

ARAKELOV CLASS GROUPS OF RANDOM NUMBER FIELDS

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ABSTRACT. The main purpose of the paper is to formulate a probabilistic model for Arakelov class groups in families of number fields, offering a correction to the Cohen–Lenstra–Martinet heuristic on ideal class groups. To that end, we show that Chinburg’s $\Omega(3)$ conjecture implies tight restrictions on the Galois module structure of oriented Arakelov class groups. As a consequence, we construct a new infinite series of counterexamples to the Cohen–Lenstra–Martinet heuristic, which have the novel feature that their Galois groups are non-abelian.

1. INTRODUCTION

It has been an area of active research over the past few decades to understand the distribution of class groups Cl_F of “random” algebraic number fields F . Specifically, we let K be a number field, and let G be a finite group. Let Λ be the quotient of the group ring $\mathbb{Z}[\frac{1}{2\cdot\#G}][G]$ by the two-sided ideal generated by $\sum_{g \in G} g$. One studies the behaviour of the Λ -module $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$, as F runs over a “natural” family \mathcal{F} of G -extensions of K .

An equally classical invariant that one attaches to F is the unit group \mathcal{O}_F^\times of the ring of integers \mathcal{O}_F of F , viewed as a $\mathbb{Z}[G]$ -module. Its isomorphism class, unlike that of the class group, has only finitely many possibilities, as F ranges over \mathcal{F} . The statistical properties of \mathcal{O}_F^\times have, however, been much less extensively studied.

In the present paper we make the case that, in this context, Cl_F and \mathcal{O}_F^\times are most naturally studied in combination, since their distributions need, by all appearances, not be independent. Their dependence is best expressed by means of the *Arakelov class group*. It is a compact abelian group attached to F , and we will recall its definition in Section 2. For number fields it plays the rôle that the Jacobian of a curve plays for function fields over finite fields. It can be broken up into two pieces, one being Cl_F and the other coming from \mathcal{O}_F^\times , but in several ways it is better than the sum of its parts. We find it convenient to replace the Arakelov class group by its Pontryagin dual Ar_F . This is a finitely generated abelian group that fits into a short

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exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{Cl}_F, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Ar}_F \longrightarrow \mathrm{Hom}(\mathcal{O}_F^\times, \mathbb{Z}) \longrightarrow 0.$$

In other words, the torsion subgroup of Ar_F is the Pontryagin dual of the class group, and its torsion-free part is the \mathbb{Z} -linear dual of the unit group. This exact sequence, being canonically associated with F , is an exact sequence of $\mathbb{Z}[G]$ -modules.

Let $G_0(\Lambda)$ denote the Grothendieck group of the category of finitely generated Λ -modules; see Section 2.3 for the definition. Let F/K be a Galois extension with Galois group G , let S_∞ be the G -set of Archimedean places of F , and let \mathbb{Z}^{S_∞} be the corresponding permutation module over $\mathbb{Z}[G]$; it is a property of \mathcal{F} that the isomorphism class of the G -set S_∞ is independent of F when $F \in \mathcal{F}$. The difference of the classes $[\Lambda \otimes_{\mathbb{Z}[G]} \mathrm{Ar}_F]$ and $[\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}]$ in $G_0(\Lambda)$ lies in the torsion subgroup $G_0(\Lambda)_{\mathrm{tors}}$ of $G_0(\Lambda)$, as can easily be deduced from Lemma 5.10. This torsion subgroup is a finite abelian group, which can be thought of as a “class group” of Λ . The following result will be proven at the end of Section 6 as a consequence of Proposition 6.5.

Theorem 1.1. *With the notation just introduced, suppose that Chinburg’s $\Omega(3)$ conjecture, Conjecture 6.3, holds for F/K . Suppose, moreover, that for every prime number p not dividing $2 \cdot \#G$, each primitive p -th root of unity in F is in K . Then the equality*

$$[\Lambda \otimes_{\mathbb{Z}[G]} \mathrm{Ar}_F] - [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}] = 0$$

holds in $G_0(\Lambda)_{\mathrm{tors}}$.

In fact, we will prove the conclusion of Theorem 1.1 under a weaker hypothesis; see Theorem 6.7.

In Section 3 we will define the families \mathcal{F} that we consider. Together, Theorem 1.1 and Propositions 3.3 and 5.3 imply that as F ranges over \mathcal{F} , the class $[\Lambda \otimes_{\mathbb{Z}[G]} \mathcal{O}_F^\times] - [\Lambda \otimes_{\mathbb{Z}[G]} \mathrm{Cl}_F]$ in $G_0(\Lambda)$ is conjectured to be constant. We have no reason to expect this to be true for either of the two terms individually.

There are cases in which the conclusion of Theorem 1.1 can be proven unconditionally. This includes Galois extensions of \mathbb{Q} of degree less than 112 (see Proposition 7.9), but also, more interestingly, a class of fields that can be used to construct a new series of counterexamples to the Cohen–Lenstra–Martinet heuristic [CLJ84, CM90] on class groups of number fields. Informally, this heuristic reads as follows.

Heuristic 1.2 (vague version). *Let F vary in a natural family of Galois number fields. Then the Galois module Ar_F , after inverting the “bad” prime numbers, behaves “randomly” with respect to a probability distribution that assigns to a suitable Galois module M a probability weight that is inversely proportional to the “size” of the automorphism group of M .*

The precise version that we shall use will be formulated as Heuristic 3.2 in Section 3. The same section explains what we mean by the term “natural family”. It also allows the ring Λ to be more general than the ring considered above.

Heuristic 3.2 represents, in several respects, a corrected version of the original Cohen–Lenstra–Martinet heuristic. Nevertheless, it is known to be invalid, counterexamples for certain abelian Galois groups G having been provided in [BLJ20, Theorem 1.1, Proposition 4.4]. One of the main results of the present paper is a new series of counterexamples, this time with non-abelian G . The other main achievement is a proposed correction to the heuristic.

The new counterexamples make use of groups G of order $2^p \cdot p$, where p is an odd prime number. Their abelianisations G/G' , which are cyclic of order $2p$, coincide with the groups on which the abelian counterexamples in [BLJ20] depend. Our groups have centres Z of order 2, and writing $Z = \langle \gamma \rangle$, we shall make use of the ring $\Lambda = \mathbb{Z}[\frac{1}{2p}][G]/(1 + \gamma)$. In Section 8 we prove the following theorem.

Theorem 1.3. *For infinitely many odd prime numbers p there is a group G with the properties just listed such that the following is true. With Λ as just defined, the group $G_0(\Lambda)_{\text{tors}}$ is non-trivial, whereas there does exist a natural family of G -extensions of \mathbb{Q} such that for all members F of the family the class of $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ in $G_0(\Lambda)_{\text{tors}}$ is trivial.*

The families in Theorem 1.3 necessarily violate Heuristic 3.2, since the latter would imply equidistribution of $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ in $G_0(\Lambda)_{\text{tors}}$ as F ranges over the family. This is proven in Section 8, to which we also refer for more information on the groups and families appearing in Theorem 1.3.

The probability weight that is inversely proportional to the “size” of the automorphism group, as referred to in Heuristic 1.2, reflects an attractive feature of the Arakelov class groups. The general principle behind many heuristics is that algebraic objects in natural families tend to be “as random as they can be”, with respect to a probability distribution that assigns to an algebraic object X a probability weight that is proportional to $1/\#\text{Aut } X$. The original Cohen–Lenstra–Martinet heuristic did, initially, look like an exception to the rule just mentioned, but this changed when it was reformulated in terms of Ar_F . This is discussed in more detail in [BLJ20]; see also Section 3 below.

Theorem 1.1 restricts how random Arakelov class groups can be. We propose a correction to Heuristic 1.2 that takes this restriction into account.

Heuristic 1.4 (vague version). *Let F vary in a natural family of Galois number fields. Then the Galois module Ar_F , after inverting the “bad” prime numbers, behaves “randomly” with respect to a probability distribution that assigns to a suitable Galois module M a probability weight that is inversely proportional to the “size” of the automorphism group of M , restricted to those modules that satisfy equation (5.12) with M in place of Ar_F .*

A precise version of Heuristic 1.4 will be formulated as Heuristic 3.4 in Section 3.

Note that the only difference between Heuristics 1.2 and 1.4 is the reference to (5.12). It expresses the restriction that Chinburg’s $\Omega(3)$ conjecture imposes on the class of Ar_F in the class group of Λ as a consequence of Theorem 1.1. It is also important to point out in which way Heuristic 1.4 differs from Conjecture 1.5 formulated in [BLJ20]. The latter conjecture is

only concerned with the local structure of Ar_F at a *finite* set of prime numbers; in that case, the class group of Λ is trivial, so that Chinburg's $\Omega(3)$ conjecture imposes no restriction. On the other hand, Heuristic 1.4 considers almost *all* prime numbers, and in this global situation Λ may have a non-trivial class group. In particular, one should be able to extract explicit information on the distribution of \mathcal{O}_F^\times as a Galois module from Heuristic 1.4, which is not possible with Conjecture 1.5 of [BLJ20].

The structure of the paper is as follows. After some preliminaries in Section 2, we formulate in Section 3 the old and the new heuristics. Some basic material on Grothendieck groups of orders is treated in Section 4. Section 5 is devoted to Arakelov class groups as Galois modules, and Section 6 to the implications of Chinburg's $\Omega(3)$ conjecture for these Galois modules. In Section 7 we collect some cases in which the conclusion of Theorem 1.1 is known, and use these in Section 8 to construct a new series of counterexamples to the Cohen–Lenstra–Martinet heuristic.

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2. PRELIMINARIES

In this section we recall material that we will use for the formulation of Heuristic 3.4. In particular, we recall from [Sch08] the definition of the Arakelov class group and of the oriented Arakelov class group of a number field.

