On some estimates for Erdös-Rényi random graph

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Abstract

We consider a number ν_n of components in a random graph $G(n,p)$ with n vertices, where the probability of an edge is equal to p. By operating with special generating functions we shows the next asymptotic relation for factorial moments of ν_n:

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} = (1+o(1))\left(\frac{1}{p}\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}(npq^n)^k\right)^s + o(1)$$

as n tends to ∞ and $q = 1-p$. And the following inequations hold:

$$1 - 2nq^{n-1} \leqslant p_n \leqslant \frac{1}{nq^n},$$

$$1 - \frac{1}{nq^n} \leqslant pi_n \leqslant nq^{n-1},$$

where p_n is the probability that $G(n,p)$ is connected and pi_n is the probability that $G(n,p)$ has an isolated vertex.

1 Notations

Let G_n be a set of undirected graphs with n labeled vertices. For any graph $g \in G_n$ let $C(g)$ be a number of connected components in the graph g and $E(g)$ be a number of edges in thr graph g. Besides we denote by $F_{s,n}$ the number of all forests in G_n, that contains exactly s trees. We also suppose that components in G_n are not ordered.

Further, let $A_{n,k,s}$ be a number of graphs in G_n, which contains n vertices, k edges and s components, $A_{n,k}$ be a number of graphs, which contains n vertices and k edges, and $B_{n,k}$ — a number of connected graphs with n vertices and k edges. For definiteness we suppose that $A_{0,k} = A_{0,k,s} = A_{n,k,0} = 0$ in all cases, except $n = k = s = 0$, where we set by definition $A_{0,0} = A_{0,0,0} = 1$. Besides, let $B_{0,k} = 0$ for all k. It's clear that $A_{n,k} = \sum_s A_{n,k,s}$, where index s runs on all integer non-nagative numbers.

Let us consider the random graph $G(n,p)$, which contains n labeled vertices, where each of $\binom{n}{2}$ edges is present with the probability p independently of other edges. Each concrete realization of random graph $G(n,p)$ is a graph from G_n.

This model of random graphs was firstly described by Erdös and Rényi in [1, 2] and then has been well studied by Béla Bollobás [3], Valentin Kolchin [4] and other authors.

It is easy to see that the parobability distribution of such random graph is defined as follows:

$$\mathsf{P}\{G(n,p) = g\} = (p/q)^{E(g)} q^{n(n-1)/2},$$

where $g \in G_n$ and $q = 1 - p$.

Let denote by ν_n the number of connected components of $G(n,p)$, i.e. $\nu_n = C(G(n,p))$, and let p_n be the probability that random graph $G(n,p)$ is connected, thus $p_n = \mathsf{P}\{\nu_n = 1\}$. It's clear that

$$\mathsf{P}\{\nu_n = s\} = \sum_{k=0}^{\infty} A_{n,k,s} (p/q)^k q^{n(n-1)/2} \tag{1}$$

and

$$p_n = \sum_{k=0}^{\infty} B_{n,k} (p/q)^k q^{n(n-1)/2}.$$

Froom the above agreements it follows that $p_0 = 0$ and $p_1 = 1$.

Below we'll need the special generated function, which we define as follows: for a sequence of functions $\{r_n(q)\}$ we put

$$R = R(x,q) = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} r_n(q),$$

where we often will skip arguments x and q, except such cases when we will use special values of them. Below in this text we will call such functions as SG-functions (SG = special generated).

It is easy to see that SG-functions are formal power series which are not converges at all. But most of all usual operations with SG-functions (such as adding, production, differentiation and integration on both arguments) does not lead to conflicts when counting coefficients before x^n.

Let denote

$$\widehat{R} = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} \frac{dr_n(q)}{dq},$$

i.e. the operator $\widehat{}$ denotes SG-function for the sequence of derivatives of $r_n(q)$.

Let also:

$$A = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}}$$

$$B = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} p_n$$

$$E = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} \mathsf{E}\nu_n$$

$$M_k = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} \mathsf{E}(\nu_n)^{\underline{k}}$$

$$\mathcal{M}_k = \sum_{n=0}^{\infty} \frac{x^n}{n! q^{n(n-1)/2}} \mathsf{E}(\nu_n - 1)^{\underline{k}},$$

where $z^{\underline{k}} = z(z-1)\dots(z-k+1)$ denotes the factorial power $k \geqslant 0$. Therefore, A is a SG-function of $\{1\}$, B is a SG-function of probabilies that graph is connected, E is a SG-function of expectations of components quantity, M_k is a SG-function of k-th factorial moments of ν_n, and $\mathcal{M}_k$ is a SG-function of k-th factorial moments of $(\nu_n - 1)$. It's easy to see that A converges only if $q = 1$. Below we'll see that all of theese series are converges in the same conditions.

2 Basical Relations

Lemma 1. *If the relation $n = k = s = 0$ does not holds, then*

$$A_{n,k,s} = \sum_{\substack{n_1+\dots+n_s=n \\ k_1+\dots+k_s=k}} \frac{n!}{s!} \frac{B_{n_1,k_1} \cdots B_{n_s,k_s}}{n_1! \cdots n_s!}, \tag{2}$$

where the summation is over all integer non-negative n_i, k_i.

Proof. Let consider the set of graphs $\bar{G}_n$ with n vertices, where the components are ordered. It is clear that the number $\bar{A}_{n,k,s}$ of such graphs with n vertices, k edges and s components is equal to $s!A_{n,k,s}$.

