

Distributed two-time-scale methods over clustered networks

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Abstract—In this paper, we consider consensus problems over a network of nodes, where the network is divided into a number of clusters. We are interested in the case where the communication topology within each cluster is dense as compared to the sparse communication across the clusters. Moreover, each cluster has one leader which can communicate with other leaders in different clusters. The goal of the nodes is to agree at some common value under the presence of communication delays across the clusters.

Our main contribution is to propose a novel distributed two-time-scale consensus algorithm, which pertains to the separation in network topology of clustered networks. In particular, one scale is to model the dynamic of the agents in each cluster, which is much faster (due to the dense communication) than the scale describing the slowly aggregated evolution between the clusters (due to the sparse communication). We prove the convergence of the proposed method in the presence of uniform, but possibly arbitrarily large, communication delays between the leaders. In addition, we provided an explicit formula for the convergence rate of such algorithm, which characterizes the impact of delays and the network topology. Our results shows that after a transient time characterized by the topology of each cluster, the convergence of the two-time-scale consensus method only depends on the connectivity of the leaders. Finally, we validate our theoretical results by a number of numerical simulations on different clustered networks.

I. INTRODUCTION

Clustered network of agents is a specific type of multi-agent systems, where the whole network is divided into distinct clusters, and usually the connection structure in each cluster is denser, while the inter-cluster connection is sparser. Each cluster might contains smaller clusters inside, resulting in a hierarchical and multi-layer clustered network. This type of system can be found in a variety of application domains including energy systems [1], robotics [2], biological and chemical engineering [3], [4], social networks [5], brain science [6], epidemic [7], etc., and has been a timely research topic in network science [8], [9].

As an example, power and energy systems are large-scale systems composing of many subsystems inside, each of them can be regarded as a cluster [1]. In another example of social networks, the opinion of each individual continuously evolves with respect to the views of the members belonging to its community in order to achieve a common agreement. In some specific conditions, at specific instants, one individual

in each community (called a leader) can change its opinion by exchanging with other leaders outside its community. They will reset their opinion taking into account the ones of other leaders. These inter-cluster interactions can be considered as resets of the opinions [10].

Hitherto, the existing literature on clustered networks of agents aims at either exploring how network structures affect to the controllability (e.g., [11]), observability [12], and control performances of the network [13], or exploiting special properties of such networks for enhancing the overall network robustness, resiliency, etc. [14]. It is worth emphasizing that in all researches on agent networks, network convergence is one of the most fundamental problems, whether infinitely or in finite-time. Network convergence could be significantly altered by latency, both intra-cluster and inter-cluster. To study clustered networks, singular perturbation theory is one popular approach, see e.g., [1], [15]–[20]. Under this approach, aggregated and reduced order models were derived, where the fast dynamics inside clusters is ignored, or lumped into that of the slow dynamics occurring across clusters. As such, *the obtained results mostly depend on what happen between clusters.*

Note also that the existing literature on distributed algorithms (stabilization, consensus, formation, etc.) for clustered networks has focused only on the network asymptotic convergence of these algorithms, while their *convergence rates are missing*, see e.g., [10], [21]–[23], in addition to the aforementioned researches. The study in [10] showed the existence of a positive decay rate to guarantee the overall network asymptotic consensus. The event-triggered resets defined for each cluster leading to asynchronous reset sequences were proposed in [21]. In our recent work [22], a robust formation controller design was proposed for clustered networks of unmanned aerial vehicles, but again the convergence was only asymptotic.

In this research, motivated by the fact that the convergence speed inside clusters is usually much faster than that between clusters, due to denser connection structures and shorter communication distances, our goal is to derive explicitly the network convergence rate in accordance to inter-cluster delays and network structures. Several recent works have investigated the inter-cluster time delays, e.g. [2], [15], [24], to achieve asymptotic network convergence. The work [2] utilized passivity theory for the cooperative control of two clusters of robots over a very long distance which naturally incurs a very large time delay between such two clusters. The study [15] designed state-feedback controllers for clustered networks using singular perturbation theory. Nevertheless, in all of above researches, *no convergence rate was considered.*

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On the other hand, it can be observed that multi-time-scale algorithms are suitable approaches for studying clustered networks, due to their nature of different time scales inside clusters and across clusters. Therefore, the current research proposes a distributed two-time-scale consensus algorithm for clustered networks with inter-cluster time delays, where a faster consensus protocol is employed intra-cluster and a slower consensus law is accounted for inter-cluster time delays. Note that several two-time-scale methods have been proposed in the literature, e.g. [25]–[36], for different problems in machine learning and reinforcement learning, however they are different from the one proposed in this paper.

