Online repeated posted price auctions with a demand side platform

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Abstract— We consider an online ad network problem in which an ad exchange auctions ad slots and intermediaries called demand side platforms (DSPs) buy these ad slots for their clients (advertisers). An intermediary represents multiple advertisers. Different types of ad slots are auctioned by the ad exchange, e.g., video ad, banner ad etc. We study repeated posted price auctions for homogeneous and heterogeneous items when there is an intermediary. In a posted price auction, the auctioneer sets a fixed reserve price. The buyer can accept the price and win the ad slot or reject the price.

We analyze the system from the auctioneer's perspective and show that the optimal reserve price is dynamic for heterogeneous items. We also investigate system from intermediary's perspective and devise algorithms for scheduling advertisers. Often the advertisers have budget constraints and impression constraints. We formulate a revenue optimization problem at the intermediary and also consider the problem of scheduling advertisers with budget and impression constraints. Finally, we present a numerical study for the single seller and advertiser model which considers various valuation distributions such as uniform, exponential and lognormal.

I. INTRODUCTION

In the age of information and internet, users often visit to different webpages for various goal, for example news (www.thehindu.com), travel (www.irctc.co.in), etc. The webpages generate revenue from having ad display slots on their page for advertisement. With increasing use of internet, mobile internet and e-commerce platforms, the online ad display market has grown significantly in last few years. The total revenue generated from internet advertising in US was \$107.5 billion for 2018, [1]. The digital advertising market in India was around \$1.3 billion for 2018, [2]. This is expected to grow further in coming years. Motivated from these developments, in this paper we study an auction mechanism for the online ad display market.

Various online content publishers have a large number of webpages. The objective of these publishers is to maximize their revenue via displaying ads. Publishers can enter into contracts with advertisers to display ads in ad slots on available on their webpages. Another way for publishers to optimize their revenue from ads is by selling ad slots using an auction mechanism, instead of fixed contracts with advertisers. These auctions are conducted at Ad exchanges. Examples of ad exchanges include Google's Double Click, OpenX, Yahoo!'s Right Media, etc.

We now briefly describe how online ad auctions work. Consider a users arriving at a webpages. The publisher approaches an ad exchange with ad display slots. Different advertisers participate in an ad-slot auction held by the ad
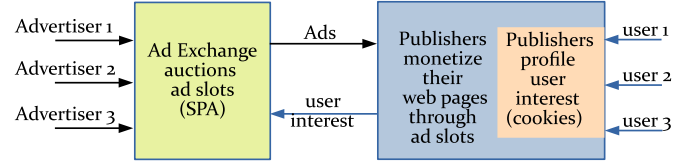

Fig. 1. Online ad auction model

exchange. The publisher shares user information (from cookies) with advertisers via the ad exchange. The advertisers that match the user's interests, participate in the auction and bid (maximum willingness to pay) for the ad slot. This bid price depends on their valuation for the ad slot. The auctioneer runs the second price auction (SPA) mechanism, where ad slot is allocated to advertiser with highest bid price and the winner pays second highest bid price. This auction is performed in a timescale of milliseconds. Each ad slot is referred to, as an impression. Online ad auction model is illustrated in Fig. 1.

Large auctioneers (ad exchanges) may sell billions of impressions in a day using the SPA mechanism. When the user clicks on an ad slot, (s)he is sent to the advertiser's webpage. The advertiser's valuation for the ad slot depends on user's click-through probability. The goal of the auctioneer is to maximize its long term revenue thorough optimal reserve pricing. Whereas the objective of advertisers is to determine the bid price for each auction such that their long term revenue is maximized. In [3], different models of ad exchanges are discussed.

Often advertisers participate in auctions through intermediaries called as demand side platforms (DSPs). This may be because most of the advertisers have low budget and might incur additional cost for maintaining their services and designing optimal bid strategies. Instead, a DSP represents the multiple advertisers in online ad auctions. These advertisers can enter into contracts with DSP for buying impressions and displaying their ads. Advertisers may have targeting constraints, budget constraints or impression constraints. Sometimes, advertisers might prefer some types of ads over others. For example, one may have preference for video ads over banner ads. Further, advertisers may have different willingness to pay for different types of ads. A DSP can have few hundreds of advertisers. *A DSP needs a policy to schedule these advertisers in online ad auctions, and, a bidding algorithm that maximizes its total utility while meeting the requirements of the advertisers.* In online ad

auctions, there can be many DSPs participating in auctions. This may influence the bid prices in auctions and affect expected revenue of the auctioneer and the publishers.

In this paper we consider a single seller and buyer (DSP or a advertiser) model. This is motivated from the ad auction problem where an advertiser or a DSP participates in large number of auctions. The SPA mechanism with a single buyer and seller is called as posted price auction (PPA) mechanism. In this mechanism, seller sets a reserve price on an item. The buyer accepts the offer if the reserve price set by auctioneer is lower than its own valuation, otherwise it rejects the offer. Repeated auctions with posted price mechanism are referred to repeated posted price auctions. This is a simplified model of SPA mechanism.

A. Related work

We now discuss some relevant work on auctions, online ad auctions and auction with intermediaries. A second price auction mechanism is first introduced by Vickery in [4], this is also referred to as Vickery auction. In SPA, there is a seller and set of potential buyers. Each buyer values a good differently and is willing to pay a different amount. Each buyer knows their valuation for the good but does not know the valuations of other buyers. The seller also does not know the valuations of buyers. But the seller and buyers know that their valuation is drawn from a probability distribution function. This distribution is independent across the buyers. This model is called as *independent private value (IPV)* model. We further consider a symmetric buyer valuation in which this probability distribution is identical. The problem of optimal auction design for IPV model is studied in [5]–[7]. Here, all buyers bid for a good. The buyer with the highest bid price wins the auction and pays the second highest bid price. In optimal auction design these authors studied equilibrium bidding behavior. It is shown that in equilibrium each buyer's dominant strategy is to report true valuation to seller. This is called as *dominant strategy incentive compatible* (DSIC) auction. Further, the seller can optimize revenue by setting an optimal reserve price for SPA and it is independent of the number of buyers in SPA. For more details on auctions, see [8].

Recently, repeated SPA has been studied in [9], [10] for budget constrained online auction problem. Here, it is shown that the optimal bidding behavior of buyer in equilibrium is to shade their bid and set the bid price less than their valuation. The buyers do not report true valuation and hence this is not DSIC auction. The various issues in online ad auction, and ad exchanges are mentioned in [3] and author discusses the perspectives of the ad exchange, the advertiser, the publisher, and, introduces the role of intermediaries in the ad network. The role of reserve prices in auction design using a field experiments are demonstrated in [11]. In [12], dynamic reserve price based learning is introduced for unknown value distribution function, in online ad auction, and, incentive compatible constraints are also studied. A variant of posted price auction model is investigated in [13] and randomized auction model is studied.

In the domain of online ad display network, there are a few works on ad auction with intermediaries (DSPs). This study has become important due to resource constraints put in by the advertisers while optimizing the online ad auction target criteria. In [14], the author studies auctions with intermediaries, where auctions are run at both the seller and the intermediary. For auction at the seller, the intermediaries are buyers and for auctions at intermediaries, the advertisers are buyers. And, the equilibrium behavior is studied. A single item auction with competing intermediary is investigated in [15], [16]. A few variants of auctions with intermediaries are studied in [17], [18] where the authors discuss dynamic bidding strategies for budget constrained advertisers. Dynamic pricing model for real time bidding in online ad display is studied in [19]. Another direction of research on auctions with intermediaries is studied in [20], where authors introduce a multi-bidding concept.

