Pricing and hedging short-maturity Asian options in local volatility models

Jaehyun Kim* Hyungbin Park† Jonghwa Park‡

April 30, 2024

Abstract

This paper discusses the short-maturity behavior of Asian option prices and hedging portfolios. We consider the risk-neutral valuation and the delta value of the Asian option having a Hölder continuous payoff function in a local volatility model. The main idea of this analysis is that the local volatility model can be approximated by a Gaussian process at short maturity T. By combining this approximation argument with Malliavin calculus, we conclude that the short-maturity behaviors of Asian option prices and the delta values are approximately expressed as those of their European counterparts with volatility

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt}\,,$$

where $\sigma(\cdot,\cdot)$ is the local volatility function and S_0 is the initial value of the stock. In addition, we show that the convergence rate of the approximation is determined by the Hölder exponent of the payoff function. Finally, the short-maturity asymptotics of Asian call and put options are discussed from the viewpoint of the large deviation principle.

Keywords Asian option, short maturity, Hölder continuous, local volatility model, Gaussian process, Malliavin calculus, large deviation principle

1 Introduction

This paper focuses on an *arithmetic average Asian option* in continuous time having a terminal payoff of the form

$$\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right).$$

*Seoul National University, Department of Mathematical Sciences, 1, Gwanak-ro, Gwanak-gu, Seoul, Republic of Korea, jaehyun107@snu.ac.kr

†Seoul National University, Department of Mathematical Sciences, 1, Gwanak-ro, Gwanak-gu, Seoul, Republic of Korea, hyungbin@snu.ac.kr, hyungbin2015@gmail.com. Corresponding author

‡Carnegie Mellon University, Department of Mathematical Sciences, Pittsburgh, Pennsylvania 15213, USA, jonghwap@andrew.cmu.edu

Here, the function $\Phi : \mathbb{R} \to \mathbb{R}$ is a prescribed payoff function, T is a constant that denotes the maturity, and $(S_t)_{t\geq 0}$ is an underlying price process. For conciseness, we refer to this option as the *Asian option*. Because of its average property, the Asian option is less exposed to a sudden plummet in stock prices just before maturity. In particular, for hedging purposes, the Asian option is attractive to many traders and financial institutions. For an overview of the role of the Asian option in the financial market, see Wilmott (2006).

Despite its popularity in the real market, the Asian option is mathematically challenging to price and hedge in general. Even when the underlying stock price $(S_t)_{t\geq 0}$ follows the classical Black–Scholes model, no simple closed-form formula for the density of the random variable $\frac{1}{T}\int_0^T S_t\,dt$ is known. In this paper, we analyze the Asian option for pricing and hedging purposes in the short-maturity regime.

We focus on the case in which the payoff function Φ is any Hölder continuous function and the process $(S_t)_{t\geq 0}$ follows a local volatility model. Detailed assumptions on the model are presented in Section 2. This paper primarily deals with two features of the Asian option. The first feature is the short-maturity behavior of the option price. The short-maturity Asian option price is shown to be determined by the *Asian volatility*, which is defined by

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t, S_0)(T-t)^2\,dt}\,,$$

where $\sigma(\cdot,\cdot)$ is a local volatility function. The second feature is the initial-value sensitivity of the Asian option. This type of sensitivity is widely referred to as *delta* in the finance literature. In the modern theory of finance, the delta value is used to hedge financial derivatives. This paper shows that the delta value can be expressed in terms of the Asian volatility for small T, as with the option price. In summary, the Asian option price $P_A(T)$ and delta value $\Delta_A(T)$ are expressed as

$$\begin{aligned}
P_A(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^{\gamma})\,,\\
\Delta_A(T) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,,
\end{aligned}$$

for a standard normal random variable Z and the Hölder exponent γ of the payoff function Φ. The asymptotic estimates established in this paper are particularly meaningful with regard to the non-linear payoff function Φ. For example, let us consider a payoff function Φ that equals $x \mapsto (x-K)_+^{\gamma}$ in some neighborhood of K for some $\frac{1}{2} < \gamma < 2$. If K equals the initial value S_0, we prove that the leading order of delta is $T^{\frac{\gamma-1}{2}}$ and we provide its coefficient in a rigorous manner. Moreover, if we consider a payoff function that equals $x \mapsto 1/(1+e^{-\kappa(x-K)})$ for some sufficiently large $\kappa > 0$, our estimates could provide us with a fast way to hedge digital options.

As a special case, estimates for the Asian call and put option delta values are enhanced. This paper supplements the asymptotic result in Pirjol and Zhu (2016) in two ways. First, we prove that the rate function of the out-of-the-money (OTM) Asian option delta value is the same as that of the OTM Asian option price. Second, a precise Taylor expansion of the in-the-money (ITM) Asian option delta value is provided.

Estimates for the price and delta of the *European option* having the terminal payoff $\Phi(S_T)$ are also investigated. Short-maturity formulas for the European option prices and delta values are obtained if the Asian volatility is replaced by

$$\sigma_E(T) := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t, S_0)\, dt}\,,$$

which we refer to as the *European volatility*, in the formulas for the Asian option prices and delta values. With regard to $\sigma_A(T)$ and $\sigma_E(T)$, we compare the Asian option with the European option in Section 7. In addition to the European option, the *geometric average Asian option* having the terminal payoff

$$\Phi\left(e^{\frac{1}{T}\int_0^T \log S_t\, dt}\right)$$

is also compared in the Black–Scholes model.

To obtain these estimates, we incorporate many well-known mathematical techniques with the approximation scheme. The main technique is L^p-approximation of the underlying stock price $(S_t)_{0\le t\le T}$ by some Gaussian process $(\hat{X}_t)_{0\le t\le T}$. Precise arguments are presented in Section 2. We adopt the method used in Pirjol and Zhu (2016, 2019); Pirjol et al. (2019), where the same idea was used to compute the short-maturity asymptotics of at-the-money(ATM) Asian call and put option prices. On the basis of this idea, our research focus shifts from the random variable $\frac{1}{T}\int_0^T S_t\, dt$ having a sophisticated density to the Gaussian random variable $\frac{1}{T}\int_0^T \hat{X}_t\, dt$. This is the key strategy that we adopt to approximate the Asian option throughout in Sections 2–4. In addition, we use Malliavin calculus theory to analyze the Asian option delta value. In Benhamou (2000); Pirjol and Zhu (2018), the authors used Malliavin calculus for their sensitivity analysis of the Asian call and put option. We use their methods to express the Asian option delta value. Furthermore, we use the large deviation principle to examine both OTM and ITM Asian call and put options. The large deviation principle was first used to investigate the short-maturity Asian option in Pirjol and Zhu (2016, 2019).

Our study is of practical interest because existing numerical methods have proven to be less efficient in the case of short maturity or low volatility. Numerical analysis of the Asian option was conducted in Geman and Yor (1993); Linetsky (2004); Broadie et al. (1999); Boyle and Potapchik (2008). However, as pointed out in Fu et al. (1999); Vecer (2002), such methods are either problematic in the short-maturity regime or computationally expensive. We expect our analysis to help overcome the numerical inefficiency in the short-maturity regime.

Recently, the short-maturity Asian option has been studied by many researchers. Under a local volatility model, the asymptotics of Asian option price have been investigated in Pirjol and Zhu (2016); Pirjol et al. (2019). In Pirjol and Zhu (2019), asymptotic analysis was conducted under the constant elasticity of the variance model. The above-mentioned studies have used the large deviation principle. They have analytically solved the rate function of the law of $\frac{1}{T}\int_0^T S_t\, dt$ for approximation. Sensitivity analysis was conducted in Pirjol and Zhu (2018) as a follow-up study under the Black–Scholes model. On the basis of the approximated option price established in Pirjol and Zhu (2016), the sensitivities have been examined in Pirjol and Zhu (2018).

Compared to the above-mentioned studies, the contributions of our study are threefold. First, our paper focuses on a model having a time-dependent diffusion term. The analysis performed in Pirjol and Zhu (2016); Pirjol et al. (2019) was based on the assumption that the diffusion is time-independent. The obtained rate function was strongly dependent on this time-independent assumption. Second, we provide the leading order and its exact coefficient for an arbitrary Hölder continuous payoff function Φ. This generalizes the results in Pirjol and Zhu (2016, 2019), where vanilla options(call and put) were mainly considered. Finally, in contrast to Pirjol and Zhu (2018), our estimates for delta do not build upon the approximated option price. Thus, our estimates are free from controlling nested errors.

The remainder of this paper is organized as follows. Section 2 outlines the model setup and introduces six auxiliary processes that are used to approximate $(S_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm. Section 3 examines the Asian option price for small T when the payoff Φ is Lipschitz continuous. Under the same assumption on Φ, Section 4 investigates the Asian option delta value. Section 5 generalizes the results from Sections 3 and 4 to the Hölder continuous payoff Φ. Section 6 performs numerical tests to justify estimations from Section 5. Section 7 concatenates the asymptotic results from Sections 3 and 4, and compares them with their European counterparts. Section 8 uses the large deviation principle to study the Asian call and put option. Finally, Section 9 concludes the paper.

2 Approximation scheme

We analyze the short-maturity asymptotic behavior of Asian options under local volatility models. Assume that the stock price process $(S_t)_{t\geq 0}$ follows a local volatility model,

$$dS_t = (r-q)S_t\,dt + \sigma(t,S_t)S_t\,dW_t\,, \quad S_0 > 0\,, \tag{2.1}$$

under risk-neutral measure $\mathbb{Q}$, where r is the short rate, q is the dividend rate, and $(W_t)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion. From Assumption 1 below, there exists a unique strong solution of Eq.(2.1).

Assumption 1. *Let us consider the following assumptions for the diffusion function.*

(i) The function $\sigma(t,x)$ is measurable in $[0,\infty)\times\mathbb{R}$ and is bounded, i.e., there are two constants $\underline{\sigma}$ and $\overline{\sigma}$ such that $0<\underline{\sigma}\leq\sigma(t,x)\leq\overline{\sigma}<\infty$ for all t and x.

(ii) For each t, the function $\sigma(t,\cdot)$ is twice differentiable in $\mathbb{R}$.

(iii) Define $\nu(t,x) := \frac{\partial[\sigma(t,x)x]}{\partial x}$ and $\rho(t,x) := \frac{\partial^2[\sigma(t,x)x]}{\partial x^2}$. Then, for each t, functions $\sigma(t,\cdot)$, $\sigma(t,\cdot)\cdot$, $\nu(t,\cdot)$, $\rho(t,\cdot)$ are Lipschitz continuous with a Lipschitz coefficient $\alpha>0$. More precisely, there is a constant $\alpha>0$ such that for any $x,y\in\mathbb{R}$,

$$\begin{aligned}
&\sup_{t\geq 0}|\sigma(t,x)-\sigma(t,y)|\leq\alpha\,|x-y|\,,\ \sup_{t\geq 0}|\sigma(t,x)x-\sigma(t,y)y|\leq\alpha\,|x-y|\,,\\
&\sup_{t\geq 0}|\nu(t,x)-\nu(t,y)|\leq\alpha\,|x-y|\,,\ \sup_{t\geq 0}|\rho(t,x)-\rho(t,y)|\leq\alpha\,|x-y|\ .
\end{aligned}$$

Clearly, this assumption covers the Black–Scholes model. In this paper, only Hölder continuous payoff Φ will be considered. Under Assumption 2, β and γ always refer to constants with regard to Φ throughout this paper.

Assumption 2. *The payoff function $\Phi:\mathbb{R}\to\mathbb{R}$ is γ-Hölder continuous with coefficient $\beta>0$. More precisely, for any $x,y\in\mathbb{R}$, $|\Phi(x)-\Phi(y)|\leq\beta|x-y|^\gamma$ with $0<\gamma\leq 1$.*

To clarify the arguments, we will first consider Lipschitz continuous payoff Φ in Sections 3 and 4. Unless stated otherwise, β always refers to the Lipschitz coefficient.

Assumption 3. *The payoff function $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with coefficient $\beta > 0$. More precisely, for any $x, y \in \mathbb{R}$, $|\Phi(x) - \Phi(y)| \leq \beta |x - y|$.*

Now, we introduce six processes that are used to approximate $(S_t)_{t\geq 0}$ in $L^p(\mathbb{Q})$:

$$X,\, Y,\, \tilde{X},\, \tilde{Y},\, \hat{X},\, \hat{Y}\,.$$

Define a process $(X_t)_{t\geq 0}$ as

$$dX_t = \sigma(t, X_t) X_t \, dW_t\,, \quad X_0 = S_0 > 0 \tag{2.2}$$

and its first variation process as

$$dY_t = \nu(t, X_t) Y_t \, dW_t\,, \quad Y_0 = 1\,. \tag{2.3}$$

These two processes will be used to approximate the underlying process $(S_t)_{t\geq 0}$ in Sections 3 and 4. We also define two geometric Gaussian processes $(\tilde{X}_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$ as

$$d\tilde{X}_t = \sigma(t, S_0)\tilde{X}_t \, dW_t\,, \quad \tilde{X}_0 = S_0\,, \qquad d\tilde{Y}_t = \nu(t, S_0)\tilde{Y}_t \, dW_t\,, \quad \tilde{Y}_0 = 1\,. \tag{2.4}$$

In Lemma 2.1, these two processes will be used to approximate $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm at short time. Finally, we define two Gaussian processes $(\hat{X}_t)_{t\geq 0}$, $(\hat{Y}_t)_{t\geq 0}$ by

$$d\hat{X}_t = \sigma(t, S_0) S_0 \, dW_t\,, \quad \hat{X}_0 = S_0\,, \qquad d\hat{Y}_t = \nu(t, S_0) \, dW_t\,, \quad \hat{Y}_0 = 1\,.$$

Furthermore, in Lemma 2.1, these two processes will be used to approximate $(\tilde{X}_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm at short time. Now, we introduce Lemma 2.1. See Appendix A.1 for the proof.

Lemma 2.1. *Under Assumption 1, for any $p > 0$, there exists a positive constant B_p depending only on p such that the following inequalities hold.*

(i) For $0 \leq t \leq 1$,

$$\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \leq B_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \leq B_p t^p. \tag{2.5}$$

(ii) For $0 \leq t \leq 1$,

$$\mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^p] \leq B_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_t - \hat{Y}_t|^p] \leq B_p t^p. \tag{2.6}$$

We now present the short-time behavior of the four processes $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ in the following lemma. All the moments of the four random variables $X_T, \tilde{X}_T, Y_T, \tilde{Y}_T$ and their integrals over $[0, T]$ converge to constants as $T \to 0$. We rephrase this argument as the following technical statement for later use. The proof is provided in Appendix A.2.

Lemma 2.2. *Under Assumption 1, consider processes $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ stated in Eqs.(2.2), (2.3), and (2.4). The process $(Z_t^{p_1,p_2,p_3,p_4})_{t\geq 0}$, which is defined by $Z_t^{p_1,p_2,p_3,p_4} := X_t^{p_1}\tilde{X}_t^{p_2}Y_t^{p_3}\tilde{Y}_t^{p_4}$ for any $p_i \in \mathbb{R}$, $i \in \{1, 2, 3, 4\}$, satisfies following two statements.*

(i) $\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\Big] < \infty$ *for any* $T>0$. *Furthermore,* $\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\Big] = S_0^{p_1+p_2}$.

