

# Monotonicity properties of the Poisson approximation to the binomial distribution

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## Abstract

Certain monotonicity properties of the Poisson approximation to the binomial distribution are established. As a natural application of these results, exact (rather than approximate) tests of hypotheses on an unknown value of the parameter  $p$  of the binomial distribution are presented.

*Keywords:* binomial distribution, Poisson distribution, approximation, monotonicity, total variation distance, Kolmogorov's distance, tests of significance

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## 1. Introduction and summary

For any natural number  $n$  and any  $p \in (0, 1)$ , let  $X_{n,p}$  denote a random variable (r.v.) having the binomial distribution with parameters  $n$  and  $p$ . For any positive real number  $\lambda$ , let  $\Pi_\lambda$  denote a r.v. having the Poisson distribution with parameter  $\lambda$ .

There are a large number of results on the accuracy of the Poisson approximation to the binomial distribution; see e.g. the survey [8]. In particular, [8, inequality (29)] (which is based on [3]) implies that

$$d_{\text{TV}}(X_{n,p}, \Pi_{np}) \leq (1 - e^{-np})p < np^2, \quad (1.1)$$

where  $d_{\text{TV}}$  is the total variation distance, defined by the formula

$$d_{\text{TV}}(X, Y) := \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|$$

for any r.v.'s  $X$  and  $Y$ , with  $\mathcal{B}(\mathbb{R})$  denoting the Borel  $\sigma$ -algebra over  $\mathbb{R}$ .

The total variation distance  $d_{\text{TV}}$  has the following easy to establish but important shift property:

$$d_{\text{TV}}(X + Z, Y + Z) \leq d_{\text{TV}}(X, Y)$$

for any r.v.'s  $X, Y, Z$  such that  $Z$  is independent of  $X$  and of  $Y$ . Since  $d_{\text{TV}}(X_{1,p} - X_{1,r}) = |p - r|$  for  $r \in (0, 1)$ , inequality (1.1), together with the pseudo-metric and shift properties of  $d_{\text{TV}}$ , immediately yields

$$d_{\text{TV}}(X_{n,p}, \Pi_\lambda) \leq d_{\text{TV}}(X_{n,p}, X_{n,\lambda/n}) + d_{\text{TV}}(X_{n,\lambda/n}, \Pi_\lambda) \leq |np - \lambda| + (1 - e^{-\lambda})\lambda/n.$$

So,  $d_{\text{TV}}(X_{n,p}, \Pi_\lambda) \rightarrow 0$  whenever  $n, p, \lambda$  vary in any manner such that  $np - \lambda \rightarrow 0$  and  $\min(\lambda, \lambda^2) = o(n)$ .

Note that

$$d_{\text{TV}}(X_{1,p}, \Pi_{\lambda_p^\circ}) \leq \Delta(p) := p + (1 - p) \ln(1 - p) \underset{p \downarrow 0}{\sim} p^2/2, \quad (1.2)$$

where

$$\lambda_p^\circ := -\ln(1 - p). \quad (1.3)$$

Using again the pseudo-metric and shift properties of  $d_{\text{TV}}$ , we immediately get

$$d_{\text{TV}}(X_{n,p}, \Pi_{n\lambda_p^\circ}) \leq n\Delta(p) \underset{p \downarrow 0}{\sim} np^2/2;$$

cf. [11, Theorem 4.1] and (1.1).

The following statement, describing the monotonicity pattern of  $d_{\text{TV}}(X_{1,p}, \Pi_\lambda)$  in  $\lambda$ , implies that the choice  $\lambda = \lambda_p^\circ$  in (1.2) is optimal if  $p \leq 1 - e^{-1}$ .

