# Harnack Inequalities and Ergodicity of Stochastic Reaction-Diffusion Equation in $I^p$

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ABSTRACT. We derive Harnack inequalities for a stochastic reaction-diffusion equation with dissipative drift driven by additive rough noise in the  $L^p$ -space, for any  $p \geq 2$ . These inequalities are used to study the ergodicity of the corresponding Markov semigroup  $(P_t)_{t\geq 0}$ . The main ingredient of our method is a coupling by the change of measure. Applying our results to the stochastic reaction-diffusion equation with a super-linear growth drift having a negative leading coefficient, perturbed by a Lipschitz term, indicates that  $(P_t)_{t\geq 0}$  possesses a unique and thus ergodic invariant measure in  $L^p$  for all  $p\geq 2$ , which is independent of the Lipschitz term.

#### 1. Introduction

We consider the stochastic reaction-diffusion equation

$$(1.1) \quad \frac{\partial X_t(\xi)}{\partial t} = \Delta X_t(\xi) + f(X_t(\xi)) + G \frac{\partial W_t(\xi)}{\partial t}, \quad (t, \xi) \in \mathbb{R}_+ \times \mathscr{O}.$$

Here the homogeneous Dirichlet boundary condition on the bounded, open subset  $\mathscr{O}$  of  $\mathbb{R}^d$  is considered, the initial value  $X_0 = x$  vanishes on the boundary  $\partial \mathscr{O}$  of  $\mathscr{O}$ , the nonlinear drift function f has polynomial growth and satisfies a certain dissipative condition, the driving process  $(W_t)_{t\geq 0}$  is a U-valued cylindrical Wiener process, and the linear operator G is a densely defined closed linear operator on U which could be unbounded (see Section 2 for more details).

Let  $x \in L^p = L^p(\mathcal{O})$ ,  $p \ge 2$ , and denote by  $(X_t^x)_{t \ge 0}$  a mild solution of Eq. (1.1) with the initial datum  $X_0 = x$ . Then  $(X_t^x)_{t \ge 0}$  is a Markov

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process and it generates a Markov semigroup  $(P_t)_{t\geq 0}$  defined as

$$(1.2) P_t \phi(x) := \mathbb{E}\phi(X_t^x), \quad t \ge 0, \ x \in L^p, \ \phi \in \mathcal{B}_b(L^p).$$

Our main concern in this paper is to investigate Harnack inequalities and ergodicity of the Markov semigroup  $(P_t)_{t\geq 0}$  defined in (1.2) in the Banach space,  $L^p$ , for all  $p\geq 2$ .

The stochastic reaction-diffusion equation (1.1) has numerous applications in material sciences and chemical kinetics [ET89]. When  $f(\xi) = \xi - \xi^3$ ,  $\xi \in \mathbb{R}$ , Eq. (1.1) is also called the stochastic Allen–Cahn equation or the stochastic Ginzburg–Landau equation. It is widely used in many fields, for example, the random interface models and stochastic mean curvature flow [Fun16]. There are many interesting and important properties for the solution of Eq. (1.1), which have been investigated in Hilbert settings. For example, the existence of invariant measures and ergodicity are studied in [BS20, Hai02, Kaw05, WXX17], the large deviation principles are investigated in [BBP17, WRD12, XZ18], and sharp interface limits are derived in [KORVE07, Web10, Yip98]. See also [BGJK, CH19, LQ20, LQ] and references therein for analysis in the numerical aspect.

In contrast to SPDEs in Hilbert spaces, only a few papers are treating the regularity, such as invariant measures and the ergodicity, of the Markov semigroup  $(P_t)_{t\geq 0}$  for SPDEs even with Lipschitz coefficients in Banach spaces. The authors in [**BLSa10**] studied invariant measures for SPDEs in martingale-type (M-type) 2 Banach spaces, under Lipschitz and dissipativity conditions, driven by regular noise. For white-noise driven stochastic heat equation (Eq. (2.3) with Lipschitz coefficients), [**BR16**] showed the uniqueness of the invariant measure, if it exists, on  $L^p(0,1)$  with p>4; the case for  $p \in (2,4]$  is unknown. Recently, their method was extended in [**BK18**] to an SPDE, arisen in stochastic finance, in a weighted  $L^p$ -space. See also [**vN01**] for the uniqueness of the invariant measure, if it exists, of the Ornstein–Uhlenbeck process (with form (2.11)) using a pure analytical method.

For SPDEs with non-Lipschitz coefficients, [BG99] obtained the existence of an invariant measure for Eq. (1.1) in the space of continuous functions under the martingale solution framework; the uniqueness of the invariant measure was derived in [Cer03, Cer05] by taking advantage of the fact that a polynomial is uniformly continuous on bounded subsets of continuous functions. Recently, [KN13] showed the existence of a unique invariant measure on the space of continuous complex functions for the stochastic complex Ginzburg–Landau (Eq.

(1.1) with  $f(u) = -\mathbf{i}|u|^2u$ , where  $\mathbf{i} = \sqrt{-1}$ , relying on some good estimates of the solution in the Hilbert–Sobolev spaces  $\dot{H}^{\beta}$  with  $\beta > d/2$ , so that the noise is spatially regular enough.

To show the uniqueness of the invariant measure, these authors mainly formulated a Bismut–Elworthy–Li formula for the derivative  $DP_t$  to get a gradient estimate, which shows the strong Feller property of  $P_t$ . Then the uniqueness of the invariant measure follows immediately by Khas'minskii and Doob theorems, provided the irreducibility holds. The difficulties for the study of the uniqueness of an invariant measure for SPDEs in Banach settings arise, mainly because the tools frequently used in the Hilbert space framework cannot be extended in a straightforward way to the Banach space settings [BR16].

In the past decade, Wang-type dimension-free inequalities have been a new and efficient tool to study diffusion semigroups. They were first introduced in [Wan97] for elliptic diffusion semigroups on non-compact Riemannian manifolds and in [Wan10] for heat semigroups on manifolds with boundary. Roughly speaking, such inequality for the Markov semigroup  $(P_t)_{t\geq 0}$  in a Banach space E is formulated as (1.3)

$$\Phi(P_t\phi(x)) \le P_t(\Phi(\phi)(y)) \exp \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in \mathcal{B}_b^+(E),$$

where  $\Phi: [0, \infty) \to [0, \infty)$  is convex,  $\Psi$  is non-negative on  $[0, \infty) \times E \times E$  with  $\Psi(t, x, x) = 0$  for all t > 0 and  $x \in E$ , and  $\mathcal{B}_b^+(E)$  denotes the family of all measurable and bounded, non-negative functions on E.

There are two frequently used choices of  $\Phi$ . One is given by a power function  $\Phi(\xi) = \xi^{\mathbf{s}}$ ,  $\xi \geq 0$ , for some  $\mathbf{s} > 1$ , then (1.3) is reduces to (1.4)

$$(P_t\phi(x))^{\mathbf{s}} \le P_t\phi^{\mathbf{s}}(y) \exp \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in \mathcal{B}_b^+(E).$$

Another is given by  $\Phi(\xi) = e^{\xi}$ ,  $\xi \in \mathbb{R}$ , in which one may use  $\log \phi$  to replace  $\phi$ , so that (1.3) becomes (1.5)

$$P_t \log \phi(x) \le \log P_t \phi(y) + \Psi(t, x, y), \quad t > 0, \ x, y \in E, \ \phi \in \mathcal{B}_b^+(E).$$

The above inequalities (1.4) and (1.5) are called power-Harnack inequality and log-Harnack inequality, respectively. Both inequalities have been investigated extensively and applied to SODEs and SPDEs via coupling by the change of measure, see, e.g., [Kaw05, Wan07, WZ13, WZ14, Zha10], the monograph [Wan13], and references therein. Besides the gradient estimate which yields strong Feller property, these Harnack inequalities also have a lot of other applications. For example, they are used to study the contractivity of the Markov

semigroup  $(P_t)_{t\geq 0}$  in [**DPRW09**, **Wan11**, **Wan17**] and to derive almost surely strictly positivity of the solution for an SPDE in [**Wan18**].

For SPDEs with polynomial growth drift driven by rough noise in Hilbert settings, we are only aware [Kaw05, Xie19] investigating Harnack inequalities in a weighted  $L^2$ -space and a subspace of  $L^2$  consisting of all non-negative functions, respectively. When the noise is of traceclass, i.e., G appearing in Eq. (1.1) is a Hilbert–Schmidt operator, then the variational solution theory can be used and multiplicative noise can also be considered, as the solution is a semi-martingale so that Itô formula can be applied; see, e.g., [HLL20, Liu09]. In the rougher white noise case, i.e., G coincides with the identity operator in  $L^2$ , the variational solution would not exist; one needs to adopt the mild solution theory instead. Generally, the mild solution is not a semi-martingale and thus Itô formula is not available. We also note that to derive Harnack inequalities for an SPDE with white noise, [WZ14] used finite-dimensional approximations to get a sequence of SODEs such that the arguments developed in [Wan11] for SODEs can be applied.

Our main idea to derive the Harnack inequalities (2.17) and (2.18) in the first main result, Theorem 2.1, for the Markov semigroup  $(P_t)_{t\geq 0}$  defined by (1.2) of Eq. (1.1) on  $(L^p)_{p\geq 2}$ -spaces, under the dissipativity condition (2.8) and the polynomial growth condition (2.9), are the construction of a coupling (see (3.9)) of the change of measure and a uniform pathwise estimate (see (3.15)) for this coupling. As by-products, a gradient estimate and the uniqueness of the invariant measure for  $(P_t)_{t\geq 0}$ , if it exists, follow immediately. To the best of our knowledge, the Harnack inequalities (2.17) and (2.18) are the first two Harnack inequalities for SPDEs in Banach settings.