By the other side, any graph from $\bar{G}_n$ with n vertices and s components we can make by getting some ordered partition of the set of n vertices with non-empty parts, which has the volumes $n_1, \dots, n_s$. The number of such partitions is equal to $n!/(n_1! \cdots n_s!)$. For every set of vertices, included in connected components, we can find the number of connected graphs with n_i vertices and k_i edges. It is equal to B_{n_i,k_i}. By choosing k_i in such a way that $k_1 + \dots + k_s = k$, and summing over all partitions of n vertices, we get the equation:

$$\bar{A}_{n,k,s} = \sum_{\substack{n_1+\dots+n_s=n \\ k_1+\dots+k_s=k}} \frac{n! B_{n_1,k_1} \cdots B_{n_s,k_s}}{n_1! \cdots n_s!}$$

From this we get (2) for positive n, n_i, s and non-negative k. Extention of this relation for zero values of n, n_i and s follows from the previous agreements. $\square$

Now we consider the next generated functions, which are exponential by parameter x:

$$A(x,y) = \sum_{n,k} \frac{A_{n,k}}{n!} x^n y^k, \qquad B(x,y) = \sum_{n,k} \frac{B_{n,k}}{n!} x^n y^k.$$

The summation is over integer non-negative n, k.

Lemma 2.

$$A(x,y) = e^{B(x,y)} \tag{3}$$

Proof. By multyplying the relation (2) by $x^n y^k / n!$ we get:

$$\frac{A_{n,k,s}}{n!} x^n y^k = \frac{1}{s!} \sum_{\substack{n_1+\cdots+n_s=n \\ k_1+\cdots+k_s=k}} \frac{B_{n_1,k_1} x^{n_1} y^{k_1} \cdots B_{n_s,k_s} x^{n_s} y^{k_s}}{n_1! \cdots n_s!} = [x^n y^k] B(x,y)^s.$$

The last notation denotes a coefficient before $x^n y^k$ in the series $B(x,y)^s$. Now, by summing over integer non-negative n, k for $s > 0$ we get the following:

$$\sum_{n,k} \frac{A_{n,k,s}}{n!} x^n y^k = \frac{1}{s!} B(x,y)^s \tag{4}$$

Note, that by virtue of the agreements this equation stays also true for $s = 0$. Finally, by summing over integer non-negative s we get:

$$A(x,y) = \sum_{s=0}^{\infty} \frac{B(x,y)^s}{s!} = e^{B(x,y)}.$$

$\square$

From the relation (3) we can obtain any exact expressions for probabilities of random graph $G(n,p)$. First of all, it is clear that:

$$A_{n,k} = \binom{n(n-1)/2}{k},$$

where we suppose that $\binom{m}{k} = 0$ for $k > m$. It is easy to see that

$$\sum_{k=0}^{\infty} \binom{n(n-1)/2}{k} y^k = \sum_{k=0}^{n(n-1)/2} \binom{n(n-1)/2}{k} y^k = (1+y)^{n(n-1)/2},$$

hence,

$$A(x,y) = \sum_{n=0}^{\infty} (1+y)^{n(n-1)/2} \frac{x^n}{n!}.$$

From this and from (3) it follows that

$$B(x,y) = \ln \sum_{n=0}^{\infty} (1+y)^{n(n-1)/2} \frac{x^n}{n!}. \tag{5}$$

One can see that $B(x,y)$ is the generated function for a sequence [5] where the nulled element is equal to zero.

By putting $y = p/q$ and from the obvious equtions

$$\sum_k A_{n,k} \left(\frac{p}{q}\right)^k q^{n(n-1)/2} = 1, \quad \sum_k B_{n,k} \left(\frac{p}{q}\right)^k q^{n(n-1)/2} = p_n,$$

we get that for previously defined series A and B the next relations are true:

$$\begin{aligned} A\left(x, \frac{p}{q}\right) &= \sum_{n,k} \frac{A_{n,k}}{n!} x^n (p/q)^k = \sum_n \frac{x^n}{q^{n(n-1)/2} n!} = A, \\ B\left(x, \frac{p}{q}\right) &= \sum_{n,k} \frac{B_{n,k}}{n!} x^n (p/q)^k = \sum_n \frac{p_n x^n}{q^{n(n-1)/2} n!} = B. \end{aligned} \tag{6}$$

Thus, we have

Lemma 3.

$$A = e^B.$$

This proved equation is the base fact, which we will use anythere below without a special link.

From (1) it follows that:

$$\sum_{n=0}^{\infty} \frac{\mathsf{P}\{\nu_n = s\}}{q^{n(n-1)/2}} \frac{x^n}{n!} = \sum_{n,k} \frac{A_{n,k,s}}{n!} x^n (p/q)^k,$$

and by (4), where we put $y = p/q$, we get following:

$$\sum_{n=0}^{\infty} \frac{x^n}{q^{n(n-1)/2} n!} \mathsf{P}\{\nu_n = s\} = \frac{1}{s!} B(x, p/q)^s = \frac{1}{s!} B^s, \tag{7}$$

i.e. the formal series $B^s/s!$ is SG-function of probabilities $\mathsf{P}\{\nu_n = s\}$ for a fixed number s of connected components.

Let us consider two SG-functions and their product:

$$R = \sum_{n=0}^{\infty} \frac{r_n x^n}{n! q^{n(n-1)/2}}, \quad T = \sum_{n=0}^{\infty} \frac{t_n x^n}{n! q^{n(n-1)/2}}, \quad RT = \sum_{n=0}^{\infty} \frac{z_n x^n}{n! q^{n(n-1)/2}}.$$

One can easily proof the following

Lemma 4 (Convolution Formula)**.** *For $n \geqslant 0$:*

$$z_n = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} r_k t_{n-k}.$$

Further we will use this formula without a special link to it. The next recursion formula for probabilities p_n is an anlogue of a recursion formula for a number of connected graphs, that was obtained in [6].

Lemma 5. *For any $n \geqslant 1$*

$$p_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k} q^{k(n-k)} p_{n-k}. \tag{8}$$

Proof. By differentiating the relation $A = e^B$ by the parameter x we get:

$$xA' = xAB',$$

hence, from the convolution formula it follows that

$$n = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} k p_k \tag{9}$$

Since $p_0 = 0$ and $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ follows

$$1 - p_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} p_k,$$

and by replacing k by $n-k$ we get the statement of Lemma. □

By analogue we can get a recursive formula for probabilities $\mathsf{P}\{\nu_n = s\}$.

