

# Proper superminimal surfaces of given conformal types in the hyperbolic four-space

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**Abstract** Let  $H^4$  denote the hyperbolic four-space. Given a bordered Riemann surface,  $M$ , we prove that every smooth conformal superminimal immersion  $\overline{M} \rightarrow H^4$  can be approximated uniformly on compacts in  $M$  by proper conformal superminimal immersions  $M \rightarrow H^4$ . In particular,  $H^4$  contains properly immersed conformal superminimal surfaces normalised by any given open Riemann surface of finite topological type without punctures. The proof uses the analysis of holomorphic Legendrian curves in the twistor space of  $H^4$ .

**Keywords** superminimal surface, hyperbolic space, twistor space, complex contact manifold, holomorphic Legendrian curve

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## 1. Introduction

Among minimal surfaces in an oriented four-dimensional Riemannian manifold  $(X, g)$  there is an interesting subclass consisting of *superminimal surfaces of positive or negative spin*. They were introduced in 1897 by Kommerell [28] and were studied by many authors; see [17] for a brief historical account. The term *superminimal surface* was coined by Bryant [12] (1982) in his seminal study of minimal surfaces in the four-sphere  $S^4$  which arise as projections of holomorphic Legendrian curves in  $\mathbb{C}\mathbb{P}^3$ , the Penrose twistor space of  $S^4$ . This *Bryant correspondence* [12, Theorems B, B'] was extended to all oriented Riemannian four-manifolds  $(X, g)$  by Friedrich [21, Proposition 4] who also showed in [22] that superminimal surfaces in the sense of Bryant coincide with those of Kommerell.

Assume that  $M \subset X$  is a smooth oriented embedded surface with the induced conformal structure in an oriented Riemannian four-manifold  $(X, g)$ . (Our considerations will also apply to immersed surfaces.) Then  $TX|_M = TM \oplus N$  where  $N = N(M)$  is the cooriented orthogonal normal bundle to  $M$ . A unit normal vector  $n \in N_x$  at a point  $x \in M$  determines a *second fundamental form*  $S_x(n) : T_x M \rightarrow T_x M$ , a self-adjoint linear operator on the tangent space of  $M$ . For a fixed tangent vector  $v \in T_x M$  we consider the closed curve

$$(1.1) \quad I_x(v) = \{S_x(n)v : n \in N_x, |n|_g = 1\} \subset T_x M.$$

**Definition 1.1.** A smooth oriented embedded surface  $M$  in an oriented Riemannian four-manifold  $(X, g)$  is *superminimal of positive (negative) spin* if for every point  $x \in M$  and unit tangent vector  $v \in T_x M$ , the curve  $I_x(v) \subset T_x M$  (1.1) is a circle centred at 0 and the map  $n \rightarrow S(n)v \in I_x(v)$  is orientation preserving (resp. orientation reversing). The last condition is void at points  $x \in M$  where the circle  $I_x(v)$  reduces to 0  $\in T_x M$ . The analogous definition applies to a smoothly immersed oriented surface  $f : M \rightarrow X$ .

Every superminimal surface is also a minimal surface; see Friedrich [22, Proposition 3] and the discussion in [17, Sect. 2]. The converse only holds in special cases. For example, every conformal minimal immersion of the two-sphere  $S^2$  into the four-sphere  $S^4$  with the spherical metric is superminimal; see [12, Theorem C] or [23, Proposition 25]. The same holds for immersions of  $S^2$  into the projective plane  $\mathbb{C}\mathbb{P}^2$  with the Fubini-Study metric (see [23, Proposition 28]). Superminimal surfaces in  $S^4$  and  $\mathbb{C}\mathbb{P}^2$  with their natural metrics have been studied extensively; see the references in [17, Sect. 2].

A motivation for the present paper is Bryant's theorem [12, Corollary H] that every compact Riemann surface,  $M$ , admits a conformal superminimal immersion into  $S^4$  with the spherical metric. In view of the Bryant correspondence, this follows from his result [12, Theorem G] saying that every such  $M$  admits a holomorphic Legendrian embedding  $M \rightarrow \mathbb{C}\mathbb{P}^3$  in the standard contact structure determined by the following 1-form on  $\mathbb{C}^4$ :

$$(1.2) \quad \alpha = z_1 dz_2 - z_2 dz_1 + z_3 dz_4 - z_4 dz_3.$$

Approximation theorems of Runge and Mergelyan type for Legendrian curves in  $\mathbb{C}\mathbb{P}^3$  have been obtained recently in [2, Corollary 7.3] and [18, Corollary 1.11].

In this paper we consider superminimal surfaces in the four dimensional *hyperbolic space*  $H^4$ , the unique simply connected complete Riemannian four-manifold of constant sectional curvature  $-1$  (see [13, Theorem 4.1]). Among the geometric models for  $H^4$  it will be most convenient for us to use the *Poincaré (conformal) ball model* given by the unit ball  $\mathbb{B} = \{x \in \mathbb{R}^4 : |x|^2 < 1\}$  endowed with the complete hyperbolic metric

$$(1.3) \quad g_h = \frac{4|dx|^2}{(1 - |x|^2)^2}, \quad x \in \mathbb{B}.$$

The ball model is related to the hyperboloid model in the Lorentz space  $\mathbb{R}^{4,1}$  by the stereographic projection (2.4); see Sect. 2.

Recall that a *bordered Riemann surface* is an open domain of the form  $M = R \setminus \bigcup_i \Delta_i$  in a compact Riemann surface  $R$ , where  $\Delta_i$  are finitely many compact pairwise disjoint discs (diffeomorphic images of  $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ ) with smooth boundaries  $b\Delta_i$ . Its closure  $\overline{M}$  is a *compact bordered Riemann surface*.

The following is our main result; it is proved in Sect. 6 as a corollary to Theorem 6.1.

**Theorem 1.2.** *Let  $M$  be a bordered Riemann surface. Every smooth conformal superminimal immersion  $f : \overline{M} \rightarrow (\mathbb{B}, g_h) = H^4$  can be approximated uniformly on compacts in  $M$  by proper conformal superminimal immersions  $\tilde{f} : M \rightarrow \mathbb{B}$ . Furthermore,  $\tilde{f}$  can be chosen to agree with  $f$  to a given finite order at finitely many points in  $M$ .*

What is new in comparison to the extant results in the literature is that we control not only the (finite) topology of proper superminimal surfaces, but also their conformal type.

Any minimal surface in  $H^4$  is open and its conformal universal covering is the disc (see [18, Corollary 6.3]). Since every open Riemann surface with finitely generated homology group  $H_1(M, \mathbb{Z})$  is conformally equivalent to a domain obtained by removing finitely many closed discs and points from a compact Riemann surface (see Stout [34, Theorem 8.1]), bordered Riemann surfaces are precisely the open Riemann surfaces of finite topology without punctures. This gives the following corollary to Theorem 1.2.

**Corollary 1.3.** *Every open Riemann surface of finite topological type without punctures is the conformal structure of a properly immersed superminimal surface in  $H^4$ .*

Although Corollary 1.3 might also hold for bordered Riemann surfaces with punctures, it is notoriously difficult to deal with this case and we leave it as an open problem.

It has recently been shown [18, Corollary 6.3] that any self-dual or anti-self dual Einstein four-manifold (this class includes  $S^4$ ,  $H^4$ , and many other Riemannian four-manifolds) also contains complete relatively compact immersed superminimal surfaces of any conformal type in Corollary 1.3, thereby solving the Calabi-Yau problem for such surfaces.

Our approach to Theorem 1.2 uses the Bryant correspondence to the effect that superminimal surfaces in an oriented Riemannian four-manifold  $(X, g)$  are the projections of horizontal holomorphic curves in total spaces of twistor bundles  $\pi^\pm : Z^\pm \rightarrow X$ , with the sign depending on the spin of the superminimal surface; see [17, Sect. 4]. Both twistor spaces  $Z^\pm$  of  $H^4 = (\mathbb{B}, g_h)$  can be identified with the domain in  $\mathbb{C}\mathbb{P}^3$  given by

$$(1.4) \quad \Omega = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathbb{P}^3 : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2\},$$

and the twistor projection  $\pi : \Omega \rightarrow \mathbb{B}$  is the restriction to  $\Omega$  of the twistor projection  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow S^4$  for the spherical metric on  $S^4$  (see Sect. 3). This is a particular instance of the general fact that the twistor bundles  $\pi^\pm : Z^\pm \rightarrow X$  of an oriented Riemannian four-manifold  $(X, g)$  depend only on the conformal class of the metric  $g$ , but the horizontal bundles  $\xi^\pm \subset TZ^\pm$  depend on the choice of a metric in that class. In the case at hand, both the spherical and the hyperbolic metric are conformally flat. The horizontal bundle  $\xi \subset T\mathbb{C}\mathbb{P}^3$  determined by the hyperbolic metric on  $\mathbb{B}$  is the holomorphic contact bundle given by the homogeneous 1-form

$$(1.5) \quad \beta = z_1 dz_2 - z_2 dz_1 - z_3 dz_4 + z_4 dz_3$$

(see Sect. 3). Compared to the 1-form  $\alpha$  (1.2), we note a change of sign in the last two terms. Although  $\xi$  is contactomorphic to the standard contact structure  $\xi_{\text{std}}$  determined by  $\alpha$  (in fact,  $\xi_{\text{std}}$  is the unique holomorphic contact structure on  $\mathbb{C}\mathbb{P}^3$  up to holomorphic contactomorphisms, see LeBrun and Salamon [29, Corollary 2.3]), these two structures behave very differently with respect to the twistor projection  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow S^4 \cong \widehat{\mathbb{R}}^4 := \mathbb{R}^4 \cup \{\infty\}$ . While  $\xi_{\text{std}}$  is orthogonal to all fibres of  $\pi$  with respect to the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^3$ ,  $\xi$  is orthogonal to the fibres of  $\pi$  over  $\mathbb{B}$  and over the complementary open ball  $\mathbb{B}' = \widehat{\mathbb{R}}^4 \setminus \mathbb{B}$  in the twistor metric induced by the hyperbolic metrics on  $\mathbb{B}$  and  $\mathbb{B}'$ , but the fibres  $\pi^{-1}(x)$  over points  $x \in b\mathbb{B}$  are  $\xi$ -Legendrian curves. Any holomorphic Legendrian immersion  $F : M \rightarrow (\mathbb{C}\mathbb{P}^3, \xi)$  whose image does not lie in a fibre of  $\pi$  determines an immersed superminimal surface in  $\mathbb{B}$  obtained by intersecting the image of  $F$  with  $\Omega$  (1.4) and projecting down to  $\mathbb{B}$ . If  $M$  is compact and  $F$  intersects  $b\Omega$  transversely, we obtain a proper superminimal surface in  $\mathbb{B}$  with smooth boundary in  $b\mathbb{B} = S^3$ , and we know by Bryant [12, Theorem G] that any compact Riemann surface embeds as a complex Legendrian curve in  $(\mathbb{C}\mathbb{P}^3, \xi)$ . However, it seems impossible to control the conformal type of the examples obtained in this way. In a related direction, Anderson [9] solved the Plateau problem for area minimizing generalized surfaces (currents) in the hyperbolic ball  $\mathbb{B}^n$ ,  $n \geq 3$ , having a given boundary manifold in the sphere  $b\mathbb{B}^n = S^{n-1}$ .