To show the existence of an invariant measure for Eq. (1.1) with super-linear growth and without strong dissipativity, e.g., q > 2 and  $\lambda$  defined in (2.20) is non-positive, we derive a uniform estimate of  $\mu_n(\|\cdot\|_{q+p-2}^{q+p-2})$  (see (4.5)) for a sequence of probability measures  $(\mu_n)_{n\in\mathbb{N}_+}$  (defined in (4.3)). However, this is not strong enough to conclude the tightness of  $(\mu_n)_{n\in\mathbb{N}_+}$ , since the embedding  $L^{q+p-2} \subset L^p$  is not compact. To overcome this difficulty, we utilize a compact embedding (see (2.5)) by a Sobolev–Slobodeckii space. This forces us to derive a uniform estimate of  $\mu_n(\|\cdot\|_{\beta,p})$  (see (4.6), with the  $\|\cdot\|_{\beta,p}$ -norm defined in (2.4)), where the aforementioned uniform estimate of  $\mu_n(\|\cdot\|_{q+p-2}^{q+p-2})$  plays a key role. Then the tightness of  $(\mu_n)_{n\in\mathbb{N}_+}$  follows and we get the existence of an invariant measure for  $(P_t)_{t\geq0}$ . In combination with the uniqueness result, we obtain the existence of a unique and thus ergodic invariant measure for  $(P_t)_{t\geq0}$  in the second main result, Theorem 2.2.

The rest of the paper is organized as follows. Some preliminaries, assumptions, and main results are given in the next section. We derive a uniform pathwise estimate to get the existence of a unique global solution to Eq. (1.1) in Section 3. In another part of Section 3, we construct the coupling and derive a uniform pathwise estimation for this coupling. These estimations ensure the well-posedness of the coupling and will be used in the last section to derive Harnack inequalities and to prove the main results, Theorems 2.1 and 2.2.

# 2. Preliminaries and Main Results

Let  $\mathscr{O} \subset \mathbb{R}^d$ ,  $d \geq 1$ , be an open, bounded Lipschitz domain. Throughout  $p \geq 2$  is a fixed constant. Denote by  $L^p = L^p(\mathscr{O})$  the usual Lebesgue space on  $\mathscr{O}$  with norm  $\|\cdot\|_p$ ,  $\mathcal{B}_b(L^p)$  the class of bounded measurable functions on  $L^p$ , and  $\mathcal{B}_b^+(L^p)$  the set of positive functions in  $\mathcal{B}_b(L^p)$ . For a function  $\phi \in \mathcal{B}_b(L^p)$ , define

$$\|\phi\|_{\infty} = \sup_{x \in L^p} |\phi(x)|, \quad \|\nabla\phi\|_{\infty} = \sup_{x \in L^p} |\nabla\phi|(x),$$

where  $|\nabla \phi|(x) = \limsup_{y \to x} |\phi(y) - \phi(x)|/||y - x||$ ,  $x \in L^p$ . In particular, when p = 2,  $L^2$  is a Hilbert space with the norm  $||\cdot|| := ||\cdot||_2$  and the inner product  $(\cdot, \cdot)$ . Let  $(W_t)_{t \geq 0}$  be a  $U := L^2$ -valued cylindrical Wiener process with respect to a complete filtered probability space  $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual condition, i.e., there exists an orthonormal basis  $(e_k)_{k=1}^{\infty}$  of  $L^2$  and a family of independent standard real-valued Brownian motions  $(\beta_k)_{k=1}^{\infty}$  such that

(2.1) 
$$W_t = \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \ge 0.$$

Define by F the Nemytskii operator associated with f, i.e.,

(2.2) 
$$F(x)(\xi) = f(x(\xi)), \quad x \in L^p, \ \xi \in \mathscr{O}.$$

The precise assumptions on f and F are given in Assumption 2.1 and Remark 2.2. Then Eq. (1.1) can be rewritten as the stochastic evolution equation

(2.3) 
$$dX_t = (AX_t + F(X_t))dt + GdW_t,$$

with the initial datum  $X_0 = x \in L^p$ , where A is the Dirichlet Laplacian operator on  $L^p$ , F is the Nemytskii operator defined in (2.2) associated with f, and W is a U-valued cylindrical Wiener process given in (2.1).

It is well-known that the Dirichlet Laplacian operator A in Eq. (2.3) generates an analytic  $C_0$ -semigroup in  $L^p$ , denoted by  $(S_t^p)_{t\geq 0}$ , for each  $p\geq 2$ ; see, e.g., [Dav89, Theorem 1.4.1]. These semigroups

are consistent, in the sense that  $S_t^{p_1}x = S_t^{p_2}x$ , for all  $t \geq 0$ ,  $x \in L^{p_1} \cap L^{p_2}$ , and  $p_1, p_2 \geq 2$ . Then we shall denote all  $(S_t^p)_{t\geq 0}$ ,  $p \geq 2$ , by  $(S_t)_{t\geq 0}$ , if there is no confusion. In Section 4, we also need the Sobolev–Slobodeckij spaces  $W^{\beta,p}$  and  $W_0^{\beta,p} := \{\phi \in W^{\beta,p} : \phi|_{\partial \mathscr{O}} = 0\}$ , with  $\beta \in (0,1)$ , whose norm is defined by

(2.4) 
$$\|\phi\|_{\beta,p} := \left( \|\phi\|_p^p + \int_{\mathscr{O}} \int_{\mathscr{O}} \frac{|\phi(\xi) - \phi(\eta)|^p}{|\xi - \eta|^{d+\beta p}} \mathrm{d}\xi \,\mathrm{d}\eta \right)^{\frac{1}{p}}.$$

It is known that the following compact embedding holds true (see, e.g., [Agr15, Theorem 2.3.4]):

$$(2.5) W_0^{\beta,p} \subset L^p, \quad \beta \in (0, d/p).$$

Moreover,  $(S_t)_{t\geq 0}$  satisfies the following ultracontractivity (see, e.g., [Cer03, Section 2.1]):

(2.6) 
$$||S_t u||_{\beta, \mathbf{r}} \le C e^{-\lambda_1 t} t^{-(\frac{\beta}{2} + \frac{d(\mathbf{r} - \mathbf{s})}{2\mathbf{r} \mathbf{s}})} ||u||_{\mathbf{s}}, \quad t > 0, \ u \in L^{\mathbf{s}},$$

for all  $\beta \in (0,1)$  and  $1 \leq \mathbf{s} \leq \mathbf{r} \leq \infty$ , where  $\lambda_1 > 0$  is the first eigenvalue of A. For convenience, here and what follows, we frequently use the generic constant C, which may be different in each appearance. When p = 2, the following Poincaré inequality holds:

where  $H_0^1 = H_0^1(\mathcal{O})$  denotes the space of weakly differentiable functions, vanishing on the boundary  $\partial \mathcal{O}$ , whose derivatives belong to  $L^2$ .

**2.1.** Main assumptions and results. Let us give the following assumptions on the data of Eq. (1.1). We begin with the conditions on the drift function f.

Assumption 2.1. There exist constants  $L_f \in \mathbb{R}$ ,  $\theta, L_f' > 0$ , and  $q \geq 2$  such that for all  $\xi, \eta \in \mathbb{R}$ ,

$$(2.8) (f(\xi) - f(\eta))(\xi - \eta) \le L_f |\xi - \eta|^2 - \theta |\xi - \eta|^q,$$

$$(2.9) |f(\xi) - f(\eta)| \le L_f'(1 + |\xi|^{q-2} + |\eta|^{q-2})|\xi - \eta|.$$

Remark 2.1. A motivating example of f such that Assumption 2.1 holds true is a polynomial of odd order q-1 with a negative leading coefficient (for the stochastic Allen–Cahn equation, q=4), perturbed with a Lipschitz continuous function; see, e.g., [**DPZ14**, Example 7.8].

REMARK 2.2. It follows from (2.8)-(2.9) that the Nemytskii operator F, defined in (2.2), associated with f is a well-defined, continuous operator from  $L^q$  to  $L^{q'}$ , with q' = q/(q-1). Moreover,

$$\langle F(u) - F(v), u - v \rangle \le L_f ||u - v||^2 - \theta ||u - v||_q^q, \quad \forall u, v \in L^q,$$

where  $\langle \cdot, \cdot \rangle$  is the dualization between  $L^{q'}$  and  $L^q$  with respect to  $L^2$ .

To perform the assumption of the noise part, as we consider the Banach spaces  $(L^p)_{p\geq 2}$ , let us first recall the required materials of stochastic calculus in Banach spaces, especially the M-type 2 spaces and the  $\gamma$ -radonifying operators. It is known that the stochastic calculations in Banach space depend heavily on the geometric structure of the underlying spaces.

We first recall the definitions of the M-type for a Banach space. Let E be a Banach space and  $(\epsilon_n)_{n\in\mathbb{N}_+}$  be a Rademacher sequence in a probability space  $(\Omega', \mathscr{F}', \mathbb{P}')$ , i.e., a sequence of independent random variables taking the values  $\pm 1$  with the equal probability 1/2. Then E is called of M-type 2 if there exists a constant  $\tau^M \geq 1$  such that

$$||f_N||_{L^2(\Omega';E)} \le \tau^M \Big( ||f_0||_{L^2(\Omega';E)}^2 + \sum_{n=1}^N ||f_n - f_{n-1}||_{L^2(\Omega';E)}^2 \Big)^{\frac{1}{2}}$$

for all E-valued square integrable martingales  $\{f_n\}_{n=0}^N$ .

