A CONSTRUCTION OF SKT MANIFOLDS USING TORIC GEOMETRY

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ABSTRACT. We produce infinite families of SKT manifolds by using methods of toric geometry like the J-construction. These SKT manifolds are total spaces of certain principal G-bundles over smooth projective toric varieties, where G is an even dimensional compact connected Lie group.

1. Introduction

An SKT structure on a complex manifold is a generalization of the more familiar notion of Kähler structure in the following precise sense. Consider a Hermitian manifold E with Riemannian metric g and integrable complex structure $\mathcal{J}$. Consider a tangential connection ∇ on E which is compatible with the Hermitian structure and define the associated torsion 3-tensor μ by

$$\mu(A, B, C) = g(T(A, B), C)\,,$$

where T denotes the torsion 2-form of ∇, and A, B, and C are arbitrary vector fields on E.

Then M admits a unique Hermitian connection ∇ such that the torsion tensor μ is totally skew symmetric. This connection is known as the KT (Kähler with torsion) or Bismut connection. Bismut used it to prove a local index formula for the Dolbeault operator when the complex manifold is not Kähler [2].

For the KT connection, we actually have

$$\mu(A, B, C) = dF(\mathcal{J}A, \mathcal{J}B, \mathcal{J}C)\,,$$

where F denotes the fundamental 2-form defined by $F(A, B) = g(\mathcal{J}A, B)$. Note that M is Kähler if $dF = 0$, or equivalently, if $\mu = 0$. A KT connection is called strong KT or SKT if $d\mu = 0$ or equivalently, if $dd^cF = 0$ (or $\partial\overline{\partial}F = 0$). Thus, we may say that E admits an SKT structure if it admits a Hermitian metric whose fundamental form is dd^c-closed or, equivalently, $\partial\overline{\partial}$-closed. Such a metric is referred to as an SKT metric.

SKT structures arise naturally in two dimensional sigma models with $(4, 0)$ supersymmetry in physics [16]. They are also closely related to generalized Kähler geometry (cf. [15, 14, 8]). A generalized Kähler structure is a pair of SKT structures with a certain compatibility condition.

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Various constructions and properties of SKT manifolds have been studied extensively (cf. [11, 7, 9, 19]). In particular, Grantcharov et. al. [12] have given a construction of SKT structures on the total space of torus principal bundles of even rank, say $2k$, over a complex Kähler manifold under the condition

$$\sum_{j=1}^{2k} w_j^2 = 0 \tag{1.1}$$

where w_j's are certain characteristic classes (see Section 5). When the base manifold of the principal bundle is of dimension 4, then the sufficient condition (1.1) may be dealt with using the intersection form on the middle cohomology of the base. In [12] some examples of SKT manifolds of dimension six are given using this principle. However, the condition is not easy to verify in general. In fact, nontrivial torus principal bundles that satisfy (1.1) seem to be relatively rare.

In these notes, we give infinite families of examples of nontrivial torus bundles over manifolds of arbitrary dimension that satisfy the condition (1.1). Our main tools are from toric geometry. We observe that (1.1) holds for certain torus bundles over every Bott manifold. The main point is that these manifolds admit nontrivial line bundles whose first Chern class has square zero. Moreover, some of these examples may be used to produce an infinite family of examples by applying the J-construction in toric geometry. Finally, using some results of [22], we produce more SKT manifolds from each of the above examples by extension of structure group.

2. Smooth toric varieties

The theory of toric varieties and their topological counterparts are widely studied and there are many good references like [10, 20, 5] etc. for them. The main purpose of this section is to introduce some definitions and notations.

A complex toric variety X of complex dimension n is a (partial) compactification of the algebraic torus $T_{\mathbb{C}}^n = (\mathbb{C}^*)^n$, which admits a natural extension of the translation action of $T_{\mathbb{C}}^n$ on itself. We will only deal with toric varieties that are smooth (nonsingular) and compact (complete). A projective toric variety is, of course, compact. For a smooth toric variety, the action of $T_{\mathbb{C}}^n$ in a neighborhood of any fixed point is the same, up to an automorphism, as its standard action on $(\mathbb{C}^*)^n$.