On the other hand, our approach provides full control of the conformal type, but we do not know whether the map  $\tilde{f} : M \rightarrow H^4$  in Theorem 1.2 can be chosen to extend continuously or smoothly to the boundary of  $M$ . This difficulty is not unique to the present situation. Indeed, even for the simplest minimal surfaces such as holomorphic curves in a bounded strongly pseudoconvex domain  $D$  in  $\mathbb{C}^n$  for  $n > 1$  it is not known whether the analogue of Theorem 1.2 holds for maps extending smoothly to the boundary  $bM$  without changing the conformal type of  $M$ . (Continuous extendibility is possible in this case.) This

holds if  $M$  is the disc (see Globevnik and the author [20]), or if the domain  $D$  is convex and  $M$  is arbitrary (see Černe and Flores [35]). The most general analogue of Theorem 1.2 in the holomorphic category, due to Drinovec Drnovšek and the author [14], pertains to holomorphic curves in any complex manifold of dimension  $> 1$  having a smooth exhaustion function whose Levi form has at least two positive eigenvalues at every point. An analogue for minimal surfaces in minimally convex domains in flat Euclidean spaces  $\mathbb{R}^n$ ,  $n \geq 3$ , was given by Alarcón et al. [1, Theorems 1.1 and 1.9].

Let us say a few words about the method of proof and the organisation of the paper.

In sections 2 and 3 we review the necessary background concerning the geometry of the hyperbolic space  $H^4$  and its twistor space. A more complete overview of the twistor space theory pertaining to superminimal surfaces is included in [17].

Our proof of Theorem 1.2 relies upon the Bryant correspondence between superminimal surfaces in  $H^4 = (\mathbb{B}, g_h)$  and holomorphic Legendrian curves in its twistor space  $(\Omega, \beta)$ . The main analytic technique used in the proof are Riemann-Hilbert modifications, using approximates solutions of certain Riemann-Hilbert boundary value problems. One of the contributions of the present paper is the development of the Riemann-Hilbert modification technique for holomorphic Legendrian curves in projective spaces  $\mathbb{C}\mathbb{P}^{2n+1}$ ; see Theorem 4.1. We expect that this result will find further applications. This classical complex-analytic method was adapted in [8, Sect. 3] to holomorphic Legendrian curves in Euclidean space  $\mathbb{C}^{2n+1}$  with the standard contact structure inherited from  $\mathbb{C}\mathbb{P}^{2n+1}$ ; however, those results do not apply to projective spaces since the relevant geometric configurations need not be contained in any particular affine chart. We also prove a general position theorem showing that any noncompact Legendrian curve in a projective space, possibly with branch points, can be approximated by holomorphic Legendrian embeddings; see Theorem 5.1.

With these newly developed tools in hand, we construct in Sect. 6 properly immersed holomorphic Legendrian curves in the twistor domain  $\Omega$  of  $\mathbb{B} = H^4$  whose projections to  $\mathbb{B}$  satisfy Theorem 1.2. The geometry of the hyperbolic space and of its twistor space (see Secs. 2–3) plays an essential role in the application of the Riemann-Hilbert method.

The Riemann-Hilbert technique was used in a recent solution of the Calabi-Yau problem for superminimal surfaces and holomorphic Legendrian curves [17], and before that in the original Calabi-Yau problem concerning minimal surfaces in Euclidean spaces; see the formulation of the problem by S.-T. Yau in [36, p. 360] and [37, p. 241], and the recent advances summarized in [6, 7]. In the paper [17] we used Riemann-Hilbert modifications with Legendrian discs of small extrinsic diameter, and in this case the required result (see [5, Theorem 1.3]) follows from the Euclidean case by the contact neighbourhood theorem given by [5, Theorem 1.1]. On the contrary, the construction of proper Legendrian curves is more demanding since one must apply Riemann-Hilbert modifications with discs of big extrinsic diameter in order to push the boundary of the surface successively closer to the boundary of the given domain, thereby obtaining a proper map in the limit.

In conclusion, we mention an open problem related to Theorem 1.2. There are constructions in the literature of infinite dimensional families of self-dual Einstein metrics with constant negative scalar curvature on the ball  $\mathbb{B} \subset \mathbb{R}^4$  inducing given conformal structures of a suitable type on the boundary sphere  $S^3 = \partial\mathbb{B}$ ; see in particular Graham and Lee [25], Hitchin [26], and Biquard [11]. The twistor space of  $\mathbb{B}$  with any such metric is a complex contact manifold. Does the analogue of Theorem 1.2 hold true for any or all of these metrics, besides the standard one considered in the present paper?

## 2. Making the acquaintance of the principal protagonist

In this section we recall a few basics of hyperbolic geometry that shall be used in the paper, referring to the monograph by Ratcliffe [32] for further information.

A geometric model of the hyperbolic  $n$ -space  $H^n$  is the hyperboloid

$$(2.1) \quad \Sigma = \Sigma^+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}.$$

This is one of two connected components of the unit sphere of radius  $i = \sqrt{-1}$  in the Lorentz space\*  $\mathbb{R}^{n,1}$ , the Euclidean space  $\mathbb{R}^{n+1}$  with the indefinite Lorentz inner product

$$(2.2) \quad x \circ y = x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1}.$$

The other connected component  $\Sigma^-$  is obtained by taking  $x_{n+1} < 0$  in (2.1). Considering  $x_{n+1}$  as the time variable,  $\Sigma^\pm$  are contained in the open cone of *time-like vectors*

$$(2.3) \quad T = \{x \in \mathbb{R}^{n+1} : x \circ x < 0\} = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1}^2 > |x'|^2\},$$

while all the nonzero tangent vectors to  $\Sigma^\pm$  are contained in the cone  $S = \{x_{n+1}^2 < |x'|^2\}$  of *space-like vectors*. Their common boundary  $bT = bS$  is the *light cone*

$$LC = \{x \in \mathbb{R}^{n+1} : x \circ x = 0\} = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1}^2 = |x'|^2\}.$$

The *Lorentz norm*  $\|x\| = \sqrt{x \circ x}$  is a positive real number for space-like vectors, a positive multiple of  $i = \sqrt{-1}$  for time-like vector, and it vanishes for *light-like vectors*  $x \in LC$ .

Consider the stereographic projection  $\sigma : \mathbb{B} = \{x \in \mathbb{R}^n : |x|^2 < 1\} \xrightarrow{\cong} \Sigma^+$  given by

$$(2.4) \quad \sigma(x) = \left( \frac{2x}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right), \quad x \in \mathbb{B}.$$

The pullback by  $\sigma$  of the Lorentz pseudometric  $\|x\|^2 = x \circ x$  on  $\mathbb{R}^{n,1}$  (see (2.2)) is the hyperbolic Riemannian metric of constant sectional curvature  $-1$  on  $\mathbb{B}$  given by (1.3). The same formula defines the hyperbolic metric on the complementary ball

$$(2.5) \quad \mathbb{B}' = \mathbb{R}^n \cup \{\infty\} \setminus \overline{\mathbb{B}}.$$

Consider the reflection  $\mathbb{B} \rightarrow \mathbb{B}'$  in the sphere  $b\mathbb{B} = b\mathbb{B}' = S^{n-1}$  given by  $\mathbb{B} \ni x \mapsto \frac{x}{|x|^2} = y \in \mathbb{B}'$ , with  $0 \mapsto \infty$ . Then  $\frac{dx}{1 - |x|^2} = \frac{dy}{1 - |y|^2}$ , and hence the reflection is an isometry. The stereographic projection  $\psi : \mathbb{R}^n \cup \{\infty\} \rightarrow S^n$  given by

$$(2.6) \quad \psi(x) = \left( \frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right), \quad x \in \mathbb{R}^n; \quad \psi(\infty) = \mathfrak{s} = (0', -1)$$

maps the balls  $\mathbb{B}, \mathbb{B}'$  onto opposite hemispheres of the Euclidean sphere  $S^n \subset \mathbb{R}^{n+1}$ .

The group of linear automorphisms of  $\mathbb{R}^{n,1}$  preserving the Lorentz inner product  $x \circ y$  is the *Lorentz group*  $O(n, 1)$ . Every Lorentz transformation preserves the light cone  $LC$  and the open cones  $T, S$  of time-like and space-like vectors, but it may interchange the two connected components  $T^\pm$  of  $T$  (2.3) defined by  $\pm x_{n+1} > 0$ . The group  $O(n, 1)$  contains the index two *positive Lorentz group*  $PO(n, 1)$  of Lorentz transformations mapping  $T^+$  (and hence also  $T^-$ ) to itself. Since  $\Sigma$  (2.1) is the component of the unit sphere of radius  $i = \sqrt{-1}$  contained in  $T^+$ , the restriction of any positive Lorentz transformation

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\*Lorentz spaces are named after Hendrik Antoon Lorentz (1853–1928), a Dutch physicist and a 1902 Nobel Prize winner who derived the transformation equations underpinning Albert Einstein's theory of special relativity on the Lorentz four-space  $\mathbb{R}^{3,1}$ , the *Minkowski space-time*. The terms *Lorentz space* and *Lorentz transformation* were introduced by Poincaré in his 1906 paper [31]. See [32, Sect. 3.6] for more information.

$A \in \text{PO}(n, 1)$  to  $\Sigma \cong H^n$  is an isometry of  $H^n$ ; conversely, every isometry of  $H^n$  extends to a unique  $A \in \text{PO}(n, 1)$  (see [32, Theorem 3.2.3]).

Via the stereographic projection  $\sigma : \mathbb{B} \rightarrow \Sigma$  given by (2.4), the group  $\text{PO}(n, 1)$  of positive Lorentz transformations of  $\mathbb{R}^{n,1}$  corresponds to the group  $\text{I}(\mathbb{B})$  of isometries of the hyperbolic  $n$ -ball  $(\mathbb{B}, g_h)$  (1.3). Note that  $\text{I}(\mathbb{B})$  coincides with the group  $\text{M}(\mathbb{B})$  of Möbius transformations of the extended Euclidean space  $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$  mapping  $\mathbb{B}$  onto itself (see Ratcliffe [32, Theorem 4.5.2 and Corollary 1]). Every Möbius transformation in  $\text{M}(\mathbb{B})$  is a composition of reflections of  $\widehat{\mathbb{R}}^n$  in spheres orthogonal to  $S^{n-1} = b\mathbb{B}$ , where spheres passing through  $\infty$  are hyperplanes through  $0 \in \mathbb{R}^n$ . In particular,  $\text{M}(\mathbb{B}) = \text{M}(\mathbb{B}')$  where  $\mathbb{B}'$  is the complementary hyperbolic ball (2.5). The restriction of the elements of  $\text{M}(\mathbb{B})$  to the sphere  $S^{n-1} = b\mathbb{B}$  is the Möbius group  $\text{M}(S^{n-1})$ .