**Proposition 1.** *For each  $p \in (0, 1)$ ,  $d_{\text{TV}}(X_{1,p}, \Pi_\lambda)$  is (strictly) decreasing in  $\lambda \in (0, \lambda_p^*]$  and (strictly) increasing in  $\lambda \in [\lambda_p^*, \infty)$ , where*

$$\lambda_p^* := \min(\lambda_p^\circ, 1) = \begin{cases} \lambda_p^\circ & \text{if } p \leq 1 - e^{-1}, \\ 1 & \text{if } p \geq 1 - e^{-1}; \end{cases}$$

hence,

$$\begin{aligned} \min_{\lambda > 0} d_{\text{TV}}(X_{1,p}, \Pi_\lambda) &= d_{\text{TV}}(X_{1,p}, \Pi_{\lambda_p^*}) = \min [d_{\text{TV}}(X_{1,p}, \Pi_{\lambda_p^\circ}), d_{\text{TV}}(X_{1,p}, \Pi_1)] \\ &= \begin{cases} p + (1 - p) \ln(1 - p) & \text{if } p \leq 1 - e^{-1}, \\ p - e^{-1} & \text{if } p \geq 1 - e^{-1}. \end{cases} \end{aligned}$$

In view of the pseudo-metric and shift properties of  $d_{\text{TV}}$ , Proposition 1 immediately yields

**Corollary 2.**

$$\begin{aligned} \min_{\lambda > 0} d_{\text{TV}}(X_{n,p}, \Pi_\lambda) &\leq d_{\text{TV}}(X_{n,p}, \Pi_{n\lambda_p^*}) \\ &\leq \begin{cases} n(p + (1 - p) \ln(1 - p)) \underset{p \downarrow 0}{\sim} np^2/2 & \text{if } p \leq 1 - e^{-1}, \\ n(p - e^{-1}) & \text{if } p \geq 1 - e^{-1}. \end{cases} \end{aligned}$$

Along with the total variation distance  $d_{\text{TV}}$ , the Kolmogorov distance, defined by the formula

$$d_{\text{K}}(X, Y) := \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|,$$

has been extensively studied. Clearly,  $d_{\text{K}} \leq d_{\text{TV}}$ . Therefore, all the upper bounds on  $d_{\text{TV}}(X, Y)$  hold for  $d_{\text{K}}(X, Y)$ .

In the sequel, we always assume that

$$m \in \{1, \dots, n\}.$$

We also use the notation  $u \vee v := \max(u, v)$  for real  $u$  and  $v$ .

**Theorem 3.** *For any  $p_n$  and  $p_{n+1}$  in the interval  $[0, 1]$  such that  $p_n > p_{n+1}$ , the following statements hold:*

- (i) *If  $(n+1)p_{n+1} \geq np_n$  and  $m \geq 1 + np_n$ , then*  

$$\mathbb{P}(X_{n+1, p_{n+1}} \geq m) > \mathbb{P}(X_{n, p_n} \geq m).$$
- (ii) *If  $(n+1)p_{n+1} \leq np_n$  and  $m \leq 1 + np_{n+1}$ , then*  

$$\mathbb{P}(X_{n+1, p_{n+1}} \geq m) < \mathbb{P}(X_{n, p_n} \geq m).$$

As is well known,  $X_{n,p}$  is stochastically monotone in  $p$ . Therefore, part (i) of Theorem 3 immediately follows from the second inequality in [1, Theorem 2.1], whereas part (ii) of Theorem 3 similarly follows from the first inequality in [1, Theorem 2.3]. In turn, the second inequality in [1, Theorem 2.1] was obtained in [1] as an immediate consequence of a more general result [5], whereas the first inequality in [1, Theorem 2.3] was proved by a different method.

In this note, we shall give a proof of Theorem 3 by a single method, which works equally well for both parts of Theorem 3.