A key role is played in toric geometry by the $T_{\mathbb{C}}^n$-invariant subvarieties of X of complex codimension one, known as the torus invariant divisors. There are finitely many of them and let these be $D_1, \ldots, D_m$. Under natural choice of orientations induced by the complex structure, the stabilizer of such a divisor D_i can be identified with a vector λ_i in the co-character lattice $N \cong \mathbb{Z}^n$ of complex one-parameter subgroups of $T_{\mathbb{C}}^n$. Topologists refer to λ_i as a (directed) characteristic vector. One can form an $n \times m$ matrix

$$\Lambda = [\lambda_1, \ldots, \lambda_m]$$

with the characteristic vectors of the invariant divisors, in some fixed order, as columns. Such a matrix will be called a characteristic matrix.

We denote the coordinates of the characteristic vector λ_i by λ_{ji}, $1 \leq j \leq n$. Let w_i denote the first Chern class of the complex line bundle L_i that corresponds to D_i under the divisor-line bundle correspondence. Then the classes $w_1, \ldots, w_m$ generate the integral cohomology ring of the smooth toric variety X. The linear relations among the w_j's are encoded in the rows of the corresponding characteristic matrix.

Each characteristic vector λ_i generates a ray in $N \otimes \mathbb{R}$, which we will still denote by λ_i for notational simplicity. These give rise to a combinatorial gadget called the fan, denoted by Σ, of the toric variety. The fan is a collection of cones. The zero vector forms the unique 0-dimensional cone in the fan. Rays corresponding to characteristic vectors constitute the 1-dimensional cones of the fan. More generally a collection $\lambda_{i_1}, \ldots, \lambda_{i_k}$ of characteristic vectors generate a k-dimensional cone in the fan if the corresponding torus invariant divisors have nonempty intersection. We denote such a cone by $\langle \lambda_{i_1}, \ldots, \lambda_{i_k} \rangle$. The fan completely determines the geometry of the toric variety. For instance, the singular cohomology ring of a smooth compact toric variety X with integer coefficients (cf. [20, pp. 134]) is given by

$$H^*(X) = \mathbb{Z}[w_1, \ldots, w_m]/\mathcal{I} \tag{2.1}$$

where the ideal $\mathcal{I}$ is generated by

(1) all products $w_{i_1} \cdots w_{i_k}$ such that $\lambda_{i_1}, \ldots, \lambda_{i_k}$ do not form a cone of Σ, and
(2) all linear combinations $\sum_{i=1}^{m} \lambda_{ji} w_i$ where $1 \leq j \leq n$.

3. Bott manifolds

A Bott tower (cf. [3, 13]) of height t,

$$M_t \to M_{t-1} \to \ldots \to M_2 \to M_1 \to M_0 = \{\text{point}\}\,,$$

is an iterated projective bundle such that each M_k is a $\mathbb{P}^1$-bundle over M_{k-1}. More precisely, $M_k = \mathbb{P}(\mathcal{O}_{M_{k-1}} \oplus \mathcal{L}_{k-1})$ where $\mathcal{L}_{k-1}$ is a line bundle over M_{k-1}. Each M_k is a smooth projective toric variety of complex dimension k and is known as a Bott manifold. Note that M_1 is just $\mathbb{P}^1$ and M_2 is a Hirzebruch surface.

We now describe the fan of M_k for $k \geq 1$. Let $e_1, \ldots, e_k$ denote the standard basis of $\mathbb{Z}^k$. We identify $e_1, \ldots, e_{k-1}$ with the standard basis of $\mathbb{Z}^{k-1}$ without confusion. The manifold M_k in a Bott tower has $2k$ characteristic vectors which we denote by $\lambda_1^{(k)}, \ldots, \lambda_{2k}^{(k)}$. These may be defined inductively as follows: Start with $\lambda_1^{(1)} = e_1$ and $\lambda_2^{(1)} = -e_1$. Then define

$$\lambda_i^{(k)} = \lambda_i^{(k-1)} \quad \text{and} \quad \lambda_{k+i}^{(k)} = \lambda_{k-1+i}^{(k-1)} + c_{i,k} e_k \quad \text{for} \quad i = 1, \ldots, k-1\,.$$

Moreover, define

$$\lambda_k^{(k)} = e_k \quad \text{and} \quad \lambda_{2k}^{(k)} = -e_k\,.$$

Here each $c_{i,k}$ is an integer. It is to be noted that different values of these constants may produce different manifolds M_k.