(ii) Moreover, for any $q_j\in\mathbb{R}$, $j\in\{1,2,\cdots 8\}$,

$$\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\left[Z_T^{q_1,q_2,q_3,q_4}\left(\frac{1}{T}\int_0^T X_t\,dt\right)^{q_5}\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)^{q_6}\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^{q_7}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{q_8}\right]$$

$$= S_0^{q_1+q_2+q_5+q_6}. \tag{2.7}$$

3 Short-maturity limit of an option price with Lipschitz continuous payoffs

Under the risk-neutral measure $\mathbb{Q}$, the arbitrage-free values of the Asian and European options are

$$P_A(T) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\right], \quad P_E(T) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)],$$

where T is the maturity. Throughout this section, we impose Assumption 3 on Φ. Our objective is to find an asymptotic formula for the Asian option price up to $\mathcal{O}(T)$; a formula for its European counterpart is also presented.

First, in Lemma 3.1, the underlying $(S_t)_{0\le t\le T}$ is approximated by $(X_t)_{0\le t\le T}$. Second, in Theorem 3.1, we approximate the process $(X_t)_{0\le t\le T}$ by $(\hat{X}_t)_{0\le t\le T}$ using Lemma 2.1. The proof of Lemma 3.1 is provided in Appendix B.1.

Lemma 3.1. *Under Assumptions 1 and 3, as* $T\to 0$, *we have*

(i) $P_A(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T)$,

(ii) $P_E(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(T)$.

As can be seen in Eqs.(2.1) and (2.2), the processes S and X have the same diffusion terms; however, the drift term of X is zero. Thus, this lemma implies that while estimating the Asian and European option prices, the drift of the underlying stock becomes negligible at small $T>0$. This is similar to Theorem 2 of Pirjol and Zhu (2016) and Theorem 5 of Pirjol and Zhu (2019). In Pirjol and Zhu (2016, 2019), the rate function that governs the short-maturity behavior of the Asian call and put option was shown to be independent of the drift term.

The main result of this section is the following theorem, which states asymptotic formulas for the Asian and European option prices.

Theorem 3.1. *Under Assumptions 1 and 3, as* $T\to 0$, *we have*

(i) $P_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\frac{1}{T^2}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt}\;Z\right)\right] + \mathcal{O}(T)$,

(ii) $P_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\int_0^T \sigma^2(t,S_0)\,dt}\,Z\right)\right] + \mathcal{O}(T)\,,$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

Proof. The statement actually directly comes from Lemmas 2.1 and 3.1. Observe that

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right]\right| \le \frac{2\beta}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|X_t - \hat{X}_t|]\,dt = \beta B_1 T$$

for the positive constant B_1 in Lemma 2.1. From a direct calculation using the Fubini theorem regarding a stochastic integral,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + \frac{S_0}{T}\int_0^T\int_0^t \sigma(s,S_0)\,dW_s\,dt\right)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + \frac{S_0}{T}\int_0^T \sigma(s,S_0)(T-s)\,dW_s\right)\right].\end{aligned}$$

Hence, we get the desired result for $P_A(T)$. Applying the same argument to $P_E(T)$, we can easily get the desired result. □

Corollary 3.2. *Under Assumptions 1 and 3, the prices of both the Asian option and the European option share the same limit $\Phi(S_0)$ as $T \to 0$ with the convergence order $\mathcal{O}(\sqrt{T})$. More precisely,*

$$P_A(T) = \Phi(S_0) + \mathcal{O}(\sqrt{T})\,, \quad P_E(T) = \Phi(S_0) + \mathcal{O}(\sqrt{T})\,.$$

We introduce two notions of volatilities called the *Asian volatility* and the *European volatility.*

Definition 3.1. *We define the Asian volatility $\sigma_A(T)$ and the European volatility $\sigma_E(T)$ as*

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt}\,, \quad \sigma_E(T) := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t,S_0)\,dt}\,. \tag{3.1}$$

In terms of the Asian volatility and the European volatility, Theorem 3.1 can be rewritten as

$$P_A(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T)\,, \quad P_E(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)] + \mathcal{O}(T)\,.$$

In Section 4, short-maturity asymptotic formulas for the delta values will be presented in terms of the Asian volatility and the European volatility. We compute asymptotic results for the call and put options as an example of Theorem 3.1. This generalizes the result in Theorem 6 of Pirjol and Zhu (2016) for the ATM case.

Example 3.1. *Let P_A^{call} and P_A^{put} be the Asian call and put prices with the strike K, i.e., the payoff functions are $\Phi(x) = (x-K)_+$, and $\Phi(x) = (K-x)_+$, respectively. Then,*

$$P_A^{\text{call}}(T) = \begin{cases} 0 + \mathcal{O}(T), & \text{if } S_0 < K\,, \\ \frac{S_0\sigma_A(T)}{\sqrt{2\pi}}\sqrt{T} + \mathcal{O}(T), & \text{if } S_0 = K\,, \\ S_0 - K + \mathcal{O}(T), & \text{if } S_0 > K\,, \end{cases}$$

$$P_A^{\text{put}}(T) = \begin{cases} K - S_0 + \mathcal{O}(T), & \text{if } S_0 < K\,, \\ \frac{S_0\sigma_A(T)}{\sqrt{2\pi}}\sqrt{T} + \mathcal{O}(T), & \text{if } S_0 = K\,, \\ 0 + \mathcal{O}(T), & \text{if } S_0 > K\,. \end{cases}$$

The prices of the European call and put option are obtained by replacing $\sigma_A(T)$ in the above-mentioned expressions with $\sigma_E(T)$.

Example 3.2. *Given any $K\,, \delta > 0$ and $1 \le \gamma < 2\,,$ define the payoff function Φ by*

$$\Phi(x) = (x-K)^{\gamma}\mathbb{1}_{\{K \le x < K+\delta\}} + \delta^{\gamma}\mathbb{1}_{\{K+\delta \le x\}}\,.$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$P_A(T) = \frac{1}{2}(S_0\sigma_A(T))^{\gamma}\, M(\gamma)T^{\frac{\gamma}{2}} + \mathcal{O}(T)\,,$$

where $M(\gamma) := \mathbb{E}^{\mathbb{Q}}[|Z|^{\gamma}]$ with a standard normal variable Z. If we replace $\sigma_A(T)$ with $\sigma_E(T)$, we get the asymptotic result for the European option price $P_E(T)$.

4 Short-maturity estimates for an option delta value with Lipschitz continuous payoffs

In this section, we present the short-maturity asymptotic for the sensitivity of the option with respect to the initial value S_0. In many studies, this sensitivity is referred to as *delta*. We follow this convention to define the Asian delta value and the European delta value as

$$\Delta_A(T) := \frac{\partial}{\partial S_0}P_A(T)\,, \quad \Delta_E(T) := \frac{\partial}{\partial S_0}P_E(T)\,.$$

Throughout this section, only the Lipschitz continuous payoff Φ will be considered. Our main objective is to obtain the short-maturity asymptotic for $\Delta_A(T)$, $\Delta_E(T)$. These asymptotic results are given in Theorems 4.1 and 4.3. In Lemma 4.1, we first approximate $(S_t)_{0\le t\le T}$ by $(X_t)_{0\le t\le T}$ in line with Lemma 3.1. See Appendix C.1 for the proof.

Lemma 4.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $\Delta_A(T) = e^{-rT}\dfrac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\dfrac{1}{T}\displaystyle\int_0^T X_t\,dt\right)\right] + \mathcal{O}(\sqrt{T})\,,$

(ii) $\Delta_E(T) = e^{-rT}\dfrac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(\sqrt{T})\,.$

Next, in Sections 4.1 and 4.2, we present a short-maturity asymptotic formula for the Asian delta value $\Delta_A(T)$. A formula for the European delta value $\Delta_E(T)$ is presented in Section 4.3.

4.1 Approximation for the Malliavin representation of the Asian delta value

By Malliavin calculus, the Asian delta value can be represented by the weighted sum of the payoffs. The computation under the Black–Scholes model has already been presented in Boyle and Potapchik (2008); Benhamou (2000). Under the local volatility model, we describe a possible representation in the following proposition.

Proposition 4.1 (Nualart (1995); Benhamou (2000)). *For the process X stated in Eq.(2.2) under Assumption 1, we have*

$$\begin{aligned}\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta\left(\frac{2Y_\cdot^2}{\sigma(\cdot,X_\cdot)X_\cdot\int_0^T Y_t\,dt}\right)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta\left(\frac{2Y_\cdot^2}{\sigma(\cdot,X_\cdot)X_\cdot}\right)\frac{1}{\int_0^T Y_t\,dt}\right]\\ &\quad - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T \frac{2{Y_s}^2}{\sigma(s,X_s)X_s}D_s\left(\frac{1}{\int_0^T Y_t\,dt}\right)ds\right],\end{aligned}$$

where $\delta(\cdot)$ is the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

With this proposition, Lemma 4.1 implies that for small $T>0$, the Asian delta value asymptotically behaves as

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right] + \mathcal{O}(\sqrt{T})\,. \tag{4.1}$$

Here, we define a process $(u_s)_{0\le s\le T}$ and a random variable F by

$$u_s := \frac{2{Y_s}^2}{\sigma(s,X_s)X_s}\,,\qquad F := \frac{1}{\int_0^T Y_t\,dt}\,.$$

To investigate $\Delta_A(T)$, we will approximate the two expectations on the right-hand side of Eq.(4.1). The approximation of the first expectation,

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right], \tag{4.2}$$

is described in Proposition 4.2, and the approximation of the second expectation,

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right], \tag{4.3}$$

is described in Proposition 4.3. To analyze Eq.(4.2), we introduce two processes $(\tilde{u}_s)_{0\le s\le T}$ and $(\hat{u}_s)_{0\le s\le T}$ and two random variables $\tilde{F}$ and $\hat{F}$, defined by

$$\tilde{u}_s := \frac{2\tilde{Y}_s^2}{\sigma(s,\tilde{X}_s)\tilde{X}_s}\,,\quad \hat{u}_s := \frac{2\hat{Y}_s^2}{\sigma(s,\hat{X}_s)\hat{X}_s}\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}}\quad \tilde{F} := \frac{1}{\int_0^T \tilde{Y}_t\,dt}\,,\quad \hat{F} := \frac{1}{\int_0^T \hat{Y}_t\,dt}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\ge\frac{1}{2}\}}\,,$$

where $\mathbb{1}_A$ denotes the indicator function of set A. In Lemma 4.2, the process $(u_s)_{0\le s\le T}$ is approximated using $(\tilde{u}_s)_{0\le s\le T}$ and $(\hat{u}_s)_{0\le s\le T}$. As for the random variable F, we use $\tilde{F}$ and $\hat{F}$. This procedure is similar to the approximation based on Lemma 2.1. The proof of Lemma 4.2 is given in Appendix C.2.

Lemma 4.2. *Under Assumption 1, for any $p > 0$, there exists a positive constant D_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|u_t - \tilde{u}_t|^p] \le D_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{u}_t - \hat{u}_t|^p] \le D_p t^p. \tag{4.4}$$

(ii) For $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|TF - T\tilde{F}|^p] \le D_p T^p, \quad \mathbb{E}^{\mathbb{Q}}[|T\tilde{F} - T\hat{F}|^p] \le D_p T^p. \tag{4.5}$$

Proposition 4.2. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] + \mathcal{O}(\sqrt{T}).$$

Proof. We may assume that $\Phi(0) = 0$. Consider a translation $\Psi(\cdot) := \Phi(\cdot) - \Phi(0)$ otherwise. Observe that

$$\begin{aligned}
&\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F - \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\\
&= \frac{1}{T}\left(\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right)\delta(u)TF\\
&+ \frac{1}{T}\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(u-\hat{u})TF + \frac{1}{T}\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\left(TF - T\hat{F}\right).
\end{aligned}$$

From Lemmas 2.1, 2.2, 4.2, the standard argument, and the fact the Skorokhod integral of u becomes the Itô integral whenever $(u_s)_{0\le s\le T}$ is adapted to the Brownian filtration $(\mathcal{F}_s^W)_{0\le s\le T}$, for $0 \le T \le 1$, we can complete the proof. $\square$

Now, we will approximate Eq.(4.3). To do this, we approximate D_sF by $D_s\tilde{F}$ and $D_s^*\hat{F}$, where $D_s^*\hat{F}$ is defined as

$$D_s^*\hat{F} := D_s\left(\frac{1}{\int_0^T \hat{Y}_t\,dt}\right)\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \ge \frac{1}{2}\}}.$$

Through some auxiliary steps, these approximations among D_sF, $D_s\tilde{F}$, and $D_s^*\hat{F}$ are presented in Lemma 4.4. Before investigating them, we first show in Lemma 4.3 that the pth moments of D_sF, $D_s\tilde{F}$, and $D_s^*\hat{F}$ are $\mathcal{O}(\frac{1}{T^p})$ in a short-maturity regime. As a necessary step, we also observe that the moments of D_sX_t, D_sY_t, $D_s\tilde{Y}_t$, and $D_s\hat{Y}_t$ are bounded. See Appendices C.3 and C.4 for the proofs.

Lemma 4.3. *Under Assumption 1, for any $p > 0$, there exists a positive constant E_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s X_t|^p] \le E_p. \tag{4.6}$$

(ii) For $0 \le t \le 1$,

$$\sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s Y_t|^p] \le E_p, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s \tilde{Y}_t|^p] \le E_p, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s \hat{Y}_t|^p] \le E_p. \tag{4.7}$$

(iii) For $0 \le T \le 1$,

$$\sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|T D_s F|^p] \le E_p, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|T D_s \tilde{F}|^p] \le E_p, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|T D_s^* \hat{F}|^p] \le E_p. \tag{4.8}$$

Lemma 4.4. *Under Assumption 1, for any $p > 0$, there exists a positive constant F_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s Y_t - D_s \tilde{Y}_t|^p] \le F_p t^{\frac{p}{2}}, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|D_s \tilde{Y}_t - D_s \hat{Y}_t|^p] \le F_p t^{\frac{p}{2}}. \tag{4.9}$$

(ii) For $0 \le T \le 1$,

$$\sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|T D_s F - T D_s \tilde{F}|^p] \le F_p T^{\frac{p}{2}}, \quad \sup_{s \ge 0} \mathbb{E}^{\mathbb{Q}}[|T D_s \tilde{F} - T D_s^* \hat{F}|^p] \le F_p T^{\frac{p}{2}}. \tag{4.10}$$

Proposition 4.3. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_s F)\,ds\right] = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^* \hat{F})\,ds\right] + \mathcal{O}(\sqrt{T}).$$

Proof. We may assume that $\Phi(0) = 0$. Observe that

$$\begin{aligned}
&\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_s F)\,ds - \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^* \hat{F})\,ds \\
&= \left(\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right)\frac{1}{T}\int_0^T u_s\left(T D_s F\right)\,ds \\
&+ \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\int_0^T (u_s - \hat{u}_s)\,T D_s F\,ds + \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\int_0^T \hat{u}_s\left(T D_s F - T D_s^* \hat{F}\right)\,ds.
\end{aligned}$$

From Lemmas 2.1, 2.2, 4.2, 4.3, 4.4 and the standard argument, we can obtain the desired result. ∎

4.2 Short-maturity asymptotic for the Asian delta value

Let us concatenate the approximations established in Propositions 4.2 and 4.3 to obtain

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] + \mathcal{O}(\sqrt{T})\,.$$

To deduce the short-maturity asymptotic formula presented in Theorem 4.1, we now estimate the following two expectations.