An important class of objects in  $H^n$  are its hyperbolic planes. A vector subspace  $V \subset \mathbb{R}^{n,1}$  is said to be *time-like* if it contains a time-like vector, i.e.,  $V$  intersects the cone  $T$  (2.3). A *hyperbolic  $m$ -plane* of  $H^n$  is the intersection of  $H^n = \Sigma$  (2.1) with an  $(m+1)$ -dimensional time-like vector subspace of  $\mathbb{R}^{n,1}$ . Hyperbolic lines are precisely the geodesics of  $H^n$  (cf. [32, p. 70]). Preimages of hyperbolic  $m$ -planes in  $H^n$  by the stereographic projection  $\sigma : \mathbb{B} \rightarrow \Sigma = H^n$  (2.4) are called hyperbolic  $m$ -planes in  $\mathbb{B}$ ; every such is the intersection of  $\mathbb{B}$  with either an  $m$ -dimensional vector subspace of  $\mathbb{R}^n$  or an  $m$ -sphere orthogonal to the unit sphere  $S^{n-1} = b\mathbb{B}$  [32, Theorem 4.5.3]. Every hyperbolic  $m$ -plane  $\Lambda \subset \mathbb{B}$  is the image of the  $m$ -ball  $\mathbb{B} \cap V \cong \mathbb{B}^m$  in an  $m$ -dimensional vector subspace  $V \subset \mathbb{R}^n$  by the orientation preserving *hyperbolic translation*  $\tau_b \in \text{M}(\mathbb{B})$  for some  $b \in \mathbb{B}$ :

$$(2.7) \quad \tau_b(x) = \frac{1}{|b|^2|x|^2 + 2x \cdot b + 1} \left( (1 - |b|^2)x + (|x|^2 + 2x \cdot b + 1)b \right).$$

(See [32, (4.5.5)]. Here,  $x \cdot b$  denotes the Euclidean inner product on  $\mathbb{R}^n$ . Note that  $\tau_b(0) = b$  and  $\tau_0 = \text{Id}$ .) Indeed, choosing  $b \in \Lambda$  with the smallest Euclidean norm  $|b|$  and letting  $V = T_b\Lambda$  considered as a vector subspace of  $\mathbb{R}^n$ , we have that  $\tau_b(\mathbb{B} \cap V) = \Lambda$ . To see this, note that for every  $x \in \mathbb{B} \cap V$  and  $h \in V$  we have that  $x \cdot b = 0$  and hence

$$\tau_b(x) = \frac{1}{|b|^2|x|^2 + 1} \left( (1 - |b|^2)x + (|x|^2 + 1)b \right), \quad (d\tau_b)_0 h = (1 - |b|^2)h.$$

Since a hyperbolic plane is uniquely determined by a pair  $(b, V)$  where  $V$  is an  $m$ -dimensional vector subspace of  $\mathbb{R}^n$  and  $b \in \mathbb{B}$  is orthogonal to  $V$ , the claim follows. We summarise this observation for a later application.

**Proposition 2.1.** *For each pair  $(b, V)$ , where  $V$  is an  $m$ -dimensional vector subspace of  $\mathbb{R}^n$  and  $b \in \mathbb{B}$  is orthogonal to  $V$ , there is a unique hyperbolic  $m$ -plane  $\Lambda(b, V) \subset \mathbb{B}$  with*

$$b \in \Lambda(b, V), \quad T_b\Lambda(b, V) = V, \quad |b| = \min\{|x| : x \in \Lambda(b, V)\}.$$

We have that  $\Lambda(0, V) = \mathbb{B} \cap V$ , and if  $b \neq 0$  then

$$(2.8) \quad \Lambda(b, V) = \tau_b(\mathbb{B} \cap V) = \mathbb{B} \cap S^m(a, r),$$

where  $a \in \mathbb{R}_+ \cdot b$  is the unique point with  $|a| = \frac{1+|b|^2}{2|b|}$ ,  $r = \frac{1-|b|^2}{2|b|}$ , and  $S^m(a, r)$  is the sphere with centre  $a$  and radius  $r$  in the  $(m+1)$ -dimensional vector subspace  $L \subset \mathbb{R}^n$  spanned by  $V$  and  $b$ . In particular,  $\Lambda(b, V)$  depends real analytically on  $(b, V)$ .

In the calculation of the centre point  $a$  and the radius  $r$  of the sphere  $S^m(a, r) \subset L \cong \mathbb{R}^{m+1}$  in the above proposition, one takes into account that  $S^m(a, r)$  intersects the unit sphere  $S^m \subset \mathbb{R}^{m+1}$  orthogonally if and only if  $|a|^2 = r^2 + 1$  [32, Theorem 4.4.2].

### 3. Twistor space of the hyperbolic four-space

We briefly recall the main facts about twistor spaces pertaining to this paper, referring to [17, Sect. 4] and the references therein for a more complete account.

To any smooth orientable Riemannian four-manifold  $(X, g)$  one associates a pair of twistor fibre bundles  $\pi^\pm : Z^\pm \rightarrow X$ . The fibre  $\pi^{-1}(x) \cong \mathbb{C}\mathbb{P}^1$  over any point  $x \in X$  consists of almost hermitian structures  $J : T_x X \rightarrow T_x X$ , that is, linear maps satisfying  $J^2 = -\text{Id}$  preserving the metric  $g$  and either agreeing or disagreeing with the orientation of  $X$ , depending on  $\pm$ . The Levi-Civita connection of  $(X, g)$  determines a horizontal subbundle  $\xi^\pm \subset TZ^\pm$  projecting by  $d\pi^\pm$  isomorphically onto the tangent bundle of  $X$ . The total spaces  $Z^\pm$  carry almost complex structures  $J^\pm$  such that  $\xi^\pm$  is a  $J^\pm$ -complex subbundle, for any point  $z \in Z^\pm$  (an almost hermitian structure on  $T_x X$  for  $x = \pi^\pm(z)$ ) we have that  $d\pi_z^\pm \circ J_z^\pm = z \circ d\pi_z^\pm$ , and  $J^\pm$  coincides with the natural complex structure on the fibres  $(\pi^\pm)^{-1}(x) \cong \mathbb{C}\mathbb{P}^1$ . The bundles  $(Z^\pm, J^\pm)$  only depend on the conformal class of  $g$ , but the horizontal bundle  $\xi^\pm$  depends on the choice of  $g$  in a given conformal class.

Let  $M$  be an open Riemann surface. The *Bryant correspondence* says that conformal superminimal immersions  $f : M \rightarrow X$  of positive or negative spin (see Def. 1.1) are the twistor projections of horizontal (tangent to  $\xi^\pm$ ) holomorphic immersions  $F^\pm : M \rightarrow Z^\pm$ , the sign depending on the spin of  $f$  (see Bryant [12, Theorems B, B'], Friedrich [21, Proposition 4], and the summary in [17, Theorem 4.6]). According to Atiyah et al. [10, Theorem 4.1], the almost complex structure  $J^\pm$  is integrable (i.e.,  $(Z^\pm, J^\pm)$  is a complex manifold) if and only if the Weyl tensor  $W = W^+ + W^-$  (the conformally invariant part of the curvature tensor of  $g$ ) satisfies  $W^+ = 0$  ( $g$  is anti-self-dual) or  $W^- = 0$  ( $g$  is self-dual), respectively. If either of these conditions hold then the corresponding horizontal subbundle  $\xi^\pm \subset TZ^\pm$  is a holomorphic subbundle if and only if  $g$  is an Einstein metric, and in such case  $\xi^\pm$  is a holomorphic contact bundle if and only if  $g$  has nonzero (constant) scalar curvature; see Salamon [33, Theorem 10.1] and Eells and Salamon [16, Theorem 4.2].

The spherical metric on  $S^4$  and the hyperbolic metric on  $H^4$  are conformally flat Einstein metrics of curvature  $\pm 1$ , so their twistor spaces are complex contact three-manifolds. It was shown by Penrose [30, Sect. VI] and Bryant [12, Sect. 1] that both twistor spaces  $Z^\pm(S^4)$  can be identified with the complex projective space  $\mathbb{C}\mathbb{P}^3$  with the Fubini-Study metric such that the horizontal bundle is the holomorphic contact bundle  $\xi_{\text{std}} \subset T\mathbb{C}\mathbb{P}^3$  given by  $\alpha$  (1.2). An elementary proof is given in [17, Sect. 6]. It is also known (see Friedrich [22]) that the twistor spaces  $Z^\pm$  of the hyperbolic space  $H^4 = (\mathbb{B}, g_h)$  (1.3) can be identified with the domain  $\Omega \subset \mathbb{C}\mathbb{P}^3$  (1.4) with the contact structure  $\xi$  defined by  $\beta$  (1.5). Since we shall need a more precise understanding of the relevant geometry, we recall the main facts.

Let  $\mathbb{H}$  denote the field of quaternions. An element of  $\mathbb{H}$  is written uniquely as

$$q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = z_1 + z_2\mathbf{j},$$

where  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ ,  $z_1 = x_1 + x_2\mathbf{i} \in \mathbb{C}$ ,  $z_2 = x_3 + x_4\mathbf{i} \in \mathbb{C}$ , and  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the quaternionic units. We identify  $\mathbb{R}^4$  with  $\mathbb{H}$  using  $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$  as the standard basis. Recall that

$$\bar{q} = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}, \quad q\bar{q} = |q|^2 = \sum_{i=1}^4 x_i^2, \quad q^{-1} = \frac{\bar{q}}{|q|^2} \text{ if } q \neq 0, \quad \overline{pq} = \bar{q}\bar{p}.$$

We identify the quaternionic plane  $\mathbb{H}^2$  with  $\mathbb{C}^4$  by

$$(3.1) \quad \mathbb{H}^2 \ni q = (q_1, q_2) = (z_1 + z_2\mathbf{j}, z_3 + z_4\mathbf{j}) = (z_1, z_2, z_3, z_4) = z \in \mathbb{C}^4.$$

Write  $\mathbb{H}_*^2 = \mathbb{H}^2 \setminus \{0\} \cong \mathbb{C}_*^4$ . The situation is described by the following diagram

$$\begin{array}{ccccc} \mathbb{H}_*^2 & \xrightarrow{\phi_1} & \mathbb{C}\mathbb{P}^3 & & \\ & \searrow \phi & \downarrow \pi & & \\ \widehat{\mathbb{R}}^4 & \xleftarrow{\cong} & \mathbb{H}\mathbb{P}^1 & \xrightarrow{\psi} & S^4 \end{array}$$

where  $\widehat{\mathbb{R}}^4 = \mathbb{R}^4 \cup \{\infty\}$ ,  $\phi_1 : \mathbb{H}_*^2 \cong \mathbb{C}_*^4 \rightarrow \mathbb{C}\mathbb{P}^3$  is the canonical projection with fibre  $\mathbb{C}^*$  sending  $q \in \mathbb{H}_*^2$  to the complex line  $\mathbb{C}q \in \mathbb{C}\mathbb{P}^3$ ,  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow \mathbb{H}\mathbb{P}^1$  is the fibre bundle sending a complex line  $\mathbb{C}q$  ( $q \in \mathbb{H}_*^2$ ) to the quaternionic line  $\mathbb{H}q = \mathbb{C}q \oplus \mathbb{C}jq$ , and  $\phi = \pi \circ \phi_1$  sends  $q \in \mathbb{H}_*^2$  to  $\mathbb{H}q \in \mathbb{H}\mathbb{P}^1$ . The fibre  $\pi^{-1}(\pi(\mathbb{C}q))$  is the linear rational curve  $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^3$  of complex lines contained in the quaternionic line  $\mathbb{H}q$ . Thus,  $\mathbb{H}\mathbb{P}^1$  is the one-dimensional quaternionic projective space which we identify with  $\mathbb{H} \cup \{\infty\} = \widehat{\mathbb{R}}^4$  such that the quaternionic line  $\{0\} \times \mathbb{H}$  corresponds to  $\infty$ . The map  $\psi : \widehat{\mathbb{R}}^4 \xrightarrow{\cong} S^4$  is the stereographic projection (2.6). With these identifications, the projection  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow \mathbb{H}\mathbb{P}^1$  is the twistor bundle  $Z^+(S^4) \rightarrow S^4$ . We get the negative twistor bundle  $Z^-(S^4) \rightarrow S^4$  by reversing the orientation on  $S^4$ ; for example, by replacing the stereographic projection  $\psi$  by the one sending  $\infty$  to  $\mathbf{n} = (0, 0, 0, 0, 1) \in S^4 \subset \mathbb{R}^5$ . Using the coordinates (3.1) we have

$$\phi(q_1, q_2) = q_1^{-1}q_2 = \frac{1}{|q_1|^2} \bar{q}_1 q_2 = \frac{1}{|z_1|^2 + |z_2|^2} (\bar{z}_1 z_3 + z_2 \bar{z}_4, \bar{z}_1 z_4 - z_2 \bar{z}_3).$$

