The partial duplication random graph with edge deletion

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Abstract

We study a random graph model in continuous time. Each vertex is partially copied with the same rate, i.e. an existing vertex is copied and every edge leading to the copied vertex is copied with independent probability p. In addition, every edge is deleted at constant rate, a mechanism which extends previous partial duplication models. In this model, we obtain results on the degree distribution, which shows a phase transition such that either – if p is small enough – the frequency of isolated vertices converges to 1, or there is a positive fraction of vertices with unbounded degree. We derive results on the degrees of the initial vertices as well as on the sub-graph of non-isolated vertices. In particular, we obtain expressions for the number of star-like subgraphs and cliques.

1 Introduction

Various random graph models have been studied in the last decades. Frequently, such models try to mimic the behavior of social networks (see e.g. Cooper and Frieze, 2003 and Barabási et al., 2002) or interactions within biological networks (see e.g. Wagner, 2001, Albert, 2005 and Jeong et al., 2000). For a general introduction to random graphs see the monographs Durrett (2008), van der Hofstad (2016) and references therein.

In this paper, we study and extend a model introduced in Bhan et al. (2002), Chung et al. (2003), Pastor-Satorras et al. (2003), Chung et al. (2003), Bebek et al. (2006a), Bebek et al. (2006b), Hermann and Pfaffelhuber (2016) and Jordan (2018). Here, a vertex models a protein and an edge denotes some form of interaction. Within the genome, the DNA encoding for a protein can be duplicated (which in fact is a long evolutionary process), such that the interactions of the copied protein are partially inherited to the copy; see Ohno (1970) for some more biological explanations. Within the random graph model, a vertex is p-copied, i.e. a new vertex is introduced and every edge of the parent vertex is independently copied with the same probability p. We will extend this model by assuming mutation (at some constant rate), which can destroy interactions, leading to the loss of edges in the random graph at constant rate.
In the model without edge deletion, which we will call *pure partial duplication model*, Hermann and Pfaffelhuber (2016) have determined a critical parameter $p^* \approx 0.567143$, the unique solution of $pe^p = 1$, below which approximately all vertices are isolated. Moreover, almost sure asymptotics and limit results for the number of k-cliques and k-stars in the random graph as well as for the degree of a fixed node were obtained. Recently, Jordan (2018) has shown that for $p < e^{-1}$ the degree distribution of the connected component, i.e. of the subgraph of non-isolated vertices, has a limit with tail behavior close to a power-law with exponent β solving $\beta - 3 + p^{\beta-2} = 0$ (cf. Jordan, 2018, Theorem 1(c)).

In the model with edge deletion, we will add results on the degree distribution of the full graph (see Theorem 1), the sub-graph of non-isolated vertices (see Proposition 2.5) as well as the number

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of star-like graphs, cliques, and degrees of initial vertices (see Theorem 2). We will see that the degree distribution of this model is closely related to a branching process as in Jordan (2018), but now with death-rate δ, the rate of edge deletion. In turn, such branching processes can be studied by using piecewise-deterministic Markov jump processes, a tool which we will introduce in Lemma 2.3; see also Section 3.1 for the connection to branching processes. These insights allow us to transfer results on branching processes to the degree distribution, generalizing the results of Theorem 2.7 in Hermann and Pfaffelhuber (2016) to the model with edge deletion; see Section 3. Here, we derive a phase transition such that if p is small, the fraction of vertices with positive degree vanishes, while for larger p, vertices are either isolated (i.e. have vanishing degree) or have unbounded degree. In Section 4, we prove Theorem 2 and derive almost sure asymptotics and limit results for binomial moments, cliques and the degree of a fixed node mainly by applying martingale theory, generalizing Theorems 2.9 and 2.14 in Hermann and Pfaffelhuber (2016).

2 Model and main results

Definition 2.1 (Partial duplication graph process with edge deletion)**.** *Let $p \in [0,1]$, $\delta \geq 0$ and $G_0 = (V_0, E_0)$ be a deterministic undirected graph without loops with vertex set $V_0 = \{v_1, \dots, v_{|V_0|}\}$ and non-empty edge set E_0. Let $PD(p,\delta) = (G_t)_{t\geq 0}$ be the continuous-time graph-valued Markov process starting in G_0 and evolving in the following way:*

- *Every node $v \in V_t$ is p-partially duplicated (or p-*copied *for short) at rate $\kappa_t := (|V_t|+1)/|V_t|$, i.e. a new node $v_{|V_t|+1}$ is added and for each $w \in V_t$ with $(v,w) \in E_t$, $v_{|V_t|+1}$ is connected to w independently with probability p.*
- *Every edge in E_t is removed at rate δ.*

Then, $PD(p,\delta)$ is a partial duplication graph process with edge deletion *with* initial graph G_0*,* edge-retaining probability p *and* deletion rate δ*. Within $PD(p,\delta)$, we define the following quantities:*

1. *Let $D_i(t) := \deg_{G_t}(v_i) \cdot \mathbb{1}_{\{i \leq |V_t|\}}$ be the* degree of v_i*, i.e. the number of its neighbors, at time t.*
2. *Let $F(t) := (F_k(t))_{k=0,1,2,\dots}$ with $F_k(t) := |\{1 \leq i \leq |V_t| : D_i(t) = k\}|/|V_t|$ for $k = 0,1,2,\dots$ be the* degree distribution *at time t. Furthermore, let $F_+(t) := 1 - F_0(t)$ be the proportion of vertices of positive degree.*
3. *For $k = 1,2,\dots$ let $B_k(t) := \sum_{\ell \geq k} \binom{\ell}{k} F_\ell(t)$ be the* kth binomial moment *of the degree distribution. Note that $B_k(t)$ is the expected number of subgraphs of V_t with k nodes, all of which share one randomly chosen center, i.e. the expected number of star-like trees with k leaves.*
4. *For $k = 1,2,\dots$ let $C_k(t)$ be the* number of k-cliques *at time t, i.e. the number of complete sub-graphs of size k.*

Remark 2.2. 1. Note that $PD(p,\delta)$ is a time-continuous duplication model and duplication events (total rate $|V_t| + 1$) at time t and the deletion events (total rate $\delta|E_t|$ at time t) depend on different statistics. An alternative, discrete model would add a node or delete an edge with some fixed probability. However, such a model would behave differently from the model above.

2. We choose the duplication rate $\kappa_t := (|V_t|+1)/|V_t|$ in order to get a closed recurrence relation for the degree distribution; see Lemma 3.3. Alternatively, we would set $\widetilde{\kappa}_t := 1$, i.e. all vertices are copied at unit rate. Since $V_t \sim e^t$ (see Lemma 4.4), and $\int_0^t \kappa_s - \widetilde{\kappa}_s ds \sim \int_0^t e^{-s} ds \xrightarrow{t\to\infty} 1 \ll \int_0^t \kappa_s ds$, we expect the random graph with our choice of κ_t to behave the same qualitatively for $t \to \infty$, while differences can occur quantitatively.

In order to formulate our results, we need an auxiliary process, which is connected to $PD(p,\delta)$. It will appear below in Theorem 1.1 and in Proposition 2.5. The proof of the following Lemma is found in Section 3.2.

Lemma 2.3 (Connection of $PD(p,\delta)$ and a piecewise-deterministic Markov process)**.** *Let $X = (X_t)_{t\geq 0}$ be a Markov process on $[0,1]$ jumping at rate 1 from $X_t = x$ to px, in between jumps evolving according to $\dot{X}_t = pX_t(1-X_t) - \delta X_t$. Furthermore, let*

$$H_x(t) := \sum_{k=0}^{\infty}(1-x)^k F_k(t), \tag{2.1}$$

i.e. the probability generating function of the degree distribution at time t. Then, for all $t \geq 0$ and $x \in [0,1]$, writing $\mathbb{E}_x[.] := \mathbb{E}[.|X_0 = x]$,

$$\mathbb{E}[H_x(t)] = \mathbb{E}_x[H_{X_t}(0)] = \sum_{k=0}^{\infty} F_k(0)\cdot \mathbb{E}_x[(1-X_t)^k]. \tag{2.2}$$

Theorem 1 (Limit of the degree distribution)**.** *Let $p \in (0,1)$ and $\delta \geq 0$.*

1. *If $\delta \geq p - p\log\frac{1}{p}$, $F(t) \xrightarrow{t\to\infty} (1,0,0,\dots)$ almost surely with (recall $F_+ = 1 - F_0$, and writing $a_t \sim b_t$ iff $a_t/b_t \xrightarrow{t\to\infty} 1$)*

$$\mathbb{E}[F_+(t)] \sim ce^{-t(1+\delta-2p)},$$

   where X is as in Lemma 2.3 and

$$c = \exp\Big(-p\int_0^\infty \frac{\mathbb{E}_1[X_s^2]}{\mathbb{E}_1[X_s]}ds\Big) \in (0, B_1(0)).$$

2. *If $p - p\log\frac{1}{p} \geq \delta \geq p - \log\frac{1}{p}$, $F(t) \xrightarrow{t\to\infty} (1,0,0,\dots)$ almost surely with*

$$-\tfrac{1}{t}\log\mathbb{E}[F_+(t)] \xrightarrow{t\to\infty} 1 - \tfrac{1}{\gamma}(1+\log\gamma),$$

   where $\gamma = \log\frac{1}{p}/(p-\delta)$.