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right]\,,\quad \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right]\,. \tag{4.11}$$

In the first step, $\delta(\hat{u})$ is approximated to a normal variable $\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)$ in Lemma 4.5. See Appendix C.5 for the proof.

Lemma 4.5. *Under Assumption 1, for any $p > 0$, there exists a positive constant G_p depending only on p such that the following inequalities hold for $0 \le T \le 1$:*

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{u})|^p] \le G_p T^{\frac{p}{2}},\quad \mathbb{E}^{\mathbb{Q}}\left[\left|\delta(\hat{u}) - \delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right|^p\right] \le G_p T^p. \tag{4.12}$$

Then, using this lemma, we can directly estimate the expectations in Eq.(4.11) only in terms of multivariate normal random variables. Consequently, we propose the following asymptotic relations in Proposition 4.4. Further details are provided in Appendix C.6.

Proposition 4.4. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] - 2\frac{\Phi(S_0)}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\end{aligned} \tag{4.13}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] = -2\frac{\Phi(S_0)}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\,, \tag{4.14}$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

We now reach one of the two main results in this section. The two estimates in Proposition 4.4 directly give the short-maturity asymptotic for the Asian delta value $\Delta_A(T)$. Recall the definition of the Asian volatility $\sigma_A(T)$ in Eq.(3.1).

Theorem 4.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})\,,$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

Corollary 4.2. *Under Assumptions 1 and 3,*

(i) if $\Delta_A(T)$ converges as $T \to 0$, then

$$\lim_{T\to 0} \Delta_A(T) = \lim_{\epsilon \searrow 0} \mathbb{E}^{\mathbb{Q}} \left[\frac{\Phi(S_0 + \epsilon Z) - \Phi(S_0)}{\epsilon Z} Z^2 \right] ,$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$;

(ii) if both the right derivative $D\Phi(S_0+)$ and the left derivative $D\Phi(S_0-)$ exist, then $\Delta_A(T)$ converges and

$$\lim_{T\to 0} \Delta_A(T) = \frac{D\Phi(S_0+) + D\Phi(S_0-)}{2} .$$

We present some examples of Theorem 4.1.

Example 4.1. *Let Δ_A^{call} and Δ_A^{put} be the Asian call and put delta value with the strike K, i.e., the payoff functions are $\Phi(x) = (x-K)_+$ and $\Phi(x) = (K-x)_+$, respectively. Then,*

$$\Delta_A^{\text{call}}(T) = \begin{cases} 0 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 < K\,, \\ \frac{1}{2} + \mathcal{O}(\sqrt{T}), & \text{if } S_0 = K\,, \\ 1 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 > K\,, \end{cases} \quad \Delta_A^{\text{put}}(T) = \begin{cases} -1 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 < K\,, \\ \frac{1}{2} + \mathcal{O}(\sqrt{T}), & \text{if } S_0 = K\,, \\ 0 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 > K\,. \end{cases}$$

Example 4.2. *Given any $K\,, \delta > 0$, and $1 \leq \gamma < 2\,$, define the payoff function Φ by*

$$\Phi(x) = (x-K)^{\gamma} \mathbb{1}_{\{K \leq x < K+\delta\}} + \delta^{\gamma} \mathbb{1}_{\{K+\delta \leq x\}} \,.$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$\Delta_A(T) = \frac{1}{2}(S_0 \sigma_A(T))^{\gamma-1} M(\gamma+1) T^{\frac{\gamma-1}{2}} + \mathcal{O}(\sqrt{T}) \,,$$

where $M(\gamma+1) := \mathbb{E}^{\mathbb{Q}}[|Z|^{\gamma+1}]$ with a standard normal variable Z. In this example, the leading order of $\Delta_A(T)$ is $T^{\frac{\gamma-1}{2}}$ as $T \to 0$.

4.3 Short-maturity asymptotic for the European delta value

In the remainder of this section, we will investigate the short-maturity behavior of the European delta value $\Delta_E(T)$. The desired asymptotic formula is presented in Theorem 4.3. To prove this, we follow the same approximation steps used to derive the asymptotic formula for the Asian delta value $\Delta_A(T)$ with a slight modification. First, we use Malliavin calculus to rewrite the European delta value as the weighted sum of the payoffs.

Proposition 4.5 (Nualart (1995); Benhamou (2000))**.** *For the process X stated in Eq.(2.2) under Assumption 1, we have*

$$\begin{aligned} \frac{\partial}{\partial S_0} \mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] &= \frac{1}{S_0 T} \mathbb{E}^{\mathbb{Q}} \left[\Phi(X_T)\, \delta\left(\frac{Y_\cdot}{\sigma(\cdot, X_\cdot) X_\cdot} \right) \frac{X_T}{Y_T} \right] - \frac{1}{S_0} \mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] \\ &\quad + \frac{1}{S_0 T} \mathbb{E}^{\mathbb{Q}} \left[\Phi(X_T) \int_0^T \frac{Y_s}{\sigma(s, X_s) X_s} \frac{X_T (D_s Y_T)}{{Y_T}^2} ds \right], \end{aligned}$$

where $\delta(\cdot)$ denotes the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

Thus, we can observe from Corollary 3.2 and Lemma 4.1 that for small $T > 0$,

$$\Delta_E(T) = \frac{1}{S_0 T}\,\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big] - \frac{\Phi(S_0)}{S_0} + \frac{1}{S_0 T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T h_s H_s\,ds\right] + \mathcal{O}(\sqrt{T})\,, \quad (4.15)$$

where the processes $(h_s)_{0\le s\le T}$, $(H_s)_{0\le s\le T}$ and a random variable G are defined by

$$h_s := \frac{Y_s}{\sigma(s, X_s)X_s}\,, \quad H_s := \frac{X_T(D_s Y_T)}{{Y_T}^2}\,, \quad G := \frac{X_T}{Y_T}\,.$$

In the second step, we approximate the two expectations on the right-hand side of Eq.(4.15), i.e.,

$$\frac{1}{S_0 T}\,\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big]\,, \quad \frac{1}{S_0 T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T h_s H_s\,ds\right]\,. \quad (4.16)$$

For the approximation, we define four auxiliary processes $(\tilde{h}_s)_{0\le s\le T}$, $(\hat{h}_s)_{0\le s\le T}$, $(\tilde{H}_s)_{0\le s\le T}$, $(\hat{H}_s)_{0\le s\le T}$ by

$$\tilde{h}_s := \frac{\tilde{Y}_s}{\sigma(s, \tilde{X}_s)\tilde{X}_s}\,, \quad \hat{h}_s := \frac{\hat{Y}_s}{\sigma(s, \hat{X}_s)\hat{X}_s}\mathbb{1}_{\{\hat{X}_s \ge \frac{S_0}{2}\}}\,, \quad \tilde{H}_s := \frac{\tilde{X}_T(D_s\tilde{Y}_T)}{\tilde{Y}_T^2}\,, \quad \hat{H}_s := \frac{\hat{X}_T(D_s\hat{Y}_T)}{\hat{Y}_T^2}\mathbb{1}_{\{\hat{Y}_T \ge \frac{1}{2}\}}\,.$$

In Lemma 4.6, $(\tilde{h}_s)_{0\le s\le T}$, $(\hat{h}_s)_{0\le s\le T}$ are used to approximate $(h_s)_{0\le s\le T}$. Similarly, $(\tilde{H}_s)_{0\le s\le T}$, $(\hat{H}_s)_{0\le s\le T}$ are used to approximate $(H_s)_{0\le s\le T}$. We also define two new random variables $\tilde{G}$, $\hat{G}$ by

$$\tilde{G} := \frac{\tilde{X}_T}{\tilde{Y}_T}\,, \quad \hat{G} := \frac{\hat{X}_T}{\hat{Y}_T}\mathbb{1}_{\{\hat{Y}_T \ge \frac{1}{2}\}}$$

to approximate G.

Lemma 4.6. *Under Assumption 1, for any $p > 0$, there exists a positive constant I_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|h_t - \tilde{h}_t|^p] \le I_p t^p\,, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{h}_t - \hat{h}_t|^p] \le I_p t^p\,.$$

(ii) For $0 \le T \le 1$,

$$\sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|H_s - \tilde{H}_s|^p] \le I_p T^{\frac{p}{2}}\,, \quad \sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|\tilde{H}_s - \hat{H}_s|^p] \le I_p T^{\frac{p}{2}}\,.$$

(iii) For $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|G - \tilde{G}|^p] \le I_p T^p\,, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{G} - \hat{G}|^p] \le I_p T^p\,.$$

Using this lemma, we can approximate the two expectations in Eq.(4.16). The approximation results are given in the following proposition.

Proposition 4.6. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $\dfrac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big] = \dfrac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right] + \mathcal{O}(\sqrt{T})$.

(ii) $\dfrac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\displaystyle\int_0^T h_s H_s\,ds\right] = \dfrac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\displaystyle\int_0^T \hat{h}_s\hat{H}_s\,ds\right] + \mathcal{O}(\sqrt{T})$.

Since the proofs for Lemma 4.6 and Proposition 4.6 are obtained by merely duplicating those for Lemma 4.2 and Propositions 4.2, 4.3, we omit them.

Then, it is easy to observe from Proposition 4.6 that for small $T > 0$,

$$\Delta_E(T) = \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right] - \frac{\Phi(S_0)}{S_0} + \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s\hat{H}_s\,ds\right] + \mathcal{O}(\sqrt{T})\,. \quad (4.17)$$

To obtain the asymptotic formula for $\Delta_E(T)$, we need to estimate the following two expectations for the last step.

$$\frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right]\,, \quad \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s\hat{H}_s\,ds\right]\,. \quad (4.18)$$

As a necessary lemma, we approximate $\delta(\hat{h})$ to a normal random variable $\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)$ in the following. The proof is similar to the proof for Lemma 4.5; hence, it is omitted.

Lemma 4.7. *Under Assumption 1, for any $p > 0$, there exists a positive constant J_p depending only on p such that the following inequalities hold for $0 \le T \le 1$:*

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{h})|^p] \le J_p T^{\frac{p}{2}}, \quad \mathbb{E}^{\mathbb{Q}}\left[\left|\delta(\hat{h}) - \delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)\right|^p\right] \le J_p T^p\,.$$

As Lemma 4.5 helps estimate the expectations in Eq.(4.11), this lemma enables us to estimate the expectations in Eq.(4.18) in relation to multivariate normal random variables. Direct calculations with regard to normal random variables yield the following two estimates. See Appendix C.7 for details.

Proposition 4.7. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\begin{aligned}&\frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \frac{\Phi(S_0)}{S_0T}\int_0^T \frac{\sigma(s,S_0) - \nu(s,S_0)}{\sigma(s,S_0)}\,ds + \mathcal{O}(\sqrt{T})\end{aligned} \quad (4.19)$$

and

$$\frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s\hat{H}_s\,ds\right] = \frac{\Phi(S_0)}{S_0T}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}\,ds + \mathcal{O}(\sqrt{T})\,, \quad (4.20)$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

Therefore, we obtain the desired asymptotic formula for $\Delta_E(T)$ by straightforward use of Proposition 4.7 in Eq.(4.17).

Theorem 4.3. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

By comparing Theorem 4.1 with Theorem 4.3, we can observe that the limits of $\Delta_A(T)$ and $\Delta_E(T)$ are the same.

Corollary 4.4. *Under Assumptions 1 and 3, if $\Delta_E(T)$ converges as $T \to 0$, then $\Delta_A(T)$ also converges and vice versa. Moreover,*

$$\lim_{T\to 0} \Delta_A(T) = \lim_{T\to 0} \Delta_E(T) \,.$$

5 Short-maturity options with Hölder continuous payoffs

In this section, we generalize Theorems 3.1, 4.1, and 4.3 to the Hölder continuous function Φ. Under Assumption 1, the first variation process of S is the unique solution of Eq.(C.1):

$$dZ_t = (r-q)Z_t\,dt + \nu(t,S_t)Z_t\,dW_t\,, \quad Z_0 = 1\,.$$

Analogous to Lemma 2.1, the lemma below is crucial in the following.

Lemma 5.1. *Under Assumption 1, for any $p > 0$, as $t \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^p] = \mathcal{O}(t^p)\,, \quad \mathbb{E}^{\mathbb{Q}}[|Z_t - Y_t|^p] = \mathcal{O}(t^p)\,.$$

Proof. The first equality with $p = 2$ has already been proved in the proof of Lemma 3.1 in Appendix B.1. The remainder of the proof is the same as that of Lemma 2.1 in Appendix A.1. □

This section considers Hölder continuous payoffs in the following order. In Section 5.1, the asymptotic for option prices in Theorem 3.1 will be generalized. Estimates for the option delta value in Theorems 4.1 and 4.3 are generalized to Hölder continuous payoffs in Section 5.2.

5.1 Estimates for option prices

The following lemma is a generalization of Lemma 3.1.

Lemma 5.2. *Under Assumptions 1 and 2, as $T \to 0$, we have*

(i) $P_A(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T^\gamma)\,,$

(ii) $P_E(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(T^\gamma)\,.$

Here, γ is the Hölder exponent in Assumption 2.

Proof. Choose $q > 1$ such that $\gamma q > 1$. Then, by the Jensen inequality and Assumption 2,

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] \le \beta\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^{\gamma q}]\,dt\right)^{\frac{1}{q}}.$$

Then, the remainder of the proof comes from Lemma 3.1. □

Now, we get the asymptotic estimates for the prices of options having Hölder continuous payoffs.

Theorem 5.1. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$P_A(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^{\gamma}), \quad P_E(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)] + \mathcal{O}(T^{\gamma}).$$

Here, γ is the Hölder exponent in Assumption 2.

Proof. The proof is straightforward from the proofs of Theorem 3.1 and Lemma 5.2. □

Compared to the Lipschitz continuous case ($\gamma = 1$), the convergence order is degraded for $\gamma < 1$. However, the following example shows that this asymptotic relation is optimal in general.

Example 5.1. *Given any K and $0 < \gamma \le 1$, define the payoff function Φ by*

$$\Phi(x) = (x - K)_+^{\gamma}.$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$P_A(T) = \frac{1}{2}(S_0\sigma_A(T))^{\gamma} M(\gamma) T^{\frac{\gamma}{2}} + \mathcal{O}(T^{\gamma}),$$

where $M(\gamma) := \mathbb{E}^{\mathbb{Q}}[|Z|^{\gamma}]$ with a standard normal variable Z. If we replace $\sigma_A(T)$ with $\sigma_E(T)$, we get the asymptotic result for the European option price $P_E(T)$.

5.2 Estimates for option delta values

In this section, we investigate the short-maturity option delta values when Φ is any Hölder continuous function. The main results are as follows:

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}} Z\right] + \mathcal{O}(T^{\gamma - \frac{1}{2}})$$

and

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}} Z\right] + \mathcal{O}(T^{\gamma - \frac{1}{2}}),$$

where Z denotes a standard normal variable and γ is the Hölder exponent in Assumption 2.

The proof begins by recognizing the changes in Lemma 4.1 from Section 4. In the proof of Lemma 4.1, we make use of the fact that any Lipschitz continuous function is almost everywhere differentiable with respect to the Lebesgue measure. However, this condition fails for arbitrary γ-Hölder continuous functions in general. Therefore, we should rely only on the Malliavin representation of the option delta (see Proposition 4.1) for the approximation. First, we examine the following Malliavin representation for $\Delta_A(T)$.

