DYNAMICAL INSTABILITY OF MINIMAL SURFACES AT FLAT SINGULAR POINTS

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Abstract. Suppose that a countably n-rectifiable set Γ_0 is the support of a multiplicity-one stationary varifold in $\mathbb{R}^{n+1}$ with a point admitting a flat tangent plane T of density $Q \geq 2$. We prove that, under a suitable assumption on the decay rate of the blow-ups of Γ_0 towards T, there exists a *non-constant*, genuinely time-dependend Brakke flow starting with Γ_0. The result, which applies, in particular, to a large class of (possibly stable) minimal immersions with branch singularities, shows non-uniqueness of Brakke flow under these conditions. Furthermore, it suggests that stationary varifolds which are *dynamically stable*, i.e. stable with respect to mean curvature flow, may be free from flat singularities.

Keywords: mean curvature flow, varifolds, singularities of minimal surfaces.

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1. Introduction

A family of surfaces is said to move by mean curvature flow (abbreviated hereafter as MCF) if the velocity of motion is equal to the mean curvature at each point and time. The MCF is one of the simplest geometric evolution problems, and it has been studied intensively by

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numerous researchers over the last few decades. In the early stages of the development of the theory of MCF, Brakke introduced in [6] a notion of MCF - which is nowadays referred to as the *Brakke flow* - within the framework of geometric measure theory. It is a generalized notion of MCF where the evolving surfaces are not required to be classical, regular submanifolds, but rather *varifolds*, and where the classical parabolic PDE describing the evolution law is replaced by an *ad hoc* inequality which is adapted to the language of varifolds while still being able to capture the geometric features of MCF; see section 2 for a brief introduction to the subject, and [6, 36] for further references. The advantage of such a seemingly abstract approach is that it allows one to describe the evolution by mean curvature of *singular* surfaces (e.g. a moving network of curves in the plane with multiple junction points or a moving cluster of bubbles in the three-dimensional space), as well as to continue the evolution of classical surfaces also after singularities arise. At the same time, a possible drawback is that the solution to Brakke flow for a given initial datum may not be unique in general.

Recently, the authors of the present paper proved a general theorem concerning the existence of Brakke flows starting from any given closed countably n-rectifiable set Γ_0 in a strictly convex domain in $\mathbb{R}^{n+1}$ and with the additional property that the (topological) boundary of the evolving varifolds is fixed throughout the flow [32]. This existence result gives rise to a number of questions pertaining to the nature of the Brakke flow. One such question to be discussed in the present paper is the following:

$$\begin{array}{c}\text{Does there exist a } \textit{stationary} \text{ initial datum } \Gamma_0 \\ \text{admitting a } \textit{non-trivial} \text{ Brakke flow starting with it?}\end{array} \tag{$\mathcal{Q}$}$$

Here, "stationary" means that the first variation of the associated multiplicity one varifold vanishes, and "non-trivial" means that the flow is genuinely time-dependent: note that a stationary Γ_0 itself is a time-independent Brakke flow with no motion. Thus, the question is equivalent to inquiring about the non-uniqueness of Brakke flow starting from a given stationary Γ_0. To avoid instantaneous vanishing, it is also natural to require the continuity of the surface measures associated to the Brakke flow at $t = 0+$. If Γ_0 is smooth, then one expects that all Brakke flows starting with it should be trivial as a consequence of the regularity theory for Brakke flows, both in the interior and at the boundary (see [17, 35, 33, 10] for the former and [15] for the latter) and the uniqueness theorem of smooth mean curvature flows, thus it is interesting to focus on stationary Γ_0 with singularities. In fact, this observation leads to the following refinement of question ($\mathcal{Q}$):

$$\begin{array}{l}\text{Which types of singularities, if any, of a stationary } \Gamma_0 \textit{ necessarily} \text{ determine} \\ \text{the existence of a non-trivial Brakke flow starting with } \Gamma_0\text{?}\end{array} \tag{$\mathcal{Q}'$}$$

The main result of the paper answers affirmatively to question ($\mathcal{Q}$), by identifying a type of singularity with the property described in question ($\mathcal{Q}'$). The result can be roughly stated as follows (see Theorem 3.5 for the precise statement).

Theorem A. Suppose that a closed countably n-rectifiable set Γ_0 is stationary, and that there exists $x_0 \in \Gamma_0$ with the following properties:

(1) one of the tangent cones to Γ_0 at x_0 is a flat plane T with multiplicity $Q \geq 2$, and
(2) the rescalings $(\Gamma_0 - \{x_0\})/r$ locally converge to T at a rate faster than $(\log(1/r))^{-1/2}$ as $r \to 0^+$.

Then, there exists a non-trivial Brakke flow starting from Γ_0.

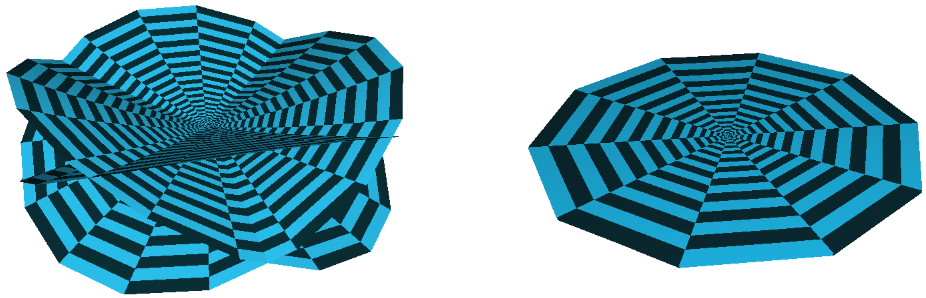

FIGURE 1. On the left, a surface Γ_0 with a flat singularity of multiplicity $Q = 3$. The homothetic rescalings $(\Gamma_0 - \{x_0\})/r$ converge to the unique tangent plane with rate $\mathrm{O}(r^{1/3})$ as $r \to 0^+$. On the right, a figure obtained zooming in the picture on the left at the singularity, thus showing (a portion of) the unique tangent plane T to Γ_0 at x_0.

We remark that, under a natural assumption to be specified later, the Brakke flow obtained in Theorem A is continuous at $t = 0+$. We observe explicitly that the assumption (2) implies, in particular, that the varifold associated with the plane T and carrying multiplicity Q is the *unique* varifold tangent cone to Γ_0 at x_0. There is a plethora of examples of stationary Γ_0 admitting the kind of singularities described in (1) and (2). Consider, for instance, the class of minimal immersions with a branch point singularity: these are immersed minimal surfaces Γ_0 in $\mathbb{R}^3$ which admit a parametrization $X \colon \Omega \to \mathbb{R}^3$ of the form

$$X(z) = \mathrm{Re}[f(z)]\,,$$

where $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ is a neighborhood of $z_0 = 0$, and $f \colon \Omega \to \mathbb{C}^3$ is a holomorphic curve in $\mathbb{C}^3$ with $f = (f_1, f_2, f_3) \in \mathbb{C}^3$ satisfying

$$(f_1')^2 + (f_2')^2 + (f_3')^2 = 0 \quad \text{in } \Omega\,, \qquad \text{and} \qquad f'(0) = 0\,.$$

In a suitable system of coordinates (x^1, x^2, x^3) of $\mathbb{R}^3$, such a surface is represented by

$$\begin{aligned} x^1(z) + i\,x^2(z) &= (x_0^1 + i\,x_0^2) + a\,z^Q + \mathrm{O}(|z|^{Q+1})\,, \\ x^3(z) &= x_0^3 + \mathrm{O}(|z|^{Q+1})\,, \end{aligned}$$

for some $Q \geq 2$, and thus it behaves like the (multivalued) graph of a complex root: setting $x_0 = (x_0^1, x_0^2, x_0^3)$, the blow-ups $r^{-1}(\Gamma_0 - \{x_0\})$ converge to the plane $T = \{x_3 = 0\}$ with multiplicity Q with a rate $\mathrm{O}(r^\alpha)$ for some $\alpha > 0$, and thus much faster than the slow logarithmic decay required in (2); see [11, Section 3.2] for a more detailed analysis of the behavior of minimal surfaces near branch points, and Figure 1 for a graphical representation. More generally, Theorem A applies to the class of those stationary varifolds of arbitrary dimension n that are graphs of multiple valued solutions to the minimal surfaces equation (*$C^{1,\alpha}$ multiple valued minimal graphs*), which have been extensively studied in the literature; see, e.g. [31, 25, 26, 27, 20, 14].

Subject to the validity of (1) and (2), Theorem A concludes the *dynamical instability* of Γ_0. We warn the reader that dynamical stability/instability is an independent notion with respect to the classical notion of stability/instability defined by the spectrum of the second variation operator on Γ_0: as observed for instance in [22], there exist *stable* (yet, by the discussion above and Theorem A, dynamically unstable) branched minimal surfaces of the type of the disk in $\mathbb{R}^3$. We will come back to give further details on the connection between the two notions of stability after a preliminary discussion concerning the conditions in (1) and (2) and their role in the regularity theory for stationary varifolds, and a review of the existing literature on the topic.

Firstly, when Γ_0 is stationary, the classical regularity theorem by Allard [2] guarantees the existence of a closed set $\mathcal{S} \subset \Gamma_0$ with n-dimensional Hausdorff measure $\mathcal{H}^n(\mathcal{S}) = 0$ such that $\Gamma_0 \setminus \mathcal{S}$ is an embedded real-analytic minimal hypersurface. With no further assumptions, presently there are no known properties of the singular set $\mathcal{S}$ other than the fact that it is $\mathcal{H}^n$-negligible (except for $n = 1$, in which case $\mathcal{S}$ is a locally $\mathcal{H}^0$-finite set, see [3]). A crucial missing piece towards a refinement of this result is an estimate on the size of the set of points $x_0 \in \Gamma_0$ which admit tangent cones that are n-dimensional flat planes with multiplicity $Q \geq 2$, that is precisely the points in Γ_0 satisfying (1). We shall call these points the *flat singularities* of Γ_0. If one knew that (2) is satisfied at every flat singularity of Γ_0, then Theorem A would provide a *dynamical* condition to exclude the presence of flat singularities altogether: explicitly, one would be able to conclude that if Γ_0 is dynamically stable (that is, if the only Brakke flow starting with Γ_0 is the trivial one), then no singular point $x_0 \in \mathcal{S}$ has a tangent cone which is supported on an n-dimensional plane. In turn, this would imply that the singular set $\mathcal{S}$ of Γ_0 has Hausdorff dimension $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-1$, and in fact that $\mathcal{S}$ is countably $(n-1)$-rectifiable by the pioneering work of Naber and Valtorta, see [24]. In fact, it would be interesting to investigate the validity of results analogous to that of Theorem A also under different assumptions on the geometry of the tangent cones to Γ_0 at a singular point x_0 (that is, cones which are not supported on a plane), with the goal of providing further insight into question $(\mathcal{Q}')$ and correspondingly deducing further information on the fine properties of the singularities of dynamically stable stationary varifolds.

As far as the authors are aware of, there are no known examples of stationary Γ_0 with a flat singularity for which the decay rate (2) fails. On the other hand, proving that (2) always holds true at flat singularities is arguably a very hard problem, since, as already noticed, it would in particular imply the uniqueness of the tangent plane, which is still a major unsolved problem in geometric measure theory, see [1, Problem 5.10]. In fact, one may wonder whether a decay rate as in (2) holds true at least assuming *a-priori* that the tangent plane is unique, but even this result is out of reach of the currently available techniques. Indeed, it is worth mentioning that presently all available results concerning uniqueness of tangent cones to a stationary Γ_0 at a singular point x_0 have been obtained under the further assumption that one of the tangent cones to the associated multiplicity one varifold V_0 at x_0 is a (necessarily non-flat) *multiplicity one* cone $\mathbf{C}_0$: for instance, in this setting Simon concluded uniqueness of $\mathbf{C}_0$ whenever $\mathbf{C}_0$ is regular in $\mathbb{R}^{n+1} \setminus \{0\}$ in [28], and also when $\mathbf{C}_0$ is a cylinder of the form $\mathbf{C}_0 = \hat{\mathbf{C}}_0 \times \mathbb{R}^{n-k}$ with $\hat{\mathbf{C}}_0$ regular in $\mathbb{R}^{k+1} \setminus \{0\}$ under additional hypotheses of integrability of the Jacobi fields of the cross section $\hat{\mathbf{C}}_0 \cap \mathbb{S}^k$ and "absence of holes" in the singular set of Γ_0, see [30]. In these cases, the multiplicity one assumption is crucial in order to locally parametrize Γ_0 over $\mathbf{C}_0$ with *single-valued* functions, for which PDE techniques are available. It is important to note that both in the cylindrical case treated in [30] and in the non-cylindrical case under integrability of Jacobi fields of the cross section (see [4]), the homothetic rescalings of the varifold V_0 at x_0 converge towards the unique tangent cone $\mathbf{C}_0$ with rate r^α for some $\alpha > 0$, that is, the aforementioned parametrization is of class $C^{1,\alpha}$. Recently, a uniqueness result similar to the one of [28] was obtained in the setting of almost area minimizing currents by

Engelstein, Spolaor, and Velichkov [12], at the price of producing $C^{1,\log}$ parametrizations. In other words, if an almost area minimizing current has a multiplicity one tangent cone $\mathbf{C}_0$ with singularity only at the origin, then the homothetic rescalings of the current converge to $\mathbf{C}_0$ at a rate $(\log(1/r))^{-\alpha}$ for some $\alpha > 0$ as $r \to 0^+$, see [12, Theorem 1.5]. The similarity between the decay rate of [12] and our assumption (2) is interesting, and will be object of further investigation.

Coming back to flat singularities, much more can be said on whether the condition in (1) implies the decay in (2) if stability of the regular part $\mathrm{Reg}(\Gamma_0)$ is assumed. Precisely, very recently Minter and Wickramasekera proved in [23] the following result: if $\mathrm{Reg}(\Gamma_0)$ is stable (that is, if every two-sided portion of $\mathrm{Reg}(\Gamma_0)$ has non-negative second variation with respect to the mass functional for compactly supported normal deformations), if a point $x_0 \in \Gamma_0$ satisfies (1) for a plane T and an integer $Q \geq 2$, and furthermore if in a neighborhood $B_{2r_0}(x_0)$ there are no classical singularities of Γ_0 of density $< Q$ [1], then T is the unique varifold tangent cone to Γ_0 at x_0 and, in the cylinder $((x_0 + T) \cap B_{r_0}(x_0)) \times T^\perp$, Γ_0 coincides with a (generalized) $C^{1,\alpha}$ Q-valued graph, so that the rescalings $r^{-1}(\Gamma_0 - \{x_0\})$ converge to T at a rate $\mathrm{O}(r^\alpha)$ for some $\alpha \in (0,1)$ depending only on (n, Q); refer to [5, 9] for the notion of multiple valued functions used in [23]. Theorem A then immediately implies the dynamical instability of Γ_0. We will record this result in Corollary 3.7.

Notice that, in spite of the fact that the condition on the absence of classical singularities of density $< Q$ in a neighborhood of x_0 is, in principle, difficult to guarantee, we point out two classical instances when its validity can be easily checked:

- if Γ_0 carries the structure of *rectifiable current*, which we denote $[\![\Gamma_0]\!]$, it is stationary as a varifold, $\mathrm{Reg}(\Gamma_0)$ is stable, and x_0 is an *interior* point (that is, $x_0 \notin \mathrm{spt}\|\partial [\![\Gamma_0]\!]\|$) satisfying (1) for a plane T and $Q = 2$, then (2) holds as a consequence of [23], and Γ_0 is dynamically unstable; see also [21];
- if Γ_0 carries the structure of rectifiable current, it is *area minimizing* $\mathrm{mod}(p)$ for an *even* integer $p = 2Q$, and it admits a flat tangent plane T at $x_0 \notin \mathrm{spt}\|\partial^p [\![\Gamma_0]\!]\|$, then the multiplicity of the plane must be Q, (2) holds as a consequence of [23], and Γ_0 is dynamically unstable; see [13, section 4.2.26] and [8] for the definition of area minimizing currents $\mathrm{mod}(p)$ and the corresponding notation, and [7] for a sharp estimate on the dimension of the set of singularities of area minimizing currents $\mathrm{mod}(2Q)$ for which (1) holds.