None of the above models consider repeated posted price auction with an intermediary, and the problem of advertiser-scheduling faced by the intermediary.

B. Our contributions

In this paper, we study repeated posted price auction mechanisms for the auction of homogeneous and hetero-geneous items. We show that the optimal reserve price of seller is fixed all round of auctions for homogeneous items and the optimal reserve price of seller for heterogeneous item is dynamic. Later, we use this insight to study an online ad auction system with a DSP as an intermediary. We provide optimization formulation as revenue maximization and scheduling scheme for the advertisers at the DSP under impression constraints and budget constraints. We provide numerical examples for a single seller and advertiser model for uniform, exponential and lognormal distribution on val-uation of advertiser. We also illustrate numerical examples with DSP model.

II. PRELIMINARIES AND MODEL

Consider an online ad auction system with a seller (ad exchange) and a DSP. The seller has multiple units of K types of items for sale and he auctions items sequentially[1]. Suppose that there are T rounds of auctions. We index the each round in the sequence by t and $t = 1, 2, \cdots T$. Ad-vertisers participate in auctions through the DSP. The seller uses posted price auction mechanism. The DSP schedules an advertiser from its set of client advertisers in each round of auction by accepting the price offered by the seller. The seller knows the type of the item which is being autioned. The DSP also has this information (type of item being auctioned).

Users arrive at various websites sequentially. Different websites have different types of ads slots available for display[2]. Let p_k be the probability that item of type k is

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[1]Examples of different types of items include banner ad, video ad, etc. The ad exchange can observe the types of ads that can be displayed at the publisher's site.

[2]Example: Youtube has video ad slots as well as display banner ad slots, for the arriving user.

auctioned by seller in auction round t, and $\sum_{k=1}^{K} p_k = 1$. We assume that the type of items auctioned in each round are independent of the other rounds. Let $X_{k,t}$ be the indicator random variable for round t, type k, with $1 \le k \le K$ and $1 \le t \le T$. It takes value $X_{k,t} = 1$ if item of type k is auctioned in round t; otherwise $X_{k,t} = 0$. Let q_t be the price set by seller in the round t. This price could depend on type of item. We shall explicitly mention this dependence of price q_t on the type of item whenever necessary.

Suppose that the DSP has a set of N advertisers (clients) denoted by $\mathcal{N} = \{1, 2, \cdots, N\}$. The advertisers are risk neutral.

We assume that the seller does not know how much each advertiser is willing to pay for different types of items. Let $\tilde{v}_{i,k,t}$ be the value estimate of advertiser i for type k item in round t, i.e., the maximum amount an advertiser is willing to pay for an item. The seller's uncertainty about the value estimate of advertiser's value is represented by a continuous probability distribution function over a finite interval. Let $F_{i,k}$ be the probability distribution function for advertiser i's valuation given that type k item is auctioned.

We now discuss posted price auction mechanism.

A. Posted price auction mechanism

In the standard posted price auction, there is a seller and a buyer. The buyer's value estimate v is unknown to the seller, but he knows the probability distribution of the value estimate, say, F over $[\underline{v}, \overline{v}]^3$. The seller sets a reserve price, say q and announces it to buyer. If $v > q$, the buyer accepts the offer, otherwise the offer is rejected. The revenue of the seller is q when the offer is accepted; it is 0 otherwise. The payoff of the buyer is $v - q$ when the offer accepted, and zero otherwise. This strategy of the buyer is dominant strategy incentive compatible (DSIC). From the seller's perspective, the objective is to optimize the revenue by setting optimal reserve price. This price is the solution of following problem.

$$q^* = \arg\max_q q \times \Pr(v > q) = \arg\max_q q(1 - F(q)).$$

Repeated posted price auction (RPPA) mechanism: In this case there are T rounds of auctions. And each round follows the PPA mechanism. Now consider RPPA with a single buyer. Let v_t be the valuation of the buyer for round t; it is drawn from probability distribution F independently for each round. Let q_t be the reserve price set by the seller in round t. The buyer accepts the offer whenever $v_t \ge q_t$ and rejects it otherwise. The expected revenue of a buyer over T rounds of auctions is

$$R_{B,PPA} = \mathbb{E}\left[\sum_{t=1}^{T}(v_t - q_t)1_{\{q_t < v_t\}}\right]. \tag{1}$$

The strategy of the buyer is DSIC. The seller's expected revenue over T rounds of auctions is

$$R_{S,PPA} = \mathbb{E}\left[\sum_{t=1}^{T} q_t 1_{\{q_t < v_t\}}\right]. \tag{2}$$

The objective of a seller is to set an optimal reserve price to maximize (2).

B. DSP's problem in a repeated posted price auction

We now consider the problem from the perspective of the DSP. It has N advertisers (clients). Let v_{k,j_t} be the valuation of advertiser $j_t = i$ for item of type k in auction t. Suppose that $q_{k,t}$ be the price set by seller for type k item in round t. DSP uses algorithm $\mathcal{A}$ to schedule one of the advertisers in each round of auctions. Let $\{a_t\}_{t=1}^{T}$ be the scheduled sequence of advertisers according to $\mathcal{A}$ and $a_t \in \mathcal{N}$. The strategy of the DSP is DSIC. The expected revenue of the DSP under algorithm $\mathcal{A}$ over T rounds of auctions is

$$R_{D,PPA}^{\mathcal{A}} = \mathbb{E}^{\mathcal{A}}\left[\sum_{t=1}^{T}\sum_{k=1}^{K}(v_{k,a_t} - q_{k,t})X_{k,t}1_{\{(q_{k,t} < v_{k,a_t})\}}\right].$$

The expected revenue of the seller over T rounds of auctions when DSP uses scheduling algorithm $\mathcal{A}$ is given by

$$R_{S,PPA}^{\mathcal{A}} = \mathbb{E}^{\mathcal{A}}\left[\sum_{t=1}^{T}\sum_{k=1}^{K} q_{k,t}X_{k,t}1_{\{(q_{k,t} < v_{k,a_t})\}}\right].$$

The objective of DSP is to come up with a scheduling algorithm to meet the criteria of the advertisers. The goal of the seller is to set the optimal reserve price to maximize its revenue.

III. ANALYSIS

In this section, we first analyze the repeated posted price auction mechanism with a single buyer (advertiser). We derive some properties of RPPAs. Later we study RPPAs involving a DSP.

A. Repeated posted price auction with single buyer

1) Homogeneous RPPA: In a homogeneous RPPA the seller auctions identical items in all rounds. We begin with an example; suppose that the buyer has fixed valuation v for each PPA (round). The best response of the buyer to the seller's reserve price q, is to accept the offer if $v > q$, and reject the offer otherwise. In each round of auctions, the buyer follows this strategy. The total revenue of the buyer over T rounds is $(v - q)T1_{\{v>q\}}$. The seller's revenue is $qT1_{\{v>q\}}$. In this example the buyer does not have incentive to lie and accept the offer when $v < q$, because the payoff is negative and the seller uses fixed reserve price for all T auctions. This strategy of the buyer is called dominant strategy incentive compatible (DSIC).