For stochastic calculus in Banach spaces, the so-called  $\gamma$ -radonifying operators play the important roles instead of Hilbert–Schmidt operators in Hilbert settings. Let  $(\gamma_n)_{n\in\mathbb{N}_+}$  be a sequence of independent  $\mathcal{N}(0,1)$ -random variables in a probability space  $(\Omega', \mathscr{F}', \mathbb{P}')$ . Denote by  $\mathcal{L}(U,E)$  the space of linear operators from U to E. An operator  $R \in \mathcal{L}(U,E)$  is called  $\gamma$ -radonifying if there exists an orthonormal basis  $(h_n)_{n\in\mathbb{N}_+}$  of U such that the Gaussian series  $\sum_{n\in\mathbb{N}_+} \gamma_n R h_n$  converges in  $L^2(\Omega';E)$ . In this situation, it is known that the number

$$||R||_{\gamma(U,E)} := \left\| \sum_{n \in \mathbb{N}_+} \gamma_n R h_n \right\|_{L^2(\Omega';E)}$$

does not depend on the sequence  $(\gamma_n)_{n\in\mathbb{N}_+}$  and the basis  $(h_n)_{n\in\mathbb{N}_+}$ , and it defines a norm on the space  $\gamma(U, E)$  of all  $\gamma$ -radonifying operators from U to E. In particular, if E reduces to a Hilbert space, then  $\gamma(U, E)$  coincides with the space of all Hilbert–Schmidt operators from U to E.

Let T > 0 and  $(E, \|\cdot\|_E)$  be an M-type 2 space. For any  $\gamma(U, E)$ -valued adapted process  $\Phi \in L^{\mathbf{s}}(\Omega; L^2(0, T; \gamma(U, E)))$  with  $\mathbf{s} \geq 2$ , the following one-sided Burkholder inequality for the E-valued stochastic integral  $\int_0^t \Phi_r dW_r$  holds for some constant  $C = C(\mathbf{s})$  (see, e.g., [Brz97, Theorem 2.4] or [HHL19, (3)]):

(2.10) 
$$\mathbb{E} \sup_{t \in [0,T]} \left\| \int_0^t \Phi_r dW_r \right\|_E^{\mathbf{s}} \le C \mathbb{E} \left( \int_0^T \|\Phi\|_{\gamma(U,E)}^2 dt \right)^{\frac{\mathbf{s}}{2}}.$$

Coming back to our case, it is known that  $(L^p)_{p\geq 2}$  are M-type 2 spaces. For more details about definitions and properties of M-type 2 spaces and  $\gamma$ -radonifying operators, we refer to [**Brz97**].

With these preliminaries, now we can give our assumptions on G appearing in Eq. (2.3). Let  $\mathcal{L}_S(U)$  be the set of all densely defined closed linear operators (L, Dom(L)) on U such that for every s > 0,  $S_sL$  extends to a unique operator in  $\gamma(U, L^p)$ , which is again denoted by  $S_sL$ . Assume that  $G \in L_S(U)$  and denote by  $W_A$  the stochastic convolution, also known as the Ornstein-Uhlenbeck process, associated with Eq. (2.3), i.e.,

(2.11) 
$$W_A(t) = \int_0^t S_{t-r} G dW_r, \quad t \ge 0.$$

It is clear that  $W_A$  is the mild solution of the linear equation

$$dZ_t = AZ_t dt + G dW_t, \quad Z_0 = 0,$$

i.e., Eq. (2.3) with F = 0, with vanishing initial datum. We always assume that, for each fixed T > 0,  $\int_0^T ||S_t G||^2_{\gamma(U,L^p)} dt < \infty$ , so it follows from the Burkholder inequality (2.10) that

$$\mathbb{E}||W_A(T)||_p^2 \le C \int_0^T ||S_t G||_{\gamma(U,L^p)}^2 dt < \infty,$$

which shows that  $W_A(T)$  possesses the bounded second moment in  $L^p$ . Moreover, we perform the following stronger assumption, as we shall handle the polynomial drift function f which satisfies Assumption 2.1.

Assumption 2.2.  $W_A$  has a continuous version in  $L^{q+p-2}$  such that

(2.13) 
$$\sup_{t>0} \mathbb{E} \|W_A(t)\|_{q+p-2}^{q+p-2} < \infty.$$

The above Assumption 2.2 on the Ornstein-Uhlenbeck process  $W_A$  (2.11) is necessary for the study of the well-posedness, Harnack inequalities, and ergodicity in Sections 3 and 4. To indicate that Assumption 2.2 is natural, we remark that the condition (2.13) is valid in various of applications, even when G is an unbounded operator.

EXAMPLE 2.1. Consider 1D Eq. (2.3) with  $\mathcal{O} = (0,1)$ ,  $U = L^2(0,1)$  (with a uniformly bounded orthonormal basis  $(e_k = \sqrt{2}\sin(k\pi x))_{k\in\mathbb{N}_+})$ , and  $G := (-\Delta)^{\theta/2}$  for some  $\theta < 1/2$ . In particular,  $\theta = 0$  corresponds to white noise and G is an unbounded operator when  $\theta \in (0,1/2)$ .

It is clear that G is a densely defined closed linear operator on U. To show  $G \in L_S(U)$ , we note that for  $\mathbf{r} \geq 2$ ,  $L^{\mathbf{r}}$  is a Banach function space with finite cotype, so an operator  $\Phi \in \gamma(U, L^{\mathbf{r}})$  if and only if

 $(\sum_{k=1}^{\infty} (\Phi e_k)^2)^{1/2}$  belongs to  $L^{\mathbf{r}}$  and there exist a constant C > 0 such that (see [vNVW08, Lemma 2.1])

(2.14)

$$\frac{1}{C} \left\| \sum_{k=1}^{\infty} (\Phi e_k)^2 \right\|_{\mathbf{r}/2} \le \|\Phi\|_{\gamma(U,L^{\mathbf{r}})}^2 \le C \left\| \sum_{k=1}^{\infty} (\Phi e_k)^2 \right\|_{\mathbf{r}/2}, \quad \Phi \in \gamma(U,L^{\mathbf{r}}).$$

It follows from the above estimates with  $\Phi = S_r G$  for r > 0 that

$$||S_r G||_{\gamma(U,L^{\mathbf{r}})}^2 \le C ||\sum_{k=1}^{\infty} (S_r G e_k)^2||_{\mathbf{r}/2} \le C \sum_{k=1}^{\infty} e^{-2\lambda_k r} \lambda_k^{\theta},$$

which is convergent if and only if  $\theta < 1/2$ , where  $\lambda_k = \pi^2 k^2$ ,  $k \in \mathbb{N}_+$ . This shows  $S_r G \in \gamma(U, L^p)$  for every r > 0 and thus  $G \in L_S(U)$ .

Finally, one can use the Burkholder inequality (2.10) and the above equivalence relation (2.14) to show that  $W_A$  belongs to  $\mathcal{C}([0,\infty); L^{\mathbf{s}}(\Omega; L^{\mathbf{r}}))$  for any  $\mathbf{s}, \mathbf{r} \geq 2$ , following an argument used in [LQ20, (2.17) in Lemma 2.2]. Indeed, for any  $t \geq 0$ ,

$$\sup_{t\geq 0} \mathbb{E} \|W_A(t)\|_{\mathbf{r}}^{\mathbf{s}} \leq C(\mathbf{s}) \sup_{t\geq 0} \left( \int_0^t \|S_r(-\Delta)^{\frac{\theta}{2}}\|_{\gamma(L^2,L^{\mathbf{r}})}^2 \mathrm{d}r \right)^{\frac{\mathbf{s}}{2}}$$

$$\leq C(\mathbf{s}) \left( \int_0^\infty \left\| \sum_{k=1}^\infty e^{-2\lambda_k r} \lambda_k^{\theta} e_k^2 \right\|_{\mathbf{r}/2} \mathrm{d}t \right)^{\frac{\mathbf{s}}{2}}$$

$$\leq C(\mathbf{s}) \left( \sum_{k=1}^\infty \lambda_k^{\theta} \|e_k\|_{\mathbf{r}}^2 \left( \int_0^\infty e^{-2\lambda_k r} \mathrm{d}t \right) \right)^{\frac{\mathbf{s}}{2}}$$

$$\leq C(\mathbf{s}) \left( \sum_{k=1}^\infty \frac{1}{2\lambda_k^{1-\theta}} \right)^{\frac{\mathbf{s}}{2}} < \infty.$$

This shows (2.13) with  $\mathbf{s} = \mathbf{r} = q + p - 2$ .

In the derivation of the existence of an invariant measure in Theorem 4.2 when q > 2, we need the following Sobolev regularity of the Ornstein–Uhlenbeck process  $(W_A(t))_{t>0}$ .