We remark that the above points identify classes of stationary varifolds which are stable for the second variation operator but *not* dynamically stable, further exploring the striking difference between the two notions.

The paper is organized as follows. In section 2 we fix the relevant notation and terminology; in section 3 we discuss the precise assumptions on the set Γ_0 and state precisely the main result, Theorem 3.5; in section 4 we describe how to suitably modify Γ_0 in an ε-neighborhood of x_0 in order to obtain a new set Γ_0^ε which has strictly less mass than Γ_0 (we shall say, informally, that Γ_0^ε has a "hole" at x_0), and then we take advantage of [32] to produce a Brakke flow starting with Γ_0^ε; sections 5 and 6 are the technical core of the paper, as they contain the main

[1] A point $x_0 \in \Gamma_0$ is called a *classical singularity* if Γ_0 is, locally, the union of finitely many, and at least three, embedded C^1 submanifolds-with-boundary M_j having the same $(n-1)$-dimensional boundary $\partial M_j = L \ni x_0$ and with $M_i \cap M_j = L$ for $i \neq j$ and with M_i and M_j intersecting transversely at every point in L for at least one pair of indexes $i \neq j$.

estimates needed to show that, along the Brakke flow evolution, the hole in Γ_0^ε *expands* in a precisely quantifiable way. Performing this operation of hole nucleation / hole expansion along a suitable sequence ε_j produces a sequence of Brakke flows which converges, as $j \to \infty$, to a limiting Brakke flow of surfaces starting with Γ_0 and having a definite mass drop (with respect to Γ_0) at a later time, thus completing the proof of Theorem A: this is achieved in section 7.

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2. Notation and terminology

2.1. **Basic notation.** The ambient space we will be working in is Euclidean space $\mathbb{R}^{n+1}$. We write $\mathbb{R}^+$ for $[0,\infty)$. For $A \subset \mathbb{R}^{n+1}$, $\operatorname{clos} A$ (or $\overline{A}$) is the topological closure of A in $\mathbb{R}^{n+1}$, $\operatorname{int} A$ is the set of interior points of A and $\operatorname{conv} A$ is the convex hull of A. The standard Euclidean inner product between vectors in $\mathbb{R}^{n+1}$ is denoted $x \cdot y$, and $|x| := \sqrt{x \cdot x}$. If $L, S \in \mathscr{L}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ are linear operators in $\mathbb{R}^{n+1}$, their (Hilbert-Schmidt) inner product is $L \cdot S := \operatorname{trace}(L^t \circ S)$, where L^t is the transpose of L and $\circ$ denotes composition. The corresponding (Euclidean) norm in $\mathscr{L}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is then $|L| := \sqrt{L \cdot L}$, whereas the operator norm in $\mathscr{L}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is $\|L\| := \sup\{|L(x)| \,:\, x \in \mathbb{R}^{n+1} \text{ with } |x| \le 1\}$. If $u, v \in \mathbb{R}^{n+1}$ then $u \otimes v \in \mathscr{L}(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is defined by $(u \otimes v)(x) := (x \cdot v)\, u$, so that $\|u \otimes v\| = |u|\,|v|$. The symbols $U_r(x)$ and $B_r(x)$ denote the open and closed balls in $\mathbb{R}^{n+1}$ centered at x and with radius $r > 0$, respectively. The Lebesgue measure of a set $A \subset \mathbb{R}^{n+1}$ is denoted $\mathcal{L}^{n+1}(A)$ or $|A|$. If $1 \le k \le n+1$ is an integer, $U^k_r(x)$ denotes the open ball with center x and radius r in $\mathbb{R}^k$. We will set $\omega_k := \mathcal{L}^k(U^k_1(0))$. The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, normalized in such a way that $\mathcal{H}^{n+1}$ and $\mathcal{L}^{n+1}$ coincide as measures.

We write $\mathbf{G}(n+1,k)$ to denote the Grassmannian of (unoriented) k-dimensional linear planes in $\mathbb{R}^{n+1}$. Given $T \in \mathbf{G}(n+1,k)$, we shall often identify T with the orthogonal projection operator onto it, and let $T^\perp := \mathrm{I} - T$, with I the identity operator in $\mathbb{R}^{n+1}$, denote the projection operator onto the orthogonal complement of T in $\mathbb{R}^{n+1}$. If $x \in \mathbb{R}^{n+1}$, $r > 0$, and $T \in \mathbf{G}(n+1,k)$, then $\mathrm{C}(T,x,r)$ denotes the cylinder orthogonal to T, centered at x with radius r, namely the set

$$\mathrm{C}(T,x,r) := \Big\{ y \in \mathbb{R}^{n+1} \,:\, |T(y-x)| < r \Big\}\,.$$

We will simply write $\mathrm{C}(x,r)$ in all contexts where the plane T is clear, and $\mathrm{C}(r)$ when $x = 0$.

A Radon measure μ in an open set $U \subset \mathbb{R}^{n+1}$ is always also regarded as a linear functional on the space $C_c(U)$ of continuous and compactly supported functions on U, with the pairing denoted $\mu(\phi)$ for $\phi \in C_c(U)$. The restriction of μ to a Borel set A is denoted $\mu\llcorner_A$, so that $(\mu\llcorner_A)(E) := \mu(A \cap E)$ for any Borel $E \subset U$. The support of μ is denoted $\operatorname{spt}\mu$, and it is the relatively closed subset of U defined by

$$\operatorname{spt}\mu := \{x \in U \,:\, \mu(U_r(x)) > 0 \text{ for every } r > 0\}\,.$$

The upper and lower k-dimensional densities of a Radon measure μ at $x \in U$ are

$$\Theta^{*k}(\mu, x) := \limsup_{r \to 0^+} \frac{\mu(U_r(x))}{\omega_k\, r^k}\,, \qquad \Theta^k_*(\mu, x) := \liminf_{r \to 0^+} \frac{\mu(U_r(x))}{\omega_k\, r^k}\,,$$

respectively. If $\Theta^{*k}(\mu, x) = \Theta^k_*(\mu, x)$ then the common value is denoted $\Theta^k(\mu, x)$, and is called the *k-dimensional density* of μ at x. For $1 \leq p \leq \infty$, the space of p-integrable (resp. locally p-integrable) functions with respect to μ is denoted $L^p(\mu)$ (resp. $L^p_{\mathrm{loc}}(\mu)$). For a set $E \subset U$, χ_E is the characteristic function of E. If E is a set of finite perimeter in U, then $\nabla\chi_E$ is the associated Gauss-Green measure in U, and its total variation $\|\nabla\chi_E\|$ in U is the perimeter measure; by De Giorgi's structure theorem, $\|\nabla\chi_E\| = \mathcal{H}^n \llcorner_{\partial^* E}$, where $\partial^* E$ is the reduced boundary of E in U.

2.2. **Varifolds.** Let $U \subset \mathbb{R}^{n+1}$ be open. The symbol $\mathbf{V}_k(U)$ will denote the space of k-dimensional *varifolds* in U, namely the space of Radon measures on $\mathbf{G}_k(U) := U \times \mathbf{G}(n+1, k)$ (see [2, 29] for a comprehensive treatment of varifolds). To any given $V \in \mathbf{V}_k(U)$ one associates a Radon measure $\|V\|$ on U, called the *weight* of V, and defined by projecting V onto the first factor in $\mathbf{G}_k(U)$, explicitly:

$$\|V\|(\phi) := \int_{\mathbf{G}_k(U)} \phi(x)\, dV(x, S) \qquad \text{for every } \phi \in C_c(U)\,.$$

A set $\Gamma \subset \mathbb{R}^{n+1}$ is *countably k-rectifiable* if it can be covered by countably many Lipschitz images of $\mathbb{R}^k$ into $\mathbb{R}^{n+1}$ up to an $\mathcal{H}^k$-negligible set. We say that Γ is (locally) *$\mathcal{H}^k$-rectifiable* if it is $\mathcal{H}^k$-measurable, countably k-rectifiable, and $\mathcal{H}^k(\Gamma)$ is (locally) finite. If $\Gamma \subset U$ is locally $\mathcal{H}^k$-rectifiable, and $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner_\Gamma)$ is a positive function on Γ, then there is a k-varifold canonically associated to the pair (Γ, θ), namely the varifold $\mathbf{var}(\Gamma, \theta)$ defined by

$$\mathbf{var}(\Gamma, \theta)(\varphi) := \int_\Gamma \varphi(x, T_x\Gamma)\, \theta(x)\, d\mathcal{H}^k(x) \qquad \text{for every } \varphi \in C_c(\mathbf{G}_k(U))\,, \tag{2.1}$$

where $T_x\Gamma$ denotes the approximate tangent plane to Γ at x, which exists $\mathcal{H}^k$-a.e. on Γ. Any varifold $V \in \mathbf{V}_k(U)$ admitting a representation as in (2.1) is said to be *rectifiable*, and the space of rectifiable k-varifolds in U is denoted by $\mathbf{RV}_k(U)$. If $V = \mathbf{var}(\Gamma, \theta)$ is rectifiable and $\theta(x)$ is an integer at $\mathcal{H}^k$-a.e. $x \in \Gamma$, then we say that V is an *integral* k-dimensional varifold in U: the corresponding space is denoted $\mathbf{IV}_k(U)$.

2.3. **First variation of a varifold.** If $V \in \mathbf{V}_k(U)$ and $f \colon U \to U'$ is C^1 and proper (that is, $f^{-1}(K)$ is compact for any compact subset $K \subset U'$), then we let $f_\sharp V \in \mathbf{V}_k(U')$ denote the push-forward of V through f. Recall that the weight of $f_\sharp V$ is given by

$$\|f_\sharp V\|(\phi) = \int_{\mathbf{G}_k(U)} \phi \circ f(x)\, |\wedge_k \nabla f(x) \circ S|\, dV(x, S) \qquad \text{for every } \phi \in C_c(U')\,, \tag{2.2}$$

where

$$|\wedge_k \nabla f(x) \circ S| := |\nabla f(x) \cdot v_1 \wedge \ldots \wedge \nabla f(x) \cdot v_k| \quad \text{for any orthonormal basis } \{v_1, \ldots, v_k\} \text{ of } S$$

is the Jacobian of f along $S \in \mathbf{G}(n+1, k)$. Given a varifold $V \in \mathbf{V}_k(U)$ and a vector field $g \in C^1_c(U; \mathbb{R}^{n+1})$, the *first variation* of V in the direction of g is the quantity

$$\delta V(g) := \left.\frac{d}{dt}\right|_{t=0} \|(\Phi_t)_\sharp V\|(\tilde{U})\,, \tag{2.3}$$

where $\Phi_t(\cdot) = \Phi(t, \cdot)$ is any one-parameter family of diffeomorphisms of U defined for sufficiently small $|t|$ such that $\Phi_0 = \mathrm{id}_U$ and $\partial_t \Phi(0, \cdot) = g(\cdot)$. The $\tilde U$ is chosen so that $\operatorname{clos} \tilde U \subset U$ is compact and $\operatorname{spt} g \subset \tilde U$, and the definition of (2.3) does not depend on the choice of $\tilde U$. It is well known that δV is a linear and continuous functional on $C^1_c(U; \mathbb{R}^{n+1})$, and in fact that

$$\delta V(g) = \int_{\mathbf{G}_k(U)} \nabla g(x) \cdot S \, dV(x, S) \qquad \text{for every } g \in C^1_c(U; \mathbb{R}^{n+1})\,, \tag{2.4}$$

where, after identifying $S \in \mathbf{G}(n+1, k)$ with the orthogonal projection operator $\mathbb{R}^{n+1} \to S$,

$$\nabla g \cdot S = \mathrm{trace}(\nabla g^t \circ S) = \sum_{i,j=1}^{n+1} S_{ij} \, \frac{\partial g_i}{\partial x_j} =: \mathrm{div}^S g$$

is the tangential divergence of g along S. If δV can be extended to a linear and continuous functional on $C_c(U; \mathbb{R}^{n+1})$, we say that V has *bounded first variation* in U. In this case, δV is naturally associated with a unique $\mathbb{R}^{n+1}$-valued measure on U by means of the Riesz representation theorem. If such a measure is absolutely continuous with respect to the weight $\|V\|$, then there exists a $\|V\|$-measurable and locally $\|V\|$-integrable vector field $h(\cdot, V)$ such that

$$\delta V(g) = -\int_U g(x) \cdot h(x, V) \, d\|V\|(x) \qquad \text{for every } g \in C_c(U; \mathbb{R}^{n+1}) \tag{2.5}$$

by the Lebesgue-Radon-Nikodým differentiation theorem. The vector field $h(\cdot, V)$ is called the *generalized mean curvature vector* of V. For any $V \in \mathbf{IV}_k(U)$ with generalized mean curvature $h(\cdot, V)$, *Brakke's perpendicularity theorem* [6, Chapter 5] says that

$$S^\perp(h(x, V)) = h(x, V) \qquad \text{for } V\text{-a.e. } (x, S) \in \mathbf{G}_k(U)\,. \tag{2.6}$$

This means that the generalized mean curvature vector is perpendicular to the approximate tangent plane almost everywhere. A special mention is due to integral varifolds V for which $h(\cdot, V) = 0$ $\|V\|$-almost everywhere: such a varifold will be called *stationary*. If V is stationary in U and $x \in \operatorname{spt}(\|V\|)$, then the function $r \in (0, \operatorname{dist}(x, \partial U)) \mapsto (\omega_k r^k)^{-1} \|V\|(U_r(x))$ is increasing, so that the density $\Theta_V(x) := \Theta^k(\|V\|, x)$ exists at every $x \in \operatorname{spt}(\|V\|)$. Furthermore, for every sequence $r_h \to 0^+$ there are a subsequence $r_{h'}$ and a stationary integral k-varifold $\mathbf{C}$ in $\mathbb{R}^{n+1}$ such that, setting $\eta_{x,r}(y) := r^{-1}\,(y - x)$, the varifolds $(\eta_{x, r_{h'}})_\sharp V$ converge to $\mathbf{C}$ in the sense of Radon measures on $\mathbf{G}_k(\mathbb{R}^{n+1})$ as $h' \to \infty$. The varifold $\mathbf{C}$ will be called a *tangent cone* to V at x, a terminology justified by the homogeneity property $(\eta_{0,\lambda})_\sharp \mathbf{C} = \mathbf{C}$ for all $\lambda > 0$.

Other than the first variation δV discussed above, we shall also use a *weighted first variation*, defined as follows. For given $\phi \in C^1_c(U; \mathbb{R}^+)$, $V \in \mathbf{V}_k(U)$, and $g \in C^1_c(U; \mathbb{R}^{n+1})$, we modify (2.3) to introduce the ϕ-weighted first variation of V in the direction of g, denoted $\delta(V, \phi)(g)$, by setting

$$\delta(V, \phi)(g) := \left.\frac{d}{dt}\right|_{t=0} \|(\Phi_t)_\sharp V\|(\phi)\,, \tag{2.7}$$

where Φ_t denotes the one-parameter family of diffeomorphisms of U induced by g as above. Proceeding as in the derivation of (2.4), one then obtains the expression

$$\delta(V, \phi)(g) = \int_{\mathbf{G}_k(U)} \phi(x) \, \nabla g(x) \cdot S \, dV(x, S) + \int_U g(x) \cdot \nabla \phi(x) \, d\|V\|(x)\,. \tag{2.8}$$

Using $\phi\nabla g = \nabla(\phi g) - g \otimes \nabla\phi$ in (2.8) and (2.4), we obtain

$$\begin{aligned}\delta(V,\phi)(g) &= \delta V(\phi g) + \int_{\mathbf{G}_k(U)} g(x)\cdot(\nabla\phi(x) - S(\nabla\phi(x)))\,dV(x,S) \\ &= \delta V(\phi g) + \int_{\mathbf{G}_k(U)} g(x)\cdot S^{\perp}(\nabla\phi(x))\,dV(x,S)\,.\end{aligned} \tag{2.9}$$

If δV has generalized mean curvature $h(\cdot,V)$, then we may use (2.5) in (2.9) to obtain

$$\delta(V,\phi)(g) = -\int_U \phi(x)g(x)\cdot h(x,V)\,d\|V\|(x) + \int_{\mathbf{G}_k(U)} g(x)\cdot S^{\perp}(\nabla\phi(x))\,dV(x,S). \tag{2.10}$$

The definition of Brakke flow requires considering weighted first variations in the direction of the mean curvature. Suppose $V \in \mathbf{IV}_k(U)$, δV is locally bounded and absolutely continuous with respect to $\|V\|$ and $h(\cdot,V)$ is locally square-integrable with respect to $\|V\|$. In this case, it is natural from the expression (2.10) to define for $\phi \in C^1_c(U;\mathbb{R}^+)$

$$\delta(V,\phi)(h(\cdot,V)) := \int_U \{-\phi(x)\,|h(x,V)|^2 + h(x,V)\cdot\nabla\phi(x)\}\,d\|V\|(x). \tag{2.11}$$

Observe that here we have used (2.6) in order to replace the term $h(x,V)\cdot S^{\perp}(\nabla\phi(x))$ with $h(x,V)\cdot\nabla\phi(x)$.