With respect to the standard basis, the characteristic matrix of M_k has the following form.

$$\Lambda(M_k) = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & c_{1,2} & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & c_{1,k} & c_{2,k} & \dots & -1 \end{bmatrix} \tag{3.1}$$

There are 2^k cones in the fan of M_k of the top dimension k. These are also easy to identify inductively. The top dimensional cones for M_1 are simply $\langle e_1\rangle$ and $\langle -e_1\rangle$. Corresponding to each $(k-1)$-dimensional cone

$$\sigma = \langle \lambda_{i_1}^{(k-1)}, \dots, \lambda_{i_{k-1}}^{(k-1)} \rangle$$

in the fan of M_{k-1}, there are two k-dimensional cones, namely $\langle \widetilde{\sigma}, e_k\rangle$ and $\langle \widetilde{\sigma}, -e_k\rangle$, in the fan of M_k where

$$\widetilde{\sigma} = \langle \lambda_{\tilde{i}_1}^{(k)}, \dots, \lambda_{\tilde{i}_{k-1}}^{(k)} \rangle \quad \text{and } \tilde{i}_j = \begin{cases} i_j & \text{if } i_j < k \\ i_j + 1 & \text{if } i_j \geq k \end{cases}$$

Lemma 3.1. *There exists a nontrivial line bundle on M_k whose first Chern class has square zero, for any $k \geq 1$. There exist at least two such independent line bundles on M_k if $k \geq 2$.*

Proof. Let L_j denote the holomorphic line bundle associated to the torus invariant divisor of M_k corresponding to $\lambda_j^{(k)}$. Let w_j be the first Chern class of L_j. By (2.1), the abelian group $H^2(M_k)$ is freely generated by $\{w_j \mid j = 1, \dots, k\}$.

We also have the following linear relations corresponding to the rows of the characteristic matrix (3.1),

$$\begin{aligned} & w_{k+1} = w_1 \\ & w_{k+2} = w_2 + c_{1,2} w_{k+1} \\ & w_{k+3} = w_3 + c_{1,3} w_{k+1} + c_{2,3} w_{k+2} \\ & \dots \\ & w_{2k} = w_k + c_{1,k} w_{k+1} + c_{2,k} w_{k+2} + \dots + c_{k-1,k} w_{2k-1} \,. \end{aligned} \tag{3.2}$$

We note that $\lambda_i^{(k)}$ and $\lambda_{k+i}^{(k)}$ never form a cone in the fan of M_k. This implies that $w_i w_{k+i} = 0$ for $i = 1, \dots, k$. Then using the first equation in (3.2), observe that

$$w_1^2 = w_1 w_{k+1} = 0 \,.$$

Next, assume $k \geq 2$. Then multiplying the second equation of (3.2) by w_2 and using $w_{k+1} = w_1$, we get

$$w_2^2 = -c_{1,2} w_1 w_2 \,.$$

It follows that $(xw_1 + yw_2)^2 = 0$ if $2x = c_{1,2} y$. Hence, $(c_{1,2} w_1 + 2w_2)^2 = 0$. Thus, the line bundles L_1 and $L_1^{c_{1,2}} \otimes L_2^2$ over M_k satisfy the desired property. ☐

Remark 3.2. Proceeding inductively in the above proof, we can express w_j^2 as a linear combination of the $w_i w_j$'s with $i < j$. Moreover, the classes $w_i w_j$ with $i < j \leq k$ form a basis of $H^4(M_k)$. However, it is necessary to impose conditions on the defining constants $c_{i,j}$'s of M_k to ensure existence of further independent line bundles whose first Chern class squared is zero. For instance, $(xw_1 + y_2 + zw_3)^2 = 0$ if and only if

$$2x = c_{1,2}\, y, \quad 2y = c_{2,3}\, z, \quad \text{and} \quad 2x = (c_{1,3} + c_{1,2}\, c_{2,3})\, z\,.$$

In addition, if $z \neq 0$, it follows that $2c_{1,3} + c_{1,2}\, c_{2,3} = 0$ is required.