Contribution. Our main contribution is to propose a novel distributed two-time-scale consensus algorithm, which pertains to the separation in network topology of clustered networks. In particular, one scale is to model the dynamic of the agents in each cluster while one scale is to present the slowly aggregated evolution between the leaders of the clusters. We prove the convergence of the proposed method in the presence of uniform, but possibly arbitrarily large, communication delays between the leaders. In addition, we provided an explicit formula for the convergence rate of such algorithm, which characterizes the impact of delays and the network topology. Our results shows that after a transient time characterized by the topology of each cluster, the convergence of the two-time-scale consensus method only depends on the connectivity of the leaders. Moreover, the results in this paper complements for the existing consensus literature over clustered networks where such a formula of convergence rates is missing. Finally, we validate our theoretical results by a number of numerical simulations on different clustered networks.

The rest of this paper is organized as follows. The problem setting and the proposed algorithm are formalized in Section II. Next, the main result of this paper is presented in Section III. Finally, a number of numerical simulations are provided in Section IV to illustrate the theoretical results of this paper.

II. DISTRIBUTED TWO-TIME-SCALE CONSENSUS METHODS

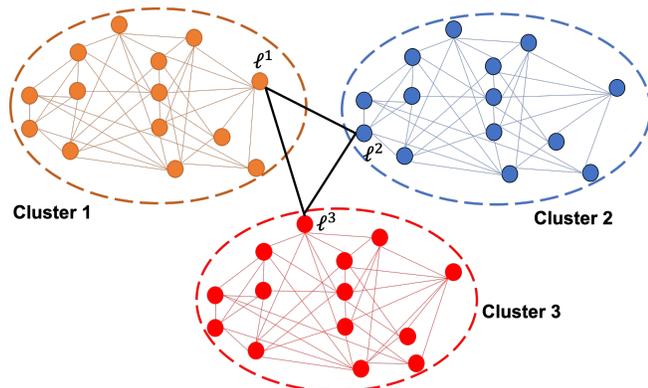


Fig. 1. A network partitioned into 3 densely connected clusters.

We consider a network of N nodes divided into r disjoint connected clusters \mathcal{C}^a , $a = 1, \dots, r$. The communication pattern between nodes within each cluster \mathcal{C}^a is modeled by a densely connected and undirected graph $\mathcal{G}^a = (\mathcal{V}^a, \mathcal{E}^a)$, where \mathcal{V}^a is the set of node indexes and \mathcal{E}^a is the set of edges. Thus, we have $\mathcal{V}^a \cap \mathcal{V}^b = \emptyset$ for all $a \neq b$ and $\sum_{a=1}^r |\mathcal{V}^a| = N$, where $|\mathcal{V}^a|$ denotes the cardinality of the set \mathcal{V}^a . In addition, the communication between the nodes in different clusters is described by a sparse time-varying graph $\mathcal{G}^C(k) = (\mathcal{V}^C, \{\mathcal{E}^C(k)\})$, that is, $(i, j) \in \mathcal{E}^C(k)$ if and only if the nodes i and j are in different clusters and there exists an edge between them at time k . We assume that $|\mathcal{V}^a \cap \mathcal{V}^C| = 1$, $a = 1, \dots, r$, i.e., each cluster has only one node, called *the leader* of \mathcal{C}^a , that is connected to (some) leaders of other clusters. Similarly, we name other nodes as followers. For simplicity and notational convenience, we assume that all the followers in each cluster is connected to its leader. However, we note that our method derived later can work for more general setting. Finally, the N nodes are connected by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \cup_{a=1}^r \mathcal{V}^a$ and $\mathcal{E} = \cup_{a=1}^r \mathcal{E}^a \cup \mathcal{E}^C$.