2.1. Pontryagin duality. We briefly recall some facts on Pontryagin duality and refer the reader to [NSW08, Chapter 1, §1] for a more detailed overview. If A and B are abelian topological groups, then $\text{Hom}_{\text{cts}}(A, B)$ denotes the group of continuous group homomorphisms from A to B . Let \mathcal{C} be the category of Hausdorff locally compact abelian topological groups. If A is an object of \mathcal{C} , then its Pontryagin dual is defined to be $\text{Hom}_{\text{cts}}(A, \mathbb{R}/\mathbb{Z})$. This defines an anti-equivalence of \mathcal{C} with itself, of which the square is isomorphic to the identity functor. It induces an anti-equivalence between the full subcategories of compact abelian groups and of discrete abelian groups. If M is a finitely generated abelian group, then $\text{Hom}_{\text{cts}}(M \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ is canonically isomorphic to $\text{Hom}(M, \mathbb{Z})$. In particular, we have $\text{Hom}_{\text{cts}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$.

2.2. The (oriented) Arakelov class group. Let F be a number field. Let Id_F be the group of fractional ideals of the ring of integers \mathcal{O}_F of F , let S_∞ denote the set of Archimedean places of F , and let $F_{\mathbb{R}}$ denote the étale \mathbb{R} -algebra $F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{w \in S_\infty} F_w$, where F_w denotes the completion of F at w . We have canonical maps $\mathfrak{N}: \text{Id}_F \rightarrow \mathbb{R}_{>0}$ and $|\text{Nm}|: F_{\mathbb{R}}^\times \rightarrow \mathbb{R}_{>0}$, the first given by the ideal norm, and the second given by the absolute value of the

\mathbb{R} -algebra norm. Let $\text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ denote the fibre product with respect to these maps. The *oriented Arakelov class group* $\widetilde{\text{Pic}}_F^0$ of F is defined as the cokernel of the map $F^{\times} \rightarrow \text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ that sends $\alpha \in F^{\times}$ to $(\alpha \mathcal{O}_F, \alpha)$. It follows from Dirichlet's unit theorem and the finiteness of the class group of \mathcal{O}_F , that this is a compact abelian group.

For every $w \in S_{\infty}$ we have a direct product decomposition $F_w^{\times} \cong \mathbb{R}_{>0} \times c(F_w^{\times})$, where $c(F_w^{\times})$ is the maximal compact subgroup of F_w^{\times} , which is equal to $\{\pm 1\}$ if w is real, and to the circle group in F_w if w is complex. The maximal compact subgroup $c(F_{\mathbb{R}}^{\times}) = \prod_{w \in S_{\infty}} c(F_w^{\times})$ of $F_{\mathbb{R}}^{\times}$ is contained in the kernel of the map $|\text{Nm}|$. Define the *Arakelov class group* Pic_F^0 of F to be the quotient of $\widetilde{\text{Pic}}_F^0$ by the image of $\{1\} \times c(F_{\mathbb{R}}^{\times}) \subset \text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ in $\widetilde{\text{Pic}}_F^0$.

As in the introduction, we write Ar_F for the Pontryagin dual of Pic_F^0 , and we write $\widetilde{\text{Ar}}_F$ for the Pontryagin dual of $\widetilde{\text{Pic}}_F^0$. Let μ_F denote the group of roots of unity in F . Note that Ar_F , $\widetilde{\text{Ar}}_F$, and μ_F are all finitely generated left $\mathbb{Z}[\text{Aut } F]$ -modules; for Ar_F this will follow from the next result, and for $\widetilde{\text{Ar}}_F$ this follows from [Sch08, Proposition 5.3].

Proposition 2.1. *There is a short exact sequence*

$$0 \longrightarrow \text{Hom}(\text{Cl}_F, \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Ar}_F \longrightarrow \text{Hom}(\mathcal{O}_F^{\times}/\mu_F, \mathbb{Z}) \longrightarrow 0.$$

Proof. By [Sch08, Proposition 2.2] there is an exact sequence

$$0 \longrightarrow (\mathcal{O}_F^{\times}/\mu_F) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \longrightarrow \text{Pic}_F^0 \longrightarrow \text{Cl}_F \longrightarrow 0.$$

The desired result follows by taking the Pontryagin dual of this sequence and recalling from Section 2.1 that we have an isomorphism $\text{Hom}_{\text{cts}}((\mathcal{O}_F^{\times}/\mu_F) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(\mathcal{O}_F^{\times}/\mu_F, \mathbb{Z})$. \square

Remark 2.2. Note that the canonical map $\text{Hom}(\mathcal{O}_F^{\times}/\mu_F, \mathbb{Z}) \rightarrow \text{Hom}(\mathcal{O}_F^{\times}, \mathbb{Z})$ is an isomorphism.

2.3. Modules and Grothendieck groups. Henceforth all modules will be assumed to be left modules unless stated otherwise. If G is a group, S is a finite G -set, and R is a ring, in the remainder of the paper R^S will denote the free R -module on the set S with the induced R -linear $R[G]$ -action. We will refer to such $R[G]$ -modules as *permutation modules*.

Recall that for a ring T , the Grothendieck group $G_0(T)$ of the category of finitely generated T -modules is the additive group generated by expressions $[M]$, one for each isomorphism class of finitely generated T -modules M , with a relation $[L] + [N] = [M]$ whenever there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of finitely generated T -modules.

If P is a set of prime numbers, then we define

$$\mathbb{Z}_{(P)} = \{a/b : a, b \in \mathbb{Z}, b \notin \bigcup_{p \in P \cup \{0\}} p\mathbb{Z}\}.$$

If $T \rightarrow T'$ is a ring homomorphism such that T' is a flat right T -module, then the functor $T' \otimes_T \bullet$ from the category of finitely generated T -modules to that of finitely generated T' -modules induces a group homomorphism $G_0(T) \rightarrow G_0(T')$. The following two examples of this construction will be

relevant to us. If G is a finite group and P is a set of prime numbers, then the flat localisation map $\mathbb{Z} \rightarrow \mathbb{Z}_{(P)}$ induces a group homomorphism $G_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}_{(P)}[G])$. Moreover, if we have a direct product decomposition $T \cong U \times W$ of rings, then the right T -module W is projective, and in particular flat, so the quotient map $T \rightarrow W$ induces a group homomorphism $G_0(T) \rightarrow G_0(W)$.

3. COHEN–LENSTRA–MARTINET HEURISTIC

In this section we propose a correction of the Cohen–Lenstra–Martinet heuristic [CLJ84, CM90]. The notation and assumptions introduced in the next three paragraphs will remain in force throughout this section.

Let G be a finite group, let P be a set of prime numbers not dividing $2 \cdot \#G$, let the \mathbb{Q} -algebra A be a quotient of $\mathbb{Q}[G]$ by a two-sided ideal containing $\sum_{g \in G} g$, and let Λ be the image of $\mathbb{Z}_{(P)}[G]$ in A ; note that the ring Λ in the introduction is a special case of this. Next, let V be a finitely generated $\mathbb{Q}[G]$ -module, and, for brevity, set $V_A = A \otimes_{\mathbb{Q}[G]} V$. Let \mathcal{M}_V be a set of finitely generated Λ -modules M that satisfy $A \otimes_{\Lambda} M \cong_A V_A$, and with the property that for every finitely generated Λ -module M' satisfying $A \otimes_{\Lambda} M' \cong_A V_A$ there exists a unique $M \in \mathcal{M}_V$ such that $M' \cong M$. Note that the set \mathcal{M}_V is countable. If M is a finitely generated Λ -module satisfying $A \otimes_{\Lambda} M \cong_A V_A$, and f is a function defined on \mathcal{M}_V , then we write $f(M)$ for the value of f on the unique element of \mathcal{M}_V that is isomorphic to M . In [BLJ17] it was shown that there is a unique “automorphism index” function $\text{ia}: \mathcal{M}_V \times \mathcal{M}_V \rightarrow \mathbb{Q}_{>0}$ that behaves, in a precise sense explained in [BLJ17, Theorem 1.1], like $(L, M) \mapsto \frac{\#\text{Aut } M}{\#\text{Aut } L}$, even when the automorphism groups of M and of L are infinite. Fix $M \in \mathcal{M}_V$. If \mathcal{N} is a subset of \mathcal{M}_V and X is a positive real number, let \mathcal{N}_X be the finite set of all $L \in \mathcal{N}$ whose torsion subgroup L_{tors} has order less than X . For $\mathcal{N} \subset \mathcal{M}_V$ and for a function $f: \mathcal{N} \rightarrow \mathbb{C}$, define the expected value of f on \mathcal{N} by

$$\mathbb{E}_{\mathcal{N}}(f) = \lim_{X \rightarrow \infty} \left(\sum_{L \in \mathcal{N}_X} \text{ia}(L, M) f(L) / \sum_{L \in \mathcal{N}_X} \text{ia}(L, M) \right)$$

when the limit exists. One of the defining properties of the function ia is that for all L, M , and $N \in \mathcal{M}_V$ we have $\text{ia}(L, M) \text{ia}(M, N) = \text{ia}(L, N)$, whence it follows that whether or not $\mathbb{E}_{\mathcal{N}}(f)$ is defined is independent of the choice of M , and so is its value when it is defined.

