Game-based coalescence over multi-agent systems

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Abstract

Coalescence, as a kind of ubiquitous group behavior in the nature and society, means that agents, companies or other substances keep consensus in states and act as a whole. This paper considers coalescence for n rational agents with distinct initial states. Considering the rationality and intellectuality of the population, the coalescing process is described by a bimatrix game which has the unique mixed strategy Nash equilibrium solution. Since the process is not an independent stochastic process, it is difficult to analyze the coalescing process. By using the first Borel-Cantelli Lemma, we prove that all agents will coalesce into one group with probability one. Moreover, the expected coalescence time is also evaluated. For the scenario where payoff functions are power functions, we obtain the distribution and expected value of coalescence time. Finally, simulation examples are provided to validate the effectiveness of the theoretical results.

Index Terms

Group behavior, coalescence, bimatrix game, expected coalescence time.

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I. Introduction

Recently, group behaviors of individuals have attracted the attention of many disciplines, such as sociology[1], economics[2], [3], biology[4] and engineering[5]. Roughly speaking, group behaviors of multiple agents include consensus[6], flocking[7], containment[8], leader emergence[9], [10] and so on. Among above aspects of group behavior, how to make a group of individuals reaching consensus is a fundamental and important issue. Consensus means that agents reach an agreement upon certain quantities of interest, such as opinions of social individuals[1], [11], and speeds of mobile autonomous robots[12]. Many results were obtained, to name but a few, leader-following consensus[13], consensus problems for multiple double-integrator agents[14], [15] and for agents with different dynamics[16], [17], [18].

The above literature mostly assumed that each agent is simple, and thereby just obeies a uniform rule without figuring out its own interest. However, in the real world, one noteworthy feature is that agents are diverse. For example, they have different objectives or interests. Another prominent feature of agents is of high intelligence — they choose the best possible response based on their interests. Thus, the relationships among agents might be noncooperative, even competitive, and the interaction of them might be playing games instead of obeying the fixed protocols. Based on game theory, some complex group behavior, such as competitive propagation[2], [19], network formation[20], collective learning[4] and coalescence [21], were studied. To achieve a global task, agents need to coalesce, i.e., to form a group where they can make decisions together and act as a whole. Coalescence is common seen in real world, such as coalescence for robot groups [22] and coalescence of opinions in social networks[21], [23].

Inspired by the above references, we consider coalescence of n agents with distinct initial states. It means that agents will finally keep consensus in states and act as a whole. It is necessary to mention the difference between consensus and coalescence. Consensus means agents' states reach or asymptotically converge to an identical value. Whereas, coalescence is more complicated than consensus. To reach coalescence, agents need to reach an agreement on states in the finite time, and from then on alway keep consensus not only in states but also in action. Therefore, the essential question we face is, how to design a mechanism to make agents coalesce into a group. We assume that each agent is rational and accesses complete information, i.e., each agent chooses the best response based on its interest and the global information of the population.

Based on this assumption, we propose a kind of bimatrix games where each player has two strategies to choose – cooperation (C) and defection (D). Cooperation means players sacrifice part of interests and change their states to achieve coalescence. On the contrary, defection means players tend to keep their states regardless of whether coalescing or not. By playing this game, agents coalesce into groups, then the agents in the same group act as a whole and play games with those in other groups. By merging groups and groups, they eventually coalesce into one group. We find that the game has the unique mixed strategy Nash equilibrium — players choose strategies in a probabilistic sense, which makes the coalescing process be a stochastic process. Because it depends on payoff functions, it is not an independent stochastic process. As a result, it is not easy to analyze the coalescence of the population. The contributions of this paper are summarized as follows.

- We establish a kind of bimatrix game model to show the interaction among agents. We prove that the game has the unique mixed strategy Nash equilibrium solution.
- By virtue of the first Borel-Cantelli Lemma, we prove that all the agents coalesce into one group with probability one.
- The distribution and the expected of coalescence time are evaluated.

The rest of this paper is organized as follows. In Section II, we introduce some basic notions of bimatrix game. Section III shows our main results. Numerical simulations are given in Section IV to illustrate the effectiveness of theoretical results. Some conclusions are drawn in Section V.

Throughout this paper, the following notations will be used: let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ be the sets of real numbers and nonnegative real numbers, respectively. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. $\mathcal{I}_n = \{1, \cdots, n\}$ is an index set. For a random event A, $\mathbb{P}(A)$ means the probability of event A. For a random variable S, $\mathbb{E}(S)$ and $\mathbb{D}(S)$ mean the expected value and the variance of S respectively.

II. Preliminaries

A. A brief introduction for bimatrix games

In this subsection, we introduce some basic notions about bimatrix game. For more details, interested readers are referred to [24].