Letting  $p_n := \lambda/n$  in Theorem 3, one immediately obtains

**Corollary 4.** ([1, Corollary 2.1]) *Take any  $\lambda \in (0, \infty)$ .*

- (i) *If  $m \geq 1 + \lambda$ , then  $\mathbb{P}(X_{n, \lambda/n} \geq m)$  is (strictly) increasing in natural  $n \geq \lambda \vee m = m$  to  $\mathbb{P}(\Pi_\lambda \geq m)$ ; in particular, it follows that*

$$\mathbb{P}(X_{n, \lambda/n} \geq m) < \mathbb{P}(\Pi_\lambda \geq m) \tag{1.4}$$

*for such  $\lambda, n, m$ .*

- (ii) *If  $m \leq \lambda$ , then  $\mathbb{P}(X_{n, \lambda/n} \geq m)$  is (strictly) decreasing in natural  $n \geq \lambda \vee m = \lambda$  to  $\mathbb{P}(\Pi_\lambda \geq m)$ ; in particular, it follows that*

$$\mathbb{P}(X_{n, \lambda/n} \geq m) > \mathbb{P}(\Pi_\lambda \geq m) \tag{1.5}$$

*for such  $\lambda, n, m$ .*

In turn, Corollary 4 immediately yields the following monotonicity of concentration property.

**Corollary 5.** *Take any  $\lambda \in (0, \infty)$ . If natural numbers  $m_1$  and  $m_2$  are such that  $m_1 \leq \lambda \leq m_2$ , then  $\mathbb{P}(m_1 \leq X_{n,\lambda/n} \leq m_2)$  is decreasing in natural  $n \geq m_2 + 1$  to  $\mathbb{P}(m_1 \leq \Pi_\lambda \leq m_2)$ ; in particular, it follows that*

$$\mathbb{P}(m_1 \leq X_{n,\lambda/n} \leq m_2) > \mathbb{P}(m_1 \leq \Pi_\lambda \leq m_2) \quad (1.6)$$

for such  $\lambda, n, m_1, m_2$ .

Another monotonicity result is

**Theorem 6.** *If  $p_n = 1 - e^{-\lambda/n}$  for all natural  $n$ , then*

$$\mathbb{P}(X_{n+1,p_{n+1}} \geq m) > \mathbb{P}(X_{n,p_n} \geq m) \quad (1.7)$$

for all natural  $n$  and all natural  $m \in [2, n + 1]$ . For  $m = 1$ , inequality (1.7) turns into the equality.

The choice  $p_n = 1 - e^{-\lambda/n}$  of  $p$  corresponds to (1.3); cf. also Proposition 1.

Noting that  $p_n = 1 - e^{-\lambda/n}$  implies  $np_n \rightarrow \lambda$ , we immediately have the following corollary of Theorem 6:

**Corollary 7.** *Take any  $\lambda \in (0, \infty)$  and any natural  $m \geq 2$ . If  $p_n = 1 - e^{-\lambda/n}$  for all natural  $n$ , then  $\mathbb{P}(X_{n,p_n} \geq m)$  is (strictly) increasing in natural  $n \geq m - 1$  to  $\mathbb{P}(\Pi_\lambda \geq m)$ ; in particular, it follows that*

$$\mathbb{P}(X_{n,p_n} \geq m) < \mathbb{P}(\Pi_\lambda \geq m)$$

for such  $\lambda, n, m$ .

It follows from Theorem 6 that the family  $(X_{n,p_n})_{n=1}^\infty$  is stochastically monotone; more specifically, it is stochastically nondecreasing. A natural way to establish the stochastic monotonicity (SM) of a family of r.v.'s is to derive it from the monotone likelihood ratio property (MLR), which implies the monotone tail ratio property (MTR), which in turn implies the SM; for the discrete case, see e.g. Theorems 1.7(b) and 1.6 and Corollary 1.4 in [6].

However, subtler tools than the MLR are needed to prove Theorems 3 and 6. Indeed, the family  $(X_{n,\lambda/n})_{n=1}^\infty$ , considered in Corollary 4 of Theorem 3, cannot have the MLR – because then, in view of the aforementioned implications  $\text{MLR} \implies \text{MTR} \implies \text{SM}$ , inequalities (1.4) and (1.5) would have to go in the same direction.