Proposition 5.1 (Nualart (1995); Benhamou (2000)). *For the process S stated in Eq.*(2.1)*, under Assumption 1, we have*

$$\Delta_A(T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\delta\left(\frac{2Z_\cdot^2}{\sigma(\cdot,S_\cdot)S_\cdot}\right)\frac{1}{\int_0^T Z_t\,dt}\right]$$
$$- e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\int_0^T \frac{2{Z_u}^2}{\sigma(u,S_u)S_u}D_u\left(\frac{1}{\int_0^T Z_t\,dt}\right)du\right],$$

where $\delta(\cdot)$ is the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

As we approximated u by $\tilde{u}$ and $\hat{u}$ in Lemma 4.2, we would approximate a process $\frac{2Z_\cdot^2}{\sigma(\cdot,S_\cdot)S_\cdot}$ by u. Likewise, a random variable $\frac{1}{\int_0^T Z_t\,dt}$ is approximated to F. Examine following Lemmas 5.3 and 5.4 in comparison to Lemmas 4.2, 4.3, and 4.4.

Lemma 5.3. *Under Assumption 1, for any $p > 0$, as $t \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2Z_t^2}{\sigma(t,S_t)S_t} - u_t\right|^p\right] = \mathcal{O}(t^p).$$

Proof. Same as the proof of Lemma 4.2. Use Lemma 5.1 instead of Lemma 2.1. □

Lemma 5.4. *Under Assumption 1, for any $p > 0$, as $T \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{1}{\frac{1}{T}\int_0^T Z_t\,dt} - TF\right|^p\right] = \mathcal{O}(T^p) \qquad \sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}\left[\left|D_s\left(\frac{1}{\frac{1}{T}\int_0^T Z_t\,dt}\right) - TD_sF\right|^p\right] = \mathcal{O}(T^p)$$

Proof. Examine the following Malliavin derivatives:

$$D_uS_l = \frac{Z_l}{Z_u}\sigma(u,S_u)S_u\mathbb{1}_{\{u\leq l\}},$$

$$D_uZ_t = Z_t\left[\nu(u,S_u) - \int_0^t \nu(l,S_l)\rho(l,S_l)D_uS_l\,dl + \int_0^t \rho(l,S_l)D_uS_l\,dW_l\right]\mathbb{1}_{\{u\leq t\}}.$$

The remainder of the proof is similar to the proofs of Lemmas 4.2–4.4. Use Lemma 5.1 instead of Lemma 2.1. □

Through minor changes in Propositions 4.2 and 4.3, we obtain the generalized version of Lemma 4.1.

Lemma 5.5. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$\Delta_A(T) = e^{-rT}\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}}).$$

Proof. Apply Lemmas 5.3 and 5.4 to the proofs of Propositions 4.2 and 4.3 instead of Lemmas 4.2–4.4. □

This section is devoted to the following result, which is the generalization of Theorems 4.1 and 4.3.

Theorem 5.2. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})$$

and

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}}).$$

Proof. Minor changes to the convergence rates in Propositions 4.2–4.4 give the first asymptotic. For the European delta, we duplicate the arguments in this section. □

Remark 5.1. *The borderline is $\gamma = \frac{1}{2}$ in these formulas. If $\gamma < \frac{1}{2}$, the estimates in Theorem 5.2 are meaningless; however, for $\frac{1}{2} < \gamma \leq 1$, they provide us with the short-maturity estimate with the convergence rate $\gamma - \frac{1}{2} \leq \frac{1}{2}$.*

Example 5.2. *Given any K and $\frac{1}{2} < \gamma < 1$, define the payoff function Φ by*

$$\Phi(x) = (x-K)_+^\gamma .$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$\Delta_A(T) = \frac{M(\gamma+1)}{2(S_0\sigma_A(T))^{1-\gamma}} \times \frac{1}{T^{\frac{1-\gamma}{2}}} + \mathcal{O}(T^{\gamma-\frac{1}{2}}),$$

where $M(\gamma+1) := \mathbb{E}^{\mathbb{Q}}[|Z|^{\gamma+1}]$ with a standard normal variable Z.

6 Numerical tests

In this section, we conduct some numerical tests of the asymptotic formulas in Theorems 5.1 and 5.2. We consider the two following local volatility models.

(1) **Extended constant elasticity of variance(CEV) model:**

$$dS_t = (r-q)S_t dt + e^{-\lambda t}\xi S_t^\theta dW_t, \tag{6.1}$$

with $r = q = 0$, $S_0 = 100$. Thus, we get $\sigma(t,x) = e^{-\lambda t}\xi x^{\theta-1}$. In our tests, we choose $\lambda = 1$, $\xi = 0.2$ and $\theta = 0.5$. In Gatheral et al. (2012); Sachs and Schneider (2014), the same set of parameters except for $\lambda = 0$ was used to test estimations of implied volatility.

(2) **Quadratic model:**

$$dS_t = (r-q)S_t dt + e^{-\lambda t}\sigma\left[\psi S_t + (1-\psi)S_0 + \frac{\eta}{2}\frac{(S_t - S_0)^2}{S_0}\right]dW_t, \tag{6.2}$$

with $r = q = 0$, $S_0 = 100$. Here, the volatility function is given by

$$\sigma(t,x) = \frac{e^{-\lambda t}\sigma}{x}\left[\psi x + (1-\psi)S_0 + \frac{\eta}{2}\frac{(x-S_0)^2}{S_0}\right].$$

Our choices of parameters are $\lambda = 1$, $\sigma = 0.2$, $\psi = 0.5$ and $\eta = 10$. With the same choice of $(r, q, S_0, \sigma, \psi, \eta)$ as above, Andersen (2011) investigated implied volatility smile for time-independent cases ($\lambda = 0$). Here, we put $\lambda = 1$ to study time-dependent cases as proposed in Gatheral et al. (2012).

value of γ	$T = 1/365$		$T = 1/10$		$T = 1$	
	MC	Asymptotics	MC	Asymptotics	MC	Asymptotics
$\gamma = 0.1$	0.35586	0.35647	0.42815	0.4257	0.46778	0.46856
$\gamma = 0.6$	0.074969	0.075051	0.21755	0.21767	0.38484	0.38708
$\gamma = 1$	0.02389	0.024095	0.14123	0.14213	0.36948	0.37097
$\gamma = 1.5$	0.0063799	0.006383	0.091161	0.091443	0.38319	0.38559
$\gamma = 1.9$	0.0023342	0.0023313	0.067727	0.067922	0.42306	0.42038

Table 1: ATM Asian option prices for $\Phi(x) = (x - K)^\gamma$ under the CEV model.

value of γ	$T = 1/365$		$T = 1/10$		$T = 1$	
	MC	Asymptotics	MC	Asymptotics	MC	Asymptotics
$\gamma = 0.6$	1.3541	1.3547	0.67115	0.66609	0.44313	0.45381
$\gamma = 1$	0.4939	0.5	0.49631	0.5	0.50188	0.5
$\gamma = 1.5$	0.1518	0.15154	0.36411	0.36806	0.59254	0.59462
$\gamma = 1.9$	0.060072	0.060393	0.2991	0.2983	0.71432	0.70736

Table 2: ATM Asian option deltas for $\Phi(x) = (x - K)^\gamma$ under the CEV model.

Observe that Assumption 1 is not fulfilled in CEV and Quadratic models. However, only finite observations of (t, x) are used for numerical test. Since the volatility functions $\sigma(t, x)$ in both models satisfy Assumption 1 in compact neighborhoods of $(t, x) = (0, S_0)$, we will regard the volatility functions $\sigma(t, x)$ as bounded and smooth outside some compact neighborhoods of $(t, x) = (0, S_0)$ so that Assumption 1 is satisfied. Sachs and Schneider (2014) took the same approach to solve the discrepancy between technical assumptions for their estimations of implied volatility and CEV model.

6.1 Powers of call options

Under these two models, we will first compare asymptotic formulas for ATM ($K = S_0$) Asian option prices and deltas having the payoff $\Phi(x) = (x - K)_+^\gamma$ presented in Examples 3.2, 4.2, 5.1 and 5.2 with results from the Monte Carlo simulation. While these options are not traded in a real market, numerical tests could show how accurate our asymptotic formulas are.

In Tables 1, 2, 3 and 4, numerical results from asymptotic formulas presented in this paper are given in *Asymptotics* columns. Results from the Monte Carlo simulation are given in *MC* columns. We consider $M = 10^5$ paths and $N = 10^3$ time steps during simulations. For

value of γ	$T = 1/365$		$T = 1/10$		$T = 1$	
	MC	Asymptotics	MC	Asymptotics	MC	Asymptotics
$\gamma = 0.1$	0.44697	0.44877	0.52977	0.53592	0.57678	0.58988
$\gamma = 0.6$	0.29916	0.29878	0.8577	0.86658	1.5252	1.541
$\gamma = 1$	0.23978	0.24095	1.428	1.4213	3.7326	3.7097
$\gamma = 1.5$	0.20237	0.20185	2.9437	2.8917	12.7076	12.1934
$\gamma = 1.9$	0.18516	0.18518	5.487	5.3952	36.3354	33.3923

Table 3: ATM Asian option prices for $\Phi(x) = (x - K)^\gamma$ under the Quadratic model.

value of γ	$T=1/365$		$T=1/10$		$T=1$	
	MC	Asymptotics	MC	Asymptotics	MC	Asymptotics
$\gamma=0.6$	0.53762	0.5393	0.26232	0.26517	0.1791	0.18066
$\gamma=1$	0.49713	0.5	0.50838	0.5	0.51115	0.5
$\gamma=1.5$	0.48055	0.47923	1.1917	1.1639	1.9573	1.8804
$\gamma=1.9$	0.47943	0.47972	2.4184	2.3695	6.2038	5.6188

Table 4: ATM Asian option deltas for $\Phi(x)=(x-K)^\gamma$ under the Quadratic model.

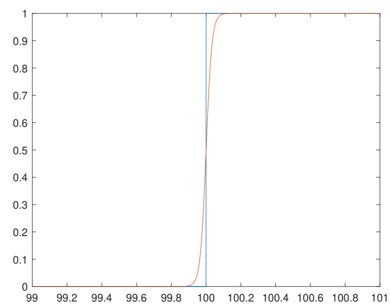

Figure 1: Plots of Φ_{binary} and $\Phi_{\text{logistic}}(\cdot:\kappa)$ with $K=100$ and $\kappa=60$.

simulating deltas, we use the Malliavin representation in Proposition 5.1.

Tables 1, 2, 3 and 4 show that asymptotic formulas are being more accurate as the maturity T gets shorter. It is also noteworthy that asymptotic formulas are more accurate under the CEV model (6.1) than under the Quadratic model (6.2). These differences, however, are natural considering that the Quadratic model implies relatively higher (local) volatilities $\sigma(t,S_t)$ than the CEV model.

6.2 Approximations of digital options

Next, we consider the way to hedge *Digital options* which are defined as having the terminal payoff $\Phi_{\text{binary}}(x)=\mathbb{1}_{\{x\geq K\}}$. The main technical issue regarding digital options is that their delta values are being unrealistically large at small T, especially when a spot price S_0 is close to a strike K.

One way to deviate this problem is to replace digital options by more manageable options for hedging purposes. Consider options having logistic functions $\Phi_{\text{logistic}}(x;\kappa)=1/(1+e^{-\kappa(x-K)})$, $\kappa>0$ as the terminal payoff. (Say these options as *Logistic options.*) Logistic functions have following properties:

Value of T	Digital option(MC)	Logistic option(MC)
$T=1/10^6$	0.50081	0.50004
$T=1/10^4$	0.49976	0.49987
$T=1/10^2$	0.49813	0.49827

Table 5: Price approximation of digital options by logistic options under the CEV model.

Value of T	Digital option(MC)	Logistic option	
		MC	Asymptotics
$T = 1/10^6$	343.3846	14.4961	14.982
$T = 1/10^4$	34.5661	13.5341	13.5279
$T = 1/10^2$	3.4577	3.3438	3.3521

Table 6: Delta approximation of digital options by logistic options under the CEV model.

Value of T	Digital option(MC)	Logistic option(MC)
$T = 1/10^8$	0.4996	0.50002
$T = 1/10^6$	0.49968	0.49996
$T = 1/10^4$	0.50112	0.50044

Table 7: Price approximation of digital options by logistic options under the Quadratic model.

(1) They are Lipschitz continuous and $\Phi_{\text{logistic}}(x;\kappa) \to (1/2)\mathbb{1}_{\{x=K\}} + \mathbb{1}_{\{x>K\}}$ as $\kappa \to \infty$.

(2) $0 < \Phi'_{\text{logistic}}(\cdot;\kappa) \le \kappa/4$ and $\Phi'_{\text{logistic}}(K,\kappa) = \kappa/4$.

These properties indicate two advantages of replacing binary options by logistic options for hedging purposes. First, prices of logistic options are close to prices of binary options for sufficiently large κ. Second, hedging logistic options is feasible in the sense that delta values do not explode at small T. This feature makes hedging logistic options in replace of binary options attractive. Corollary 4.2 implies that the upper bounded of delta values of logistic options is close to $\kappa/4$ when $K = S_0$. Therefore by controlling κ, practitioners can manage the cost of hedging logistic options.

We perform numerical test for $\kappa = 60$ and $K = S_0$ under the CEV model (6.1) and the Quadratic model (6.2). In Tables 5 and 7, computations of digital option prices and logistic options prices by the Monte Carlo simulation are listed in *Digital option(MC)* and *Logistic option(MC)* columns, respectively. Test results justify that risk-neutral valuations of logistic options are similar to that of binary options. Similarly in Tables 6 and 8, delta values of digital options and logistic options from the Monte Carlo method are respectively given in the first and second columns. The third columns contain estimated delta values of logistic options obtained from Theorem 4.1. As one can see, delta values of binary options explode at small T where as those of logistic options seem to be bounded above by $\kappa/4 = 15$. Also, estimations from Theorem 4.1 and the Monte Carlo simulation are in good agreement. Therefore, estimations of delta values of logistic options by Theorem 4.1 provide us with the new way to hedge digital options.

Value of T	Digital option(MC)	Logistic option	
		MC	Asymptotics
$T = 1/10^8$	344.38	14.6368	14.982
$T = 1/10^6$	34.5449	13.4539	13.5279
$T = 1/10^4$	3.4775	3.3679	3.3443

Table 8: Delta Approximation of digital options by logistic options under the Quadratic model.

7 Comparison between volatilities at short maturity

To emphasize the dependence on the payoff function Φ and the volatility function $\sigma(t,x)$, we denote the option price $P_A(T)$, $P_E(T)$ and the option delta value $\Delta_A(T)$, $\Delta_E(T)$ by $P_A(T;\Phi\,,\sigma)$, $P_E(T;\Phi\,,\sigma)$ and $\Delta_A(T;\Phi\,,\sigma)$, $\Delta_E(T;\Phi\,,\sigma)$, respectively. Since the Asian and European volatilities $\sigma_A(T)$, $\sigma_E(T)$ defined in Eq.(3.1) also depend on the volatility function $\sigma(t,x)$, we denote them by $\sigma_A(T;\sigma)$, $\sigma_E(T;\sigma)$.

7.1 Comparison under the general local volatility model

In practice, Asian-style options are mainly quoted by their prices, not by their implied volatilities. This is due to the lack of a simple closed-form formula for the density of $\frac{1}{T}\int_0^T S_t\,dt$. Instead, many practitioners estimate the implied volatilities of European-style options for pricing and hedging purposes.