3. *If $\delta < p - \log\frac{1}{p}$, $F(t) \xrightarrow{t\to\infty} (F_0,0,0,\dots)$ almost surely, where F_0 is non-deterministic and*

$$\mathbb{E}[F_0] = 1 - \Big(1 - \frac{\delta}{p} - \frac{1}{p}\log\frac{1}{p}\Big)\sum_{k=1}^{\infty} B_k(0)(-1)^{k-1}\prod_{\ell=1}^{k-1}\Big(1 - \frac{\delta}{p} - \frac{1-p^\ell}{p\ell}\Big),$$

   where we define $\prod_\emptyset := 1$.

Remark 2.4 (Interpretations)**.**

1. Clearly, the quantity F_0 is increasing in δ and decreasing in p (and F_+ is decreasing in δ and increasing in p). See also the illustrations of Theorem 1 in Figure 1. We note that for $\delta = 0$, the three cases can be distinguished using p^*, the solution of $pe^p = 1$ (or $\frac{1}{p}\log\frac{1}{p} = 1$). The three cases are then $p \leq 1/e$, $1/e \leq p \leq p^*$ and $p > p^*$. For 3., we will see in the proof that the right hand side is the hitting probability of a stochastic process, and in particular is in $(0,1)$. This interpretation shows that $0 < \mathbb{E}[F_0] < 1$ as long as the initial graph is not trivial (i.e. $F_0(0) < 1$).

2. The asymptotics given in case $\delta \geq p - p\log\frac{1}{p}$ is more exact than the one given for $p - p\log\frac{1}{p} \geq \delta \geq p - \log\frac{1}{p}$ (in the sense that $-\frac{1}{t}\log\mathbb{E}[F_+(t)] \sim 1 + \delta - 2p$ is a consequence of 1. of the above Theorem). The reason is that we can give a formula for c in this case, which does not carry over to 2.; see the proof of Proposition 2.5.

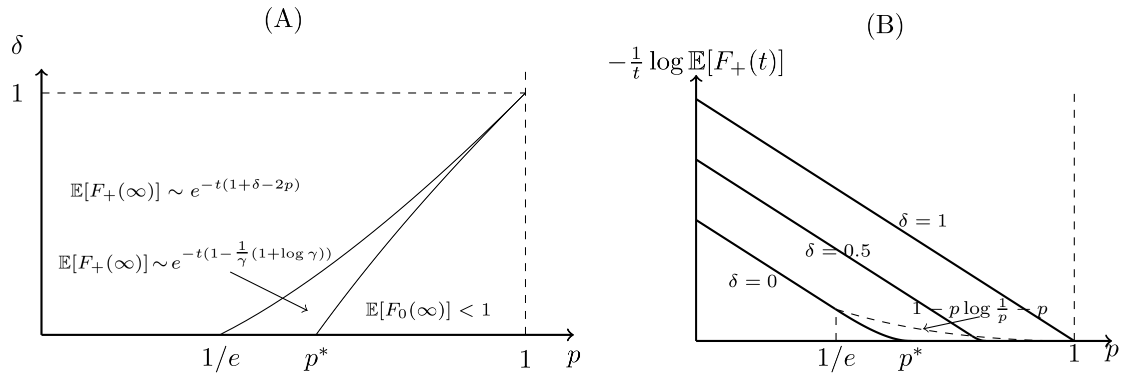

Figure 1:
Illustration of Theorem 1. In (A), the three cases are shown in the $p-\delta$-plane. In (B), we draw the different exponential rates of decrease in $\mathbb{E}[F_+]$.

In the case $\delta \geq p - \log\frac{1}{p}$, the frequency of isolated vertices converges to 1. Hence, it is interesting to study the (degree distribution of the) sub-graph of non-isolated vertices. In order to do so, note that $F_+(t) = 1 - H_1(t)$, with H from (2.1). Also note that, if at some time t a duplication event is triggered, $1 - H_p(t)$ denotes the probability that the new node is not isolated. The next result gives asymptotics of the generating function of the degree distribution of the sub-graph of non-isolated vertices,

$$\frac{\sum_{k=1}^{\infty}(1-x)^k\mathbb{E}[F_k(t)]}{\mathbb{E}[F_+(t)]} = \frac{\mathbb{E}[H_x(t) - H_1(t)]}{\mathbb{E}[1 - H_1(t)]} = 1 - \frac{\mathbb{E}[1 - H_x(t)]}{\mathbb{E}[1 - H_1(t)]}.$$

Proposition 2.5 (Limit of degree distribution on the set of non-isolated vertices)**.** *Let X be the process given in Lemma 2.3.*

1. *If $\delta > p - p\log\frac{1}{p}$, then $\mathbb{E}[1 - H_x(t)] \sim \mathbb{E}_x[X_t] \cdot B_1(0)$.*
2. *If $\delta = p - p\log\frac{1}{p}$ and if the limits $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k]$ as $t \to \infty$ exist for all $x \in (0,1]$ and $k = 1, 2, ...$, then $\mathbb{E}[1 - H_x(t)] \sim \mathbb{E}_x[X_t] \cdot B_1(0)$.*
3. *If $p - \log\frac{1}{p} < \delta < p - p\log\frac{1}{p}$ and if the limits $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k]$ as $t \to \infty$ exist for all $x \in (0,1]$ and $k = 1, 2, ...$, then*

$$\mathbb{E}[1 - H_x(t)] \sim \mathbb{E}_x[X_t] \cdot \sum_{\ell=1}^{\infty} B_\ell(0)(-1)^{\ell+1}\prod_{k=1}^{\ell-1}\frac{\log\frac{1}{\gamma p^k} + \gamma p^k - 1}{k\gamma p},$$

   where $\gamma = \log\frac{1}{p}/(p - \delta)$.

In both cases,

$$\frac{\mathbb{E}[1 - H_x(t)]}{\mathbb{E}[F_+(t)]} \sim \frac{\mathbb{E}_x[X_t]}{\mathbb{E}_1[X_t]} \xrightarrow{t\to\infty} x\exp\Big(p\int_0^\infty \frac{\mathbb{E}_1[X_s^2]}{\mathbb{E}_1[X_s]} - \frac{\mathbb{E}_x[X_s^2]}{\mathbb{E}_x[X_s]}ds\Big). \tag{2.3}$$

Remark 2.6 (Interpretation)**.** Since the right hand side of (2.3) is non-trivial, the Proposition shows that the degree distribution of the sub-graph of non-isolated vertices converges to some non-trivial distribution. However, for a closer analysis, more insight into the process X would be necessary.

Remark 2.7 (Convergence of moments)**.** In 2. and 3., we have to assume that the limits of $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k]$ exist. We conjecture that this assumption is not necessary since we can at least prove the weaker convergence in the sense of Césaro. Precisely, we can show that for $p - \log \frac{1}{p} < \delta < p - p\log\frac{1}{p}$, it holds that for $k = 1, 2, ...$ and any $x \in (0, 1]$

$$\frac{1}{t}\int_0^t \frac{\mathbb{E}_x[X_s^{k+1}]}{\mathbb{E}_x[X_s^k]}ds \xrightarrow{t\to\infty} c_k(p,\delta) := \frac{\log\frac{1}{\gamma p^k} + \gamma p^k - 1}{k\gamma p} \in (0,1]. \tag{2.4}$$

Indeed, since the moments of X satisfy, for $k = 1, 2, ...,$

$$\begin{aligned}\frac{d}{dt}\log\mathbb{E}_x[X_t^k] &= \frac{1}{\mathbb{E}_x[X_t^k]}\Big(p^k\mathbb{E}_x[X_t^k] - \mathbb{E}_x[X_t^k] + (p-\delta)k\mathbb{E}_x[X_t^k] - pk\mathbb{E}_x[X_t^{k+1}]\Big)\\ &= p^k + (p-\delta)k - 1 - \frac{pk\mathbb{E}_x[X_t^{k+1}]}{\mathbb{E}_x[X_t^k]}\end{aligned} \tag{2.5}$$

and thus, integrating, dividing by $-t$, and using Corollary 2.4 of Hermann and Pfaffelhuber (2018),

$$\begin{aligned}\frac{1}{t}\int_0^t \mathbb{E}_x[X_s^{k+1}]/\mathbb{E}_x[X_s^k]ds &= -\frac{1}{pk}\Big(\frac{1}{t}(\log\mathbb{E}_x[X_t^k] - \log(x^k)) - p^k - (p-\delta)k + 1\Big)\\ &\xrightarrow{t\to\infty} \frac{p - \delta + p^k - (1+\log\gamma)/\gamma}{pk} = c_k(p,\delta).\end{aligned}$$

Clearly, if the limit as assumed in Proposition 2.5 exists, it must hold that $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k] \xrightarrow{t\to\infty} c_k(p,\delta)$.
Similarly, for the critical case $\delta = p - p\log\frac{1}{p}$, where $\gamma = 1/p$, it follows from (2.4) for $k \geq 2$

$$\frac{1}{t}\int_0^t \frac{\mathbb{E}_x[X_s^k]}{\mathbb{E}_x[X_s]}ds \leq \frac{1}{t}\int_0^t \frac{\mathbb{E}_x[X_s^2]}{\mathbb{E}_x[X_s]}ds \xrightarrow{t\to\infty} c_1(p,\delta) = 0,$$

such that $\mathbb{E}[X_t^k] = o(\mathbb{E}[X_t])$ if the limits exist.

We now investigate the limiting behavior of certain functionals of the graph.