4. J-construction on nonsingular toric varieties

The origins of the J-construction can be traced back to [23] and [17] who referred to it as *simplicial wedge* and *dual wedge* respectively. It was used by Ewald in [6], a work we will invoke below, who called it *canonical extension*. It was referred to as the *doubling construction* in [18]. But it was introduced in its current avatar in [1]. The relevance of this technique to us is that it produces a projective toric variety of higher dimension starting from one of lower dimension, and the cohomology ring generators of the two varieties have a simple bijective correspondence.

Let X be a toric variety of dimension n with m torus invariant divisors. Let J be an m-tuple of natural numbers, $J = (j_1, \dots, j_m)$. A J-construction on X produces a toric variety $X(J)$ of dimension

$$d_J := n + \sum_{k=1}^{m} (j_k - 1)\,.$$

By definition, $X(J) = X$ when $J = (1, \dots, 1)$. In general, $X(J)$ can be obtained from X by a sequence of *atomic* steps that increase d_J by one. More precisely, such an atomic step produces $X(J)$ for $J = (j_1, \dots, j_i, \dots, j_m)$, where $j_i \geq 2$, from $X(J')$ where $J' = (j_1, \dots, j_i - 1, \dots, j_m)$. The order in which these atomic steps are performed is not important for the end result. An atomic step is also commonly referred to as a *simplicial wedge construction*.

Without loss of generality, we demonstrate how to perform the simplicial wedge construction to produce $X(J)$, where $J = (2, 1, \dots, 1)$, from X. We may conveniently refer to this as performing the simplicial wedge construction along the characteristic vector λ_1. In the language of fans, it amounts to building a complete fan $\Sigma(J)$ of one higher dimension than Σ. To accomplish this, the fan Σ, with the exception of λ_1, is embedded in a coordinate hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$, the support of $\Sigma(J)$. Naturally, we define $\lambda_i(J) = (\lambda_i, 0)$ for $1 < i \leq m$. The vector λ_1 is modified to $\lambda_1(J) = (\lambda_1, -1)$. The fan $\Sigma(J)$ has one additional characteristic vector, $\lambda_{m+1}(J) = (0, \dots, 0, 1)$.

We may write $\widetilde{\lambda}_i$ for $\lambda_i(J)$ without confusion for notational simplicity. Moreover, we use the notation $\lambda_i = (\lambda_{1i}, \dots, \lambda_{ni})$ as in Section 2. Thus, we have the following characteristic matrix for $\Sigma(J)$.

$$\Lambda(J) = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1m} & 0 \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2m} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nm} & 0 \\ -1 & 0 & \dots & 0 & 1 \end{bmatrix}$$

The top or $(n+1)$-dimensional cones of the fan $\Sigma(J)$ are as follows. For every n-dimensional cone $<\lambda_{i_1},\ldots,\lambda_{i_n}>$ of Σ that does not contain λ_1, there are two $(n{+}1)$-dimensional cones in $\Sigma(J)$ namely, $<\widetilde{\lambda}_{i_1},\ldots,\widetilde{\lambda}_{i_n},\widetilde{\lambda}_1>$ and $<\widetilde{\lambda}_{i_1},\ldots,\widetilde{\lambda}_{i_n},\widetilde{\lambda}_{m+1}>$. Moreover, for every n-dimensional cone that contains λ_1, say $<\lambda_1,\lambda_{i_2},\ldots,\lambda_{i_n}>$, there is an $(n{+}1)$-dimensional cone in $\Sigma(J)$, $<\widetilde{\lambda}_1,\widetilde{\lambda}_{i_2},\ldots,\widetilde{\lambda}_{i_n},\widetilde{\lambda}_{m+1}>$. The subcones of these top dimensional cones constitute, and thus determine, the fan $\Sigma(J)$.

The toric variety $X(J)$ corresponding to $\Sigma(J)$ is nonsingular if X is so. We denote the torus invariant divisor of $X(J)$ corresponding to $\lambda_j(J)$ by $D_j(J)$, and its first Chern class by $w_j(J)$, or simply by $\widetilde{w}_j$ when there is no scope for confusion. We can read off the linear relations among the generators $\widetilde{w}_j$'s from the rows of the matrix $\Lambda(J)$. Note that $\widetilde{w}_1=\widetilde{w}_{m+1}$. It follows that the classes $\widetilde{w}_1,\ldots,\widetilde{w}_m$ generate the cohomology ring of $X(J)$. This easily generalises to the case of an arbitrary J by induction.