We cannot use the same kind of quick argument concerning Theorem 6, because it does imply the SM of the family  $(X_{n,p_n})_{n=1}^\infty$  (with  $p_n = 1 - e^{-\lambda/n}$ ). Yet, we still have

**Proposition 8.** *In general, the family  $(X_{n,p_n})_{n=1}^\infty$  with  $p_n = 1 - e^{-\lambda/n}$ , considered in Theorem 6, does not have the MLR.*

A natural application of inequalities (1.4), (1.5), and (1.6) in Corollaries 4 and 5 (taking also into account the previously mentioned stochastic monotonicity of  $X_{n,p}$  in  $p$ ) is to exact, conservative – rather than approximate – tests of hypotheses on an unknown value of the parameter  $p$  of the binomial distribution:

**Corollary 9.** *Take any natural  $n$  and any  $p_0 \in (0, 1)$ . Let  $\mathbb{I}\{\cdot\}$  denote the indicator function.*

- (i) *For any natural  $m \geq np_0 + 1$  and  $n \geq m$ , the test  $\delta_+(X_{n,p}) := \mathbb{I}\{X_{n,p} \geq m\}$  of the null hypothesis  $H_0: p = p_0$  (or  $H_0: p \leq p_0$ ) versus the (right-sided) alternative  $H_1: p > p_0$  is of level  $\alpha_+ := \mathbb{P}(\Pi_{np_0} \geq m)$ ; that is,  $\mathbb{E} \delta_+(X_{n,p}) \leq \alpha_+$  for all  $p \leq p_0$ .*
- (ii) *For any natural  $m \leq np_0 + 1$ , the test  $\delta_-(X_{n,p}) := \mathbb{I}\{X_{n,p} \leq m\}$  of the null hypothesis  $H_0: p = p_0$  (or  $H_0: p \geq p_0$ ) versus the (left-sided) alternative  $H_1: p < p_0$  is of level  $\alpha_- := \mathbb{P}(\Pi_{np_0} \leq m)$ ; that is,  $\mathbb{E} \delta_-(X_{n,p}) \leq \alpha_-$  for all  $p \geq p_0$ .*
- (iii) *For any natural  $m_1, m_2$ , and  $n$  such that  $m_1 \leq np_0 \leq m_2$  and  $n \geq m_2 + 1$ , the test  $\delta_{\pm}(X_{n,p}) := 1 - \mathbb{I}\{m_1 \leq X_{n,p} \leq m_2\}$  of the null hypothesis  $H_0: p = p_0$  versus the (two-sided) alternative  $H_1: p \neq p_0$  is of level  $\alpha_{\pm} := 1 - \mathbb{P}(m_1 \leq \Pi_{np_0} \leq m_2)$ ; that is,  $\mathbb{E} \delta_{\pm}(X_{n,p_0}) \leq \alpha_{\pm}$ .*

Corollary 9 follows immediately from (1.4), (1.5), and (1.6), in view of the stochastic monotonicity of  $X_{n,p}$  in  $p$ . Here one may note that parts (i) and (ii) of Corollary 9 do not immediately follow from each other, because of the absence of the required symmetry.

*Remark 10.* It is well known (see e.g. [7, Theorem 3.4.1]) that the test  $\delta_+(X_{n,p}) = \mathbb{I}\{X_{n,p} \geq m\}$  of  $H_0: p = p_0$  (or  $H_0: p \leq p_0$ ) versus  $H_1: p > p_0$  is a uniformly most powerful (UMP) test but of level  $\mathbb{P}(X_{n,p_0} \geq m)$  rather than  $\mathbb{P}(\Pi_{np_0} \geq m)$ . The test  $\delta_-(X_{n,p}) = \mathbb{I}\{X_{n,p} \leq m\}$  in part (ii) of Corollary 9 has the similar property.

Concerning Remark 10, Corollaries 4 and 7, and otherwise, one may also note the following result [4]: for all  $A \subseteq \mathbb{R}$  and  $p \in [0, 1)$ ,

$$\mathbb{P}(X_{n,p} \in A) \leq \frac{\mathbb{P}(\Pi_{np} \in A)}{1 - p},$$

which implies

$$\mathbb{P}(X_{n,p} \in A) \geq \frac{\mathbb{P}(\Pi_{np} \in A) - p}{1 - p},$$

again for all  $A \subseteq \mathbb{R}$  and  $p \in [0, 1)$ . Other bounds on the tail probabilities of  $X_{n,p}$  were given e.g. in [2].