2.4. **Brakke flow.** In order to motivate the weak formulation of the MCF introduced by Brakke in [6], note that a smooth family of k-dimensional surfaces $\{\Gamma(t)\}_{t\geq 0}$ in U is a MCF if and only if the following inequality holds true for all $\phi = \phi(x,t) \in C^1_c(U\times[0,\infty);\mathbb{R}^+)$:

$$\frac{d}{dt}\int_{\Gamma(t)} \phi\,d\mathcal{H}^k \leq \int_{\Gamma(t)} \left\{-\phi\,|h(\cdot,\Gamma(t))|^2 + \nabla\phi\cdot h(\cdot,\Gamma(t)) + \frac{\partial\phi}{\partial t}\right\}\,d\mathcal{H}^k\,. \tag{2.12}$$

In fact, the "only if" part holds with equality in place of inequality. For a more comprehensive treatment of the Brakke flow, see [36, Chapter 2]. Formally, if $\partial\Gamma(t)\subset\partial U$ is fixed in time, with $\phi = 1$, we also obtain

$$\frac{d}{dt}\mathcal{H}^k(\Gamma(t)) \leq -\int_{\Gamma(t)} |h(x,\Gamma(t))|^2\,d\mathcal{H}^k(x)\,, \tag{2.13}$$

which states the well-known fact that the L^2-norm of the mean curvature represents the dissipation of area along the MCF. Motivated by (2.12) and (2.13), we have defined in [32] the following notion of Brakke flow with fixed boundary.

Definition 2.1. Let $U\subset\mathbb{R}^{n+1}$ be an open set. We say that a family of varifolds $\{V_t\}_{t\geq 0}$ in U is a *k-dimensional Brakke flow in U* if all of the following hold:

(a) For a.e. $t\geq 0$, $V_t\in\mathbf{IV}_k(U)$;
(b) For a.e. $t\geq 0$, δV_t is bounded and absolutely continuous with respect to $\|V_t\|$;
(c) The generalized mean curvature $h(x,V_t)$ (which exists for a.e. t by (b)) satisfies for all $s>0$

$$\|V_s\|(U) + \int_0^s dt\int_U |h(x,V_t)|^2\,d\|V_t\|(x) \leq \|V_0\|(U); \tag{2.14}$$

(d) For all $0\leq t_1<t_2<\infty$ and $\phi\in C^1_c(U\times\mathbb{R}^+;\mathbb{R}^+)$,

$$\|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \delta(V_t,\phi(\cdot,t))(h(\cdot,V_t)) + \|V_t\|(\frac{\partial\phi}{\partial t}(\cdot,t))\,dt\,, \tag{2.15}$$

having set $\|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} := \|V_{t_2}\|(\phi(\cdot,t_2)) - \|V_{t_1}\|(\phi(\cdot,t_1))$.

Furthermore, if ∂U is not empty and $\Sigma \subset \partial U$, we say that $\{V_t\}_{t\geq 0}$ has *fixed boundary* Σ if, together with conditions (a)-(d) above, it holds

(e) For all $t \geq 0$, $(\mathrm{clos}\,(\mathrm{spt}\,\|V_t\|)) \setminus U = \Sigma$.

Notice that, formally, we obtain the analogue of (2.14) by integrating (2.13) from 0 to s. By integrating (2.12) from t_1 to t_2, we also obtain the analogue of (2.15) via the expression (2.11). We recall that the closure is taken with respect to the topology of $\mathbb{R}^{n+1}$ while the support of $\|V_t\|$ is in U. Thus (e) geometrically means that "the boundary of V_t (or $\|V_t\|$) is Σ".

3. Main results

As anticipated in the introduction, as an *initial datum* we are going to consider a closed countably n-rectifiable set Γ_0 in $\mathbb{R}^{n+1}$. In order to guarantee the existence of a Brakke flow starting with Γ_0 we are going to require that Γ_0 satisfies the same set of assumptions under which the theory in [32] was developed. For the reader's convenience, we record those assumptions here.

Assumption 3.1. Let us fix integers $n \geq 1$ and $N \geq 2$. We consider U, Γ_0, and $\{E_{0,i}\}_{i=1}^N$ such that:

- (A1) $U \subset \mathbb{R}^{n+1}$ is a strictly convex bounded domain with boundary ∂U of class C^2;
- (A2) $\Gamma_0 \subset U$ is a relatively closed, countably n-rectifiable set with $\mathcal{H}^n(\Gamma_0) < \infty$;
- (A3) $E_{0,1}, \ldots, E_{0,N}$ are non-empty, open, and mutually disjoint subsets of U such that $U \setminus \Gamma_0 = \bigcup_{i=1}^N E_{0,i}$;
- (A4) $\partial\Gamma_0 := \mathrm{clos}(\Gamma_0) \setminus U$ is not empty, and for each $x \in \partial\Gamma_0$ there exist at least two indexes $i_1 \neq i_2$ in $\{1, \ldots, N\}$ such that $x \in \mathrm{clos}\Big(\mathrm{clos}(E_{0,i_j}) \setminus (U \cup \partial\Gamma_0)\Big)$ for $j = 1, 2$;
- (A5) $\mathcal{H}^n(\Gamma_0 \setminus \bigcup_{i=1}^N \partial^* E_{0,i}) = 0$.

Remark 3.2. Under the validity of assumptions (A1)-(A4), there exists a Brakke flow $\{V_t\}_{t\geq 0}$ with fixed boundary $\partial\Gamma_0$ and such that $\|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}$, and if also (A5) holds then the surface measures associated to such Brakke flow are continuous at $t = 0^+$, that is also $\lim_{t\to 0^+} \|V_t\| = \mathcal{H}^n \llcorner_{\Gamma_0}$; see [32, Theorem 2.2]. Moreover, $\{V_t\}_{t\geq 0}$ can be made canonical in the sense of [34]. As it will become apparent in the sequel, the construction of the present paper is purely local, and based at a fixed point $x_0 \in \Gamma_0$. Hence, the fact that the boundary $\partial\Gamma_0$ is kept fixed throughout the evolution is not important here, and we could potentially also work in the setting of [18], where U is replaced by the whole Euclidean space $\mathbb{R}^{n+1}$, the finiteness of the $\mathcal{H}^n$-measure of Γ_0 can be assumed to hold locally, and (A4) is dropped. Nonetheless, in order to fix the ideas we will always work in the "constrained" fixed boundary case, and leave to the reader the necessary modifications to treat the "unconstrained" case.

Next, we focus on the main assumption of this paper.

Assumption 3.3. Let U, Γ_0, and $\{E_{0,i}\}_{i=1}^N$ satisfy Assumption 3.1, and let $V_0 := \mathbf{var}(\Gamma_0, 1)$. We suppose that

(H0) V_0 is a stationary varifold, with $\mathrm{spt}(\|V_0\|) = \Gamma_0$,

and that there exists a point $x_0 \in \Gamma_0$, without loss of generality $x_0 = 0$, with the following properties:

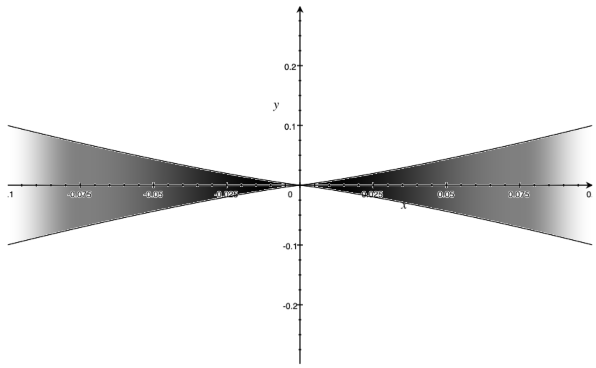

FIGURE 2. The growth condition required in (H2): the shaded area is the region $\{|x_{n+1}| \leq G(x')\}$ ($\alpha = 0.51$) in the cylinder $\mathrm{C}(T, 0, r_0)$ with $r_0 = 0.1$.

(H1) one of the tangent cones to V_0 at $x_0 = 0$ is of the form $\mathbf{var}(T, Q)$, for some n-dimensional plane $T \in \mathbf{G}(n+1, n)$ and an integer $Q \geq 2$;

(H2) there exists a radius $r_0 \in (0, 1)$ such that, writing $x = (x', x_{n+1}) \in \mathbb{R}^{n+1} = T \oplus T^\perp$ we have

$$\Gamma_0 \cap \mathrm{C}(T, 0, r_0) \cap \{|x_{n+1}| < r_0\} \subset \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1} : \; |x_{n+1}| \leq G(x')\}, \tag{3.1}$$

where G is the positive, radial function $G(x') = g(|x'|)$ defined by

$$g(s) = \frac{s}{\log^\alpha(1/s)} \qquad \text{for } s > 0, \text{ with } \alpha > \tfrac{1}{2}, \tag{3.2}$$

see Figure 2.

Some comments on the hypotheses (H0)-(H1)-(H2) are now in order.

Remark 3.4. A fundamental observation stemming directly from the definitions is that, in general, a stationary varifold may have multiple different tangent cones at a given point. In particular, (H0) and (H1) alone do not imply that $\mathbf{var}(T, Q)$ is the *only* tangent cone to V_0 at x_0 (nor that all tangent cones to V_0 at x_0 are actually flat). As a matter of fact, the only conclusions that one may draw from (H0) and (H1) are that the density $\Theta_{V_0}(x_0)$ is an integer $Q \geq 2$ and that if $\mathbf{C}$ is tangent to V_0 at x_0 and $\mathrm{spt}(\|\mathbf{C}\|)$ is contained in an n-plane T' then $\mathbf{C} = \mathbf{var}(T', Q)$ (as a consequence of the constancy lemma for stationary varifolds). The hypothesis (H2) resolves the ambiguity, so that in our setting $\mathbf{var}(T, Q)$ is the *unique* tangent cone to V_0 at x_0.

The following is the main result of this paper.

Theorem 3.5. *If Assumption 3.3 holds, then there exists a Brakke flow $\{V_t\}_{t \geq 0}$ with fixed boundary $\partial\Gamma_0$ such that:*

(i) $\lim_{t \to 0^+} \|V_t\| = \|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}$;

(ii) $\|V_t\|(U) < \|V_0\|(U)$ *for all* $t > 0$.

We observe explicitly that if V_0 is stationary then the constant flow $V_t = V_0$ for all $t \geq 0$ is an n-dimensional Brakke flow with fixed boundary $\partial\Gamma_0$: the only condition to verify is the validity of Brakke's inequality (2.15), which can be readily deduced by

$$\begin{aligned} \|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} &= \|V_0\|(\phi(\cdot,t_2)-\phi(\cdot,t_1)) \\ &= \|V_0\|\left(\int_{t_1}^{t_2}\frac{\partial\phi}{\partial t}(\cdot,t)\,dt\right) \\ &\le \int_{t_1}^{t_2}\|V_0\|(\frac{\partial\phi}{\partial t}(\cdot,t))\,dt\,. \end{aligned}$$

Hence, for an initial datum as in Assumption 3.3, Theorem 3.5 is a statement of non-uniqueness of Brakke flow.

Remark 3.6. In the light of the above discussion, one may consider the subset of $\mathbf{IV}_n(U)$ consisting of those stationary varifolds V_0 such that the assignment $V_t = V_0$ for all $t \geq 0$ defines the *only* Brakke flow starting with V_0. We will say that such a stationary varifold is *dynamically stable*. A natural question is whether dynamically stable stationary varifolds enjoy better regularity properties than what stationarity alone is able to guarantee. By Theorem 3.5, if $V_0 = \mathbf{var}(\Gamma_0, 1)$ with Γ_0 as in Assumption 3.1 and if V_0 is dynamically stable then the set of points satisfying (H1) *and* (H2) is empty. As anticipated in the introduction, it is an open question whether this implies that the set of points satisfying (H1) alone is empty, too (in other words, whether (H0) and (H1) imply (H2)), although that appears to be the case in all known examples. Notice that if V_0 is such that the set of points as in (H1) is empty then, as a consequence of the celebrated regularity theorem by Allard [2] and of the recent results by Naber and Valtorta [24], Γ_0 is an embedded real analytic n-dimensional minimal hypersurface in U outside of a countably $(n-1)$-rectifiable singular set $\mathcal{S}$ (hence, in particular, such that $\mathcal{H}^{n-1+\delta}(\mathcal{S}) = 0$ for every $\delta > 0$).

As noted in the Introduction, the following is an immediate corollary of Theorem 3.5 and the work [23] by Minter and Wickramasekera. We refer to the Introduction and to [23] for the definitions of stability of $\mathrm{Reg}(V_0)$ and of classical singularities of V_0.

Corollary 3.7. *Let U, Γ_0, and $\{E_{0,i}\}_{i=1}^N$ satisfy Assumption 3.1, and let $V_0 = \mathbf{var}(\Gamma_0, 1)$. Suppose that V_0 satisfies (H0) and (H1), that the regular part $\mathrm{Reg}(V_0)$ is stable, and that in a neighborhood of x_0 the set of classical singularities y of V_0 with $\Theta_{V_0}(y) < Q$ is empty. Then, there exists a non-trivial Brakke flow $\{V_t\}_{t\geq 0}$ starting with V_0, and satisfying the conclusions of Theorem 3.5. In particular, this applies if*

(a) *Γ_0 has the structure of rectifiable current, denoted $[\![\Gamma_0]\!]$, $x_0 \in U \setminus \mathrm{spt}\|\partial[\![\Gamma_0]\!]\|$, and $Q = 2$,*
(b) *or Γ_0 has the structure of rectifiable current, $[\![\Gamma_0]\!]$ is area minimizing $\mathrm{mod}(2Q)$, and $x_0 \in U \setminus \mathrm{spt}\|\partial^p[\![\Gamma_0]\!]\|$.*

The rest of the paper is devoted to the proof of Theorem 3.5. The proof is constructive, and it roughly proceeds as follows. First, we modify the set Γ_0 in a small ball of radius ε centered at $x_0 = 0$, so to have a quantifiable drop of its mass: we shall call this modification a "hole nucleation" in Γ_0. Then, we use our existence theorem from [32] to produce a Brakke flow (with fixed boundary $\partial\Gamma_0$) starting with this modified set Γ_0^ε. Using an iterative procedure which hinges upon Brakke's "expanding hole lemma" [6, Lemma 6.5] (of which we present a detailed proof for the reader's convenience), we show that this hole "expands" at future times: in particular, a hole of size $\sim \varepsilon$ at time $t = 0$ becomes *almost* a hole of size $\sim 2^j\,\varepsilon$ at time $t \sim 2^{2j}\,\varepsilon^2$. The "*almost*" above accounts for an error occurring at each iteration which needs to be estimated in order to make sure that the evolving varifolds never re-gain the initial mass drop. The growth assumption (H2) is crucial to perform this estimate. The final product of this iterative process is a Brakke flow (depending on the size ε of the initial hole nucleation) which, at a time $\bar t > 0$, has strictly less mass than Γ_0, with both $\bar t$ and the mass loss *independent* of ε: the Brakke flow in the statement is then obtained in the limit as $\varepsilon \to 0^+$.