Assumption 2.3. There exists a  $\beta_0 > 0$  such that

(2.15) 
$$\sup_{t\geq 0} \mathbb{E} \|W_A(t)\|_{\beta_0,p} < \infty.$$

Example 2.2. As in Example 2.1,

$$\sup_{t \ge 0} \mathbb{E} \|W_A(t)\|_{\beta_0, \mathbf{r}}^{\mathbf{s}} \le C(\mathbf{s}) \left( \int_0^\infty \|(-A)^{\frac{\beta_0 + \theta}{2}} S_r G\|_{\gamma(L^2, L^{\mathbf{r}})}^2 dr \right)^{\frac{\mathbf{s}}{2}} \\
\le C(\mathbf{s}) \left( \sum_{k=1}^\infty \frac{1}{\lambda_k^{1 - (\beta_0 + \theta)}} \right)^{\frac{\mathbf{s}}{2}},$$

which is convergent if and only if  $\beta_0 + \theta < 1/2$ . As  $\theta < 1/2$  in Example 2.1, this shows (2.15) with  $\mathbf{r} = p$  and  $\mathbf{s} = 1$  for any  $\beta_0 \in (0, 1/2 - \theta)$ .

To derive Wang-type Harnack inequalities and the ergodicity for  $(P_t)_{t\geq 0}$  in Section 4, we need the following standard elliptic condition.

ASSUMPTION 2.4.  $GG^*$  is invertible, with inverse  $(GG^*)^{-1}$ , such that  $G^{-1} := G^*(GG^*)^{-1}$  is a bounded linear operator on U:

(2.16) 
$$||G^{-1}||_{\infty} := \sup_{x \in U: \ x \neq 0} \frac{||G^{-1}x||}{||x||} < \infty.$$

REMARK 2.3. Let  $\theta \in [0, 1/2)$ . Then the operator  $G := (-\Delta)^{\theta/2}$  defined in Example 2.1 is invertible with bounded inverse  $G^{-1} := (-\Delta)^{-\theta/2}$ .

**2.2.** Main results. Now we are in the position to present our main results. Let us first recall some definitions. The Markov semigroup  $(P_t)_{t\geq 0}$  defined in (1.2) is called of strong Feller if  $P_T\phi \in \mathcal{C}_b(L^p)$  for any T>0 and  $\phi \in \mathcal{B}_b(L^p)$ . A probability measure  $\mu$  on  $L^p$  is said to be an invariant measure of  $(P_t)_{t\geq 0}$  or of Eq. (2.3), if

$$\int_{L^p} P_t \phi(x) \mu(\mathrm{d}x) = \int_{L^p} \phi(x) \mu(\mathrm{d}x), \quad \phi \in \mathcal{B}_b(L^p), \ t \ge 0.$$

### df of ergodicity

Our first main result is the following log-Harnack inequality and power-Harnack inequality, from which the uniqueness of the invariant measure, if it exists, for  $(P_t)_{t>0}$  follows.

THEOREM 2.1. Let Assumptions 2.1, 2.2, and 2.4 hold. For any T > 0,  $\mathbf{s} > 1$ ,  $x, y \in L^p$ , and  $\phi \in \mathcal{B}_b^+(L^p)$ ,

(2.17) 
$$P_T \log \phi(y) \le \log P_T \phi(x) + \frac{\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2,$$

$$(2.18) (P_T \phi(y))^{\mathbf{s}} \le P_T \phi^{\mathbf{s}}(x) \exp\left(\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x - y\|_p^2}{(\mathbf{s} - 1)(e^{2\lambda T} - 1)}\right).$$

Consequently,  $(P_t)_{t\geq 0}$  has at most one invariant measure.

The next main result is the existence of an invariant measure for  $(P_t)_{t\geq 0}$ . In combination with the uniqueness result in Theorem 2.1,  $(P_t)_{t\geq 0}$  possesses exactly one ergodic invariant measure.

Theorem 2.2. Let Assumptions 2.1 and 2.2 hold. Assume that q>2 such that

(2.19) 
$$d < \frac{2p(q+p-2)}{(p-1)(q-2)},$$

and Assumption 2.3 holds, or  $\lambda$  defined by

(2.20) 
$$\lambda := -L_f + \theta \chi_{q=2, p\neq 2} + \lambda_1 \chi_{q\neq 2, p=2} + (\lambda_1 + \theta) \chi_{q=p=2},$$

is positive. Then  $(P_t)_{t\geq 0}$  has an invariant measure  $\mu$  with full support on  $L^p$  such that  $\mu(\|\cdot\|_{q+p-2}^{q+p-2})<\infty$ . Assume furthermore that Assumption 2.4 holds, then  $\mu$  is the unique ergodic invariant measure of  $(P_t)_{t\geq 0}$ .

# 3. Coupling and Moments' Estimations

The main aims of this section are to show the existence of a unique global solution to Eq. (1.1) and to construct a well-defined coupling process for this solution process. We also derive several uniform a priori estimates on moments of these two processes, which will be used in Section 4 to derive Wang-type Harnack inequalities and the ergodicity of  $(P_t)_{t>0}$ .

**3.1.** Well-posedness and moments' estimations. Let us first recall that an  $L^p$ -valued process  $(X_t)_{t\in[0,T]}$  is called a mild solution of Eq. (2.3) with the initial datum  $X_0 = x$  if  $\mathbb{P}$ -a.s.

(3.1) 
$$X_t = S_t x + \int_0^t S_{t-r} F(X_r) dr + W_A(t), \quad t \in [0, T].$$

From Remark 2.2, the deterministic convolution in Eq. (3.1) makes sense. Define  $Z = X - W_A$ . It is clear that X is a mild solution Eq. (2.3) if and only if Z is a mild solution of the random PDE

(3.2) 
$$\partial_t Z_t = \Delta Z_t + F(Z_t + W_A(t)), \quad Z_0 = x.$$

The following results show the existence of a unique mild solution of Eq. (2.3) which is a Markov process and depends on the initial data continuously in pathwise sense.

LEMMA 3.1. Let T>0,  $x\in L^p$ , and Assumptions 2.1 and 2.2 hold. Eq. (1.1) with the initial datum  $X_0=x$  possesses a unique mild solution, in  $\mathcal{C}([0,T];L^p)\cap L^{q+p-2}(0,T;L^{q+p-2})$   $\mathbb{P}$ -a.s., which is a Markov process. Moreover, there exists a constant  $\lambda$  defined in (2.20) such that

(3.3) 
$$||X_t^x - X_t^y||_p \le e^{-\lambda t} ||x - y||_p, \quad t \ge 0, \ x, y \in L^p.$$

PROOF. From (2.9), f is locally Lipschitz continuous, so it is clear that both Eq. (3.2) with  $Z = X - W_A$  and Eq. (2.3) exist local solutions on  $[0, T_0)$  for some  $T_0 \in (0, T]$ . To extend this local solution to the whole time interval [0, T], it suffices to give a priori uniform estimations for Z and X.

As Eq. (3.2) is a pathwise random PDE, we test  $p|Z_t|^{p-2}Z_t$  on this equation with  $t \in [0, T_0)$  and use the conditions (2.8)-(2.9) and Young inequality to obtain

$$\partial_{t} \|Z_{t}\|_{p}^{p} + p(p-1) \int_{\mathscr{O}} |Z_{t}|^{p-2} |\nabla Z_{t}|^{2} d\xi$$

$$= p \langle |Z_{t}|^{p-2} Z_{t}, F(Z_{t} + W_{A}(t)) - F(W_{A}(t)) \rangle + p \langle |Z_{t}|^{p-2} Z_{t}, F(W_{A}(t)) \rangle$$

$$\leq p L_{f} \|Z_{t}\|_{p}^{p} - p\theta \|Z_{t}\|_{q+p-2}^{q+p-2} + p \langle |Z_{t}|^{p-2} Z_{t}, F(W_{A}(t)) \rangle$$

$$\leq p L_{f} \|Z_{t}\|_{p}^{p} - p\theta_{1} \|Z_{t}\|_{q+p-2}^{q+p-2} + C \|F(W_{A}(t))\|_{\frac{q+p-2}{q-1}}^{\frac{q+p-2}{q-1}}$$

$$\leq C(1 + \|W_{A}(t)\|_{q+p-2}^{q+p-2}) + p L_{f} \|Z_{t}\|_{p}^{p} - p\theta_{1} \|Z_{t}\|_{q+p-2}^{q+p-2},$$

where  $\theta_1$  could be chosen as any positive number which is smaller than  $\theta$ . This yields

(3.4)

$$||Z_t||_p^p + p\theta_1 \int_0^t ||Z_t||_{q+p-2}^{q+p-2} dr + p(p-1) \int_0^t \int_{\mathscr{O}} |Z_t|^{p-2} |\nabla Z_t|^2 d\xi dr$$

$$\leq ||x||_p^p + C \int_0^t (1 + ||W_A(r)||_{q+p-2}^{q+p-2}) dr + pL_f \int_0^t ||Z_t||_p^p dr.$$

Using Gronwall inequality, we obtain

$$||Z_t||_p^p + p\theta_1 \int_0^t ||Z_t||_{q+p-2}^{q+p-2} dr + p(p-1) \int_0^t \int_{\mathscr{O}} |Z_t|^{p-2} |\nabla Z_t|^2 d\xi dr$$

$$\leq e^{pL_f t} \Big( ||x||_p^p + C \int_0^t (1 + ||W_A(r)||_{q+p-2}^{q+p-2}) dr \Big).$$

The above uniform estimate, in combination with the condition (2.13) of  $W_A$  in Assumption 2.2, implies the global existence of a mild solution Z to Eq. (3.2) on [0,T] in  $\mathcal{C}([0,T];L^p) \cap L^{q+p-2}(0,T;L^{q+p-2})$   $\mathbb{P}$ -a.s. Taking into account the relation  $X = Z + W_A$  and the condition (2.13), we obtain a global mild solution X to Eq. (2.3).