Suppose that two players P_1 and P_2 play a game. P_1 has strategies $r_1, r_2, \cdots, r_m$, and P_2 has strategies $c_1, c_2, \cdots, c_n$. If P_1 adopts the strategy r_i and P_2 adopts the strategy c_j, then (r_i, c_j) is a pair of pure strategies, and a_{ij} (respectively, b_{ij}) denotes the profit incurred to P_1 (respectively, P_2). Each player seeks to maximum its own profit by independent and simultaneous decision. This game is comprised of two $(m \times n)$-dimensional matrices, $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, with each pair of entries (a_{ij}, b_{ij}) denoting the payoff of the game corresponding to a particular pair of decisions made by the players. Thus, this game is called the bimatrix game (A, B). A pair of strategies $\{r_{i^*}, c_{j^*}\}$ is said to constitute a pure strategy Nash equilibrium solution to a bimatrix game (A, B) if the following pair of inequalities is satisfied for all $i \in \mathcal{I}_m$, $j \in \mathcal{I}_n$: $\begin{cases} a_{i^*j^*} \geq a_{ij^*}, \\ b_{i^*j^*} \geq b_{i^*j}. \end{cases}$ Furthermore, the pair $(a_{i^*j^*}, b_{i^*j^*})$ is known as a pure strategy Nash equilibrium of the bimatrix game. In many cases, pure strategy Nash equilibrium strategies might not exist. Hence, we now enlarge the concepts of strategy and Nash equilibrium, which are defined as the set of all probability distributions on the set of pure strategies of each player. We call $\Gamma_1 = \{r_1, r_2, \cdots, r_m\}$ and $\Gamma_2 = \{c_1, c_2, \cdots, c_n\}$ are the strategy spaces of players P_1 and P_2, respectively. Let $\alpha = [\alpha_1, \cdots, \alpha_m]^T$ be a non-negative vector satisfying $\sum_{i=1}^m \alpha_i = 1$, where α_i denotes player P_1 will choose strategy r_i with probability α_i. Obviously, α is the probability distribution of the strategy space Γ_1. We define that α is a mixed strategy of P_1. Likewise, $\beta = [\beta_1, \cdots, \beta_n]^T$ is a mixed strategy of P_2. Suppose that the game is played repeatedly, and the outcomes which are maximized by players is determined by averaging the outcomes of the player. Hence, we call (α, β) as a pair of mixed strategies, and $U_1(\alpha, \beta) = \alpha^T A \beta$ and $U_2 = (\alpha, \beta) = \alpha^T B \beta$ as the corresponding utilities of P_1 and P_2, respectively. Each player decides its mixed strategy independently to maximize its utility. Subsequently, we give the definition of mixed strategy Nash equilibrium[24].

Definition 1: A mixed strategy pair $\{\alpha^*, \beta^*\}$ is said to constitute a mixed strategy Nash equilibrium solution to a bimatrix game (A, B), if the following inequalities are satisfied for all mixed strategy pairs:

$$(\alpha^*)^T A \beta^* \geq \alpha^T A \beta^*,$$

$$(\alpha^*)^T B \beta^* \geq (\alpha^*)^T B \beta.$$

B. The first Borel-Cantelli lemma

At the end of this section, we introduce the first Borel-Cantelli lemma which will be used in our paper.

Lemma 1: (The first Borel-Cantelli lemma[25])Let $\{A_k, k \geq 1\}$ be arbitrary events. If the sum of the probabilities of the A_k is finite, then the probability that infinitely many of them occur is 0, that is, $\mathbb{P}(\cup_{l=1}^{\infty} \cup_{k \geq l}^{\infty} A_k) = 0$.

III. **Coalescence of multiple agents**

Consider a system with n agents labeled $1, 2, \cdots, n$ where each agent i ($i \in \mathcal{I}_n$) has the state $x_i(k) \in \mathbb{R}^m$ at time $k = 0, 1, \cdots$. Throughout this paper, we assume that

Assumption 1: All agents are rational and complete information accessible.

Assumption 2: At time $k = 0$, each agent composes one group and has a distinctive state, i.e., $x_i(0) \neq x_j(0)$ for all $i \neq j$.

Assumption 3: Agents who are in the same group will make decisions together, share information simultaneously and keep consensus on states.

In this paper, we consider how to make n agents coalescing, i.e., merging into one group where they can make decisions together, share information simultaneously and keep consensus on states. We first propose the notion of coalescence for the system.

Definition 2: For a multi-agent system composed of n agents with distinct initial states, if there exists a minimum time K^* such that, starting from time K^*, all agents make decisions together, share information simultaneously and keep consensus on states, then the system is said to reach coalescence at time K^*. Random variable K^* is called the coalescence time of the system. $\mathbb{E}(K^*)$ is called the expected coalescence time.

A. The interaction among groups

In this subsection, we propose a bimatrix game $\mathbb{G}$ to model the interaction of groups. Moreover, the unique mixed strategy Nash equilibrium solution of game $\mathbb{G}$ is obtained.