When the buyer's valuation in each round is drawn from probability distribution F over $[\underline{v}, \overline{v}]$. Each round is an independent PPA; hence, the buyer's strategy is to accept the offer when current valuation is higher than the reserve price of the seller. We make the following assumption on probability distribution function F.

Definition 1 (Assumption): The distribution function F with density f is regular and $\left(v - \frac{1-F(v)}{f(v)}\right)$ is strictly increasing in v over $[\underline{v}, \overline{v}]$.

We present the following proposition which is simple variant of [5].

Proposition 1: In homogeneous RPPA, the seller sets a single optimal reserve price q^*. The optimal revenue of the seller over T rounds of auctions is

$$Tq^*\Pr(v > q^*), \tag{3}$$

and the optimal revenue of the buyer over T rounds of auction is

$$T\mathbb{E}[(v - q^*)1_{\{v > q^*\}}]. \tag{4}$$

Sketch of the proof: In repeated PPA, each round is PPA, where seller can determine the optimal reserve price and it has a unique solution because F is regular. Taking derivative of $q(1 - F(q))$ with respect to q, and equating it to 0, we get the optimal reserve price q^*. This is a solution of

$$q - \frac{1 - F(q)}{f(q)} = 0. \tag{5}$$

Buyer accepts PPA with probability $\Pr(v > q^*)$. Then, we obtain optimal revenue of the seller from T rounds of auctions as $Tq^*\Pr(v > q^*)$.

The buyer accepts an offer if value v_t drawn from F in auction t is higher than q^*. The identical items are auctioned in round. Thus, the optimal revenue of the buyer is $T\mathbb{E}[(v - q^*)1_{\{v > q^*\}}]$.

2) Heterogeneous RPPA: Here, the seller has different types of items for auction. In each round one type of item is auctioned. Both the seller and buyer know the type of item that is put in for auction. Our objective here is to show that, in heterogeneous RPPA, the optimal strategy for a seller is to have dynamic reserve price. To gain insight into this, let us look at the following example.

Example: Consider a buyer and seller model with two types of items to be auctioned by the seller. Suppose that v_1 and v_2 are valuations of the buyer for these two types of items; and $v_1 < v_2$. In auction t, the item of type i is auctioned with prob. p_i, and $0 < p_1, p_2 < 1$, $p_1 + p_2 = 1$. Now suppose that the seller sets a reserve price q_1, for all T round of auctions and $q_1 < v_1 < v_2$. The revenue of the seller over T rounds of auction under reserve price q_1 is that $q_1 T$. If seller sets reserve price higher than v_2, then clearly seller's revenue is 0. If the seller sets the reserve price q_2 for all T and $v_1 < q_2 < v_2$, then clearly there is no revenue of the seller from type 1 items auction because reserve price is higher than valuation but the seller can have revenue from type 2 item. Thus the revenue of the seller is $q_2 p_2 T$. Now suppose the seller has a dynamic reserve price such that, when item of type i is auctioned, the reserve price set is q_i, and $q_1 < v_1 < q_2 < v_2$. Total revenue of the seller under dynamic reserve price is $(p_1 q_1 + p_2 q_2)T$. Clearly, we have $(p_1 q_1 + p_2 q_2)T > q_2 p_2 T$ and $(p_1 q_1 + p_2 q_2)T > q_1 T$. Thus, the seller uses dynamic reserve price for multi-type of items.

We now extend this to K-different types of items. Their valuation is drawn from distribution F_k with density f_k over support $[\underline{v}_k, \overline{v}_k]$ and $\underline{v}_{k-1} < \overline{v}_k$.

Proposition 2: In heterogeneous RPPA, the optimal reserve price of the seller is dynamic. The optimal revenue

of a seller using dynamic reserve price over T rounds of auctions is

$$\sum_{t=1}^{T}\sum_{k=1}^{K} p_k q_k^* \Pr(v_t > q_k^*).$$

Sketch of the proof: In auction t item of type k is auctioned with probability p_k, $\sum_{k=1}^{K} p_k = 1$. The buyer has valuation v_t which is drawn from probability distribution F_k over $[\underline{v}_k, \overline{v}_k]$, given that type k item is auctioned. The optimal reserve price set by the seller is q_k^* which is solution of $q_k - \frac{1 - F_k(q_k)}{f_k(q_k)} = 0$. The seller's revenue conditioned on type k item is $q_k^* \Pr_{F_k}(v > q_k^*)$. For K different type of items, there are different optimal reserve prices. $q^* = [q_1^*, \cdots, q_K^*]$ is vector of optimal reserve prices for K type of items. The optimal revenue from a auctions is

$$\sum_{k=1}^{K} p_k q_k^* \Pr_{F_k}(v_t > q_k^*).$$

Thus total revenue over T rounds of auctions is

$$\sum_{t=1}^{T}\sum_{k=1}^{K} p_k q_k^* \Pr_{F_k}(v_t > q_k^*).$$

From the buyer's perspective, the revenue from auction t conditioned on the auction of item of type k, is $\mathbb{E}_{F_k}\left[(v_t - q_k^*)1_{\{v_t > q_k^*\}}\right]$ Therefore, buyer's total revenue over T rounds is

$$\sum_{t=1}^{T}\sum_{k=1}^{K} p_k \mathbb{E}_{F_k}\left[(v_t - q_k^*)1_{\{v_t > q_k^*\}}\right].$$

B. Repeated posted price auction with a DSP

1) Homogeneous RPPA: There is a seller and a DSP. The seller auctions identical items in each round of auctions. DSP has N advertisers (clients). We first study the DSP's problem of advertiser scheduling, assuming perfect hindsight of valuation for advertisers. $v_{n,t}$ is the valuation of advertiser n and q^* is the reserve price set by the seller. A naive scheduling scheme based on highest valuation is described in Algorithm. 1. The advertiser with the highest valuation in round t is scheduled if it is higher than reserve price q^*. We denote the allocation matrix as $\{x_{n,t}\}$; $x_{n,t} = 0$ if $v_{n,t} < q^*$, for $n \in \mathcal{N}$, $1 \le t \le T$ and $x_{n,t} \in \{0, 1\}$ if $v_{n,t} \ge q^*$, for $n \in \mathcal{N}$, $1 \le t \le T$. If two or more advertisers share the highest valuation, then one of them is scheduled randomly.

When the valuation of the advertiser $v_{n,t} = v$ for all $1 \le t \le T$ and all $n \in \mathcal{N}$ then advertisers are scheduled uniform randomly. If $v_{1,t} > \max_{n \in \mathcal{N}, n \neq 1} v_{n,t}$, for all $1 \le t \le T$, then according to Algorithm 1, DSP schedules only advertiser 1. Algorithm 1 is not a suitable choice for the DSP as it is not fair; some advertisers are disproportionately favored. Advertisers often have impression constraints, i.e., they want to display their ad for a fixed number of times. To account for this, we formulate the following problem under

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Algorithm 1: DSP Scheduling algorithm for advertisers.