An illustrative example of clustered networks is given in Fig. 1 where the nodes are divided into r clusters. The connection between nodes in each cluster is modeled by a densely connected graph, while each cluster \mathcal{C}^a has one leader l^a , $a = 1, \dots, r$, connected to other leaders in other clusters (described by black edges). A concrete motivating example for this problem is coordinated control of multi-robot systems across long distances [2]. As the spaces between robots grow they experience larger delays in their communication. The presence of delays, if not properly addressed, may lead to oscillatory behaviors and even instabilities among the robots. To handle the impact of delays, distributed consensus algorithm based on the so-called scattering transformation is proposed in [2]. By utilizing passivity theory, the authors can show an asymptotic convergence of their proposed method under the presence of communication delays. Our focus in this paper is to understand the rate of convergence of consensus algorithm over clustered networks, which is missing in [2]. Indeed, while passivity theory can help to study the stability (or asymptotic convergence) of consensus algorithms under delays, it may not be applicable to study the finite-time convergence. Therefore, we propose a novel distributed two-time-scale algorithm for solving consensus problems under delays over clustered networks. Moreover, we carefully characterize the finite-time performance of our proposed methods discussed in details in Section III.

Our proposed method, formally stated in Algorithm 1, is explained as follows. Suppose that each node $i \in \mathcal{V}$ is initialized with some arbitrary value p_i . This variable may encode the initial local information at each node in \mathcal{G} , e.g., the position and velocity of a robot in networked formation control problems. The goal of the nodes is to cooperatively agree at some common value, i.e., they achieve a consensus. Since the graphs \mathcal{G}^a are dense while $\mathcal{G}^C(k)$ is sparse, information shared between the nodes within each cluster

is mixed much faster than the one between the clusters. For example, in Fig. 1 the rate of information sent from ℓ^1 to its followers in \mathcal{C}^1 is much faster than the time it gets to the nodes in \mathcal{C}^r . To model this difference, we propose a novel distributed two-time-scale consensus method, where the updates of the followers in any cluster is implemented at a faster time-scale as compared to the one between the leaders of the clusters. This is described by the use of two different step sizes in Eqs. (1) and (2).

Algorithm 1: Distributed two-time-scale consensus methods under delays

Initialization: Each follower $i \in \mathcal{V}^a$ maintains x_i^a and each leader a maintains $x_\ell^a, \forall a \in [1, r]$
Each node $i \in \mathcal{V}$ initializes its variable at a constant p_i
The nodes initialize proper step sizes $\beta \ll \gamma \in (0, 1)$.

for $k=0,1,2,\dots$ **do**

for each cluster $a = 1, \dots, r$ **do**

for each follower $i \in \mathcal{V}^a$ **do**

1. Receive $x_\ell^a(k)$ from its leaders

2. Exchange $x_i^a(k)$ with neighbors $j \in \mathcal{N}_i^a$

3. Implement

$$x_i^a(k+1) = (1 - \gamma) \sum_{j \in \mathcal{N}_i^a} w_{ij}^a x_j^a(k) + \gamma x_\ell^a(k). \quad (1)$$

end

Leader ℓ^a : Exchange x_ℓ^a to other leaders $b \in \mathcal{N}_a^C(k)$ and update

$$x_\ell^a(k+1) = (1 - \beta)x_\ell^a(k) + \beta \sum_{b \in \mathcal{N}_a^C(k)} v_{ab}(k)x_\ell^b(k - \tau). \quad (2)$$

end

end

In particular, for each cluster \mathcal{C}^a each follower $i \in \mathcal{V}^a$ maintains a variable x_i^a and the leader maintains x_ℓ^a , both are set at their initial values. The nodes in each $\mathcal{C}^a, a = 1, \dots, r$, then consider the updates in (1) and (2), where \mathcal{N}_i^a is the neighboring set of node i in the cluster \mathcal{C}^i and $\mathcal{N}_a^C(k)$ is the neighboring set of leader a in the graph $\mathcal{G}^C(k)$ of the clusters at time k . The step sizes γ and β are in $(0, 1)$. Moreover, w_{ij}^a and $v_{ab}(k)$ are some positive (time-varying) weights, which will be specified shortly. Here, Eq. (1) is a consensus step between the followers in each cluster \mathcal{C}^a and Eq. (2) is the consensus step between the leaders of the clusters. We allow that the updates of the followers may take into account the value of its leaders but not vice versa. Finally, the constant τ in (2) represents the communication delays in the information exchange between the leaders.