Remark 3.1. Expected values behave well under passing to quotients in the following sense. Let $G_1 \rightarrow G_2$ be a surjective group homomorphism, let P be a set of prime numbers not dividing $2 \cdot \#G_1$, let A_2 be a quotient of $\mathbb{Q}[G_2]$ as above, and let V_2 be a finitely generated $\mathbb{Q}[G_2]$ -module; let A_1 be the same as A_2 but viewed as a quotient of $\mathbb{Q}[G_1]$, and let V_1 be the same as V_2 but viewed as a $\mathbb{Q}[G_1]$ -module. For $i \in \{1, 2\}$, let Λ_i be the image of $\mathbb{Z}_{(P)}[G_i]$ in A_i . Note that in particular the map $G_1 \rightarrow G_2$ induces a ring isomorphism $\Lambda_1 \rightarrow \Lambda_2$. For $i \in \{1, 2\}$, define sets \mathcal{M}_{V_i} of Λ_i -modules as above. For brevity, write $\mathcal{M}_i = \mathcal{M}_{V_i}$ for $i \in \{1, 2\}$. Let $f_2: \mathcal{M}_2 \rightarrow \mathbb{C}$ be a function, and let $f_1: \mathcal{M}_1 \rightarrow \mathbb{C}$ be given by $M \mapsto f_2(\Lambda_2 \otimes_{\Lambda_1} M)$. Then one has $\mathbb{E}_{\mathcal{M}_1}(f_1) = \mathbb{E}_{\mathcal{M}_2}(f_2)$.

Let K be a number field, and let \bar{K} be an algebraic closure of K . Given a pair (F, i) , where $F \subset \bar{K}$ is a Galois extension of K and i is an isomorphism between the Galois group of F/K and G , we view $\text{Gal}(F/K)$ -modules as G -modules via i . Let \mathcal{F} be the set of all such pairs (F, i) for which F contains no primitive p -th root of unity for any prime number $p \in P$, and for which there is an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} \cong V$ of $\mathbb{Q}[G]$ -modules. Assume that \mathcal{F} is infinite. Such an \mathcal{F} is what we called a “natural family” in the introduction. Note that this family is a special case of the families considered in [BLJ20, §2].

For $(F, i) \in \mathcal{F}$, let $c_{F/K}$ be the ideal norm of the product of the prime ideals of \mathcal{O}_K that ramify in F/K . For a positive real number B , let $\mathcal{F}_{c \leq B} = \{(F, i) \in \mathcal{F} : c_{F/K} \leq B\}$. The following version of the Cohen–Lenstra–Martinet heuristic is a variant of [BLJ20, Heuristic 2.1] phrased in terms of Arakelov class groups. It differs in several ways from the heuristic formulated in [CLJ84, CM90], but none of those differences shall concern us in the present paper.

Heuristic 3.2. *Let f be a “reasonable” \mathbb{C} -valued function on \mathcal{M}_V . Then the limit*

$$\lim_{B \rightarrow \infty} \frac{\sum_{(F, i) \in \mathcal{F}_{c \leq B}} f(\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F)}{\#\mathcal{F}_{c \leq B}}$$

exists, and is equal to $\mathbb{E}_{\mathcal{M}_V}(f)$.

The notion of a reasonable function is left intentionally vague. The functions considered in [CM87] give rise to many examples of presumably reasonable functions on \mathcal{M}_V that factor through $M \mapsto M_{\text{tors}}$. An example of a function not of that form that we would consider reasonable, and which depends on the Galois module structure of both the class group and the unit group of the ring of integers, is $M \mapsto \#\text{Hom}_{\Lambda}(M/M_{\text{tors}}, M_{\text{tors}})$.

If the set P is infinite, then Conjecture 5.11 can be an obstruction to the conclusions of Heuristic 3.2. For example it was shown in [BLJ20, §4], as a consequence of a proven special case of Conjecture 5.11, that the conclusion of Heuristic 3.2 does not, in general, hold for functions of the form $M \mapsto \chi([M])$, where $\chi: G_0(\Lambda) \rightarrow \mathbb{C}^{\times}$ is a homomorphism of finite order. In [BLJ20] a corrected heuristic was proposed in which P was assumed to be finite. In the remainder of the section, we formulate a Cohen–Lenstra–Martinet heuristic without the hypothesis that P be finite.

Proposition 3.3. *Let (F, i) and $(F', i') \in \mathcal{F}$, and let S_{∞} and S'_{∞} be the sets of Archimedean places of F and F' , respectively. Then the equality*

$$[\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_{\infty}}] = [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S'_{\infty}}]$$

holds in $G_0(\Lambda)$.

Proof. By definition of the family \mathcal{F} , we have an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{F'}^{\times}$ of $\mathbb{Q}[G]$ -modules. By a theorem of Herbrand there is an isomorphism $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times}) \oplus \mathbb{Q} \cong \mathbb{Q}^{S_{\infty}}$ of $\mathbb{Q}[G]$ -modules, see for example [Tat84, Chapter I, 4.3], and similarly for $\mathcal{O}_{F'}^{\times}$. Thus, we have an isomorphism $\mathbb{Q}^{S_{\infty}} \cong \mathbb{Q}^{S'_{\infty}}$.

Since all point stabilisers for S_{∞} and S'_{∞} are inertia groups at Archimedean places, they are all cyclic. It follows from Artin’s induction theorem (e.g. by

combining [Ser77, §13.1, Corollary 1 and Theorem 30] and comparing dimensions) that if S and S' are finite G -sets with cyclic point stabilisers such that there is an isomorphism $\mathbb{Q}^S \cong \mathbb{Q}^{S'}$ of $\mathbb{Q}[G]$ -modules, then the G -sets S and S' are isomorphic. In particular, there is then an isomorphism $\mathbb{Z}^S \cong \mathbb{Z}^{S'}$ of $\mathbb{Z}[G]$ -modules. The result follows by applying this observation to the G -sets S_∞ and S'_∞ . \square

We define $C(\mathcal{F})$ to be the common class of $\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}$ in $G_0(\Lambda)$ for all $(F, i) \in \mathcal{F}$, where S_∞ is the G -set of Archimedean places of F .

Heuristic 3.4. *Let $\mathcal{N} = \{M \in \mathcal{M}_V : [M] = C(\mathcal{F}) \text{ in } G_0(\Lambda)\}$, and let f be a “reasonable” \mathbb{C} -valued function on \mathcal{M}_V . Then the limit*

$$\lim_{B \rightarrow \infty} \frac{\sum_{(F, i) \in \mathcal{F}_{c \leq B}} f(\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F)}{\#\mathcal{F}_{c \leq B}}$$

exists, and is equal to $\mathbb{E}_{\mathcal{N}}(f)$.

4. GROTHENDIECK GROUPS OF ORDERS

In this section we review some standard facts about Grothendieck groups of orders, and examine the effect of some duality operations upon these Grothendieck groups.

Let R be a Dedekind domain and let k be the field of fractions of R . An R -order is an R -algebra that is finitely generated and projective as an R -module. For example if G is a finite group, then the group ring $\Lambda = R[G]$ is an R -order.

Let Λ be an R -order. A finitely generated Λ -module that is projective over R will be referred to as a Λ -lattice. Let $G_0^R(\Lambda)$ denote the Grothendieck group of the category of Λ -lattices. By definition, $G_0^R(\Lambda)$ is the additive group generated by expressions $[M]$, one for each isomorphism class of Λ -lattices M , with a relation $[L] + [N] = [M]$ whenever there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of Λ -lattices. Recall from Section 2.3 that if T is a ring, then we similarly define the Grothendieck group $G_0(T)$ of the category of finitely generated T -modules by replacing, in the above definition, “ Λ -lattice” by “finitely generated T -module”. By [CR87, Theorem (38.42)], the inclusion of the category of Λ -lattices into the category of all finitely generated Λ -modules induces a canonical isomorphism $G_0^R(\Lambda) \cong G_0(\Lambda)$.

Let Λ^{op} denote the opposite ring of Λ . If M is a Λ -lattice, then $M^* = \text{Hom}_R(M, R)$ is a Λ^{op} -lattice. This defines a contravariant functor from the category of Λ -lattices to the category of Λ^{op} -lattices, given on objects by $M \mapsto M^*$ for every Λ -lattice M , and on morphisms by $f \mapsto (\nu \mapsto \nu \circ f) \in M^*$ for every morphism $f: M \rightarrow N$ of Λ -lattices and every $\nu \in N^*$. This functor is easily seen to be exact, and to induce a group isomorphism $G_0^R(\Lambda) \rightarrow G_0^R(\Lambda^{\text{op}})$, and hence an isomorphism

$$j: G_0(\Lambda) \rightarrow G_0(\Lambda^{\text{op}}).$$

For a Λ -module N , we define $N^\vee = \text{Hom}_R(N, k/R)$, which is also a Λ^{op} -module. If N is a finitely generated Λ -module that is R -torsion, then N^\vee is finitely generated over Λ^{op} and R -torsion.

In the special case that $\Lambda = R[G]$, where G is a finite group, the ring Λ is equipped with an involution ι induced by $g \mapsto g^{-1}$ for all $g \in G$. If M is a finitely generated $R[G]$ -module, then we view the $R[G]^{\text{op}}$ -module M^* as a finitely generated $R[G]$ -module via ι , and we view the map j as an automorphism of $G_0(R[G])$. Specialising further to $R = \mathbb{Z}$, if A is a Hausdorff locally compact abelian topological group on which G acts by continuous automorphisms, then we view its Pontryagin dual $\text{Hom}_{\text{cts}}(A, \mathbb{R}/\mathbb{Z})$ as a $\mathbb{Z}[G]$ -module via ι . If N is a $\mathbb{Z}[G]$ -module of finite cardinality, then so is $\text{Hom}_{\text{cts}}(N, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) = N^\vee$.

Proposition 4.1. *Let N be a finitely generated Λ -module that is R -torsion. Then the equality*

$$[N^\vee] = -j[N]$$

holds in $G_0(\Lambda^{\text{op}})$.

Proof. Since R is a Dedekind domain, every Λ -submodule of a Λ -lattice is itself a Λ -lattice. Hence there exists a presentation $0 \rightarrow M_1 \rightarrow M_2 \rightarrow N \rightarrow 0$ of N by Λ -lattices, so that $[N] = [M_2] - [M_1]$ in $G_0(\Lambda)$. We claim that N^\vee is canonically isomorphic as a Λ^{op} -module to M_1^*/M_2^* .