4. Hole nucleation

In this section we show that, given U, Γ_0, and $\{E_{0,i}\}_{i=1}^N$ as in Assumption 3.3, there exist Brakke flows starting from a set Γ_0^ε obtained by modifying Γ_0 in a small ball $U_{2\varepsilon}$ around $x_0 = 0$ in a way to obtain a quantifiable drop of its mass, and we discuss the limits of such Brakke flows as $\varepsilon \to 0^+$. Informally, we may say that Γ_0^ε is obtained by "making a hole" of radius $\sim \varepsilon$ in Γ_0. The details of the construction of Γ_0^ε are contained in the following lemma.

Lemma 4.1. *Let U, Γ_0, $\{E_{0,i}\}_{i=1}^N$, and T be as in Assumption 3.3. There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, there exist a relatively closed and $\mathcal{H}^n$-rectifiable set $\Gamma_0^\varepsilon \subset U$, a family $\{E_{0,i}^\varepsilon\}_{i=1}^N$ of pairwise disjoint non-empty open subsets of U with finite perimeter such that:*

1. *$\Gamma_0^\varepsilon \setminus U_{2\varepsilon} = \Gamma_0 \setminus U_{2\varepsilon}$ and $E_{0,i}^\varepsilon \setminus U_{2\varepsilon} = E_{0,i} \setminus U_{2\varepsilon}$ for each $i = 1, \ldots, N$;*
2. *$\Gamma_0^\varepsilon = U \setminus \cup_{i=1}^N E_{0,i}^\varepsilon$;*
3. *$\Gamma_0^\varepsilon \cap U_{2\varepsilon} \subset \{(x', x_{n+1}) : |x_{n+1}| \le G(x')\}$;*
4. *$\mathcal{H}^n(\Gamma_0^\varepsilon \cap U_{2\varepsilon}) \le (4\varepsilon)^n \omega_n (Q+1)$;*
5. *$\mathcal{H}^n(\mathrm{C}(T, 0, \varepsilon) \cap U_{2\varepsilon} \cap \Gamma_0^\varepsilon) \le \omega_n \varepsilon^n$.*

Proof. As usual, we assume without loss of generality that $T = \mathbb{R}^n \times \{0\}$, and thus we write $x = (x', x_{n+1}) \in \mathbb{R}^{n+1} = T \oplus T^\perp$. Furthermore, we let $V_0 = \mathbf{var}(\Gamma_0, 1)$, so that, setting $\eta_r(x) = x/r$, we have $\lim_{r\to 0^+} (\eta_r)_\sharp V_0 = \mathbf{var}(T, Q)$ as varifolds. Define $\Gamma^\varepsilon := \eta_\varepsilon(\Gamma_0)$, $V^\varepsilon := (\eta_\varepsilon)_\sharp V_0$ and $E_i^\varepsilon := \eta_\varepsilon(E_{0,i})$ for each $i = 1, \ldots, N$ for simplicity. By (H2), there exists a sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have

$$U_2 \cap \Gamma^\varepsilon \subset \{|x_{n+1}| \le 1/20\}. \tag{4.1}$$

We may additionally assume that $E_i^\varepsilon \setminus U_2 \neq \emptyset$ for all i. In the following, let $\delta = 1/5$ and define the following function $\mathrm{g}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as in [19, Proof of Lemma 4.7]. For $|x'| \le 1$, we set

$$\mathrm{g}(x', x_{n+1}) = \begin{cases} (x', x_{n+1}) & \text{if } |x_{n+1}| \ge \delta\,, \\ (x', 0) & \text{if } |x_{n+1}| \le \frac{\delta}{2}\,, \\ (x', 2\,x_{n+1} - \delta) & \text{if } \frac{\delta}{2} \le x_{n+1} \le \delta\,, \\ (x', 2\,x_{n+1} + \delta) & \text{if } -\delta \le x_{n+1} \le -\frac{\delta}{2}\,, \end{cases} \tag{4.2}$$

whereas in the region $1 \le |x'| \le 1 + \delta$ we set

$$\mathrm{g}(x', x_{n+1}) = \begin{cases} (x', x_{n+1}) & \text{if } |x_{n+1}| \ge \delta \text{ or } |x_{n+1}| \le |x'| - 1\,, \\ (x', |x'| - 1) & \text{if } |x'| - 1 \le x_{n+1} \le \frac{|x'|-1}{2} + \frac{\delta}{2}\,, \\ (x', 2\,x_{n+1} - \delta) & \text{if } \frac{|x'|-1}{2} + \frac{\delta}{2} \le x_{n+1} \le \delta\,, \\ (x', 1 - |x'|) & \text{if } \frac{1-|x'|}{2} - \frac{\delta}{2} \le x_{n+1} \le 1 - |x'|\,, \\ (x', 2\,x_{n+1} + \delta) & \text{if } -\delta \le x_{n+1} \le \frac{1-|x'|}{2} - \frac{\delta}{2}\,. \end{cases} \tag{4.3}$$

Finally, we set

$$\mathrm{g}(x', x_{n+1}) = (x', x_{n+1}) \qquad \text{if } |x'| > 1 + \delta\,; \tag{4.4}$$

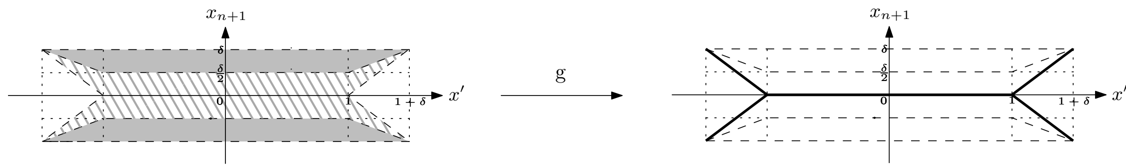

FIGURE 3. The map g.

see Figure 3. One may check that g is a Lipschitz map with $\mathrm{Lip}(\mathrm{g}) \le 2$. Next, define

$$\tilde{E}_i^\varepsilon := \mathrm{int}(\mathrm{g}(E_i^\varepsilon)) \quad \text{and} \quad E_{0,i}^\varepsilon := \eta_{1/\varepsilon}(\tilde{E}_i^\varepsilon) \tag{4.5}$$

for each $i = 1, \ldots, N$, as well as

$$\tilde{\Gamma}^\varepsilon := \eta_\varepsilon(U) \setminus \cup_{i=1}^N \tilde{E}_i^\varepsilon \quad \text{and} \quad \Gamma_0^\varepsilon := \eta_{1/\varepsilon}(\tilde{\Gamma}^\varepsilon)\,. \tag{4.6}$$

Since g is a retraction map, one can check that $\tilde{E}_1^\varepsilon, \ldots, \tilde{E}_N^\varepsilon$ are mutually disjoint open sets, and so are $E_{0,1}^\varepsilon, \ldots, E_{0,N}^\varepsilon$. It follows from the definition of g that $E_i^\varepsilon \setminus U_2 = \tilde{E}_i^\varepsilon \setminus U_2$ for $i = 1, \ldots, N$. Thus (1) is satisfied for Γ_0^ε and $E_{0,i}^\varepsilon$, and (2) holds by construction. We next check

$$\tilde{\Gamma}^\varepsilon \subset \mathrm{g}(\Gamma^\varepsilon). \tag{4.7}$$

We only need to prove the inclusion on the set on which the map g is not one-to-one, namely on $\{(x', 1-|x'|), (x', |x'|-1) : 1 \le |x'| < 1+\delta\} \cup (T \cap U_1)$. Note that any point x of this set has the property that $\mathrm{g}^{-1}(x)$ is a closed line segment, say I, perpendicular to T. If $x \notin \mathrm{g}(\Gamma^\varepsilon)$, that is, $I \cap \Gamma^\varepsilon = I \cap \cup_{i=1}^N \partial E_i^\varepsilon = \emptyset$, then there must exist some E_i^ε such that $I \subset E_i^\varepsilon$. Then one can see that x is an interior point of $\mathrm{g}(E_i^\varepsilon)$, so that $x \in \tilde{E}_i^\varepsilon$ and not in $\tilde{\Gamma}^\varepsilon$. This proves (4.7). The inclusion (4.7) moreover proves that $\tilde{\Gamma}^\varepsilon$ is countably n-rectifiable. From the definition of g, we have $|T^\perp(\mathrm{g}(x))| \le |T^\perp(x)|$, and (3) follows from this fact. Since $\mathrm{Lip}(\mathrm{g}) \le 2$ and $\mathcal{H}^n(U_2 \cap \Gamma^\varepsilon) \le 2^n \omega_n (Q+1)$ for all small ε, the area formula guarantees that $\mathcal{H}^n(U_2 \cap \tilde{\Gamma}^\varepsilon) \le 4^n \omega_n (Q+1)$. This gives (4) after change of variables. In particular, $\tilde{\Gamma}^\varepsilon$ has no interior points and finite $\mathcal{H}^n$ measure, it holds $\tilde{\Gamma}^\varepsilon = \cup_{i=1}^N \partial \tilde{E}_i^\varepsilon$, and the sets $E_{0,1}^\varepsilon, \ldots, E_{0,N}^\varepsilon$ have bounded perimeter. Finally, from the definition of g, we have (writing $\mathrm{C}(r)$ for $\mathrm{C}(T, 0, r)$)

$$\begin{aligned} &\mathrm{g}(\mathrm{C}(1) \cap \{\delta/2 \le x_{n+1} \le \delta\}) = \mathrm{C}(1) \cap \{0 \le x_{n+1} \le \delta\}, \\ &\mathrm{g}(\mathrm{C}(1) \cap \{-\delta \le x_{n+1} \le -\delta/2\}) = \mathrm{C}(1) \cap \{-\delta \le x_{n+1} \le 0\}\,. \end{aligned} \tag{4.8}$$

Since $\cup_{i=1}^N \partial E_i^\varepsilon = \Gamma^\varepsilon$ in U_2 and $E_1^\varepsilon, \ldots, E_N^\varepsilon$ are mutually disjoint, (4.1) shows that there exist $i_1, i_2 \in \{1, \ldots, N\}$ such that

$$(U_2 \cap \{x_{n+1} \ge \delta/2\}) \subset E_{i_1}^\varepsilon \quad \text{and} \quad (U_2 \cap \{x_{n+1} \le -\delta/2\}) \subset E_{i_2}^\varepsilon. \tag{4.9}$$

We claim that

$$\mathrm{C}(1) \cap U_2 \cap \tilde{\Gamma}^\varepsilon \subset T \cap B_1. \tag{4.10}$$

Note that (4.8) and (4.9) imply that $\mathrm{C}(1) \cap \{0 < x_{n+1} \le \delta\} \subset \tilde{E}_{i_1}^\varepsilon$ and $\mathrm{C}(1) \cap \{-\delta \le x_{n+1} < 0\} \subset \tilde{E}_{i_2}^\varepsilon$. Thus, we have $\tilde{\Gamma}^\varepsilon \cap \mathrm{C}(1) \cap \{-\delta \le x_{n+1} \le \delta\} \subset T \cap B_1$. Since $U_2 \cap \Gamma^\varepsilon \cap \{|x_{n+1}| \ge \delta\} = \emptyset$, this proves (4.10), and consequently we have (5). $\square$

By [32, Theorems 2.2 & 2.3] we have then immediately the following existence of Brakke flows starting with Γ_0^ε.

Proposition 4.2. *With Γ_0^ε and $\{E_{0,i}^\varepsilon\}_{i=1}^N$ given in Lemma 4.1, there exists a Brakke flow $\{V_t^\varepsilon\}_{t\geq 0}$ with fixed boundary $\partial\Gamma_0$ and $\|V_0^\varepsilon\| = \mathcal{H}^n \llcorner_{\Gamma_0^\varepsilon}$. For each $i = 1, \ldots, N$, there exists a one-parameter family $\{E_i^\varepsilon(t)\}_{t\geq 0}$ of open sets $E_i^\varepsilon(t) \subset U$ with the properties described in* [32, Theorem 2.3].

Proof. To apply [32, Theorem 2.2 & 2.3] we only need to check that (A1)-(A4) in Assumption 3.1 are satisfied with U, Γ_0^ε, and $\{E_{0,i}^\varepsilon\}_{i=1}^N$ for each $\varepsilon \in (0, \varepsilon_0]$. These follow from Lemma 4.1 (1),(2). ☐

Proposition 4.3. *For any sequence $\{\varepsilon_j\}_{j=1}^\infty \subset (0, \varepsilon_0]$ converging to 0, there exist a subsequence (denoted by the same index) and a Brakke flow $\{V_t\}_{t\geq 0}$ with fixed boundary $\partial\Gamma_0$ such that $\lim_{j\to\infty} \|V_t^{\varepsilon_j}\| = \|V_t\|$ in U for each $t \geq 0$ and $\lim_{t\to 0+} \|V_t\| = \|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}$.*

Proof. Since $\|V_t^\varepsilon\|(U) \leq \|V_0^\varepsilon\|(U) = \mathcal{H}^n(\Gamma_0^\varepsilon) = \mathcal{H}^n(\Gamma_0) + o(1)$ as $\varepsilon \to 0^+$ for all $t > 0$, we have a uniform mass bound for the family. Such a family of Brakke flows is known to be compact (see [16] and [36, Section 3.2]), thus there exists a subsequence (denoted by the same index) and a limit Brakke flow $\{V_t\}_{t\geq 0}$ such that $\lim_{j\to\infty} \|V_t^{\varepsilon_j}\| = \|V_t\|$ as Radon measures on U for all $t \geq 0$. By Lemma 4.1(4), we also have $\|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}$. For each $B_r(x) \times [t_1, t_2] \subset\subset U \times (0, \infty)$ and $i \in \{1, \ldots, N\}$, the argument for the proof of [18, Theorem 3.5(6)] (which is equally valid for the fixed boundary case away from ∂U) shows that $\mathcal{L}^{n+1}(E_i^\varepsilon(t) \cap B_r(x))$ is $\frac{1}{2}$-Hölder continuous as a function of $t \in [t_1, t_2]$, with uniformly bounded Hölder norm independently of ε. Also, if $0 \notin B_{r+2\varepsilon}(x)$, the same proof shows that $\mathcal{L}^{n+1}(B_r(x) \cap E_i^\varepsilon(t))$ is continuous at $t = 0$ with uniform modulus of continuity with respect to ε. Since $\|\partial^* E_i^\varepsilon(t)\| \leq \|V_t^\varepsilon\|$ for all $t \geq 0$ (see [32, Theorem 2.3(8)]), and since the latter is uniformly bounded, by a suitable diagonal argument and the uniform continuity in t, one can prove that there exists a further subsequence (denoted by the same index) and a family of sets of finite perimeter $\{\tilde{E}_i(t)\}_{t\geq 0}$ in U for $i = 1, \ldots, N$ such that $\lim_{j\to\infty} \mathcal{L}^{n+1}(E_i^{\varepsilon_j}(t) \triangle \tilde{E}_i(t)) = 0$ for all $t \geq 0$ and $\mathcal{L}^{n+1}(B_r(x) \cap \tilde{E}_i(t))$ is $C^{1/2}((0, \infty)) \cap C([0, \infty))$ as a function of t for any $B_r(x) \subset\subset U$. For each $t \geq 0$, $\{\tilde{E}_i(t)\}_{i=1}^N$ satisfies $\mathcal{L}^{n+1}(\tilde{E}_i(t) \cap \tilde{E}_{i'}(t)) = 0$ for $i \neq i'$ and $\mathcal{L}^{n+1}(U \setminus \cup_{i=1}^N \tilde{E}_i(t)) = 0$. By Lemma 4.1(1), we also have $\tilde{E}_i(0) = E_{0,i}$ for each $i = 1, \ldots, N$.

