A Kenmotsu metric as a conformal η-Einstein soliton

By

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Abstract

The object of the present paper is to study some properties of Kenmotsu manifold whose metric is conformal η-Einstein soliton. We have studied some certain properties of Kenmotsu manifold admitting conformal η-Einstein soliton. We have also constructed a 3-dimensional Kenmotsu manifold satisfying conformal η-Einstein soliton.

Key words : Einstein soliton, η-Einstein soliton, conformal η-Einstein soliton, η-Einstein manifold, Kenmotsu manifold.

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1. **Introduction**

The notion of Einstein soliton was introduced by G. Catino and L. Mazzieri [3] in 2016, which generates self-similar solutions to Einstein flow,

$$\frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g), \tag{1.1}$$

where S is Ricci tensor, g is Riemannian metric and r is the scalar curvature.
The equation of the η-Einstein soliton [2] is given by,

$$\pounds_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where $\pounds_\xi$ is the Lie derivative along the vector field ξ, S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g, and λ and μ are real constants. For $\mu = 0$, the data (g, ξ, λ) is called Einstein soliton.
In 2018, Mohd Danish Siddiqi [5] introduced the notion of conformal η-Ricci soliton [7] as:

$$\pounds_\xi g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $\pounds_\xi$ is the Lie derivative along the vector field ξ , S is the Ricci tensor, λ, μ are constants, p is a scalar non-dynamical field(time dependent scalar field)and n is the dimension of manifold. For $\mu = 0$, conformal η-Ricci soliton becomes conformal Ricci soliton [6].

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In [8], Roy, Dey and Bhattacharyya have defined conformal Einstein soliton, which can be written as:

$$\pounds_V g + 2S + [2\lambda - r + (p + \frac{2}{n})]g = 0, \tag{1.4}$$

where $\pounds_V$ is the Lie derivative along the vector field V , S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g, λ is real constant, p is a scalar non-dynamical field(time dependent scalar field)and n is the dimension of manifold.
So we introduce the notion of conformal η-Einstein soliton as:
Definition 1.1: A Riemannian manifold (M, g) of dimension n is said to admit conformal η-Einstein soliton if

$$\pounds_\xi g + 2S + [2\lambda - r + (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where $\pounds_\xi$ is the Lie derivative along the vector field ξ ,λ, μ are real contants and S, r, p, n are same as defined in (1.4).

In the present paper we study conformal η-Einstein soliton on Kenmotsu manifold. The paper is organized as follows:
After introduction, section 2 is devoted for preliminaries on $(2n+1)$ dimensional Kenmotsu manifold. In section 3, we have studied conformal η-Einstein soliton on Kenmotsu manifold. Here we proved if a $(2n+1)$ dimensional Kenmotsu manifold admits conformal η-Einstein soliton then the manifold becomes η-Einstein. We have also characterized the nature of the manifold if the manifold is Ricci symmetric and the Ricci tensor is η-recurrent. Also we have discussed about the condition when the manifold has cyclic Ricci tensor. Then we have obtained the conditons in a $(2n+1)$ dimensional Kenmotsu manifold admitting Conformal η-Einstein soliton when a vector field V is pointwise co-linear with ξ and a (0,2) tensor field h is parallel with respect to the Levi-Civita connection associated to g. We have also examined the nature of a Ricci-recurrent Kenmotsu manifold admitting conformal η-Einstein soliton.
In last section we have given an example of a 3-dimensional Kenmotsu manifold satisfying conformal η-Einstein soliton.

2. Preliminaries

Let M be a $(2n+1)$ dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for all vector fields $X, Y \in \chi(M)$.
An almost contact metric manifold is said to be a Kenmotsu manifold [4] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold the following relations hold [1]:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

where R is the Riemannian curvature tensor.

$$S(X, \xi) = -2n\eta(X), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.12}$$

for all vector fields $X, Y, Z \in \chi(M)$.
Now we know,

$$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \tag{2.13}$$

for all vector fields $X, Y, \in \chi(M)$.
Then using (2.6) and (2.13), we get,

$$(\pounds_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.14}$$

3. Conformal η-Einstein soliton on Kenmotsu manifold

Let M be a $(2n+1)$ dimensional Kenmotsu manifold. Consider the conformal η-Einstein soliton (1.5) on M as:

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) + [2\lambda - r + (p + \frac{2}{2n+1})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.1}$$

for all vector fields $X, Y, \in \chi(M)$.
Then using (2.14),the above equation becomes,

$$S(X, Y) = -[\lambda - \frac{r}{2} + \frac{(p + \frac{2}{2n+1})}{2} + 1]g(X, Y) - (\mu - 1)\eta(X)\eta(Y). \tag{3.2}$$

Taking $Y=\xi$ in the above equation and using (2.10), we get,

$$r=(p+\frac{2}{2n+1})-4n+2\lambda+2\mu, \tag{3.3}$$

since $\eta(X)\neq 0$, for all $X\in\chi(M)$.
Also from (3.2), it follows that the manifold is η- Einstein.
This leads to the following:

Theorem 3.1. *If the metric of a (2n+1) dimensional Kenmotsu manifold is a conformal η-Einstein soliton then the manifold becomes η- Einstein and the scalar curvature is $(p+\frac{2}{2n+1})-4n+2\lambda+2\mu$.*

We know,

$$(\nabla_X S)(Y,Z)=X(S(Y,Z))-S(\nabla_X Y,Z)-S(Y,\nabla_X Z), \tag{3.4}$$

for all vector fields X,Y,Z on M and ∇ is the Levi-Civita connection associated with g.
Now replacing the expression of S from (3.2), we obtain,

$$(\nabla_X S)(Y,Z)=-(\mu-1)[\eta(Z)(\nabla_X\eta)Y+\eta(Y)(\nabla_X\eta)Z]. \tag{3.5}$$

for all vector fields X,Y,Z on M.

Let the manifold M be Ricci symmetric i.e $\nabla S=0$.
Then from (3.5), we get,

$$-(\mu-1)[\eta(Z)(\nabla_X\eta)Y+\eta(Y)(\nabla_X\eta)Z]=0, \tag{3.6}$$

for all vector fields $X,Y,Z\in\chi(M)$.
Taking $Z=\xi$ in the above equation and using (2.12), (2.1), we obtain,

$$\mu=1. \tag{3.7}$$

Then from (3.3), we get,

$$r=(p+\frac{2}{2n+1})-4n+2\lambda+2. \tag{3.8}$$

So we can state the following theorem:

Theorem 3.2. *If the metric of a (2n+1) dimensional Ricci symmteric Kenmotsu manifold is a conformal η-Einstein soliton then $\mu=1$ and the scalar curvature is $(p+\frac{2}{2n+1})-4n+2\lambda+2$.*

Now if the Ricci tensor S is η-recurrent, then we have,

$$\nabla S=\eta\otimes S, \tag{3.9}$$

which implies that,

$$(\nabla_X S)(Y,Z)=\eta(X)S(Y,Z), \tag{3.10}$$

for all vector fields X,Y,Z on M.
Using (3.5), the above equation reduces to,

$$-(\mu-1)[\eta(Z)(\nabla_X\eta)Y+\eta(Y)(\nabla_X\eta)Z]=\eta(X)S(Y,Z). \tag{3.11}$$

Taking $Y=\xi, Z=\xi$ in the above equation and using (2.12),(3.2), we get,

$$[\lambda+\mu-\frac{r}{2}+\frac{p+\frac{2}{2n+1}}{2}]\eta(X)=0, \tag{3.12}$$

which implies that,

$$r=2\lambda+2\mu+(p+\frac{2}{2n+1}). \tag{3.13}$$

Then we can state the following:

Theorem 3.3 *If the metric of a (2n+1) dimensional Kenmotsu manifold is a conformal η-Einstein soliton and the Ricci tensor S is η- Recurrent, then the scalar curvature is* $2\lambda+2\mu+(p+\frac{2}{2n+1})$.

Similarly from (3.5), we get,

$$(\nabla_Y S)(Z,X)=-(\mu-1)[\eta(X)(\nabla_Y\eta)Z+\eta(Z)(\nabla_X\eta)Y], \tag{3.14}$$

and

$$(\nabla_Z S)(X,Y)=-(\mu-1)[\eta(Y)(\nabla_Z\eta)X+\eta(X)(\nabla_Z\eta)Y]. \tag{3.15}$$

for all vector fields X,Y,Z on M.
Then adding (3.5),(3.14), (3.15) and using (2.12), (2.2), we obtain,

$$\begin{aligned}(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y) &= -2(\mu-1)[\eta(X)g(\phi Y,\phi Z)\\ &+ \eta(Y)g(\phi Z,\phi X)\\ &+ \eta(Z)g(\phi X,\phi Y)].\end{aligned} \tag{3.16}$$