Since M_1, M_2 are projective over R , applying the functors $\text{Hom}_R(M_i, \bullet)$ for $i = 1, 2$ to the short exact sequence

$$0 \longrightarrow R \longrightarrow k \longrightarrow k/R \longrightarrow 0$$

yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M_2, R) & \longrightarrow & \text{Hom}_R(M_2, k) & \longrightarrow & \text{Hom}_R(M_2, k/R) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_R(M_1, R) & \longrightarrow & \text{Hom}_R(M_1, k) & \longrightarrow & \text{Hom}_R(M_1, k/R) \longrightarrow 0, \end{array}$$

where the vertical maps are induced by the injection $M_1 \rightarrow M_2$. Of these, the middle map $\text{Hom}_R(M_2, k) \rightarrow \text{Hom}_R(M_1, k)$ is an isomorphism. Indeed, it is the k -linear dual of the map $k \otimes_R M_1 \rightarrow k \otimes_R M_2$, which is clearly an isomorphism, since the cokernel N of $M_1 \rightarrow M_2$ is R -torsion. The snake lemma therefore gives an isomorphism of right Λ -modules from the kernel of $\text{Hom}_R(M_2, k/R) \rightarrow \text{Hom}_R(M_1, k/R)$ to the cokernel of $\text{Hom}_R(M_2, R) \rightarrow \text{Hom}_R(M_1, R)$. Since $\text{Hom}_R(\bullet, k/R)$ is left exact, that kernel is exactly N^\vee , while the cokernel is precisely M_1^*/M_2^* , as claimed. The proposition immediately follows. \square

5. ORIENTED ARAKELOV CLASS GROUPS AS GALOIS MODULES

In this section we prove some properties of oriented Arakelov class groups as Galois modules, and formulate the main working hypothesis that motivates the statistical heuristic in Section 3.

Lemma 5.1. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let S_∞ be the set of Archimedean places of F , and let d*

be the degree of K over \mathbb{Q} . Then the equality

$$[\mathrm{Hom}_{\mathrm{cts}}(\mathrm{c}(F_{\mathbb{R}}^{\times}), \mathbb{R}/\mathbb{Z})] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}^{S_{\infty}}]$$

holds in $G_0(\mathbb{Z}[G])$, where $\mathrm{c}(F_{\mathbb{R}}^{\times})$ is as defined in Section 2.2.

Proof. If v is an Archimedean place of K , let $I_v \subset G$ denote an inertia subgroup at v , and let τ_v be a $\mathbb{Z}[I_v]$ -module defined as follows: if v is real and I_v is the trivial group, then $\tau_v = \mathbb{F}_2$; if v is real and I_v has order 2, then τ_v is free over \mathbb{Z} of rank 1, and with the generator of I_v acting by -1 ; and if v is complex, so that I_v is necessarily trivial, then $\tau_v = \mathbb{Z}$. Then it is easy to see that we have an isomorphism

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{c}(F_{\mathbb{R}}^{\times}), \mathbb{R}/\mathbb{Z}) \cong \bigoplus_v \mathrm{Ind}_{I_v}^G \tau_v$$

of $\mathbb{Z}[G]$ -modules, where the direct sum runs over the Archimedean places of K , and $\mathrm{Ind}_{I_v}^G$ denotes induction from I_v to G .

If v is a real place of K such that I_v is trivial, then the exact sequence

$$0 \longrightarrow \mathrm{Ind}_{I_v}^G \mathbb{Z} \xrightarrow{\times 2} \mathrm{Ind}_{I_v}^G \mathbb{Z} \longrightarrow \mathrm{Ind}_{I_v}^G \mathbb{F}_2 \longrightarrow 0$$

shows that $[\mathrm{Ind}_{I_v}^G \tau_v] = 0$ in $G_0(\mathbb{Z}[G])$. We deduce that for all Archimedean places v of K we have

$$[\mathrm{Ind}_{I_v}^G \tau_v] = \delta_v [\mathbb{Z}[G]] - [\mathbb{Z}^{G/I_v}], \quad (5.2)$$

where $\delta_v = 1$ if v is real, and $\delta_v = 2$ if v is complex. The result follows by summing (5.2) over all Archimedean places v of K . \square

For every permutation $\mathbb{Z}[G]$ -module M , we have $M \cong M^*$, where M^* is as defined in Section 4 with $R = \mathbb{Z}$, and therefore $j[M] = [M]$ in $G_0(\mathbb{Z}[G])$. We will repeatedly use this observation. If S is a set of places of F containing all Archimedean places, let $\mathcal{O}_{F,S}$ denote the ring of S -integers in F , let $\mathcal{O}_{F,S}^{\times}$ denote its unit group, and let $\mathrm{Cl}_{F,S}$ denote its class group. If S is G -stable, then $\mathcal{O}_{F,S}^{\times}$ and $\mathrm{Cl}_{F,S}$ are $\mathbb{Z}[G]$ -modules. Recall that Ar_F and $\widetilde{\mathrm{Ar}}_F$ denote the Pontryagin duals of Pic_F^0 and $\widetilde{\mathrm{Pic}}_F^0$, respectively, and μ_F denotes the group of roots of unity in F .

Proposition 5.3. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let d be the degree of K over \mathbb{Q} , and let S be a finite G -stable set of places of F containing all Archimedean places. Then Ar_F and $\widetilde{\mathrm{Ar}}_F$ are finitely generated $\mathbb{Z}[G]$ -modules. Moreover the equalities*

$$\begin{aligned} [\mathrm{Ar}_F] &= j[\mathcal{O}_F^{\times}/\mu_F] - j[\mathrm{Cl}_F], \\ [\widetilde{\mathrm{Ar}}_F] &= d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}^S] + j[\mathcal{O}_{F,S}^{\times}] - j[\mathrm{Cl}_{F,S}] \end{aligned}$$

hold in $G_0(\mathbb{Z}[G])$.

Proof. It was already noted in Section 2.2 that Ar_F and $\widetilde{\mathrm{Ar}}_F$ are finitely generated $\mathbb{Z}[\mathrm{Aut} F]$ -modules, and this immediately implies the first assertion. The expression for $[\mathrm{Ar}_F]$ follows from Propositions 2.1 and 4.1.

The rest of the proof is devoted to the derivation of the expression for $[\widetilde{\mathrm{Ar}}_F]$. Let $S_f = S \setminus S_{\infty}$ denote the set of non-Archimedean places in S . The subgroup of Id_F generated by the prime ideals corresponding to the places

in S_f is free abelian on the set S_f . Below, when we write \mathbb{Z}^{S_f} , we will mean that subgroup. Let $\text{Id}_{F,S}$ be the quotient of Id_F by the subgroup \mathbb{Z}^{S_f} . It is naturally isomorphic to the group of fractional ideals of $\mathcal{O}_{F,S}$. The preimage of

$$\mathbb{Z}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times} \subset \text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$$

under the inclusion map $F^{\times} \rightarrow \text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ is $\mathcal{O}_{F,S}^{\times}$. There is thus a commutative diagram of $\mathbb{Z}[G]$ -modules with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{F,S}^{\times} & \longrightarrow & \mathbb{Z}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times} & \longrightarrow & T_{F,S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{\times} & \longrightarrow & \text{Id}_F \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times} & \longrightarrow & \widetilde{\text{Pic}}_F^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{\times} / \mathcal{O}_{F,S}^{\times} & \longrightarrow & \text{Id}_{F,S} & \longrightarrow & \text{Cl}_{F,S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

where $T_{F,S}$ is defined by the exactness of the last column, and the exactness of the first row follows from the snake lemma.

The group $\mathbb{Z}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ can be explicitly described as follows: we have

$$\mathbb{Z}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times} = \left\{ ((a_p)_p, b) \in \mathbb{Z}^{S_f} \times F_{\mathbb{R}} : \prod_{p \in S_f} (\mathfrak{N}p)^{a_p} = |\text{Nm}|(b) \right\},$$

where \mathfrak{N} and $|\text{Nm}|$ are as defined in Section 2.2. That group naturally embeds into $\mathbb{R}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$, where the fibre product is taken with respect to the map

$$\begin{aligned} \mathbb{R}^{S_f} &\longrightarrow \mathbb{R}_{>0}, \\ (a_p)_{p \in S_f} &\mapsto \prod_{p \in S_f} (\mathfrak{N}p)^{a_p}, \end{aligned}$$

and to the same map $|\text{Nm}|: F_{\mathbb{R}}^{\times} \rightarrow \mathbb{R}_{>0}$ as before, so that we have an exact sequence

$$0 \longrightarrow T_{F,S} \longrightarrow \frac{\mathbb{R}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}}{\mathcal{O}_{F,S}^{\times}} \longrightarrow (\mathbb{R}/\mathbb{Z})^{S_f} \longrightarrow 0. \quad (5.4)$$

The preimage of $\{0\} \times c(F_{\mathbb{R}}^{\times}) \subset \mathbb{R}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}$ in $\mathcal{O}_{F,S}^{\times}$ is μ_F . We therefore deduce from Dirichlet's (S -)unit theorem [Lan94, Chapter V, §1], that the middle term of (5.4) is an extension of the form

$$0 \longrightarrow c(F_{\mathbb{R}}^{\times}) / \mu_F \longrightarrow \frac{\mathbb{R}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}}{\mathcal{O}_{F,S}^{\times}} \longrightarrow (\mathcal{O}_{F,S}^{\times} / \mu_F) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \longrightarrow 0. \quad (5.5)$$

In particular, it is a compact abelian group whose Pontryagin dual

$$\mathrm{Hom}_{\mathrm{cts}} \left(\frac{\mathbb{R}^{S_f} \times_{\mathbb{R}_{>0}} F_{\mathbb{R}}^{\times}}{\mathcal{O}_{F,S}^{\times}}, \mathbb{R}/\mathbb{Z} \right)$$

is finitely generated, therefore the same is true of the closed subgroup $T_{F,S}$.