In Eqs. (1) and (2), we use two different step sizes γ, β to

represent the difference in information propagation between the nodes within each cluster and across different clusters. Indeed, since γ is associated with the fast-time scale of the followers' updates in each cluster we choose $\gamma \gg \beta$, which corresponds to the slow-time scale of the updates between the clusters. As shown in our numerical experiments in Section IV, the followers' iterates move toward to the leaders' values, which is slowly pushed to a consensus value through step (2). Finally, the step size β is also chosen properly to handle the delays as considered in the previous work [25].

Remark 1: Recall that in (1) we assume that the followers in each cluster is connected to its leader. Such an assumption only helps to reduce the burden notation in our algorithm. Our two-time-scale approach, however, can be applied for solving the consensus problem over general cluster networks without requiring this assumption.

III. MAIN RESULTS

In this section, we analyze the convergence properties of the proposed distributed two-time-scale methods under delays presented in the previous section. Specifically, our results show that under some proper choice of step size $\gamma \gg \beta$ the nodes in the network \mathcal{G} reach a consensus at a rate β/γ . In addition, we provide an explicit formula to show the dependence of this convergence on the network topology and the constant delays τ . By using the two-time-scale approach, we can show that the convergence of the followers within each cluster \mathcal{C}^a only depends on the topology of \mathcal{G}^a while the convergence of the leaders only depends on the sparse graph \mathcal{G}^C . Since the former convergence happens much faster than the latter, we observe that after a transient time the convergence of Algorithm 1 only depends on the connectivity of \mathcal{G}^C .

We begin our analysis by introducing more notation. We denote by $\mathbf{W}^a = [w_{ij}^a] \in \mathbb{R}^{|\mathcal{V}^a| \times |\mathcal{V}^a|}$ the weighted adjacency matrix corresponding to \mathcal{G}^a at cluster \mathcal{C}^a . Similarly, let $\mathbf{V}(k) = [v_{ab}(k)] \in \mathbb{R}^{r \times r}$ be the time-varying weighted adjacency matrix corresponding to the sequence of graphs $\{\mathcal{G}^C(k)\}$ between the clusters. For convenience, we use the following notation

$$\mathbf{X}^a \triangleq \begin{pmatrix} (x_1^a)^T \\ \vdots \\ (x_n^a)^T \end{pmatrix} \in \mathbb{R}^{|\mathcal{V}^a| \times d}, \quad \mathbf{X}_\ell \triangleq \begin{pmatrix} (x_\ell^1)^T \\ \vdots \\ (x_\ell^r)^T \end{pmatrix} \in \mathbb{R}^{r \times d}.$$

Using this notation, the matrix forms of (1) and (2) are given

$$\begin{aligned} \mathbf{X}^a(k+1) &= (1 - \gamma)\mathbf{W}^a\mathbf{X}^a(k) + \gamma\mathbf{1}x_\ell^a(k)^T \\ \mathbf{X}_\ell(k+1) &= (1 - \beta)\mathbf{X}_\ell(k) + \beta\mathbf{V}(k)\mathbf{X}_\ell(k - \tau), \end{aligned} \quad (3)$$

where we denote by $\mathbf{1}$ a vector with proper dimension whose entries are all equal to the constant 1. Finally, let $\bar{x}^a \in \mathbb{R}^d, a = 1, \dots, r$ and $\bar{x}_\ell \in \mathbb{R}^d$ be the average of the row vectors of \mathbf{X}^a and \mathbf{X}_ℓ , respectively, i.e.,

$$\bar{x}^a = \frac{1}{|\mathcal{V}^a|} \sum_{i \in \mathcal{V}^a} x_i^a \quad \text{and} \quad \bar{x}_\ell = \frac{1}{r} \sum_{a=1}^r x_\ell^a.$$

Next, we make an assumption on \mathbf{W}^a , $a = 1, \dots, r$, and $\mathbf{V}(k)$ which is fairly standard in the consensus literature to guarantee the convergence of the nodes' estimates to a consensus point [37]. The assumption given below also imposes a constraint on the communication between the followers at each cluster and between the leaders of the clusters, in which the nodes are only allowed to exchange messages with neighboring nodes, i.e., those directly connected to them.