Lemma 6.

$$\mathsf{P}\{\nu_n = s\} = \sum_{k=s-1}^{n-1} \binom{n-1}{k} \mathsf{P}\{\nu_k = s-1\} p_{n-k} q^{k(n-k)} \tag{10}$$

for $n \geqslant s > 1$.

Proof. Let us denote

$$B_s = \sum_{n=0}^{\infty} \frac{x^n}{q^{n(n-1)/2} n!} \mathsf{P}\{\nu_n = s\},$$

then by (7) we get:

$$s! B_s(x) = B(x)^s,$$

then by differentiating by x it follows that:

$$s!B'_s = sB^{s-1}B' = s(s-1)!B_{s-1}B',$$

hence,

$$xB'_s = xB_{s-1}B'.$$

From this and according to $\mathsf{P}\{\nu_k = s-1\} = 0$ as $k < s-1$ we get Lemma statement. $\square$

Lemma 7. *The following relations hold:*

$$p_{n+1} = \sum_{s=1}^{n} \sum_{k_1+\dots+k_s=n} \frac{n!(1-q^{k_1})\dots(1-q^{k_s})}{s!k_1!\dots k_s!} \mathsf{P}\{\nu_n = s\}, \tag{11}$$

$$p_{n+1} \geqslant (1-q^n)p_n. \tag{12}$$

Proof. If we put x/q instead of x in the definition of series A, we get that $A' = A(x/q) = e^{B(x/q)}$. On the other side, $A' = B'e^B$. Therefore,

$$B'e^B = e^{B(x/q)}, \qquad B' = e^{B(x/q)-B(x)},$$

hence,

$$B' = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\sum_{n=0}^{\infty} \frac{p_n x^n (1-q^n)}{n! q^{n(n-1)/2}} \right)^s.$$

Now we take the corresponding coefficients before x^n in theese series and get the relation (11). The ineqution (12) follows from (11) if we left in this summa only the summand with $s = 1$. $\square$

3 Several Equations

Lemma 8. *For $s \geqslant 0$*

$$M_s = AB^s,$$

and in particulary, $E = AB$.

Proof. By definition,

$$\mathsf{E}(\nu_n)^{\underline{s}} = \sum_{k=0}^{n} k^{\underline{s}} \mathsf{P}\{\nu_n = k\},$$

hence by (7) we get:

$$M_s = \sum_{n=0}^{\infty} \frac{\mathsf{E}(\nu_n)^{\underline{s}} x^n}{n! q^{n(n-1)/2}} = \sum_{k=0}^{\infty} k^{\underline{s}} \sum_{n=0}^{\infty} \frac{\mathsf{P}\{\nu_n = k\} x^n}{n! q^{n(n-1)/2}} =$$

$$= \sum_{k=0}^{\infty} k^{\underline{s}} B^s / s! = B^s \sum_{k=s}^{\infty} \frac{B^{k-s}}{(k-s)!} = AB^s.$$

$\square$

Now we consider the connection between moments of ν_n and $\nu_n - 1$.

Lemma 9. *For $s \geqslant 1$*

$$M_s = \mathcal{M}_s + s\mathcal{M}_{s-1}$$
$$\frac{(-1)^s}{s!}\mathcal{M}_s = \sum_{k=0}^{s} \frac{(-1)^k}{k!} M_k$$

Proof. The first equation is follows from

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} = \mathsf{E}(\nu_n - 1)\dots(\nu_n - s) = \mathsf{E}(\nu_n)^{\underline{s}} - s\mathsf{E}(\nu_n - 1)^{\underline{s-1}},$$

and the second one not hard to proof by induction with the obvious start eqation $\mathcal{M}_0 = A = M_0$. □

Lemma 10. *For $s \geqslant 1$*

$$\frac{M_s'}{s!} = B'\left(\frac{M_s}{s!} + \frac{M_{s-1}}{(s-1)!}\right)$$
$$\frac{\mathcal{M}_s'}{s!} = B'\left(\frac{\mathcal{M}_s}{s!} + \frac{\mathcal{M}_{s-1}}{(s-1)!}\right) = B'\frac{M_s}{s!}$$

Now we ready to use the operator $\widehat{\ }$ for SG-functions of moments. First of all, we get:

Lemma 11 (Derivative Relashionship Formula)**.** *If R is a SG-function, then:*

$$\widehat{R} = R_q' + \frac{x^2}{2q}R''$$

Here and below the single quote without a parameter notation denotes the derivative by x, and the derivative by q is marked by index q.

The following equations hold.

Lemma 12.

$$\frac{\widehat{M_s}}{s!} = \frac{x^2}{2q}(B')^2\left(\frac{M_{s-1}}{(s-1)!} + \frac{M_{s-2}}{(s-2)!}\right) = \frac{x^2}{2q}B'\frac{M_{s-1}'}{(s-1)!}$$
$$\frac{\widehat{\mathcal{M}_s}}{s!} = \frac{x^2}{2q}(B')^2\left(\frac{\mathcal{M}_{s-1}}{(s-1)!} + \frac{\mathcal{M}_{s-2}}{(s-2)!}\right) = \frac{x^2}{2q}(B')^2\frac{M_{s-1}}{(s-1)!} = \frac{x^2}{2q}B'\frac{\mathcal{M}_{s-1}'}{(s-1)!}$$

Proof. By the convolution formula and from $\widehat{A} = 0$ we get:

$$A_q' = -\frac{x^2}{2q}A''.$$

From here it follows that:

$$(M_s)_q' = (AB^s)_q' = A_q'B^s + sAB^{s-1}B_q' = A_q'(B^s + sB^{s-1}) = -\frac{x^2}{2q}A''(B^s + sB^{s-1})$$
$$M_s'' = (AB^s)'' = (A'B^s + sB^{s-1}A')' = A''(B^s + sB^{s-1}) + A'B'(sB^{s-1} + s(s-1)B^{s-2})$$

Hence by Derivative Relashionship Formula we get that:

$$\widehat{M_s} = (M_s)'_q + \frac{x^2}{2q}M''_s = \frac{x^2}{2q}A'B'(sB^{s-1} + s(s-1)B^{s-2}) =$$
$$= \frac{x^2}{2q}(B')^2(sM_{s-1} + s(s-1)M_{s-2}),$$

so we have the first equation of statement.