## 2. Proofs

*Proof of Proposition 1.* We have

$$d(\lambda) := 2d_{\text{TV}}(X_{1,p}, \Pi_{\lambda}) = |1 - p - e^{-\lambda}| + |p - \lambda e^{-\lambda}| + 1 - e^{-\lambda} - \lambda e^{-\lambda}. \quad (2.1)$$

Let

$$\lambda_1 := \lambda_1(p) := -\ln(1-p) = \lambda_p^\circ, \quad (2.2)$$

so that

$$\lambda_1 \leq 1 \iff p \leq 1 - e^{-1}. \quad (2.3)$$

Note that  $\lambda e^{-\lambda}$  is continuously increasing in  $\lambda \in (0, 1]$  from 0 to  $e^{-1}$  and continuously decreasing in  $\lambda \in [1, \infty)$  from  $e^{-1}$  back to 0. Therefore,

$$\lambda e^{-\lambda} > p \iff (p \leq e^{-1} \ \& \ \lambda_2 < \lambda < \lambda_3), \quad (2.4)$$

where  $\lambda_2 = \lambda_2(p)$  and  $\lambda_3 = \lambda_3(p)$  are the unique roots  $\lambda$  of the equation  $\lambda e^{-\lambda} = p$  in the intervals  $(0, 1]$  and  $[1, \infty)$ , respectively.

Further, for all  $\lambda > 0$  the inequality  $e^\lambda > 1 + \lambda$  can be rewritten as  $-\ln(1 - \lambda e^{-\lambda}) < \lambda$ . Hence,  $\lambda_1 = -\ln(1 - p) = -\ln(1 - \lambda_2 e^{-\lambda_2}) < \lambda_2$ , so that for all  $p \leq e^{-1}$

$$0 < \lambda_1 < \lambda_2 \leq 1 \leq \lambda_3 < \infty. \quad (2.5)$$

So, to complete the proof of Proposition 1, it suffices to show that

- (I) for  $p \leq e^{-1}$ ,  $d(\lambda)$  is decreasing in  $\lambda \in (0, \lambda_1]$  and increasing in  $\lambda \in [\lambda_1, \lambda_2]$ , in  $\lambda \in [\lambda_2, \lambda_3]$ , and in  $\lambda \in [\lambda_3, \infty)$ ;
- (II) for  $p \in (e^{-1}, 1 - e^{-1}]$ ,  $d(\lambda)$  is decreasing in  $\lambda \in (0, \lambda_1]$  and increasing in  $\lambda \in [\lambda_1, \infty)$ ;
- (III) for  $p > 1 - e^{-1}$ ,  $d(\lambda)$  is decreasing in  $\lambda \in (0, 1]$  and increasing in  $\lambda \in [1, \lambda_1]$  and in  $\lambda \in [\lambda_1, \infty)$ .

Thus, we have to consider the following corresponding cases.

*Case I.1:*  $p \leq e^{-1}$  and  $\lambda \in (0, \lambda_1]$ . Then, by (2.3),  $\lambda_1 \leq 1$  and, in view of (2.1), (2.2), (2.4), and (2.5),

$$d(\lambda) = e^{-\lambda} - 1 + p + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2p - 2\lambda e^{-\lambda},$$

which is decreasing in  $\lambda \in (0, 1]$  and hence in  $\lambda \in (0, \lambda_1]$ .

*Case I.2:*  $p \leq e^{-1}$  and  $\lambda \in [\lambda_1, \lambda_2]$ . Then

$$d(\lambda) = 1 - p - e^{-\lambda} + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2(1 - (1 + \lambda)e^{-\lambda}),$$

which is (easily seen to be) increasing in  $\lambda \geq 0$  and hence in  $\lambda \in [\lambda_1, \lambda_2]$ .