Define the space-time Radon measure $d\mu := d\|V_t\|dt$ on $U \times (0, \infty)$ and $(\operatorname{spt}\mu)_t := \{x \in U : (x, t) \in \operatorname{spt}\mu\}$ for each $t > 0$. By the fact that $\{V_t\}_{t\geq 0}$ is a Brakke flow, we have for all $t > 0$

$$\operatorname{spt}\|V_t\| \subset (\operatorname{spt}\mu)_t \text{ and } \mathcal{H}^n((\operatorname{spt}\mu)_t \cap \tilde{U}) < \infty \text{ for all } \tilde{U} \subset\subset U \tag{4.11}$$

by [18, Lemma 10.1] and [18, Corollary 10.8], respectively. Since

$$\|\partial^* \tilde{E}_i(t)\| \leq \liminf_{j\to\infty} \|\partial^* E_i^{\varepsilon_j}(t)\| \leq \lim_{j\to\infty} \|V_t^{\varepsilon_j}\| = \|V_t\| \tag{4.12}$$

for all $t \geq 0$, (4.11) shows that $\operatorname{spt}\|\partial^* \tilde{E}_i(t)\| \subset (\operatorname{spt}\mu)_t$ for all $t > 0$. In particular, on each connected component of $U \setminus (\operatorname{spt}\mu)_t$, $\chi_{\tilde{E}_i(t)}$ is constant. Then, it follows using the continuity property of $\mathcal{L}^{n+1}(B_r(x) \cap \tilde{E}_i(t))$ that the open set $(U \times (0, \infty)) \setminus \operatorname{spt}\mu$ can be decomposed into mutually disjoint open sets $E_1, \ldots, E_N$ such that $\cup_{i=1}^N E_i = (U \times (0, \infty)) \setminus \operatorname{spt}\mu$ and such that $\mathcal{L}^{n+1}(\tilde{E}_i(t) \triangle \{x \in U : (x, t) \in E_i\}) = 0$ for all $t > 0$. We may redefine $E_i(t) = \{x \in U : (x, t) \in E_i\}$, which is open and $\mathcal{L}^{n+1}(\tilde{E}_i(t) \triangle E_i(t)) = 0$. By definition, we have $U \setminus \cup_{i=1}^N E_i(t) = (\operatorname{spt}\mu)_t$ and (4.11) shows that $U \setminus \cup_{i=1}^N E_i(t)$ has no interior points, so that we have $\cup_{i=1}^N \partial E_i(t) = (\operatorname{spt}\mu)_t$ for all $t > 0$. The continuity at $t = 0$ shows that $\lim_{t\to 0+} \mathcal{L}^{n+1}(E_i(t) \triangle E_{0,i}) = 0$ for $i = 1, \ldots, N$.

By [32, Theorem 2.3(5)(11)], for each j and all $t \geq 0$, $\mathrm{spt}\,\|V_t^{\varepsilon_j}\| \subset \mathrm{conv}(\Gamma_0^{\varepsilon_j} \cup \partial\Gamma_0)$. Since the difference of $\Gamma_0^{\varepsilon_j}$ and Γ_0 lies within $U_{2\varepsilon_j}$, we may conclude that $\mathrm{spt}\|V_t\| \subset \mathrm{conv}(\Gamma_0 \cup \partial\Gamma_0)$ for all $t \geq 0$, and one may deduce that $(\mathrm{spt}\,\mu)_t \subset \mathrm{conv}(\Gamma_0 \cup \partial\Gamma_0)$ for $t > 0$. We also can see from the last claim that

$$E_i(t) \setminus \mathrm{conv}(\Gamma_0 \cup \partial\Gamma_0) = E_{0,i} \setminus \mathrm{conv}(\Gamma_0 \cup \partial\Gamma_0) \tag{4.13}$$

for all $t \geq 0$ and $i = 1, \ldots, N$.

Next, we prove that V_t has a fixed boundary $\partial\Gamma_0$, i.e., $(\mathrm{clos}\,(\mathrm{spt}\|V_t\|)) \setminus U = \partial\Gamma_0$. The inclusion $\subset$ follows from $\mathrm{spt}\|V_t\| \subset \mathrm{conv}(\Gamma_0 \cup \partial\Gamma_0)$, the definition of $\partial\Gamma_0$ and the strict convexity of U. For the converse inclusion, assume that we have $x \in \partial\Gamma_0$ and there exists $r > 0$ such that $\mathrm{spt}\,\|V_t\| \cap U_r(x) = \emptyset$. Then we have $\|V_t\|(U_r(x)) = 0$. But then (4.12) shows $\|\partial^* E_i(t)\|(U_r(x) \cap U) = 0$ for $i = 1, \ldots, N$. On the other hand, by (4.13) and (A4) of Assumption 3.1, we must have some $i_1 \neq i_2$ such that $U_r(x) \cap E_{i_k}(t) \neq \emptyset$ for $k = 1, 2$. These are not compatible. Thus we have $(\mathrm{clos}\,(\mathrm{spt}\|V_t\|)) \setminus U = \partial\Gamma_0$.

Finally, $\lim_{t\to 0+} \|V_t\| = \mathcal{H}^n \llcorner_{\Gamma_0}$ follows from the argument in [32, Proposition 6.10] under the condition (A5) of Assumption 3.1 that $\mathcal{H}^n(\Gamma_0 \setminus \cup_{i=1}^N \partial^* E_{0,i}) = 0$. Indeed, for any $\phi \in C_c(U; \mathbb{R}^+)$,

$$\limsup_{t\to 0+} \|V_t\|(\phi) \leq \|V_0\|(\phi) = \mathcal{H}^n \llcorner_{\Gamma_0} (\phi)$$

follows from the property of Brakke flow, and

$$\begin{aligned} 2\mathcal{H}^n \llcorner_{(\cup_{i=1}^N \partial^* E_{0,i})} (\phi) &= \sum_{i=1}^N \|\nabla \chi_{E_{0,i}}\|(\phi) \leq \sum_{i=1}^N \liminf_{t\to 0+} \|\nabla \chi_{E_i(t)}\|(\phi) \\ &\leq \liminf_{t\to 0+} \sum_{i=1}^N \|\nabla \chi_{E_i(t)}\|(\phi) \leq 2 \liminf_{t\to 0+} \|V_t\|(\phi). \end{aligned} \tag{4.14}$$

These show that $\lim_{t\to 0+} \|V_t\|(\phi) = \mathcal{H}^n \llcorner_{\Gamma_0} (\phi)$ if $\mathcal{H}^n(\Gamma_0 \setminus \cup_{i=1}^N \partial^* E_{0,i}) = 0$. □

5. Brakke's expanding holes lemma

In this section, we discuss Brakke's expanding holes lemma [6, Lemma 6.5], which is a key tool towards the proof of Theorem 3.5. Given its importance in the following arguments, and for the reader's convenience, we provide a detailed proof. The lemma is valid for Brakke flow of any codimension, and we will state it and prove it in such generality. Hence, in this section k will be a fixed integer in $\{1, \ldots, n\}$, and T will be a plane in $\mathbf{G}(n+1, k)$. Before stating the lemma, we will need some preliminary notation.

Definition 5.1. Let $\chi \colon \mathbb{R}^k \to \mathbb{R}^+$ be a smooth cut-off function $0 \leq \chi \leq 1$, such that:

(a) $\chi(x)$ is a decreasing function of the radial variable $r = |x|$;
(b) $\mathrm{spt}(\chi) \subset U_1^k(0)$;
(c) $\chi(x) = 1$ if $0 \leq |x| < 1 - \zeta$ for a small positive number ζ.

We will denote

$$\rho := \sup_{x \in \mathbb{R}^k} \left\{ |\nabla \chi(x)| + 2\|D^2 \chi(x)\| \right\}, \tag{5.1}$$

and we shall often use the fact that $|\nabla \chi(x)|^2 / \chi(x) \leq 2 \sup \|D^2 \chi\| \leq \rho$.

For a radius $R > 0$, we set

$$\chi_{T,R}(x) := \chi(T(x)/R). \tag{5.2}$$

Since the plane T will always be kept fixed, we will drop the subscript $_T$ in (5.2) and denote the cylindrical cut-off at scale R simply by χ_R. Along the same lines, we also recall that $\mathrm{C}(R)$ denotes the infinite cylinder orthogonal to T centered at the origin and with radius R.

Let us also collect the following well-known facts concerning the orthogonal projection operators onto planes in $\mathbf{G}(n+1,k)$. The reader can consult [17, Lemma 11.1] for their proofs.

Lemma 5.2. *For $S, T \in \mathbf{G}(n+1,k)$ and $v \in \mathbb{R}^{n+1}$, the following holds.*

$$\mathrm{I} \cdot T = k\,, \quad T^t = T\,, \quad T \circ T = T\,, \quad T \circ T^\perp = 0\,. \tag{5.3}$$

$$0 \le k - S\cdot T = S^\perp \cdot T \le k\,\|S-T\|^2\,. \tag{5.4}$$

$$0 \le \|S-T\|^2 \le (S-T)\cdot(S-T) = 2\,T^\perp \cdot S\,. \tag{5.5}$$

$$|T(S^\perp(v))| \le \|T-S\|\,|v|\,. \tag{5.6}$$

$$|T(S^\perp(T(v)))| \le \|T-S\|^2\,|v|\,. \tag{5.7}$$

Finally, the following lemma estimates the tilt of tangent planes to an integral varifold with respect to a reference plane T in terms of the L^2-excess of V with respect to T and the L^2 norm of the generalized mean curvature vector. The proof can be found in [17, Lemma 11.2].

Lemma 5.3. *Let $U \subset \mathbb{R}^{n+1}$ be open, suppose that $V \in \mathbf{IV}_k(U)$ admits generalized mean curvature vector $h(\cdot, V)$, and let $T \in \mathbf{G}(n+1,k)$ and $\phi \in C^1_c(U;\mathbb{R}^+)$. Then*

$$\begin{aligned}\int_{\mathbf{G}_k(U)} &\|S-T\|^2\,\phi^2(x)\,dV(x,S) \\ &\le 4\left(\int_U |h(x,V)|^2\,\phi^2(x)\,d\|V\|(x)\right)^{\frac12}\left(\int_U |T^\perp(x)|^2\,\phi^2(x)\,d\|V\|(x)\right)^{\frac12} \\ &\quad + 16\int_U |T^\perp(x)|^2\,|\nabla\phi(x)|^2\,d\|V\|(x)\,.\end{aligned} \tag{5.8}$$

Remark 5.4. In the following we will need to apply a slightly modified version of the estimate (5.8). Precisely, if $\phi \in C^2_c(U;\mathbb{R}^+)$ is such that $\{\phi = 0\} \subset \{\nabla\phi = 0\}$ then

$$\begin{aligned}\int_{\mathbf{G}_k(U)} &\|S-T\|^2\,|\nabla\phi(x)|^2\,dV(x,S) \\ &\le 4\left(\int_U |h(x,V)|^2\,\phi^2(x)\,d\|V\|(x)\right)^{\frac12}\left(\int_U |T^\perp(x)|^2\,|\nabla\phi(x)|^4\,\phi^{-2}(x)\,d\|V\|(x)\right)^{\frac12} \\ &\quad + 16\int_U |T^\perp(x)|^2\,|\nabla|\nabla\phi(x)||^2\,d\|V\|(x)\,.\end{aligned} \tag{5.9}$$

The proof can be obtained by repeating *verbatim* the proof of [17, Lemma 11.2] with ϕ replaced by $|\nabla\phi|$ until the last inequality of formula (11.9): there, first multiply and divide by ϕ (on the set where $\nabla\phi \neq 0$) and then apply the Cauchy-Schwarz inequality to deduce (5.9).

Lemma 5.5 (Brakke's expanding holes lemma)**.** *Let $T \in \mathbf{G}(n+1,k)$, $0 \le t_1 < t_2 < \infty$, $0 < R_1 < R_2$, $0 < \hat R_1 < \hat R_2$ and set*

$$\sigma := \frac{R_2^2 - R_1^2}{t_2 - t_1}\,, \qquad R(t)^2 := R_1^2 + \sigma\,(t - t_1)\,, \qquad \phi_t(x) := \chi_{R(t)}(x) = \chi(T(x)/R(t))\,.$$

Let $\{V_t\}_{t\in[t_1,t_2]}$ be a k-dimensional Brakke flow in $C(R_2) \cap \{|T^\perp(x)| < \hat R_2\}$ such that

$$\mathrm{spt}\,\|V_t\| \cap \{\hat R_1 < |T^\perp(x)| < \hat R_2\} = \emptyset \qquad \textit{for all } t \in [t_1,t_2]. \tag{5.10}$$

For a.e. $t \in [t_1, t_2]$, define the functions $\alpha(t)$ and $\mu(t)$ by

$$\mu(t)^2 := \int_{\mathrm{C}(R(t))} |T^\perp(x)|^2 \, d\|V_t\|(x) \, , \tag{5.11}$$

$$\alpha(t)^2 := \int |h(x, V_t)|^2 \, \phi_t^2(x) \, d\|V_t\|(x) \, . \tag{5.12}$$

Then, we have for a.e. $t \in [t_1, t_2]$

$$\delta(V_t, \phi_t^2)(h(\cdot, V_t)) < -\frac{\alpha(t)^2}{2} + 320 \, \rho^2 R(t)^{-4} \mu(t)^2 \, . \tag{5.13}$$

Furthermore, there is $M = M(k, \sigma, \rho) < \infty$ such that if $\mu \in [0, \infty)$ satisfies

$$\mu(t)^2 \leq \mu^2 \, R(t)^{k+2} \qquad \textit{for a.e. } t \in [t_1, t_2] \, , \tag{5.14}$$

then

$$R_2^{-k} \|V_{t_2}\|(\phi_{t_2}^2) \leq R_1^{-k} \|V_{t_1}\|(\phi_{t_1}^2) + M \mu^2 \log(R_2/R_1) \, . \tag{5.15}$$

Remark 5.6. The main conclusion of the lemma, equation (5.15), establishes an upper bound on the gain of mass density ratio for Brakke flow at times t_1, t_2 in enlarging cylinders $\mathrm{C}(R_1) \subset \mathrm{C}(R_2)$. The difference in mass density ratios is bounded above by the supremum, in the interval $[t_1, t_2]$, of the (scale invariant) L^2-excess of V_t with respect to the plane T, namely the function $R(t)^{-(k+2)} \mu(t)^2$.

Proof. In the following, we use test functions ϕ_t^2 in (2.15), which do not have compact support in $\mathrm{C}(R_2)$. On the other hand, due to (5.10), we may multiply ϕ_t^2 by a suitable cut-off function which is identically equal to 1 on $\{|T^\perp(x)| \leq \hat{R}_1\}$ and which vanishes on $\{|T^\perp(x)| \geq \hat{R}_2\}$ and so that the resulting functions belong to $C_c^\infty(\mathrm{C}(R_2))$. The computation is not affected at all so that it is understood in the following that we implicitly modify ϕ_t^2 as such without changing the notation. First, we show the validity of the dissipation inequality (5.13). Applying (2.11) with $\phi = \phi_t^2$, using that

$$\nabla[\phi_t^2] = 2\phi_t \nabla\phi_t \, , \qquad \nabla\phi_t(x) = R(t)^{-1} T[\nabla\chi(T(x)/R(t))] \in T \, , \tag{5.16}$$

and exploiting Brakke's perpendicularity Theorem 2.6 we calculate

$$\begin{aligned} \delta(V_t, \phi_t^2)(h(\cdot, V_t)) &= -\int |h(x, V_t)|^2 \, \phi_t^2(x) \, d\|V_t\|(x) + 2 \int h(x, V_t) \cdot S^\perp(\nabla\phi_t(x)) \, \phi_t(x) \, dV_t(x, S) \\ &\leq -\alpha(t)^2 + 2 \int |h(x, V_t)| \, \|S - T\| \, |\nabla\phi_t(x)| \, \phi_t(x) \, dV_t(x, S) \\ &\leq -\frac{3}{4}\alpha(t)^2 + 4 \int \|S - T\|^2 \, |\nabla\phi_t(x)|^2 \, dV_t(x, S) \, . \end{aligned} \tag{5.17}$$

In order to estimate the second term, we apply (5.9) to deduce

$$
\begin{aligned}
&\int \|S-T\|^2\,|\nabla\phi_t(x)|^2\,dV_t(x,S)\\
&\le 16\int |T^\perp(x)|^2\,|\nabla|\nabla\phi_t(x)||^2\,d\|V_t\|(x)\\
&\qquad + 4\left\{\left(\int |h(x,V_t)|^2\,\phi_t^2(x)\,d\|V_t\|(x)\right)\left(\int |T^\perp(x)|^2\,|\nabla\phi_t(x)|^4\,\phi_t^{-2}(x)\right)\right\}^{1/2} \qquad (5.18)\\
&\le 16\rho^2 R(t)^{-4}\mu(t)^2 + 4\rho R(t)^{-2}\alpha(t)\mu(t)\\
&\le \frac{\alpha(t)^2}{16} + 80\rho^2 R(t)^{-4}\mu(t)^2\,.
\end{aligned}
$$

Equations (5.17) and (5.18) together prove (5.13).