To get the equations for $\mathcal{M}_s$ it is sufficient to use Lemmas 9, 10 and previous relation. □

4 Several Inequations

Let denote by $\gg$ that the inequation $\geqslant$ holds for all coefficient before x^n in the considering series. For example, the notation $\sum a_n x^x \gg \sum b_n x^n$ means that for all n the inequation $a_n \geqslant b_n$ holds. It is easy to verify that:

if $X \gg Y$ and $Z \gg 0$, then $XZ \gg YZ$;

if $X \gg Y$ and $V \gg W$, then $X + V \gg Y + W$.

Lemma 13. *For $n > 0$*

$$q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}} \leqslant \mathsf{E}(\nu_n - 1)^{\underline{s}} \leqslant \mathsf{E}(\nu_{n-1})^{\underline{s}}$$

Proof. Left inequation follows from:

$$\mathcal{M}'_s = B'M_s \gg M_s$$

with help of convolution formula and because of $B' \gg 1$. Right inequation follows from:

$$\mathcal{M}'_s = B'M_s = B'AB^s = A'B^s = A(x/q)B^s \ll A(x/q)B(x/q)^s = M_s(x/q).$$

□

Lemma 14. *For all $n \geqslant 1$ and $s \geqslant 1$ the following inequations hold:*

$$(n-1)^{\underline{s}} \cdot q^{(n-1)s} \leqslant \mathsf{E}(\nu_n - 1)^{\underline{s}} \leqslant 2(n-1)^{\underline{s}} q^{(n-1)(s+1)/2}$$

Proof. Left inequation.

$$\frac{x\mathcal{M}'_s}{s!} = \frac{xB'M_s}{s!} = xA'\frac{B^s}{s!} = xA'B_s,$$

hence, by the convolution formula we get:

$$\frac{n\mathsf{E}(\nu_n - 1)^{\underline{s}}}{s!} = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} k\mathsf{P}\{\nu_{n-k} = s\},$$

where the last summation we can estimate by the summand as $k = n - s$, and therefore we have:

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} \geqslant \binom{n}{n-s} s! q^{s(n-s)} q^{s(s-1)/2} \cdot \frac{n-s}{n} = (n-1)^{\underline{s}} \cdot q^{(n-1)s} q^{-s(s-1)/2},$$

here we get the left equation of Lemma statement.

Right inequation. Following relations one can get from the results that were proved above.

$$\frac{x(\widehat{\mathcal{M}}_s)'}{s!} = x\left(\frac{x^2}{2q}(B')^2\frac{M_{s-1}}{(s-1)!}\right)' = \frac{x^2(B')^2 + x^3B'B''}{q}\frac{M_{s-1}}{(s-1)!} + \frac{x^3(B')^2}{2q}\frac{M'_{s-1}}{(s-1)!} =$$
$$= \frac{x^2}{q}\left((B')^2\frac{M_{s-1}}{(s-1)!} + xB'B''\frac{M_{s-1}}{(s-1)!} + \frac{x}{2}(B')^3\frac{M_{s-1}}{(s-1)!} + \frac{x}{2}(B')^3\frac{M_{s-2}}{(s-2)!}\right)$$
$$\frac{x^2}{qs!}\mathcal{M}''_s = \frac{x^2}{qs!}(B'M_s)' = \frac{x^2}{q}\left(B''\frac{M_s}{s!} + (B')^2\frac{M_s}{s!} + (B')^2\frac{M_{s-1}}{(s-1)!}\right)$$

$$\frac{x(\widehat{\mathcal{M}}_s)'}{s!} - s\frac{x^2}{qs!}\mathcal{M}''_s = \frac{x^2}{q}(B')^2\frac{M_{s-1}}{(s-1)!}(1-s) + \frac{x^2}{q}B''M_{s-1}\left(\frac{xB'}{(s-1)!} - \frac{sB}{s!}\right)$$
$$+ \frac{x^2}{q}(B')^2M_{s-1}\left(\frac{xB'}{2(s-1)!} - \frac{sB}{s!}\right) + \frac{x^3}{2q}(B')^3\frac{M_{s-2}}{(s-2)!}$$

$$\begin{aligned} x(\widehat{\mathcal{M}}_s)' - \frac{sx^2}{q}\mathcal{M}''_s =& s\frac{x^2}{q}B''M_{s-1}(xB'-B) + \frac{sx^2}{q}(B')^2M_{s-1}(xB'/2-B) \\ &+ s(s-1)\frac{x^2}{q}(B')^2M_{s-2}(xB'/2-B) = \\ =& s\frac{x^2}{q}B''M_{s-1}(xB'-B) + \frac{s!x^2}{q}(B')^2\left(\frac{M_{s-1}}{(s-1)!} + \frac{M_{s-2}}{(s-2)!}\right)(xB'/2-B) = \\ =& s\frac{x^2}{q}B''M_{s-1}(xB'-B) + 2\widehat{M}_s(xB'/2-B). \end{aligned} \tag{13}$$

Since $n-1 \geqslant 0$, $n/2-1 \geqslant 0$ for $n \geqslant 2$, $n/2-1 \geqslant -1/2$ for $n=1$ it follows that

$$xB' - B \gg 0; \qquad \frac{xB'}{2} - B \gg -\frac{x}{2},$$

and from the equations (13) we get the next inequation:

$$x(\widehat{\mathcal{M}}_s)' + x\widehat{M}_s \gg \frac{sx^2}{q}\mathcal{M}''_s.$$

Now we get coefficients before x^n:

$$n(\mathsf{E}(\nu_n-1)^{\underline{s}})'_q + nq^{n-1}(\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q \geqslant \frac{sn(n-1)}{q}\mathsf{E}(\nu_n-1)^{\underline{s}}.$$