Assumption 1: \mathbf{W}^a , $a = 1, \dots, r$, is a doubly stochastic matrix, i.e., $\sum_i w_{ij}^a = \sum_j w_{ij}^a = 1$. Moreover, $w_{ii}^a > 0$, $\forall i$, and $w_{ij}^a > 0$ if and only if $(i, j) \in \mathcal{E}^a$ otherwise $w_{ij}^a = 0$.

Assumption 2: There exists a positive constant γ such that $\mathbf{V}(k)$ satisfies the following conditions for all $k \geq 0$:

- (a) $v_{aa}(k) \geq \alpha$, for all $a = 1, \dots, r$.
- (b) $v_{ab}(k) \in [\alpha, 1]$ if $(a, b) \in \mathcal{N}_a^C(k)$ otherwise $v_{ab}(k) = 0$.
- (c) $\sum_{a=1}^r v_{ab}(k) = \sum_{b=1}^r v_{ab}(k) = 1$, for all a, b .

In addition, we assume that the graph $\mathcal{G}^C(k)$ is connected at any time $k \geq 0$, formally stated as follows.

Assumption 3: For all $k \geq 0$, $\mathcal{G}^C(k) = (\mathcal{V}, \mathcal{E}(k))$ is connected and undirected.

By Assumption 1 and since \mathcal{G}^a is connected, each \mathbf{W}^a has 1 as the largest singular value. Let σ_a be the second largest eigenvalue of \mathbf{W}^a , which by the Perron-Frobenius theorem [38] we have $\sigma_a \in (0, 1)$. Similarly, we denote by $\sigma(\mathbf{V}(k))$ the second largest singular value of $\mathbf{V}(k)$. Furthermore, let δ_C be a parameter representing the spectral properties of the time-varying graph $\mathcal{G}^C(k)$ defined as

$$\delta_C = \max_{k \geq 0} \sigma(\mathbf{V}(k)). \quad (4)$$

Assumptions 1 and 3 imply that $\delta_C \in (0, 1)$. We now present the main steps in our analysis. We note that some of the analysis below is quite standard in the existing literature. We include them in this paper for completeness. Our first result is to show that the followers in each cluster reach a consensus exponentially, stated in the following lemma.

Lemma 1: The sequence $\{\mathbf{X}^a(k)\}$ generated by (3) satisfies

$$\|\mathbf{X}^a(k) - \bar{x}^a(k)^T \mathbf{1}\| \leq ((1 - \gamma)\sigma_a)^{k+1} \|\mathbf{X}^a(0)\|, \forall a. \quad (5)$$

Proof: Denote by $\mathbf{Y}^a = \mathbf{X}^a - \mathbf{1}(\bar{x}^a)^T = (\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{X}^a$. Since \mathbf{W}^a is a doubly stochastic matrix, using (3) we have

$$\begin{aligned} \bar{x}^a(k+1) &= \frac{1}{|\mathcal{V}^a|} \sum_{i \in \mathcal{V}^a} x_i^a(k+1) \\ &= (1 - \gamma)\bar{x}^a(k) + \gamma x_{\ell}^a(k). \end{aligned} \quad (6)$$

By (3) and (6) we consider for each $a = 1, \dots, r$

$$\begin{aligned} \mathbf{Y}^a(k+1) &= \mathbf{X}^a(k+1) - \mathbf{1}\bar{x}^a(k+1)^T \\ &= (1 - \gamma)\mathbf{W}^a\mathbf{X}^a(k) - (1 - \gamma)\mathbf{1}\bar{x}^a(k)^T \\ &= (1 - \gamma)\mathbf{W}^a\mathbf{Y}^a(k), \end{aligned}$$

which by using $\|\mathbf{Y}^a\| \leq \|\mathbf{X}^a\|$ yields (5), i.e.,

$$\begin{aligned} \|\mathbf{Y}^a(k+1)\| &= \|(1 - \gamma)\mathbf{W}^a\mathbf{Y}^a(k)\| \leq (1 - \gamma)\sigma_a \|\mathbf{Y}^a(k)\| \\ &\leq ((1 - \gamma)\sigma_a)^{k+1} \|\mathbf{Y}^a(0)\|, \end{aligned}$$

where the second inequality is to Assumption 1, i.e.,

$$\|\mathbf{W}\mathbf{Y}^a\| = \left\| \mathbf{W} \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^T \right) \mathbf{X}^a \right\| \leq \sigma_a \|\mathbf{X}^a\|. \quad \blacksquare$$