In order to prove (5.15), we observe that, heuristically,

$$\partial_t\left[R(t)^{-k}\,\|V_t\|(\phi_t^2)\right] = -kR'(t)R(t)^{-k-1}\,\|V_t\|(\phi_t^2) + R(t)^{-k}\partial_t\left[\|V_t\|(\phi_t^2)\right]\,,$$

which can be expressed rigorously in terms of the inequality

$$R(t)^{-k}\|V_t\|(\phi_t^2)\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} R(t)^{-k}D\left[\|V_t\|(\phi_t^2)\right] - kR'(t)R(t)^{-k-1}\,\|V_t\|(\phi_t^2)\,dt\,, \qquad (5.19)$$

where $D\left[\|V_t\|(\phi_t^2)\right]$ is the distributional derivative of $t\in[t_2,t_2]\mapsto\|V_t\|(\phi_t^2)$. On the other hand, since $\{V_t\}_{t\ge0}$ is a Brakke flow, Brakke's inequality (2.15) implies that, in the sense of distributions,

$$D\left[\|V_t\|(\phi_t^2)\right] \le \delta(V_t,\phi_t^2)(h(\cdot,V_t)) + \|V_t\|(\partial_t[\phi_t^2])\,. \qquad (5.20)$$

Since (5.13) controls the first addendum, we only have to estimate the second one. We first compute explicitly the time derivative $\partial_t[\phi_t^2]$, namely

$$
\begin{aligned}
\partial_t[\phi_t^2] &= 2\,\phi_t\,\partial_t\phi_t = -2\,\phi_t\,R(t)^{-2}R'(t)\,T(x)\cdot\nabla\chi(T(x)/R(t)) =\\
&= -2\,\phi_t\,R(t)^{-2}R'(t)\,x\cdot T\left[\nabla\chi(T(x)/R(t))\right]\\
&= -2\,\phi_t\,R(t)^{-1}R'(t)\,x\cdot\nabla\phi_t(x)\,,
\end{aligned}
$$

where we have used that T is an orthogonal projection operator and the expression for $\nabla\phi_t$ in (5.16). Hence, we have

$$\|V_t\|(\partial_t[\phi_t^2]) = R(t)^{-1}R'(t)\underbrace{\int\{-2\,\phi_t(x)\,x\cdot\nabla\phi_t(x)\}\,d\|V_t\|(x)}_{A}\,. \qquad (5.21)$$

Now, observe that, by the definition of first variation and generalized mean curvature,

$$
\begin{aligned}
-\int \phi_t^2(x)\,T(x)\cdot h(x,V_t)\,d\|V_t\|(x) &= \delta V_t(\phi_t^2\,T(\cdot))\\
&= \int \nabla[\phi_t^2\,T(x)]\cdot S\,dV_t(x,S)\\
&= k\,\|V_t\|(\phi_t^2) + \int 2\,\phi_t\,[T(x)\otimes\nabla\phi_t]\cdot S\,dV_t(x,S)\\
&\qquad + \int \phi_t^2\,(T\cdot S-k)\,dV_t(x,S)\,,
\end{aligned}
$$

and since $[u \otimes v] \cdot S = S(v) \cdot u = S(u) \cdot v$ for a symmetric S, we have that

$$\begin{aligned}&\int \{-2\,\phi_t\,\nabla\phi_t \cdot S(T(x))\}\,dV_t(x,S)\\&\qquad = k\,\|V_t\|(\phi_t^2) + \underbrace{\int \phi_t^2(x)\,T(x)\cdot h(x,V_t)\,d\|V_t\|(x)}_{I_1} + \underbrace{\int \phi_t^2\,(T\cdot S - k)\,dV_t(x,S)}_{I_2}\,.\end{aligned}$$

On the other hand, it also holds

$$\int \{-2\,\phi_t\,\nabla\phi_t \cdot S(T(x))\}\,dV_t(x,S) = A - \underbrace{\int \{-2\,\phi_t(x)\,\nabla\phi_t(x)\cdot[x - S(T(x))]\}\,dV_t(x,S)}_{I_3}\,,$$

so that

$$A = k\,\|V_t\|(\phi_t^2) + I_1 + I_2 + I_3\,, \tag{5.22}$$

and we can estimate the three pieces one at a time.

In order to estimate the term I_1, we use Brakke's perpendicularity theorem to write

$$I_1 = \int \phi_t^2(x)\,S^\perp(T(x))\cdot h(x,V_t)\,dV_t(x,S)\,,$$

so that, using $|T(x)| \le R(t)$ on $\mathrm{spt}(\phi_t)$ together with $2\,R(t)R'(t) = \sigma$, we get from (5.3) that

$$\begin{aligned}|I_1| &\le \int \phi_t^2(x)\,\|S-T\|\,|T(x)|\,|h(x,V_t)|\,dV_t(x,S)\\&\le \frac{R(t)}{R'(t)}\,\frac{\alpha(t)^2}{4} + \frac{\sigma}{2}\int \phi_t^2(x)\,\|S-T\|^2\,dV_t(x,S)\,.\end{aligned} \tag{5.23}$$

Using (5.4), we have, instead:

$$|I_2| \le k\int \phi_t^2(x)\,\|S-T\|^2\,dV_t(x,S)\,. \tag{5.24}$$

Because $\nabla\phi_t \in T$ and $T\circ T = T$, $\nabla\phi_t\cdot[x - S(T(x))] = \nabla\phi_t \cdot T(S^\perp(T(T(x))))$, and thus (5.7) and Young's inequality give

$$\begin{aligned}|I_3| &\le 2\int \phi_t(x)\,|\nabla\phi_t(x)|\,\|S-T\|^2\,|T(x)|\,dV_t(x,S)\\&\le \int \phi_t^2(x)\,\|S-T\|^2\,dV_t(x,S) + R(t)^2\int |\nabla\phi_t(x)|^2\,\|S-T\|^2\,dV_t(x,S)\,.\end{aligned} \tag{5.25}$$

In turn, by (5.8) we can further estimate

$$\int \phi_t^2(x)\,\|S-T\|^2\,dV_t(x,S) \le 4\alpha(t)\mu(t) + 4\rho R(t)^{-2}\mu^2(t)\,, \tag{5.26}$$

so that plugging (5.18) and (5.26) into (5.23), (5.24), and (5.25) yields

$$|I_1| + |I_2| + |I_3| \le \frac{R(t)}{R'(t)}\,\frac{\alpha(t)^2}{4} + C\,(\alpha(t)\mu(t) + R(t)^{-2}\mu(t)^2)\,, \tag{5.27}$$

where C is a constant depending only on k, σ and ρ.

In particular, from (5.22) and the definition of A we conclude the following bound:

$$\begin{aligned}
&\|V_t\|(\partial_t[\phi_t^2]) - k\,R(t)^{-1}R'(t)\,\|V_t\|(\phi_t^2) \\
&\le \frac{\alpha(t)^2}{4} + C\,R'(t)R(t)^{-1}\left(\alpha(t)\mu(t) + R(t)^{-2}\mu(t)^2\right) \\
&\le \frac{\alpha(t)^2}{2} + C\,|R'(t)R(t)^{-1}|^2\,\mu(t)^2 + C\,R'(t)R(t)^{-3}\,\mu(t)^2 \\
&\le \frac{\alpha(t)^2}{2} + CR(t)^{-4}\mu(t)^2\,.
\end{aligned} \tag{5.28}$$

In the last line, we used the identity $\sigma = 2\,R'(t)R(t)$: the constants C are different from line to line throughout the calculation, but they all depend only on k, σ and ρ. Now, we first use (5.20), (5.13) and (5.28), and then we multiply by $R(t)^{-k}$ in order to gain, thanks to $\sigma = 2\,R'(t)R(t)$ and the definition of μ in (5.14), the estimate

$$\begin{aligned}
R(t)^{-k}D\left[\|V_t\|(\phi_t^2)\right] - kR'(t)R(t)^{-k-1}\,\|V_t\|(\phi_t^2)& \\
\le |R'(t)R(t)^{-1}|\,M\,\mu^2\,,&
\end{aligned} \tag{5.29}$$

where M is a constant depending only on k, σ an ρ.

The conclusion (5.15) then follows plugging (5.29) into (5.19). □

6. L^2 excess estimates

This section contains the technical results which will be needed in the proof of Theorem 3.5 in order to estimate the L^2 excess terms in the iterative applications of (5.15), representing the possible gains of mass density ratio at each iteration. A careful estimate of these terms is crucial to show that the limiting Brakke flow is not trivial.

We begin with the following result, which is an adaptation of [17, Proposition 6.5]. It states that the (scale invariant) L^2 excess of varifolds evolving according to Brakke flow in a given ball can be estimated *uniformly in time* with the L^2 excess of the initial datum in a larger ball, with an error terms which decays to zero exponentially fast as the magnifying factor of the ball diverges to infinity, *provided* said varifolds have uniformly bounded mass density ratio in such larger ball.

Proposition 6.1. *Let $R > 0$, $2 \le L < \infty$, and let $\{V_t\}_{0\le t\le R^2}$ be a k-dimensional Brakke flow in U_{LR}. Then, for every $T \in \mathbf{G}(n+1,k)$, and for all $t \in [0, R^2]$ we have*

$$\begin{aligned}
R^{-(k+2)}\int_{U_R}|T^\perp(x)|^2\,d\|V_t\| \le e^{1/4}\,R^{-(k+2)}\int_{U_{LR}}|T^\perp(x)|^2\,d\|V_0\|& \\
+ c(n,k)\,L^{k+2}\,\exp\Big(-(L-1)^2/8\Big)\sup_{t\in[0,R^2]}\frac{\|V_t\|(U_{LR})}{(LR)^k}\,.&
\end{aligned} \tag{6.1}$$

Proof. Without loss of generality, we can assume $R = 1$. Let $\psi \in C_c^\infty(U_L)$ be a radially symmetric cut-off function with $0 \le \psi \le 1$, $\psi \equiv 1$ in B_{L-1}, and $|\nabla\psi|, \|D^2\psi\| \le c(n)$. Using that $\{V_t\}_{0\le t\le 1}$ is a Brakke flow, we test Brakke's inequality (2.15) with

$$\phi(x,t) := |T^\perp(x)|^2\,\psi(x)\,\varrho(x,t)\,, \tag{6.2}$$

where $\varrho(x,t) := \varrho_{(0,2)}(x,t)$ is the k-dimensional backward heat kernel

$$\varrho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{k/2}}\, \exp\left(-\frac{|x-y|^2}{4(s-t)}\right)\,, \tag{6.3}$$

with $y=0$ and $s=2$ and thus we obtain, writing $\psi=\psi(x)$, $\varrho=\varrho(x,t)$, and $h=h(x,V_t)$, and for any $\tau\le 1$,

$$\begin{aligned}\int |T^\perp(x)|^2\,\psi\varrho\, d\|V_t\|\Big|_{t=0}^{\tau} \le \int_0^\tau \int & \left\{-|h|^2\,\psi\varrho\,|T^\perp(x)|^2 + h\cdot\nabla(\psi\varrho|T^\perp(x)|^2)\right\} \\ & + \psi\,|T^\perp(x)|^2\,\frac{\partial\varrho}{\partial t}\, d\|V_t\| dt\,. \end{aligned}\tag{6.4}$$

Using the perpendicularity of the mean curvature (2.6), and consequently the fact that

$$0 \le \left| h - \frac{S^\perp(\nabla\varrho)}{\varrho}\right|^2 = |h|^2 - \frac{2}{\varrho}(h\cdot\nabla\varrho) + \frac{|S^\perp(\nabla\varrho)|^2}{\varrho^2} \qquad \text{for } V_t\text{-a.e. } (x,S),\ t\in[0,\tau]\,,$$

we can estimate the integrand on the right-hand side of (6.4) by

$$\begin{aligned} & -|h|^2\,\psi\varrho\,|T^\perp(x)|^2 + (h\cdot\nabla\varrho)\,\psi\,|T^\perp(x)|^2 + \varrho\, h\cdot\nabla(\psi\,|T^\perp(x)|^2) + \psi\,|T^\perp(x)|^2\,\frac{\partial\varrho}{\partial t} \\ \le\ & -(h\cdot\nabla\varrho)\,\psi\,|T^\perp(x)|^2 + \frac{|S^\perp(\nabla\varrho)|^2}{\varrho}\,\psi\,|T^\perp(x)|^2 + \varrho\, h\cdot\nabla(\psi\,|T^\perp(x)|^2) + \psi\,|T^\perp(x)|^2\,\frac{\partial\varrho}{\partial t}\,. \end{aligned}$$

On the other hand, we have by the definition of generalized mean curvature vector and the properties of Brakke flow that for a.e. $t\in(0,\tau)$

$$\begin{aligned} & \int \left\{-(h\cdot\nabla\varrho)\,\psi\,|T^\perp(x)|^2 + \varrho\, h\cdot\nabla(\psi\,|T^\perp(x)|^2)\right\}\, d\|V_t\| \\ =\ & \int \left\{\nabla(\psi\,|T^\perp(x)|^2\,\nabla\varrho)\cdot S - \nabla(\varrho\,\nabla(\psi\,|T^\perp(x)|^2))\cdot S\right\}\, dV_t(x,S) \\ =\ & \int \left\{(D^2\varrho\cdot S)\,\psi\,|T^\perp(x)|^2 - \varrho\, D^2(\psi\,|T^\perp(x)|^2)\cdot S\right\}\, dV_t(x,S)\,. \end{aligned}$$

It is easy to see by direct calculation that, for any $S\in\mathbf{G}(n+1,k)$

$$(D^2\varrho\cdot S) + \frac{|S^\perp(\nabla\varrho)|^2}{\varrho} + \frac{\partial\varrho}{\partial t} \equiv 0\,. \tag{6.5}$$

Hence, we conclude from (6.4) that

$$\int |T^\perp(x)|^2\,\psi\varrho\, d\|V_t\|\Big|_{t=0}^{\tau} \le -\int_0^\tau\int \varrho\, D^2(\psi\,|T^\perp(x)|^2)\cdot S\, dV_t(x,S)\, dt\,. \tag{6.6}$$

Now, using that S is symmetric we can directly compute

$$-D^2(\psi\,|T^\perp(x)|^2)\cdot S = -(D^2\psi\cdot S)\,|T^\perp(x)|^2 - 4\,(\nabla\psi\otimes T^\perp(x))\cdot S - 2\,\psi\,(T^\perp\cdot S)\,. \tag{6.7}$$

Notice that $T^\perp\cdot S\ge 0$ by (5.4), and that (5.5) and (5.6) allow to estimate

$$4\,|(\nabla\psi\otimes T^\perp(x))\cdot S| \le 4\sqrt{2}\,|\nabla\psi|\,|T^\perp(x)|\,\sqrt{T^\perp\cdot S} \le 2\,\psi\,(T^\perp\cdot S) + 4\,\frac{|\nabla\psi|^2}{\psi}\,|T^\perp(x)|^2\,, \tag{6.8}$$

so that, using $|\nabla\psi|^2/\psi\le c(n)$, (6.6) yields

$$\begin{aligned}\int |T^\perp(x)|^2\,\psi\varrho\,d\|V_t\|\Big|_{t=0}^{\tau} &\le c(n)\int_0^1\int \varrho\,\|D^2\psi\|\,|T^\perp(x)|^2\,d\|V_t\|dt\\ &\le c(n,k)\,L^2\,\exp\left(-\frac{(L-1)^2}{8}\right)\int_0^\tau \|V_t\|(U_L)\,dt\,,\end{aligned}\tag{6.9}$$

where in the last inequality we have used that $\|D^2\psi\| = 0$ in the complement of $A = U_L \setminus B_{L-1}$, that $|T^\perp(x)| \le L$ for $x \in U_L$, and that $\varrho(x,t) \le c(k)\,\exp(-(L-1)^2/8)$ for $t \in [0,1]$ and $x \in A$. Since $L \ge 2$, $\psi \equiv 1$ in B_1. Using furthermore that $\psi \le \chi_{U_L}$, that $\varrho(x,0) \le (8\pi)^{-k/2}$ everywhere, and that $\varrho(x,\tau) \ge (8\pi)^{-k/2}\,e^{-1/4}$ for $x \in B_1$ and $\tau \le 1$, we obtain (6.1) from (6.9). ☐

When $\{V_t\}$ is the Brakke flow $\{V_t^\varepsilon\}$ of Proposition 4.2, the last term of (6.1) can be controlled by the localized Huisken's monotonicity formula.