Players: There are two players P_1 and P_2. Players decide whether or not to change their states by playing games. Let the states of P_1 and P_2 before game be y_1 and y_2 and after game be y_1' and y_2' respectively ($y_r, y_r' \in \mathbb{R}^m, r = 1, 2$).

Strategies: Each player has two strategies to choose from— cooperation (C) and defection (D). If a player chooses C, it means this player will change its state to coalesce with the other player. If a player chooses D, this agent will not change its state regardless of whether they can coalesce or not. Therefore, there are four strategy pairs and the corresponding out-comings (presented in Table I and Fig. 1):

- *If both two players choose C, i.e., the strategy pair (C, C), both of them will update states to the middle of their states to coalesce into a group;*
- *If one player chooses C and the other chooses D, i.e., the strategy pair (C, D) or (D, C), only the cooperative player will change its state to that of the other one's and they coalesce into a group;*
- *If both two players choose D, i.e., the strategy pair (D, D), no one will change its state and thereby merging fails.*

Payoff: Each player will face two kinds of interests— cost of state changing and profit of coalescence. For player $P_r(r = 1, 2)$, the cost of state changing is $f(||y'_r - y_r||)$, and the profit of coalescence is $g(||y_1 - y_2||)$, where $f : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ and $g : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ are strictly monotone increasing continuous functions with $f(0) = g(0) = 0$. Therefore, the payoff of player P_r is $g(||y_1 - y_2||) - f(||y'_r - y_r||)$.

Suppose that two players choose strategies independently and simultaneously. Let $\xi = ||y_1 - y_2||$. The strategy pairs, outcomes, and payoffs of the game are listed in Table I.

TABLE I

THE OUTCOMES AND PAYOFFS OF THE GAME $\mathbb{G}$.

(s_1, s_2)	Outcomes of states	Payoff of P_1	Payoff of P_2
(C, C)	$y'_1 = y'_2 = \frac{y_1+y_2}{2}$	$g(\xi) - f(\frac{\xi}{2})$	$g(\xi) - f(\frac{\xi}{2})$
(C, D)	$y'_1 = y'_2 = y_2$	$g(\xi) - f(\xi)$	$g(\xi)$
(D, C)	$y'_1 = y'_2 = y_1$	$g(\xi)$	$g(\xi) - f(\xi)$
(D, D)	$y'_1 = y_1, y'_2 = y_2$	0	0

Remark 1: Easy to find that the game will represent a prisoner's dilemma if $f(\xi) > g(\xi)$, i.e., each player will choose D, which means that all agents always keep their initial states. Therefore, in the remaining parts of the paper we assume that $f(\xi) < g(\xi)$.

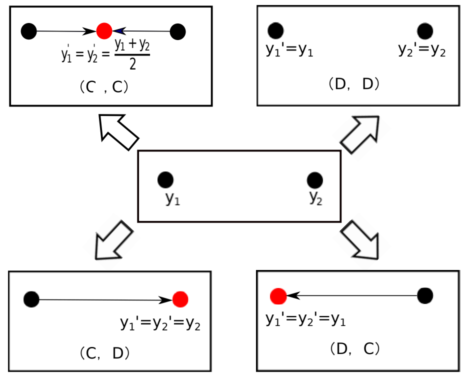

Fig. 1. Four strategy pairs and out-comings of Game $\mathbb{G}$

Theorem 1: Game $\mathbb{G}$ has the unique mixed strategy Nash equilibrium solution, that is each player choosing C with probability $\frac{f(\xi)-g(\xi)}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$ and D with probability $\frac{f(\frac{\xi}{2})}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$.

Proof. According to the definition of bimatrix game, game $\mathbb{G}$ is a bimatrix game where $\Gamma_1 = \Gamma_2 = \{C, D\}$ and

$$A = \begin{pmatrix} g(\xi) - f(\frac{\xi}{2}) & g(\xi) - f(\xi) \\ g(\xi) & 0 \end{pmatrix}, B = A^T.$$

Let $\alpha = [p, 1-p]^T$ and $\beta = [q, 1-q]^T$ be the mixed strategies of P_1 and P_2, respectively. Then, the utilities of P_1 and P_2 are $U_1(p, q) = [p, 1-p]A[q, 1-q]^T$ and $U_2(p, q) = [p, 1-p]B[q, 1-q]^T$.