Let us focus on the situation in which the volatility function $\sigma(t,x)$ of the underlying process Eq.(2.1) is calibrated to match market data on European-style options. Denote this calibrated volatility by $\sigma_{\text{Implied}}(t,x)$. In this section, we aim to price and hedge the Asian option under the volatility function $\sigma_{\text{Implied}}(t,x)$. Note from the asymptotic formulas established in Theorems 3.1, 4.1, and 4.3 that if a function $\tau(t,x)$ satisfies Assumption 1 and the equality

$$\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau) \tag{7.1}$$

for sufficiently small $T>0$, then for any Lipschitz continuous function $\Phi(\cdot)$,

$$P_A(T;\Phi,\sigma_{\text{Implied}}) = P_E(T;\Phi,\tau) + \mathcal{O}(T)\,, \quad \Delta_A(T;\Phi,\sigma_{\text{Implied}}) = \Delta_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,. \tag{7.2}$$

Meanwhile, if the equality Eq.(7.1) fails in any neighborhood of $T=0$ and, heuristically speaking, τ deviates excessively from σ_{Implied} , we will see in Propositions 7.1 and 7.2 that the convergence rates in Eq.(7.2) are degraded in general. In this sense, we may regard the European option under the volatility $\tau(t,x)$ satisfying Eq.(7.1) as a short-maturity proxy for the Asian option under the volatility $\sigma_{\text{Implied}}(t,x)$.

Proposition 7.1. *Consider any volatility function τ that satisfies Assumption 1.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$P_A(T;\Phi,\sigma_{\text{Implied}}) = P_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,.$$

(ii) Suppose that $\limsup_{T\to 0}|\sigma_A(T;\sigma_{\text{Implied}}) - \sigma_E(T;\tau)| \neq 0$. Then, for any $0<\epsilon\le\frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A(T;\Phi_\epsilon,\sigma_{\text{Implied}}) - P_E(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon})\,. \tag{7.3}$$

(iii) By contrast, suppose that $\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau)$ for small $T>0$. Then, for any Φ that satisfies Assumption 3,

$$P_A(T;\Phi,\sigma_{\text{Implied}}) = P_E(T;\Phi,\tau) + \mathcal{O}(T)\,.$$

Proof. Except for Eq.(7.3), the remainder of the proof is obvious from Theorem 3.1. Take Φ_ϵ as

$$\Phi_\epsilon(x) := (x-K)^{1+\epsilon}\mathbb{1}_{\{K\le x<2K\}} + (2K)^{1+\epsilon}\mathbb{1}_{\{2k\le x\}}\,,$$

with $K = S_0$. Then, Eq.(7.3) easily follows. ☐

Proposition 7.2. *Consider any volatility function τ that satisfies Assumption 1.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$\Delta_A(T;\Phi,\sigma_{\text{Implied}}) = \Delta_E(T;\Phi,\tau)\ + \mathcal{O}(1)\,.$$

(ii) Suppose that $\limsup_{T\to 0}|\sigma_A(T;\sigma_{\text{Implied}}) - \sigma_E(T;\tau)| \neq 0$. Then, for any $0<\epsilon\le\frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|\Delta_A(T;\Phi_\epsilon,\sigma_{\text{Implied}}) - \Delta_E(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^\epsilon)\,.$$

(iii) By contrast, suppose that $\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau)$ for small $T>0$. Then, for any Φ that satisfies Assumption 3,

$$\Delta_A(T;\Phi,\sigma_{\text{Implied}}) = \Delta_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,.$$

Proof. Same as that of Proposition 7.1. ☐

Remark 7.1. *Suppose that $s\mapsto\sigma_{\text{Implied}}(s,S_0)$ and $s\mapsto\tau(s,S_0)$ are both continuous at $s=0$. Then, the condition $\limsup_{T\to 0}|\sigma_A(T;\sigma_{\text{Implied}}) - \sigma_E(T;\tau)| \neq 0$ is equivalent to*

$$\tau(0,S_0) \neq \frac{1}{\sqrt{3}}\sigma_{\text{Implied}}(0,S_0)\,,$$

because $\lim_{T\to 0}\sigma_A(T;\sigma_{\text{Implied}}) = \frac{1}{\sqrt{3}}\sigma_{\text{Implied}}(0,S_0)$ and $\lim_{T\to 0}\sigma_E(T;\tau) = \tau(0,S_0)$. These limits coincide with the well-known result that ATM Asian vol $=\frac{1}{\sqrt{3}}\cdot$ ATM European vol. See Pirjol and Zhu (2016).

If some suitable technical condition is satisfied for $s\mapsto\sigma_{\text{Implied}}(s,S_0)$, then Eq.(7.1) forces τ to be determined uniquely. Hence, whenever σ_{Implied} is given, we can always approximate the Asian option having volatility σ_{Implied} by the European option having volatility τ.

Proposition 7.3. *Suppose that τ satisfies Assumption 1 and Eq.(7.1). If $s\mapsto\sigma_{\text{Implied}}(s,S_0)$ is continuous in some neighborhood of $s=0$, say $[0,\delta]$, then $s\mapsto\tau(s,S_0)$ is uniquely determined in $[0,\delta]$ by*

$$\tau(s,S_0) = \left[\frac{2}{s^3}\int_0^s \sigma^2_{\text{Implied}}(u,S_0)(us-u^2)\,du\right]^{\frac{1}{2}}.$$

Conversely, if $s\mapsto\tau(s,S_0)$ is of $C^2[0,\delta]$, then the only choice of a continuous function $s\mapsto\sigma_{\text{Implied}}(s,S_0)$ in $[0,\delta]$ is

$$\sigma_{\text{Implied}}(s,S_0) = \tau(s,S_0)\left[3 + 6s\frac{\tau'(s,S_0)}{\tau(s,S_0)} + s^2\left(\frac{\tau'(s,S_0)}{\tau(s,S_0)}\right)^2 + s^2\frac{\tau''(s,S_0)}{\tau(s,S_0)}\right]^{\frac{1}{2}}.$$

Proof. Differentiate both sides of Eq.(7.1) by T. ☐

7.2 Comparison under the Black–Scholes model

In this section, we focus on the Black–Scholes model, i.e., $\sigma(t,x) \equiv \sigma$. Observe that under the Black–Scholes model, $\sigma_A(T;\sigma) = \frac{\sigma}{\sqrt{3}}$ and $\sigma_E(T;\sigma) = \sigma$. Now, consider a new option, i.e., the so-called *geometric average Asian option*, whose price is given by

$$P_G^{\mathrm{BS}}(T;\Phi,\sigma) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(e^{\frac{1}{T}\int_0^T \log S_t\,dt}\right)\right].$$

Here, the superscript BS is used to emphasize the Black–Scholes model. We denote its delta value by $\Delta_G^{\mathrm{BS}}(T;\Phi,\sigma)$.

Let us confine ourselves to the situation in which a constant $\sigma_{\mathrm{Implied}}$ is obtained from the European option. We want to approximate the Asian option price $P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}})$ and its delta value $\Delta_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}})$ by their European and geometric average Asian counterparts. In Propositions 7.4 and 7.5, we observe that the European option having volatility $\frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}$ and the geometric average Asian option having volatility $\sigma_{\mathrm{Implied}}$ are optimal choices for the asymptotic approximation.

Proposition 7.4. *Consider any constant volatility $\tau > 0$.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = P_E^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(\sqrt{T}),$$
$$P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = P_G^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(\sqrt{T}).$$

(ii) Suppose that $\tau \neq \frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}$. Then, for any $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - P_E^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon}).$$

Likewise, if $\tau \neq \sigma_{\mathrm{Implied}}$ and $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - P_G^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon}).$$

(iii) By contrast, for any Φ that satisfies Assumption 3,

$$P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = P_E^{\mathrm{BS}}\left(T;\Phi_\epsilon,\frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}\right) + \mathcal{O}(T),$$
$$P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = P_G^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) + \mathcal{O}(T).$$

Proof. Under the Black–Scholes model, $e^{\frac{1}{T}\int_0^T \log S_t\,dt}$ is a log-normal random variable. Hence, it is easy to check from Lemma 3.1 that

$$P_G^{\mathrm{BS}}(T;\Phi,\tau) = P_E^{\mathrm{BS}}\left(T;\Phi,\frac{1}{\sqrt{3}}\tau\right) + \mathcal{O}(T)$$

for any positive constant τ and Φ satisfying Assumption 3. The remainder of the proof is the same as that of Proposition 7.1. □

Proposition 7.5. *Consider any constant volatility* $\tau > 0$.

(i) *For any payoff function* Φ *that satisfies Assumption 3,*

$$\Delta_A^{\rm BS}(T;\Phi,\sigma_{\rm Implied}) = \Delta_E^{\rm BS}(T;\Phi,\tau) + \mathcal{O}(1)\,,$$
$$\Delta_A^{\rm BS}(T;\Phi,\sigma_{\rm Implied}) = \Delta_G^{\rm BS}(T;\Phi,\tau) + \mathcal{O}(1)\,.$$

(ii) *Suppose that* $\tau \neq \frac{1}{\sqrt{3}}\sigma_{\rm Implied}$. *Then, for any* $0 < \epsilon \leq \frac{1}{2}$, *there exists a payoff function* Φ_ϵ *that satisfies Assumption 3 such that*

$$|\Delta_A^{\rm BS}(T;\Phi_\epsilon,\sigma_{\rm Implied}) - \Delta_E^{\rm BS}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^\epsilon)\,.$$

Likewise, if $\tau \neq \sigma_{\rm Implied}$ *and* $0 < \epsilon \leq \frac{1}{2}$, *there exists a payoff function* Φ_ϵ *that satisfies Assumption 3 such that*

$$|\Delta_A^{\rm BS}(T;\Phi_\epsilon,\sigma_{\rm Implied}) - \Delta_G^{\rm BS}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^\epsilon)\,.$$

(iii) *By contrast, for any* Φ *that satisfies Assumption 3,*

$$\Delta_A^{\rm BS}(T;\Phi_\epsilon,\sigma_{\rm Implied}) = \Delta_E^{\rm BS}\left(T;\Phi_\epsilon,\frac{1}{\sqrt{3}}\sigma_{\rm Implied}\right) + \mathcal{O}(\sqrt{T})\,,$$
$$\Delta_A^{\rm BS}(T;\Phi_\epsilon,\sigma_{\rm Implied}) = \Delta_G^{\rm BS}(T;\Phi_\epsilon,\sigma_{\rm Implied}) + \mathcal{O}(\sqrt{T})\,.$$

Proof. From Lemma 4.1 and Lemma C.1 in its proof, it is easy to check that

$$\Delta_G^{\rm BS}(T;\Phi,\tau) = \Delta_E^{\rm BS}\left(T;\Phi,\frac{1}{\sqrt{3}}\tau\right) + \mathcal{O}(\sqrt{T})$$

for any positive constant τ and Φ satisfying Assumption 3. The remainder of the proof is the same as that of Proposition 7.2. ◻

Remark 7.2. *Consider any positive constant* τ . *Even though* $P_A^{\rm BS}(T;\Phi,\tau)$, $P_G^{\rm BS}(T;\Phi,\tau)$, $P_E^{\rm BS}(T;\Phi,\tau)$ *as well as* $\Delta_A^{\rm BS}(T;\Phi,\tau)$, $\Delta_G^{\rm BS}(T;\Phi,\tau)$, $\Delta_E^{\rm BS}(T;\Phi,\tau)$ *share the same limit as* $T\to 0$ *from Corollaries 3.2 and 4.2, Propositions 7.4 and 7.5 argue that the Asian option is more "close" to the geometric average Asian option than to the European option.*

8 Special case: Approximation for call and put options

This section only considers the Asian call option, i.e., $\Phi(x) = (x-K)_+$, and the Asian put option, i.e., $\Phi(x) = (K-x)_+$. The meanings of the following notations are self-explanatory:

$$P_A^{\rm call}(T)\,,\ P_A^{\rm put}(T)\,,\ \Delta_A^{\rm call}(T)\,,\ \Delta_A^{\rm put}(T)\,.$$

The short-maturity behaviors of these four quantities have already been examined in Examples 3.1 and 4.1. However, this section uses the large deviation principle to provide additional information about their behaviors.

8.1 Application of the large deviation principle

Consider the model where the volatility function $\sigma(t,x)$ is independent of t. In other words, $\sigma(t,x) \equiv \sigma(x)$. Besides Assumption 1, let us impose the following assumption on the volatility function $\sigma(x)$.

Assumption 4. *There are constants $M > 0$ and $\gamma > 0$ such that for any $x, y \in \mathbb{R}$,*

$$|\sigma(e^x) - \sigma(e^y)| \leq M|x-y|^{\gamma}.$$

Under Assumptions 1 and 4, the following short-maturity asymptotic results for $P_A^{\text{call}}(T)$ and $P_E^{\text{put}}(T)$ were first proved in Pirjol and Zhu (2016).

Theorem 8.1 (Pirjol and Zhu (2016)). *Under Assumptions 1 and 4, the following hold.*

(i) For an OTM Asian call option, i.e., $K > S_0$,

$$\lim_{T \to 0} T \log(P_A^{\text{call}}(T)) = -\mathcal{I}(K, S_0).$$

(ii) For an OTM Asian put option, i.e., $S_0 > K$,

$$\lim_{T \to 0} T \log(P_A^{\text{put}}(T)) = -\mathcal{I}(K, S_0).$$

Here, for any $x\,, y > 0$, the rate function $\mathcal{I}$ is defined by

$$\mathcal{I}(x\,, y) := \inf_{\substack{\int_0^1 e^{g(t)}\,dt = x, \\ g(0)=\log y,\, g \in \mathcal{AC}[0,1]}} \frac{1}{2} \int_0^1 \left(\frac{g'(t)}{\sigma(e^{g(t)})} \right)^2 dt\,, \tag{8.1}$$

where $\mathcal{AC}[0,1]$ is the space of absolutely continuous functions on $[0,1]$.

Remark 8.1. *Note that the rate function $\mathcal{I}$ in Pirjol and Zhu (2016) comes from the large deviation principle. According to the large deviation principle, for any Borel set A in $\mathbb{R}^+$,*

$$\begin{aligned}
-\inf_{x \in A^\circ} \mathcal{I}(x, S_0) \leq \liminf_{T \to 0} T \log\left(\mathbb{Q}\left\{ \frac{1}{T} \int_0^T S_t\,dt \in A \right\} \right) \\
\leq \limsup_{T \to 0} T \log\left(\mathbb{Q}\left\{ \frac{1}{T} \int_0^T S_t\,dt \in A \right\} \right) \leq -\inf_{x \in \overline{A}} \mathcal{I}(x, S_0)\,,
\end{aligned}$$

where A° is the interior of A and $\overline{A}$ is the closure of A. See Dembo (1998); Pirjol and Zhu (2016) for details.

By solving the variational problem on the right-hand side of Eq.(8.1), the following property of the rate function $\mathcal{I}$ was proposed in Pirjol and Zhu (2016).

Proposition 8.1 (Pirjol and Zhu (2016)). *Given any $y > 0$, $x \mapsto \mathcal{I}(x,y)$ is a continuous map that is monotone decreasing in $(-\infty, y]$ and monotone increasing in $[y, \infty)$.*

8.2 Short-maturity asymptotic for the Asian call and put option delta value

Similarly to Pirjol and Zhu (2016), we use the large deviation theory to examine the short-maturity asymptotic for $\Delta_A^{\text{call}}(T)$ and $\Delta_E^{\text{put}}(T)$. See Appendix D.1 for the proof.