Input: The valuations of the advertisers, $v_{n,t}$ for all $n \in \mathcal{N}$, $1 \leq t \leq T$ and seller's reserve price q^*.
Output: Allocation matrix $x_{n,t}$
for $1 \leq t \leq T$ **do**
> $v_{\max,t} = \max_{n \in \mathcal{N}} v_{n,t}$
> $a_t = \arg\max_{n \in \mathcal{N}} v_{n,t}$
> $x_{n,t} = 1$ if $a_t = n$, & $v_{\max,t} \geq q^*$
> $x_{n,t} = 0$ else
> **end**

end

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perfect hindsight of valuation of advertiser.

$$\begin{aligned} \max_x \quad & G(x) = \sum_{t=1}^{T} \sum_{n=1}^{N} \left[(v_{n,t} - q^*) x_{n,t} \right] \\ \text{s.t.} \quad & \sum_{n=1}^{N} x_{n,t} \leq 1, \ \text{for } 1 \leq t \leq T \\ & \sum_{t=1}^{T} \sum_{n=1}^{N} x_{n,t} \leq T \\ & \sum_{t=1}^{T} x_{n,t} \geq \Delta_n \quad \text{for } n \in \mathcal{N} \\ & x_{n,t} = 0 \ \text{ if } v_{n,t} < q^*, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T \\ & x_{n,t} \in \{0,1\} \ \text{ if } v_{n,t} \geq q^*, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T. \end{aligned}$$
(P1)

Note that problem (P1) is a linear program with integer constraints. The first constraint is that only an advertiser can be scheduled, the second is a constraint on the total number of auctions won by the DSP, the third constraint presents the demand of impressions from the advertiser, Δ_n, we assume that $\Delta_n < T$. Fourth and fifth constraints are on the values of allocation variable $x_{n,t}$. The solution of the problem can be approached using Lagrangian relaxation method. This provides an upper bound on the solution of problem (P1), [21, Chapter 10]. One can further relax the integer constraint; this is the relaxed Lagrangian problem. Then, a subgradient scheme can be used to update the Lagrangian variable.

We now suppose that the valuations of advertisers are drawn from identical distribution F with density f and follow the regularity property. Then, the seller can set optimal reserve price q^*. Note that this reserve price is fixed for all the rounds of auctions. The DSP sequentially schedules an advertiser from the set of advertisers. It solves the following optimization problem.

$$\begin{aligned} \max_x \quad & \sum_{t=1}^{T} \sum_{n=1}^{N} \mathbb{E}_F \left[(v_{n,t} - q^*) 1_{\{v_{n,t} > q^*\}} x_{n,t} \right] \\ \text{s.t.} \quad & \sum_{n=1}^{N} x_{n,t} \leq 1, \ \text{for } 1 \leq t \leq T \\ & x_{n,t} = 0 \ \text{ with prob. } F(q^*) \text{ for all } n, t \\ & x_{n,t} \in \{0,1\} \ \text{ with prob. } 1 - F(q^*) \text{ for all } n, t \end{aligned}$$
(P2)

In this optimization formulation, the expectation is w.r.t. distribution F. The first constraint indicates that only one advertiser can be scheduled in each round. The condition in the second constraint implies that $v_{n,t} < q^*$ and hence $x_{n,t} = 0$. This occurs with prob. $F(q^*)$. The third condition implies that $v_{n,t} \geq q^*$ and hence $x_{n,t} \in \{0,1\}$. This occurs with prob. $1 - F(q^*)$.

A simple strategy for the DSP would be to schedule the advertiser with maximum valuation in the current round if it is higher than optimal reserve price, otherwise reject the

offer of the seller. This allocation mechanism is randomized because the valuation of each advertiser is drawn according to F from $[\underline{v}, \overline{v}]$. This is described in Algorithm 2.

Algorithm 2: DSP's randomized scheduling algorithm for advertisers.

Input: Seller's reserve price is q^*.
Output: Allocation matrix $x_{n,t}$
for $1 \leq t \leq T$ **do**
> Draw $v_{n,t}$ according to F from $[\underline{v}, \overline{v}]$
> $v_{\max,t} = \max_{n \in \mathcal{N}} v_{n,t}$
> $a_t = \arg\max_{n \in \mathcal{N}} v_{n,t}$
> $x_{n,t} = 1$ if $a_t = n$ and $v_{max,t} > q^*$
> $x_{n,t} = 0$ else

end

The probability of scheduling an advertiser in round t is given by

$$\begin{aligned} \Pr(\max_n v_{n,t} > q^*) &= 1 - \Pr(\max_n v_{n,t} \leq q^*) \\ &= 1 - \prod_{n=1}^{N} \Pr(v_{n,t} \leq q^*) \\ &= 1 - (F(q^*))^N. \end{aligned}$$

The expected number of impressions (auctions) won over T rounds of auctions is $T(1 - (F(q^*))^N)$. Hence, the expected number of impressions assigned to an advertiser is $\frac{T(1 - (F(q^*))^N)}{N}$.

The advertisers may also have different demands for impressions. Thus, the DSP may need to consider the following impression constraint optimization problem, where constraints are on expected impressions.

$$\begin{aligned} \max_x \quad & \sum_{t=1}^{T} \sum_{n=1}^{N} \mathbb{E}_F \left[(v_{n,t} - q^*) 1_{\{v_{n,t} > q^*\}} x_{n,t} \right] \\ \text{s.t.} \quad & \sum_{n=1}^{N} x_{n,t} \leq 1, \ \text{for } 1 \leq t \leq T \\ & \mathbb{E}_F \left[\sum_{t=1}^{T} 1_{\{v_{n,t} > q^*\}} x_{n,t} \right] \geq \Delta_n \quad \text{for } n \in \mathcal{N} \\ & x_{n,t} = 0 \ \text{ with prob. } F(q^*) \text{ for all } n, t \\ & x_{n,t} \in \{0,1\} \ \text{ with prob. } 1 - F(q^*) \text{ for all } n, t. \end{aligned}$$
(P3)

Here, second condition gives the expected demand constraints from advertisers. Again, solution to problem (P3) can be obtained using Lagrangian relaxation approach.

2) Heterogeneous RPPA: We now analyze the DSP's advertiser scheduling problem for a heterogeneous RPPA. At the beginning of each round, the DSP and seller know the type of item that is put in for auction. Using this information, the seller sets an optimal reserve price and it is dependent on type of item.

We assume that DSP has perfect hindsight of valuation for advertisers. Let $v_{n,k,t}$ be the valuation of advertiser n if item of type k is auctioned in round t. Let q_k^* be the optimal reserve price for type k item, set by the seller. As before, we first study a naive scheduling scheme based on highest valuation; this is given in Algorithm. 3. The advertiser with the highest valuation in round t is scheduled if its valuation is higher than reserve price q_k^*. If two or more advertisers share the highest valuation, then one of them is scheduled

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Algorithm 3: DSP Scheduling algorithm of heterogeneous items for advertisers.

Input: The valuations of the advertisers, $v_{n,k,t}$ for all $n \in \mathcal{N}$, $1 \leq t \leq T$. and $\{b_t\}_{t=1}^{T}$

Output: Allocation matrix $x_{n,t}$

for $1 \leq t \leq T$ **do**
    Type of item for auction is observed, i.e., k
    $v_{\max,t} = \max_{n \in \mathcal{N}} v_{n,k,t}$
    $a_t = \arg \max_{n \in \mathcal{N}} v_{n,t}$
    Seller sets the optimal reserve price $b_t = q_k^*$
    $x_{n,t} = 1$ if $a_t = n$, & $v_{\max,t} \geq b_t$
    $x_{n,t} = 0$ else
end
end

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randomly. We use $\{b_t\}_{t=1}^{T}$ to denote the sequence of the type of item auctioned; $b_t \in \{q_1^*, \cdots q_K^*\}$.