Proposition 6.2. *Let $\{V_t^\varepsilon\}_{t\ge 0}$ be the Brakke flow obtained in Proposition 4.2. Then there exists a constant E_0 depending only on $4R_0 := \mathrm{dist}(0,\partial U), \Gamma_0$, n, and Q such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\left(\sup_{R\in[\varepsilon,R_0],\,s\in[0,R_0^2]}\frac{\|V_s^\varepsilon\|(U_R)}{\omega_n R^n}\right) \le E_0\,.\tag{6.10}$$

Proof. Let $\psi \in C_c(U_{2R_0})$ be a radially symmetric function such that $0 \le \psi \le 1$, $\psi = 1$ on B_{R_0} and $\|\psi\|_{C^2} \le c(R_0)$. We use $\psi(x)\varrho_{(0,s+R^2)}(x,t)$ in (2.15) (with $k = n$) and proceed as in the proof of Proposition 6.1. Then we obtain for $s \in (0, R_0^2]$ and $R \in [\varepsilon, R_0]$

$$\int \varrho_{(0,s+R^2)}\,\psi\,d\|V_t^\varepsilon\|\Big|_{t=0}^{s} \le c(R_0)\sup_{t\in[0,s]}\|V_t^\varepsilon\|(U_{R_0}) \le c(R_0)\mathcal{H}^n(\Gamma_0^\varepsilon),\tag{6.11}$$

where $c(R_0)$ is an another constant depending only on R_0 and the last inequality is due to (2.14) and $\|V_0^\varepsilon\| = \mathcal{H}^n \llcorner_{\Gamma_0^\varepsilon}$. Since $\mathcal{H}^n(\Gamma_0^\varepsilon)$ is close to $\mathcal{H}^n(\Gamma_0)$ for small ε, the right-hand side of (6.11) is uniformly bounded. For the left-hand side, the evaluation of $t = s$ gives

$$\begin{aligned}\int \varrho_{(0,s+R^2)}(x,s)\,\psi(x)\,d\|V_s^\varepsilon\|(x) &\ge \int_{U_R}\frac{\psi(x)}{(4\pi R^2)^{n/2}}\exp\Big(-\frac{|x|^2}{4R^2}\Big)\,d\|V_s^\varepsilon\|(x)\\ &\ge \frac{e^{-1/4}}{(4\pi)^{n/2}}R^{-n}\|V_s^\varepsilon\|(U_R),\end{aligned}\tag{6.12}$$

where we used $\psi = 1$ on $U_R \subset B_{R_0}$. The evaluation of $t = 0$ may be estimated using Fubini's Theorem as

$$\int \varrho_{(0,s+R^2)}\psi\,d\|V_0^\varepsilon\| \le (4\pi(s+R^2))^{-n/2}\int_0^1 f(\lambda)\,d\lambda\tag{6.13}$$

with $f(\lambda) := \|V_0^\varepsilon\|(\{x \in U_{2R_0} : \exp\big(-\frac{|x|^2}{4(s+R^2)}\big) \ge \lambda\})$. We next evaluate $f(\lambda)$ depending on the value of λ in

(a) $(0, \exp\big(-4R_0^2/4(s+R^2)\big))$,
(b) $[\exp\big(-4R_0^2/4(s+R^2)\big), \exp\big(-4\varepsilon^2/4(s+R^2)\big))$ and
(c) $[\exp\big(-4\varepsilon^2/4(s+R^2)\big), 1)$.

In the case of (a), one can see that $f(\lambda) = \|V_0^\varepsilon\|(U_{2R_0})$. For (b), we use the fact that $\mathcal{H}^n(\Gamma_0^\varepsilon \cap B_r) \le c(\Gamma_0) r^n$ for $r \in [2\varepsilon, 2R_0]$ (which follows from the monotonicity formula for Γ_0 and Lemma 4.1(1)(4)) and obtain

$$\begin{aligned} f(\lambda) &= \mathcal{H}^n(\Gamma_0^\varepsilon \cap B_{\sqrt{4(s+R^2)\log(1/\lambda)}}) \\ &\le c(\Gamma_0)(4(s+R^2)\log(1/\lambda))^{n/2}. \end{aligned}$$

For (c), the set in question is included in $U_{2\varepsilon}$, so that $f(\lambda)$ is bounded by $(4\varepsilon)^n \omega_n (Q+1)$ due to Lemma 4.1(4). Combining these estimates, we have

$$\begin{aligned} \int_0^1 f(\lambda)\, d\lambda \le e^{-R_0^2/(s+R^2)} \|V_0^\varepsilon\|(U_{2R_0}) + c(\Gamma_0)(4(s+R^2))^{n/2} \int_0^1 \log(1/\lambda)^{n/2}\, d\lambda \\ + (4\varepsilon)^n \omega_n (Q+1). \end{aligned} \tag{6.14}$$

Since $(4\pi(s+R^2))^{-n/2} e^{-R_0^2/(s+R^2)}$ and $\|V_0^\varepsilon\|(U_{2R_0})$ are bounded uniformly and $R \ge \varepsilon$, (6.13) and (6.14) show that $\int \varrho_{(0,s+R^2)} \psi \, d\|V_0^\varepsilon\|$ is bounded depending only on R_0, Γ_0, n and Q. The estimates (6.11)-(6.14) now show (6.10). ☐

With a simple geometric argument, we are allowed to replace balls with cylinders in (6.1) if the initial datum is sufficiently flat. For the proof, we show that there is an "empty spot" just above and below the origin for all sufficiently small scale.

Proposition 6.3. *Let $\{V_t^\varepsilon\}_{t \ge 0}$ be the Brakke flow obtained in Proposition 4.2. Then there exists $r_1 = r_1(n, r_0, \alpha)$ such that, for $R \in [\varepsilon, r_1)$, $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, 4R^2]$, we have*

$$\mathrm{C}(\sqrt{2}R) \cap \{\sqrt{2}R \le |T^\perp(x)| \le 2R\} \cap \operatorname{spt}\|V_t^\varepsilon\| = \emptyset\,. \tag{6.15}$$

Moreover, for $2 \le L < \infty$ with $2LR < r_0$, we have

$$\begin{aligned} \int_{\mathrm{C}(\sqrt{2}R) \cap \{|T^\perp(x)| < \sqrt{2}R\}} |T^\perp(x)|^2 \, d\|V_t^\varepsilon\| \le e^{1/4} \int_{U_{2LR}} |T^\perp(x)|^2 \, d\|V_0^\varepsilon\| \\ + c(n)(RL)^{n+2} \exp(-(L-1)^2/8) E_0\,, \end{aligned} \tag{6.16}$$

where E_0 is as in Proposition 6.2.

Proof. Set $\delta_1 := (8n+2)/(\sqrt{2}-1)$ and fix a sufficiently small $r_1 = r_1(n, \alpha, r_0) > 0$ so that

$$\frac{\delta_1}{\log^\alpha(1/(r_1\delta_1))} < 1\,, \text{ and} \tag{6.17}$$

$$(\sqrt{2} + 2\delta_1) r_1 < r_0\,. \tag{6.18}$$

Assume $R < r_1$. Let $A(t)$ be the closed ball with center at $(x', x_{n+1}) = (0, R(\sqrt{2} + \sqrt{\delta_1^2 - 8n - 2}))$ and the radius given by $\sqrt{(R\delta_1)^2 - 2nt}$. The radius is chosen so that $\partial A(t)$ is a MCF and

$$A(0) \subset \{|T(x)| \le R\delta_1\} \cap \{R < x_{n+1} < r_0\}, \tag{6.19}$$

$$\mathrm{C}(\sqrt{2}R) \cap \{\sqrt{2}R \le x_{n+1} \le 2R\} \subset A(4R^2). \tag{6.20}$$

Indeed, the minimum of $T^\perp(A(0))$ satisfies

$$R\left(\sqrt{2} + \sqrt{\delta_1^2 - 8n - 2} - \delta_1\right) = R\left(\sqrt{2} - \frac{8n+2}{\delta_1 + \sqrt{\delta_1^2 - 8n - 2}}\right) > R\left(\sqrt{2} - \frac{8n+2}{\delta_1}\right) = R$$

by the definition of δ_1 and the maximum satisfies

$$R(\sqrt{2}+\sqrt{\delta_1^2-8n-2}+\delta_1) < R(\sqrt{2}+2\delta_1) < r_0$$

by (6.18) and $R < r_1$. These show (6.19). One can check by calculation that (6.20) holds as well. By (6.17), Lemma 4.1(1)(4) and (3.1), one can show that

$$\Gamma_0^\varepsilon \cap \{|T(x)| \le R\delta_1\} \cap \{|x_{n+1}| < r_0\} \subset \{|x_{n+1}| < R\}$$

and thus, by (6.19), $A(0)\cap\Gamma_0^\varepsilon = \emptyset$. By Brakke's sphere barrier to external varifold lemma, see [6, § 3.7] and [18, Lemma 10.12], one can conclude that $A(t)\cap \operatorname{spt}\|V_t^\varepsilon\| = \emptyset$ for $t \in [0,(R\delta_1)^2/2n)$ and in particular for $t \in [0, 4R^2]$. This combined with (6.20) shows (6.15) for the case of $x_{n+1} > 0$, and the case of $x_{n+1} < 0$ is symmetric. Finally, we have

$$\mathrm{C}(\sqrt{2}R) \cap \{|T^\perp(x)| < \sqrt{2}R\} \subset U_{2R}.$$

We can then deduce (6.16) from (6.1) with R there replaced by $2R$ and $k = n$, and thanks to (6.10). ☐

7. Proof of Theorem 3.5

We are now in the position of proving Theorem 3.5. We fix U, Γ_0, and $\{E_{0,i}\}_{i=1}^N$ so that Assumption 3.3 holds. By choosing a smaller $r_0 > 0$, we may assume that $r_0 < \operatorname{dist}(0,\partial U)/4$ (cf. Proposition 6.2), that

$$Q+1 \ge \frac{\mathcal{H}^n(\Gamma_0\cap B_r)}{\omega_n\, r^n} \ge Q \qquad \text{with } Q \ge 2 \text{ for all } r \le r_0\,, \tag{7.1}$$

and that the growth conditions (3.1)-(3.2) hold. We also fix a small $\zeta > 0$ (cf. Definition 5.1 and (5.2)) depending only on Q and n such that

$$\frac{1}{\omega_n r^n}\int_{\{|T^\perp(x)|<\sqrt{2}r\}} \chi_r^2\, d\mathcal{H}^n\llcorner_{\Gamma_0} \ge 1 + \frac{Q-1}{2} \tag{7.2}$$

for all $r \le r_0$ (by choosing an even smaller r_0 if necessary).

We shall divide the proof into three steps. Throughout the proof we are going to use the following notation. Recall that, given $\lambda > 0$, $\eta_{x,\lambda}$ denotes the function $\eta_{x,\lambda}(y) := \lambda^{-1}(y-x)$. For the sake of simplicity, we will set $\eta_\lambda := \eta_{0,\lambda}$. Furthermore, if $\mathscr{V} = \{V_t\}_{t\ge0}$ is a family of n-varifolds, we will let $\mathscr{W} = (\eta_\lambda)_\sharp\mathscr{V}$ denote the family $\{W_\tau\}_{\tau\ge0}$ of n-varifolds defined by

$$W_\tau := (\eta_\lambda)_\sharp V_{\lambda^2\tau}\,, \tag{7.3}$$

where the varifold on the right-hand side is the push-forward of the varifold $V_{\lambda^2\tau}$ through the dilation map η_λ. It is easy to check by direct calculation that if $\mathscr{V}$ is an n-dimensional Brakke flow in U then $(\eta_\lambda)_\sharp\mathscr{V}$ is an n-dimensional Brakke flow in $\eta_\lambda(U) = \lambda^{-1}U$; see [36, Section 3.4].

7.1. **Step one: hole nucleation.** Let ε_0 be given by Lemma 4.1, let $\varepsilon \in (0,\varepsilon_0]$, and let $\mathscr{V}^\varepsilon = \{V_t^\varepsilon\}_{t\ge0}$ be the Brakke flow with fixed boundary $\partial\Gamma_0$ and initial datum Γ_0^ε as in Proposition 4.2. Correspondingly, consider the Brakke flow

$$\hat{\mathscr{V}}^{\varepsilon,1} := (\eta_\varepsilon)_\sharp\mathscr{V}^\varepsilon\,.$$

By the conclusion in Proposition 4.2, we have that, denoting $\hat{\mathscr{V}}^{\varepsilon,1} = \{\hat{V}_t^{\varepsilon,1}\}_{t\ge0}$,

$$\|\hat{V}_0^{\varepsilon,1}\| = \mathcal{H}^n\llcorner_{\hat{\Gamma}_0^{\varepsilon,1}}\,, \tag{7.4}$$

where, as a result of Lemma 4.1 and (3.1)-(3.2), $\hat{\Gamma}_0^{\varepsilon,1} := \varepsilon^{-1}\,\Gamma_0^\varepsilon$ satisfies the following properties:

(1) $\mathcal{H}^n(\mathrm{C}(1)\cap U_2\cap\hat\Gamma_0^{\varepsilon,1})\le\omega_n$.

(2) $\hat\Gamma_0^{\varepsilon,1}\cap\mathrm{C}(r_0/\varepsilon)\cap\{|x_{n+1}|<r_0/\varepsilon\}\subset\{(x',x_{n+1})\,:\,|x_{n+1}|\le\frac{|x'|}{\log^\alpha(1/(\varepsilon|x'|))}\}$.

Moreover, (6.15) with $R=\varepsilon$ and after rescaling by η_ε gives

(3) $\mathrm{C}(\sqrt2)\cap\{\sqrt2\le|x_{n+1}|\le2\}\cap\mathrm{spt}\,\|\hat V_t^{\varepsilon,1}\|=\emptyset$ for $t\in[0,4]$.