Suppose that $\{[p^*, 1-p^*]^T, [q^*, 1-q^*]^T\}$ is the mixed strategy Nash equilibrium solution of the game $\mathbb{G}$. By the definition of mixed strategy Nash equilibrium, we have

$$\begin{cases} U_1(p^*, q^*) \geq U_1(p, q^*) \\ U_2(p^*, q^*) \geq U_2(p^*, q) \end{cases},$$

which means that

$$\begin{cases} \left.\dfrac{\partial U_1(p, q^*)}{\partial p}\right|_{p=p^*} = 0, \\ \left.\dfrac{\partial U_2(p^*, q)}{\partial q}\right|_{q=q^*} = 0. \end{cases} \tag{1}$$

By solving (1), we have $p^* = q^* = \frac{g(\xi)-f(\xi)}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$. Hence, we know that the Nash equilibrium solution in the mixed strategies is that each player chooses C with probability $\frac{g(\xi)-f(\xi)}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$ and D with probability $\frac{f(\frac{\xi}{2})}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$. ■

Corollary 1: Two players will coalesce into one bigger group with probability $1-\left(\frac{f(\frac{\xi}{2})}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}\right)^2$.

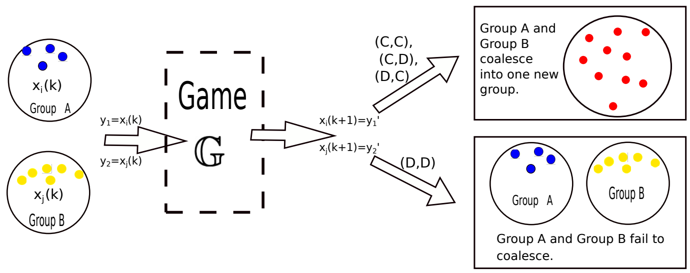

Fig. 2. the dynamic of the system

Proof. By the definition of the game, we know that two players will coalesce if and only if $y_1' = y_2'$. It is easy to find from Table 1 that

$$\mathbb{P}(y_1' = y_2') = 1 - \left(\frac{f(\frac{\xi}{2})}{g(\xi) - f(\xi) + f(\frac{\xi}{2})} \right)^2 . \blacksquare$$

The interaction among groups can be described in the following manner (See Fig. 2): at each time k, two groups are chosen to play the game $\mathbb{G}$, where all involved agents will update their states according to the rules of game $\mathbb{G}$. If merger occurs in the game $\mathbb{G}$, two groups will coalesce into one group.

Remark 2: Since each member of a group has the same state, they have the same interest. Therefore, we assume that strategy selection is determined by all agents of the group. Agents, who make decisions together, obtain the identical payoff simultaneously. Suppose that two groups consist of s_1 agents and s_2 agents respectively. Define

$$U^{(k)}(p^*, q^*) = s_1 U_1(p^*, q^*) + s_2 U_2(p^*, q^*)$$

as the aggregate expectational payoff of two groups at time k.

B. Coalescence of multiple agents: general cases

Let $\xi_k = |y_1(k) - y_2(k)|$ where $y_1(k)$ and $y_2(k)$ represent the states of two players before playing game $\mathbb{G}$ at time k. Then, we can calculate

$$U^{(k)}(p^*, q^*) = (s_1 + s_2) \frac{(g(\xi_k) - f(\xi_k))g(\xi_k)}{g(\xi_k) - f(\xi_k) + f(\frac{\xi_k}{2})}.$$

Let p_k indicate the probability of "two players coalesce at time k". Then, we have $p_k = 1 - \left(\frac{f(\frac{\xi_k}{2})}{g(\xi_k) - f(\xi_k) + f(\frac{\xi_k}{2})} \right)^2$.

Lemma 2: For the initial states $x_1(0), \dots, x_n(0)$, there exist two positive constants $0 < p_{low} < p_{up} < 1$ such that $p_{low} \leq p_k \leq p_{up}$ for all $k = 1, 2, \dots, K^*$.

Proof. For the initial states $x_1(0), \dots, x_n(0)$, we have

$$\xi_{min} \leq \xi_k \leq \xi_{max}.$$

where $\xi_{min} = \min_{i,j \in \{1,2,\dots,n\}} |x_i(0) - x_j(0)|$ and

$$\xi_{max} = \max_{i,j \in \{1,2,\dots,n\}} |x_i(0) - x_j(0)|.$$

Since $f(.)$ and $g(.)$ are continuous, $\frac{f(\frac{\xi}{2})}{g(\xi)-f(\xi)+f(\frac{\xi}{2})}$ is also continuous in $[\xi_{min}, \xi_{max}]$. It is easy to obtain that, there exist two positive constants $0 < \nu < \mu < 1$ such that $0 < \nu \leq \frac{f(\frac{\xi}{2})}{g(\xi)-f(\xi)+f(\frac{\xi}{2})} \leq \mu < 1$ for all $\xi_k \in [\xi_{min}, \xi_{max}]$. Thus, we have $1 - \mu^2 = p_{low} < p_k < p_{up} = 1 - \nu^2$. ■