To show the continuous dependence (3.3), let us note that

$$\partial_t (X_t^x - X_t^y) = A(X_t^x - Y_t^y) + F(X_t^x) - F(Y_t^y).$$

Testing  $p|X_t^x - X_t^y|^{p-2}(X_t^x - X_t^y)$  on the above equation, using integration by parts formula, and applying the condition (2.8), we obtain

$$||X_t^x - X_t^y||_p^p + p\theta \int_0^t ||X_r^x - X_r^y||_{q+p-2}^{q+p-2} dr + p(p-1) \int_0^t \int_{\mathcal{C}} |X_r^x - X_r^y|^{p-2} |\nabla(X_r^x - X_r^y)|^2 d\xi dr$$

$$(3.5) \leq ||x - y||_p^p + pL_f \int_0^t ||X_r^x - X_r^y||_p^p dr.$$

We conclude (3.3) with  $\lambda = -L_f$  by Gronwall inequality. When q = 2, then (3.3) holds with  $\lambda = \theta - L_f$ , as one can subtract the first integral on the left-hand side of the above inequality; while p = 2, then using the Poincaré inequality (2.7) yields (3.3) with  $\lambda = \lambda_1 - L_f$ . Similarly, (3.3) holds with  $\lambda = \lambda_1 + \theta - L_f$  when q = p = 2. These statements show (3.3) with  $\lambda$  given by (2.20).

The pathwise continuous dependence clearly implies the uniqueness of the solution to Eq. (2.3). One can also show the Markov property for this solution using a standard method, see, e.g., [DPZ14, Theorem 9.21]. This completes the proof.

REMARK 3.1. The pathwise estimate (3.3) immediately yields the following estimate between any two solutions in **r**-Wasserstein distance for any  $\mathbf{r} \geq 1$ :

$$W_{\mathbf{r}}(\mu_t^1, \mu_t^2) := \inf(\mathbb{E}||X_t^{x_1} - X_t^{x_2}||^{\mathbf{r}})^{\frac{1}{\mathbf{r}}} \le e^{-\lambda t} W_{\mathbf{r}}(\mu_0^1, \mu_0^2), \quad t \ge 0,$$

where  $(\mu_0^i)_{i=1}^2$  are two measures on  $L^p$ ,  $(X_t^{x_i})_{i=1}^2$  are the solutions to Eq. (2.3) starting from  $(x_i)_{i=1}^2$  of laws  $(\mu_0^i)_{i=1}^2$ , and the infimum runs over all random variables  $(X_t^{x_i})_{i=1}^2$  with laws  $(\mu_t^i)_{i=1}^2$ ,  $t \geq 0$ , respectively. Similar contraction-type estimate in 2-Wasserstein distance on  $\mathbb{R}^d$  had been investigated in [BGG12].

3.2. Construction of coupling and moments' estimations. Let T > 0 be fixed throughout the rest of Section 3 and set

(3.6) 
$$\gamma_t := \int_0^{T-t} e^{2\lambda r} dr = \frac{e^{2\lambda(T-t)} - 1}{2\lambda}, \quad t \in [0, T],$$

where  $\lambda$  is given in (2.20). For convention, if  $\lambda = 0$ , we set  $\frac{e^{2\lambda t}-1}{2\lambda} := t$  for  $t \in [0,T]$ . Then  $\gamma$  is smooth, strictly positive, and strictly decreasing on [0,T) (with  $\gamma_T = 0$ ) such that

$$(3.7) \gamma_t' + 2\lambda \gamma_t + 1 = 0.$$

Moreover, the integral of  $\gamma^{-1}$  on [0,T) diverges:

$$\int_0^T \frac{1}{\gamma_t} dt = \infty.$$

Now we can define the coupling Y of X as the mild solution of the coupling equation

(3.9) 
$$dY_t = (AY_t + F(Y_t) + \gamma_t^{-1}(X_t - Y_t))dt + GdW_t,$$

with an initial datum  $Y_0 = y \in L^p$ . Since the additional drift term  $\gamma_t^{-1}(X_t - Y_t)$  is Lipschitz continuous for each fixed  $t \in [0, T)$  and  $\omega \in \Omega$ , one can use similar arguments in Lemma 3.1 to show that Y is a well-defined continuous process on [0, T).

REMARK 3.2. As  $\gamma^{-1}$  is continuous and thus integrable on  $[0, T_0] \subset [0, T)$  for any  $T_0 \in (0, T)$ , one can use the arguments in Lemma 3.1 to extend the local solution to [0, T). However, it is difficult to get a uniform a priori estimation, following the idea in Lemma 3.1, to conclude the well-posedness of Y at T as  $\gamma_T^{-1}$  is singular satisfying (3.8).

For each  $s \in [0, T)$ , we set

(3.10) 
$$v_s := \frac{G_s^{-1}(X_s - Y_s)}{\gamma_t}, \quad \widetilde{W}_s := W_s + \int_0^s v_r dr,$$

and define

(3.11) 
$$M_s := \exp\left(-\int_0^s (v_r, dW_r) - \frac{1}{2} \int_0^s ||v_r||^2 dr\right).$$

From (3.10) and (3.11), M can also be rewritten as

(3.12) 
$$M_s := \exp\left(-\int_0^s (v_r, d\widetilde{W}_r) + \frac{1}{2} \int_0^s ||v_r||^2 dr\right), \quad s \in [0, T).$$

It is clear that  $\mathbb{Q}_s := M_s \mathbb{P}$  is a probability measure. By the representation (3.10) and the non-degenerate condition (2.16) in Assumption 2.4, we have

(3.13) 
$$\frac{1}{2} \int_0^s \|v_r\|^2 dr \le \frac{\|G^{-1}\|_\infty^2}{2} \int_0^s \frac{\|X_r - Y_r\|_p^2}{\gamma_r^2} dr.$$

We first show that for any  $s \in (0, T)$ ,  $(\widetilde{W}_t)_{t \in [0, s]}$  is a U-valued cylindrical Wiener process under the probability measure  $\mathbb{Q}_s$  through Girsanov theorem ensured by the Novikov condition (3.16).

LEMMA 3.2. Let Assumptions 2.1, 2.2, and 2.4 hold. For any  $s \in (0,T)$ ,  $(\widetilde{W}_t)_{t \in [0,s]}$  is a *U*-valued cylindrical Wiener process under  $\mathbb{Q}_s$ .

PROOF. It follows from Eq. (2.3) and Eq. (3.9) on [0, s] that (3.14)

$$\partial_t (X_t - Y_t) = A(X_t - Y_t) + F(X_t) - F(Y_t) - \gamma_t^{-1} (X_t - Y_t), \quad \mathbb{P}\text{-a.s.}$$

As in the proof of the inequality (3.5), we test  $p|X_t - Y_t|^{p-2}(X_t - Y_t)$  on the above equation, use integration by parts formula, and apply the

condition (2.8) to obtain

$$\partial_t \|X_t - Y_t\|_p^p + p(p-1) \int_{\mathcal{O}} |X_t - Y_t|^{p-2} |\nabla(X_t - Y_t)|^2 d\xi$$

$$\leq pL_f \|X_t - Y_t\|_p^p - p\theta \|X_t - Y_t\|_{q+p-2}^{q+p-2} - p\gamma_t^{-1} \|X_t - Y_t\|_p^p.$$

It follows from the chain rule that

$$\begin{split} \partial_t \|X_t - Y_t\|_p^2 &= \frac{2}{p} \|X_t - Y_t\|_p^{2-p} \partial_t \|X_t - Y_t\|_p^p \\ &\leq -2(p-1) \|X_t - Y_t\|_p^{2-p} \int_{\mathscr{O}} |X_t - Y_t|^{p-2} |\nabla (X_t - Y_t)|^2 \mathrm{d}\xi \\ &+ 2L_f \|X_t - Y_t\|_p^2 - 2\theta \|X_t - Y_t\|_p^{2-p} \|X_t - Y_t\|_{q+p-2}^{q+p-2} - 2\gamma_t^{-1} \|X_t - Y_t\|_p^2, \\ \text{and thus} \end{split}$$

$$\partial_t ||X_t - Y_t||_p^2 \le -2\lambda ||X_t - Y_t||_p^2 - 2\gamma_t^{-1} ||X_t - Y_t||_p^2$$

The product rule of differentiation and the equality (3.7) yield that

$$\begin{split} \partial_t (\gamma_t^{-1} \| X_t - Y_t \|_p^2) &= \gamma_t^{-1} \partial_t \| X_t - Y_t \|_p^2 - \gamma_t^{-2} \gamma_t' \| X_t - Y_t \|_p^2 \\ &\leq -\gamma_t^{-2} (\gamma_t' + 2\lambda \gamma_t + 2) \| X_t - Y_t \|_p^2 \\ &= -\gamma_t^{-2} \| X_t - Y_t \|_p^2. \end{split}$$

Integrating on both sides from 0 to s, we obtain

(3.15) 
$$\frac{\|X_s - Y_s\|_p^2}{\gamma_s} + \int_0^s \frac{\|X_t - Y_t\|_p^2}{\gamma_t^2} dt \le \frac{\|x - y\|_p^2}{\gamma_0}, \quad \mathbb{P}\text{-a.s.}$$

This pathwise estimate, in combination with the estimate (3.13), particularly implies the Novikov condition

(3.16) 
$$\mathbb{E} \exp\left(\frac{1}{2} \int_0^s \|v_t\|^2 dt\right) \le \exp\left(\frac{\|G^{-1}\|_{\infty}^2 \|x - y\|_p^2}{2\gamma_0}\right) < \infty.$$

By Girsanov theorem,  $(\widetilde{W}_t)_{t\in[0,s]}$  is a cylindrical Wiener process under the probability measure  $\mathbb{Q}_s$ .