Taking the Pontryagin dual of the right column of the commutative diagram above, we deduce that there is an equality

$$[\widetilde{\mathrm{Ar}}_F] = [\mathrm{Cl}_{F,S}^{\vee}] + [\mathrm{Hom}_{\mathrm{cts}}(T_{F,S}, \mathbb{R}/\mathbb{Z})] \quad (5.6)$$

in $G_0(\mathbb{Z}[G])$.

Combining (5.6) with the Pontryagin duals of (5.4) and (5.5) and applying Lemma 5.1, we see that there is an equality

$$[\widetilde{\mathrm{Ar}}_F] = [\mathrm{Cl}_{F,S}^{\vee}] + d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}^{S_{\infty}}] - [\mu_F^{\vee}] + [(\mathcal{O}_{F,S}^{\times}/\mu_F)^*] - [(\mathbb{Z}^{S_f})^*]$$

in $G_0(\mathbb{Z}[G])$. We now deduce the result by applying Proposition 4.1 and noting that there are $\mathbb{Z}[G]$ -module isomorphisms $(\mathbb{Z}^{S_f})^* \cong \mathbb{Z}^{S_f}$ and $\mathbb{Z}^S \cong \mathbb{Z}^{S_{\infty}} \oplus \mathbb{Z}^{S_f}$. \square

Corollary 5.7. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let d be the degree of K over \mathbb{Q} , and let S be a finite G -stable set of places of F containing all Archimedean places, and large enough for $\mathrm{Cl}_{F,S}$ to be trivial. Then the equality*

$$[\widetilde{\mathrm{Ar}}_F] = d \cdot [\mathbb{Z}[G]] - j([\mathbb{Z}^S] - [\mathcal{O}_{F,S}^{\times}])$$

holds in $G_0(\mathbb{Z}[G])$.

Proof. The result follows by combining Proposition 5.3 with the observation that $j[\mathbb{Z}^S] = [\mathbb{Z}^S]$. \square

Corollary 5.8. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, and let S and S' be two finite G -stable sets of places of F , both containing all Archimedean places. Then the equality*

$$[\mathbb{Z}^{S'}] - [\mathcal{O}_{F,S'}^{\times}] + [\mathrm{Cl}_{F,S'}] = [\mathbb{Z}^S] - [\mathcal{O}_{F,S}^{\times}] + [\mathrm{Cl}_{F,S}]$$

holds in $G_0(\mathbb{Z}[G])$.

Proof. The result follows by combining Proposition 5.3 with the observation that $j[\mathbb{Z}^S] = [\mathbb{Z}^S]$ and $j[\mathbb{Z}^{S'}] = [\mathbb{Z}^{S'}]$. \square

As we will see in the next section, the $\Omega(3)$ conjecture, a standard conjecture in the theory of Galois module structures, implies the following simpler expressions for the classes of Ar_F and $\widetilde{\mathrm{Ar}}_F$.

Conjecture 5.9. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let d be the degree of K over \mathbb{Q} , let S_{∞} be the set of Archimedean places of F , let μ_F be the group of roots of unity in F , and let j be the automorphism of $G_0(\mathbb{Z}[G])$ induced by the involution $g \mapsto g^{-1}$ on $\mathbb{Z}[G]$, as defined in Section 4. Then the following equalities hold in $G_0(\mathbb{Z}[G])$:*

- (a) $[\widetilde{\mathrm{Ar}}_F] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}];$
- (b) $[\mathrm{Ar}_F] = [\mathbb{Z}^{S_{\infty}}] - [\mathbb{Z}] - j[\mu_F].$

The next result shows that, in the two equations in Conjecture 5.9, the difference between the left hand side and the right hand side is the same. Hence each of (a) and (b) implies the other, and is therefore equivalent to the entire conjecture.

Lemma 5.10. *With the same notation as in Conjecture 5.9, we have*

$$[\widetilde{\text{Ar}}_F] - [\text{Ar}_F] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}^{S_\infty}] + j[\mu_F]$$

in $G_0(\mathbb{Z}[G])$. Moreover, the common difference between the left hand side and the right hand side of the two equations in Conjecture 5.9 lies in the torsion subgroup $G_0(\mathbb{Z}[G])_{\text{tors}}$ of $G_0(\mathbb{Z}[G])$.

Proof. The first assertion follows from Proposition 5.3 with $S = S_\infty$.

We now prove the second assertion. By a theorem of Herbrand there is an isomorphism $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times) \oplus \mathbb{Q} \cong \mathbb{Q}^{S_\infty}$ of $\mathbb{Q}[G]$ -modules, see for example [Tat84, Chapter I, 4.3]. Let $\theta : G_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Q}[G])$ be the map induced by the flat ring homomorphism $\mathbb{Z}[G] \rightarrow \mathbb{Q}[G]$. For every finitely generated $\mathbb{Z}[G]$ -module M we have $\theta(j[M]) = \theta([M])$. Therefore applying θ to the first equality of Proposition 5.3, we deduce the equality

$$[\mathbb{Q} \otimes_{\mathbb{Z}} \text{Ar}_F] - [\mathbb{Q}^{S_\infty}] + [\mathbb{Q}] = [\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times] - [\mathbb{Q}^{S_\infty}] + [\mathbb{Q}] = 0$$

in $G_0(\mathbb{Q}[G])$. This shows that the difference between the left hand side and the right hand side of Conjecture 5.9(b) lies in $\ker(\theta)$, which is equal to $G_0(\mathbb{Z}[G])_{\text{tors}}$ by [CR87, Theorem (39.14)]. The conclusion for Conjecture 5.9(a) follows from this and the first assertion. \square

For every ring R that is flat over \mathbb{Z} , the functor $R \otimes_{\mathbb{Z}} \bullet$ induces a homomorphism $G_0(\mathbb{Z}[G]) \rightarrow G_0(R[G])$. Hence, Conjecture 5.9 implies the following conjecture, which will suffice for the applications to the Cohen–Lenstra–Martinet heuristic in Section 3.

Conjecture 5.11. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let d be the degree of K over \mathbb{Q} , and let P be a set of prime numbers not dividing $2 \cdot \#G$. Then the equalities*

$$\begin{aligned} [\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \widetilde{\text{Ar}}_F] &= d \cdot [\mathbb{Z}_{(P)}[G]] - [\mathbb{Z}_{(P)}], \\ [\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \text{Ar}_F] &= [(\mathbb{Z}_{(P)})^{S_\infty}] - [\mathbb{Z}_{(P)}] - j[\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \mu_F] \end{aligned} \quad (5.12)$$

hold in $G_0(\mathbb{Z}_{(P)}[G])$.

Remark 5.13. Conjecture 5.11 is equivalent to an affirmative answer to [BLJ20, Question 5.5]. Thus in the special case that G is abelian and $K = \mathbb{Q}$, Conjecture 5.11 was proven unconditionally in [BLJ20, Theorem 5.4].

Remark 5.14. Let F/K be a finite Galois extension of number fields and let G be the Galois group. Lemma 5.10 shows that if $G_0(\mathbb{Z}[G])_{\text{tors}}$ vanishes, then Conjecture 5.9 holds for F/K . There are many cases in which $G_0(\mathbb{Z}[G])_{\text{tors}}$ has been calculated, and among these are numerous instances in which it is trivial. In the case that G is abelian, a formula for $G_0(\mathbb{Z}[G])$ is given in [Len81]; this result has been extended to the case that G is nilpotent in [HTW88]. Analogous observations also apply to Conjecture 5.11.

6. THE RELATION TO CHINBURG'S $\Omega(3)$ CONJECTURE

In this section we explain that Conjecture 5.9, and a fortiori Conjecture 5.11, follow from Conjecture 6.4 below, which is a weaker variant of Chinburg's $\Omega(3)$ conjecture, Conjecture 6.3 [Chi85, Conjecture 3.1].

We first briefly review some material from algebraic K-theory. We refer the reader to [CR87, §38, §39, §49] for further details.

Let R be a Dedekind domain and let Λ be an R -order. Let $K_0(\Lambda)$ denote the Grothendieck group of the category of finitely generated projective Λ -modules. By definition, $K_0(\Lambda)$ is the additive group generated by expressions $[P]$, one for each isomorphism class of finitely generated projective Λ -modules P , with relations $[P_1 \oplus P_2] = [P_1] + [P_2]$ for all such modules P_1, P_2 . By [CR87, Theorem (38.50)], the group $K_0(\Lambda)$ can be identified with the Grothendieck group of the category of finitely generated Λ -modules of finite projective dimension.

For each maximal ideal \mathfrak{p} of R , let $R_{\mathfrak{p}}$ denote the localisation of R at \mathfrak{p} , and define $\Lambda_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R \Lambda$. A Λ -lattice M is said to be *locally free* if $\Lambda_{\mathfrak{p}} \otimes_{\Lambda} M$ is free over $\Lambda_{\mathfrak{p}}$ for every such \mathfrak{p} . Note that every locally free Λ -lattice is projective by [CR81, Proposition (8.19)]. Let $K_0(\Lambda) \rightarrow K_0(\Lambda_{\mathfrak{p}})$ be the map induced by the ring homomorphism $\Lambda \rightarrow \Lambda_{\mathfrak{p}}$. Then the *locally free class group* $\text{Cl}(\Lambda)$ is defined to be the kernel of the homomorphism $K_0(\Lambda) \rightarrow \prod_{\mathfrak{p}} K_0(\Lambda_{\mathfrak{p}})$, where the product runs over all maximal ideals \mathfrak{p} of R (see [CR87, Definition (39.12)]). By [CR87, (39.13)] we have

$$\text{Cl}(\Lambda) = \{[\Lambda] - [L] \in K_0(\Lambda) : L \text{ is a locally free } \Lambda\text{-lattice of rank 1}\}. \quad (6.1)$$

Note that there are several equivalent definitions of $\text{Cl}(\Lambda)$ (see [CR87, §49A], particularly [CR87, p. 223]).