*Case I.3:*  $p \leq e^{-1}$  and  $\lambda \in [\lambda_2, \lambda_3]$ . Then

$$d(\lambda) = 1 - p - e^{-\lambda} + \lambda e^{-\lambda} - p + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2 - 2p - 2e^{-\lambda},$$

which is increasing in  $\lambda \geq 0$  and hence in  $\lambda \in [\lambda_2, \lambda_3]$ .

*Case I.4:*  $p \leq e^{-1}$  and  $\lambda \in [\lambda_3, \infty)$ . Then

$$d(\lambda) = 1 - p - e^{-\lambda} + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2(1 - (1 + \lambda)e^{-\lambda}),$$

the same as the expression for  $d(\lambda)$  in Case I.2, where this expression was seen to be increasing in  $\lambda \geq 0$  and hence in  $\lambda \in [\lambda_3, \infty)$ .

*Case II.1:*  $p \in (e^{-1}, 1 - e^{-1}]$  and  $\lambda \in (0, \lambda_1]$ . Then  $\lambda_1 \leq 1$  and

$$d(\lambda) = e^{-\lambda} - 1 + p + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2p - 2\lambda e^{-\lambda},$$

which is decreasing in  $\lambda \in (0, 1]$  and hence in  $\lambda \in (0, \lambda_1]$ .

*Case II.2:*  $p \in (e^{-1}, 1 - e^{-1}]$  and  $\lambda \in [\lambda_1, \infty)$ . Then  $\lambda_1 \leq 1$  and

$$d(\lambda) = 1 - p - e^{-\lambda} + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2(1 - (1 + \lambda)e^{-\lambda}),$$

the same as the expression for  $d(\lambda)$  in Case I.2, where this expression was seen to be increasing in  $\lambda \geq 0$  and hence in  $\lambda \in [\lambda_1, \infty)$ .

*Case III.1:*  $p > 1 - e^{-1}$  and  $\lambda \in (0, 1]$ . Then, by (2.3),  $\lambda_1 > 1$  and

$$d(\lambda) = e^{-\lambda} - 1 + p + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2p - 2\lambda e^{-\lambda},$$

which is decreasing in  $\lambda \in (0, 1]$ .

*Case III.2:*  $p > 1 - e^{-1}$  and  $\lambda \in [1, \lambda_1]$ . Then

$$d(\lambda) = e^{-\lambda} - 1 + p + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2p - 2\lambda e^{-\lambda},$$

which is increasing in  $\lambda \geq 1$  and hence in  $\lambda \in [1, \lambda_1]$ .

*Case III.3:*  $p > 1 - e^{-1}$  and  $\lambda \in [\lambda_1, \infty)$ . Then

$$d(\lambda) = 1 - p - e^{-\lambda} + p - \lambda e^{-\lambda} + 1 - e^{-\lambda} - \lambda e^{-\lambda} = 2(1 - (1 + \lambda)e^{-\lambda}),$$

the same as the expression for  $d(\lambda)$  in Case I.2, where this expression was seen to be increasing in  $\lambda \geq 0$  and hence in  $\lambda \in [\lambda_1, \infty)$ .

The proof of Proposition 1 is now complete.  $\square$

*Proof of Theorem 3.* It is well known that

$$Q_n := \mathbb{P}(X_{n,p_n} \geq m) = \frac{n!}{(m-1)!(n-m)!} J_n, \quad (2.6)$$

where

$$J_n := \int_{1-p_n}^1 t^{n-m}(1-t)^{m-1} dt; \quad (2.7)$$

see e.g. [9, formula (3)]. (The expression for  $Q_n$  in (2.6) can be obtained by (say) repeated integration by parts for the integral in (2.7).) Therefore,

$$Q_{n+1} - Q_n \stackrel{\text{sign}}{=} \Delta_n := (n+1)J_{n+1} - (n-m+1)J_n \quad (2.8)$$

$$= (n-m+1)I_1 - (n+1)I_2, \quad (2.9)$$

where  $A \stackrel{\text{sign}}{=} B$  means  $\text{sign } A = \text{sign } B$ ,

$$I_1 := \int_0^{1-p_n} t^{n-m}(1-t)^{m-1} dt, \quad \text{and} \quad I_2 := \int_0^{1-p_{n+1}} t^{n-m+1}(1-t)^{m-1} dt \quad (2.10)$$