Next, we will give two uniform moments' estimations (3.17) and (3.22) for certain functionals of  $(M_s)$  for all  $s \in [0,T)$ . The first moments' estimation (3.17) will indicate that  $(M_s)_{s \in [0,T]}$  defined in (3.12) is indeed a uniformly integrable martingale.

LEMMA 3.3. Let Assumptions 2.1, 2.2, and 2.4 hold. Then

(3.17) 
$$\sup_{s \in [0,T)} \mathbb{E}[M_s \log M_s] \le \frac{\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2, \quad x, y \in L^p.$$

Consequently,  $M_T := \lim_{s \uparrow T} M_s$  exists and  $(M_s)_{s \in [0,T]}$  is a martingale.

PROOF. Let  $s \in [0, T)$  be fixed. By the construction (3.10), we can rewrite Eq. (2.3) and Eq. (3.9) on [0, s] as

(3.18) 
$$dX_t = (AX_t + F(X_t) - \gamma_t^{-1}(X_t - Y_t))dt + Gd\widetilde{W}_t,$$

(3.19) 
$$dY_t = (AY_t + F(Y_t))dt + Gd\widetilde{W}_t,$$

with initial values  $X_0 = x$  and  $Y_0 = y$ , respectively. Then Eq. (3.14) about X - Y also holds  $\mathbb{Q}_s$ -a.s. Therefore, the pathwise estimate (3.15) is valid  $\mathbb{Q}_s$ -a.s., which in combination with the equality (3.12) and the estimate (3.13) implies that

$$\log M_s = -\int_0^s (v_r, d\widetilde{W}_r) + \frac{1}{2} \int_0^s ||v_r||^2 dr$$

$$\leq -\int_0^s (v_r, d\widetilde{W}_r) + \frac{||G^{-1}||_{\infty}^2 ||x - y||_p^2}{2\gamma_0}, \quad \mathbb{Q}_s\text{-a.s.}$$

Taking into account the fact in Lemma 3.2 that  $(\widetilde{W}_t)_{t\in[0,s]}$  is a *U*-valued cylindrical Wiener process under  $\mathbb{Q}_s$ , we arrive at

$$\mathbb{E}[M_s \log M_s] = \mathbb{E}_s \log M_s \le \frac{\|G^{-1}\|_{\infty}^2 \|x - y\|_p^2}{2\gamma_0},$$

and thus obtain (3.17), noting that  $\gamma_0$  is given in (3.6) with t = 0, where  $\mathbb{E}_s$  denotes the expectation with respect to  $\mathbb{Q}_s$ . By the martingale convergence theorem,  $M_T := \lim_{s \uparrow T} M_s$  exists and  $(M_t)_{t \in [0,T]}$  is a well-defined martingale.

Lemmas 3.2 and 3.3 ensures that  $(\widetilde{W}_t)_{t\in[0,T]}$  is a U-valued cylindrical Wiener process under the probability measure  $\mathbb{Q}:=M_T\mathbb{P}$  and

(3.21) 
$$\sup_{s \in [0,T]} \mathbb{E}[M_s \log M_s] \le \frac{\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2, \quad x, y \in L^p.$$

Then by (2.3) and (3.19), the coupling  $(X_t, Y_t)$  is well constructed under  $\mathbb{Q}$  for  $t \in [0, T]$ .

LEMMA 3.4. Let Assumptions 2.1, 2.2, and 2.4 hold. For any  $x, y \in L^p$  and s > 1,

(3.22) 
$$\sup_{s \in [0,T]} \mathbb{E} M_s^{\frac{\mathbf{s}}{\mathbf{s}-1}} \le \exp\left(\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x-y\|_p^2}{(\mathbf{s}-1)^2 (e^{2\lambda T}-1)}\right).$$

PROOF. Let  $s \in [0, T]$ . Denote by  $v_r^{\mathbf{s}} := -\frac{1}{\mathbf{s}-1}v_r$  for  $r \in [0, s] \subset [0, T]$ . The representation (3.12) and the pathwise estimate (3.15) with  $\gamma_0$  given in (3.6) yield that

$$M_s^{\frac{1}{\mathbf{s}-1}} = \exp\left(-\frac{1}{\mathbf{s}-1} \int_0^s \langle v_r, d\widetilde{W}_r \rangle + \frac{1}{2(\mathbf{s}-1)} \int_0^s \|v_r\|^2 dr\right)$$

$$= \exp\left(\int_0^s \langle v_r^{\mathbf{s}}, d\widetilde{W}_r \rangle - \frac{1}{2} \int_0^s \|v_r^{\mathbf{s}}\|^2 dr\right)$$

$$\times \exp\left(\frac{\mathbf{s}}{2(\mathbf{s} - 1)^2} \int_0^s \|v_r\|^2 dr\right)$$

$$\leq \widetilde{M}_s \exp\left(\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x - y\|_p^2}{(\mathbf{s} - 1)^2 (e^{2\lambda T} - 1)}\right), \quad \mathbb{Q}_s\text{-a.s.}$$

where  $\widetilde{M}_s := \exp(\int_0^s \langle v_r^{\mathbf{s}}, d\widetilde{W}_r \rangle - \frac{1}{2} \int_0^s ||v_r^{\mathbf{s}}||^2 dr)$ . It follows that

$$\mathbb{E}M_s^{\frac{\mathbf{s}}{\mathbf{s}-1}} = \mathbb{E}_s M_s^{\frac{1}{\mathbf{s}-1}} \le \exp\left(\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x-y\|_p^2}{(\mathbf{s}-1)^2 (e^{2\lambda T}-1)}\right) \mathbb{E}_s \widetilde{M}_s.$$

Taking into account the fact that  $(\widetilde{W}_t)_{t\in[0,T]}$  is a U-valued cylindrical Wiener process under  $\mathbb{Q}$ , which shows that  $(\widetilde{M}_t)_{t\in[0,T]}$  is a martingale under  $\mathbb{Q}$ , we have  $\mathbb{E}_s\widetilde{M}_s = \mathbb{E}_s\widetilde{M}_0 = 1$  and obtain (3.22).

# 4. Harnack Inequalities and Ergodicity

In the last section, we derive Harnack inequalities and the ergodicity for the Markov semigroup  $(P_t)_{t\geq 0}$ , in the following three parts. At the first two parts, we give the proof of our main results, Theorems 2.1 and 2.2, espectively. Other applications, including several estimates for the density of  $(P_t)_{t\geq 0}$ , are also derived.

**4.1.** Harnack inequalities. We begin with the following Harnack inequalities.

THEOREM 4.1. Let Assumptions 2.1, 2.2, and 2.4 hold. Then (2.17) and (2.18) hold for any T > 0,  $\mathbf{s} > 1$ ,  $x, y \in L^p$ , and  $\phi \in \mathcal{B}_b^+(L^p)$ .

PROOF. We first show that  $X_T = Y_T$  Q-a.s. From Lemma 3.3,  $(M_t)_{t \in [0,T]}$  is a uniformly integrable martingale and  $(\widetilde{W}_t)_{[0,T]}$  is a cylindrical Wiener process under the probability measure Q. So  $Y_t$  can be solved up to time T. Let

$$\tau := \inf\{t \in [0,T]: \ X_t = Y_t\} \quad \text{with} \quad \inf\emptyset := \infty.$$

Suppose that for some  $\omega \in \Omega$  such that  $\tau(\omega) > T$ , then the continuity of the process X - Y, in Lemma 3.1 and Remark 3.2, yields

$$\inf_{t \in [0,T]} ||X_t - Y_t||_p^2(\omega) > 0.$$

By the divergence relation (3.8),

$$\int_0^T \frac{\|X_t - Y_t\|_p^2(\omega)}{\gamma_t^2} dt = \infty$$

holds on the set  $(\tau > T) := \{\omega : \tau(\omega) > T\}$ . But according to the pathwise estimate (3.15) which holds  $\mathbb{Q}$ -a.s.,

$$\mathbb{E}_{\mathbb{Q}} \int_0^T \frac{\|X_t - Y_t\|_p^2(\omega)}{\gamma_t^2} dt \le \frac{\|x - y\|_p^2}{\gamma_0} < \infty,$$

where  $\mathbb{E}_{\mathbb{Q}}$  denotes the expectation with respect to  $\mathbb{Q}$ . It follows from the above two estimates that  $\mathbb{Q}(\tau > T) = 0$ , i.e.,  $\tau \leq T$   $\mathbb{Q}$ -a.s. By the definition of  $\tau$ ,  $X_T = Y_T$   $\mathbb{Q}$ -a.s.

Therefore, we get a coupling (X, Y) by the change of measure, with changed probability  $\mathbb{Q} = M_T \mathbb{P}$ , such that  $X_T = Y_T \mathbb{Q}$ -a.s. Consequently, the inequalities (2.17) and (2.18) follow from the following known inequalities (see, e.g., [Wan13, Theorem 1.1.1]):

$$P_T \log \phi(y) \le \log P_T \phi(x) + \mathbb{E}[M_t \log M_t],$$
$$(P_T \phi(y))^{\mathbf{s}} \le (P_T \phi^{\mathbf{s}}(x))(\mathbb{E}M_t^{\frac{\mathbf{s}}{\mathbf{s}-1}})^{\mathbf{s}-1},$$

and the estimations (3.17) and (3.22), in Lemmas 3.3 and 3.4, respectively.