We now recall the statement of Chinburg's $\Omega(3)$ conjecture. For the rest of the section, let F/K be a finite Galois extension of number fields and let G be the Galois group. For any finite G -stable set S of places of F , let X_S be the kernel of the augmentation map $\mathbb{Z}^S \rightarrow \mathbb{Z}$ of $\mathbb{Z}[G]$ -lattices. Henceforth let S be a finite G -stable set of places of F such that

- (i) S contains the set S_{∞} of Archimedean places of F ,
- (ii) S contains the ramified places of F/K , and
- (iii) for every subfield N of F containing K , the ideal class group of N is generated by the classes $\{[\mathfrak{p} \cap \mathcal{O}_N] : \mathfrak{p} \in S \setminus S_{\infty}\}$.

Tate [Tat66, p. 711] defined a canonical class $\alpha = \alpha_S \in \text{Ext}_{\mathbb{Z}[G]}^2(X_S, \mathcal{O}_{F,S}^{\times})$, and showed the existence of so-called Tate sequences [Tat84, II, Théorème 5.1], that is, four term exact sequences of finitely generated $\mathbb{Z}[G]$ -modules

$$0 \longrightarrow \mathcal{O}_{F,S}^{\times} \longrightarrow A \longrightarrow B \longrightarrow X_S \longrightarrow 0 \quad (6.2)$$

representing α , where A and B are of finite projective dimension. In [Chi85], Chinburg defined $\Omega(F/K, 3) = [A] - [B] \in K_0(\mathbb{Z}[G])$. Moreover, he showed that $\Omega(F/K, 3)$ lies in the locally free class group $\text{Cl}(\mathbb{Z}[G])$, and depends only on the extension F/K ; in particular, it does not depend on the choice of S or on the choice of exact sequence (6.2).

The root number class $W_{F/K} \in \text{Cl}(\mathbb{Z}[G])$ was defined by Ph. Cassou-Noguès in the case that F/K is at most tamely ramified, and was generalised to wildly ramified extensions F/K by Fröhlich [Frö78]. It is an element of

order at most 2, and is defined in terms of the Artin root numbers of the irreducible symplectic characters of G . Moreover, if G has no irreducible symplectic characters (for example, if G is abelian or of odd order), then $W_{F/K}$ is trivial by definition.

Conjecture 6.3 (Chinburg's $\Omega(3)$ conjecture). *The equality*

$$\Omega(F/K, 3) = W_{F/K}$$

holds in $\text{Cl}(\mathbb{Z}[G])$.

Fix, for the rest of the section, a maximal order \mathcal{M} in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$, let $\rho: \text{Cl}(\mathbb{Z}[G]) \rightarrow \text{Cl}(\mathcal{M})$ be the map induced by the ring homomorphism $\mathbb{Z}[G] \rightarrow \mathcal{M}$, and define the *kernel subgroup* $D(\mathbb{Z}[G])$ of $\text{Cl}(\mathbb{Z}[G])$ to be the kernel of ρ . If \mathcal{M}' is any other maximal order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$, and $\rho': \text{Cl}(\mathbb{Z}[G]) \rightarrow \text{Cl}(\mathcal{M}')$ is the analogous map, then by [Ull77, Theorem 1.6] the kernel of ρ is equal to that of ρ' .

Conjecture 6.4 (Chinburg's $\Omega(3)$ conjecture modulo the kernel group). *We have $\Omega(F/K, 3) \equiv W_{F/K} \pmod{D(\mathbb{Z}[G])}$.*

The next result will be the main ingredient in the proof of Theorem 1.1 from the introduction.

Proposition 6.5. *Conjecture 6.4 for F/K implies Conjecture 5.9 for F/K .*

In order to prove this result, we will make use of the following lemma. Let

$$\Psi(F/K) = [\mathcal{O}_F^\times] - [X_{S_\infty}] - [\text{Cl}_F] \in G_0(\mathbb{Z}[G]).$$

Lemma 6.6. *Conjecture 5.9 for F/K is equivalent to the vanishing of $\Psi(F/K)$, while Conjecture 5.11 is equivalent to the assertion that for all sets P of prime numbers not dividing $2 \cdot \#G$ the image of $\Psi(F/K)$ under the map $G_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}_{(P)}[G])$ is 0. Moreover, we always have $\Psi(F/K) \in G_0(\mathbb{Z}[G])_{\text{tors}}$.*

Proof. By Lemma 5.10, Conjectures 5.9(a) and 5.9(b) are equivalent to each other. By Corollary 5.7, Conjecture 5.9(a) is equivalent to the assertion that we have

$$[\mathbb{Z}] = j([\mathbb{Z}^S] - [\mathcal{O}_{F,S}^\times])$$

in $G_0(\mathbb{Z}[G])$. Since $j[\mathbb{Z}] = [\mathbb{Z}]$, this is equivalent to

$$[\mathcal{O}_{F,S}^\times] = [\mathbb{Z}^S] - [\mathbb{Z}] = [X_S].$$

By Corollary 5.8 with $S' = S_\infty$, this in turn is equivalent to $\Psi(F/K) = 0$.

The proof of the claim regarding Conjecture 5.11 is completely analogous.

The last claim follows from Lemma 5.10. \square

In light of Lemma 6.6, Proposition 6.5 amounts to the statement that Conjecture 6.4 implies the vanishing of $\Psi(F/K)$. This statement is well known to experts in Galois module theory; see [Chi83, III], [CNCFT91, §4, Proposition 6] or [CKPS98, §1]. We will recall the argument in the following proof and give some additional references.

Proof of Proposition 6.5. Let $\mu : \text{Cl}(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}[G])$ denote the restriction of the Cartan map $K_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}[G])$, which is induced by letting $\mu([M]) = [M]$ if M is a finitely generated projective $\mathbb{Z}[G]$ -module.

Let $\xi \in \text{Cl}(\mathbb{Z}[G])$. Write $\xi = [\mathbb{Z}[G]] - [L]$, where L is a locally free left ideal of $\mathbb{Z}[G]$, which we may do by (6.1). By [CR81, Exercise 31.10] we have

$$\xi = [\mathbb{Z}[G]] - [L] = [\mathcal{M}] - [\mathcal{M} \otimes_{\mathbb{Z}[G]} L] \text{ in } G_0(\mathbb{Z}[G]).$$

Hence we have the following commutative diagram of abelian groups:

$$\begin{array}{ccc} \text{Cl}(\mathbb{Z}[G]) & \xrightarrow{\mu} & G_0(\mathbb{Z}[G]) \\ \rho \downarrow & & \uparrow \alpha \\ \text{Cl}(\mathcal{M}) & \xrightarrow{\mu'} & G_0(\mathcal{M}), \end{array}$$

where α is induced by restriction, and μ' is defined analogously to μ .

Let A and B be as in the exact sequence (6.2). Then it follows from the definition of $\Omega(F/K, 3)$, from the exact sequence (6.2), and from Corollary 5.8 with $S' = S_\infty$, that we have the equalities

$$\mu(\Omega(F/K, 3)) = [A] - [B] = [\mathcal{O}_{F,S}^\times] - [X_S] = \Psi(F/K).$$

Moreover, a special case of a result of Queyruat [Que85, Proposition 2.3] shows that $\mu(W_{F/K}) = 0$.

Now suppose that Conjecture 6.4 holds for F/K . By definition of $D(\mathbb{Z}[G])$, this is equivalent to $\rho(\Omega(F/K, 3)) = \rho(W_{F/K})$. Since μ factors via ρ , we have

$$\Psi(F/K) = \mu(\Omega(F/K, 3)) = \mu(W_{F/K}) = 0.$$

The result now follows from Lemma 6.6. \square

We close the section by proving Theorem 1.1 from the introduction. In fact, we prove the following stronger statement.

Theorem 6.7. *Let F/K be a Galois extension of number fields, let G be the Galois group, let S_∞ be the G -set of Archimedean places of F , and let Λ be the quotient of the group ring $\mathbb{Z}[\frac{1}{2 \cdot \#G}][G]$ by the two-sided ideal generated by $\sum_{g \in G} g$. Suppose that Conjecture 6.4 holds for F/K , and that for every prime number p not dividing $2 \cdot \#G$, each primitive p -th root of unity in F is in K . Then the equality*

$$[\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F] - [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}] = 0$$

holds in $G_0(\Lambda)_{\text{tors}}$.

Proof. Note that the fact that $[\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F] - [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}]$ indeed lies in $G_0(\Lambda)_{\text{tors}}$ follows from Lemma 5.10.

The ring homomorphism $\mathbb{Z}[G] \rightarrow \Lambda$ is a composition of two homomorphisms of the types discussed at the end of Section 2.3, so that it induces a group homomorphism $G_0(\mathbb{Z}[G]) \rightarrow G_0(\Lambda)$. By Proposition 6.5, the hypotheses of the theorem imply that we have the equality

$$[\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F] = [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}^{S_\infty}] - [\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}] - j[\Lambda \otimes_{\mathbb{Z}[G]} \mu_F]$$

in $G_0(\Lambda)$, where j is the automorphism of $G_0(\Lambda)$ induced by the involution $g \mapsto g^{-1}$ on G . Write temporarily $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2 \cdot \#G}]$. Since the element

$\frac{1}{\#G} \sum_{g \in G} g$ of $\mathbb{Z}'[G]$ acts trivially on the $\mathbb{Z}'[G]$ -module \mathbb{Z}' , but is 0 in the quotient Λ , we have $[\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}] = 0$.

