

- $(w_1, w_2) = (1, 0)$: by the similar analysis, $|Q_c| |Q_{c+2}| = 2(|S'_c| |S'_{c+2}|)$.
- $(w_1, w_2) = (0, 1)$: $|Q_c| |Q_{c+2}| = 2(|S'_c| |S'_{c+2}|)$.
- $(w_1, w_2) = (1, 1)$: $|Q_c| |Q_{c+2}| = -2(|S'_c| |S'_{c+2}|)$.

Hence, we can reduce the counting of $|Q_c| - |Q_{c+2}|$ over (x_1, \dots, x_n) to the counting of $|S'_c| - |S'_{c+2}|$ over (x_3, \dots, x_n) , and gradually after g-many such reduction, we can derive

$$|Q_c| - |Q_{c+2}| = (-1)^m 2^g (|Q'_c| - |Q'_{c+2}|),$$

where $Q'_c = \{(x_{r+1}, \dots, x_n) | 2 \sum_{i=r+1}^n w_i x_i = c\}$ and *m* is the number of (w_{2j-1}, w_{2j}) pairs in $f(\mathbf{x})$ such that $(w_{2j-1}, w_{2j}) = (1, 1)$, hence $2w_{2j-1}w_{2j} = 2$.

Now it is left to argue that $||Q'_c| - |Q'_{c+2}||$ is either zero or a power of 2. Let $q(x) = \sum_{i=r+1}^n 2w_i x_i$. Since $w_i \in \{0, 1\}$, q(x) is linear with coefficient from $\{0, 2\}$. Then we can reduce it to the 1st and 2nd cases in Lemma 4.1. In the 2nd case, it gives that $|Q'_c| - |Q'_{c+2}| = 0$ if $w_i = 1$ for some $i \in \{r+1, \dots, n\}$. Now assume non-zero case. Then we have $w_i = 0$ for all $i \in \{r+1, \dots, n\}$, which gives

$$Q'_{c}| - |Q'_{c+2}| = (-1)^m 2^g 2^{n-r} = (-1)^m 2^{n-g}$$

and hence it completes the proof.

Proof for Lemma 4.3. Since f is non-alternating, by Corollary 3.3, there exists a basis for V over K, determining the coordinates (x_1, \dots, x_n) , such that

$$f(\mathbf{x}) = \sum_{j=1}^{r} x_j + 2\sum_{i=1}^{n} w_i x_i,$$

for some $\mathbf{w} = (w_1, \cdots, w_n) \in K^n$. By rearranging, we have

$$f(\mathbf{x}) = \sum_{j=1}^{r} (1+2w_j)x_j + 2\sum_{i=r+1}^{n} w_i x_i = \sum_{j=1}^{r} w'_j x_j + 2\sum_{i=r+1}^{n} w_i x_i,$$

where $w'_j = 1 + 2w_j$. Note that w'_j can only be 1 or 3. Then we can reduce it to the 2nd, 3rd, 4th and 5th cases in Lemma 4.1.

The 2nd case gives the trivial case where both $N_0 - N_2$ and $N_1 - N_3$ are zero. Now assume non-zero case. Then we have $w'_i, w_i \in \{0, 1, 3\}$.

Define c to be the number of w_i 's such that $w_i = 0$ with $i \in \{r + 1, \dots, n\}$ and d to be the number of pairs such that $(1+2w_j, 1+2w_{j'}) = (1,3)$ with $j, j' \in \{1, \dots, r\}$. Also let m = n - c - 2d and rewrite m = 4a + b, and define η such that $\eta = 0$ if the rest m-many coefficients are all 1's but $\eta = 1$ if they are all 3's. Then the differences $N_0 - N_2$ and $N_1 - N_3$ are taking one of the following values:

• if b = 0, then $N_0 - N_2 = (-1)^a 2^{(n+c)/2}$, $N_1 - N_3 = 0$;

• if
$$b = 1$$
, then $N_0 - N_2 = (-1)^a 2^{(n+c-1)/2}$, $N_1 - N_3 = (-1)^{a+\eta} 2^{(n+c-1)/2}$;

• if
$$b = 2$$
, then $N_0 - N_2 = 0$, $N_1 - N_3 = (-1)^{a+\eta} 2^{(n+c)/2}$;

• if b = 3, then $N_0 - N_2 = (-1)^{a+1} 2^{(n+c-1)/2}$, $N_0 - N_2 = (-1)^{a+\eta} 2^{(n+c-1)/2}$.