The log-Harnack inequality (2.17) imply the following gradient estimate and regularity properties for the Markov semigroup  $(P_t)_{t>0}$ .

COROLLARY 4.1. Let Assumptions 2.1, 2.2, and 2.4 hold. For any T > 0 and  $\phi \in \mathcal{B}_b(L^p)$ ,

(4.1) 
$$||DP_T\phi|| \le \sqrt{\frac{2\lambda ||G^{-1}||_{\infty}^2}{e^{2\lambda T} - 1}} \sqrt{P_T\phi^2 - (P_T\phi)^2}.$$

Consequently,  $(P_t)_{t\geq 0}$  is strong Feller and has at most one invariant measure, and if it has one, the density of  $(P_t)_{t\geq 0}$  with respect to the invariant measure is strictly positive.

PROOF. The gradient estimate (4.1) and the uniqueness of the invariant measure for  $(P_t)_{t\geq 0}$  with a strictly positive density, if it exists, are direct consequence of the log-Harnack inequality (2.17), see Proposition 1.3.8 and Theorem 1.4.1 in [Wan13], respectively. Finally, the strong Feller property of  $(P_t)_{t\geq 0}$  follows easily from the gradient estimate (4.1).

REMARK 4.1. The uniqueness of the invariant measure, if it exists, for 1D stochastic heat equation (Eq. (2.3) with Lipschitz coefficients) driven by white noise on  $L^p(0,1)$  with p>4 was shown in [BR16]. So Corollary 4.1 can be seen as filling the gap for  $p \in (2,4]$  in the additive white noise case.

REMARK 4.2. Under the conditions in Corollary 4.1, there exist constants  $C, T_0 > 0$  such that

$$(4.2) \|\mathcal{L}(X_t^x) - \mathcal{L}(X_t^y)\|_{\text{TV}} \le Ce^{-\lambda t} \|x - y\|_p, t \ge T_0, \ x, y \in L^p,$$

where  $\mathcal{L}(X)$  denotes the distribution of X on  $L^p$ ,  $\lambda$  is given in (2.20), and  $\|\cdot\|_{\mathrm{TV}}$  denotes the total variation norm betweem two signed measures, i.e.,  $\|\mu - \nu\|_{\mathrm{TV}} := \sup_{\|\phi\|_{\infty} \le 1} |\int_{L^p} \phi \mathrm{d}\mu - \int_{L^p} \phi \mathrm{d}\nu|$  for two signed measures  $\mu$  and  $\nu$ .

PROOF OF THEOREM 2.1. Theorem 2.1 follows from Theorem 4.1 and Corollary 4.1.  $\Box$ 

**4.2. Ergodicity.** In this part, we show the existence of an invariant measure for the Markov semigroup  $(P_t)_{t\geq 0}$ . In combination with the uniqueness of the invariant measure, as shown in Corollary 4.1, we derive the existence of a unique and thus ergodic invariant measure. We also note that [**BG99**, Theorem 6.1] used the factorization approach to obtain the existence of an invariant measure in  $L^p$  with  $p \geq 2$  under the martingale solution framework.

THEOREM 4.2. Let Assumptions 2.1, 2.2, and 2.3 hold. Assume that q > 2 and (2.19) holds. Then  $(P_t)_{t\geq 0}$  has an invariant measure  $\mu$  with full support on  $L^p$  such that  $\mu(\|\cdot\|_{q+p-2}^{q+p-2}) < \infty$ . Assume furthermore that Assumption 2.4 holds, then  $\mu$  is the unique and thus ergodic invariant measure of  $(P_t)_{t\geq 0}$ .

PROOF. The uniqueness of the invariant measure and the strong Feller Markov property for  $(P_t)_{t\geq 0}$  have been shown in Corollary 4.1. Thus, to show the existence of an invariant measure, by Krylov–Bogoliubov theorem, it suffices to verify the tightness of the sequence of probability measures  $(\mu_n)_{n\in\mathbb{N}_+}$  defined by

(4.3) 
$$\mu_n := \frac{1}{n} \int_0^n \delta_0 P_t dt, \quad n \in \mathbb{N}_+,$$

where  $\delta_0 P_t$  is the distribution of  $X_t^0$ , the solution of Eq. (2.3) with the initial datum  $X_0 = 0$ .

It follows from the relation  $X = Z + W_A$ , the estimate (3.4) with x = 0, and Young inequality that (4.4)

$$||X_t^0||_p^p \le 2^{p-1} ||Z_t^0||_p^p + 2^{p-1} ||W_A(t)||_p^p$$

$$\le C \int_0^t (1 + ||W_A(r)||_{q+p-2}^{q+p-2}) dr + 2^{p-1} p L_f \int_0^t ||Z_r^0||_p^p dr$$

$$-2^{p-1}p\theta_{1} \int_{0}^{t} \|Z_{r}^{0}\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_{A}(t)\|_{p}^{p}$$

$$\leq C \int_{0}^{t} (1 + \|W_{A}(r)\|_{q+p-2}^{q+p-2}) dr - \theta_{4} \int_{0}^{t} \|Z_{r}^{0}\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_{A}(t)\|_{p}^{p}$$

$$\leq C \int_{0}^{t} (1 + \|W_{A}(r)\|_{q+p-2}^{q+p-2}) dr - \theta_{5} \int_{0}^{t} \|X_{r}^{0}\|_{q+p-2}^{q+p-2} dr + 2^{p-1} \|W_{A}(t)\|_{p}^{p},$$

for some constants  $\theta_4, \theta_5 > 0$ , where we have used the elementary inequality  $|\xi - \eta|^{\mathbf{r}} \geq 2^{1-r}\xi^r - \eta^r$  for  $\xi, \eta \geq 0$  and  $\mathbf{r} \geq 1$ , in the last inequality. Then we have

$$\theta_5 \int_0^t \|X_r^0\|_{q+p-2}^{q+p-2} dr \le C \int_0^t (1 + \|W_A(r)\|_{q+p-2}^{q+p-2}) dr + 2^{p-1} \|W_A(t)\|_p^p.$$

The above estimate, in combination with the condition (2.13), yields that there exists a constant C such that for all  $n \ge 1$ ,

$$\mu_{n}(\|\cdot\|_{q+p-2}^{q+p-2}) = \frac{1}{n} \int_{0}^{n} \mathbb{E} \|X_{r}^{0}\|_{q+p-2}^{q+p-2} dr$$

$$\leq \frac{C}{\theta_{5}} \left(1 + \frac{\mathbb{E} \|W_{A}(n)\|_{p}^{p}}{n} + \frac{1}{n} \int_{0}^{n} \mathbb{E} \|W_{A}(r)\|_{q+p-2}^{q+p-2} dr\right) \leq C.$$

It follows from the ultracontractivity (2.6), with  $\mathbf{r} = p$  and  $\mathbf{s} = \frac{q+p-2}{a-1}$ , and Young convolution inequality that

$$\int_{0}^{n} \left\| \int_{0}^{t} S_{t-r} F(X_{r}^{0}) dr \right\|_{\beta,p} dt 
\leq C \int_{0}^{n} \int_{0}^{t} e^{-\lambda_{1}(t-r)} (t-r)^{-\alpha} (1 + \|X_{r}^{0}\|_{q+p-2}^{q-1}) dr dt 
\leq C \left( \int_{0}^{n} e^{-\lambda_{1} t} t^{-\alpha} dt \right) \left( \int_{0}^{n} (1 + \|X_{t}^{0}\|_{q+p-2}^{q-1}) dt \right),$$

where  $\alpha = \frac{\beta}{2} + \frac{d(p-1)(q-2)}{2p(q+p-2)} \in (0,1)$  provided that  $\beta > 0$  is sufficiently small, since  $d < \frac{2p(q+p-2)}{(p-1)(q-2)}$ . The fact that

$$\sup_{n\geq 1} \left( \int_0^n e^{-\lambda_1 t} t^{-\alpha} dt \right) \leq \int_0^\infty e^{-\lambda_1 t} t^{-\alpha} dt < \infty,$$

for all  $\lambda_1 > 0$  and  $\alpha \in (0,1)$ , and Young inequality imply that

$$\int_0^n \left\| \int_0^t S_{t-r} F(X_r^0) dr \right\|_{\beta,p} dt \le C \int_0^n (1 + \|X_t^0\|_{q+p-2}^{q+p-2}) dt.$$

By Fubini theorem, the estimate (4.5), and the condition (2.15), we arrive at

(4.6)

$$\mu_{n}(\|\cdot\|_{\beta,p}) = \frac{1}{n} \int_{0}^{n} \mathbb{E} \|X_{r}^{0}\|_{\beta,p} dr$$

$$\leq \frac{1}{n} \mathbb{E} \int_{0}^{n} \|\int_{0}^{t} S_{t-r} F(X_{r}^{0}) dr \|_{\beta,p} dr + \frac{1}{n} \int_{0}^{n} \mathbb{E} \|W_{A}(t)\|_{\beta,p} dr$$

$$\leq \frac{C}{n} \int_{0}^{n} (1 + \mathbb{E} \|X_{t}^{0}\|_{q+p-2}^{q+p-2}) dt + \frac{1}{n} \int_{0}^{n} \mathbb{E} \|W_{A}(t)\|_{\beta,p} dr \leq C < \infty,$$

for all  $n \geq 1$  and  $\beta < (1 - \frac{d(p-1)(q-2)}{2p(q+p-2)}) \wedge \beta_0$ . For any fixed  $p \geq 2$ , we take  $\beta < (1 - \frac{d(p-1)(q-2)}{2p(q+p-2)}) \wedge \beta_0 \wedge \frac{d}{p}$ , so that the embedding  $W_0^{\beta,p} \subset L^p$  in (2.5) is compact. Consequently, the above estimate (4.6) shows that  $\{u \in L^p : \|u\|_{\beta,p} \leq N\}$  is relatively compact in  $L^p$  for any N > 0, and thus  $(\mu_n)_{n \in \mathbb{N}_+}$  is tight. This shows the existence of an invariant measure, denoted by  $\mu$ , of  $(P_t)_{t>0}$ .