The advertisers can have constraints on number of impressions. Also, each advertiser may prefer different types of items with different frequency. Hence, the DSP considers the following optimization formulation.

$$
\begin{aligned}
\max_x \quad & \textstyle\sum_{t=1}^{T} \sum_{n=1}^{N} \sum_{k=1}^{K} \left[(v_{n,k,t} - b_t) x_{n,t} \right] \\
\text{s.t.} \quad & \textstyle\sum_{n=1}^{N} x_{n,t} \leq 1, \text{ for } 1 \leq t \leq T \\
& \textstyle\sum_{t=1}^{T} 1_{\{v_{n,k,t} > q_k^*\}} x_{n,t} \geq y_{n,k} \text{ for } n \in \mathcal{N}, 1 \leq t \leq T \\
& \textstyle\sum_{t=1}^{T} x_{n,t} \geq \Delta_n \quad \text{for } n \in \mathcal{N} \\
& x_{n,t} = 0 \text{ if } v_{n,k,t} < b_t, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T \\
& x_{n,t} \in \{0, 1\} \text{ if } v_{n,k,t} \geq b_t, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T.
\end{aligned}
$$
(P4)

Note that all constraints in problem (P4) are similar to earlier problem (P1), except the second constraint which indicates demands of advertisers for the number of impressions for different types of items.

We next study heterogeneous RPPA when valuation of the advertisers is a realization drawn from fixed distributions. An item of type k is auctioned in slot t with probability p_k. The seller and DSP has knowledge of type of items at the beginning of every auction. The DSP can schedule advertisers based on the highest valuation. The probability of scheduling an advertiser in round t given that type k item is auctioned, is given by

$$
\Pr(\max_n v_{n,k,t} > q_k^* \mid \text{ type } k \text{ item }) = 1 - (F_k(q_k^*))^N.
$$

The expected number of impressions (auctions) won over T rounds of auctions are $T \sum_{k=1}^{K} p_k (1 - (F_k(q_k^*))^N)$. Hence, the expected number of impressions assigned to an advertiser are $\frac{T \sum_{k=1}^{K} p_k (1 - (F_k(q_k^*))^N)}{N}$.

The advertisers may have an expectation constraint on the number on impressions. In this case the DSP considers the following impression constraint optimization problem.

$$
\begin{aligned}
\max_x \quad & \left[\textstyle\sum_{t=1}^{T} \sum_{n=1}^{N} \sum_{k=1}^{K} p_k \mathbb{E}_{F_k} \left[(v_{n,k,t} - q_k^*) 1_{\{v_{n,k,t} > q_k^*\}} x_{n,t} \right] \right] \\
\text{s.t.} \quad & \textstyle\sum_{n=1}^{N} x_{n,t} \leq 1, \text{ for } 1 \leq t \leq T \\
& \mathbb{E}_{F_k} \left[\textstyle\sum_{t=1}^{T} 1_{\{v_{n,k,t} > q_k^*\}} x_{n,t} \right] = y_{n,k} \text{ for } n \in \mathcal{N}, 1 \leq t \leq T \\
& \textstyle\sum_{t=1}^{T} \sum_{k=1}^{K} \mathbb{E}_{F_k} \left[1_{\{v_{n,k,t} > q_k^*\}} x_{n,t} \right] \geq \Delta_n \quad \text{for } n \in \mathcal{N} \\
& x_{n,t} = 0 \quad \text{with prob. } \textstyle\sum_{k=1}^{K} p_k F_k(q_k^*) \text{ for all } n, t \\
& x_{n,t} \in \{0, 1\} \quad \text{with prob. } 1 - \textstyle\sum_{k=1}^{K} p_k F_k(q_k^*)) \text{ for all } n, t.
\end{aligned}
$$
(P5)

Here, conditioned on type of item k is auctioned in round t, $x_{n,t} = 0$ with prob. $F_k(q_k^*)$. Hence unconditional this, $x_{n,t} = 0$ with prob. $\sum_{k=1}^{K} p_k F_k(q_k^*)$. Similarly, $x_{n,t} \in \{0, 1\}$ with prob. $1 - \sum_{k=1}^{K} p_k F_k(q_k^*)$.

IV. BUDGET CONSTRAINED ADVERTISERS

Let us now look at the budget management strategy for advertiser, i.e., buyer's perspective in posted price auction. Here, the seller sets a fixed reserve price, say q. To gain insight into this, consider a single advertiser and seller model with posted price mechanism. Let v be the fixed valuation of advertiser with total budget B and he is interacting with the seller over T rounds. Let $v > q$. If total budget $B > Tq$, then the advertiser can win all impressions. Suppose that $B < Tq$, the advertiser can win a fraction M of impressions from T, i.e., $M = \frac{B}{q}$. A simple strategy of the advertiser could be to buy the first M impressions out of T. But, this may not be suitable for an advertiser who wants to display ads uniformly over T auctions. Hence, the advertiser uses a probabilistic throttling scheme in which he participates in each round of auctions with probability ξ, and $0 < \xi < \frac{M}{T} < 1$. This simple scenario suggests that an advertiser with tight budget would prefer to participate in auctions selectively.

Now consider a DSP with clients who are budget constrained. Let us now look at the problem of budget-constrained-advertiser scheduling at the DSP in a homogeneous RPPA. Also, the advertisers have distinct budget constraints. Let B_n be the budget of advertiser n. For simplicity assume that all advertisers have the same valuation v and this is higher than seller's reserve price q. And, $\sum_{n=1}^{N} B_n \leq Tq$. The DSP can use probabilistic throttling scheme in which it schedules advertiser n in each round with probability $0 < \xi_n < \frac{B_i}{q} < 1$, and $\sum_{n=1}^{N} \xi_n = 1$.

For the case where valuation changes for every round, the DSP considers following budget constrained optimization problem under perfect hindsight of valuation of advertisers.

$$
\begin{aligned}
\max_x \quad & \textstyle\sum_{t=1}^{T} \sum_{n=1}^{N} \left[(v_{n,t} - q^*) x_{n,t} \right] \\
\text{s.t.} \quad & \textstyle\sum_{n=1}^{N} x_{n,t} \leq 1, \text{ for } 1 \leq t \leq T \\
& \textstyle\sum_{t=1}^{T} x_{n,t} \leq \frac{B_n}{q^*} \quad \text{for } n \in \mathcal{N} \\
& x_{n,t} = 0 \text{ if } v_{n,t} < q^*, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T \\
& x_{n,t} \in \{0, 1\} \text{ if } v_{n,t} \geq q^*, \text{ for } n \in \mathcal{N}, 1 \leq t \leq T.
\end{aligned}
$$
(P6)

Note that in the above formulation, B_n is budget of advertiser n, and the second constraint indicates the budget constraint of that advertiser.