Identifying  $\mathbb{H}\mathbb{P}^1 \cong \mathbb{R}^4 \cup \{\infty\} = \mathbb{C}^2 \cup \{\infty\} =: \widehat{\mathbb{C}}^2$  and using complex coordinates  $w = (w_1, w_2) \in \mathbb{C}^2$ , this shows that the twistor projection  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow \widehat{\mathbb{C}}^2$  is given by

$$(3.2) \quad w_1 = \frac{\bar{z}_1 z_3 + z_2 \bar{z}_4}{|z_1|^2 + |z_2|^2}, \quad w_2 = \frac{\bar{z}_1 z_4 - z_2 \bar{z}_3}{|z_1|^2 + |z_2|^2}, \quad |w|^2 = \frac{|z_3|^2 + |z_4|^2}{|z_1|^2 + |z_2|^2} = \frac{|q_2|^2}{|q_1|^2}.$$

The maximal subgroup  $G_s \subset \mathrm{GL}_4(\mathbb{C})$  which passes down to the group of biholomorphic isometries of  $\mathbb{C}\mathbb{P}^3$  in the Fubini-Study metric, and further down to the group of isometries of  $\mathbb{H}\mathbb{P}^1 \cong \widehat{\mathbb{R}}^4$  in the spherical metric  $g_s = \frac{4|x|^2}{(1+|x|^2)^2}$  ( $x \in \mathbb{R}^4$ ), is the group preserving the quaternionic inner product on  $\mathbb{H}^2$  given by

$$(3.3) \quad \mathbb{H}^2 \times \mathbb{H}^2 \ni (p, q) \longmapsto p\bar{q}^t = p_1 \bar{q}_1 + p_2 \bar{q}_2 \in \mathbb{H}.$$

(We consider elements of  $\mathbb{H}^2$  as row vectors acted upon by right multiplication.) Writing

$$(3.4) \quad p = (z_1 + z_2\mathbf{j}, z_3 + z_4\mathbf{j}) = z, \quad q = (v_1 + v_2\mathbf{j}, v_3 + v_4\mathbf{j}) = v,$$

a calculation gives

$$(3.5) \quad p\bar{q}^t = z\bar{v}^t + \alpha_0(z, v)\mathbf{j}, \quad \alpha_0(z, v) = z_2 v_1 - z_1 v_2 + z_4 v_3 - z_3 v_4.$$

Then  $\alpha_0(z, dz) = \alpha$  is the contact form (1.2). Denoting by  $J_0 \in \mathrm{SU}(4)$  the matrix having diagonal blocks  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and zero off-diagonal blocks, we have  $\alpha_0(z, v) = zJ_0v^t$  and hence

$$(3.6) \quad G_s = \{A \in \mathrm{U}(4) : AJ_0A^t = J_0\} = \mathrm{U}(4) \cap \mathrm{Sp}_2(\mathbb{C}),$$

where  $\mathrm{Sp}_2(\mathbb{C})$  denotes the complexified symplectic group.

We now consider the twistor space  $Z^+$  of the hyperbolic space  $H^4 = (\mathbb{B}, g_h)$ . From (3.2) we see that the preimage of  $\mathbb{B}$  by the twistor projection  $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow \widehat{\mathbb{C}}^2$  is the domain

$$(3.7) \quad \Omega = \pi^{-1}(\mathbb{B}) = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathbb{P}^3 : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2\}.$$

Likewise, the preimage  $\Omega' = \pi^{-1}(\mathbb{B}') \subset \mathbb{C}\mathbb{P}^3$  obtained by reversing the inequality in (3.7) is the twistor space of the complementary four-ball  $\mathbb{B}'$  (2.5) with the hyperbolic metric. The common boundary of these two domains is the cone

$$(3.8) \quad \Gamma = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathbb{P}^3 : |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\}$$

whose projection  $\pi(\Gamma)$  is the unit sphere  $b\mathbb{B} = b\mathbb{B}' = S^3 \subset \mathbb{R}^4$ . The maximal subgroup  $G_h$  of  $\mathrm{GL}_4(\mathbb{C})$  descending to a group of holomorphic automorphisms of  $\mathbb{C}\mathbb{P}^3$ , and further down to the group of isometries  $I(\mathbb{B}) = I(\mathbb{B}') \subset M(\widehat{\mathbb{R}}^4)$  of the hyperbolic balls  $\mathbb{B}$  and  $\mathbb{B}'$ , consists of all  $A \in \mathrm{GL}_4(\mathbb{C})$  preserving the indefinite quaternionic inner product

$$\mathbb{H}^2 \times \mathbb{H}^2 \ni (p, q) \longmapsto p * q = p_1\bar{q}_1 - p_2\bar{q}_2 \in \mathbb{H}.$$

Writing  $(p, q)$  in the complex notation (3.4) we have that

$$(3.9) \quad p * q = (z_1\bar{v}_1 + z_2\bar{v}_2 - z_3\bar{v}_3 - z_4\bar{v}_4) + (z_2v_1 - z_1v_2 - z_4v_3 + z_3v_4)j.$$

The subgroup of  $\mathrm{GL}_4(\mathbb{C})$  preserving the first component on the right hand side is  $U(2, 2)$ . Let  $\beta_0(z, v)$  denote the coefficient of  $j$  in (3.9). Note that  $\beta_0(z, dz) = \beta$  is the form (1.5). Let  $J_1 \in \mathrm{SU}(4)$  be the matrix having the diagonal blocks  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and zero off-diagonal blocks. Then  $\beta_0(z, v) = zJ_1v^t$ , so the group we are looking for is

$$(3.10) \quad G_h = \{A \in U(2, 2) : AJ_1A^t = J_1\}.$$

This also shows that the horizontal bundle of the twistor projections  $\pi : \Omega \rightarrow \mathbb{B}$  and  $\pi : \Omega' \rightarrow \mathbb{B}'$  is the kernel  $\xi \subset T\mathbb{C}\mathbb{P}^3$  of the 1-form  $\beta$  (1.5), a contact bundle.

For any  $p = (p_1, p_2) \in \mathbb{H}_*^2$  the fibre  $\pi^{-1}(\phi(p)) \subset \mathbb{C}\mathbb{P}^3$  is the space of all complex lines contained in the quaternionic line  $\mathbb{H}p$ . The tangent space to this fibre at any point is spanned by a vector  $q = ap$  for some imaginary quaternion  $a \in \mathbb{H}$  with  $|a| = 1$ . From (3.9) we get

$$p * q = p_1\overline{ap_1} - p_2\overline{ap_2} = p_1\bar{p}_1\bar{a} - p_2\bar{p}_2\bar{a} = (p * p)\bar{a}.$$

This vanishes for all  $a \in \mathbb{H}$  precisely when  $|p_1|^2 = |p_2|^2$  which is equivalent to  $\phi_1(p) \in \Gamma = \pi^{-1}(b\mathbb{B})$  (3.8). It follows that for every point  $x \in b\mathbb{B} = b\mathbb{B}'$  the fibre  $\pi^{-1}(x) \subset \Gamma$  is a  $\xi$ -Legendrian curve. This is in strong contrast to the situation for the contact bundle  $\xi_{\mathrm{std}}$  which is transverse to all fibres of  $\pi$ . This difference reflects the fact that the hyperbolic metrics on  $\mathbb{B}$  and  $\mathbb{B}'$  blow up along their common boundary sphere.

The above discussion is illustrated by the following diagram, where  $G_h$  is the group (3.10) and  $M(\mathbb{B})$  is the Möbius group (the isometry group) of  $\mathbb{B}$  introduced in Sect. 2.

$$\begin{array}{ccccc} \mathbb{H}_*^2 & \xrightarrow{A \in G_h} & \mathbb{H}_*^2 & & \\ \downarrow \phi_1 & & \downarrow \phi_1 & & \\ \Omega & \xrightarrow{\quad} & \mathbb{C}\mathbb{P}^3 & \xrightarrow{\tilde{A} \in \mathrm{PGL}(4)} & \mathbb{C}\mathbb{P}^3 & \xrightarrow{\quad} & \Omega \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ \mathbb{B} & \xrightarrow{\quad} & \widehat{\mathbb{R}}^4 & \xrightarrow{A' \in M(\mathbb{B})} & \widehat{\mathbb{R}}^4 & \xrightarrow{\quad} & \mathbb{B} \end{array}$$

The negative twistor bundle  $Z^-(\mathbb{B})$  is the positive twistor bundle of  $\mathbb{B}$  with the opposite orientation; it can still be identified with  $(\Omega, \beta)$ . There is however no need to consider it since an orientation reversing isometry  $\tau : \mathbb{B} \rightarrow \mathbb{B}$  (for example, a reflection in a hyperplane of  $\mathbb{R}^4$  through the origin) maps any conformal superminimal surface  $f : M \rightarrow \mathbb{B}$  of negative spin to a conformal superminimal surface  $\tau \circ f : M \rightarrow \mathbb{B}$  of positive spin and vice versa; hence it suffices to consider superminimal surfaces of positive spin.