To show that the invariant measure  $\mu$  has full support on  $L^p$ , let us choose  $\mathbf{s} = 2$ ,  $\phi = \chi_{\Gamma}$ , in (2.18), with  $\Gamma$  being a Borel set in  $L^p$ , and get

$$(P_T \chi_{\Gamma}(x))^2 \int_{L^p} \exp\left(-\frac{2\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2\right) \mu(\mathrm{d}y)$$

$$\leq \int_H P_T \chi_{\Gamma}(y) \mu(\mathrm{d}y) = \int_H \chi_{\Gamma}(y) \mu(\mathrm{d}y) = \mu(\Gamma), \quad T > 0, \ x \in L^p.$$

This shows that the transition kernel of  $(P_t)_{t\geq 0}$  is absolutely continuous with respect to  $\mu$  so that it has a density  $p_T(x,y)$ . Suppose that supp  $\mu \neq L^p$ , then there exist  $x_0 \in L^p$  and r > 0 such that  $\mu(B(x_0,r)) = 0$ , where  $B(x_0,r)$  is a ball in  $L^p$  with radius r and center  $x_0$ . Then  $p_T(x_0, B(x_0,r)) = 0$  and  $\mathbb{P}(\|X_T^{x_0} - x_0\|_p \leq r) = 0$  for all T > 0. This contradicts with the fact that  $X_T^{x_0}$  is a continuous process on  $L^p$  as shown in Lemma 3.1.

Similarly to (4.5), we have (with n = 1 and  $X_0 = x$ )

$$\int_{0}^{1} P_{t} \| \cdot \|_{q+p-2}^{q+p-2}(x) dt = \int_{0}^{1} \mathbb{E} \| X_{t}^{x} \|_{q+p-2}^{q+p-2} dt \le C(1 + \|x\|_{p}^{p}).$$

Integrating on  $L^p$  with respect to the invariant measure  $\mu$  and using Fubini theorem, we obtain

$$\mu(\|\cdot\|_{q+p-2}^{q+p-2}) = \int_0^1 \int_H P_t \|\cdot\|_{q+p-2}^{q+p-2}(x) \mu(\mathrm{d}x) \mathrm{d}t \le C(1 + \mu(\|\cdot\|_p^p)) < \infty.$$

This shows that  $\mu(\|\cdot\|_{q+p-2}^{q+p-2}) < \infty$  and completes the proof.  $\square$ 

REMARK 4.3. In the case q > 2 = p, the condition (2.19) is equivalent to d < 4 + 8/(q - 2), which will be always valid in d = 1, 2, 3-dimensional cases.

REMARK 4.4. Under Assumptions 2.1 and 2.2, if  $\lambda$  defined in (2.20) is positive, one can use the standard remote control method to show that  $(P_t)_{t\geq 0}$  has an invariant measure  $\mu$ , once we derive similar estimates as (3.3) and (3.4), without the restriction (2.19); see, e.g., [**DPZ96**, Theorem 6.3.2]. In this case,  $\mu$  also has full support on  $L^p$  such that  $\mu(\|\cdot\|_{p+q-2}^{p+q-2}) < \infty$  holds.

REMARK 4.5. Under the conditions of Theorem 4.2 or Remark 4.4,  $(P_t)_{t\geq 0}$  has a unique invariant measure  $\mu$  with full support on  $L^p$ , which shows that  $(P_t)_{t\geq 0}$  is irreducible, i.e.,  $P_T\chi_{\Gamma}(x)>0$  for arbitrary nonempty open set  $\Gamma\subset L^p$ ,  $x\in L^p$ , and T>0. Indeed, the power-Harnack inequality (2.18) with  $f=\chi_{\Gamma}$  yields that

$$(4.7) \quad (P_T \chi_{\Gamma}(y))^{\mathbf{s}} \le P_T \chi_{\Gamma}(x) \exp\left(\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x - y\|_p^2}{(\mathbf{s} - 1)(e^{2\lambda T} - 1)}\right), \quad y \in L^p.$$

The facts that  $\mu$  is  $P_T$ -invariant and has full support on  $L^p$  imply

$$\int_{L^p} P_T \chi_{\Gamma}(y) \mu(\mathrm{d}y) \le \int_{L^p} \chi_{\Gamma}(y) \mu(\mathrm{d}y) = \mu(\Gamma) > 0,$$

which shows that there is a  $y \in L^p$  such that  $P_T \chi_{\Gamma}(y) > 0$ . Then (4.7) yields that  $P_T \chi_{\Gamma}(x) > 0$  for all  $x \in L^p$ , so that the irreducibility holds.

PROOF OF THEOREM 2.2. Theorem 2.2 follows from Theorem 4.2 and Remark 4.4.  $\hfill\Box$ 

**4.3. Estimates of density.** Finally, we use the Harnack inequalities (2.17) and (2.18) to derive an estimate of the density, denoted by  $p_T(x, y)$ , with respect to the invariant measure  $\mu$  of  $(P_t)_{t\geq 0}$ .

COROLLARY 4.2. Let Assumptions 2.1 and 2.2 hold. Assume that q > 2 such that (2.19) hold and Assumption 2.3 hold, or  $\lambda$  defined in (2.20) is positive. Then for all T > 0,  $x \in L^p$ , and s > 1,

$$(4.8) \|p_T(x,\cdot)\|_{L^{\mathbf{s}}(\mu)} \le \left( \int_H \exp\left(-\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^2 \|x-y\|_p^2}{e^{2\lambda T}-1}\right) \mu(\mathrm{d}y) \right)^{-\frac{\mathbf{s}-1}{\mathbf{s}}}.$$

PROOF. Using (2.18) with s replaced by  $\frac{s}{s-1}$ , and noting that  $\mu$  is  $(P_t)_{t>0}$ -invariant, we obtain

$$||p_T(x,\cdot)||_{L^{\mathbf{s}}(\mu)}$$

$$= \sup\{\langle p_T(x,\cdot), \phi \rangle_{\mu} : \phi \in B_b^+(L^p), \ \mu(\phi^{\frac{\mathbf{s}}{\mathbf{s}-1}}) \le 1\}$$

$$= \sup\{P_T \phi(x) : \ \phi \in B_b^+(L^p), \ \mu(\phi^{\frac{s}{s-1}}) \le 1\}$$

$$\leq \left( \int_{H} \phi^{\frac{\mathbf{s}}{\mathbf{s}-1}}(y) \mu(\mathrm{d}y) \right)^{\frac{\mathbf{s}-1}{\mathbf{s}}} \left( \int_{H} \exp\left( -\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^{2} \|x-y\|_{p}^{2}}{e^{2\lambda T}-1} \right) \mu(\mathrm{d}y) \right)^{\frac{\mathbf{s}-1}{\mathbf{s}}}$$

$$= \mu(\phi^{\frac{\mathbf{s}}{\mathbf{s}-1}}) \left( \int_{H} \exp\left( -\frac{\mathbf{s}\lambda \|G^{-1}\|_{\infty}^{2} \|x-y\|_{p}^{2}}{e^{2\lambda T}-1} \right) \mu(\mathrm{d}y) \right)^{\frac{\mathbf{s}-1}{\mathbf{s}}}.$$

This shows the density estimate (4.8).

REMARK 4.6. According to [Wan10, Proposition 2.4], the log-Harnack inequality (2.17) and the power-Harnack inequality (2.18) are equivalent to the following two heat kernel inequalities, respectively, provided  $P_T$  have a strictly positive density  $p_T(x, y)$  with respect to a Radon measure of  $P_T$ :

$$\int_{L^p} p_T(x,z) \log \frac{p_T(x,z)}{p_T(y,z)} \mu(\mathrm{d}z) \le \frac{\lambda \|G^{-1}\|_{\infty}^2}{e^{2\lambda T} - 1} \|x - y\|_p^2,$$

$$\int_{L^p} p_T(x,z) \left(\frac{p_T(x,z)}{p_T(y,z)}\right)^{\frac{1}{s-1}} \mu(\mathrm{d}z) \le \frac{s\lambda \|G^{-1}\|_{\infty}^2}{(s-1)^2 (1 - e^{-2\lambda T})} \|x - y\|_p^2.$$

Under the conditions in Corollary 4.2,  $(P_t)_{t\geq 0}$  has a unique invariant measure  $\mu$  such that  $p_T(x,y)$  is strictly positive. Then the above two heat kernel inequalities are direct consequence of Theorem 4.1, Theorem 4.2, and Remark 4.4.

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