Theorem 2: If Assumptions 1 - 3 hold and all groups interact by playing game $\mathbb{G}$, then the probability with which the coalescence time equals to $T(\geq n-1)$ can be estimated by

$$C_{T-1}^{n-2}(1-p_{up})^{T+1-n} p_{low}^{n-1} \leq \mathbb{P}(K^* = T) \leq C_{T-1}^{n-2}(1-p_{low})^{T+1-n} p_{up}^{n-1}. \tag{2}$$

Moreover, the expected coalescence time can be estimated by

$$(n-1)\frac{p_{low}^{n-1}}{p_{up}^{n}} \leq \mathbb{E}(K^*) \leq (n-1)\frac{p_{up}^{n-1}}{p_{low}^{n}}.$$

Proof. We know that the number of groups will decrease by 1 at the time k if two players coalesce into one group. Let Δ_k indicate whether two players coalesce at time $k = 1, 2, \cdots$ or not, i.e.,

$$\Delta_k = \begin{cases} 1, \textit{players coalesce at time } k, \\ 0, \textit{otherwise}. \end{cases}$$

By the definition of Δ_k and K^*, it is easy to know that

$$\{K^* = T\} = \{\Delta_T = 1, \sum_{k=1}^{T-1} \Delta_k = n-2\}.$$

Consequently,

$$\mathbb{P}(K^* = T) = \mathbb{P}(\Delta_T = 1 | \sum_{k=1}^{T-1} \Delta_k = n-2)\mathbb{P}(\sum_{k=1}^{T-1} \Delta_k = n-2). \tag{3}$$

By Lemma 2, we have

$$p_{low} \leq \mathbb{P}(\Delta_k = 1 | \Delta_{k-1} = \delta_{k-1}, \dots, \Delta_1 = \delta_1) \leq p_{up}$$

and

$$1 - p_{up} \leq \mathbb{P}(\Delta_k = 0 | \Delta_{k-1} = \delta_{k-1}, \ldots, \Delta_1 = \delta_1) \leq 1 - p_{low}$$

for all $T - 1 \leq k \leq 1$. *Because*

$$\begin{aligned} &\mathbb{P}(\Delta_{T-1} = \delta_{T-1}, \ldots, \Delta_1 = \delta_1) \\ &= \mathbb{P}(\Delta_{T-1} = \delta_{T-1} | \Delta_{T-2} = \delta_{T-2}, \ldots, \Delta_1 = \delta_1) \cdots \mathbb{P}(\Delta_2 = \delta_2 | \Delta_1 = \delta_1)\mathbb{P}(\Delta_1 = \delta_1), \end{aligned}$$

we have

$$(1 - p_{up})^{T+1-n} p_{low}^{n-2} \leq \mathbb{P}(\Delta_{T-1} = \delta_{T-1}, \ldots, \Delta_1 = \delta_1) \leq (1 - p_{low})^{T+1-n} p_{up}^{n-2}$$

for all $\sum_{k=1}^{T-1} \delta_k = n - 2$. *Then, it follows from*

$$\mathbb{P}(\sum_{k=1}^{T-1} \Delta_k = n - 2) = \sum_{\sum_{k=1}^{T-1} \delta_k = n-2} \mathbb{P}(\Delta_{T-1} = \delta_{T-1}, \ldots, \Delta_1 = \delta_1)$$

that

$$C_{T-1}^{n-2}(1 - p_{up})^{T+1-n} p_{low}^{n-2} \leq \mathbb{P}(\sum_{k=1}^{T-1} \Delta_k = n - 2) \leq C_{T-1}^{n-2}(1 - p_{low})^{T+1-n} p_{up}^{n-2}.$$

By Lemma 2, we have

$$p_{low} \leq \mathbb{P}(\Delta_T = 1 | \sum_{k=1}^{T-1} \Delta_k = n - 2) \leq p_{up}.$$

Therefore, (2) holds.

One knows that

$$\sum_{T=n-1}^{\infty} T C_{T-1}^{n-2}(1 - p_{up})^{T+1-n} p_{low}^{n-1} \leq \mathbb{E}(K^*) \leq \sum_{T=n-1}^{\infty} T C_{T-1}^{n-2}(1 - p_{low})^{T+1-n} p_{up}^{n-1}.$$

Denote $s = T - n + 1$. *It follows from*

$$\sum_{T=n-1}^{\infty} C_T^{n-1}(1 - p_{up})^{T+1-n} = \sum_{s=0}^{\infty} C_{s+n-1}^{s}(1 - p_{up})^s = p_{up}^{-n}$$

that

$$\sum_{T=n-1}^{\infty} T C_{T-1}^{n-2}(1 - p_{up})^{T+1-n} p_{low}^{n-1} = \sum_{T=n-1}^{\infty} (n-1) C_T^{n-1} p_{low}^{n-1} (1 - p_{up})^{T+1-n} = (n-1)\frac{p_{low}^{n-1}}{p_{up}^{n}}.$$

Similarly, we have

$$(n-1)\frac{p_{low}^{n-1}}{p_{up}^{n}} \leq \mathbb{E}(K^*) \leq (n-1)\frac{p_{up}^{n-1}}{p_{low}^{n}}. \blacksquare$$

Theorem 3: If Assumptions 1 - 3 hold and all groups interact by playing game $\mathbb{G}$, then the system reaches coalescence with probability 1.