We note that this lemma only states that the followers in each cluster agree at a common point. However, these points might be different for different clusters, i.e., the nodes in different clusters might not agree with each other. Our next result is to study the consensus between the leaders under communication delays τ . We note that under communication delays the estimate in (2) depends on the time interval $[k - \tau, k]$ for all $k \geq 0$. We, therefore, utilize the discrete-time variant of the *Grönwall-Bellman* Inequality [39], to hand such a dependence. The following lemma is to show that the leaders reach an agreement under some proper choice of step sizes. The analysis is motivated by the one in [25].

Lemma 2: Let $\beta \in \left(0, 1 - e^{-\frac{\ln(1/\delta_C)}{\tau}}\right)$ and η be defined as

$$\eta \triangleq 1 - \beta + \frac{\delta_C \beta}{(1 - \beta)^\tau}. \quad (7)$$

Then the sequence $\{\mathbf{X}_\ell(k)\}$ of the leaders satisfies

$$\|\mathbf{X}_\ell(k) - \mathbf{1}\bar{x}_\ell(k)^T\| \leq 2\eta^k \|\mathbf{X}_\ell(0)\|. \quad (8)$$

Proof: We denote by $\mathbf{Y}_\ell = \mathbf{X}_\ell - \mathbf{1}\bar{x}_\ell^T = (\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\mathbf{X}_\ell$. Since $\mathbf{V}(k)$ satisfies Assumption 2(c), using (3) we have

$$\begin{aligned} \mathbf{Y}_\ell(k+1) &= (1 - \beta)\mathbf{X}_\ell(k) + \beta\mathbf{V}(k)\mathbf{X}_\ell(k - \tau) \\ &\quad - (1 - \beta)\mathbf{1}\bar{x}_\ell(k)^T + \beta\mathbf{1}\bar{x}_\ell(k - \tau)^T \\ &= (1 - \beta)\mathbf{Y}_\ell(k) + \beta\mathbf{V}(k)\mathbf{Y}_\ell(k - \tau) \\ &= (1 - \beta)^{k+1}\mathbf{Y}_\ell(0) + \beta \sum_{t=0}^k \mathbf{V}(t)\mathbf{Y}_\ell(t - \tau)(1 - \beta)^{k-t}, \end{aligned}$$

which by (4) implies that

$$\begin{aligned} \|\mathbf{Y}_\ell(k+1)\| &\leq (1 - \beta)^{k+1} \|\mathbf{Y}_\ell(0)\| \\ &\quad + \beta \sum_{t=0}^k (1 - \beta)^{k-t} \|\mathbf{V}(t)\mathbf{Y}_\ell(t - \tau)\| \\ &\leq (1 - \beta)^{k+1} \|\mathbf{Y}_\ell(0)\| \\ &\quad + \delta_C \beta \sum_{t=0}^k (1 - \beta)^{k-t} \|\mathbf{Y}_\ell(t - \tau)\|, \end{aligned} \quad (9)$$

where the second inequality is due to Assumptions 2 and 3

$$\|\mathbf{V}(t)\mathbf{Y}_\ell(t - \tau)\| \leq \sigma_C \|\mathbf{Y}_\ell(t - \tau)\|.$$

We now apply the discrete-time variant of the *Grönwall-Bellman* Inequality [39] to handle the right-hand side of (9). Let $z(k)$ be defined as

$$z(k) = \sum_{t=0}^k (1 - \beta)^{-t} \|\mathbf{Y}_\ell(t - \tau)\|$$

Thus we have $z(-1) = 0$ and $z(k)$ is a nondecreasing nonnegative function of time. Moreover, by (9) we have

$$\|\mathbf{Y}_\ell(k+1)\| \leq (1 - \beta)^{k+1} \|\mathbf{Y}_\ell(0)\| + \delta_C \beta (1 - \beta)^k z(k).$$