Now if the manifold M has cyclic Ricci tensor i.e $(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0$, then from (3.16), we have,

$$(\mu-1)[\eta(X)g(\phi Y,\phi Z)+\eta(Y)g(\phi Z,\phi X)+\eta(Z)g(\phi X,\phi Y)]=0. \tag{3.17}$$

Taking $X=\xi$ in the above equation and using (2.1), we get,

$$\mu=1. \tag{3.18}$$

Again if we take $\mu=1$ in (3.16), we obtaion $(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0$, i.e the manifold M has cyclic Ricci tensor.
Hence we can state the following:

Theorem 3.4 *Let the metric of a (2n+1) dimensional Kenmotsu manifold M is a conformal η-Einstein soliton. Then M has cyclic Ricci tensor iff $\mu=1$.*

Now if $\mu=1$, then from (3.3) we obtain,

$$r=(p+\frac{2}{2n+1})-4n+2\lambda+2. \tag{3.19}$$

Then we have,
Corollary 3.5. *If a (2n+1) dimensional Kenmotsu manifold M has a cyclic Ricci tensor and the metric is a conformal η-Einstein soliton then the scalar curvature is* $(p+\frac{2}{2n+1})-4n+2\lambda+2$.

Let a conformal η-Einstein soliton is defined on a $(2n+1)$ dimensional Kenmotsu manifold M as,

$$\pounds_V g + 2S + [2\lambda - r + (p + \frac{2}{2n+1})]g + 2\mu\eta \otimes \eta = 0, \tag{3.20}$$

where $\pounds_V$ is the Lie derivative along the vector field V , S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g, λ, μ are real contants, p is a scalar non-dynamical field(time dependent scalar field).
Let V be pointwise co-linear with ξ, i.e $V = b\xi$, where b is a function on M.
Then (3.20) becomes,

$$(\pounds_{b\xi} g)(X,Y) + 2S(X,Y) + [2\lambda - r + (p + \frac{2}{2n+1})]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.21}$$

for all vector fields X, Y on M.
Applying the property of Lie derivative and Levi-Civita connection we have,

$$\begin{aligned} bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X,Y) \\ + [2\lambda - r + (p + \frac{2}{2n+1})]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.22}$$

Now using (2.6), we get,

$$\begin{aligned} 2bg(X,Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) \\ + [2\lambda - r + (p + \frac{2}{2n+1})]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.23}$$

Taking $Y = \xi$ in the above equation and using (2.1),(2.4),(2.10), we obtain,

$$(Xb) + (\xi b)\eta(X) - 4n\eta(X) + [2\lambda - r + (p + \frac{2}{2n+1})]\eta(X) + 2\mu\eta(X) = 0. \tag{3.24}$$

Then by putting $X = \xi$, the above equation reduces to,

$$\xi b = 2n + \frac{r}{2} - \lambda - \mu - \frac{(p + \frac{2}{2n+1})}{2}. \tag{3.25}$$

Using (3.25), (3.24) becomes,

$$(Xb) + [\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2} - 2n - \frac{r}{2}]\eta(X) = 0. \tag{3.26}$$

Applying exterior differentiation in (3.26), we have,

$$[\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2} - 2n - \frac{r}{2}]d\eta = 0. \tag{3.27}$$

Now we know,

$$d\eta(X,Y) = \frac{1}{2}[(\nabla_X \eta)Y - (\nabla_Y \eta)X], \tag{3.28}$$

for all vector fields X, Y on M.
Using (2.12),the above equation becomes,

$$d\eta = 0. \tag{3.29}$$

Hence the 1-form η is closed.
So from (3.27), either $r = 2\lambda+2\mu+(p+\frac{2}{2n+1})-4n$ or $r \neq 2\lambda+2\mu+(p+\frac{2}{2n+1})-4n$.
If $r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$, (3.26) reduces to,

$$(Xb) = 0. \tag{3.30}$$

This implies that b is constant.