(in fact,  $Q_{n+1} - Q_n = \binom{n}{m-1} \Delta_n$ ); the equality in (2.9) holds because  $I_1 + J_n = B(n - m + 1, m)$  and  $I_2 + J_{n+1} = B(n - m + 2, m)$ , where  $B(k, m) := \int_0^1 t^{k-1} (1-t)^{m-1} dt = (k-1)!(m-1)!/(k+m-1)!$ , so that  $(n-m+1)(I_1 + J_n) = (n+1)(I_2 + J_{n+1})$ . Next,

$$I_1 = I_{11} + I_{12}, \quad (2.11)$$

where

$$I_{11} := \int_0^{1-p_n} t^{n-m+1} (1-t)^{m-1} dt \quad \text{and} \quad I_{12} := \int_0^{1-p_n} t^{n-m} (1-t)^m dt; \quad (2.12)$$

this follows because the sum of the integrands in  $I_{11}$  and  $I_{12}$  equals the integrand in  $I_1$ . Further, integrating by parts, we see that

$$(n-m+1)I_{12} = (1-p_n)^{n-m+1} p_n^m + mI_{11}. \quad (2.13)$$

Collecting now (2.9), (2.11), (2.13), (2.12), and (2.10), we have

$$\begin{aligned} \Delta_n &= (1-p_n)^{n-m+1} p_n^m + (n+1)(I_{11} - I_2) \\ &= (1-p_n)^{n-m+1} p_n^m - (n+1) \int_{1-p_n}^{1-p_{n+1}} g(t) dt, \end{aligned} \quad (2.14)$$

where  $g(t) := t^{n-m+1} (1-t)^{m-1}$ . The function  $g$  is (strictly) increasing on the interval  $[0, 1 - \frac{m-1}{n}]$  and decreasing on  $[1 - \frac{m-1}{n}, 1]$ . So, the condition  $m \geq 1 + np_n$ , which is equivalent to the condition  $1 - \frac{m-1}{n} \leq 1 - p_n$ , implies that  $g(t) < g(1-p_n) = (1-p_n)^{n-m+1} p_n^{m-1}$  for  $t \in (1-p_n, 1-p_{n+1})$ , whence, by (2.14),

$$\begin{aligned} \Delta_n &> (1-p_n)^{n-m+1} p_n^m - (n+1)(p_n - p_{n+1})(1-p_n)^{n-m+1} p_n^{m-1} \\ &\stackrel{\text{sign}}{=} p_n - (n+1)(p_n - p_{n+1}) = (n+1)p_{n+1} - np_n. \end{aligned}$$

Now part (i) of Theorem 3 follows from the relation  $\stackrel{\text{sign}}{=}$  in (2.8) and the definition of  $Q_n$  in (2.6).

The proof of part (ii) of Theorem 3 is completed quite similarly. Here, we note that the condition  $m \leq 1 + np_{n+1}$  is equivalent to the condition  $1 - \frac{m-1}{n} \geq 1 - p_{n+1}$ , which latter implies that  $g(t) > g(1-p_n) = (1-p_n)^{n-m+1} p_n^{m-1}$  for  $t \in (1-p_n, 1-p_{n+1})$ .  $\square$

*Proof of Theorem 6.* The case  $m = 1$  is trivial, because for  $p_n = 1 - e^{-\lambda/n}$  we have  $P(X_{n,p_n} \geq 1) = 1 - (1-p_n)^n = 1 - e^{-\lambda}$  for all natural  $n$ .

The case  $m = n + 1$  is also trivial.