Consider

$$z(k+1) - z(k) = (1 - \beta)^{-k-1} \|\mathbf{Y}_\ell(k+1 - \tau)\|,$$

which implies that

$$\begin{aligned} z(k+1) &= (1 - \beta)^{-k-1} \|\mathbf{Y}_\ell(k+1 - \tau)\| + z(k) \\ &\leq (1 - \beta)^{-\tau} \|\mathbf{Y}_\ell(0)\| + \frac{\delta_C \beta}{(1 - \beta)^{\tau+1}} z(k - \tau) + z(k) \\ &\leq (1 - \beta)^{-\tau} \|\mathbf{Y}_\ell(0)\| + \left(1 + \frac{\delta_C \beta}{(1 - \beta)^{\tau+1}}\right) z(k) \\ &\leq \frac{(1 - \beta) \|\mathbf{Y}_\ell(0)\|}{\delta_C \beta} \left(1 + \frac{\delta_C \beta}{(1 - \beta)^{\tau+1}}\right)^{k+1}, \end{aligned}$$

where we use $w(-1) = 0$ in the third inequality. Substituting the previous relation into (9) we have

$$\begin{aligned} &\|\mathbf{Y}_\ell(k+1)\| \\ &\leq (1 - \beta)^{k+1} \|\mathbf{Y}_\ell(0)\| \\ &\quad + \|\mathbf{Y}_\ell(0)\| (1 - \beta)^{k+1} \left(1 + \frac{\delta_C \beta}{(1 - \beta)^{\tau+1}}\right)^{k+1} \\ &= (1 - \beta)^{k+1} \|\mathbf{Y}_\ell(0)\| \\ &\quad + \|\mathbf{Y}_\ell(0)\| \left(1 - \beta + \frac{\delta_C \beta}{(1 - \beta)^\tau}\right)^{k+1}. \end{aligned} \quad (10)$$

Since $\beta \in \left(0, 1 - e^{-\frac{\ln(1/\delta_C)}{\tau}}\right)$ we have

$$\eta \triangleq 1 - \beta + \frac{\delta_C \beta}{(1 - \beta)^\tau} < 1.$$

Thus, by (10) and since $\|\mathbf{Y}_\ell\| \leq \|\mathbf{X}_\ell\|$ we have (8). \blacksquare Lemma 2 states that under a proper choice of step size β , the leaders' iterates converge exponentially to the same value even under the presence of delays. On the other hand, Lemma 1 states that the followers in each cluster agree at the same value. We now present a result to show that the followers' iterates in each cluster follows its leader' value.

Lemma 3: Let $P = \max_i \|p_i\|$. We have

$$\|\bar{x}^a(k) - x_\ell^a(k)\| \leq (1 - \gamma)^k \|\bar{x}^a(0) - x_\ell^a(0)\| + \frac{2P\beta}{\gamma}. \quad (11)$$

Proof: We first note that $x_i(k) = p_i$ for all $i \in \mathcal{V}$ and $k \in [-\tau, 0]$. In addition, since each node $i \in \mathcal{V}$ only considers the consensus updates (1) and (2), the nodes' iterates are always bounded, i.e.,

$$\begin{cases} \|x_i^a(k)\| \leq (1 - \gamma)P + \gamma P = P \\ \|x_\ell^a(k)\| \leq (1 - \beta)P + \beta P = P, \end{cases} \quad \forall i \in \mathcal{V}^a, \forall a, \forall k \geq 0. \quad (12)$$

Next, for any cluster a , by (6) and (2) we have

$$\begin{aligned} \bar{x}^a(k+1) - x_\ell^a(k+1) &= (1 - \gamma)(\bar{x}^a(k) - x_\ell^a(k)) + \beta x_\ell^a(k) \\ &\quad - \beta \sum_{b \in \mathcal{N}_a^C(k)} v_{ab}(k) x_\ell^b(k - \tau), \end{aligned}$$

which by using (12) yields

$$\begin{aligned} &\|\bar{x}^a(k+1) - x_\ell^a(k+1)\| \\ &\leq (1 - \gamma) \|\bar{x}^a(k) - x_\ell^a(k)\| + 2P\beta \\ &\leq (1 - \gamma)^{k+1} \|\bar{x}^a(0) - x_\ell^a(0)\| + 2P\beta \sum_{t=0}^k (1 - \gamma)^{k-t} \\ &\leq (1 - \gamma)^{k+1} \|\bar{x}^a(0) - x_\ell^a(0)\| + \frac{2P\beta}{\gamma}. \end{aligned}$$

By putting the results in Lemmas 1–3, one can show that the nodes in the network achieve a consensus even under communication delays. The following theorem is to formally state this result.