Proof. Let A_k be the event that all agents do not coalesce into one group at time k. It follows that the event "all agents do not coalesce into one group" is $\cup_{l=1}^{\infty} \cup_{k\geq l}^{\infty} A_k$. It is easy to find that $A_k = \{\sum_{t=1}^{k} \Delta_t < n-1\}$. By Theorem 2, we have

$$\begin{cases} \mathbb{P}(A_k) = 1, k = 0, 1, \cdots, n-2, \\ \mathbb{P}(A_k) = \mathbb{P}\left(\sum_{t=1}^{k} \Delta_t < n-1\right) \leq \sum_{s=0}^{n-2} C_k^s p_{up}^s (1-p_{low})^{k-s}, \\ \qquad k = n-1, n, \cdots . \end{cases}$$

For $k > n-1$,

$$\begin{aligned} \sum_{s=0}^{n-2} C_{k+1}^s p_{up}^s (1-p_{low})^{k+1-s} &= \sum_{s=0}^{n-2} (1-p_{low}) \frac{k+1}{k+1-s} C_k^s p_{up}^s (1-p_{low})^{k-s} \\ &< \frac{(k+1)(1-p_{low})}{k+1-(n-2)} \sum_{s=0}^{n-2} C_k^s p_{up}^s (1-p_{low})^{k-s}. \end{aligned} \tag{4}$$

We know that

$$(1-p_{low}) \frac{k+1}{k+1-(n-2)} < 1 - \frac{p_{low}}{2} < 1 \tag{5}$$

holds for all $k > (n-2)(\frac{2}{p_{low}} - 1) - 1$. Let $K_0 = \max\{(n-2)(\frac{2}{p_{low}} - 1), n-1\}$. It follows from (4) and (5) that

$$\begin{aligned} \sum_{k=K_0}^{\infty} \sum_{s=0}^{n-2} C_k^s p_{up}^s (1-p_{low})^{k-s} &< \sum_{l=0}^{\infty} \sum_{s=0}^{n-2} (1 - \frac{p_{low}}{2})^l C_{K_0}^s p_{up}^s (1-p_{low})^{K_0-s} \\ &= \frac{2}{p_{low}} \sum_{s=0}^{n-2} C_{K_0}^s p_{up}^s (1-p_{low})^{K_0-s} < \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{P}(A_k) &= \sum_{k<K_0} \mathbb{P}(A_k) + \sum_{k\geq K_0} \mathbb{P}(A_k) \\ &\leq \sum_{k<K_0} \mathbb{P}(A_k) + \sum_{k=K_0}^{\infty} \sum_{s=0}^{n-2} C_k^s p_{up}^s (1-p_{low})^{k-s} \\ &< \sum_{k<T} \mathbb{P}(A_k) + \frac{2}{p_{low}} \sum_{s=0}^{n-2} C_{K_0}^s p_{up}^s (1-p_{low})^{K_0-s} < \infty. \end{aligned}$$

By Lemma 1, we know that $\mathbb{P}(\cup_{l=1}^{\infty} \cup_{k\geq l}^{\infty} A_l) = 0$, which means that the system will reach coalescence with probability 1. ■

C. Coalescence of multiple agents: special cases

Generally speaking, $\Delta_1, \Delta_2, \ldots, \Delta_{K^*}$ are not independent, i.e., the results of game $\mathbb{G}$ at time $1, 2, \ldots, k-1$ influence that of at time k. However, if p_k is independent from ξ_k, then $\Delta_1, \Delta_2, \ldots, \Delta_{K^*}$ are independent. We have the following results.

Theorem 4: If $g(\xi) = \theta\xi^\lambda$ and $f(\xi) = cg(\xi) (\lambda > 0, \theta > 0, 0 < c < 1)$, then

1) *$\Delta_1, \Delta_2, \ldots, \Delta_{K^*}$ are independent, identically distributed (i.i.d.) random variables, and the distribution of Δ_k is*

$$\mathbb{P}(\Delta_k = 1) = \hat{p}, \mathbb{P}(\Delta_k = 0) = \hat{q},$$

where $\hat{p} = 1 - \left(\frac{c}{2^\lambda(1-c)+c}\right)^2$ and $\hat{q} = 1 - \hat{p}$;

2) *the distribution of K^* is*

$$\mathbb{P}(K^* = T) = C_{T-1}^{n-2}\hat{p}^{n-1}\hat{q}^{T-(n-1)},\ T = n-1, n, \cdots;$$

3) $\mathbb{E}(K^*) = \frac{n-1}{\hat{p}}, \mathbb{D}(K^*) = \frac{(n-1)\hat{q}}{\hat{p}^2}$.