Suppose now that  $1 < m < n + 1$ . In view of the definitions of  $\Delta_n$  and  $J_n$  in (2.8) and (2.7), for  $\Delta_n(\lambda)$  denoting  $\Delta_n$  with  $p_n = 1 - e^{-\lambda/n}$ , we have

$$\Delta'_n(\lambda) \frac{e^\lambda}{(e^{\lambda/n} - 1)^{m-1}} = \Delta_{n,1}(\lambda) := \left( \frac{e^{\lambda/(n+1)} - 1}{e^{\lambda/n} - 1} \right)^{m-1} - \frac{n-m+1}{n}.$$

Next,

$$\frac{(e^{\lambda/(n+1)} - 1)'_{\lambda}}{(e^{\lambda/n} - 1)'_{\lambda}} = \frac{n}{n+1} e^{-\lambda/(n^2+n)}$$

is decreasing in  $\lambda > 0$ . So, by the special-case l'Hospital-type rule for monotonicity (see e.g. [10, Proposition 4.1]),  $\frac{e^{\lambda/(n+1)} - 1}{e^{\lambda/n} - 1}$  is decreasing in  $\lambda > 0$  and hence  $\Delta'_n(\lambda)$  can only switch its sign from  $+$  to  $-$  as  $\lambda$  is increasing from 0 to  $\infty$ . So, for all real  $\lambda > 0$ ,

$$\Delta_n = \Delta_n(\lambda) \geq \min[\Delta_n(0), \Delta_n(\infty-)] = 0, \quad (2.15)$$

since  $\Delta_n(0) = 0 = \Delta_n(\infty-)$ .

Moreover,

$$\Delta_{n,1}(0+) = g(m) := \left(\frac{n}{n+1}\right)^{m-1} - \frac{n-m+1}{n} > 0$$

for  $m > 1$ , because  $g(1) = 0$ ,  $g'(1) = \frac{1}{n} - \ln(1 + \frac{1}{n}) > 0$ , and the function  $g$  is convex. Also,

$$\Delta_{n,1}(\infty-) = -\frac{n-m+1}{n} < 0$$

for  $m \in (1, n]$ . So,  $\Delta_n(\lambda)$  is actually strictly increasing in  $\lambda$  in a right neighborhood of 0 and strictly decreasing in  $\lambda$  in a left neighborhood of  $\infty$ . So, the inequality in (2.15) is actually strict. Now (1.7) follows by the  $\stackrel{\text{sign}}{=}$  relation in (2.8) and the definition of  $Q_n$  in (2.6).  $\square$

*Proof of Proposition 8.* The MLR of the family  $(X_{n,p_n})_{n=1}^{\infty}$  with  $p_n = 1 - e^{-\lambda/n}$  consistent with the stochastic monotonicity (1.7) means that for all natural  $n$  and all integers  $k$  such that  $0 \leq k \leq n-1$  we have

$$\delta_{n,k} := P_{n+1,k+1}P_{n,k} - P_{n,k+1}P_{n+1,k} \geq 0, \quad (2.16)$$

where  $P_{n,k} := \mathbb{P}(X_{n,p_n} = k)$ . It is not hard to see that

$$\delta_{n,k} \stackrel{\text{sign}}{=} \tilde{\delta}_{n,k} := -(n-k)(e^{\lambda/n} - e^{\lambda/(n+1)}) + e^{\lambda/(n+1)} - 1$$

Letting now, for instance,  $k \sim an$  and  $\lambda \sim cn$  as  $n \rightarrow \infty$ , with constant  $a \in (0, 1)$  and  $c \in (0, \infty)$ , we see that  $\tilde{\delta}_{n,k} \rightarrow \ell(a, c) := (a - h(c))ce^c < 0$  for  $a \in (0, h(c))$ , where  $h(c) := \frac{e^{-c}-1+c}{c}$ , which latter is increasing in  $c \in (0, \infty)$  from 0 to 1. Thus, inequality (2.16) will fail to hold when  $k \sim an$ ,  $\lambda \sim cn$ ,  $c \in (0, \infty)$ ,  $a \in (0, h(c))$ , and  $n$  is large enough. This completes the proof of Proposition 8.  $\square$

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