**Proposition 3.1.** *Every oriented hyperbolic 2-plane  $\Lambda(b, V) \subset \mathbb{B} = \mathbb{B}^4$  in Proposition 2.1 is a totally geodesic superminimal surface in  $(\mathbb{B}, g_h)$ , hence a superminimal surface of both positive and negative spin. Its twistor lift to the domain  $\Omega \subset \mathbb{C}\mathbb{P}^3$  (3.7) is the intersection of  $\Omega$  with a linear  $\xi$ -Legendrian rational curve  $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^3$ .*

*Proof.* For any two-plane  $0 \in V \subset \mathbb{R}^4$ ,  $\Lambda(0, V) = \mathbb{B} \cap V$  is a hyperbolic disc in the metric  $g_h$ . Given a circle  $C \subset V$  intersecting  $b\mathbb{B} \cap V$  orthogonally in  $V$ ,  $C$  also intersects  $b\mathbb{B}$  orthogonally in  $\mathbb{R}^4$ , so  $C \cap \mathbb{B}$  is a geodesic of  $(\mathbb{B}, g_h)$ . This shows that  $\mathbb{B} \cap V$  is a totally geodesic surface in  $\mathbb{B}$ , hence superminimal with all circles  $I_x(v)$  in Def. 1.1 reducing to points. (In particular,  $\mathbb{B} \cap V$  with any orientation is superminimal of both  $\pm$  spin.) Taking  $V = \mathbb{R}^2 \times \{0\}^2 = \mathbb{C} \times \{0\}$  and the parameterization  $f(\zeta) = (\zeta, 0) \in \mathbb{B} \cap V$  for  $\zeta \in \mathbb{D} = \{|\zeta| < 1\}$ , we see from (3.2) that the holomorphic  $\xi$ -Legendrian embedding

$$F : \mathbb{C}\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \hookrightarrow \mathbb{C}\mathbb{P}^3, \quad F(\zeta) = [1 : 0 : \zeta : 0]$$

restricted to the disc  $\mathbb{D}$  is the twistor lift of  $f$ . (Note that  $F$  is also  $\xi_{\text{std}}$ -Legendrian, so this particular map  $f$  is also a superminimal surface in  $S^4$  with the spherical metric.) Reversing the orientation on  $V = \mathbb{R}^2 \times \{0\}^2$ , a conformal orientation preserving parameterization of  $\mathbb{B} \cap V$  is  $f(\zeta) = (\bar{\zeta}, 0)$  ( $\zeta \in \mathbb{D}$ ) with the twistor lift  $F(\zeta) = [0 : 1 : 0 : \zeta] \in \Omega$ .

Any other hyperbolic surface  $\Lambda(b, V)$  can be obtained from  $\mathbb{B} \cap (\mathbb{R}^2 \times \{0\}^2)$  by an orientation preserving isometry of  $\mathbb{B}$ . Indeed, we get other planes through 0 by orthogonal rotations in  $\text{SO}(4)$ , and for  $0 \neq b \in \mathbb{B}$  we have that  $\Lambda(b, V) = \tau_b(\mathbb{B} \cap V)$  (cf. (2.8)) where  $\tau_b \in \text{M}(\mathbb{B})$  is the orientation preserving hyperbolic translation (2.7). Since every orientation preserving isometry of  $\mathbb{B}$  lifts to a holomorphic contactomorphism of  $(\mathbb{C}\mathbb{P}^3, \xi)$  preserving the domain  $\Omega$ , the same conclusion holds for the twistor lift of every surface  $\Lambda(b, V)$ .  $\square$

#### 4. The Riemann-Hilbert method for Legendrian curves

In this section we develop the Riemann-Hilbert deformation method for holomorphic Legendrian curves in complex projective spaces; see Theorem 4.1.

The Riemann-Hilbert deformation method for holomorphic curves and related geometric objects is a very useful tool in global constructions of such object having certain additional properties. A particularly useful feature of this technique is that it offers a precise geometric control of the placement of the object into the ambient space; this is especially helpful if one aims to preserve its conformal (complex) structure without having to cut away pieces of it during an inductive construction. It is therefore not surprising that this technique has been used in constructions of proper holomorphic maps from bordered Riemann surfaces into an optimal class of complex manifolds and complex spaces (see [14] and the references therein), of complete holomorphic curves which are either proper in a given domain or contained in a small neighbourhood of a given curve (see [3]), in the Poletsky theory of disc functionals (see [15]), and others. In recent years this method has been adapted to several other geometries, in particular to conformal minimal surfaces in Euclidean spaces  $\mathbb{R}^n$  and holomorphic null curves in  $\mathbb{C}^n$  for any  $n \geq 3$  (see the survey [6]) and to holomorphic Legendrian curves in  $\mathbb{C}^{2n+1}$  with its standard complex contact structure (see [8]).

The following is the main result of this section. Since the contact structure on  $\mathbb{C}\mathbb{P}^{2n+1}$  is unique up to holomorphic contactomorphisms (see [29, Corollary 2.3]), the precise choice of the contact bundle is irrelevant.

**Theorem 4.1** (The Riemann-Hilbert method for Legendrian curves in  $\mathbb{C}\mathbb{P}^{2n+1}$ ). *Assume that  $M$  is a compact bordered Riemann surface,  $I \subset bM$  is an arc which is not a boundary component of  $M$ ,  $f : M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  is a Legendrian map of class  $\mathcal{A}^1(M) = \mathcal{C}^1(M) \cap \mathcal{O}(M)$ , and for every  $u \in bM$  the map  $\overline{\mathbb{D}} \ni v \mapsto F(u, v) \in \mathbb{C}\mathbb{P}^{2n+1}$  is a Legendrian disc of class  $\mathcal{A}^1(\overline{\mathbb{D}})$  depending continuously on  $u \in bM$  such that  $F(u, 0) = f(u)$  for all  $u \in bM$  and  $F(u, v) = f(u)$  for all  $u \in bM \setminus I$  and  $v \in \overline{\mathbb{D}}$ . Assume that there is a projective hyperplane  $H \subset \mathbb{C}\mathbb{P}^{2n+1}$  which avoids the compact set  $\bigcup_{u \in I} F(u, \overline{\mathbb{D}})$ . Given a number  $\epsilon > 0$  and a neighbourhood  $U \subset M$  of the arc  $I$ , there exist a holomorphic Legendrian map  $\tilde{f} : M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  and a neighbourhood  $V \Subset U$  of  $I$  with a smooth retraction  $\tau : V \rightarrow V \cap bM$  such that the following conditions hold.*

- (i)  $\text{dist}(\tilde{f}(u), f(u)) < \epsilon$  for all  $u \in M \setminus V$ .
- (ii)  $\text{dist}(\tilde{f}(u), F(u, b\mathbb{D})) < \epsilon$  for all  $u \in bM$ .
- (iii)  $\text{dist}(\tilde{f}(u), F(\tau(u), \overline{\mathbb{D}})) < \epsilon$  for all  $u \in V$ .
- (iv)  $\tilde{f}$  agrees with  $f$  to a given finite order on a given finite set of points in  $\overset{\circ}{M}$ .

Recall that a map from a compact bordered Riemann surface  $M$  to a complex manifold  $X$  is called holomorphic if it extends to a holomorphic map  $U \rightarrow X$  from an open neighbourhood of  $M$  in an ambient Riemann surface  $R$ .

*Proof.* We adapt the proof of [8, Theorem 3.3] (which applies to holomorphic Legendrian curves in  $\mathbb{C}^{2n+1}$ ) to the projective case. For simplicity of notation we consider the case  $n = 1$ ; the same proof applies in general with obvious modifications.

We begin with a few reductions. We may assume that  $M$  is connected,  $f$  is nonconstant, and its image  $f(M)$  is not contained in the affine chart  $\mathbb{C}\mathbb{P}^3 \setminus H$ , for otherwise the result follows from [8, Theorem 3.3]. The special case when  $M = \overline{\mathbb{D}}$  and the entire configuration is contained in an affine chart  $\mathbb{C}^3 \subset \mathbb{C}\mathbb{P}^3$  is furnished by [8, Lemma 3.2].

By Bertini's theorem (see [24, p. 150] or [27] and note that this is an application of the transversality theorem) we can move the hyperplane  $H$  slightly to ensure that it intersects  $f(M)$  transversely and it does not meet the compact set  $f(bM) \cup \bigcup_{p \in I} F(p, b\mathbb{D})$ . In particular,  $f$  is an immersion at any point  $p \in \overset{\circ}{M}$  with  $f(p) \in H$ ; the set  $P$  of all such points is finite and contained in  $\overset{\circ}{M}$ . Choose a closed smoothly bounded simply connected domain  $D \subset U$  such that  $D$  is a neighbourhood of the arc  $I$  and  $f(D) \cap H = \emptyset$ . By denting  $bM$  inward along a neighbourhood of the arc  $I$  we find a smoothly bounded compact domain  $M' \subset M$  diffeotopic to  $M$  and such that

$$M = M' \cup D \quad \text{and} \quad \overline{M' \setminus D} \cap \overline{D \setminus M'} = \emptyset.$$

Thus,  $(M', D)$  is a *Cartan pair* (cf. [19, Definition 5.7.1]). Note that  $P \subset M'$ .

By [2, Proposition 2.2] there are homogeneous coordinates  $[z_0 : z_1 : z_2 : z_3]$  on  $\mathbb{C}\mathbb{P}^3$  with  $H = \{z_0 = 0\}$  such that the contact form on  $\mathbb{C}\mathbb{P}^3 \setminus H \cong \mathbb{C}^3 = \{z_0 = 1\}$  is given by  $dz_1 + z_2 dz_3 - z_3 dz_2$ , and in these coordinates  $f$  is of the form

$$(4.1) \quad f = \mathcal{F}(g, h) = [1 : e : g : h], \quad e = gh - 2 \int g dh = 2 \int h dg - gh,$$

where  $g, h : M \rightarrow \mathbb{C}\mathbb{P}^1$  are meromorphic functions on  $M$  having at most simple poles at the points in  $P$  (this reflects the fact that  $f$  intersects  $H$  transversely at these points and hence is an immersion there) and of class  $\mathcal{C}^1$  near the boundary of  $M$  (in particular,

$g$  and  $h$  are holomorphic on  $\mathring{M} \setminus P$ ), and  $gdh$  is an exact meromorphic 1-form with a meromorphic primitive  $\int gdh$  determined up to an additive constant. In fact, every holomorphic Legendrian map into  $\mathbb{CP}^3$  intersecting the hyperplane  $H$  transversely is of this form, and such an  $f$  is an immersion if and only if the map  $(g, h) : M \setminus P \rightarrow \mathbb{C}^2$  is an immersion (cf. [2, Corollary 2.3]).

The meromorphic 1-form  $gdh$  on  $M$  is exact if and only if  $\int_C gdh = 0$  for every closed curve  $C$  in  $\mathring{M} \setminus P$ . There are two types of curves to consider: those in a homology basis of  $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^l$ , say  $C_1, \dots, C_l$ , and small loops around the poles of  $gdh$ . Since  $M'$  is a deformation retract of  $M$ , the curves  $C_i$  forming a homology basis of  $M$  can be chosen in  $\mathring{M}' \setminus P$  and such that they meet at a single point  $p_0 \in \mathring{M}'$  and their union  $\bigcup_{i=1}^l C_i$  is Runge in  $M$  (i.e., holomorphically convex in  $M$ ). The integral of  $gdh$  along a small Jordan curve around a pole  $a \in P$  equals  $2\pi i \operatorname{Res}_a(gdh)$ . Assuming that  $a$  is a simple pole of  $g$  or  $h$  (as is the case in our situation), vanishing of this integral is equivalent to

$$(4.2) \quad c_{-1}(h, a)c_1(g, a) - c_{-1}(g, a)c_1(h, a) = \operatorname{Res}_a(gdh) = 0,$$

where  $c_k(h, a)$  denotes the coefficient of  $(\zeta - a)^k$  in the Laurent series for  $h$  at  $a$  in a local holomorphic coordinate  $\zeta$  (so  $c_{-1}(h, a) = \operatorname{Res}_a h$ ); see [2, Proposition 2.4]. Clearly these vanishing conditions are preserved if we replace  $(g, h)$  by any pair  $(g', h')$  of meromorphic functions which agrees with  $(g, h)$  to the second order at every point  $a \in P$ .