Proof. From Corollary 1, we have

$$p_k = 1 - \left(\frac{f(\frac{\xi_k}{2})}{g(\xi_k) - f(\xi_k) + f(\frac{\xi_k}{2})}\right)^2.$$

Easy to find that $p_k = \hat{p}$, which is independent from ξ_k. As a result, $\Delta_1, \Delta_2, \ldots, \Delta_{K^*}$ are independent and $\mathbb{P}(\Delta_k = 1) = \hat{p}$.

Since Δ_1 Δ_2, $\cdots, \Delta_{K^*}$ are i.i.d. random variables. We have

$$\begin{aligned}\mathbb{P}(K^* = T) &= \mathbb{P}(\Delta_T = 1, \sum_{i=1}^{T-1}\Delta_i = n-2) \\ &= \mathbb{P}(\Delta_T = 1)\mathbb{P}(\sum_{i=1}^{T-1}\Delta_i = n-2) \\ &= \begin{cases} 0, & k = 0, 1, \cdots, n-2, \\ C_{T-1}^{n-2}\hat{p}^{n-1}\hat{q}^{T-1-(n-2)}, & T = n-1, n, \cdots. \end{cases}\end{aligned}$$

It follows that the expectation of K^* is

$$\mathbb{E}(K^*) = \sum_{T=n-1}^{\infty} TC_{T-1}^{n-2}\hat{p}^{n-1}\hat{q}^{T-(n-1)}.$$

Using the similar argument in Theorem 2, we have

$$\mathbb{E}(K^*) = \frac{n-1}{\hat{p}}.$$

We can also show that

$$\mathbb{E}((K^*)^2) = \sum_{T=n-1}^{\infty} T^2 C_{T-1}^{n-2} \hat{p}^{n-1} \hat{q}^{T-(n-1)}.$$

Let $s = T - n + 1$*, we obtain*

$$\begin{aligned}\mathbb{E}((K^*)^2) &= (n-1)\hat{p}^{n-1} \sum_{s=0}^{\infty} (s+n-1) C_{s+n-1}^{s} \hat{q}^s \\ &= (n-1)\hat{p}^{n-1} \left[n \sum_{s=0}^{\infty} C_{s+n}^{s} \hat{q}^s - \sum_{s=0}^{\infty} C_{s+n-1}^{s} \hat{q}^s \right] \\ &= \frac{n(n-1)}{\hat{p}^2} - \frac{n-1}{\hat{p}}.\end{aligned}$$

Therefore, we have $\mathbb{D}(K^*) = \frac{(n-1)\hat{q}}{\hat{p}^2}$. ■

The expected coalescence time can measure how fast all agents coalesce into one group. By Theorem 4, we have the following result.

Corollary 2: If $g(\xi) = \theta\xi^{\lambda}$ *and* $f(\xi) = cg(\xi)$*(*$\lambda > 0, \theta > 0, 0 < c < 1$*), then* $\mathbb{E}(K^*)$ *is a strictly monotone increasing function of* c*.*

Proof. By $\hat{p} = 1 - \left(\frac{c}{2^{\lambda}(1-c)+c}\right)^2$*, we can find that* $\hat{p}$ *is a strictly monotonic decreasing function of* c*. And* $\mathbb{E}(K^*)$ *is a strictly monotone decreasing function of* $\hat{p}$*. Therefore,* $\mathbb{E}(K^*)$ *is a strictly monotone increasing function of* c*.* ■

At time k*, the game* $\mathbb{G}$ *is played by two groups with size* s_1 *and* s_2*. When* $g(\xi) = \theta\xi^{\lambda}$ *and* $f(\xi) = cg(\xi)$*(*$\lambda > 0, \theta > 0, 0 < c < 1$*), the aggregate expectational payoff of all agents is* $U^{(k)}(p^*, q^*) = (s_1 + s_2)\frac{2^{\lambda}(1-c)\theta\xi^{\lambda}}{2^{\lambda}(1-c)+c}$.

IV. Simulations

Suppose that there are 20 agents with distinct initial states. Firstly, we let $g(\xi) = 0.8\xi$ *and* $f(\xi) = \frac{6}{8}g(\xi)$*. In Fig.3, we show the process of coalescing by presenting the groups at time when merging event happens. Since agents from the same group have the same state, each dot indicates one group. In order to show the process clearly, we use bigger dots to indicate groups with more agents. It is shown that, when two groups play game* $\mathbb{G}$ *and coalesce into a bigger one, the number of groups shrinks by 1. Moreover, some groups become bigger and bigger as time goes by. The system reaches coalescence at time 30.*

Secondly, We simulate 20000 times with the same initial states. It is shown that each time the system always achieves coalescence in the finite time. Moreover, we also get the frequency