Dividing by n we get:

$$(\mathsf{E}(\nu_n-1)^{\underline{s}})'_q + q^{n-1}(\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q \geqslant \frac{s(n-1)}{q}\mathsf{E}(\nu_n-1)^{\underline{s}} \quad \text{as } n>0. \tag{14}$$

It is esy to see that

$$q^{n-1}(\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q = (q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q - \frac{(n-1)}{q}q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}},$$

where we use derivative of product. Therefore from this and (14) we get

$$(\mathsf{E}(\nu_n-1)^{\underline{s}}+q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q \geqslant \frac{s(n-1)}{q}\mathsf{E}(\nu_n-1)^{\underline{s}}+\frac{(n-1)}{q}q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}} \tag{15}$$

Hence, dividing by $\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}$ we find the inequation

$$\frac{(\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q}{\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \geqslant \frac{n-1}{q}\cdot\frac{s\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}}{\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \tag{16}$$

Note, that the function $f(t) = (s+t)/(1+t)$ not increases as t increases, if $s \geqslant 1$ and $t > 0$. From Lemma 13 it follows that:

$$\mathsf{E}(\nu_n-1)^{\underline{s}} \geqslant q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}} \quad \text{as } n>0$$

Therefore from this and (16) we get:

$$\frac{(\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q}{\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \geqslant \frac{(s+1)(n-1)}{2q} \tag{17}$$

Let $q \leqslant q_1 \leqslant 1$, and let $\mathsf{E}_1 = \mathsf{E}|_{q=q_1}$. By integrating (17) on the interval $[q;q_1]$ we get follows:

$$\ln\left(\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}\right)\Big|_q^{q_1} \geqslant \frac{(s+1)(n-1)}{2}\ln q\Big|_q^{q_1}$$

Then we put both sides of this inequation into the argument of function e^x, and get:

$$\mathsf{E}(\nu_n-1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}} \leqslant \left(\mathsf{E}_1(\nu_n-1)^{\underline{s}} + q_1^{n-1}\mathsf{E}_1(\nu_{n-1})^{\underline{s}}\right)\left(\frac{q^{n-1}}{q_1^{n-1}}\right)^{(s+1)/2}$$

or

$$\mathsf{E}(\nu_n-1)^{\underline{s}} \leqslant \left(\mathsf{E}_1(\nu_n-1)^{\underline{s}} + q_1^{n-1}\mathsf{E}_1(\nu_{n-1})^{\underline{s}}\right)\left(\frac{q^{n-1}}{q_1^{n-1}}\right)^{(s+1)/2}.$$

Then we set $q_1 = 1$ and finally find that

$$\mathsf{E}(\nu_n-1)^{\underline{s}} \leqslant 2(n-1)^{\underline{s}}q^{(n-1)(s+1)/2},$$

because as $q_1 = 1$ we have $\mathsf{E}_1(\nu_n-1)^{\underline{s}} = (n-1)^{\underline{s}}$.

This proofs the right inequation of Lemma. □

Note, that if we put $s = n-1$, then we have an equation

$$\mathsf{E}(\nu_n-1)^{\underline{n-1}} = (n-1)!q^{n(n-1)/2} = (n-1)^{\underline{n-1}}q^{n(n-1)/2},$$

where the right hand side is equal to half of the just proved estimation.

5 Asymptotic behavior of ν_n

In this section we consider an asymptotics of moments $\mathsf{E}(\nu_n - 1)^{\underline{s}}$ as s is fixed and positive. We will study a behavior of moments in the following zones of parameters:

1. $p \to 0$, $n = \mathrm{const}$;
2. $q^n \to e^{-\alpha}$, where fixed $\alpha \geqslant 0$ and $n \to \infty$;
3. $q^n \to 0$ as $n \to \infty$

   3.1 $nq^n \to \infty$,

   3.2 $nq^n \to \alpha > 0$,

   3.3 $nq^n \to 0$ (in this case p can be a positive constant < 1).

5.1 Asymptotics for $n = \mathrm{const}$

It is easy to prove the following Lemma, besause the minimal graph with n vertices and s components is a forest with s trees.

Lemma 15. *If $p \to 0$ and $n = \mathrm{const}$, then for any $s \leqslant n$ the following equation holds: $\mathsf{P}\{\nu_n = s\} = F_{s,n}p^{n-s} + O(p^{n-s+1})$. In particular, $p_n = n^{n-2}p^{n-1} + O(p^n)$.*

Hence we have the following

Theorem 1. *If $p \to 0$ and $n = \mathrm{const}$, then for any $s \leqslant n$:*

$$\mathsf{E}\nu_n^s = n^s(1 + o(1)), \quad \mathsf{E}(\nu_n - 1)^{\underline{s}} \to n^{\underline{s}}.$$

5.2 Asymptotics for $q^n \to e^{-\alpha}$

Let

$$\beta(x) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} x^k.$$

This series converges as $|x| \leqslant e^{-1}$ and this is a generated function for sequence of numbers of labelled trees [4].

Theorem 2. *If $q^n \to e^{-\alpha}$ as $n \to \infty$, where $\alpha \geqslant 0$, then for $s \geqslant 0$*

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} = \left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s (1 + o(1)).$$

In particular, for $\alpha = 0$ we have the relation $\mathsf{E}(\nu_n - 1)^{\underline{s}} \sim n^s$.

Proof. We will use a mathematical induction on the parameter s. It is clear that the statement of Theorem holds for $s = 0$. Let us suppose that it holds for $s - 1$ and will show it for $s \geqslant 1$.