Next, suppose that the valuations of advertisers are drawn from identical distribution F with density f and seller sets

the optimal reserve price q^*. The DSP sequentially schedules budget constrained advertisers by solving the following optimization problem.

$$
\begin{aligned}
\max_x \quad & \mathbb{E}_F\left[\sum_{t=1}^{T}\sum_{n=1}^{N}\left[(v_{n,t}-q^*)1_{\{v_{n,t}>q^*\}}x_{n,t}\right]\right] \\
\text{s.t.} \quad & \sum_{n=1}^{N}x_{n,t}\leq 1, \text{ for } 1\leq t\leq T \\
& \mathbb{E}_F\left[\sum_{t=1}^{T}1_{\{v_{n,t}>q^*\}}x_{n,t}\right]\leq\frac{B_n}{q^*} \quad \text{for } n\in\mathcal{N} \\
& x_{n,t}=0 \quad \text{with prob. } F(q^*) \text{ for all } n,t \\
& x_{n,t}\in\{0,1\} \quad \text{with prob. } 1-F(q^*) \text{ for all } n,t
\end{aligned}
$$
(P7)

Second constraint indicates that the expected budget constraint on each advertiser.

In heterogeneous RPPA, single seller and an advertiser problem is non trivial. The advertiser may have preferences for different type of items. Hence he would like balance budget for distinct items. This is again constrained optimization problem. Finally, we consider DSP with budget constrained advertisers and DSP solves the following budget constrained optimization problem.

$$
\begin{aligned}
\max_x \quad & \sum_{t=1}^{T}\sum_{k=1}^{K}p_k\sum_{n=1}^{N}\mathbb{E}_{F_k}\left[(v_{n,k,t}-q_k^*)1_{\{v_{n,k,t}>q_k^*\}}x_{n,t}\right] \\
\text{s.t.} \quad & \sum_{n=1}^{N}x_{n,t}\leq 1, \text{ for } 1\leq t\leq T \\
& \mathbb{E}_{F_k}\left[\sum_{t=1}^{T}1_{\{v_{n,k,t}>q_k^*\}}x_{n,t}\right]\geq y_{n,k} \text{ for } n\in\mathcal{N},\ 1\leq t\leq T \\
& \sum_{t=1}^{T}\sum_{k=1}^{K}p_k\mathbb{E}_{F_k}\left[q_k^*1_{\{v_{n,k,t}>q_k^*\}}x_{n,t}\right]\leq B_n \quad \text{for } n\in\mathcal{N} \\
& x_{n,t}=0 \quad \text{with prob. } \sum_{k=1}^{K}p_kF(q_k^*) \text{ for all } n,t \\
& x_{n,t}\in\{0,1\} \quad \text{with prob. } \sum_{k=1}^{K}p_k(1-F(q_k^*)) \text{ for all } n,t.
\end{aligned}
$$
(P8)

We note that in above formulation, second constraint indicated different demands for different type of items and the third constraint represent the expected budget constraint.

V. NUMERICAL RESULTS

In this section, we first provide numerical examples for a single seller and single buyer model. We obtain the optimal reserve price when the valuation of the buyer is a random variable with uniform , exponential, and lognormal distribution. Also, we compute the expected revenue of the buyers over T rounds of auctions. Next we consider a numerical example for the DSP under impression constraints. Finally, we provide simulations for a DSP with multiple budget constraint advertisers.

- Uniform distribution $U[\underline{v},\overline{v}]$, then cdf is

$$
F(v)=\begin{cases} 0 & \text{if } v<\underline{v} \\ \frac{v-\underline{v}}{\overline{v}-\underline{v}} & \text{if } v\in[\underline{v},\overline{v}) \\ 1 & \text{if } v>\overline{v}. \end{cases}
$$

and pdf is

$$
f(v)=\begin{cases} \frac{1}{\overline{v}-\underline{v}} & \text{if } v\in[\underline{v},\overline{v}] \\ 0 & \text{Otherwise} \end{cases}
$$

Then using eqn. (5), we obtain

$$
v-\frac{1-\frac{v-\underline{v}}{\overline{v}-\underline{v}}}{\frac{1}{\overline{v}-\underline{v}}}=0.
$$

and hence we can have optimal reserve price $q^*=\frac{\overline{v}}{2}$. In the example $U[0,1]$, optimal reserve price $1/2$. From Eqn. (3) and (4), the expected revenue of a seller is $\frac{T}{4}$ and expected revenue of a buyer is $\frac{T}{8}$.

- Exponential distribution $Exp[\lambda]$, then cdf is

$$
F(v)=1-e^{-\lambda v} \text{ for } v\in[0,\infty).
$$

and pdf is

$$
f(v)=\lambda e^{-\lambda v}
$$

Then using eqn. (5), we obtain

$$
v-\frac{1-(1-e^{-\lambda v})}{\lambda e^{-\lambda v}}=0.
$$

and hence we can have optimal reserve price $v^*=\frac{1}{\lambda}$. Then

$$
\Pr(v^*>\frac{1}{\lambda})=1-\Pr(v^*\leq\frac{1}{\lambda})
$$
$$
=1-\left(1-e^{-\lambda\times\frac{1}{\lambda}}\right)=1-1+e^{-1}=e^{-1}
$$

Thus the seller's revenue is $T/\lambda e$. Buyers expected revenue is

$$
T\int_{1/\lambda}^{\infty}\left(x-\frac{1}{\lambda}\right)\lambda e^{-\lambda x}dx=\frac{T}{\lambda e}.
$$

Note that mean of exponential random variable is $\mu=\frac{1}{\lambda}$. This suggest that the higher the mean implies higher expected revenue for both buyer and seller.

- Lognormal distribution: The cdf with mean μ and variance σ is

$$
F(v)=\Phi\left(\frac{\ln v-\mu}{\sigma}\right),
$$

where Φ is the cumulative distribution function of the standard normal distribution (i.e. $N(0,1)$). The probability density function is

$$
f(v)=\frac{1}{v\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln v-\mu)^2}{2\sigma^2}\right).
$$

Then using eqn. (5), we get

$$
v-\frac{1-\Phi\left(\frac{\ln v-\mu}{\sigma}\right)}{\frac{1}{v\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln v-\mu)^2}{2\sigma^2}\right)}=0.
$$

After further simplification we get

$$
\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln v-\mu)^2}{2\sigma^2}\right)-1+\Phi\left(\frac{\ln v-\mu}{\sigma}\right)=0.
$$

Analytical solution of this equation is difficult to obtain. We compute the solution using numerical methods and get the optimal reserve price q^*. Buyers expected revenue is

$$
T\int_{q^*}^{\infty}(v-q^*)\frac{1}{\sigma v\sqrt{2\pi}}\exp\left(-\frac{(\ln v-\mu)^2}{2\sigma^2}\right)dv.
$$

Seller's expected revenue is

$$
Tq^*\left(1-\Phi\left(\frac{\ln v-\mu}{\sigma}\right)\right).
$$

It is difficult to write the above expressions in closed form. Hence, we provide results through numerical

μ	σ	q^*	Seller R	Buyer R
0	1	1.4	$0.5156T$	$0.7512T$
0.25	1	1.8	$0.8380T$	$0.9175T$
0.5	1	2.3	$1.23T$	$1.18T$
2	1	10.1	$9.61T$	$5.37T$

methods, in Table I. From Table I observe that as mean μ increases, the optimal reserve price of the seller increases. This means, the sellers expected revenue also increases. Further, the expected revenue of buyer also increases with μ.