The notations used above may be conveniently applied to the situation of a general J as well. We refer the reader to [1, Section 3] for a more comprehensive description of the J-construction. We content ourselves here by providing below the characteristic matrix $\Lambda(J)$ when $J=(3,2,1,\ldots,1)$ as an example:

$$\Lambda(J)=\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_m & \vec{0} & \vec{0} & \vec{0} \\ -1 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition 4.1. Let X be a nonsingular toric variety with fan Σ. Suppose w_p in $H^2(X)$ is the cohomology class corresponding to the torus invariant divisor D_p. We say that w_p has the *isolation property* if it admits a decomposition $w_p=\sum_{k\in I}a_kw_k$, where $p\notin I$, such that λ_p and λ_k do not form a cone in Σ for each $k\in I$.

Lemma 4.2. *Let X be a nonsingular toric variety. Suppose w_p admits the isolation property. Then $w_p^2=0$.*

Proof. Consider a decomposition $w_p=\sum_{k\in I}a_kw_k$, where $p\notin I$, such that λ_p and λ_k do not form a cone in Σ for each $k\in I$. Note that $w_p^2=\sum_{k\in I}a_kw_kw_p$, and each $w_kw_p=0$ as λ_k,λ_p do not form a cone. □

Lemma 4.3. *Let X be a smooth toric variety. Let $X(J)$ be a nonsingular toric variety obtained from X by a J-construction. Suppose $w_p\in H^2(X)$ admits the isolation property. Then $w_p(J)$ also admits the isolation property.*

Proof. By induction, it is enough to verify this for an atomic J-construction. Without loss of generality, let $J=(2,1,\ldots,1)$. Then,

$$\sum a_iw_i=0\implies\sum a_iw_i(J)=0\,.$$

Moreover, if λ_i and λ_j do not form a cone in Σ, neither do $\lambda_i(J)$ and $\lambda_j(J)$ in $\Sigma(J)$. The lemma follows. □

Corollary 4.4. *Suppose X is a Bott manifold M_k. Let $X(J)$ be a nonsingular toric variety obtained from X by a J-construction. Then there exist nontrivial line bundles on $X(J)$ whose first Chern class squared is zero.*

Proof. Note that the class w_p in a Bott manifold M_k, where $p \leq k$, has the isolation property if the constants $c_{1,p}, \ldots, c_{p-1,p}$ are zero (see (3.2)) as $\lambda_p^{(k)}$ and $\lambda_{p+k}^{(k)}$ do not form a cone. In particular, the class w_1 in a Bott manifold always has the isolation property. Then the result follows from Lemmas 4.2 and 4.3. □

5. Construction of SKT metrics

Suppose $\pi : E \to X$ is principal torus bundle over a complex manifold X with a $2k$-dimensional real torus $T \cong (S^1)^{2k}$ as fiber. The Lie algebra $\mathfrak{t}$ of T has a natural lattice given by the circle subgroups of T. A choice of an integral basis of $\mathfrak{t}$ gives a decomposition of T into a product of S^1's.

Let Θ be a connection 1-form on the bundle $E \to X$ and $(\theta_1, \ldots, \theta_{2k})$ be a representation of Θ corresponding to a basis of $\mathfrak{t}$. Define an almost complex structure $\mathcal{J}_E$ on E by lifting the complex structure $\mathcal{J}_X$ on X to the horizontal space of Θ, and by defining

$$\mathcal{J}_E\, \theta_{2j-1} = \theta_{2j},\ 1 \leq j \leq k\,,$$

along the vertical directions. The Chern classes of $\pi : E \to X$ are generated by $\omega_j \in \Omega^2(X)$ where $\pi^*(\omega_j) = d\theta_j$. If the classes $w_j := [\omega_j] \in H^2(X, \mathbb{R})$ are of type $(1,1)$, then $\mathcal{J}_E$ is integrable and π is holomorphic with respect to it, see [12, Lemma 1].

If the bundle $\pi : E \to X$ is obtained by reduction of structure group from a holomorphic principal bundle over X, then the above construction of integrable complex structure on E is applicable, see [22, Section 5]. In the sequel, we will apply this construction when the torus bundle is obtained by reduction from a direct sum of holomorphic line bundles.