Theorem 2 (Binomial moments, cliques and degrees)**.** *As $t \to \infty$, the following statements hold almost surely:*

1. *For $k = 1, 2, ...$, $e^{t\beta_k}B_k(t) \to B_k(\infty)$, where $B_k(\infty) \in \mathcal{L}^1$ and*

$$\beta_k = \begin{cases} 1 + \delta - 2p, & \textit{if } \delta \geq p - \frac{p(1-p^{k-1})}{k-1},\\ 1 + \delta k - pk - p^k, & \textit{otherwise.}\end{cases}$$

2. *For $k = 2, 3, ...$, $\exp(-t(kp^{k-1} - \delta\binom{k}{2}))C_k(t) \to C_k(\infty)$, where $C_k(\infty) \in \mathcal{L}^1$.*
   - *(a) If $C_k(0) > 0$ and $\delta < 2p^{k-1}/(k-1)$, the convergence also holds in $\mathcal{L}^1$ and $\mathbb{P}(C_k(\infty) > 0) > 0$.*
   - *(b) Otherwise, if $\delta \geq 2p^{k-1}/(k-1)$, $C_k(t) = 0$ for all $t \geq T^{C_k}$ for some finite random variable T^{C_k} and $\mathbb{P}(C_k(\infty) = 0) = 1$.*
3. *For $i \leq |V_0|$, $e^{-t(p-\delta)}D_i(t) \to D_i(\infty)$, where $D_i(\infty) \in \mathcal{L}^1$.*
   - *(a) If $D_i(0) > 0$ and $\delta < p$, the convergence also holds in $\mathcal{L}^r$ for all $r \geq 1$ and*

$$\mathbb{E}[D_i(\infty)] = D_i(0)\Big(1 + \frac{p}{|V_0|}\Big). \tag{2.6}$$

(b) *Otherwise, if $\delta \geq p$, $D_i(t) = 0$ for all $t \geq T^{D_i}$ for some finite random variable T^{D_i} and* $\mathbb{P}(D_i(\infty) = 0) = 1$.

Remark 2.8 (Interpretations)**.** 1. For Theorem 2.1, we have $\beta_1 = 1 + \delta - 2p$ and for $\delta = 0$, we have $\beta_k = 1 - pk - p^k$, $k = 1, 2, ...$ In all cases, we can also write $\beta_k = (1 + \delta - 2p) \wedge (1 + \delta k - pk - p^k)$, which immediately shows that β_k is continuous in p and δ. In addition, for $k = 2, 3, ...$, we find $p \geq \frac{p(1-p^{k-1})}{k-1}$, i.e. we can choose $\delta \geq 0$ such that either of the two cases can in fact occur. Moreover, $\beta_k \leq \beta_{k-1}$, which can be seen as follows: First, note that $\frac{(1-p^{k-2})/(k-2)}{(1-p^{k-1})/(k-1)} = \frac{(1+\cdots+p^{k-3})/(k-2)}{(1+\cdots+p^{k-2})/(k-1)} \geq 1$. So, if $\delta \geq p - \frac{p(1-p^{k-1})}{k-1}$, both β_{k-1} and β_k do not depend on k anyway. Then, if $p - \frac{p(1-p^{k-1})}{k-1} \geq \delta \geq p - \frac{p(1-p^{k-2})}{k-2}$, we have that

$$\beta_{k-1} - \beta_k = 1 + \delta - 2p - \min(1 + \delta - 2p, 1 + \delta k - pk - p^k) \geq 0.$$

Finally, for $p - \frac{p(1-p^{k-2})}{k-2} \geq \delta$, we have

$$\begin{aligned}\beta_{k-1} - \beta_k &= p - \delta - p^{k-1} + p^k \geq \frac{p(1-p^{k-1})}{k-1} - p(1-p)p^{k-2} \\ &= p(1-p)\Big(\frac{1 + \cdots + p^{k-2}}{k-1} - p^{k-2}\Big) \geq 0.\end{aligned}$$

The fact that $\beta_{k-1} \geq \beta_k$ implies that there are much less star-like subgraphs with $k-1$ leaves than star-like subgraphs with k leaves, $k = 2, 3, ...$ This can only be explained by nodes with high degree.

2. Noting that $B_1(t) = 2|V(t)| \cdot C_2(t)$ and $\frac{1}{t}\log|V(t)| \xrightarrow{t\to\infty} 1$, we see that the results in 1. and 2. imply the same growth rate for the number of edges.

3. Interestingly, we find that $\delta \geq p$ implies that all vertices of the initial graph will eventually be isolated (i.e. have degree 0). However, the total number of edges, denoted by C_2, only dies out for $\delta \geq 2p$. So, for $p < \delta < 2p$, all initial vertices become isolated, but are copied often enough such that the number of edges is positive for all times with positive probability.

Remark 2.9 (Connection to previous work)**.** We analyzed the case $\delta = 0$ previously in Hermann and Pfaffelhuber (2016). We note that Theorem 2.7 in that paper is extended by Theorem 1, which not only treats the case $\delta > 0$, but also gives precise exponential decay rates in 1. and 2. Moreover, we add here the almost sure convergence of each component of the degree distribution in 3.

Theorem 2.9 in Hermann and Pfaffelhuber (2016) is dealing with cliques and k-stars in the case $\delta = 0$ and is extended by Theorem 2. More precisely, since $|V_t| \sim e^t$, and Hermann and Pfaffelhuber (2016) treats the time-discrete model, we note that Theorem 2.9(1) of Hermann and Pfaffelhuber (2016) aligns with Theorem 2.2, but only gives $\mathcal{L}^1$ (rather than $\mathcal{L}^2$)-convergence. In Theorem 2.9(2) of Hermann and Pfaffelhuber (2016), S_k°, the number of k-stars in the network at time t relative to the network size, was analyzed, which coincided with the factorial moments of the degree distribution. There, a k-star was not defined as a sub-graph of G_t, since it depended on the order of the nodes. $|V_t| \cdot B_k(t)$ now gives the number of star-like sub-graphs in the network at time t consisting of $k+1$ nodes. Since the only difference between S_k° and B_k, as given in Theorem 2.1 is a factor of $k!$, the results of Hermann and Pfaffelhuber (2016) easily apply also for B_k if $\delta = 0$. Theorem 2.14 of Hermann and Pfaffelhuber (2016) treats the degrees of initial vertices and thus can be compared to Theorem 2.3.

3 Proof of Theorem 1

Our analysis of the random graph $PD(p, \delta)$ is based on some main observations: First, the expected degree distribution can be represented by a birth-death process with binomial disasters Z, such that the distribution of Z_t equals the expected degree distribution of G_t; see (3.2). Second, asymptotics for the survival probability of such processes were studied in Hermann and Pfaffelhuber (2018).

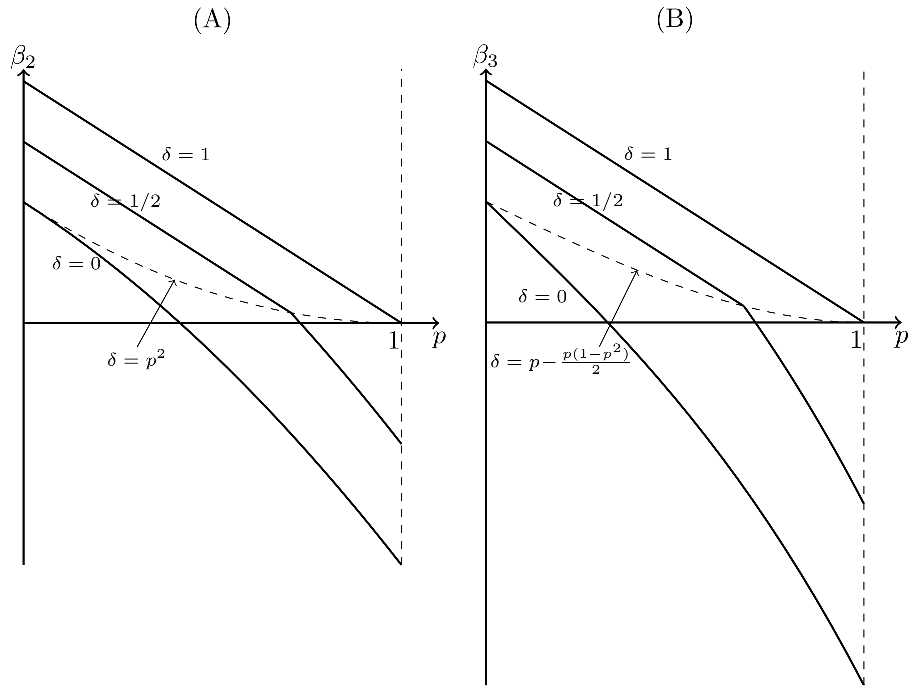

Figure 2:
Illustration of Theorem 1.1. We display the rates of decay of $\mathbb{E}[B_k(t)]$ for $k = 2$ (A) and $k = 3$ (B).

3.1 Birth-death processes with disasters and p-jump processes

Definition 3.1. *Let $b > 0$, $d \geq 0$ and $p \in [0, 1]$. Let $Z(b, d, p) = (Z_t)_{t \geq 0}$ be a continuous-time Markov process on $\mathbb{N}_0$ that evolves as follows: Given $Z_0 = z$, the process jumps*

- *to $z + 1$ at rate bz;*
- *to $z - 1$ at rate dz;*
- *to a binomially distributed random variable with parameters z and p at rate 1.*

Then we call $Z(b, d, p)$ a birth-death process subject to binomial disasters *with* birth-rate b*,* death-rate d *and* survival probability p*.*

Remark 3.2. 1. A birth-death process with binomial disasters, $Z(b, d, p)$ models the size of a population where each individual duplicates with rate b and dies with rate d, subjected to binomial disasters at rate 1. These disasters are global events that kill off each individual independently of each other with probability $1-p$, which generates the binomial distribution in the third part of Definition 3.1.