From the relation $x\mathcal{M}'_s = xB'M_s = xB'BM_{s-1} = Bx\mathcal{M}'_{s-1}$ (see Lemma 10) and from the convolution formula we get:

$$n\mathsf{E}(\nu_n - 1)^{\underline{s}} = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} p_k (n-k) \mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}} =$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} q^{k(n-k)} p_k \mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}} = n(S_1 + S_2),$$

where

$$S_1 = \sum_{k=0}^{k_0} \binom{n-1}{k} q^{k(n-k)} p_k \mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}},$$

$$S_2 = \sum_{k=k_0+1}^{n-1} \binom{n-1}{k} q^{k(n-k)} p_k \mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}}$$

As k is fixed, one can get next relations: $\binom{n-1}{k} q^{k(n-k)} \sim (nq^n)^k/k!$, $p_k \sim k^{k-2}p^{k-1}$ (Lemma 15). And from the induction hypothesis we get: $\mathsf{E}(\nu_{n-k})^{\underline{s-1}} \sim \left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^{s-1}$. Hence:

$$S_1 = \sum_{k=0}^{k_0} \frac{(npq^n)^k}{pk!} k^{k-2} \left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^{s-1} (1+o(1)) = \sum_{k=0}^{k_0} \frac{n}{\alpha} \frac{k^{k-2}}{k!} (\alpha e^{-\alpha})^k \left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^{s-1} (1+o(1)),$$

where we use the asymptotics $np \to \alpha$ and $npq^n \to \alpha e^{-\alpha}$, which is follows from Theorem conditions.

So, it is easy to see that $S_1/\left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s$ as closed to 1 as k_0 is bigger, because of the convergence of the series $\beta(x)$ for $x = \alpha e^{-\alpha}$.

Let we estimate S_2. It is clear that $\mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}} \leqslant n^{s-1}$. From this and from the equation (9) we get

$$1 \geqslant \sum_{k=k_0+1}^{n-1} \binom{n-1}{k} q^{k(n-1-k)} p_k \frac{k}{n-1} \geqslant \frac{k_0}{n-1} \sum_{k=k_0+1}^{n} \binom{n-1}{k} q^{k(n-k)} p_k \geqslant$$

$$\geqslant \frac{k_0}{n^s} \sum_{k=k_0+1}^{n-1} \binom{n-1}{k} q^{k(n-k)} p_k \mathsf{E}(\nu_{n-k} - 1)^{\underline{s-1}} = \frac{k_0}{n^s} S_2.$$

Therefore,

$$S_2 = \frac{1}{k_0} O\left(n^s\right) = \frac{1}{k_0} O\left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s,$$

i.e. the ratio $S_2/\left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s$ tends to 0 as $k_0 \to \infty$.

Thus, $\mathsf{E}(\nu_n - 1)^{\underline{s}} = \left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s (1 + o(1))$.

In the case of $\alpha = 0$ the proof of Theorem is similary, but instead of $\beta(\alpha e^{-\alpha})/\alpha$ we should write 1 at all places. $\square$

5.3 Asymptotics for $q^n \to 0$

Theorem 3. *Let $q^n \to 0$ and $nq^n \geqslant C$ as $n \to \infty$, where fixed $C > 0$, then*

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} = (nq^n)^s(1 + o(1)).$$

Proof. We will use an induction by s. It is clear that the statement of Theorem holds for $s = 0$. Let us suppose that it holds for $s-1$ and will show it for $s \geqslant 1$.

The following relations hold:

$$\sum_{n=0}^{\infty} \frac{(\mathsf{E}(\nu_n)^{\underline{s}} - \mathsf{E}(\nu_{n+1} - 1)^{\underline{s}})x^n}{n!q^{n(n-1)/2}} = M_s - \mathcal{M}'_s(xq) = M_s - B'(xq)M_s(xq) =$$
$$= AB^s - A'(xq)B^s(xq) = A(B^s - B^s(xq)) = A(s!B_s - s!B_s(xq)) =$$
$$= A\sum_{n=0}^{\infty}(1 - q^n)\frac{x^n\mathsf{P}\{\nu_n = s\}s!}{n!q^{n(n-1)/2}} \ll A\sum_{n=0}^{\infty} np\frac{x^n\mathsf{P}\{\nu_n = s\}s!}{n!q^{n(n-1)/2}} =$$
$$= Apx(B^s)' = spxAB'B^{s-1} = spxB'M_{s-1} = spx\mathcal{M}'_{s-1} = sp\sum_{n=0}^{\infty}\frac{\mathsf{E}(\nu_n - 1)^{\underline{s-1}}nx^n}{n!q^{n(n-1)/2}},$$

where we use the fact, that $(1 - q^n) \leqslant n(1 - q) = np$. Therefore we get that

$$\mathsf{E}(\nu_n)^{\underline{s}} - \mathsf{E}(\nu_{n+1} - 1)^{\underline{s}} \leqslant spn\mathsf{E}(\nu_n - 1)^{\underline{s-1}}$$

or

$$\mathsf{E}(\nu_{n-1})^{\underline{s}} \leqslant \mathsf{E}(\nu_n - 1)^{\underline{s}} + spn\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}}.$$

From the equation (16) it follows that

$$\frac{(\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q}{\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \geqslant \frac{n-1}{q} \cdot \frac{s\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}}{\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \geqslant$$
$$\frac{n-1}{q} \cdot \frac{s\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}(\mathsf{E}(\nu_n - 1)^{\underline{s}} + snp\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}})}{\mathsf{E}(\nu_n - 1)^{\underline{s}} + q^{n-1}(\mathsf{E}(\nu_n - 1)^{\underline{s}} + snp\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}})} =$$
$$= \frac{n-1}{q} \cdot \frac{s + q^{n-1} + snpq^{n-1}\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}}/\mathsf{E}(\nu_n - 1)^{\underline{s}}}{1 + q^{n-1} + snpq^{n-1}\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}}/\mathsf{E}(\nu_n - 1)^{\underline{s}}}. \quad (18)$$