Theorem 8.2. *Under Assumptions 1 and 4, the following hold for the rate function $\mathcal{I}$ defined by Eq.*(8.1)*.*

(i) For an OTM Asian call option, i.e., $K > S_0$,

$$\lim_{T\to 0} T\log(\Delta_A^{\text{call}}(T)) = -\mathcal{I}(K, S_0)\,. \tag{8.2}$$

(ii) For an OTM Asian put option, i.e., $S_0 > K$,

$$\lim_{T\to 0} T\log(-\Delta_A^{\text{put}}(T)) = -\mathcal{I}(K, S_0)\,.$$

As a corollary of Theorem 8.2, we can approximate ITM Asian call and put option delta values.

Corollary 8.3. *Under Assumptions 1 and 4, the following asymptotic relations hold as $T \to 0$.*

(i) For an ITM Asian call option, i.e., $S_0 > K$,

$$\Delta_A^{\text{call}}(T) = 1 - \frac{1}{2}(r+q)T + \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)\,.$$

(ii) For an ITM Asian put option, i.e., $K > S_0$,

$$\Delta_A^{\text{put}}(T) = -1 + \frac{1}{2}(r+q)T - \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)\,.$$

Proof. From Lemma C.1 in Appendix C.1, we can get the put-call parity for the Asian option delta value:

$$\Delta_A^{\text{call}}(T) - \Delta_A^{\text{put}}(T) = \frac{e^{-rT}}{S_0}\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[S_t]\,dt = \frac{e^{-rT}}{S_0}\frac{1}{T}\int_0^T S_0 e^{(r-q)t}\,dt = \frac{e^{-qT}-e^{-rT}}{(r-q)T}\,.$$

From Theorem 8.2, the OTM Asian delta value vanishes at an exponential rate. Therefore, the Taylor expansion

$$\frac{e^{-qT}-e^{-rT}}{(r-q)T} = 1 - \frac{1}{2}(r+q)T + \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)$$

gives Corollary 8.3. □

Remark 8.2. *Theorem 8.2 and its Corollary 8.3 are obtained from direct use of the large deviation theory. This is different from the method used in Pirjol and Zhu (2018). More precisely, Pirjol and Zhu (2018) involved a sensitivity analysis of approximated option prices, not true option prices.*

Remark 8.3. *Observe that Corollary 8.3 extends the result in Example 4.1. The drift term determines the order greater than $\sqrt{T}$.*

9 Conclusion

This paper described the short-maturity asymptotic analysis of the Asian option having an arbitrary Hölder continuous payoff in the local volatility model. We were mainly interested in the Asian option price and the Asian option delta value. The short-maturity behaviors of the option price and the delta value were both expressed in terms of the *Asian volatility*, which was defined by

$$\sigma_A(T) = \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t, S_0)(T-t)^2\,dt}\,.$$

For sufficiently small $T > 0$, we proved that

$$\begin{aligned} P_A(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^\gamma)\,, \\ \Delta_A(T) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,, \end{aligned}$$

for a standard normal random variable Z and the Hölder exponent γ of the payoff function Φ. These estimations can be applied to hedge digital options.

These asymptotic results were based on the idea that an underlying process $(S_t)_{t\geq 0}$ under the local volatility model can be approximated by some suitable Gaussian processes in the $L^p(\mathbb{Q})$ norm. To implement this main idea in the approximation, we used Malliavin calculus theory to represent the delta of the Asian option. In addition, we used the large deviation principle to investigate an asymptotic for the Asian call and put option.

For comparison with the Asian option, we examined the short-maturity behavior of the European option. In contrast to the Asian volatility, we proved that at short maturity T, the European option is expressed by the *European volatility*. In terms of these volatilities, we observed the resemblance between Asian and European options at short maturity.

Acknowledgments.

Hyungbin Park was supported by the National Research Foundation of Korea (NRF) grants funded by the Ministry of Science and ICT (No. 2021R1C1C1011675 and No. 2022R1A5A6000840). Financial support from the Institute for Research in Finance and Economics of Seoul National University is gratefully acknowledged.

A Proofs of the results in Section 2

A.1 Proof for Lemma 2.1

Proof. First, we prove the second inequality of Eq.(2.5). It suffices to show this for $p \geq 2$ since once this is proven, for $0 < p < 2$, we can show by the Jensen inequality, Now, for $p \geq 2$, observe that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] &\le C_p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t |\sigma(s,S_0)\tilde{X}_s - \sigma(s,S_0)S_0|^2\, ds\right)^{\frac{p}{2}}\right] \\
&\le C_p \overline{\sigma}^p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - S_0|^p]\, ds
\end{aligned} \tag{A.1}
$$

for some constant $C_p > 0$. For these inequalities, we used the Burkholder–Davis–Gundy inequality, Assumption 1, and the Jensen inequality. Using the Jensen inequality and Theorem 3.4.3 of Zhang and Zhang (2017), it follows that for $t \le 1$,

$$
\begin{aligned}
t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - S_0|^p]\, ds &\le 2^{p-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds + 2^{p-1} t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\hat{X}_s - S_0|^p]\, ds \\
&\le 2^{p-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds + 2^{p-1}\overline{\sigma}^p {S_0}^p \tilde{C}_p \frac{1}{\frac{p}{2}+1} t^p
\end{aligned} \tag{A.2}
$$

for some constant $\tilde{C}_p > 0$. Hence, from Eqs.(A.1) and (A.2), we get

$$
\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \le f_p(t) + A_p \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds\,,
$$

where $f_p(t) := C_p \overline{\sigma}^p 2^{p-1} \overline{\sigma}^p {S_0}^p \tilde{C}_p \frac{1}{\frac{p}{2}+1} t^p$ and $A_p := C_p \overline{\sigma}^p 2^{p-1}$. Then, by the Gronwall inequality, for any $0 \le t \le 1$, we can find a constant B_p for $\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \le B_p t^p$.

For the first inequality of Eq.(2.5), we also present the proof for $p \ge 2$. By the Burkholder–Davis–Gundy inequality and the Jensen inequality, we get

$$
\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \le C_p t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,X_s)X_s - \sigma(s,S_0)\tilde{X}_s|^p]\, ds
$$

for some constant $C_p > 0$. Using the Jensen inequality,

$$
\begin{aligned}
t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,X_s)X_s - \sigma(s,S_0)\tilde{X}_s|^p]\, ds &\le 3^{p-1} t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,X_s)X_s - \sigma(s,\tilde{X}_s)\tilde{X}_s|^p]\, ds \\
&\quad + 3^{p-1} t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,\tilde{X}_s)\tilde{X}_s - \sigma(s,\hat{X}_s)\tilde{X}_s|^p]\, ds \\
&\quad + 3^{p-1} t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,\hat{X}_s)\tilde{X}_s - \sigma(s,S_0)\tilde{X}_s|^p]\, ds\,.
\end{aligned}
$$

First, observe under Assumption 1 that if $t \le 1$,

$$
t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,X_s)X_s - \sigma(s,\tilde{X}_s)\tilde{X}_s|^p]\, ds \le \alpha^p \int_0^t \mathbb{E}^{\mathbb{Q}}[|X_s - \tilde{X}_s|^p]\, ds\,.
$$

Second, by Assumption 1, the Hölder inequality and the second inequality of Eq. (2.5) with $t \leq 1$,

$$
\begin{aligned}
t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,\tilde{X}_s)\tilde{X}_s - \sigma(s,\hat{X}_s)\tilde{X}_s|^p]\,ds &\leq t^{\frac{p}{2}-1}\int_0^t (\mathbb{E}^{\mathbb{Q}}[|\sigma(s,\tilde{X}_s)-\sigma(s,\hat{X}_s)|^{2p}])^{\frac{1}{2}}(\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s|^{2p}])^{\frac{1}{2}}\,ds \\
&\leq t^{\frac{p}{2}-1}\alpha^p \int_0^t (\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s-\hat{X}_s|^{2p}])^{\frac{1}{2}} {S_0}^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds \\
&\leq \alpha^p (B_{2p})^{\frac{1}{2}} {S_0}^p t^{\frac{p}{2}-1}\int_0^t s^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds\,. \\
&\leq \alpha^p (B_{2p})^{\frac{1}{2}} {S_0}^p t^p \int_0^1 e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds\,. \qquad \text{(A.3)}
\end{aligned}
$$

Observe that Eq.(A.3) holds under $t \leq 1$ with $p \geq 2$. Third, we can easily see that

$$
\begin{aligned}
t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,\hat{X}_s)\tilde{X}_s - \sigma(s,S_0)\tilde{X}_s|^p]\,ds &\leq \alpha^p\overline{\sigma}^p {S_0}^{2p}(\tilde{C}_{2p})^{\frac{1}{2}} t^{\frac{p}{2}-1}\int_0^t s^{\frac{p}{2}} e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds \\
&\leq \alpha^p\overline{\sigma}^p {S_0}^{2p}(\tilde{C}_{2p})^{\frac{1}{2}} \frac{1}{\frac{p}{2}+1} t^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2}\vee 0}\,. \qquad \text{(A.4)}
\end{aligned}
$$

From $t \leq 1$, Eq.(A.4) follows. By combining the three inequalities and Gronwall inequality, for $t \leq 1$, we obtain $\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \leq B'_p t^p$.

The proof for the second inequality of Eq.(2.6) is nearly a repetition of the proof for the second inequality of Eq.(2.5); hence, it is omitted. Here, we examine only the first inequality of Eq.(2.6). Observe that

$$
\begin{aligned}
Y_t - \tilde{Y}_t = \int_0^t \nu(s,X_s)Y_s - \nu(s,X_s)\tilde{Y}_s\,ds &+ \int_0^t \nu(s,X_s)\tilde{Y}_s - \nu(s,\hat{X}_s)\tilde{Y}_s\,ds \\
&+ \int_0^t \nu(s,\hat{X}_s)\tilde{Y}_s - \nu(s,S_0)\tilde{Y}_s\,ds\,.
\end{aligned}
$$

The remainder of the proof is similar to the proof for Eq.(2.5). we can easily obtain the results. □

Remark A.1. *In the sequel, any argument using Jensen, and Hölder inequality such as the ones used in the proof of Lemma 2.1 will be referred to as 'a standard argument'.*

A.2 Proof for Lemma 2.2

Before we prove Lemma 2.2, we state and prove the generalized version of it. Later, we will show that the following lemma is actually a sufficient condition.

Lemma A.1. *Given a measure space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a Brownian motion $(W_t)_{t\geq 0}$, suppose that a process $(\theta_t)_{t\geq 0}$ is adapted to the Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$ and is uniformly bounded. More precisely, there is a constant $C > 0$ such that $|\theta_t(\omega)| \leq C$ for any $t \geq 0$ and $\omega \in \Omega$. Define a continuous martingale process $(M_t)_{t\geq 0}$ as*

$$
M_t := M_0 e^{-\frac{1}{2}\int_0^t \theta_s^2\,ds + \int_0^t \theta_s\,dW_s}, \quad M_0 > 0\,.
$$

Then, for any $\xi \in \mathbb{R}$, the following three statements hold.

(i) $\lim\limits_{T\to 0} \mathbb{E}^{\mathbb{Q}}[M_T^\xi] = M_0^\xi$.

(ii) $\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] < \infty$ *, for any* $T>0$ *. Furthermore,* $\lim\limits_{T\to 0} \mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] = M_0^\xi$.

(iii) $\lim\limits_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T M_t\,dt\right)^\xi\right] = M_0^\xi$.

Proof. Observe that

$$M_0^\xi e^{k(\xi)T} e^{-\frac{1}{2}\int_0^T \xi^2\theta_s^2\,ds + \int_0^T \xi\theta_s\,dW_s} \le M_T^\xi \le M_0^\xi e^{K(\xi)T} e^{-\frac{1}{2}\int_0^T \xi^2\theta_s^2\,ds + \int_0^T \xi\theta_s\,dW_s}\,, \tag{A.5}$$

where $k(\xi) := \frac{\xi(\xi-1)}{2} \wedge 0$, $K(\xi) := \frac{\xi(\xi-1)}{2} \vee 0$. By taking the expectation $\mathbb{E}^{\mathbb{Q}}$ on both sides, we obtain $M_0^\xi e^{k(\xi)T} \le \mathbb{E}^{\mathbb{Q}}[M_T^\xi] \le M_0^\xi e^{K(\xi)T}$. Take $T\to 0$ to get the limit $\lim\limits_{T\to 0}\mathbb{E}^{\mathbb{Q}}[M_T^\xi] = M_0^\xi$.

Suppose that $\xi > 0$. Choose p such that $p\ge 1$ and $\xi p > 1$. Use the Jensen inequality and the Doob L^p inequality to obtain

$$\left(\mathbb{E}^{\mathbb{Q}}\left[\Big(\max_{0\le t\le T} M_t\Big)^\xi\right]\right)^p \le \mathbb{E}^{\mathbb{Q}}\left[\Big(\max_{0\le t\le T} M_t\Big)^{\xi p}\right] \le \left(\frac{\xi p}{\xi p - 1}\right)^{\xi p} \mathbb{E}^{\mathbb{Q}}[M_T^{\xi p}]\,.$$

Since $\mathbb{E}^{\mathbb{Q}}[M_T^{\xi p}] \le M_0^{\xi p} e^{K(\xi p)T} < \infty$, we get that $\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] < \infty$ for any $T>0$. Note that $\max\limits_{0\le t\le T} M_t^\xi \searrow M_0^\xi$ almost surely as $T\to 0$. Thus, from the Lebesgue dominated convergence theorem, $\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] = M_0^\xi$. Next, suppose that $\xi < 0$. Observe that

$$\frac{1}{M_t} = \frac{1}{M_0} e^{\frac{1}{2}\int_0^t \theta_s^2\,ds - \int_0^t \theta_s\,dW_s} = \frac{1}{M_0} e^{\int_0^t \theta_s^2\,ds} e^{-\frac{1}{2}\int_0^t \theta_s^2\,ds - \int_0^t \theta_s\,dW_s} \le \frac{1}{M_0} e^{C^2 t} e^{-\frac{1}{2}\int_0^t (-\theta_s)^2\,ds + \int_0^t -\theta_s\,dW_s}\,.$$

Then $\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] < \infty$ for any $T>0$ and $\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} M_t^\xi\Big] = M_0^\xi$.

Finally, we prove that $\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T M_t\,dt\right)^\xi\right] = M_0^\xi$. This is straightforward from the Fatou lemma and a inequality $\left(\frac{1}{T}\int_0^T M_t\,dt\right)^\xi \le \max\limits_{0\le t\le T} M_t^\xi$ for $\xi\in\mathbb{R}$. □

Proof of Lemma 2.2. We now prove Lemma 2.2. Observe that the processes $(X_t)_{t\ge 0}$, $(\tilde{X}_t)_{t\ge 0}$, $(Y_t)_{t\ge 0}$, $(\tilde{Y}_t)_{t\ge 0}$ all satisfy the condition in Lemma A.1. Thus, from Lemma A.1 and the Hölder inequality we obtain $\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\Big] < \infty$ for any $T>0$. Moreover, it is followed by the upper bound $\limsup_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\Big] \le S_0^{p_1+p_2}$. The Fatou lemma yields the lower bound $S_0^{p_1+p_2} \le \liminf_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max\limits_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\Big]$. The inequality Eq.(2.7) can be proved similarly. □

B Proofs of the results in Section 3

B.1 Proof for Lemma 3.1

Proof. By Theorem 3.4.3 of Zhang and Zhang (2017) and the standard argument,

$$\mathbb{E}^{\mathbb{Q}}[|X_t - S_t|^2] \leq C_2\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T (r-q)X_u\,du\right)^2\right] \leq C_2 T\int_0^T |r-q|^2\,\mathbb{E}^{\mathbb{Q}}[|X_u|^2]\,du \leq \mathcal{O}(T^2)$$

for some constant $C_2 > 0$. Thus, from the inequalities and Lemma 2.2, we obtain the desired proof. $\square$

C Proofs of the results in Section 4

C.1 Proof for Lemma 4.1

To prove the Lemma 4.1, we need following Lemma.