2. Hermann and Pfaffelhuber (2018) provides several limit results for such branching processes with disasters. As reference for the following, let $Z = Z(b, d, p)$ be as above. Then, Corollary 2.7 of Hermann and Pfaffelhuber (2018) states:

   (a) If $b - d \leq p \log \frac{1}{p}$, Z goes extinct almost surely and

$$\lim_{t \to \infty} -\tfrac{1}{t} \log \mathbb{P}(Z_t > 0) = (1 - p) - (b - d).$$

(b) If $p\log\frac{1}{p} < b-d \le \log\frac{1}{p}$, Z goes extinct almost surely and

$$\lim_{t\to\infty} -\tfrac{1}{t}\log\mathbb{P}(Z_t>0) = 1-\frac{b-d}{\log\frac{1}{p}}\Big(1+\log\Big(\frac{\log\frac{1}{p}}{b-d}\Big)\Big).$$

(c) If $b-d>\log\frac{1}{p}$, then $\mathbb{P}_k(\lim_{t\to\infty} Z_t = 0) + \mathbb{P}_k(\lim_{t\to\infty} Z_t=\infty) = 1$ and

$$\mathbb{P}_k(\lim_{t\to\infty} Z_t=\infty) = \Big(1-\frac{d+\log\frac{1}{p}}{b}\Big)\sum_{\ell=1}^{k}\binom{k}{\ell}(-1)^{\ell-1}\prod_{m=1}^{\ell-1}\Big(1-\frac{dm+(1-p^m)}{bm}\Big).$$

By constructing a relationship between $PD(p,\delta)$ and $Z(p,\delta,p)$ in Lemma 3.3, we are able to transfer these results to our duplication graph processes.

Lemma 3.3. *Let $p\in(0,1)$, $\delta\ge 0$ and recall $F_k(t)$ from Definition 2.1. As $h\to 0$, the entries F_k of the degree distribution yield*

$$\begin{aligned}&\tfrac{1}{h}\mathbb{E}[F_k(t+h)-F_k(t)\mid G_t]\\ &\quad= -(1+pk+\delta k)F_k(t)+p(k-1)F_{k-1}(t)+\delta(k+1)F_{k+1}(t)+\sum_{\ell\ge k}\binom{\ell}{k}p^k(1-p)^{\ell-k}F_\ell(t)+o(1).\end{aligned}\tag{3.1}$$

Moreover, recall $Z:=Z(p,\delta,p)$ from Definition 3.1 (i.e. the binomial distribution of the disasters has the birth rate as a parameter) and let $\mathbb{P}(Z_0=k)=F_k(0)$ for all k be its initial distribution. Then, for all $t\ge 0$ and k, it holds

$$\mathbb{P}(Z_t=k)=\mathbb{E}[F_k(t)],\tag{3.2}$$

i.e. the distribution of Z_t equals the expected degree distribution of G_t.

Proof. Letting $\Phi_k(t):=|V_t|F_k(t)$, the absolute number of nodes with degree k at time t, we obtain for $h\to 0$ that

$$\begin{aligned}\tfrac{1}{h}\mathbb{E}[\Phi_k(t+h)-\Phi_k(t)\mid G_t] &= -(pk\kappa_t+\delta k)\Phi_k(t)+p(k-1)\kappa_t\Phi_{k-1}(t)+\delta(k+1)\Phi_{k+1}(t)\\ &\quad+\sum_{\ell\ge k}\kappa_t\Phi_\ell(t)\binom{\ell}{k}p^k(1-p)^{\ell-k}+O(h),\end{aligned}$$

where the first term on the right hand side stands for the events at which a node can lose the degree k by either obtaining a new neighbor (by one of its k neighbors being copied, which happens with rate $k\kappa_t$, retaining at least the one relevant edge, which has probability p) or one of its k edges being deleted, which happens at rate δk. The second and third terms describe the corresponding gain of a node with degree k by analogous events. Finally, the sum equals the rate of a new node arising with degree k, which can only happen if a node of degree $\ell\ge k$ is copied (with rate $\kappa_t\Phi_\ell(t)$) and the copy retains exactly k edges (which then has a binomial probability).

Now, since $|V_t|$ only increases if a new node is added, i.e. on an event related to κ_t, it follows

$$
\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[F_k(t+h) - F_k(t) \mid F(t)] \\
&= \tfrac{1}{h}\mathbb{E}\Big[\frac{\Phi_k(t+h)}{|V_{t+h}|} - \frac{\Phi_k(t)}{|V_{t+h}|}\Big| F(t)\Big] + \tfrac{1}{h}\mathbb{E}\Big[\frac{\Phi_k(t)}{|V_{t+h}|} - \frac{\Phi_k(t)}{|V_t|}\Big| F(t)\Big] \\
&= \frac{\kappa_t}{|V_t|+1}\Big(- pk\Phi_k(t) + p(k-1)\Phi_{k-1}(t) + \sum_{\ell \geq k} \Phi_\ell(t)\binom{\ell}{k}p^k(1-p)^{\ell-k}\Big) \\
&\qquad + \frac{-\delta k \Phi_k(t) + \delta(k+1)\Phi_{k+1}(t)}{|V_t|} + \kappa_t |V_t| \Phi_k(t)\Big(\frac{1}{|V_t|+1} - \frac{1}{|V_t|}\Big) + O(h) \\
&= -pkF_k(t) + p(k-1)F_{k-1}(t) + \sum_{\ell\geq k} F_\ell(t)\binom{\ell}{k}p^k(1-p)^{\ell-k} \\
&\qquad - \delta k F_k(t) + \delta(k+1)F_{k+1}(t) - F_k(t) + O(h),
\end{aligned}
$$

and (3.1) holds. Computing the Kolmogorov forwards equation for Z shows that for $k = 0, 1, 2, ...$

$$
\begin{aligned}
\tfrac{d}{dt}\mathbb{P}(Z_t = k) &= -(1 + pk + \delta k)\mathbb{P}(Z_t = k) + p(k-1)\mathbb{P}(Z_t = k-1) + \delta(k+1)\mathbb{P}(Z_t = k+1) \\
&\qquad + \sum_{\ell \geq k} \mathbb{P}(Z_t = \ell)\binom{\ell}{k}p^k(1-p)^{\ell-k},
\end{aligned}
$$

which is the same relation as (3.1) after taking expectation and letting $h \to 0$. This shows (3.2). □

3.2 Properties of the piecewise deterministic jump process X

We have seen the connection of $PD(p, \delta)$ to a branching process with disasters in Lemma 3.3. Such branching processes are in turn closely connected to piecewise deterministic jump processes as in Lemma 2.3 (Hermann and Pfaffelhuber, 2018). Hence, we can now prove Lemma 2.3.

Proof of Lemma 2.3. Lemma 3.3 implies that $\mathbb{E}[H_x(t)] = \mathbb{E}[(1-x)^{Z_t}]$. Recognizing $(Z_t) = Z(p, \delta, p)$ as a homogeneous branching process with disasters $Z^h_{\lambda,q,1,p}$ in the sense of Hermann and Pfaffelhuber, 2018, Definition 2.5, with death-rate $\lambda = p + \delta$ and offspring distribution $q = (q_0, 0, q_2, 0, \ldots)$ holding $q_2 = \frac{p}{p+\delta} = 1 - q_0$, the result follows from Lemma 4.1 in Hermann and Pfaffelhuber (2018). □

For the process X, we now obtain a property which is needed in the proofs of Theorem 1 and Proposition 2.5.

Lemma 3.4 (Moments of X)**.** *Let X be as in Lemma 2.3. If $\delta > p - p\log\frac{1}{p}$, then $\mathbb{E}_x[X_t^k] = o(\mathbb{E}_x[X_t])$ for all $k = 2, 3, ...$ and $\mathbb{E}_x[X_t] \sim ce^{-t(1+\delta-2p)}$, where*

$$c := x \cdot \exp\Big(- p\int_0^\infty \frac{\mathbb{E}_x[X_s^2]}{\mathbb{E}_x[X_s]}ds\Big) \in (0, 1).$$

Proof. Recall $\gamma := \log\frac{1}{p}/(p - \delta)$. Indeed, for $p - p\log\frac{1}{p} < \delta \leq p - p^2\log\frac{1}{p}$, such that $\gamma \in (p^{-1}, p^{-2})$, it follows from Corollary 2.4 of Hermann and Pfaffelhuber (2018) that – independent of x –

$$-\frac{1}{t}\log\Big(\frac{\mathbb{E}_x[X_t^2]}{\mathbb{E}_x[X_t]}\Big) \xrightarrow{t\to\infty} 1 - \frac{1 + \log\gamma}{\gamma} - (1 + \delta - 2p) = 2p \cdot c_2(p, \delta) > 0.$$

On the other hand, if $\delta \geq p - p^2 \log \frac{1}{p}$, the same corollary gives

$$-\frac{1}{t} \log \Big(\frac{\mathbb{E}_x[X_t^2]}{\mathbb{E}_x[X_t]} \Big) \xrightarrow{t\to\infty} 1 + 2\delta - 2p - p^2 - (1 + \delta - 2p)$$

$$= \delta - p^2 \geq p^2(\tfrac{1}{p} - 1 - \log \tfrac{1}{p}) > 0.$$

In either case, there is an $\varepsilon > 0$ such that $0 < r(s) := \mathbb{E}_x[X_s^2]/\mathbb{E}_x[X_s] = O(e^{-\varepsilon s})$ and it follows from (2.5)

$$\mathbb{E}_x[X_t] = x \exp\Big(- t(1 + \delta - 2p) - p \int_0^t r(s)ds \Big),$$

conluding the proof. □

3.3 Proof of Theorem 1

By (3.1) in Lemma 3.3, we get that, as $h \to 0$

$$\tfrac{1}{h}\mathbb{E}[F_0(t+h) - F_0(t)|G_t] \to -F_0(t) + \delta F_1(t) + \sum_{\ell=0}^{\infty}(1-p)^\ell F_\ell(t) = \delta F_1(t) + \sum_{\ell=1}^{\infty}(1-p)^\ell F_\ell(t) \geq 0.$$

Hence, $(F_0(t))_t$ is a bounded sub-martingale and converges almost surely and in $\mathcal{L}^1$. Consequently, the left hand side has to converge to 0 almost surely. Since $(1-p)^\ell$ is always positive, that can only be the case if $F_\ell(t) \to 0$ almost surely for all $\ell = 1, 2, ...$, which guarantees almost sure convergence of $F(t)$ to a vector of the form $F(\infty) = (F_0, 0, 0, \dots)$ in all cases.