Suppose that g_X is a Hermitian metric on X with fundamental form F_X. Following [12], consider a Hermitian metric g_E on E defined by

$$g_E = \pi^* g_X + \sum_{j=1}^{2k} \theta_j \otimes \theta_j\,.$$

The fundamental form of g_E is given by

$$F_E = \pi^* F_X + \sum_{j=1}^{k} \theta_{2j-1} \wedge \theta_{2j}\,.$$

It follows that

$$dd^c F_E = \pi^* dd^c F_X - \sum_{j=1}^{2k} \pi^*(\omega_j \wedge \omega_j)\,.$$

Thus E admits an SKT structure if

$$dd^c F_X = \sum_{j=1}^{2k} \omega_j \wedge \omega_j \,. \tag{5.1}$$

Therefore, if X is Kähler, the vanishing of $\sum \omega_j \wedge \omega_j$ is sufficient for existence of an SKT structure on E. The following lemma, which is based on a trick in the proof of Theorem 15 of [12], shows that the vanishing of $\sum [\omega_j] \wedge [\omega_j] = \sum w_j^2$ is sufficient.

Lemma 5.1. *If X is Kähler and $\sum_{j=1}^{2k} w_j^2 = 0$, then E admits an SKT structure.*

Proof. As $\sum w_j^2 = 0$, the (2,2)-form $\sum \omega_j \wedge \omega_j$ is d-exact and d-closed. The complex structure $\mathcal{J}_X$ on X preserves $(2,2)$-forms. Hence, $\sum \omega_j \wedge \omega_j$ is d^c-closed. So, by the dd^c-lemma, there exists a real $(1,1)$-form α on X such that $\sum \omega_j \wedge \omega_j = dd^c \alpha$. Choose an appropriate multiple β of a Kähler form on X such that

$$\min_{p \in X} \big(\min_{||\zeta||=1} \beta_p(\zeta, \mathcal{J}_X \zeta) \big) > - \min_{p \in X} \big(\min_{||\zeta||=1} \alpha_p(\zeta, \mathcal{J}_X \zeta) \big) \,.$$

Then the positive definite form $\alpha + \beta$ defines a Hermitian metric g_X on X which satisfies (5.1). □

Theorem 5.2. *Suppose X is a Bott manifold M_k. Let $X(J)$ be a nonsingular toric variety obtained from X by a J-construction. Then there exist nontrivial principal torus bundles $E(J)$ over $X(J)$ such that the total space of $E(J)$ admits an SKT structure.*

Proof. As noted in Corollary 4.4 , there exist nontrivial line bundles on $X(J)$ whose first Chern class squared is zero. (This also follows from Lemma 3.1 when $X(J) = X$.) Consider $E(J)$ to be the torus bundle over $X(J)$ obtained by the reduction of structure group from the direct sum of an even number of such line bundles. As X is projective, $X(J)$ is also projective (cf. [6, Theorem 2]), and hence Kähler. Then $E(J)$ admits an SKT structure by Lemma 5.1. □

Note that the above procedure produces a family of SKT metrics on $E(J)$ parametrised by an open subset of the space of Kähler metrics on M_k. A good reference for Kähler metrics on Bott manifolds is the recent article [4] by Boyer et al.

Compact connected Lie groups of even dimension admit invariant complex structures (cf. [21] or [22, Section 2]). Let G be such a group whose rank $\geq 2k$. Then there exists an injective homomorphism of Lie groups, $\phi : T \to G$ such that the image of ϕ is a closed subgroup of G. A choice of a complex structure on T has been made above while defining $\mathcal{J}_E$. Assume that we have used an integral basis of $\mathfrak{t}$ in defining Θ. It is then explained in [22, Section 5] how an invariant complex structure may be chosen on G so that the map ϕ is holomorphic. The following result then follows from the proof of [22, Theorem 5.2].

Corollary 5.3. *Let $E(J)$ be a principal T-bundle as in Theorem 5.2, and let $\phi : T \to G$ be a holomorphic monomorphism of Lie groups as above. Then the total space of the principal G-bundle $E(J) \times^{\phi} G$ admits an SKT structure.*

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