By the induction hypothesis and from Lemma 14 we get that

$$\frac{\mathsf{E}(\nu_{n-1} - 1)^{\underline{s-1}}}{\mathsf{E}(\nu_n - 1)^{\underline{s}}} \leqslant C_1\frac{(nq^n)^{s-1}}{(n-1)^{\underline{s}}q^{(n-1)s}} \leqslant \frac{C_2}{nq^n},$$

where the positive constants C_k, generally speaking, are depends on the parameter s. By putting this inequation into (18) we get:

$$\frac{(\mathsf{E}(\nu_n-1)^{\underline{s}}+q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}})'_q}{\mathsf{E}(\nu_n-1)^{\underline{s}}+q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}} \geqslant \frac{n-1}{q}\cdot\frac{s+q^{n-1}+C_4npq^{n-1}/(nq^n)}{1+q^{n-1}+C_4npq^{n-1}/(nq^n)} \geqslant$$
$$\geqslant \frac{n-1}{q}(s-sq^{n-1}-C_5npq^{n-1}/(nq^n)) \geqslant \frac{n-1}{q}(s-sq^{n-1}-C_6p) \geqslant$$
$$\geqslant (n-1)sq^{-1}-C_7(n-1)q^{n-2}-C_8np. \quad (19)$$

Let $q_1=\varepsilon^{1/(n-1)}$, where ε is an arbitrary small positive number, hence $q_1^{n-1}=\varepsilon$ and $q<q_1$ (it follows from $q^n\to 0$). Besides let denote $\mathsf{E}_1=\mathsf{E}|_{q=q_1}$ as it was above.

Now, we integrate the inequation (19) on the interval $[q;q_1]$ and get that

$$\ln\left(\mathsf{E}(\nu_n-1)^{\underline{s}}+q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}}\right)\Big|_q^{q_1} \geqslant s(n-1)\ln q\Big|_q^{q_1}-(n-1)C_7\frac{q^{n-1}}{n-1}\Big|_q^{q_1}-C_8n(q-q^2/2)\Big|_q^{q_1}$$

or

$$\mathsf{E}(\nu_n-1)^{\underline{s}}+q^{n-1}\mathsf{E}(\nu_{n-1})^{\underline{s}} \leqslant$$
$$\leqslant \left(\mathsf{E}_1(\nu_n-1)^{\underline{s}}+q_1^{n-1}\mathsf{E}_1(\nu_{n-1})^{\underline{s}}\right)\left(\frac{q}{q_1}\right)^{s(n-1)} e^{C_7(q_1^{n-1}-q^{n-1})}e^{C_8n(q_1-q+q^2/2-q_1^2/2)} \leqslant$$
$$\leqslant \frac{\mathsf{E}_1(\nu_n-1)^{\underline{s}}+q_1^{n-1}\mathsf{E}_1(\nu_{n-1})^{\underline{s}}}{\varepsilon^s}q^{s(n-1)}e^{C_9\varepsilon}, \quad (20)$$

because $q_1^{n-1}=\varepsilon$ and $n(q_1-q+q^2/2-q_1^2/2)=n(q_1-q)(1-q/2-q_1/2)=n(p-p_1)(p/2+p_1/2)\leqslant np^2\to 0$. The last expression is follows from $np^2\cdot nq^n=(np)^2e^{n\ln q}\to 0$ and from the conditions of Theorem.

By Theorem 2 we get that

$$\mathsf{E}_1(\nu_n-1)^{\underline{s}}=\left(\frac{n}{\alpha}\beta(\alpha e^{-\alpha})\right)^s(1+o(1))=n^s\beta(\alpha\varepsilon)^s/\alpha^s(1+o(1)).$$

where $\alpha=-\ln\varepsilon$.

Besides that,

$$q_1^{n-1}\mathsf{E}_1(\nu_{n-1})^{\underline{s}}=\varepsilon(\mathsf{E}_1(\nu_{n-1}-1)^{\underline{s}}+s\mathsf{E}_1(\nu_{n-1}-1)^{\underline{s-1}})=$$
$$=\varepsilon(n^s\beta(\alpha\varepsilon)^s/\alpha^s+sn^{s-1}\beta(\alpha\varepsilon)^{s-1}/\alpha^{s-1})(1+o(1)),$$

because $q_1^{n-1}=\varepsilon$ and again from Theorem 2. From this and from (20) it follows that

$$\mathsf{E}(\nu_n-1)^{\underline{s}} \leqslant \frac{(1+\varepsilon)\beta(\alpha\varepsilon)^s/\alpha^s+\varepsilon s\beta(\alpha\varepsilon)^{s-1}/\alpha^{s-1}}{\varepsilon^s}n^sq^{sn}e^{C_{10}\varepsilon}(1+o(1)).$$

Therefore, by choosing an arbitrary small $\varepsilon > 0$ and using the relationship $\beta(x) \sim x$ as $x \to 0$ we get the relation:

$$\limsup_{n\to\infty} \frac{\mathsf{E}(\nu_n - 1)^{\underline{s}}}{(nq^n)^s} \leqslant \lim_{\varepsilon\to 0} \frac{(1+\varepsilon)\beta(\alpha\varepsilon)^s + s\varepsilon\alpha\beta(\alpha\varepsilon)^{s-1}}{\alpha^s\varepsilon^s} e^{C_{10}\varepsilon} = 1.$$

From Lemma 14 we have $\mathsf{E}(\nu_n - 1)^{\underline{s}} \geqslant (n-1)^{\underline{s}} q^{s(n-1)} = (nq^n)^s(1 + o(1))$. Now we see that Theorem follows from theese both equations. $\square$

Let $nq^n \to \alpha$, where α is a positive constant. From Theorem 3 we see that $\mathsf{E}(\nu_n - 1)^{\underline{s}} \to \alpha^s$.

It is known that in this case random variable $(\nu_n - 1)$ tends to Poisson distribution with the parameter α.

Thus we have

Theorem 4. *If $nq^n \to \alpha$ as $n \to \infty$ and α is a fixed positive constant, then for any fixed integer $k \geqslant 1$:*

$$\mathsf{P}\{\nu_n = k\} \to \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha}.$$

From Lemma 14 it follows that if $nq^n \to 0$, then $\mathsf{E}(\nu_n - 1) \asymp nq^{n-1}$. So we can conclude that ν_n tends to 1. Below we'll show an estimation of p_n in this case.