Let $Z := (Z_t)_{t\geq 0} := Z(p, \delta, p)$ be as in Definition 3.1. We note that $\mathbb{E}[F_+(t)] = \mathbb{P}(Z_t > 0)$ by (3.2). For 1., we see from Lemma 2.3 and Lemma 3.4 that

$$\mathbb{E}[F_+(t)] = 1 - \mathbb{E}[H_1(t)] = 1 - \sum_{k=0}^{\infty} F_k(0)\mathbb{E}_1[(1-X_t)^k]$$

$$= \sum_{k=1}^{\infty} kF_k(0)\mathbb{E}_1[X_t] + o(\mathbb{E}_1[X_t]) \sim B_1(0) \cdot ce^{-t(1+\delta-2p)}$$

with c as in Theorem 1.1. Moreover, 2. follows directly from Corollary 2.7 in Hermann and Pfaffelhuber (2018); see Remark 3.2.2. by setting $b = p$ and $d = \delta$. For 3., we again use Corollary 2.7 in Hermann and Pfaffelhuber (2018), but use in addition that

$$\mathbb{E}[F_0(t)] = \mathbb{P}(Z_t = 0) = \sum_{k=0}^{\infty} \mathbb{P}_k(Z_t = 0) \cdot \mathbb{P}(Z_0 = k),$$

and $\sum_{k\geq \ell} F_k(0)\binom{k}{\ell} = B_\ell(0)$. □

3.4 Proof of Proposition 2.5

We set

$$I_x(t) := 1 - H_x(t) = 1 - \sum_{k=0}^{\infty}(1-x)^k F_k(t).$$

Using the duality relation in (2.2) and Bernoulli's formula we compute

$$\begin{aligned}
\mathbb{E}[I_x(t)] &= 1 - \sum_{k=0}^{\infty} F_k(0)\mathbb{E}_x[(1-X_t)^k] \\
&= 1 - \sum_{\ell=0}^{\infty} \mathbb{E}_x[X_t^\ell](-1)^\ell \sum_{k\geq \ell} F_k(0)\binom{\ell}{k} \\
&= 1 - \sum_{\ell=0}^{\infty} (-1)^\ell \mathbb{E}_x[X_t^\ell] \cdot B_\ell(0) \\
&= \sum_{\ell=1}^{\infty} B_\ell(0)(-1)^{\ell+1}\mathbb{E}_x[X_t^\ell].
\end{aligned}$$

For 1., we obtain $\mathbb{E}[I_x(t)] \sim B_1(0)\mathbb{E}_x[X_t]$ from the last display together with Lemma 3.4.
For 3. suppose that the limits do exist. Then, we see from Remark 2.7 that $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k] \xrightarrow{t\to\infty} c_k(p,\delta)$ with $c_k(p,\delta)$ as in (2.4). Hence,

$$\frac{\mathbb{E}[I_x(t)]}{\mathbb{E}_x[X_t]} = \sum_{\ell=1}^{\infty} B_\ell(0)(-1)^{\ell+1} \prod_{k=1}^{\ell-1} \frac{\mathbb{E}_x[X_t^{k+1}]}{\mathbb{E}_x[X_t^k]} \xrightarrow{t\to\infty} \sum_{\ell=1}^{\infty} B_\ell(0)(-1)^{\ell+1} \prod_{k=1}^{\ell-1} c_k(p,\delta).$$

Also, the last part of Remark 2.7 shows, given that $\mathbb{E}_x[X_t^{k+1}]/\mathbb{E}_x[X_t^k] \xrightarrow{t\to\infty} c_k(p,\delta)$, that $\mathbb{E}[X_t^k] = o(\mathbb{E}[X_t])$ for all $k \geq 2$ such that 2. follows analogously to 1.
Noting that in any case the limit of $\mathbb{E}[I_x(t)]/\mathbb{E}_x[X_t]$ does not depend on x and using that $I_1(t) = F_+(t)$, we see that $\mathbb{E}[I_x(t)]/\mathbb{E}[F_+(t)] \sim \mathbb{E}_x[X_t]/\mathbb{E}_1[X_t]$ and finally, (2.3) is a consequence of (2.5).

4 Proof of Theorem 2

The proof of Theorem 2, which is carried out in Section 4.4, will be based on the analysis of several martingales, which are derived in Proposition 4.5 in Section 4.3. In Section 4.2, we will analyze the total size of G_t.

4.1 Two auxiliary functions

We will need two specific functions in the sequel, which we now analyze.

Lemma 4.1. *Let $p \in (0,1)$, $\delta \geq 0$ and*

$$g : \begin{cases} [0,\infty) & \to \mathbb{R}, \\ x & \mapsto 1 + \delta x - px - p^x. \end{cases}$$

Then, g is strictly concave and thus, $x \mapsto g(x)/x$ strictly decreases. Also, the following holds:

1. *If $\delta \geq p$, g is strictly increasing and,*

$$g(x) \xrightarrow{x\to\infty} \begin{cases} \infty, & \text{if } p < \delta, \\ 1, & \text{if } p = \delta. \end{cases}$$

2. *If $p - \log\frac{1}{p} < \delta < p$,*

$$g \text{ is } \begin{cases} \text{strictly increasing on } (0,\xi), \\ \text{strictly decreasing on } (\xi,\infty) \end{cases}$$

for $\xi := \log\gamma/\log\frac{1}{p}$ with $\gamma := \log\frac{1}{p}/(p-\delta)$. The global maximum is $g(\xi) = 1 - \frac{1}{\gamma}(1+\log\gamma)$.

3. If $\delta \le p - \log \frac{1}{p}$, g strictly decreases and its maximum is $g(0) = 0$.

Proof. All results are straight-forward to compute. First, $g'(x) = \delta - p + \log \frac{1}{p} \cdot p^x$ for all cases. Since the right hand side strictly increases, g is strictly concave. 1. follows from the form of g'. For 3., we have that $g'(x) \le \delta - p + \log \frac{1}{p} \le 0$, implying the result. For 2., we have that $g'(x) = 0$ iff $p^{-x} = \log \frac{1}{p}/(p-\delta) = \gamma$ iff $x = \log \gamma / \log \frac{1}{p} = \xi = \log \gamma/(\gamma(p-\delta))$ and the rest follows. □

Lemma 4.2. *Let $g^r(n) := \frac{\Gamma(n+r)}{\Gamma(n)}$ and $n_0 \ge 2$. Then, there are $0 < c_r \le 1 < C_r < \infty$, such that*

$$c_r n^r \le g^r(n) \le C_r n^r \text{ for all } n \ge n_0. \tag{4.1}$$

Proof. First, we note that $g^r(n) \sim n^r$ as $n \to \infty$ (see e.g. 6.1.46. of Abramowitz and Stegun, 1964) and hence, the result follows. □

4.2 The size of the graph

For the asymptotics of the functionals of the random graph in Theorem 2 it will be helpful to understand the asymptotics of the process $(|V_t|)$. Here and below, we will frequently use the following well-known lemma.

Lemma 4.3. *Let $X = (X_t)_{t\ge 0}$ be a Markov process with complete and separable state space (E, r), and $f : E \to \mathbb{R}$ continuous and bounded and such that*

$$\lim_{h\to 0} \tfrac{1}{h}\mathbb{E}[f(X_{t+h}) - f(X_t)|X_t = x] = \lambda f(x), \qquad x \in E$$

for some $\lambda \in \mathbb{R}$, then $(e^{-t\lambda} f(X_t))_{t\ge 0}$ is a martingale.