A. DSP Problem

We present numerical example for RPPA with a DSP, and use the naive scheduling scheme (see Algorithm 1) when the valuations of advertisers are known in hindsight. We use following parameters: number of auctions $T = 10000$, $N = 5$. We use fixed valuation $v_{n,t}$ and generate $[[v_{n,t}]]$ using lognormal distribution with mean $\mu = 0$ and variance $\sigma = 1$. For different reserve prices q, we provide simulations to illustrate effect of reserve price on the expected revenue of seller ($\frac{\sum_{t=1}^{T} q 1_{\{\max_n v_{n,t} > q\}}}{T}$), the expected revenue of buyers ($\frac{\sum_{t=1}^{T} (v_{n,t} - q) x_{n,t}}{T}$), and number of impressions win ($\sum_{t=1}^{T} x_{n,t}$). The results are summarized in the tables below The above numerical examples suggests that, setting of

Reserve price q	$q = 1$	$q = 2$	$q = 4$
Seller's revenue	0.9727	1.5104	1.3812

Adv.	$q = 1$		$q=2$		$q=4$	
	Impressions	Revenue	Imp.	Rev.	Imp.	Rev.
1	2004	0.6449	1551	0.4651	707	0.2506
2	1906	0.5951	1484	0.4226	683	0.2137
3	1882	0.5759	1457	0.4060	643	0.2050
4	1950	0.6125	1517	0.4363	735	0.2221
5	1985	0.6239	1543	0.4454	685	0.2316

optimal reserve price by the seller impacts its expected revenue. High reserve price may lead to low revenue. And, too low reserve price also leads to low revenue. But by setting optimal reserve price, a seller can optimize its revenue. As reserve price of seller increases, the number of impressions won by DSP for advertiser decreases and also the expected revenue of advertisers decreases.

More simulation results are provided in [22].

VI. CONCLUDING REMARKS

This paper has studied repeated posted price auction (RPPA) in online advertising. We considered the problem of a single seller and an advertiser where items are auctioned sequentially. This was studied for both homogeneous and heterogeneous items. We also computed the optimal reserve price for seller under homogeneous items. We have shown that for heterogeneous items, the seller's optimal reserve price is dynamic. Later, we studied an RPPA involving a DSP, and devised a simple scheduling scheme under assumption of hindsight on valuation for repeated auctions. We also formulated the advertiser scheduling problem of the DSP when the advertisers have impression and budget constraints. This was again done for both homogeneous and heterogeneous items. Finally, we presented some numerical examples for a single seller and advertiser model with PPA and considered various valuation distributions such as uniform, exponential and lognormal. We also provided numerical examples for a single seller and DSP model with a naive scheduling scheme based on valuation of advertisers.

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VII. Appendix

A. Numerical Example and Extension of Table I

In this, we provide a numerical example of a single seller and advertiser model with lognormal distribution function for different values of variance σ. This is given in Table II.

TABLE II

THE OPTIMAL RESERVE PRICE, EXPECTED REVENUE OF SELLER (SELLER R) AND EXPECTED REVENUE OF BUYER (BUYER R) FOR DIFFERENT σ.

μ	σ	q^*	Seller R	Buyer R
0	0.25	0.76	$0.07T$	$0.28T$
0	0.5	0.78	$0.2T$	$0.41T$
0	1	1.36	$0.5T$	$0.73T$
0	2	23.2	$10.05T$	$3.57T$

We observe that as variance σ increases, expected revenue of the seller and advertiser increases. Further, the seller's expected revenue increase is significantly higher because the reserve price q^* increase is steep.

B. Numerical Example of DSP Problem

We here present additional numerical examples with a single seller and DSP problem. We use the same parameters as in Section V-A. Here, we study a round-robin and randomized policy for scheduling of advertiser.

In round-robin (RR) policy, DSP selects an advertisers are selected in round-robin fashion, and are scheduled for participation in auction if their valuation is higher than the reserve price. Otherwise they are not scheduled. Numerical results are presented in Table III and IV. Since, in some auctions no advertiser is scheduled, advertisers win fewer number of impressions as compared to given in Sec. V-A. Here, as the reserve price increases, the expected revenue of the seller and of the advertisers decreases.

TABLE III

THE SELLER'S REVENUE FOR DIFFERENT RESERVE PRICE WHEN DSP USES ROUND ROBIN POLICY

Reserve price q	$q = 1$	$q = 2$	$q = 4$
Seller's revenue	0.502	0.4954	0.3324

TABLE IV

ADVERTISER'S REVENUE FOR DIFFERENT RESERVE PRICE WHEN DSP USES ROUND ROBIN POLICY

Adv.	$q = 1$ Impressions	Revenue	q=2 Imp.	Rev.	q=4 Imp.	Rev.
1	993	0.1713	499	0.1011	164	0.0440
2	1016	0.1813	483	0.1104	165	0.0524
3	994	0.1748	482	0.1062	154	0.0504
4	1003	0.1787	514	0.1073	175	0.0481
5	1014	0.1825	499	0.1115	173	0.0513

We now study a randomized policy. In this policy, DSP selects advertisers in uniform random fashion. These randomly selected advertisers are scheduled for participation in auctions if their valuation is higher than the seller's reserve price q. Numerical examples with different reserve price values are given in Table V and VI. In this policy, if DSP selects an advertiser whose valuation is smaller than the reserve price, no advertiser is scheduled in auction. Observe that as reserve price increases, expected revenues of the seller and the advertisers decrease.

TABLE V

THE SELLER'S REVENUE FOR DIFFERENT RESERVE PRICE WHEN DSP USES UNIFORM RANDOMIZED POLICY

Reserve price q	$q = 1$	$q = 2$	$q = 4$
Seller's revenue	0.493	0.482	0.3364

TABLE VI

ADVERTISER'S REVENUE FOR DIFFERENT RESERVE PRICE WHEN DSP USES UNIFORM RANDOMIZED POLICY

Adv.	$q = 1$ Impressions	Revenue	q=2 Imp.	Rev.	q=4 Imp.	Rev.
1	961	0.183	489	0.1027	160	0.0528
2	1017	0.18	459	0.0846	189	0.0543
3	960	0.17	476	0.1072	134	0.0432
4	948	0.16	480	0.1033	192	0.0577
5	1042	0.185	506	0.1054	166	0.0442

The performance of this policy is very similar to round robin policy In both these policies, in some fraction of auctions, selected advertisers are not scheduled because their valuation is lower than the reserve price.

It is possible that one can modify these policies in the following manner. The set of advertisers is selected in auction based on their valuation, i.e., if their valuation is higher than reserve price. Later, round robin or randomized policy can be used to select and schedule one advertiser for each round of auction. Now, it can be expected that the DSP wins more number of impressions for advertisers. This increases the expected revenue of both the seller and the advertisers.