So we can state the following theorem:

Theorem 3.6. *Let M be a (2n+1) dimensional Kenmotsu manifold admitting a conformal η-Einstein soliton (g, V), V being a vector field on M. If V is pointwise co-linear with ξ, a vector field on M, then V is a constant multiple of ξ, provided the scalar curvature is $2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$.*

Let h be a symmetric tensor field of (0,2) type which we suppose to be parallel with respect to the Levi-Civita connection ∇ i.e $\nabla h = 0$.
Applying the Ricci commutation identity, we have,

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0. \tag{3.31}$$

for all vector fields X, Y, Z, W on M.
From (3.31), we obtain the relation,

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{3.32}$$

Replacing $Z = W = \xi$ in the above equation and using (2.8), we get,

$$\eta(X)h(Y, \xi) - \eta(Y)h(X, \xi) = 0. \tag{3.33}$$

Replacing $X = \xi$ and using (2.1), the above equation reduces to,

$$h(Y, \xi) = \eta(Y)h(\xi, \xi), \tag{3.34}$$

for all vector fields Y on M.
Differentiating the above equation covariantly with respect to X, we get,

$$\nabla_X(h(Y, \xi)) = \nabla_X(\eta(Y)h(\xi, \xi)). \tag{3.35}$$

Now expanding the above eqution by using (3.34), (2.6),(2.12) and the property that $\nabla h = 0$, we obtain,

$$h(X, Y) = h(\xi, \xi)g(X, Y), \tag{3.36}$$

for all vector fields X, Y on M.
Let us take,

$$h = \pounds_\xi g + 2S + 2\mu\eta \otimes \eta. \tag{3.37}$$

Then from (2.14),(3.2), we get,

$$h(\xi, \xi) = -2\lambda - (p + \frac{2}{2n+1}) + r. \tag{3.38}$$

Then using (3.37), (3.36) becomes,

$$(\pounds_\xi g)(X, Y)+2S(X, Y)+[2\lambda-r+(p+\frac{2}{2n+1})]g(X, Y)+2\mu\eta(X)\eta(Y) = 0, \tag{3.39}$$

which is the Conformal η-Einstein soliton. This leads to,

Theorem 3.7. *In a (2n+1) dimensional Kenmotsu manifold assume that a symmetric (0,2) tensor field $h = \pounds_\xi g + 2S + 2\mu\eta\otimes\eta$ is parallel with respect to the Levi-Civita connection associated to g. Then (g,ξ) yields a conformal η-Einstein soliton.*

Definition 3.8 A Kenmotsu manifold is said to be Ricci-recurrent manifold if there exists a non-zero 1-form A such that

$$(\nabla_W S)(Y,Z) = A(W)S(Y,Z), \tag{3.40}$$

for any vector fields W, Y, Z on M.
Replacing Z by ξ in the above equation and using (2.10), we get,

$$(\nabla_W S)(Y,\xi) = -2nA(W)\eta(Y), \tag{3.41}$$

which implies that,

$$WS(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) = -2nA(W)\eta(Y). \tag{3.42}$$

Using (2.10) and (2.6), the above equation becomes,

$$2n(\nabla_W \eta)Y + 2n\eta(W)\eta(Y) + S(Y,W) = 2nA(W)\eta(Y). \tag{3.43}$$

Again using (2.12), the above equation reduces to,

$$2ng(W,Y) + S(Y,W) = 2nA(W)\eta(Y). \tag{3.44}$$

Taking $W = \xi$ in the above equation and using (3.2), we obtain,

$$r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) + 4n(A(\xi) - 1). \tag{3.45}$$

So we can state,

Theorem 3.9. *If the metric of a (2n+1) dimensional Ricci-recurrent Kenmotsu manifold is a conformal η-Einstein soliton with the 1-form A, then the scalar curvature becomes $2\lambda + 2\mu + (p + \frac{2}{2n+1}) + 4n(A(\xi) - 1)$.*

4. Example of a 3-dimensional Kenmotsu manifold admitting conformal η-Einstein soliton:

We consider the three-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, (x,y,z) \neq (0,0,0)\}$, where (x,y,z) are standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0,$$
$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$,where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the (1, 1)-tensor field defined by,

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we have,

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have,

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The connection ∇ of the metric g is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszuls formula.
Using Koszuls formula, we can easily calculate,

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above it follows that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu Manifold.
Also, the Riemannian curvature tensor R is given by,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Hence,

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0.$$

Then, the Ricci tensor S is given by,

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2.$$

From (3.2), we have,

$$S(e_3, e_3) = -[\lambda + \mu - \frac{r}{2} + \frac{(p + \frac{2}{3})}{2}], \tag{4.1}$$

which implies that,

$$r = 2\lambda + 2\mu - 4 + (p + \frac{2}{3}). \tag{4.2}$$

Hence λ and μ satisfies equation (3.3) and so g defines a conformal η-Einstein soliton on the 3-dimensional Kenmotsu manifold M.

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