Next, the hypothesis on roots of unity implies that for all prime numbers p not dividing $2 \cdot \#G$ and all $k \in \mathbb{Z}_{\geq 0}$, every p^k -th root of unity in F is in K . This implies that G acts trivially on $\mathbb{Z}' \otimes_{\mathbb{Z}} \mu_F$, hence, by the same argument as above, we have $[\Lambda \otimes_{\mathbb{Z}[G]} \mu_F] = 0$. This completes the proof. \square

7. SOME KNOWN CASES OF THE CONJECTURES

In the present section we collect some situations in which Conjecture 5.11 is known. The main result of the section is Theorem 7.2.

Definition 7.1. For a complex irreducible character χ of a finite group G , let $\mathbb{Q}(\chi)$ denote the field generated by the values of χ , and let $C(\chi)$ be the narrow class group of $\mathbb{Q}(\chi)$ if χ is symplectic, and the usual ideal class group of $\mathbb{Q}(\chi)$ otherwise.

Theorem 7.2. *Let F/K be a finite Galois extension of number fields, let G be the Galois group, let G' be its commutator subgroup, and let P be a set of prime numbers not dividing $2 \cdot \#G$. Suppose that $F^{G'}/\mathbb{Q}$ is abelian, and that for every complex irreducible character χ of G satisfying $\chi(1) > 1$, the group $C(\chi)$ is generated by the classes of the non-zero prime ideals of $\mathcal{O}_{\mathbb{Q}(\chi)}$ not dividing any element of P . Then Conjecture 5.11 holds for F/K and P .*

The rest of the section is devoted to the proof of Theorem 7.2 and some consequences.

Theorem 7.3. *Let F/K be a finite Galois extension of number fields and let G be the Galois group. Suppose that F/\mathbb{Q} is abelian. Then we have $\Omega(F/K, 3) = W_{F/K} = 0$ in $\text{Cl}(\mathbb{Z}[G])$.*

Proof. This is a special case of [BF06, Corollary 1.4]. \square

Corollary 7.4. *Let F/K be a finite Galois extension of number fields such that F/\mathbb{Q} is abelian. Then Conjecture 5.9 holds for F/K .*

Proof. The assertion immediately follows from Theorem 7.3 and Proposition 6.5. \square

If G is a finite group, we denote by $\text{Irr}_{\text{na}}(G)$ the set of complex irreducible characters of G of degree greater than 1, and for $\chi, \chi' \in \text{Irr}_{\text{na}}(G)$ we write $\chi \sim \chi'$ if χ and χ' are in the same Galois orbit, i.e., if there exists $\tau \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\chi = \tau \circ \chi'$.

Lemma 7.5. *Let G be a finite group and let G' be its commutator subgroup. Then there is a direct product decomposition of \mathbb{Q} -algebras*

$$\mathbb{Q}[G] \cong \mathbb{Q}[G/G'] \times \prod_{\chi \in \text{Irr}_{\text{na}}(G)/\sim} A_{\chi}, \quad (7.6)$$

where the product is taken over a full set of representatives of Galois orbits of non-abelian characters of G , and each A_{χ} is a simple \mathbb{Q} -algebra. Moreover, for a set P of prime numbers not dividing $\#G$, there is a corresponding direct product decomposition

$$\mathbb{Z}_{(P)}[G] \cong \mathbb{Z}_{(P)}[G/G'] \times \prod_{\chi \in \text{Irr}_{\text{na}}(G)/\sim} R_{\chi}, \quad (7.7)$$

where each R_χ is a maximal $\mathbb{Z}_{(P)}$ -order in A_χ .

Proof. The decomposition (7.6) is standard. By [CR81, Proposition (27.1)], the $\mathbb{Z}_{(P)}$ -order $\mathbb{Z}_{(P)}[G]$ is maximal in $\mathbb{Q}[G]$, so the decomposition (7.7) follows from [Rei75, Theorem (10.5)]. \square

Lemma 7.8. *Let G be a finite group, let P be a set of prime numbers, let χ be a complex irreducible character of G , let A_χ be the simple quotient of $\mathbb{Q}[G]$ corresponding to the Galois orbit of χ , let R_χ be the image of $\mathbb{Z}_{(P)}[G]$ in A_χ , and let $\mathbb{Q}(\chi)$ and $C(\chi)$ be as in Definition 7.1. Then $G_0(R_\chi)_{\text{tors}}$ is isomorphic to the quotient of $C(\chi)$ by the subgroup generated by the non-zero prime ideals of $\mathcal{O}_{\mathbb{Q}(\chi)}$ not dividing any element of P .*

Proof. This follows immediately from [CR87, Theorem (38.67)]. \square

Proof of Theorem 7.2. For an element m of $G_0(\mathbb{Z}[G])$, let $\mathbb{Z}_{(P)} \otimes m$ denote the image of m in $G_0(\mathbb{Z}_{(P)}[G])$. Then Lemma 6.6 implies that we have $\mathbb{Z}_{(P)} \otimes \Psi(F/K) \in G_0(\mathbb{Z}_{(P)}[G])_{\text{tors}}$, and that the claim is equivalent to the assertion that $\mathbb{Z}_{(P)} \otimes \Psi(F/K) = 0$. Moreover, by Lemma 7.5, there is a direct sum decomposition of abelian groups

$$G_0(\mathbb{Z}_{(P)}[G])_{\text{tors}} \cong G_0(\mathbb{Z}_{(P)}[G/G'])_{\text{tors}} \oplus \bigoplus_{\chi \in \text{Irr}_{\text{na}}(G)/\sim} G_0(R_\chi)_{\text{tors}}.$$

Therefore the assertion that $\mathbb{Z}_{(P)} \otimes \Psi(F/K) = 0$ is in turn equivalent to the claim that the image of $\Psi(F/K)$ in each of the above summands is 0.

Since P does not contain any prime divisors of $\#G'$, it is easily seen that the image of $\Psi(F/K)$ in $G_0(\mathbb{Z}_{(P)}[G/G'])_{\text{tors}}$ is equal to the image of $\Psi(F^{G'}/K)$. Since $F^{G'}/\mathbb{Q}$ is abelian, that image is 0 by Corollary 7.4.

By Lemma 7.8, the hypotheses of the theorem imply that $G_0(R_\chi)_{\text{tors}}$ is trivial for every $\chi \in \text{Irr}_{\text{na}}(G)$, so the result follows. \square

We conclude the section with a summary of what we know about Conjecture 5.11 for small groups.

Proposition 7.9. *Let F/\mathbb{Q} be a Galois extension of degree less than 112, let G be the Galois group, and let P be a set of prime numbers not dividing $2 \cdot \#G$. Then Conjecture 5.11 holds for F/K and P , with $K = \mathbb{Q}$.*

Proof. A direct computation, e.g. using the computational algebra system MAGMA [BCP97], shows that the assumption that $\#G < 112$ implies that the hypotheses of Theorem 7.2, with $K = \mathbb{Q}$ and P equal to the set of prime numbers not dividing $2 \cdot \#G$, are satisfied for F/K . A fortiori, the assumption is also satisfied with P being any smaller set. \square

Remark 7.10. There are exactly two groups of order 112 that have an irreducible character χ of degree greater than 1 such that, in the notation of the proof of Theorem 7.2, the group $G_0(R_\chi)_{\text{tors}}$ is non-trivial. Each has exactly one Galois orbit of such characters, in both cases of degree 2. The two groups are both semidirect products of a normal cyclic subgroup C of order 56 and a group H of order 2. Let $x \in C$ be an element of order 7, and let $y \in C$ be an element of order 8. In one semidirect product, the non-trivial element of H acts on C by $x \mapsto x^{-1}$ and $y \mapsto y^5$; and in the other it acts

on C by $x \mapsto x^{-1}$ and $y \mapsto y^3$. These are the two smallest Galois groups G for which we do not currently know Conjecture 5.11 with $K = \mathbb{Q}$ and with P being the set of all prime numbers not dividing $2 \cdot \#G$.

8. COUNTEREXAMPLES TO THE COHEN–LENSTRA–MARTINET HEURISTIC

In this section we prove Theorem 1.3 from the introduction and explain how it disproves Heuristic 3.2. The following notation will remain in force throughout the section.

Notation 8.1. Let p be an odd prime number, let C_2 and C_p be cyclic groups of orders 2 and p , respectively, and let $G = C_2 \wr C_p = C_2^p \rtimes C_p$, where C_p acts on C_2^p via its regular permutation action. Let Z be the centre of G , which is cyclic of order 2, and let $\gamma \in Z$ be the unique non-trivial central element. Let P be the set of all prime numbers not dividing $2p$, and let Λ be the quotient of $\mathbb{Z}_{(P)}[G]$ by the ideal generated by $1 + \gamma$. Let \mathcal{F} be the family of all pairs (F, i) , where F is a Galois number field satisfying $\mu_F = \{\pm 1\}$, and i is an isomorphism from G to the Galois group of F sending Z to the inertia group of every Archimedean place.