C. Impression constraint advertisers at DSP

Using Lagrangian relaxation approach for problem (P1), we obtain

$$L(x,\lambda) = \sum_{t=1}^{T}\sum_{n=1}^{N}(v_{n,t}-q)x_{n,t} + \sum_{n=1}^{N}\lambda_n\left(\sum_{t=1}^{T}x_{n,t}-\Delta_n\right).$$

Here, $\lambda_n \geq 0$ is Lagrangian variable associated with demand of advertiser n. After simplification, we have

$$L(x,\lambda) = \sum_{t=1}^{T}\sum_{n=1}^{N}(v_{n,t}+\lambda_n-q)x_{n,t} - \sum_{n=1}^{N}\lambda_n\Delta_n.$$

We represent the additional constraint as follows.

$$C = \left\{x \;\middle|\; \sum_{n=1}^{N}x_{n,t} \leq 1, 1 \leq t \leq T; \right.$$
$$\left. x_{n,t} \in \{0,1\} \;\; \text{for } n \in \mathcal{N}, \, 1 \leq t \leq T \right\}.$$

We have following inequality $\text{Opt}(P1) \leq \max_{x \in C} L(x, \lambda)$ and Let

$$
\begin{aligned}
h(\lambda) &= \max_{x \in C} L(x, \lambda) \\
&= \max_{x \in C} \left[\sum_{t=1}^{T} \sum_{n=1}^{N} (v_{n,t} + \lambda_n - q) x_{n,t} \right] - \sum_{n=1}^{N} \lambda_n \Delta_n.
\end{aligned}
$$

Moreover DSP may schedule nth advertiser if $v_{n,t} + \lambda_n - q \geq 0$, otherwise DSP does not that advertiser. This is because, for not scheduling, the payoff is zero instead of negative. This gives us an intuition that, higher demand of nth advertiser may correspond to higher values of λ_n. This boosts the valuation to $(v_{n,t} + \lambda_n)$. This introduces aggressive bidding behavior of advertisers with high demand of impressions and hence these advertisers are scheduled more frequently than others.

Note that this is a linear program with integer constraints. Thus, it is difficult to solve. An approach is to further relax the integer constraints and consider $0 \leq x_{n,t} \leq 1$. Introduction of this relaxation increases the size of constraints. Hence, we can have

$$
\widetilde{C} = \left\{ x \ \middle| \ \sum_{n=1}^{N} x_{n,t} \leq 1, 1 \leq t \leq T; \right.
$$
$$
\left. 0 \leq x_{n,t} \leq 1 \ \text{ for } n \in \mathcal{N}, \ 1 \leq t \leq T, \right\}.
$$

and $C \subset \widetilde{C}$. Then relaxed Lagrangian problem is

$$
\begin{aligned}
h^R(\lambda) &= \max_{x \in \widetilde{C}} L(x, \lambda) \\
&= \max_{x \in \widetilde{C}} \left[\sum_{t=1}^{T} \sum_{n=1}^{N} (v_{n,t} + \lambda_n - q) x_{n,t} \right] - \sum_{n=1}^{N} \lambda_n \Delta_n.
\end{aligned}
$$

Also, $h(\lambda) \leq h^R(\lambda)$. Then, Lagrangian dual of the relaxation of problem (P1) is given by

$$
h_{LD}^R = \min_{\lambda \geq 0} h^R(\lambda).
$$

Note that $h^R(\lambda)$ is piece-wise linear and convex in λ. Then subgradient algorithm for λ is as follows.

$$
\lambda_n^{k+1} = \max \left\{ 0, \lambda_n^k + s^k \left(\sum_{t=1}^{T} x_{n,t}(\lambda^k) - \Delta_n \right) \right\},
$$

for $n = 1, 2, \cdots, N$. Here, s^k is step sizes. The convergence of subgradient algorithm is obtained for suitably selected step sizes. More detail on step size selection is given in [21, Chapter 10, page no. 502].

1) Numerical examples: We next present few numerical examples with simple heuristic greedy algorithms. In the first example, we consider the valuation of all advertisers to be equal and it is constant for all rounds of auctions. Moreover, this valuation is higher than reserve price q. We use the following parameters. $v = 2.5$, $q = 1$, $N = 5$ and $T = 10000$ and demand for impressions $\Delta = [400, 800, 4800, 400, 1600]$. Define $\xi_n = \frac{\Delta_n}{\sum_{n=1}^{N} \Delta_n}$. Then $\xi = [0.05, 0.1, 0.6, 0.05, 0.2]$. We use a greedy algorithm in

which an advertiser is selected according to their demand, i.e., advertiser n is selected with prob. ξ_n. Table VII shows the expected revenue of seller and the number of impressions for different advertisers. Expected revenue of the seller is 1. Notice that a simple greedy algorithm at DSP meets the demands of all advertisers.

TABLE VII
ADVERTISER'S REVENUE WITH FIXED VALUATION $v = 2.5$, $q = 1$ AND IMPRESSION DEMAND

| Adv. | $q = 1$ | |
	Impressions	Revenue
1	485	0.07
2	984	0.15
3	6002	0.91
4	498	0.07
5	2031	0.3

In second numerical example, the valuation of advertisers is fixed and it is drawn according to lognormal distribution function. In this example, DSP selects advertiser n with prob. ξ_n and it is scheduled in auction t if $v_{n,t} > q$. Otherwise no advertiser is scheduled in auction t. The expected revenue of advertisers and number of impressions is given in Table VIII. The seller's expected revenue is 0.5. Observe that using this policy, the impression demands of some advertisers are not fulfilled.

TABLE VIII
ADVERTISER'S REVENUE WITH FIXED VALUATION ACCORDING TO LOGNORMAL DISTRIBUTION FUNCTION AND IMPRESSION DEMAND

| Adv. | $q = 1$ | |
	Impressions	Revenue
1	260	0.04
2	519	0.09
3	2981	0.52
4	235	0.04
5	1000	0.17

TABLE IX
ADVERTISER'S REVENUE WITH FIXED VALUATION ACCORDING TO LOGNORMAL DISTRIBUTION FUNCTION AND IMPRESSION DEMAND: EXAMPLE-3

| Adv. | $q = 1$ | |
	Impressions	Revenue
1	1024	0.19
2	1607	0.28
3	3749	0.64
4	1036	0.19
5	2317	0.40

In the third example we consider another variant of the preceding algorithm. Here, advertisers with valuation above the reserve price are selected. Then, DSP schedules advertisers according to their demand of impressions. The results are shown in Table IX. Now observe that the impression demand of all advertisers is fulfilled, except advertiser 3 which is very higher than other advertisers. This simple variation performs better than the preceding algorithm. Thus,

the expected revenue of advertisers is higher and the expected revenue of a seller is 0.97.

TABLE X

ADVERTISER'S REVENUE WITH FIXED VALUATION ACCORDING TO LOGNORMAL DISTRIBUTION FUNCTION AND IMPRESSION DEMAND: EXAMPLE-4

Adv.	$q = 1$	
	Impressions	Revenue
1	814	0.1557
2	1177	0.1972
3	5046	0.8235
4	810	0.1534
5	1931	0.3342

In our final numerical example, we use the insight developed from Lagrangian relaxation of the problem. If the demand of advertiser is high, then the valuation of that advertiser is boosted by having higher Lagrangian multiplier. Motivated from this, in this example the valuation of advertiser 3 is increase marginally for all auctions by adding $\lambda_3 = 0.25$ and for other advertisers $\lambda_n = 0$, $n \neq 3$. Here, we use the algorithm studied in example 3. This is given in Table X. The seller's expected revenue is 0.98. Note that using this simple variation, DSP can fulfill the impression demand of all advertisers.