Let  $\mathcal{A}^1(M; P, g)$  denote the space of meromorphic functions on  $M$  which are of class  $\mathcal{C}^1$  up to the boundary, they have poles only at the points of  $P$ , and they agree with  $g$  to the second order at each point of  $P$  (i.e., the difference has a second order zero). Likewise,  $\mathcal{A}^1(M; P, h)$  denotes the corresponding space for  $h$ . Consider the *period map*

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_l) : \mathcal{A}^1(M; P, g) \times \mathcal{A}^1(M; P, h) \rightarrow \mathbb{C}^l$$

whose  $j$ -th component equals

$$(4.3) \quad \mathcal{P}_j(x, y) = \int_{C_j} x dy, \quad x \in \mathcal{A}^1(M; P, g), \quad y \in \mathcal{A}^1(M; P, h).$$

Note that  $\mathcal{P}(x, y) = 0$  if and only if the 1-form  $x dy$  is exact if and only if the map  $\mathcal{F}(x, y) : M \rightarrow \mathbb{CP}^3$  (4.1) is a holomorphic Legendrian curve. Exactness at the points of  $P$  is ensured by (4.2) and the definition of the spaces  $\mathcal{A}^1(M; P, g)$  and  $\mathcal{A}^1(M; P, h)$ .

The idea of proof is to first use the Riemann-Hilbert deformation method for holomorphic curves without paying attention to the Legendrian condition. Applying this technique to the central curve  $f : M \rightarrow \mathbb{CP}^3$  and the family of boundary discs  $F(u, \cdot)$  yields a new holomorphic curve  $\tilde{f} : M \rightarrow \mathbb{CP}^3$  which satisfies the conditions in Theorem 4.1 but is not necessarily Legendrian. In fact, as shown in [8, proof of Lemma 3.2] the deviation from the Legendrian condition is not even pointwise small due to the fast turning of the curve  $\tilde{f}$  from being close to  $f$  on  $M \setminus V$  (see condition (i)) to being close to the union of the boundary discs  $F(u, \cdot)$  when the point of  $M$  approaches the boundary arc  $I$  (see conditions (ii) and (iii)). However, what makes the method feasible is that the integral of the error is uniformly small, and hence it is possible to correct it and find nearby a Legendrian solution. For technical reasons which will become apparent in the proof, we shall apply the Riemann-Hilbert deformation method not only to a single data, but to a holomorphically varying collection (a spray) of data of the same kind which we shall now construct. By doing things right, the new family of curves will still satisfy the approximation conditions for small values of the parameter, and the family will contain a Legendrian curve. We now explain the details of this idea.

Since the map  $f = [1 : e : g : h]$  (4.1) is nonconstant, one of the components  $g, h$  is nonconstant. Assume that  $h$  is nonconstant; the other case can be handled by a symmetric argument. Then,  $h|_{C_j}$  is nonconstant for any  $j = 1, \dots, l$  by the identity principle. Since the compact set  $\bigcup_{j=1}^l C_j$  is Runge in  $M$  and every pair of curves  $C_i, C_j$  with  $i \neq j$  only meet at a point, we can find holomorphic functions  $g_1, \dots, g_l$  on  $M$  vanishing to the second order at every point of  $P$  such that for every  $j, k = 1, \dots, l$  the number  $\int_{C_j} g_k dh \approx \delta_{j,k}$  is arbitrarily close to 1 if  $j = k$  and to 0 if  $j \neq k$ . (Here,  $\delta_{j,k}$  is the Kronecker symbol. To find such function, we first construct smooth functions  $g_k$  on  $\bigcup_{j=1}^l C_j$  such that  $\int_{C_j} g_k dh = \delta_{j,k}$  and then use Mergelyan's approximation theorem and Weierstrass's interpolation theorem to approximate them by holomorphic functions with the stated properties on  $M$ . The elementary details are left to the reader; see [4, Lemma 5.1] or [6, Lemma 3.2] for the details in a similar situation when constructing minimal surfaces in  $\mathbb{R}^n$ .) Let  $\zeta = (\zeta_1, \dots, \zeta_l) \in \mathbb{C}^l$ . Consider the meromorphic function  $\hat{g} : M \times \mathbb{C}^l \rightarrow \mathbb{C}\mathbb{P}^1$  given by

$$(4.4) \quad \hat{g}(u, \zeta) = g(u) + \sum_{k=1}^l \zeta_k g_k(u), \quad u \in M, \zeta \in \mathbb{C}^l.$$

Note that  $\hat{g}(\cdot, \zeta) \in \mathcal{A}^1(M; P, g)$  for every fixed  $\zeta$ . For all  $j, k \in \{1, \dots, l\}$  we have

$$(4.5) \quad \left. \frac{\partial}{\partial \zeta_k} \right|_{\zeta=0} \int_{C_j} \hat{g}(\cdot, \zeta) dh = \int_{C_j} g_k dh \approx \delta_{j,k}.$$

If the above approximations are close enough then

$$\left. \frac{\partial}{\partial \zeta} \right|_{\zeta=0} \mathcal{P}(\hat{g}(\cdot, \zeta), h) : \mathbb{C}^l \longrightarrow \mathbb{C}^l \text{ is an isomorphism.}$$

(A map (4.4) with this property is called a *period dominating holomorphic spray* with the core  $\hat{g}(\cdot, 0) = g$ .) By the inverse function theorem there is a ball  $r\mathbb{B} \subset \mathbb{C}^l$  around the origin such that the period map  $r\mathbb{B} \ni \zeta \mapsto \mathcal{P}(\hat{g}(\cdot, \zeta), h) \in \mathbb{C}^l$  is biholomorphic onto its image, the latter being a neighbourhood of the origin in  $\mathbb{C}^l$ .

Fix a point  $u_0 \in \mathring{D}$ . Consider the function  $\tilde{e} : D \times \mathbb{C}^l \rightarrow \mathbb{C}$  given by

$$\tilde{e}(u, \zeta) = \hat{g}(u, \zeta)h(u) - 2 \int_{u_0}^u \hat{g}(\cdot, \zeta) dh + c_0, \quad u \in D, \zeta \in \mathbb{C}^l,$$

where the constant  $c_0 \in \mathbb{C}$  is chosen such that  $\tilde{e}(u_0, 0) = e(u_0)$ , and hence  $\tilde{e}(\cdot, 0) = e|_D$ . (The integral is independent of the path in the disc  $D$ . It is however impossible to extend  $\tilde{e}(\cdot, \zeta)$  to all of  $M$  since the 1-form  $\hat{g}(\cdot, \zeta) dh$  has nonvanishing periods for  $\zeta \neq 0$ .) Let  $\hat{f} : D \times \mathbb{C}^l \rightarrow \mathbb{C}^3$  be the family of holomorphic Legendrian discs

$$(4.6) \quad D \ni u \mapsto \hat{f}(u, \zeta) = [1 : \tilde{e}(u, \zeta) : \hat{g}(u, \zeta) : h(u)] \in \mathbb{C}\mathbb{P}^3$$

of the form (4.1) and depending holomorphically on  $\zeta \in \mathbb{C}^l$ . Note that  $\hat{f}(u, 0) = f(u)$  for  $u \in D$ . Since these discs lie in the affine chart  $\mathbb{C}\mathbb{P}^3 \setminus H$ , we delete the initial component 1 and consider them as discs in  $\mathbb{C}^3$ . Using the same affine coordinates, we write the given Legendrian discs  $F(u, \cdot)$  in the theorem as

$$F(u, v) = (Z(u, v), X(u, v), Y(u, v)), \quad u \in bM, v \in \overline{\mathbb{D}}.$$

In view of (4.1) we have that

$$Z(u, v) = X(u, v)Y(u, v) - 2 \int_0^v X(u, t)dY(u, t) + e(u) - g(u)h(u).$$

For each point  $u \in bD \cap bM$  and for every  $\zeta \in \mathbb{C}^l$  we let

$$\overline{\mathbb{D}} \ni v \mapsto \widehat{F}(u, v, \zeta) = (\widehat{Z}(u, v, \zeta), \widehat{X}(u, v, \zeta), Y(u, v)) \in \mathbb{C}^3$$

be the Legendrian disc of class  $\mathcal{A}^1(\overline{\mathbb{D}})$  given by

$$\begin{aligned} \widehat{X}(u, v, \zeta) &= X(u, v) + \widehat{g}(u, \zeta) - g(u), \\ \widehat{Z}(u, v, \zeta) &= \widehat{X}(u, v, \zeta)Y(u, v) - 2 \int_{t=0}^{t=v} \widehat{X}(u, t, \zeta) dY(u, t) + \widehat{e}(u, \zeta) - \widehat{g}(u, \zeta)h(u). \end{aligned}$$

Note that  $\widehat{F}(u, v, 0) = F(u, v)$  and

$$\widehat{F}(u, 0, \zeta) = \widehat{f}(u, \zeta), \quad u \in bD \cap bM, \quad \zeta \in \mathbb{C}^l.$$

Finally, for every point  $u \in bD \cap bM \setminus I$  and for all  $\zeta \in \mathbb{C}^l$  we have that

$$\widehat{F}(u, v, \zeta) = \widehat{F}(u, 0, \zeta) = \widehat{f}(u, \zeta), \quad v \in \overline{\mathbb{D}},$$

so  $\widehat{F}(u, \cdot, \zeta)$  is the constant disc. We extend  $\widehat{F}$  to all points  $u \in bD$  by setting

$$\widehat{F}(u, v, \zeta) = \widehat{f}(u, \zeta) \text{ for all } u \in bD \setminus I, v \in \overline{\mathbb{D}} \text{ and } \zeta \in \mathbb{C}^l.$$

Note that for every fixed  $\zeta \in \mathbb{C}^l$  the Legendrian disc  $\widehat{f}(\cdot, \zeta): D \rightarrow \mathbb{C}^3$  and the family of Legendrian discs  $\widehat{F}(u, \cdot, \zeta): \overline{\mathbb{D}} \rightarrow \mathbb{C}^3$  ( $u \in bD$ ) satisfy the assumptions of [8, Lemma 3.2] on  $D$  (which is conformally diffeomorphic to the standard disc  $\overline{\mathbb{D}}$ ), and both families depend holomorphically on  $\zeta \in \mathbb{C}^l$ . Hence, [8, Lemma 3.2] furnishes a family of Legendrian discs

$$\widehat{G}(\cdot, \zeta) = (\widehat{G}_1, \widehat{G}_2, \widehat{G}_3): D \rightarrow \mathbb{C}^3$$

depending holomorphically on  $\zeta$  and satisfying the estimates in the lemma uniformly with respect to  $\zeta \in r\mathbb{B}$ . (These estimates are of the same type as those in conditions (i)–(iii) of Theorem 4.1 with  $M$  replaced by  $D$ . The observation regarding holomorphic dependence and uniformity of the estimates with respect to  $\zeta$  is evident from [8, proof of Lemma 3.2] and has also been used in [8, proof of Theorem 3.3].)