Proof. See Lemma 4.3.2 of Ethier and Kurtz (1986). □

Lemma 4.4 (Graph size)**.** *Let $g^r(n) := \Gamma(n + r)/\Gamma(n)$. For all $r > -(|V_0| + 1)$, the process $(e^{-tr} g^r(|V_t| + 1))_{t\ge 0}$ is a non-negative martingale. Moreover, there is a random variable V_∞ such that the following holds:*

$$\begin{aligned}
&e^{-t}|V_t| \xrightarrow{t\to\infty} V_\infty \text{ almost surely and in } \mathcal{L}^r \text{ for all } r \ge 1,\\
&e^{t}/|V_t| \xrightarrow{t\to\infty} 1/V_\infty \text{ almost surely and in } \mathcal{L}^r \text{ for } 1 \le r < |V_0| + 1,\\
&V_\infty \text{ is } \Gamma(|V_0| + 1, 1)\text{-distributed.}
\end{aligned}$$

Proof. Let $0 < c_r < 1 < C_r < \infty$ be as in Lemma 4.2. The process $V := (|V_t|)_{t\ge 0}$ is a Markov process which jumps from v to $v + 1$ at rate $v + 1$. Setting $g^r(v) = \Gamma(v + r)/\Gamma(v)$, we see that the process $(g^r(|V_t| + 1))_{t\ge 0}$ is well-defined and non-negative if $|V_t| + 1 + r > 0$ for all t, i.e. if $r > -(|V_0| + 1)$. Then, as $h \to 0$,

$$\begin{aligned}
\tfrac{1}{h}\mathbb{E}[g^r(V_{t+h}) - g^r(V_t)|V_t = v] &= (v + 1)(g^r(v + 2) - g^r(v + 1)) + o(1)\\
&= (v + 1)g^r(v + 1)\Big(\frac{v + 1 + r}{v + 1} - 1\Big) + o(1) = rg^r(v + 1) + o(1)
\end{aligned}$$

and Lemma 4.3 implies that $(e^{-tr} g^r(|V_t| + 1))_{t\ge 0}$ is a (non-negative) martingale for all $r > -(|V_0| + 1)$, and therefore $\mathcal{L}^1$-bounded. By the martingale convergence theorem, this martingale converges almost surely. Using (4.1), the martingale $(e^{-t} g^1(|V_t| + 1))_t = (e^{-t}(|V_t| + 1))_t$ is $\mathcal{L}^r$-bounded for every $r \ge 1$ and therefore converges in $\mathcal{L}^r$. Analogously, for $r = -1$, the martingale $(e^t g^{-1}(|V_t| + 1))_t = (e^t/|V_t|)_t$ is $\mathcal{L}^r$-bounded for $1 \le r < |V_0| + 1$ and converges in $\mathcal{L}^r$.

Noting that $(|V_t| + 1)_{t\ge 0}$ is a Yule-process starting in $|V_0| + 1$, we have that $|V_t| + 1$ is distributed as the sum of $|V_0| + 1$ independent, geometrically distributed random variables with success probabilities e^{-t} (see e.g. p. 109 of Athreya and Ney, 1972). Hence, as $t \to \infty$, we find that $e^{-t}|V_t|$ converges in distribution to the sum of $|V_0| + 1$ independent, exponentially distributed random variables with unit rate. This is a $\Gamma(|V_0| + 1, 1)$ distribution. □

4.3 Some martingales

Similarly to the discrete-time pure duplication graph in Hermann and Pfaffelhuber (2016) we obtain martingales for the functionals of $PD(p,\delta)$.

Proposition 4.5 (Martingales)**.**

1. *Considering the function g of Lemma 4.1, it holds*
   (a) *if $g(1) \leq g(k)$, $(e^{tg(1)}B_k(t))_{t\geq 0}$ is a martingale that almost surely converges to a limit $B_k(\infty) \in \mathcal{L}^1$.*
   (b) *if $g(1) > g(k)$, there is a process $R_k(t)$ such that $(e^{tg(k)}(B_k(t)+R_k(t)))_{t\geq 0}$ is a positive martingale that almost surely converges to a limit $B_k(\infty) \in \mathcal{L}^1$ and $e^{tg(k)}R_k(t) \xrightarrow{t\to\infty} 0$. In particular, $e^{tg(k)}B_k(t) \to B_k(\infty)$ almost surely as $t \to \infty$.*

   Combining (a) and (b), we find $e^{t(g(1)\wedge g(k))}B_k(t) \xrightarrow{t\to\infty} B_k(\infty) \in \mathcal{L}^1$.

2. *For $k = 2,3,...$, $(e^{-t(kp^{k-1}-1-\delta\binom{k}{2})}C_k(t)/|V_t|)_{t\geq 0}$ is a martingale that converges almost surely to a limit $\tilde{C}_k(\infty)$. If additionally $C_k(0) > 0$ and $\delta < 2p^{k-1}/(k-1)$, the convergence also holds in $\mathcal{L}^2$.*

3. *Let $i \leq |V_0|$. Then, $(e^{-t(p-\delta-1)}D_i(t)/|V_t|)_{t\geq 0}$ is a martingale that converges almost surely to a limit $\tilde{D}_i(\infty) \in \mathcal{L}^1$. Moreover,*
$$\mathbb{E}[e^{-t(p-\delta)}D_i(t)] = D_i(0)\Big(1+(1-e^{-t})\frac{p}{|V_0|}\Big) \tag{4.2}$$
   and for $r \geq 2$, there is $C > 0$, depending only on r, p, δ such that
$$\mathbb{E}[(e^{-t(p-\delta)}D_i(t))^r] \leq \mathbb{E}[(D_i(0))^r] + C\int_0^t e^{-s(p-\delta)}\mathbb{E}[(e^{-s(p-\delta)}D_i(s))^{r-1}]ds. \tag{4.3}$$

Proof. 1. Since the sum in $B_k(t)$ is almost surely finite for every k and t, it follows for $h \to 0$ using equation (3.1), that

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[B_k(t+h)-B_k(t)|G_t]\\
&\quad= \sum_{\ell\geq k}\binom{\ell}{k}\Bigg(-(1+p\ell+\delta\ell)F_\ell(t)+p(\ell-1)F_{\ell-1}(t)+\delta(\ell+1)F_{\ell+1}(t)\\
&\qquad\qquad + \sum_{m\geq\ell}\binom{m}{\ell}p^\ell(1-p)^{m-\ell}F_m(t)\Bigg)+o(1)\\
&\quad= -B_k(t)+p(k-1)F_{k-1}(t)+\sum_{m\geq k}F_m(t)\sum_{\ell=k}^m\binom{\ell}{k}\binom{m}{\ell}p^\ell(1-p)^{m-\ell}\\
&\qquad + \sum_{\ell\geq k}F_\ell(t)\Bigg(\underbrace{-(p+\delta)\ell\binom{\ell}{k}+p\ell\binom{\ell+1}{k}+\delta\ell\binom{\ell-1}{k}}_{=:a(\ell,k)}\Bigg)+o(1).
\end{aligned}$$

Considering that $\binom{n+1}{m}-\binom{n}{m}=\binom{n}{m-1}$ and $\frac{n}{m}\cdot\binom{n-1}{m-1}=\binom{n}{m}$, we deduce

$$\begin{aligned}
a(\ell,k) &= p\ell\binom{\ell}{k-1}-\delta\ell\binom{\ell-1}{k-1} = (p-\delta)\ell\binom{\ell-1}{k-1}+p\ell\binom{\ell-1}{k-2}\\
&= (p-\delta)k\binom{\ell}{k}+p(k-1)\binom{\ell}{k-1},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[B_k(t+h)-B_k(t)|G_t]\\
&=-B_k(t)+p(k-1)B_{k-1}(t)+(p-\delta)kB_k(t)\\
&\qquad\qquad+\sum_{m\geq k}F_m(t)\underbrace{\sum_{\ell=0}^{m-k}\binom{m-k}{\ell}\binom{m}{k}p^{\ell+k}(1-p)^{m-k-\ell}}_{=\binom{m}{k}p^k}+o(1)\\
&=B_k(t)\big((p-\delta)k-(1-p^k)\big)+p(k-1)B_{k-1}(t)+o(1)\\
&=-g(k)B_k(t)+p(k-1)B_{k-1}(t)+o(1),
\end{aligned}$$

recalling the function $g: x\mapsto 1+\delta x-px-p^x$ from Lemma 4.1. In any case we see that $(e^{tg(1)}B_1(t))_{t\geq 0}$ is a non-negative martingale converging almost surely to a limit $B_1(\infty)\in\mathcal{L}^1$. For $k=2,3,...$ let $g_{\min}(k):=\min_{1\leq\ell\leq k}g(k)$ the running minimum of g. Then, there are two cases to consider:

1. $g(1)\leq g(k)$: It holds by strict concavity of g (see Lemma 4.1) that in this case $g(1)=g_{\min}(k)<g(\ell)$ for all $1<\ell<k$. Thus, letting

$$\lambda_m^k:=\frac{g(1)}{g(k)}\prod_{\ell=m}^{k-1}\frac{p\ell}{g(\ell)-g(1)},\qquad m=2,...,k$$

and $\lambda_1^k:=1+\frac{1}{g(1)}p\lambda_2^k$, these coefficients are well-defined and positive. Considering the linear combination $Q_k(t):=\sum_{m=1}^k\lambda_m^kB_m(t)$ we obtain

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[Q_k(t+h)-Q_k(t)|G_t]\\
&\quad=-g(1)B_k(t)+\sum_{m=1}^{k-1}B_m(t)(-\lambda_m^kg(m)+\lambda_{m+1}^kpm)+o(1)\\
&\quad=-g(1)B_k(t)+\sum_{m=2}^{k-1}B_m(t)\lambda_m^k\Big(-g(m)+pm\frac{g(m)-g(1)}{pm}\Big)+B_1(t)\Big(-g(1)\lambda_1^k+p\lambda_2^k\Big)+o(1)\\
&\quad=-g(1)Q_k(t)+o(1).
\end{aligned}$$

So now, $(e^{tg(1)}Q_k(t))_{t\geq 0}$ is a non-negative martingale for every k. Since $B_k(t)$ can be represented as a linear combination of $(Q_\ell(t))_{1\leq\ell\leq k}$, also $(e^{tg(1)}B_k(t))_{t\geq 0}$ has to be a non-negative martingale and thus converges to some $B_k(\infty)\in\mathcal{L}^1$.