Proposition 8.2. *All complex irreducible characters χ of G with $\chi(1) > 1$ satisfy $\mathbb{Q}(\chi) = \mathbb{Q}$.*

Proof. Let χ be a complex irreducible character of G with $\chi(1) > 1$. Then by [Ser77, §8.2] we have $\chi = \text{Ind}_{C_2^p}^G \psi$ for some irreducible character ψ of C_2^p . Every such character ψ satisfies $\mathbb{Q}(\psi) = \mathbb{Q}$, so we also have $\mathbb{Q}(\chi) = \mathbb{Q}$. \square

Corollary 8.3. *Let $K = \mathbb{Q}$, and let F be a Galois extension of \mathbb{Q} with Galois group isomorphic to G . Then Conjecture 5.11 holds for F/K and P .*

Proof. This follows from Proposition 8.2 and Theorem 7.2. \square

Lemma 8.4. *There exist, up to isomorphism, infinitely many Galois number fields F with Galois group isomorphic to G such that the inertia groups at infinity map to Z and such that $\mu_F = \{\pm 1\}$.*

Proof. Let L/\mathbb{Q} be a cyclic extension of degree p and let H be the Galois group. Since $[L : \mathbb{Q}]$ is odd and L/\mathbb{Q} is Galois, L must be totally real. Let l be a prime number that splits completely in L , and let \mathfrak{l} be a place of L above l . By weak approximation, there exists $a \in L^\times$ such that a is totally negative, has \mathfrak{l} -adic normalised valuation 1, and for all $\sigma \in H \setminus \{1\}$ has $\sigma(\mathfrak{l})$ -adic valuation 0. In particular, a has the property that for every non-empty subset $\Sigma \subseteq H$ the product $\prod_{\sigma \in \Sigma} \sigma(a)$ is not a square in L^\times . The Galois closure $F = L(\{\sqrt{\sigma(a)} : \sigma \in H\})$ of $L(\sqrt{a})$ then has Galois group isomorphic to G such that the inertia groups at infinity map to Z . Moreover, the maximal abelian extension inside F is $F^{\text{ab}} = L(\sqrt{\text{Norm}(a)})$ where $\text{Norm}(a) = \prod_{\sigma \in H} \sigma(a) \in \mathbb{Q}^\times$, which has l -adic valuation 1. Thus F^{ab}/\mathbb{Q} is ramified at l , so as l varies, we obtain infinitely many extensions F . Of these, only finitely many can contain a non-trivial cyclotomic field, which completes the proof. \square

Proposition 8.5. *For all $(F, i) \in \mathcal{F}$ the class of $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ in $G_0(\Lambda)$ is trivial.*

Proof. Let $(F, i) \in \mathcal{F}$. Under the hypotheses the extension F/F^Z is a totally imaginary quadratic extension of a totally real field, and $\mu_F = \{\pm 1\}$, which imply that $\Lambda \otimes_{\mathbb{Z}[G]} \mathcal{O}_F^\times$ is trivial. Now Corollary 8.3 implies that Conjecture 5.11 holds in for F/K and P , and thus one has $[\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \text{Ar}_F] = [(\mathbb{Z}_{(P)})^{S_\infty}] - [\mathbb{Z}_{(P)}]$ in $G_0(\mathbb{Z}_{(P)}[G])$. Since γ acts trivially on the two terms on the right, this implies that $[\Lambda \otimes_{\mathbb{Z}[G]} \text{Ar}_F]$ is trivial in $G_0(\Lambda)$. By the first equality of Proposition 5.3 we have

$$[\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \text{Ar}_F] = j[\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times] - j[\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \text{Cl}_F]$$

in $G_0(\mathbb{Z}_{(P)}[G])$. Since j is the automorphism of $G_0(\mathbb{Z}_{(P)}[G])$ induced by the involution $\sigma \mapsto \sigma^{-1}$ on $\mathbb{Z}[G]$, and since we have $\gamma^{-1} = \gamma$, we conclude that

$$[\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F] = [\Lambda \otimes_{\mathbb{Z}[G]} \mathcal{O}_F^\times] = 0$$

in $G_0(\Lambda)$, as desired. \square

Lemma 8.6. *Let $p > 19$ be a prime number satisfying $p \equiv \pm 3 \pmod{8}$ and let ζ_p denote a primitive p -th root of unity. Then $\text{Cl}(\mathbb{Z}[\zeta_p, \frac{1}{2p}])$ is non-trivial.*

Proof. Let C be the class group of $\mathbb{Q}(\zeta_p)$ and let C^+ be the class group of the maximal real subfield $\mathbb{Q}(\zeta_p)^+$. By [Was97, Theorem 4.14], the natural map $C^+ \rightarrow C$ is an injection, and so we can and do view C^+ as a subgroup of C . By [Was97, Theorem 2.13], 2 splits into $(p-1)/f$ distinct primes in $\mathbb{Q}(\zeta_p)$, where f is the multiplicative order of 2 mod p . The condition $p \equiv \pm 3 \pmod{8}$ is equivalent to 2 being a quadratic non-residue mod p , and thus f must be even. Since $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is cyclic, we conclude that any prime of $\mathbb{Q}(\zeta_p)^+$ lying above 2 must be inert in $\mathbb{Q}(\zeta_p)/\mathbb{Q}(\zeta_p)^+$. Thus the class in C of any prime of $\mathbb{Q}(\zeta_p)$ above 2 must in fact lie in C^+ . Since $p > 19$, we have that $\#C/\#(C^+) > 1$ by [Was97, Corollary 11.18]. Moreover, the unique prime of $\mathbb{Q}(\zeta_p)$ above p is principal (generated by $1 - \zeta_p$). Therefore the quotient of C by the subgroup generated by the classes of primes above 2 and p is non-trivial, which gives the desired result. \square

Proposition 8.7. *Let $p > 19$ be a prime number satisfying $p \equiv \pm 3 \pmod{8}$. Then $G_0(\Lambda)_{\text{tors}}$ is non-trivial.*

Proof. Since G/G' is cyclic of order $2p$, we have that

$$\mathbb{Z}_{(P)}[G/G'] \cong \mathbb{Z}[\frac{1}{2p}] \times \mathbb{Z}[\zeta_p, \frac{1}{2p}] \times \mathbb{Z}[\frac{1}{2p}] \times \mathbb{Z}[\zeta_p, \frac{1}{2p}].$$

Let Λ' be the image of Λ under the projection $\mathbb{Z}_{(P)}[G] \rightarrow \mathbb{Z}_{(P)}[G/G']$. Then Λ' is a direct factor of Λ and $\Lambda' \cong \mathbb{Z}[\frac{1}{2p}] \times \mathbb{Z}[\zeta_p, \frac{1}{2p}]$. Thus $G_0(\Lambda')_{\text{tors}}$ is a direct summand of $G_0(\Lambda)_{\text{tors}}$ and $G_0(\Lambda')_{\text{tors}} \cong \text{Cl}(\mathbb{Z}[\zeta_p, \frac{1}{2p}])$ by [CR87, Theorem (38.67)]. Therefore the result follows from Lemma 8.6. \square

Remark 8.8. The conclusions of Lemma 8.6 and Proposition 8.7 hold under the weaker assumption on p that $p > 19$ be a prime number such that 2 generates a group of even cardinality in $(\mathbb{Z}/p\mathbb{Z})^\times$, and the proofs carry over almost verbatim. The set of all such primes has been investigated, and Hasse [Has66] has computed its Dirichlet density to be $17/24$.

Proof of Theorem 1.3. Let p be a prime number satisfying the hypothesis of Proposition 8.7, and let G and Λ be as in Notation 8.1. Then $G_0(\Lambda)_{\text{tors}}$ is non-trivial. The family \mathcal{F} as in Notation 8.1 is infinite by Lemma 8.4, and for all $(F, i) \in \mathcal{F}$ the class of $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ in $G_0(\Lambda)$ is trivial by Proposition 8.5. \square

Finally, the next result shows that Heuristic 3.2 is false for \mathcal{F} and Λ as above and for some natural functions f .

Proposition 8.9. *Suppose that for all homomorphisms $\phi: G_0(\Lambda)_{\text{tors}} \rightarrow \mathbb{C}^\times$, Heuristic 3.2 holds with $K = \mathbb{Q}$ and with f being the function that assigns to a finite Λ -module M the value of ϕ on the class of M in $G_0(\Lambda)_{\text{tors}}$. Then as (F, i) ranges over \mathcal{F} , the class of $\Lambda \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ in $G_0(\Lambda)_{\text{tors}}$ is equidistributed.*

Proof. By Lemma 7.5 there is a direct product decomposition

$$\mathbb{Z}_{(P)}[G] \cong \mathbb{Z}_{(P)}[G/G'] \times \prod_{\chi \in \text{Irr}_{\text{na}}(G)/\sim} R_\chi.$$

Note that the group G/G' is cyclic of order $2p$. Let $\bar{\gamma}$ be the image of γ under the projection map $G \rightarrow G/G'$, and let $\bar{\Lambda}$ be the quotient of $\mathbb{Z}_{(P)}[G/G']$ by the two-sided ideal generated by $1 + \bar{\gamma}$. Then $\Lambda \cong \bar{\Lambda} \times T$, where T is a direct factor of $\prod_{\chi \in \text{Irr}_{\text{na}}(G)/\sim} R_\chi$. Moreover, for $\chi \in \text{Irr}_{\text{na}}(G)$, we have that $\mathbb{Q}(\chi) = \mathbb{Q}$ by Proposition 8.2, and thus $G_0(R_\chi)_{\text{tors}}$ is trivial by Lemma 7.8. Hence there is a canonical isomorphism $G_0(\Lambda)_{\text{tors}} \cong G_0(\bar{\Lambda})_{\text{tors}}$. The set \mathcal{F} is infinite by Lemma 8.4. Note that $F^{G'}$ is an imaginary abelian number field that is a quadratic extension of a real field, and $\bar{\Lambda} \otimes_{\mathbb{Z}[G]} \text{Cl}_F$ is the maximal quotient of $\mathbb{Z}_{(P)} \otimes_{\mathbb{Z}} \text{Cl}_{F^{G'}}$ on which complex conjugation acts by -1 . Equidistribution of minus parts of class groups in the corresponding Grothendieck group was shown in [BLJ20, Proposition 4.4] to follow from the Cohen–Lenstra–Martinet heuristic for families of imaginary extensions *with Galois group isomorphic to G/G'* . However, it is easy to see that the prediction is the same in the present situation, despite the fields being ordered differently; see Remark 3.1. \square

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