Let  $V \subset D \setminus M'$  be a small neighbourhood of the arc  $I \subset bM$ . By [8, Lemma 3.2 (iv)] we may assume that  $\widehat{G}(\cdot, \zeta)$  is as close as desired to  $\widehat{f}(\cdot, \zeta)$  in the  $\mathcal{C}^1$  norm on  $D \setminus V$ , and hence on  $M' \cap D \subset D \setminus V$  for all  $\zeta \in r\mathbb{B}$ . In particular, given  $\delta > 0$  we may assume that

$$\|\widehat{G}(\cdot, \zeta) - \widehat{f}(\cdot, \zeta)\|_{\mathcal{C}^1(M' \cap D)} < \delta, \quad \zeta \in r\mathbb{B}.$$

Recall that the component  $\widehat{g}$  of  $\widehat{f}$  (4.6) is a meromorphic function on  $M \times \mathbb{C}^l$  with poles only on  $P \times \mathbb{C}^l$ . By solving a Cousin-I problem with bounds on the Cartan pair  $(M', D)$ , with interpolation on  $P$ , we can glue  $\widehat{g}$  and  $\widehat{G}_2$  into a function  $H_2(\cdot, \zeta): M \rightarrow \mathbb{C}P^1$  of class  $\mathcal{A}^1(M; P, g)$ , holomorphic in  $\zeta \in r\mathbb{B}$  and satisfying the estimates

$$\|H_2(\cdot, \zeta) - \widehat{g}(\cdot, \zeta)\|_{\mathcal{C}^1(M')} < C\delta, \quad \|H_2(\cdot, \zeta) - \widehat{G}_2(\cdot, \zeta)\|_{\mathcal{C}^1(D)} < C\delta,$$

where the constant  $C > 0$  only depends on the Cartan pair  $(M', D)$ . By the same token, we can glue the last component  $h$  of  $\widehat{f}$  with the function  $\widehat{G}_3(\cdot, \zeta)$  into a function  $H_3(\cdot, \zeta)$  of class  $\mathcal{A}^1(M; P, h)$ , depending holomorphically on  $\zeta \in r\mathbb{B}$  and satisfying the estimates

$$\|H_3(\cdot, \zeta) - h\|_{\mathcal{C}^1(M')} < C\delta, \quad \|H_3(\cdot, \zeta) - \widehat{G}_3(\cdot, \zeta)\|_{\mathcal{C}^1(D)} < C\delta.$$

Since  $\bigcup_{i=1}^l C_i \subset M' \setminus P$ , it follows that the period map  $\zeta \mapsto \mathcal{P}(H_2(\cdot, \zeta), H_3(\cdot, \zeta))$  (see (4.3)) approximates the biholomorphic map  $\zeta \mapsto \mathcal{P}(\widehat{g}(\cdot, \zeta), h)$  uniformly on  $\zeta \in r\mathbb{B}$ .

Assuming that  $\delta > 0$  is chosen small enough, there is a point  $\zeta' \in r\mathbb{B}$  as close to the origin as desired such that

$$\mathcal{P}(H_2(\cdot, \zeta'), H_3(\cdot, \zeta')) = 0.$$

Setting  $\tilde{g} = H_2(\cdot, \zeta')$ ,  $\tilde{h} = H_3(\cdot, \zeta')$  we obtain a holomorphic Legendrian curve

$$\tilde{f} = [1 : \tilde{e} : \tilde{g} : \tilde{h}] : M \rightarrow \mathbb{C}\mathbb{P}^3$$

of the form (4.1) with  $\tilde{e}(u_0) = e(u_0)$ . It follows from the construction that  $\tilde{f}$  satisfies conditions (i)–(iii) of Theorem 4.1 provided that the approximations were close enough.

In order to ensure also the interpolation condition (iv) at finitely many points  $A = \{a_1, \dots, a_k\} \subset \overset{\circ}{M}$ , we amend the above procedure as follows. First, we choose the hyperplane  $H$  at the beginning of the proof such that, in addition to the other stated conditions, it does not intersect the finite set  $f(A)$ , and we choose the disc  $D$  as above and contained in  $M \setminus A$ . Pick a base point  $u_0 \in \overset{\circ}{D}$ . We connect  $u_0$  to each point  $a_j \in A$  by an embedded oriented arc  $E_j \subset \overset{\circ}{M} \setminus P$  which exits  $D$  only once and such that any two of these arcs only meet at  $u_0$ . It follows that the inclusion  $M \setminus (D \cup \bigcup_{i=1}^k E_i) \subset M$  is a homotopy equivalence, and hence we can choose curves  $C_1, \dots, C_l$  forming a homology basis of  $M$  contained in the complement of  $D \cup P \cup \bigcup_{i=1}^k E_i$ . To the period map  $\mathcal{P}$  (4.3) we add  $k$  additional components given by the integrals over the arcs  $E_1, \dots, E_k$ . The rest of the proof remains unchanged. By ensuring that the integrals of the 1-form  $\tilde{g}\tilde{d}\tilde{h}$  over the arcs  $E_1, \dots, E_k$  equal those of  $gdh$ , the map  $\tilde{f}$  agrees with  $f$  at the points of  $A$ . By the same tools we can obtain finite order interpolation on  $A$ .  $\square$

## 5. A general position theorem for Legendrian curves in projective spaces

Holomorphic Legendrian curves obtained by Riemann-Hilbert modifications in the previous section typically have branch points. However, in the application of this method to the proof of Theorem 1.2 we need Legendrian immersions.

The purpose of this section is to explain the following general position theorem for holomorphic Legendrian curves in projective spaces. As was already pointed out in the previous section,  $\mathbb{C}\mathbb{P}^{2n+1}$  admits a unique complex contact structure up to holomorphic contactomorphisms a hence a concrete choice of the contact bundle is irrelevant.

- Theorem 5.1.** (a) *Let  $M$  be a compact bordered Riemann surface. Every Legendrian curve  $f : M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  of class  $\mathcal{A}^1(M)$  can be approximated in the  $\mathcal{C}^1(M, \mathbb{C}\mathbb{P}^{2n+1})$  topology by holomorphic Legendrian embeddings  $\tilde{f} : M \hookrightarrow \mathbb{C}\mathbb{P}^{2n+1}$ .*
- (b) *Every holomorphic Legendrian curve  $f : M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  from an open Riemann surface can be approximated uniformly on compacts in  $M$  by holomorphic Legendrian embeddings  $\tilde{f} : M \hookrightarrow \mathbb{C}\mathbb{P}^{2n+1}$ .*

The analogue of this result for Legendrian curves in  $\mathbb{C}^{2n+1}$  with its standard complex contact structure was proved in [8, Theorem 5.1] where it was shown in addition that the approximating embedding  $\tilde{f} : M \hookrightarrow \mathbb{C}^{2n+1}$  in case (b) can be chosen proper. (The latter condition is of course impossible in the compact manifold  $\mathbb{C}\mathbb{P}^{2n+1}$ .) The cited result also gives approximation of generalised Legendrian curves  $S \rightarrow \mathbb{C}^{2n+1}$  on compact Runge admissible sets  $S$  in an open Riemann surface  $M$ ; this can be extended to Legendrian curves in  $\mathbb{C}\mathbb{P}^{2n+1}$  as well, but we shall not need it in the present paper.

*Proof.* It was shown in [2, Corollary 3.7] that every holomorphic Legendrian immersion  $M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  from an open Riemann surface  $M$  can be approximated uniformly on compacts by holomorphic Legendrian embeddings  $M \hookrightarrow \mathbb{C}\mathbb{P}^{2n+1}$ . The proof combines [5, Theorem 1.2] to the effect that every holomorphic Legendrian immersion  $M \rightarrow X$  from an open Riemann surface to an arbitrary complex contact manifold can be approximated, uniformly on any compact subset  $K$  of  $M$ , by holomorphic Legendrian embeddings  $U \hookrightarrow X$  from open neighbourhoods  $U \subset M$  of  $K$ , and the approximation theorem for holomorphic Legendrian immersions into projective spaces given by [2, Theorem 3.4].

To prove the theorem, it remains to show that one can approximate any Legendrian map  $f : M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  of class  $\mathcal{A}^1(M)$  by holomorphic Legendrian immersions  $U \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  from open neighbourhoods  $U$  of  $M$  in an ambient Riemann surface. For the convenience of notation we consider curves in  $\mathbb{C}\mathbb{P}^3$ , although this restriction is inessential. As in the proof of Theorem 4.1 we find a projective hyperplane  $H \subset \mathbb{C}\mathbb{P}^3$  intersecting  $f(M)$  transversely in at most finitely many points  $P \subset \mathring{M}$  and not intersecting  $f(bM)$ , and homogeneous coordinates  $[z_0 : z_1 : z_2 : z_3]$  with  $H = \{z_0 = 0\}$  in which  $f = \mathcal{F}(g, h) = [1 : gh - 2 \int gdh : g : h]$  (cf. (4.1)), where  $g, h : M \rightarrow \mathbb{C}\mathbb{P}^1$  are meromorphic functions having only simple poles at the points in  $P$  and of class  $\mathcal{C}^1$  near the boundary of  $M$ . A map  $f$  of this form is an immersion if and only if  $(g, h) : M \setminus P \rightarrow \mathbb{C}^2$  is an immersion (cf. [2, Corollary 2.3]). It now suffices to approximate the map  $(g, h) : M \rightarrow (\mathbb{C}\mathbb{P}^1)^2$  as closely as desired in  $\mathcal{C}^1(M, (\mathbb{C}\mathbb{P}^1)^2)$  by a meromorphic map  $(\tilde{g}, \tilde{h}) : U \rightarrow (\mathbb{C}\mathbb{P}^1)^2$  defined on a neighbourhood  $U \subset R$  of  $M$  such that  $(\tilde{g}, \tilde{h})$  agrees with  $(g, h)$  to the second order at every point of  $P$ , it is a holomorphic immersion  $U \setminus P \rightarrow \mathbb{C}^2$ , and the meromorphic 1-form  $\tilde{g}d\tilde{h}$  is exact. We have seen in the proof of Theorem 4.1 that the interpolation condition on  $P$  ensures that  $\tilde{g}d\tilde{h}$  has a local meromorphic primitive at every point of  $P$ ; see (4.2). Therefore, exactness of  $\tilde{g}d\tilde{h}$  is equivalent to the period vanishing conditions  $\int_{C_i} \tilde{g}d\tilde{h} = 0$  ( $i = 1, \dots, l$ ) where  $C_1, \dots, C_l \subset \mathring{M} \setminus P$  is a basis of the homology group  $H_1(M, \mathbb{Z}) = \mathbb{Z}^l$ .

The construction of  $(\tilde{g}, \tilde{h})$  satisfying these conditions is made in two steps. In the first step we approximate  $(g, h)$  by a meromorphic map  $(\hat{g}, \hat{h})$  defined on a neighbourhood  $U \subset R$  of  $M$  which agrees with  $(g, h)$  to the second order at the points of  $P$ , it has no poles on  $M \setminus P$ , and such that  $\hat{g}d\hat{h}$  is exact. This is achieved by following the proof of [8, Lemma 4.3], the only addition being the presence of poles at the points in  $P$  and the interpolation condition at these points. Next, we follow the first part of the proof of [8, Lemma 4.4] in order to approximate  $(\hat{g}, \hat{h})$  on  $M$  and interpolate it to the second order on  $P$  by a meromorphic map  $(\tilde{g}, \tilde{h})$  on a neighbourhood of  $M$  which is a holomorphic immersion of  $M \setminus P$  into  $\mathbb{C}^2$  and such that  $\tilde{g}d\tilde{h}$  is an exact meromorphic 1-form. By what has been said, the associated map  $\tilde{f} = \mathcal{F}(\tilde{g}, \tilde{h}) : M \rightarrow \mathbb{C}\mathbb{P}^3$  defined by (4.1) is then a holomorphic immersion. Both proofs alluded to above easily extend to the present setting in essentially the same way as was done in the proof of Theorem 4.1, and we leave the details to the reader.  $\square$

**Problem 5.2.** Does part (a) of Theorem 5.1 hold for Legendrian curves in an arbitrary complex contact manifold  $(X, \xi)$ ?