2. $g(1)>g(k)$: Here it holds by strict concavity of g that $g(\ell)>g(k)=g_{\min}(k)$ for all $\ell=1,...,k-1$. Hence

$$\lambda_m^k:=\prod_{\ell=m}^{k-1}\frac{p\ell}{g(\ell)-g(k)},\qquad m=1,...,k,$$

are well-defined and positive. We compute analogously to the first case that, as $h\to 0$, with $Q_k(t):=\sum_{m=1}^k\lambda_m^kB_m$,

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[Q_k(t+h)-Q_k(t)|G_t]\\
&\quad=-g(k)B_k(t)+\sum_{m=1}^{k-1}B_m(t)\lambda_m^k\Big(-g(m)+pm\frac{g(m)-g(k)}{pm}\Big)+o(1)=-g(k)Q_k(t)+o(1).
\end{aligned}$$

Thus, $(e^{tg(k)}Q_k(t))_{t\geq 0}$ is a non-negative martingale and has an almost sure limit $B_k(\infty) \in \mathcal{L}^1$. Moreover, for $\ell < k$, since $g(\ell) > g(k)$, we have that $e^{tg(k)}Q_\ell(t) \to 0$, so, writing $R_k(t) = Q_k(t) - B_k(t)$, we see that $R_k(t) = \sum_{\ell=1}^{k-1} \mu_\ell^k Q_\ell(t)$ for some $\mu_1^k, ..., \mu_{k-1}^k$ and $e^{tg(k)}R_k(t) \to 0$ and $e^{tg(k)}B_k(t) \to B_k(\infty)$ follows.

2. For the cliques fix $t \geq 0$ and let $N_k(v)$ for every node $v \in V_t$ denote the number of k-cliques that node is part of. Then, $\sum_{v\in V_t} N_k(v) = kC_k(t)$. Analogously define $M_k(e)$ as the number of cliques that the edge $e \in E_t$ is contained in, such that $\sum_{e\in E_t} M_k(e) = \binom{k}{2}C_k(t)$. Also, let $\tilde{C}_k(t) := C_k(t)/|V_t|$. Note that for a new k-clique to arise, a node v inside of such a clique has to be copied. Then, every of the $N_k(v)$ cliques v is part of has a chance of p^{k-1} that the copy obtains the $k-1$ edges it needs to form a new k-clique. Also, whenever an edge e is deleted, all $M_k(e)$ k-cliques are destroyed. We deduce

$$\begin{aligned}
\tfrac{1}{h}\mathbb{E}[\tilde{C}_k(t+h) - \tilde{C}_k(t)|G_t] &= \frac{|V_t|+1}{|V_t|}\sum_{v\in V_t}\Big(\frac{C_k(t) + p^{k-1}N_k(v)}{|V_t|+1} - \tilde{C}_k(t)\Big) - \delta\sum_{e\in E_t}\frac{M_k(e)}{|V_t|} + o(1) \\
&= \sum_{v\in V_t}\Big(\tilde{C}_k(t) + \frac{p^{k-1}N_k(v)}{|V_t|} - \frac{|V_t|+1}{|V_t|}\tilde{C}_k(t)\Big) - \frac{\delta}{|V_t|}\binom{k}{2}C_k(t) + o(1) \\
&= \tilde{C}_k(t)\underbrace{\Big(kp^{k-1} - 1 - \delta\binom{k}{2}\Big)}_{=:q_k} + o(1). \qquad (4.4)
\end{aligned}$$

This shows that $(e^{-tq_k} \cdot \tilde{C}_k(t))_{t\geq 0}$ is a non-negative martingale and hence converges almost surely to an integrable random variable $\tilde{C}_k(\infty)$.

It remains to show the $\mathcal{L}^2$-convergence of the martingale $(e^{-tq_k}\tilde{C}_k(t))_{t\geq 0}$ for $q_k + 1 > 0$, i.e. $\delta < 2p^{k-1}/(k-1)$. This will be done by considering the number of (unordered) pairs of k-cliques, $CC_k(t) := \binom{C_k(t)}{2} = (C_k(t)^2 - C_k(t))/2$, and verifying that the process given by $\widetilde{CC}_k(t) := e^{-t\cdot 2q_k}\frac{CC_k(t)}{|V_t|(|V_t|-1)}$ is $\mathcal{L}^1$-bounded, which implies $\mathcal{L}^2$-boundedness of the martingale $(e^{-tq_k}\cdot\tilde{C}_k(t))_{t\geq 0}$ and concludes the proof.

Let us denote by $C_{k,\ell}(t)$ the number of (unordered) pairs of k-cliques which have exactly ℓ shared vertices. Since the *overlap* of such a pair (i.e. the sub-graph both cliques have in common) is an ℓ-clique with $\binom{\ell}{2}$ edges, the number of edges making up the pair equals $2\binom{k}{2} - \binom{\ell}{2}$. Hence, arguing as in the proof of Theorem 2.9 in Hermann and Pfaffelhuber (2016), considering that (i) one new such pair arises if one of the $2(k-\ell)$ non-shared vertices is fully copied (probability p^{k-1}), and (ii) one new pair arises if one of the ℓ shared vertices is fully copied (probability $p^{2k-\ell-1}$), by taking the copied node instead of the original one; in addition, there are two new pairs of k-cliques, one original and one copied, which share $\ell - 1$ vertices, and (iii) if one of the ℓ shared vertices is chosen, but only one of the two cliques is fully copied (probability $2p^{k-1}(1-p^{k-\ell})$) one new pair of k-cliques arises, which shares $\ell - 1$ vertices. In addition, such a pair will be destroyed if one of its edges is deleted, hence we deduce for $\ell \leq k-2$

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[C_{k,\ell}(t+h) - C_{k,\ell}(t) \mid G_t] \\
&\quad= (|V_t|+1)\cdot\Big(\frac{2(k-\ell)p^{k-1} + \ell p^{2k-\ell-1}}{|V_t|}C_{k,\ell}(t) + \frac{2(\ell+1)p^{k-1}}{|V_t|}C_{k,\ell+1}(t)\Big) \\
&\qquad\qquad - \delta\cdot\Big(2\binom{k}{2} - \binom{\ell}{2}\Big)\cdot C_{k,\ell}(t) + o(1),
\end{aligned}$$

which implies for $\widetilde{C}_{k,\ell}(t) := e^{-t\cdot 2q_k}\frac{C_{k,\ell}(t)}{|V_t|(|V_t|-1)}$, that

$$
\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[\widetilde{C}_{k,\ell}(t+h) - \widetilde{C}_{k,\ell}(t) \mid G_t] \\
&\quad = -2q_k\widetilde{C}_{k,\ell}(t) + e^{-t\cdot 2q_k}(|V_t|+1) \\
&\qquad\qquad \cdot \Bigg(\frac{(2(k-\ell)p^{k-1} + \ell p^{2k-\ell-1})C_{k,\ell}(t) + 2(\ell+1)p^{k-1}C_{k,\ell+1}(t)}{|V_t|\cdot(|V_t|+1)|V_t|} \\
&\qquad\qquad\qquad + C_{k,\ell}(t)\cdot\Big(\frac{1}{(|V_t|+1)|V_t|} - \frac{1}{|V_t|(|V_t|-1)}\Big)\Bigg) \\
&\qquad\qquad\qquad\qquad - 2\delta\binom{k}{2}\widetilde{C}_{k,\ell}(t) + \delta\binom{\ell}{2}\widetilde{C}_{k,\ell}(t) + o(1) \\
&\quad = \widetilde{C}_{k,\ell}(t)\Bigg(-2q_k - 2\delta\binom{k}{2} + \delta\binom{\ell}{2} + (2(k-\ell)p^{k-1} + \ell p^{2k-\ell-1})\cdot\frac{|V_t|-1}{|V_t|} - 2\Bigg) \\
&\qquad + \widetilde{C}_{k,\ell+1}(t)\cdot 2(\ell+1)p^{k-1}\cdot\frac{|V_t|-1}{|V_t|} + o(1) \\
&\quad \le \widetilde{C}_{k,\ell}(t)\Big(-2\ell p^{k-1} + \ell p^{2k-\ell-1} + \delta\binom{\ell}{2}\Big) + \widetilde{C}_{k,\ell+1}(t)\cdot 2(\ell+1)p^{k-1} + o(1) \\
&\quad = -\ell\widetilde{C}_{k,\ell}(t)\Big(p^{k-1}(2-p^{k-\ell}) - \tfrac{\delta}{2}(\ell-1)\Big) + 2(\ell+1)p^{k-1}\widetilde{C}_{k,\ell+1}(t) + o(1).
\end{aligned}
$$

Analogously, for $\ell = k-1$, additional pairs arise if a clique with k vertices is completely copied (probability p^{k-1}), so

$$
\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[\widetilde{C}_{k,k-1}(t+h) - \widetilde{C}_{k,k-1}(t) \mid G_t] \\
&\quad \le -(k-1)\widetilde{C}_{k,k-1}(t)\Big(p^{k-1}(2-p) - \tfrac{\delta}{2}(k-2)\Big) + 2kp^{k-1}e^{-t\cdot 2q_k}\cdot\frac{C_k(t)}{|V_t|(|V_t|-1)} + o(1).
\end{aligned}
$$

Also, letting $\widehat{C}_k(t) := e^{-t\cdot 2q_k}C_k(t)/(|V_t|(|V_t|-1))$ and combining the calculation above with the one in (4.4), it follows that

$$
\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[\widehat{C}_k(t+h) - \widehat{C}_k(t) \mid G_t] \\
&\quad = -2q_k\widehat{C}_k(t) + e^{-t\cdot 2q_k}\Bigg(\frac{kp^{k-1}C_k(t)}{|V_t|(|V_t|-1)}\cdot\frac{|V_t|-1}{|V_t|} + \frac{C_k(t)}{|V_t|} - C_k(t)\frac{|V_t|+1}{|V_t|(|V_t|-1)}\Bigg) - \delta\binom{k}{2}\widehat{C}_k(t) + o(1) \\
&\quad \le \widehat{C}_k(t)\Big(-2q_k + kp^{k-1} - 2 - \delta\binom{k}{2}\Big) + o(1) \\
&\quad = -\Big(kp^{k-1} - \delta\binom{k}{2}\Big)\widehat{C}_k(t) = -k\widehat{C}_k(t)\Big(p^{k-1}(2-p^0) - \tfrac{\delta}{2}(k-1)\Big) + o(1).
\end{aligned}
$$