Note that Lemmas 4.1, 4.2, and 4.3 and the proof method of Theorem 4.4 apply to more general input **a** and output **b** as well, so that we have the following supplementary result:

Theorem A.2. Given a stabilizer circuit C and its quadratic form $q_C(\mathbf{y}, \mathbf{z})$, assume we know \mathbf{Q}, \mathbf{D}_1 and \mathbf{D}_2 with entries in \mathbb{F}_2 such that $\mathbf{y}^\top \mathbf{Q}^\top \mathbf{A} \mathbf{Q} \mathbf{y} = \mathbf{y}^\top (\mathbf{D}_1 + 2\mathbf{D}_2) \mathbf{y}$ where

- if q_C is alternating, D₂ is a diagonal matrix with entries in {0,1} and D₁ = M₁ ⊕ · · · ⊕ M_g has even rank r = 2g over F₂;
- if q_C is non-alternating, \mathbf{D}_1 and \mathbf{D}_2 are both diagonal matrices with entries in $\{0, 1\}$.

Then we can compute $|\langle \mathbf{b} | C | \mathbf{0} \rangle|^2$ for any output vector \mathbf{b} to the circuit in O(en) time where $n = |\mathbf{y}|$ and e is the number of ones in \mathbf{y} .

Proof. Assume $\mathbf{Q} = (Q_{i,j})$ with $Q_{i,j} \in \mathbb{F}_2$ and take any output vector **b**. Then $q_C(\mathbf{y}, \mathbf{b}) = \mathbf{y}^\top \mathbf{A} \mathbf{y} + \mathbf{y}^\top \mathbf{\Delta} \mathbf{y}$ and we have

$$\begin{split} \mathbf{y}^{\top} \mathbf{Q}^{\top} \mathbf{A} \mathbf{Q} \mathbf{y} + \mathbf{y}^{\top} \mathbf{Q}^{\top} \mathbf{\Delta} \mathbf{Q} \mathbf{y} &= \mathbf{y}^{\top} (\mathbf{D}_1 + 2\mathbf{D}_2) \mathbf{y} + \sum_i 2 \mathbf{y}^{\top} \mathbf{E}_i \mathbf{y} \\ &= \mathbf{y}^{\top} \mathbf{D}_1 \mathbf{y} + \mathbf{y}^{\top} 2 (\mathbf{D}_2 + \sum_i \mathbf{E}_i) \mathbf{y} \end{split}$$

where E_i is a diagonal matrix $diag(Q_{i,1}, \dots, Q_{i,n})$ for *i* such that $\Delta_{i,i} = 1$ and \mathbf{D}_1 varies depending on whether it is alternating or non-alternating. Then each $E_i = diag(Q_{i,1}, \dots, Q_{i,n})$ can be obtained in O(n) time given the matrix \mathbf{Q} .

We also know that in both the alternating and non-alternating cases, the output probability $|\langle \mathbf{b} | C | \mathbf{0} \rangle|^2$ is determined by the rank of \mathbf{D}_1 if $|\langle \mathbf{b} | C | \mathbf{0} \rangle|^2 \neq 0$. Now we will show that with the knowledge of such \mathbf{Q} , we can tell $|\langle \mathbf{b} | C | \mathbf{0} \rangle|^2 = 0$ in O(en) time.

First suppose $q_C(\mathbf{y}, \mathbf{z})$ is alternating and $n = |\mathbf{y}|$, then for output **b** we can rewrite

$$\mathbf{y}^{\top}\mathbf{D}_{1}\mathbf{y} + \mathbf{y}^{\top}2(\mathbf{D}_{2} + \sum_{i}\mathbf{E}_{i})\mathbf{y} = \sum_{j=1}^{g} 2y_{2j-1}y_{2j} + \sum_{i=1}^{n} 2w_{i}y_{i} \mod 4,$$

where $w_i \in \{0, 1\}$. Once we finish updating the above equation (which takes O(en) time), we can by Lemma 4.2, get the value $\langle \mathbf{b} | C | \mathbf{0} \rangle$ and identify if $|\langle \mathbf{b} | C | \mathbf{0} \rangle|^2 = 0$ which happens when $w_i = 0$ for some $i \in \{r + 1, \dots, n\}$. Analogously, this also can be done in O(en) time by Lemma 4.3 for non-alternating cases.