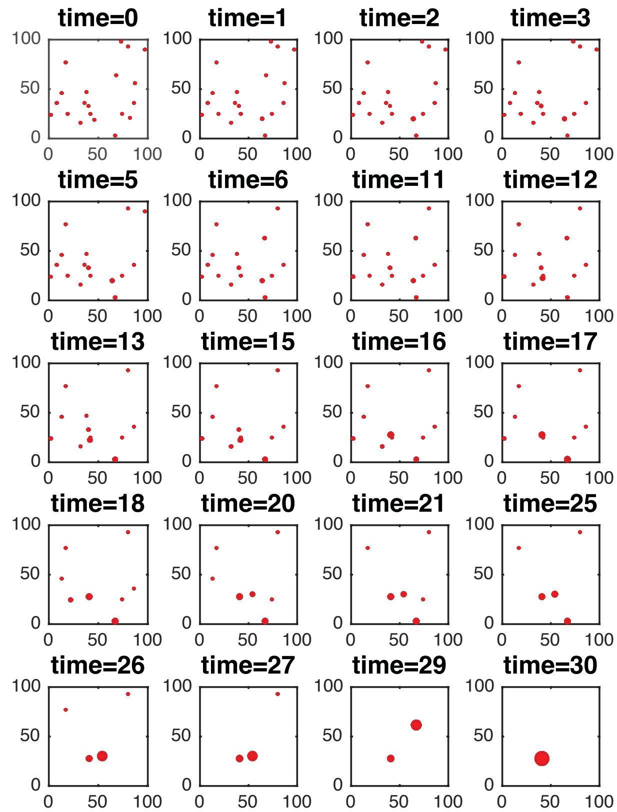

Fig. 3. The process of coalescing. Each figure indicates states of agents after two groups coalesce into one. Since the states of agents from one group are the same, each dot also indicates one group. We use bigger dots to indicate groups with more agents. It is easy to find that the numbers of groups decreases by 1 at each time. Moreover, some groups become bigger and bigger as time goes by. Finally, all agents coalesce into one big group at time 30.

of coalescence time K^* over those 20000 times simulations. The comparison between the distribution and the frequency of K^* is shown in Fig. 4. Those results manifest the effectiveness of theoretical results in Theorems 3 and 4.

Thirdly, we let $g(\xi) = 0.8\xi$ and $f(\xi) = \frac{5}{8}g(\xi)$. Then we do the same simulations. The comparison between the distribution and the frequency of K^* is shown in Fig. 5. Easy to find from Fig. 4 and Fig. 5 that the system is more likely reaching coalescence earlier when $g(\xi) = 0.8\xi$ and $f(\xi) = \frac{5}{8}g(\xi)$. Those results manifest the effectiveness of theoretical results in Corollary 2.

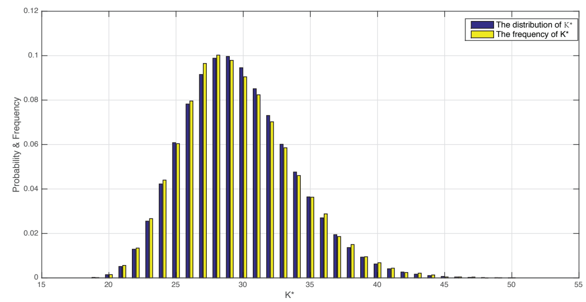

Fig. 4. The distribution and the frequency of K^* with $f(\xi) = 0.6\xi$ and $g(\xi) = 0.8\xi$

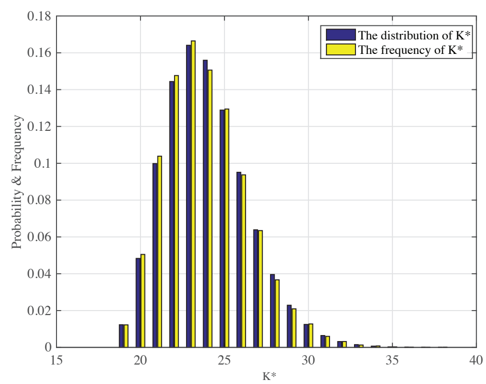

Fig. 5. The distribution and the frequency of K^* with $f(\xi) = 0.5\xi$ and $g(\xi) = 0.8\xi$

V. Conclusion

To achieve some global tasks, multiple agents need to coalesce into one group — they will make decisions together, share information instantly, keep consensus in states. This paper focused on the coalescence of a population of rational and complete information accessible agents. We modeled the coalescing process as a repeated bimatrix game. Agents form groups and groups coalesce into one bigger group. We proved that coalescence will be reached with probability one and gave an estimation for the expected coalescence time. Moreover, when payoff functions are power functions, the distribution of coalescence time was obtained. Future work might contain the coalescence under partial information or under learning mechanisms.

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