Now, since

$$\delta < \frac{2p^{k-1}}{k-1} = \min_{2\le m\le k}\Big\{\frac{2p^{k-1}(2-p^{k-m})}{m-1}\Big\},$$

the coefficients given by

$$\lambda_\ell := \prod_{m=1}^{\ell}\frac{2p^{k-1}}{p^{k-1}(2-p^{k-m}) - \frac{\delta}{2}(m-1)}$$

for $1 \le \ell \le k$ are well-defined and positive and we obtain for the linear combination

$$R_k(t) := \widetilde{C}_{k,0}(t) + \sum_{\ell=1}^{k-1}\lambda_\ell\widetilde{C}_{k,\ell}(t) + \lambda_k\widehat{C}_k(t)$$

that

$$\lim_{h\to 0} \tfrac{1}{h}\mathbb{E}[R_k(t+h) - R_k(t) \mid G_t]$$

$$\leq \sum_{\ell=1}^{k-1} \widetilde{C}_{k,\ell}(t)\cdot \ell\Big(-\lambda_\ell\big(p^{k-1}(2-p^{k-\ell}) - \tfrac{\delta}{2}(\ell-1)\big) + \lambda_{\ell-1}2p^{k-1}\Big)$$

$$+ \widehat{C}_k(t)\cdot k\Big(-\lambda_k\big(p^{k-1}(2-p^{k-k}) - \tfrac{\delta}{2}(k-1)\big) + \lambda_{k-1}2p^{k-1}\Big) = 0.$$

Thus, $(R_k(t))$ is a non-negative super-martingale, $\mathcal{L}^1$-bounded and, since $\lambda_{\min} := \min(\{1\} \cup \{\lambda_\ell; 1\leq \ell\leq k\}) > 0$ and $\widetilde{CC}_k(t) \leq R_k(t)/\lambda_{\min}$, the proof of 2. is complete.

3. For the degree $D_i(t)$, we set $\tilde{D}_i(t) = D_i(t))/|V_t|$ and compute, as $h\to 0$,

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[\tilde{D}_i(t+h) - \tilde{D}_i(t) \mid G_t] \\
&= (|V_t|+1)\Bigg(\frac{pD_i(t)}{|V_t|}\Big(\frac{D_i(t)+1}{|V_t|+1} - \frac{D_i(t)}{|V_t|}\Big) + \Big(1-\frac{pD_i(t)}{|V_t|}\Big)\Big(\frac{D_i(t)}{|V_t|+1} - \frac{D_i(t)}{|V_t|}\Big)\Bigg) \\
&\qquad + \delta D_i(t)\Big(\frac{D_i(t)-1}{|V_t|} - \frac{D_i(t)}{|V_t|}\Big) + o(1) \\
&= \frac{D_i(t)}{|V_t|}\Bigg(p\frac{|V_t| - D_i(t)}{|V_t|} - \Big(1 - \frac{pD_i(t)}{|V_t|}\Big) - \delta\Bigg) + o(1) \\
&= \tilde{D}_i(t)(p-\delta-1) + o(1).
\end{aligned}$$

Lemma 4.3 shows that $(e^{-t(p-\delta-1)}D_i(t)/|V_t|)_{t\geq 0}$ is a non-negative martingale, and hence converges almost surely. Furthermore, we write with $g^r(n) := \Gamma(n+r)/\Gamma(n)$

$$\begin{aligned}
&\tfrac{1}{h}\mathbb{E}[g^r(D_i(t+h) - g^r(D_i(t)) \mid G_t] \\
&= (|V_t|+1)\frac{pD_i(t)}{|V_t|}(g^r(D_i(t)+1) - g^r(D_i(t))) + \delta D_i(t)(g^r(D_i(t)-1) - g^r(D_i(t))) + o(1) \\
&= g^r(D_i(t))\Big(pr\frac{|V_t|+1}{|V_t|} + \delta D_i(t)\Big(\frac{D_i(t)-1}{D_i(t)+r-1} - 1\Big)\Big) + o(1) \\
&= g^r(D_i(t))r(p-\delta) + g^r(D_i(t))r\Big(p\Big(\frac{|V_t|+1}{|V_t|} - 1\Big) - \delta\Big(\frac{D_i(t)}{D_i(t)+r-1} - 1\Big)\Big) + o(1) \\
&= g^r(D_i(t))r(p-\delta) + g^r(D_i(t))r\Big(p\frac{1}{|V_t|} + \delta(r-1)\frac{1}{D_i(t)+r-1}\Big) + o(1).
\end{aligned} \tag{4.5}$$

Letting $h\to 0$, this gives (4.2) for $r=1$ since, using the martingale $(e^{-t(p-\delta-1)}D_i(t)/|V_t|)_{t\geq 0}$,

$$\begin{aligned}
\mathbb{E}[e^{-t(p-\delta)}D_i(t)] &= D_i(0) + \int_0^t p\mathbb{E}[e^{-s(p-\delta)}D_i(t)/|V_s|]ds \\
&= D_i(0) + \int_0^t pe^{-s}D_i(0)/|V_0|ds.
\end{aligned}$$

Moreover, since $\frac{g^r(D_i(t))}{D_i(t)+r-1} = g^{r-1}(D_i(t))$, (4.5) gives for some $C>0$, depending on r,p,δ

$$\frac{d}{dt}\mathbb{E}[e^{-tr(p-\delta)}g^r(D_i(t))] \leq C\cdot \mathbb{E}[e^{-tr(p-\delta)}g^{r-1}(D_i(t))],$$

and (4.3) follows with Lemma 4.2 and integration. □

4.4 Proof of Theorem 2

1. Recalling the function g from Lemma 4.1, we note that (see also Remark2.8.1 for the second equality) $\beta_k = (1+\delta-2p) \wedge 1 + (\delta - p)k - p^k = g(1) \wedge g(k)$. Hence, Lemma 4.5.1 shows that $e^{t\beta_k} B_k(t)$ is non-negative and converges to some $B_k(\infty) \in \mathcal{L}^1$. So, 1. follows.

2. We combine Lemma 4.5.2 (recall the random variable $\tilde{C}_k(\infty)$) with the almost sure convergence $e^{-t}V_t \xrightarrow{t\to\infty} V_\infty$ from Lemma 4.4. In all cases, we have that

$$\begin{aligned}\exp\Big(-t\Big(kp^{k-1} - \delta\binom{k}{2}\Big)\Big)C_k(t) &= \exp\Big(-t\Big(kp^{k-1} - 1 - \delta\binom{k}{2}\Big)\Big)C_k(t)/|V_t| \cdot e^{-t}|V_t| \\ &\xrightarrow{t\to\infty} \tilde{C}_k(\infty)\cdot V_\infty =: C_k(\infty),\end{aligned} \tag{4.6}$$

where $V_\infty > 0$ almost surely.
If $\delta \geq 2p^{k-1}/(k-1)$, it is $kp^{k-1} - \delta\binom{k}{2} \leq 0$ and the convergence can only hold if $C_k(t) \xrightarrow{t\to\infty} 0$ almost surely. Since $C_k(t) \in \mathbb{N}_0$, the first hitting time T^{C_k} of 0 has to be finite. On the other hand, for $\delta < 2p^{k-1}/(k-1)$, combining the $\mathcal{L}^2$-convergences in Lemma 4.5.2 and Lemma 4.4 we obtain that the convergence in (4.6) also holds in $\mathcal{L}^1$. Since $(e^{-t(kp^{k-1}-1-\delta\binom{k}{2})}C_k(t)/|V_t|)$ is an $\mathcal{L}^2$-convergent and thus uniformly integrable martingale, $\mathbb{P}(C_k(\infty) > 0) = \mathbb{P}(\tilde{C}_k(\infty) > 0) > 0$.

3. Finally, fix $i \in \{1, \ldots, |V_0|\}$. Again, we combine Lemma 4.5.3 (recall the random variable $\tilde{D}_i(\infty)$) with the almost sure convergence $e^{-t}V_t \xrightarrow{t\to\infty} V_\infty$ from Lemma 4.4. In all cases, we have that

$$e^{-t(p-\delta)}D_i(t) = e^{-t(p-\delta-1)}\frac{D_i(t)}{|V_t|}e^{-t}|V_t| \xrightarrow{t\to\infty} \tilde{D}_i(\infty)\cdot V_\infty =: D_i(\infty). \tag{4.7}$$

If $\delta < p$, we find by (4.2) that $(e^{-t(p-\delta)}D_i(t))_{t\geq 0}$ is $\mathcal{L}^1$-bounded. Then, inductively using (4.3) shows that $(e^{-t(p-\delta)}D_i(t))_{t\geq 0}$ is $\mathcal{L}^r$-bounded for all $r \geq 1$. In particular, this implies that the convergence in (4.7) also holds in $\mathcal{L}^r$ for all $r \geq 1$. This gives convergence of first moments, and (2.6) follows by taking $t \to \infty$ in (4.2).
If $\delta \geq p$, the almost sure convergence in (4.7) implies, since $D_i(t) \in \mathbb{N}_0$, that $D_i(\infty) = 0$, so there must be a finite hitting time T^{D_i} of 0.

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