6 Several Consequences

Generally, we can conclude that in all zones of parameters p and n

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} \sim (\beta(npq^n)/p)^s \qquad \text{as } nq^n \to \infty$$

and

$$\mathsf{E}(\nu_n - 1)^{\underline{s}} = (\beta(npq^n)/p)^s + o(1) \qquad \text{as } nq^n = O(1)$$

It is easy to verify, because if $np \to \infty$ or $np \to 0$, then it follows that $npq^n \to 0$ and $\beta(npq^n)/p \sim nq^n$.

Now we can estimate the probability p_n that graph $G(n,p)$ is connected.

$$p_n = \mathsf{P}\{\nu_n < 2 - 1/n\} = 1 - \mathsf{P}\{\nu_n - 1 \geqslant 1 - 1/n\} \geqslant 1 - \mathsf{E}(\nu_n - 1)\frac{n}{n-1}, \quad (21)$$

and from Lemma 14 we get:

$$p_n \geqslant 1 - 2nq^{n-1}. \quad (22)$$

If we put $p = \frac{c \ln n}{n}$ and $c > 1$, then we have $nq^{n-1} = n\exp\{-c\ln n + O(\ln^2 n)/n\} = n^{1-c}(1 + O(\ln^2 n)/n)$. Therefore we finally get:

$$p_n \geqslant 1 - \frac{2}{n^{c-1}}(1 + O(\ln^2 n)/n). \quad (23)$$

If $nq^n \to \alpha$ (for example, $p = (\ln n + c + o(1))/n$, where $\alpha = e^{-c}$), then from Theorem 4 we get that:

$$p_n \to e^{-\alpha}.$$

To estimate p_n as $nq^n \to \infty$ we now consider the isolating probability. Let pi_n be a probability that $G(n,p)$ has an isolated vertex. Let A_i be an event that i-th vertex is isolated, then from the Inclusion–exclusion principle we get:

$$pi_n = \mathsf{P}\{A_1 \cup \cdots \cup A_n\} = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \mathsf{P}\{A_{i_1} \ldots A_{i_k}\} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\mathsf{P}\{A_1 \ldots A_k\}.$$

It is easy to see that $\mathsf{P}\{A_1 \ldots A_k\} = q^{k(k-1)/2}q^{k(n-k)}$, so

$$pi_n = \sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}q^{k(n-k)}q^{k(k-1)/2} + 1.$$

According to convolution formula we can find that SG-function PI of $\{pi_n\}$ is equal to $RT+A$, where R and T are SG-functions of the corresponding sequences $\{r_n\}$ and $\{t_n\}$, which are defined as follows: $r_n = (-1)^{n-1}q^{n(n-1)/2}$ and $t_n = 1$.

Hence we have

$$R = \sum_{n=0}^{\infty} \frac{r_n x^n}{n! q^{n(n-1)/2}} = -e^{-x}; \qquad T = A.$$

Thus $PI = A - e^{-x}A = A(1 - e^{-x})$.

Since $(1-e^{-x}) \leqslant x$ it follows that $PI \ll Ax$, and from the convolution formula we obtain

$$pi_n \leqslant nq^{n-1}. \tag{24}$$

It is easy to see that $PI' = A'(1-e^{-x}) + Ae^{-x} = PI \cdot B' + A - PI \gg PI \cdot B'$, because $A - PI \gg 0$, and from the convolution formula we get:

$$npi_n \geqslant \sum_{k=1}^{n}\binom{n}{k}q^{k(n-k)}pi_{n-k}kp_k \geqslant n(n-1)q^{n-1}p_{n-1}$$

or

$$pi_{n+1} \geqslant nq^n p_n \tag{25}$$

So, if $nq^n \to \alpha > 0$, then $pi_n \geqslant \alpha e^{-\alpha} + o(1)$.

And also we have

$$p_n \leqslant pi_{n+1}/(nq^n) \leqslant 1/(nq^n) \tag{26}$$

Since $PI = A(1-e^{-x})$ it follows that $PIe^x = Ae^x - A$ and, therefore, $(PI - A)(e^x - 1) = -PI$. From the relation $e^x - 1 > x$ we get that $PI \gg (A - PI)x$, therefore from the convolution formula we find that $pi_n \geqslant (1 - pi_n)nq^{n-1}$, then $(1 - pi_n) \leqslant 1/(nq^{n-1})$ and we get finally

$$pi_n \geqslant 1 - \frac{1}{nq^{n-1}} \tag{27}$$

Now we can combine all obtained results (22), (26), (25), (24) and (27) in the following

Theorem 5. *For all $n \geqslant 1$*

$$\begin{aligned} 1 - 2nq^{n-1} \leqslant p_n \leqslant \frac{1}{nq^n}, \\ 1 - \frac{1}{nq^n} \leqslant pi_n \leqslant nq^{n-1}, \\ nq^n p_n \leqslant pi_{n+1} \end{aligned}$$

And if $nq^n \geqslant C > 0$ as $n \to \infty$, then we can substitute nq^n by $\mathsf{E}(\nu_n - 1)(1 + o(1))$ in theese relations.

References

[1] Erdös, P. and Rényi, A. (1959). "On Random Graphs." *Publicationes Mathematicae* **6**: 290-297.

[2] Erdös, P. and Rényi, A. (1960) "On the Evolution of Random Graphs." *Publ. Math. Inst. Hungar. Acad. Sci.* **5**, 17-61.

[3] Bollobás, B. (2001) *Random Graphs (2nd ed.).* Cambridge University Press.

[4] Kolchin, V. F. *Random Graphs.* New York: Cambridge University Press, 1998.

[5] Sloane, N. J. A. Sequence A062734 in "The On-Line Encyclopedia of Integer Sequences."

[6] Harary, Frank; Palmer, Edgar M. (1973). *Graphical Enumeration.*