Lemma C.1. *For the process S stated in* (2.1) *under Assumptions 1 and 3, we have*

$$\begin{aligned}
&\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T S_t\,dt\right)\frac{1}{T}\int_0^T Z_t\,dt\right],\\
&\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] = \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}[\Phi'(S_T)Z_T].
\end{aligned}$$

where Z_t is a unique solution of SDE

$$dZ_t = (r-q)Z_t\,dt + \nu(t,S_t)Z_t\,dW_t\,,\quad Z_0 = 1. \tag{C.1}$$

Here, the derivative Φ' is defined almost everywhere with respect to the Lebesgue measure.

Proof. By Theorem 3.4.3 of Zhang and Zhang (2017) and Assumptions 1, 3, for $p > 1$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{1}{h}\left(\Phi\left(\frac{1}{T}\int_0^T S_t^{S_0+h}\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T S_t^{S_0}\,dt\right)\right)\right|^p\right],$$

is bounded for $|h| \leq 1$. Thus From Theorems 6.21 and 6.25 of Klenke (2013), and Lemma 5.2.3 of Zhang and Zhang (2017), we obtain the results. $\square$

Proof of Lemma 4.1. Define a bounded process $\theta_t := \frac{r-q}{\sigma(t,S_t)}, t \geq 0$. Then, a process $M_t = e^{-\frac{1}{2}\int_0^t \theta_u^2\,du - \int_0^t \theta_u\,dW_u}$ is continuous martingale. By the Girsanov theorem, $dB_t := dW_t + \theta_t\,dt$, $0 \leq t \leq T$ is a Brownian motion under the measure $d\mathbb{P} := M_T\,d\mathbb{Q}$ on $\mathcal{F}_T^W$. Since $dS_t = \sigma(t,S_t)S_t\,dB_t$ under the measure $\mathbb{P}$, Lemma C.1 states that

$$\begin{aligned}
\Delta_A(T) &= e^{-rT}\mathbb{E}^{\mathbb{P}}\left[\Phi'\left(\frac{1}{T}\int_0^T S_t\,dt\right)\frac{1}{T}\int_0^T Z_t\,dt\,\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T X_t\,dt\right)\frac{1}{T}\int_0^T \tilde{Z}_t\,dt\,e^{-\frac{1}{2}\int_0^T \eta_t^2\,dt + \int_0^T \eta_t\,dW_t}\right],
\end{aligned}$$

where $\eta_t = \frac{r-q}{\sigma(t,X_t)}$ and $\tilde{Z}_t$ is the unique solution of SDE

$$d\tilde{Z}_t = \left((r-q) - \frac{(r-q)\nu(t,X_t)}{\sigma(t,X_t)}\right)\tilde{Z}_t\,dt + \nu(t,X_t)\tilde{Z}_t\,dW_t\,, \quad Z_0 = 1.$$

Therefore, by the inequality $|\Phi'| \le \beta$ from Assumption 3,

$$\begin{aligned}&\left|\Delta_A(T) - e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T X_t\,dt\right)\frac{1}{T}\int_0^T Y_t\,dt\right]\right| \\ &\le \beta e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \tilde{Z}_t\,dt\left|e^{-\frac{1}{2}\int_0^T \eta_t^2\,dt + \int_0^T \eta_t\,dW_t} - 1\right| + \frac{1}{T}\int_0^T \left|\tilde{Z}_t - Y_t\right|\,dt\right]\end{aligned}$$

Since by Theorem 3.4.3 of Zhang and Zhang (2017), $\mathbb{E}^{\mathbb{Q}}\left[\left|e^{-\frac{1}{2}\int_0^T \eta_t^2\,dt+\int_0^T \eta_t\,dW_t} - 1\right|^2 + \left|\tilde{Z}_t - X_t\right|^2\right] = \mathcal{O}(T)$, the proof for the Asian delta value is complete. The proof for the European delta value can be obtained similarly. □

C.2 Proof for Lemma 4.2

Proof. We prove the first inequality of Eq.(4.4). It is sufficient to show this for $p \ge 1$. By Assumption 1, observe that,

$$|u_t - \tilde{u}_t|^p = \left|\frac{2(Y_t + \tilde{Y}_t)(Y_t - \tilde{Y}_t)}{\sigma(t,X_t)X_t} + \frac{2\tilde{Y}_t^2(\sigma(t,\tilde{X}_t)\tilde{X}_t - \sigma(t,X_t)X_t)}{\sigma(t,X_t)X_t\sigma(t,\tilde{X}_t)\tilde{X}_t}\right|^p.$$

From Lemmas 2.1, 2.2 and the standard argument, we get

$$\mathbb{E}^{\mathbb{Q}}[|u_t - \tilde{u}_t|^p] \le D_p t^p\,,$$

The second inequality of Eq.(4.4) can be obtained similarly. We may assume that $p \ge 1$. Observe from the Jensen inequality,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}[|\tilde{u}_t - \hat{u}_t|^p] \\ &\le 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}\right|^p \mathbb{1}_{\{\hat{X}_t < \frac{S_0}{2}\}}\right] + 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t} - \frac{2\hat{Y}_t^2}{\sigma(t,\hat{X}_t)\hat{X}_t}\right|^p \mathbb{1}_{\{\hat{X}_t \ge \frac{S_0}{2}\}}\right].\end{aligned}$$

Following the argument used to prove the first inequality of Eq. (4.4), there exists some positive constant D_p such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t} - \frac{2\hat{Y}_t^2}{\sigma(t,\hat{X}_t)\hat{X}_t}\right|^p \mathbb{1}_{\{\hat{X}_t \ge \frac{S_0}{2}\}}\right] \le D_p t^p$$

for small $0 \le t \le 1$. Now, note from Assumption 1 and the Hölder inequality that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}\right|^p \mathbb{1}_{\{\hat{X}_t < \frac{S_0}{2}\}}\right] \le \frac{2^p}{\underline{\sigma}^p}\left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le s\le 1}\left(\tilde{Y}_s^{4p}\tilde{X}_s^{-2p}\right)\right]\right)^{\frac{1}{2}}\left(\mathbb{Q}\left\{\hat{X}_t < \frac{S_0}{2}\right\}\right)^{\frac{1}{2}}.$$

Let Z be the standard normal variable with respect to the measure $\mathbb{Q}$ and $N(\cdot)$ be a cumulative function of Z. Since $\hat{X}_t$ is a normal random variable, it is implied from Assumption 1 that the following inequality holds for any $t > 0$.

$$\mathbb{Q}\left\{\hat{X}_t < \frac{S_0}{2}\right\} = N\left(-\frac{1}{2\sqrt{\int_0^t \sigma(u,S_0)^2\,du}}\right) \le N\left(-\frac{1}{2\overline{\sigma}\sqrt{t}}\right).$$

However, it is easy to check from the definition of $N(\cdot)$ that $N\left(-\frac{1}{2\overline{\sigma}\sqrt{t}}\right) = o(t^q)$ for any $q > 0$ as $t \to 0$. Thus, from Lemma 2.2, we can completes the proof of Eq.(4.4).

The proof for first inequality of Eq.(4.5) can be proven by Lemmas 2.1, 2.2 and the standard argument.

For the second inequality of Eq.(4.5), we may also assume that $p \ge 1$. Note from the Jensen inequality that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|T\tilde{F} - T\hat{F}|^p] \le\ & 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p} \mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt < \frac{1}{2}\}}\right] \\ &+ 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |\tilde{Y}_t - \hat{Y}_t|\,dt\right)^{p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^{-p} \mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \ge \frac{1}{2}\}}\right].\end{aligned}$$

Observe Lemma 2.1, Lemma 2.2 and the standard argument, for $0 \le t \le T$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |\tilde{Y}_t - \hat{Y}_t|\,dt\right)^{p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^{-p} \mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \ge \frac{1}{2}\}}\right] \le D_p T^p.$$

Next, observe from the Fubini theorem with regard to a stochastic integral and Assumption 1 that

$$\begin{aligned}\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt < \frac{1}{2}\right\} &= \mathbb{Q}\left\{1 + \frac{1}{T}\int_0^T\int_0^T \nu(s,S_0)\mathbb{1}_{\{s\le t\}}\,dW_s\,dt < \frac{1}{2}\right\} \\ &= \mathbb{Q}\left\{\frac{1}{T}\int_0^T \nu(s,S_0)(T-s)\,dW_s < -\frac{1}{2}\right\} \\ &= \mathbb{Q}\left\{\frac{1}{T}\sqrt{\int_0^T \nu^2(s,S_0)(T-s)^2\,ds}\,Z < -\frac{1}{2}\right\} \\ &\le N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right),\end{aligned} \tag{C.2}$$

where Z denotes a standard normal random variable and $N(\cdot)$ denotes a cumulative function of Z. Since $\left(N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right)\right)^{\frac{1}{2}} = o(T^q)$ for any $q > 0$ as $T \to 0$, the second inequality of (4.5) is proven. $\square$

C.3 Proof for Lemma 4.3

Proof. First, we present the proof for Eq.(4.6). In Benhamou (2000), D_sX_t is explicitly expressed as

$$D_sX_t = \frac{Y_t}{Y_s}\sigma(s, X_s)X_s\mathbb{1}_{\{s\le t\}}$$

under Assumption 1. Therefore, for any $p > 0$ and $0 \le t \le 1$, by Assumption 1 and Lemma 2.2, and the standard argument, the proof of Eq.(4.6) can be complete.

Next, let us prove Eq.(4.7). The only nontrivial inequality of Eq.(4.7) is the first one. From Proposition 1.5.1 of Nualart (1995), $D_s\tilde{Y}_t$ and $D_s\hat{Y}_t$ can be computed as

$$D_s\tilde{Y}_t = \nu(s, S_0)\tilde{Y}_t\mathbb{1}_{\{s\le t\}}\,, \quad D_s\hat{Y}_t = \nu(s, S_0)\mathbb{1}_{\{s\le t\}}\,. \tag{C.3}$$

Therefore, from Assumption 1, Lemma 2.2 and the standard argument, it is easy to check that both $\sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|D_s\tilde{Y}_t|^p]$ and $\sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|D_s\hat{Y}_t|^p]$ are bounded by some constant $E_p > 0$ in $0 \le t \le 1$. To prove the first inequality of Eq.(4.7), use Malliavin calculus theory presented in Proposition 1.5.1 of Nualart (1995) to express D_sY_t by

$$D_sY_t = Y_t\left[\nu(s, X_s) - \int_0^t \nu(u, X_u)\rho(u, X_u)D_sX_u\,du + \int_0^t \rho(u, X_u)D_sX_u\,dW_u\right]\mathbb{1}_{\{s\le t\}}\,. \tag{C.4}$$

Now using the inequalities $|\nu| \le \alpha$, $|\rho| \le \alpha$, observe that for $p \ge 1$, $s \ge 0$, by the Jensen inequality,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|D_sY_t|^p] &\le 3^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}[Y_t^p] + 3^{p-1}\alpha^{2p}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t |D_sX_u|^p\,du\right]t^{p-1}\\ &\quad + 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t \rho(u, X_u)D_sX_u\,dW_u\right|^p\right].\end{aligned}$$

By Lemma 2.2, the standard argument and Eq.(4.6), that

$$\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t |D_sX_u|^p\,du\right]t^{p-1} \le \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}}(E_{2p})^{\frac{1}{2}}t^p \le C \tag{C.5}$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t \rho(u, X_u)D_sX_u\,dW_u\right|^p\right] \le \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}}(C_p\alpha^{2p}E_{2p}t^p)^{\frac{1}{2}} \le C \tag{C.6}$$

for $0 \le t \le 1$ and $s \ge 0$. Hence, we complete the proof of Eq.(4.7).

Finally, we will examine the proof of Eq.(4.8). Note from Proposition 1.5.1 of Nualart (1995) that TD_sF can be expressed as $TD_sF = -\frac{1}{T}\int_0^T D_sY_t\,dt\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^{-2}$. Assume that $p \ge 1$. From Lemma 2.2, the standard argument and Eq.(4.7), Eq.(4.8) can be established. □

C.4 Proof of Lemma 4.4

Proof. We will prove the first inequality of Eq.(4.9). Assume that $p \geq 1$. From Eqs.(C.3) and (C.4), we derive the following inequality under Assumption 1:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|D_sY_t - D_s\tilde{Y}_t|^p] \\
&\leq 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|Y_t\nu(s,X_s) - \tilde{Y}_t\nu(s,S_0)\right|^p\right]\mathbb{1}_{\{s\leq t\}} + 3^{p-1}\alpha^{2p}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t |D_sX_u|^p\,du\right]t^{p-1} \\
&+ 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t \rho(u,X_u)D_sX_u\,dW_u\right|^p\right].
\end{aligned}$$

The inequalities Eqs.(C.5) and (C.6) imply $\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t |D_sX_u|^p\,du\right]t^{p-1} + \mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t \rho(u,X_u)D_sX_u\,dW_u\right|^p\right] \leq F_pt^{\frac{p}{2}}$, in $0 \leq t \leq 1$ for some positive constant F_p. Now, observe from

$$\left|Y_t\nu(s,X_s) - \tilde{Y}_t\nu(s,S_0)\right|\mathbb{1}_{\{s\leq t\}} \leq Y_t\left|\nu(s,X_s) - \nu(s,S_0)\right|\mathbb{1}_{\{s\leq t\}} + |\nu(s,S_0)|\left|Y_t - \tilde{Y}_t\right|.$$

From Lemmas 2.1, 2.2, Theorem 3.4.3 of Zhang and Zhang (2017) and the standard argument, we can prove the first inequality of Eq.(4.9).

Next, we prove the second inequality of Eq.(4.9). This easily comes from Eq.(C.3) and straightforward use of Theorem 3.4.3 of Zhang and Zhang (2017).

Finally, we will prove Eq.(4.10). From Proposition 1.5.1 of Nualart (1995), Malliavin calculus gives

$$TD_sF = \frac{-\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^2}, \quad TD_s\tilde{F} = \frac{-\frac{1}{T}\int_0^T D_s\tilde{Y}_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2}.$$

If $p \geq 1$, by the Jensen inequality, $\mathbb{E}^{\mathbb{Q}}[|TD_sF - TD_s\tilde{F}|^p] \leq 2^{p-1}\mathbb{E}^{\mathbb{Q}}[L_T^p] + 2^{p-1}\mathbb{E}^{\mathbb{Q}}[R_T^p]$, where L_T , R_T are given by

$$L_T := \left|\frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^2} - \frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2}\right|, \quad R_T := \left|\frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2} - \frac{\frac{1}{T}\int_0^T D_s\tilde{Y}_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2}\right|.$$

Then, from Lemmas 2.1, 2.2, 4.3 and the standard argument, we can completes the proof.