Assuming that  $f : M \rightarrow X$  is a Legendrian immersion of class  $\mathcal{A}^2(M, X)$ , it was shown in [18, Theorem 1.2] that  $f$  can be approximated in  $\mathcal{C}^2(M, X)$  by holomorphic Legendrian embeddings of small open neighbourhoods of  $M$  into  $X$ ; however, the cited result does not apply to branched Legendrian maps.

## 6. Proof of Theorem 1.2

Let  $\Omega \subset \mathbb{CP}^3$  be the domain (3.7) and  $\pi : \Omega \rightarrow \mathbb{B} \subset \mathbb{R}^4$  be the twistor bundle over the hyperbolic ball  $(\mathbb{B}, g_h)$  given by (3.2). Denote by  $\xi \subset T\mathbb{CP}^3$  the holomorphic contact bundle determined by the homogeneous 1-form  $\beta$  (1.5), so  $\xi|_{T\Omega}$  is the horizontal bundle of the twistor projection  $\pi : \Omega \rightarrow \mathbb{B}$ . When speaking of Legendrian curves in  $\Omega$ , we always mean holomorphic curves tangent to  $\xi$ . By what has been said in Sect. 3, Theorem 1.2 follows immediately from the following result.

**Theorem 6.1.** *Let  $M$  be a bordered Riemann surface and  $F : \overline{M} \rightarrow \Omega$  be a Legendrian curve of class  $\mathcal{C}^1(\overline{M}, \Omega)$  which is holomorphic on  $M$ . Then,  $F$  can be approximated uniformly on compacts in  $M$  by proper holomorphic Legendrian embeddings  $\tilde{F} : M \hookrightarrow \Omega$  which can be chosen to agree with  $F$  to a given finite order at finitely many points in  $M$ .*

Indeed, by the Bryant correspondence the given superminimal immersion  $f : \overline{M} \rightarrow \mathbb{B}$  in Theorem 1.2 (which may be assumed of positive spin) lifts to a unique Legendrian immersion  $F : \overline{M} \rightarrow \Omega$  as in Theorem 6.1, and if  $\tilde{F} : M \hookrightarrow \Omega$  is a resulting proper holomorphic Legendrian embedding in Theorem 6.1 then its projection  $\tilde{f} = \pi \circ \tilde{F} : M \rightarrow \mathbb{B}$  is a proper superminimal immersion satisfying Theorem 1.2.

We begin with some preparations. Consider the exhaustion function  $\rho : \Omega \rightarrow [0, 1)$  defined in the homogeneous coordinates  $z = [z_1 : z_2 : z_3 : z_4]$  by

$$(6.1) \quad \rho([z_1 : z_2 : z_3 : z_4]) = |\pi(z)|^2 = \frac{|z_3|^2 + |z_4|^2}{|z_1|^2 + |z_2|^2}$$

(see (3.2)). Given a pair of numbers  $0 < c < c' \leq 1$  we write

$$(6.2) \quad \Omega_c = \{z \in \Omega : \rho(z) < c\}, \quad \Omega_{c,c'} = \{z \in \Omega : c < \rho(z) < c'\}.$$

For every point  $z \in \Omega \setminus \pi^{-1}(0)$  there is a unique properly embedded Legendrian disc  $L_z \subset \Omega$  with  $z \in L_z$  whose projection  $\pi(L_z) \subset \mathbb{B}$  is a hyperbolic surface  $\Lambda(\pi(z), V)$  in Proposition 2.1. Indeed, by the twistor correspondence the point  $z$  represents an almost hermitian structure on the tangent space  $T_x\mathbb{R}^4$  at the base point  $x = \pi(z) \in \mathbb{B} \setminus \{0\}$ . Let  $S_x \subset \mathbb{B}$  denote the three-sphere with centre 0 and passing through  $x$ . Then,

$$(6.3) \quad \pi(L_z) = \Lambda(x, V) \text{ where } V = T_x S_x \cap z(T_x S_x).$$

That is,  $V$  is the unique  $z$ -invariant two-plane contained in the three dimensional tangent space  $T_x S_x$  to the sphere  $S_x$  at  $x$ . (Such  $L_z$  also exists for every point  $z$  in the central fibre  $\pi^{-1}(0)$ , but it is not unique since different 2-planes  $V \subset T_0\mathbb{R}^4$  may determine the same almost hermitian structure  $z$  on  $T_0\mathbb{R}^4$ .) By Proposition 3.1,  $L_z$  is the intersection of  $\Omega$  with a linearly embedded Legendrian rational curve  $\mathbb{CP}^1 \subset \mathbb{CP}^3$ . By Proposition 2.1 we have

$$(6.4) \quad L_z \subset \{z\} \cup \Omega_{c,1} \text{ where } c = |\rho(z)| \in (0, 1),$$

where we are using the notation (6.2). It is obvious that the family of Legendrian holomorphic discs  $L_z$  depend real-analytically on the point  $z \in \Omega \setminus \pi^{-1}(0)$ .

Theorem 6.1 is obtained from the following lemma by a standard inductive procedure.

**Lemma 6.2.** *Let  $M$  be a bordered Riemann surface,  $P$  be a finite set of points in  $M$ , and  $0 < c < c' < c'' < 1$ . Assume that  $F : \overline{M} \rightarrow \Omega$  is a Legendrian map of class  $\mathcal{A}^1(\overline{M}, \Omega)$  and  $U \Subset M$  is an open subset such that  $F(M \setminus U) \subset \Omega_{c,c'}$ . Given  $\epsilon > 0$  there exists a holomorphic Legendrian embedding  $G : \overline{M} \rightarrow \Omega$  satisfying the following conditions:*

- (i)  $G(bM) \subset \Omega_{c',c''}$ ,
- (ii)  $G(M \setminus U) \subset \Omega_{c,c''}$ ,
- (iii)  $\text{dist}(G(u), F(u)) < \epsilon$  for  $u \in \overline{U}$ , and
- (iv)  $F$  and  $G$  have the same  $k$ -jets at each of the points in  $P$  for a given  $k \in \mathbb{N}$ .

The details of proof that Lemma 6.2 implies Theorem 6.1 are left to the reader. Inductive constructions of this type are ubiquitous in the literature; see e.g. [14, proof of Theorem 1.1] using [14, Lemma 6.3] and note that our situation is simpler since the exhaustion function  $\rho$  (6.1) of  $\Omega$  has no critical points in  $\Omega \setminus \pi^{-1}(0)$ . In order to ensure that the limit map of this procedure is a Legendrian embedding, we use the general position theorem (see Theorem 5.1) at each step and approximate sufficiently closely in subsequent steps.

*Proof of Lemma 6.2.* Given a pair of numbers  $0 < c < c' < 1$  and an open set  $\omega \subset b\Omega_c = \rho^{-1}(c)$  (see (6.2)), we let

$$(6.5) \quad D(\omega, c, c') := \Omega_{c'} \setminus \bigcup_{z \in b\Omega_c \setminus \omega} L_z.$$

Clearly,  $D(\omega, c, c')$  is an open set containing  $\Omega_c$  and we have that

$$(6.6) \quad z \in \Omega \setminus D(\omega, c, c') \implies L_z \subset \Omega \setminus D(\omega, c, c').$$

Moreover, it is elementary to see that there is a subdivision  $c = c_0 < c_1 < \dots < c_m = c'$  and for every  $i = 0, 1, \dots, m-1$  a finite open cover  $\omega_{i,1}, \dots, \omega_{i,k_i}$  of  $b\Omega_{c_i}$  such that

$$(6.7) \quad \bigcup_{j=1}^{k_i} D(\omega_{i,j}, c_i, c_{i+1}) = \Omega_{c_{i+1}},$$

and for every  $i$  as above and  $j = 1, \dots, k_i$  we also have that

$$(6.8) \quad \bigcup_{z \in \overline{D(\omega_{i,j}, c, c')} \setminus \Omega_{c_i}} L_z \cap \overline{\Omega_{c_{i+1}}} \text{ is contained in an affine chart of } \mathbb{C}\mathbb{P}^3.$$

We are now ready to prove the lemma. Let us begin by explaining the initial step. The assumptions imply that  $F(bM) \subset \Omega_{c,c'}$ . Consider the set

$$(6.9) \quad I'_1 = \{u \in bM : F(u) \in D_1 := D(\omega_{1,1}, c_0, c_1)\}.$$

Assume first that  $I'_1$  does not contain any boundary component of  $M$ . Then,  $I'_1$  is contained in the interior of the union  $I = \bigcup_{i=1}^j I_i$  of finitely many pairwise disjoint segments  $I_1, \dots, I_j \subset bM$  none of which is a component of  $bM$ . Choose a number  $c'_1$  with  $c_1 < c'_1 < c_2$  and close to  $c_1$ . Consider the Riemann-Hilbert problem (cf. Theorem 4.1) with the central Legendrian curve  $F : M \rightarrow \Omega$  and the family of Legendrian discs  $\widehat{L}_u := L_{F(u)} \cap \overline{\Omega_{c'_1}}$  for points  $u \in I$ . (In Theorem 4.1, the central disc is denoted  $f$  and parameterizations of the boundary discs are denoted  $F(u, \cdot)$ .) For the values  $u \in I \setminus I'_1$  we shrink the discs  $\widehat{L}_u$  within themselves (by dilations) to reach the constant discs  $\widehat{L}_u = \{F(u)\}$  as  $u$  reaches the boundary of  $I$ ; these discs remain in the complement of  $\overline{D_1}$  in view of (6.6). By (6.8) and decreasing  $c'_1 > c_1$  if necessary we can also arrange that the set  $\bigcup_{u \in I} \widehat{L}_u$  is contained in an affine chart of  $\mathbb{C}\mathbb{P}^3$ . Applying Theorem 4.1 to this configuration gives a new holomorphic Legendrian curve  $F' : \overline{M} \rightarrow \Omega$  whose boundary  $F'(bM) \subset \Omega_{c,c'}$  no longer intersects  $\overline{D_1}$  and the remaining conditions in the theorem are satisfied. If however the set  $I'_1$  (6.9) contain a boundary component of  $M$ , we perform the same procedure twice, first pushing a part of  $I'_1$  out of  $\overline{D_1}$  and thereby reducing to the previous case.

The subsequent steps are basically the same as the first one. For simplicity we denote the result of step 1 again by  $F$ , so in step 2 the assumption is that  $F(bM) \subset \Omega_{c'} \setminus \overline{D_1}$ . By following the same procedure we push the boundary of  $M$  out of the set  $\overline{D_1} \cup \overline{D_2}$  where  $D_2 := D(\omega_{1,2}, c_0, c_1)$ . Note that a point of  $F(bM)$  which is outside of  $\overline{D_1}$  will not reenter this set in subsequent steps in view of condition (6.6). We see from (6.7) that in  $k_1$  steps of this kind the image of  $bM$  is pushed into  $\Omega_{c_1, c'}$ . We then continue inductively to the next levels  $c_2, \dots, c_m = c'$ , eventually pushing the image of  $bM$  into the domain  $\Omega_{c', c''}$  by a Legendrian map  $G$  satisfying the conditions in the lemma.  $\square$

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