We apply Lemma 5.5 to the flow $\{\hat V_t^{\varepsilon,1}\}_{t\ge0}$ regarded as a Brakke flow in $\mathrm{C}(\sqrt2)\cap\{|x_{n+1}|\le2\}$ with $\hat R_1=\sqrt2$, $\hat R_2=2$ and

$$\begin{aligned} t_1&=0\,, & t_2&=1\,,\\ R_1^2&=1\,, & R_2^2&=2. \end{aligned}$$

We deduce from (5.15) as well as (3) that

$$\begin{aligned} 2^{-n/2}\,\|\hat V_1^{\varepsilon,1}\|(\chi^2_{\sqrt2}\llcorner_{\{|x_{n+1}|\le2\}}) &\le \|\hat V_0^{\varepsilon,1}\|(\chi_1^2\llcorner_{\{|x_{n+1}|\le\sqrt2\}})+M\,\mu_1^2\\ &\le \mathcal{H}^n(\hat\Gamma_0^{\varepsilon,1}\cap\mathrm{C}(1)\cap\{|x_{n+1}|\le\sqrt2\})+M\,\mu_1^2\,, \end{aligned}\tag{7.5}$$

where in the last inequality we have used (7.4) and the properties of χ, and where

$$\mu_1^2=\sup_{t\in[0,1]}R(t)^{-(n+2)}\int_{\mathrm{C}(R(t))\cap\{|x_{n+1}|\le\sqrt2\}}|T^\perp(x)|^2\,d\|\hat V_t^{\varepsilon,1}\|(x)\,,\qquad R(t)^2=1+t\,.$$

Observe that, thanks to property (1), (7.5) reads

$$2^{-n/2}\|\hat V_1^{\varepsilon,1}\|(\chi^2_{\sqrt2}\llcorner_{\{|x_{n+1}|\le2\}})\quad\le\quad\omega_n+M\,\mu_1^2\,.\tag{7.6}$$

Now fix a number $2\le L_1<\infty$ to be chosen later. We can apply Proposition 6.3 (with $R=\varepsilon$ and rescaling by η_ε) in order to estimate

$$\begin{aligned} \mu_1^2 &\le \sup_{t\in[0,1]}\int_{\mathrm{C}(\sqrt2)\cap\{|x_{n+1}|\le\sqrt2\}}|T^\perp(x)|^2\,d\|\hat V_t^{\varepsilon,1}\|(x)\\ &\le e^{1/4}\int_{U_{2L_1}}|T^\perp(x)|^2\,d\|\hat V_0^{\varepsilon,1}\|(x)+c(n)\,L_1^{n+2}\,\exp\Big(-(L_1-1)^2/8\Big)\,E_0. \end{aligned}$$

We set the following condition: we will choose L_1 in such a way that

$$2\varepsilon L_1<r_0\,.\tag{7.7}$$

If (7.7) holds, then we can use again property (2) above in order to further estimate

$$\begin{aligned} \mu_1^2 &\le e^{1/4}\,\frac{(2L_1)^2}{\log^{2\alpha}(1/(2\varepsilon L_1))}\,\mathcal{H}^n(\hat\Gamma_0^{\varepsilon,1}\cap U_{2L_1})+c(n)\,L_1^{n+2}\,\exp\Big(-(L_1-1)^2/8\Big)\,E_0\\ &\le c(n)\,L_1^{n+2}\left(\frac{1}{\log^{2\alpha}(1/(2\varepsilon L_1))}\,\frac{\mathcal{H}^n(\Gamma_0^\varepsilon\cap U_{2L_1\varepsilon})}{(2L_1\varepsilon)^n}+\exp\Big(-(L_1-1)^2/8\Big)\,E_0\right)\\ &\le c(n)\,L_1^{n+2}E_0\left(\frac{1}{\log^{2\alpha}(1/(2\varepsilon L_1))}+\exp\Big(-(L_1-1)^2/8\Big)\right), \end{aligned}$$

where we also used (6.10).

Finally, rescaling (7.6) back we conclude

$$(\sqrt2\,\varepsilon)^{-n}\|V^\varepsilon_{\varepsilon^2}\|(\chi^2_{\sqrt2\,\varepsilon}\llcorner_{\{|x_{n+1}|\le2\varepsilon\}})\le\omega_n+M\,\mu_1^2\,,\tag{7.8}$$

$$\mu_1^2\le c(n)\,L_1^{n+2}E_0\left(\frac{1}{\log^{2\alpha}(1/(2\varepsilon L_1))}+\exp\Big(-(L_1-1)^2/8\Big)\right)\,.\tag{7.9}$$

7.2. **Iteration: hole expansion.** Let $h \geq 2$ be an integer, and consider now the Brakke flow

$$\hat{\mathscr{V}}^{\varepsilon,h} := (\eta_{2^{(h-1)/2}\varepsilon})_\sharp \mathscr{V}^\varepsilon\,.$$

Again by the conclusions of Proposition 4.2, and with $\hat{\mathscr{V}}^{\varepsilon,h} = \{\hat V_t^{\varepsilon,h}\}_{t\geq 0}$, we have that

$$\|\hat V_0^{\varepsilon,h}\| = \mathcal{H}^n \llcorner_{\hat\Gamma_0^{\varepsilon,h}}\,,$$

where $\hat\Gamma_0^{\varepsilon,h} := \Gamma_0^\varepsilon/(2^{(h-1)/2}\varepsilon)$ satisfies

$$\begin{aligned}\hat\Gamma_0^{\varepsilon,h} \cap \mathrm{C}\left(\frac{r_0}{2^{(h-1)/2}\varepsilon}\right) \cap \left\{|x_{n+1}| < \frac{r_0}{2^{(h-1)/2}\varepsilon}\right\}& \\ \subset \left\{(x', x_{n+1}) :\ |x_{n+1}| \leq \frac{|x'|}{\log^\alpha\left(1/(2^{(h-1)/2}\varepsilon|x'|)\right)}\right\}&\,. \end{aligned} \tag{7.10}$$

As long as we have

$$2^{(h-1)/2}\varepsilon < r_1\,, \tag{7.11}$$

we have by (6.15)

$$\mathrm{C}(\sqrt{2}) \cap \{\sqrt{2} \leq |x_{n+1}| \leq 2\} \cap \operatorname{spt}\|\hat V_t^{\varepsilon,h}\| = \emptyset \text{ for } t \in [0,4]. \tag{7.12}$$

We apply Lemma 5.5 to the flow $\{\hat V_t^{\varepsilon,h}\}_{t\geq 0}$ with $\hat R_1 = \sqrt{2}$, $\hat R_2 = 2$ and

$$\begin{aligned} t_1 &= \frac{1}{2}\,, & t_2 &= 1\,,\\ R_1^2 &= 1\,, & R_2^2 &= 2 \end{aligned}$$

to deduce

$$2^{-n/2}\|\hat V_1^{\varepsilon,h}\|(\chi^2_{\sqrt{2}}\llcorner_{\{|x_{n+1}|\leq 2\}}) \quad\leq\quad \|\hat V_{1/2}^{\varepsilon,h}\|(\chi_1^2\llcorner_{\{|x_{n+1}|\leq\sqrt{2}\}}) + M\,\mu_h^2\,, \tag{7.13}$$

where

$$\mu_h^2 := \sup_{t\in[1/2,1]} R(t)^{-(n+2)}\int_{\mathrm{C}(R(t))\cap\{|x_{n+1}|\leq\sqrt{2}\}} |T^\perp(x)|^2\, d\|\hat V_t^{\varepsilon,h}\|\,,\ R(t)^2 = 1 + 2\left(t - \frac{1}{2}\right)\,. \tag{7.14}$$

As long as L_h (to be chosen) satisfies

$$2L_h 2^{(h-1)/2}\varepsilon < r_0, \tag{7.15}$$

by (6.16), we have

$$\begin{aligned}\mu_h^2 &\leq \sup_{t\in[1/2,1]}\int_{\mathrm{C}(\sqrt{2})\cap\{|x_{n+1}|\leq\sqrt{2}\}}|T^\perp(x)|^2\,d\|\hat V_t^{\varepsilon,h}\|\\ &\leq e^{1/4}\int_{U_{2L_h}}|T^\perp(x)|^2\,d\|\hat V_0^{\varepsilon,h}\| + c(n)\,L_h^{n+2}\exp\left(-(L_h-1)^2/8\right)E_0.\end{aligned}$$

By (7.10) and proceeding as in Step one, we further deduce

$$\mu_h^2 \quad\leq\quad c(n)\,L_h^{n+2}E_0\left(\frac{1}{\log^{2\alpha}\left(1/(2^{(h+1)/2}\varepsilon L_h)\right)} + \exp\left(-(L_h-1)^2/8\right)\right)\,.$$

Now, if we rescale (7.13) back, we have

$$\begin{aligned}&(2^{h/2}\,\varepsilon)^{-n}\|V^{\varepsilon}_{2^{h-1}\varepsilon^2}\|(\chi^2_{2^{h/2}\varepsilon}\llcorner_{\{|x_{n+1}|\le 2^{(h+1)/2}\varepsilon\}}) \\ &\quad\le (2^{(h-1)/2}\varepsilon)^{-n}\|V^{\varepsilon}_{2^{h-2}\varepsilon^2}\|(\chi^2_{2^{(h-1)/2}\varepsilon}\llcorner_{\{|x_{n+1}|\le 2^{h/2}\varepsilon\}}) + M\,\mu_h^2\,,\end{aligned} \tag{7.16}$$

$$\mu_h^2 \le c(n)\,L_h^{n+2}E_0\left(\frac{1}{\log^{2\alpha}\left(1/(2^{(h+1)/2}\varepsilon L_h)\right)} + \exp\left(-(L_h-1)^2/8\right)\right)\,. \tag{7.17}$$

7.3. **Conclusion.** Let $j \ge 1$. If we chain the inequalities (7.16) as h varies in $\{2,\dots,j\}$ together with (7.8) we conclude that

$$(2^{j/2}\,\varepsilon)^{-n}\|V^{\varepsilon}_{2^{j-1}\varepsilon^2}\|(\chi^2_{2^{j/2}\varepsilon}\llcorner_{\{|x_{n+1}|\le 2^{(j+1)/2}\varepsilon\}}) \le \omega_n + M\sum_{h=1}^{j}\mu_h^2\,, \tag{7.18}$$

where, thanks to (7.9) and (7.17),

$$\mu_h^2 \le c(n)L_h^{n+2}E_0\left(\frac{1}{\log^{2\alpha}\left(1/(2^{(h+1)/2}\varepsilon L_h)\right)} + \exp\left(-(L_h-1)^2/8\right)\right)\,, \qquad h\ge 1\,, \tag{7.19}$$

as long as (7.7), (7.11) and (7.15) are satisfied. In order to guarantee this, we will have to carefully choose ε, j, and L_h. We proceed as follows.

Let K be a large integer, fixed but to be chosen at the end, and for any $J \ge K+1$ apply (7.18) with $j = J-K$ and $\varepsilon = \varepsilon_J := 2^{-J/2}$. With these choices, (7.18) reads

$$(2^{-K/2})^{-n}\,\|V^{\varepsilon_J}_{2^{-(K+1)}}\|(\chi^2_{2^{-K/2}}\llcorner_{\{|x_{n+1}|\le 2^{-(K-1)/2}\}}) \le \omega_n + M\sum_{h=1}^{J-K}\mu_h^2\,. \tag{7.20}$$

We can now choose the constants L_h by setting

$$L_h := \log(J-h)\,, \qquad \text{for } 1\le h\le J-K\,, \tag{7.21}$$

so that (7.19) becomes

$$\mu_h^2 \le c(n)E_0\log^{n+2}(J-h)\left(\frac{1}{\log^{2\alpha}\left(\frac{2^{(J-h-1)/2}}{\log(J-h)}\right)} + \exp\left(-\frac{(\log(J-h)-1)^2}{8}\right)\right)\,, \tag{7.22}$$

which is valid assuming that

$$2^{(h-J+1)/2}\log(J-h) < r_0\,, \tag{7.23}$$

$$2^{(h-J-1)/2} < r_1 \tag{7.24}$$

for all $h\in\{1,\dots,J-K\}$, corresponding to (7.7), (7.11), and (7.15).

In order to simplify the notation, it is useful to change variable in the sum from h to $q := J-h$, so that (7.20) becomes

$$(2^{-K/2})^{-n}\,\|V^{\varepsilon_J}_{2^{-(K+1)}}\|(\chi^2_{2^{-K/2}}\llcorner_{\{|x_{n+1}|\le 2^{-(K-1)/2}\}}) \le \omega_n + c(n)E_0M\sum_{q=K}^{J-1}a_q^2\,, \tag{7.25}$$

with

$$a_q^2 := \frac{\log^{n+2}(q)}{\log^{2\alpha}\left(\frac{2^{(q-1)/2}}{\log(q)}\right)} + \log^{n+2}(q)\,\exp\left(-\frac{(\log(q)-1)^2}{8}\right)\,, \tag{7.26}$$

and the conditions (7.23) and (7.24) read

$$2^{(-q+1)/2}\log(q) < r_0, \tag{7.27}$$

$$2^{(-q-1)/2} < r_1 \tag{7.28}$$

for $q \in \{K, \dots, J-1\}$. To check the validity of (7.27), we notice that, for q large, the function $q \mapsto 2^{-(q+1)/2}\log(q)$ is decreasing towards 0. In particular, (7.27) is satisfied if we choose K large enough depending only on r_0. The condition (7.28) is also satisfied as soon as K is large enough depending on $r_1 = r_1(n, \alpha, r_0)$.

We have then validated the estimate (7.25) with a_q^2 defined by (7.26). Notice that the estimate remains valid independently of the choice of $J \geq K+1$. Hence, we can now let $J \to \infty$, so that, for a (not relabeled) subsequence of $\{\varepsilon_J\}$ satisfying the conclusion of Proposition 4.3, and with $\{V_t\}_{t\geq 0}$ the corresponding limit Brakke flow, we have

$$(2^{-K/2})^{-n}\, \|V_{2^{-(K+1)}}\|(\chi^2_{2^{-K/2}} \llcorner_{\{|x_{n+1}| < 2^{-(K-1)/2}\}}) \leq \omega_n + c(n) E_0 M \sum_{q=K}^{\infty} a_q^2\,. \tag{7.29}$$

Observe that Proposition 4.3 guarantees that $\{V_t\}_{t\geq 0}$ has fixed boundary $\partial\Gamma_0$ and that $\lim_{t\to 0^+} \|V_t\| = \|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}$. This shows that $\{V_t\}_{t\geq 0}$ satisfies the conclusion (i) of Theorem 3.5. Hence, we are only left with proving that $t \mapsto V_t$ is not identically equal to V_0. To this end, notice that if $\alpha > \frac{1}{2}$ then there exists $\gamma > 1$ such that $\lim_{q\to\infty} a_q^2\, q^\gamma = 0$, which implies that $\sum_{q=K}^\infty a_q^2$ is a convergent series: therefore, we may choose K so large (depending on $c(n)E_0M$) that

$$(2^{-K/2})^{-n}\, \|V_{2^{-(K+1)}}\|(\chi^2_{2^{-K/2}} \llcorner_{\{|x_{n+1}| < 2^{-(K-1)/2}\}}) \leq \omega_n \left(1 + \frac{Q-1}{4}\right). \tag{7.30}$$

Due to (7.2) with $r = 2^{-K/2}$ and (7.30), we see that

$$\|V_{r^2/2}\|(\chi^2_r \llcorner_{\{|x_{n+1}| < \sqrt{2}r\}}) < \|V_0\|(\chi^2_r \llcorner_{\{|x_{n+1}| < \sqrt{2}r\}}), \tag{7.31}$$

which shows $V_{2^{-(K+1)}} \neq V_0$. We may similarly argue that (7.31) holds for $r = 2^{-j/2}$ with any $j > K$. Finally, we prove $\|V_t\|(U) < \|V_0\|(U)$ for all $t > 0$. First, note that $\|V_t\|(U) \leq \|V_0\|(U)$ for all $t > 0$ by (2.14). Assume for a contradiction that there exists $t_0 > 0$ with $\|V_{t_0}\|(U) = \|V_0\|(U)$. By (2.14), for a.e. $t \in [0, t_0]$, we have $h(\cdot, V_t) = 0$. Choose $j \geq K$ such that $2^{-j} < t_0$. By the above argument, we may choose a smooth function $\phi \in C_c^\infty(U)$ with $0 \leq \phi \leq 1$ such that $\|V_{2^{-j}}\|(\phi) < \|V_0\|(\phi)$. Then, by (2.15) and $h(\cdot, V_t) = 0$ for a.e. $t \in [0, t_0]$, we also have $\|V_{t_0}\|(\phi) < \|V_0\|(\phi)$. Since $\|V_{t_0}\|(U) = \|V_0\|(U)$, we should have $\|V_{t_0}\|(1-\phi) > \|V_0\|(1-\phi)$. Then by approximation, we have a non-negative function $\hat\phi \in C_c^\infty(U)$ such that $\|V_{t_0}\|(\hat\phi) > \|V_0\|(\hat\phi)$. Since $h(\cdot, V_t) = 0$, (2.15) shows $\|V_{t_0}\|(\hat\phi) \leq \|V_0\|(\hat\phi)$, which is a contradiction. This shows $\|V_t\|(U) < \|V_0\|(U)$ for all $t > 0$ and completes the proof of the theorem. ☐

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