The second inequality of Eq.(4.10) can be obtained similarly except that we should additionally control the indicator function of $\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \geq \frac{1}{2}\}$ shown in the definition of $D_s^*\hat{F}$. However, we can resolve this subtle difference with the same technique as that used to prove Lemma 4.2. □

C.5 Proof of Lemma 4.5

Proof. Observe that the Skorokhod integral $\delta(\hat{u})$ coincides with the Itô integral of $\hat{u}_s$ in $s \in [0,T]$. By from Doob L^p inequality, Burkholder-Davis-Gundy inequality and the standard argument, we get

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{u})|^p] \leq C_p\frac{4^p}{\underline{\sigma}^pS_0^p}\left(\frac{2p}{2p-1}\right)^{2p}G_pT^{\frac{p}{2}}.$$

for any $0 \leq T \leq 1$.

Next, we will prove the second inequality of Eq.(4.12). For simplicity, we use the notation $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$ for the remainder of the proof. Observe that the Skorokhod integral $\delta(g)$ coincides with the Itô integral of g_s in $s \in [0,T]$ and g_s can be written as

$$g_s = \left(\hat{u}_s - \frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0} \mathbb{1}_{\{\hat{X}_s \geq \frac{S_0}{2}\}} \right) - \frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0} \mathbb{1}_{\{\hat{X}_s \leq \frac{S_0}{2}\}} + \left(\frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0} - \frac{2}{\sigma(s,S_0)S_0} \right).$$

Then since $\hat{Y}_s$, $\hat{X}_s$ are normal variables and $\mathbb{Q}\{\hat{X}_s < \frac{S_0}{2}\} = o(s^q)$ for any $q > 0$ as $s \to 0$, from the Theorem 3.4.3 of Zhang and Zhang (2017) and the standard argument,

$$\mathbb{E}^{\mathbb{Q}}\left[|g_s|^p\right] \leq G_p s^{\frac{p}{2}}. \tag{C.7}$$

Then we can complete the proof for Lemma 4.5. □

C.6 Proof for Proposition 4.4

Proof. First, we will prove Eq.(4.13). The proof comprises five claims: Claims C.6.1–C.6.5. Throughout the proof, we will use the notation $\Delta_A^*(T)$, which is defined as

$$\Delta_A^*(T) := \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\, dt \right) \frac{1}{T}\delta(\hat{u}) \right].$$

Claim C.6.1.

$$\Delta_A^*(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}} Z \right] + \mathcal{O}(\sqrt{T}).$$

Proof. From $\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\right] = 0$, $\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right]$, Lemma 4.5, Theorem 3.4.3 of Zhang and Zhang (2017) and the standard argument, we get

$$\begin{aligned}
&\left| \Delta_A^*(T) - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\, dt \right) \frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0} \right) \right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi(S_0)\frac{1}{T}\delta(g) \right] \right| \\
&\leq \mathbb{E}^{\mathbb{Q}}\left[\left| \Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\, dt \right) - \Phi(S_0) \right| \frac{1}{T}|\delta(g)| \right] \leq C\sqrt{T},
\end{aligned}$$

where $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$. Next, from the Fubini theorem with regard to a stochastic integral,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\, dt \right) \frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0} \right) \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + \frac{S_0}{T}\int_0^T \sigma(s,S_0)(T-s)\, dW_s \right) \frac{1}{T}\int_0^T \frac{2}{\sigma(s,S_0)S_0}\, dW_s \right].
\end{aligned}$$

We can easily see that

$$\int_0^T \sigma(s,S_0)(T-s)\, dW_s \perp \int_0^T \frac{2}{\sigma(s,S_0)S_0}\, dW_s - \frac{T^2/S_0}{\int_0^T \sigma^2(s,S_0)(T-s)^2\, ds}\int_0^T \sigma(s,S_0)(T-s)\, dW_s.$$

Therefore, by direct calculation, we can get the desired result. □

Claim C.6.2.

$$\Delta_A^*(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\geq\frac{1}{2}\}}\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. We may assume that $\Phi(0) = 0$. Otherwise, consider a translation $\Phi(\cdot) - \Phi(0)$ and follow the arguments below to get the result. From Lemma 4.5, Eq.(C.2), $N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right) = o(T^q)$ for any $q > 0$ as $T \to 0$. and the standard argument, we can get the desired result. □

Claim C.6.3.

$$\Delta_A^*(T) - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] = \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. Apply Claim C.6.2 to this Claim. Then,

$$\Delta_A^*(T) - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T}).$$

Thus, it suffices to show that

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T})\,.$$

To show this, observe from the inequality $T\hat{F} \leq 2$,, Lemma 4.5, Theorem 3.4.3 of Zhang and Zhang (2017) and the standard argument that

$$\begin{aligned}
&\left|E_T - \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]\right| \\
&\leq \frac{2}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right) - \Phi(S_0)\right|\frac{1}{\sqrt{T}}|\delta(\hat{u})|\frac{1}{T}\int_0^T |\hat{Y}_t - 1|\,dt\right] \leq C\sqrt{T}
\end{aligned}$$

where C is postivie constant. we can prove the claim. □

Claim C.6.4.

$$\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. proof easily comes from Lemma 4.5 and the standard argument. □

Claim C.6.5.

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\,.$$

Proof. Observe from Lemma 4.5, and the standard argument that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T})\,.$$

Note from the Itô isometry that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \frac{2}{\sigma(s,S_0)S_0}\,dW_s\right)\left(\frac{1}{T}\int_0^T \nu(s,S_0)(T-s)\,dW_s\right)\right] \\
&= \frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds\,.
\end{aligned}$$

This completes the proof. □

Concatenate the inequalities established through Claims C.6.1–C.6.5 to obtain Eq.(4.13).

Now, we will prove Eq.(4.14). We divide the proof into four claims.

Claim C.6.6.

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] = \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. Observe from Lemma 4.3, definition of $\hat{u}_s$ and the standard argument that

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] - \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right]\right| \\
&\leq \beta\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^2]\,dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[\hat{u}_s^2(TD_s^*\hat{F})^2]\,ds\right)^{\frac{1}{2}} \leq C\sqrt{T}
\end{aligned}$$

where C is a positive constant. Thus, we achieve Claim C.6.6. □

Claim C.6.7.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. From Lemma 4.3, definition of $\hat{u}_s$ and the standard argument,

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]\right| \\
&\leq \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^2|TD_s^*\hat{F}|^2]\,ds\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2 - 1\right)^2\right]\right)^{\frac{1}{2}} \leq C\sqrt{T}
\end{aligned}$$

where C is the positive constant. Then this complete the proof. □

Claim C.6.8.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof. From the inequality E.q (C.7), Lemma 4.3 and the standard argument,

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]\right|$$
$$\le \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[g_s^2(TD_s^*\hat{F})^2]\,ds\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[\hat{Y}_t^4]\,dt\right)^{\frac{1}{2}} \le C\sqrt{T},$$

where $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$. and C is the positive constant. This proves the claim. ☐

Claim C.6.9.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] = -\frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\,.$$

Proof. From the definition of $D_s^*\hat{F}$ and the computation of $D_s\hat{Y}_t$ in Eq.(C.3), we get

$$D_s^*\hat{F} = -\frac{\int_0^T D_s\hat{Y}_t\,dt}{\left(\int_0^T \hat{Y}_t\,dt\right)^2}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\ge\frac{1}{2}\}} = -\frac{1}{T^2}\frac{\nu(s,S_0)(T-s)}{\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\ge\frac{1}{2}\}}\,.$$

Using this identity, and $\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt < \frac{1}{2}\right\} = o(T^q)$ for any $q > 0$ as $T \to 0$. we can prove the claim. ☐

Combining Claims C.6.6–C.6.9, we finally get Eq.(4.14). Here, we complete the proof of Proposition 4.4. ☐

C.7 Proof for Proposition 4.7

Proof. Observe from First, we will study Eq.(4.19). Observe from $\mathbb{E}^{\mathbb{Q}}[|\hat{X}_T - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$, $\mathbb{E}^{\mathbb{Q}}[\hat{G}^p] = \mathcal{O}(1)$ for any $p > 0$, $\mathbb{E}^{\mathbb{Q}}[|\hat{G} - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$, $\mathbb{E}^{\mathbb{Q}}\left[\left|\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right|^p\right] = \mathcal{O}(T^{\frac{p}{2}})$ and Lemma 4.7 that

$$\frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right] - \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_0)\delta(\hat{h})\hat{G}\right]$$
$$= \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\left(\Phi(\hat{X}_T) - \Phi(S_0)\right)\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)\hat{G}\right] + \mathcal{O}(\sqrt{T})$$
$$= \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\left(\Phi(\hat{X}_T) - \Phi(S_0)\right)\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)S_0\right] + \mathcal{O}(\sqrt{T})$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})\,.$$

The same type of manipulation technique has already been used to prove Claim C.6.1 in Appendix C.6. Similarly, from $\mathbb{E}^{\mathbb{Q}}[\delta(\hat{h})] = 0$, $\mathbb{E}^{\mathbb{Q}}[|\hat{G} - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$, $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T - 1|^p] = \mathcal{O}(T^{\frac{p}{2}})$ $\mathbb{Q}\{\hat{Y}_T < \frac{1}{2}\} = o(T^p)$ for any $p > 0$, the Itô isometry and Lemma 4.7

$$\begin{aligned}
\frac{\Phi(S_0)}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{h})\hat{G}\right] &= \frac{\Phi(S_0)}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{h})(\hat{G} - S_0)\right] \\
&= \frac{\Phi(S_0)}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot, S_0)S_0}\right)(\hat{G} - S_0)\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{\Phi(S_0)}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot, S_0)S_0}\right)(\hat{G} - S_0)\hat{Y}_T\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{\Phi(S_0)}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot, S_0)S_0}\right)(\hat{X}_T - S_0\hat{Y}_T)\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{\Phi(S_0)}{S_0 T}\int_0^T \frac{\sigma(s, S_0) - \nu(s, S_0)}{\sigma(s, S_0)}\, ds + \mathcal{O}(\sqrt{T})\,.
\end{aligned}$$

From these arguments, we can prove Eq.(4.19).

Now, we analyze Eq.(4.20) from a series of asymptotic relations below. Observe from $\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}[|\hat{H}_s|^p] = \mathcal{O}(1)$, $\mathbb{E}^{\mathbb{Q}}\left[\left|\hat{h}_s - \frac{1}{\sigma(s,S_0)S_0}\right|^p\right] = \mathcal{O}(s^{\frac{p}{2}})$, as $T \to 0$. that

$$\begin{aligned}
\frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s\hat{H}_s\, ds\right] &= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{1}{\sigma(s, S_0)S_0}\hat{H}_s\, ds\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{1}{\sigma(s, S_0)S_0}\hat{H}_s\, ds\, \hat{Y}_T^2\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{\nu(s, S_0)}{\sigma(s, S_0)S_0}\, ds\, \hat{X}_T\right] + \mathcal{O}(\sqrt{T}) \\
&= \frac{\Phi(S_0)}{T}\int_0^T \frac{\nu(s, S_0)}{\sigma(s, S_0)}\, ds + \mathcal{O}(\sqrt{T})\,.
\end{aligned}$$

The proof for Eq.(4.20) is thus complete. □

D Proofs of the results in Section 8

D.1 Proof for Theorem 8.2

Proof. First, we will show Eq.(8.2). Let Z_t be the unique solution of E.q (C.1). Using Lemma C.1, and the standard argument,

$$\begin{aligned}
\Delta_A^{\text{call}}(T) &= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T Z_t\, dt \mathbb{1}_{\{\frac{1}{T}\int_0^T S_t\, dt \geq K\}}\right] \\
&\leq \frac{e^{-rT}}{S_0}\left(\frac{1}{T}\int_0^T S_0^p e^{p(r-q)t} e^{K(p)\overline{\sigma}^2 t}\, dt\right)^{\frac{1}{p}}\left(\mathbb{Q}\left\{\frac{1}{T}\int_0^T S_t\, dt \geq K\right\}\right)^{\frac{1}{p'}}.
\end{aligned}$$

For the last inequality, we use Eq.(A.5) with $K(p) := \frac{p(p-1)}{2} \vee 0$. By taking $T \to 0$, we get the following inequality from Remark 8.1,

$$\limsup_{T\to 0} T \log \Delta_A^{\text{call}}(T) \leq \frac{-\mathcal{I}(K, S_0)}{p'} .$$

Take $p' \to 1$ to get an upper bound. Next, a lower bound for Eq.(8.2) is obtained from Lemma A.1, the reverse Hölder inequality , $Z_t > 0$ and Remark 8.1. Thus we get the desired results. □

References

Leif Andersen. Option pricing with quadratic volatility: a revisit. *Finance and Stochastics*, 15 (2):191–219, 2011.

Eric Benhamou. An application of Malliavin calculus to continuous time Asian options greeks. Technical report, London School of Economics, 2000.

Phelim Boyle and Alexander Potapchik. Prices and sensitivities of Asian options: A survey. *Insurance Mathematics and Economics*, 42(1):189–211, 2008.

Mark Broadie, Paul Glasserman, and S.G. Kou. Connecting discrete and continuous path-dependent options. *Finance and Stochastics*, 3(1):55–82, 1999.

Amir Dembo. *Large Deviations Techniques and Applications*. Springer, New York, 2nd edition, 1998.

Michael C Fu, Dilip B Madan, and Tong Wang. Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance*, 2(2):49–74, 1999.

Jim Gatheral, Elton P Hsu, Peter Laurence, Cheng Ouyang, and Tai-Ho Wang. Asymptotics of implied volatility in local volatility models. *Mathematical Finance*, 22(4):591–620, 2012.

Hélyette Geman and Marc Yor. Bessel processes, Asian options and perpetuities. *Mathematical Finance*, 3(4):349–375, 1993.

Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

Vadim Linetsky. Spectral expansions for Asian (Average price) options. *Operations Research*, 52(6):856–867, 2004.

David Nualart. *The Malliavin Calculus and Related Topics*. Springer series in statistics. Probability and its applications. Springer-Verlag, New York, 1995.

Dan Pirjol and Lingjiong Zhu. Short maturity Asian options in local volatility models. *SIAM Journal on Financial Mathematics*, 7(1):947–992, 2016.

Dan Pirjol and Lingjiong Zhu. Sensitivities of Asian options in the Black–Scholes model. *International Journal of Theoretical and Applied Finance*, 21(1), 2018.

Dan Pirjol and Lingjiong Zhu. Short maturity Asian options for the CEV model. *Probability in the Engineering and Informational Sciences*, 33(2):258–290, 2019.

Dan Pirjol, Jing Wang, and Lingjiong Zhu. Short maturity forward start Asian options in local volatility models. *Applied Mathematical Finance*, 26(3):187–221, 2019.

Ekkehard W. Sachs and Marina Schneider. Reduced-order models for the implied variance under local volatility. *International Journal of Theoretical and Applied Finance*, 17(8):1450053, 2014.

Jan Vecer. Unified Asian pricing. *Risk*, 15(6):113–116, 2002.

Paul Wilmott. *Paul Wilmott on Quantitative Finance*. John Wiley Sons Inc., Hoboken, NJ, 2nd edition, 2006.

Jianfeng Zhang and Jianfeng Zhang. *Backward stochastic differential equations*. Springer, 2017.