Theorem 1: Let Assumptions 1–3 hold. Let the sequence $\{x_i^a(k)\}$ and $\{x_\ell^a(k)\}$, for all $i \in \mathcal{V}^a$ and $a = 1, \dots, r$, be generated by (1) and (2). Then we have $\forall a$ and $\forall i \in \mathcal{V}^a$

$$\begin{aligned} \|x_i^a(k) - \bar{x}_\ell^a(k)\| &\leq ((1 - \gamma)\sigma_a)^k \|\mathbf{X}^a(0)\| + 2\eta^k \|\mathbf{X}_\ell(0)\| \\ &\quad + (1 - \gamma)^k \|\bar{x}^a(0) - x_\ell^a(0)\| + \frac{2P\beta}{\gamma}. \end{aligned} \quad (13)$$

Proof: We have

$$\begin{aligned} &x_i^a(k) - \bar{x}_\ell^a(k) \\ &= x_i^a(k) - \bar{x}^a(k) + \bar{x}^a(k) - x_\ell^a(k) + x_\ell^a(k) - \bar{x}_\ell^a(k). \end{aligned}$$

Using the results in Lemmas 1–3 and the triangle inequality immediately gives our result in (13). \blacksquare

Remark 2: Here, we make some comments on the results given in Theorem 1.

1. As mentioned, to present the difference in the time scale between the dynamics of the followers and clusters, we choose $\beta \ll \gamma \in (0, 1)$. Thus, Eqs. (8) and (13) imply that the nodes converge arbitrarily close to each other exponentially.

2. Since each graph \mathcal{G}^a is denser than the graph \mathcal{G}^C of the leaders, $\sigma_a \ll \sigma_C < 1$. By (13) the exponential rate essentially depends on η , a function of σ_C and the delays τ . This show that after a transient time characterized by the topology of each \mathcal{G}^a , the convergence of the two-time-scale algorithms only depends on the connectivity of \mathcal{G}^C and the delays between leaders. This observation agrees with the one using singular perturbation theory [15]–[18]. We investigate further this observation numerically in Section IV.

3. One particular choice of the step sizes is $\gamma = \beta^{1/3}$. Under this choice, Eqs. (8) and (13) shows that the nodes in the network reach a consensus at a rate $\beta^{2/3}$. This rate is similar to the one studied in (distributed) linear two-time-scale stochastic approximation; see for example [28], [29], [36]. For example, let T be a positive integer. If we run the algorithm in T steps and let $\beta = 1/T$, then the algorithm converges at a rate $1/T^{2/3}$.

IV. NUMERICAL SIMULATIONS

In this section, we investigate the impact of time delays and network structure on the performance of the two-time

scale method, Algorithm 1, over clustered networks. In particular, we use two different step sizes γ, β in *Algorithm 1* to represent the difference in information propagation between the nodes within each cluster and across different clusters, and two examples of clustered networks to see the effect of network structure to convergence speed. Since γ is associated with the fast-time scale of the followers' updates in each cluster, we choose $\gamma \gg \beta$, where β corresponds to the slow-time scale of the updates between the clusters. Moreover, β is also chosen properly to handle the communication delays in the information exchange between the leaders.

A. Small network analysis

We first consider the performance of Algorithm 1 on a small network, i.e., a clustered network consisting of 60 agents divided into 3 clusters where nodes 1, 21, and 41 are chosen as leaders (see Fig. 2). In each cluster a , a node is connected to the nearest two nodes, i.e., $|\mathcal{N}_i^a| = 2$ for all $i \in \mathcal{V}^a$. Here, intra-cluster structure is not dense, which is purpose-built to compare with the dense cluster structure in Section IV-B.

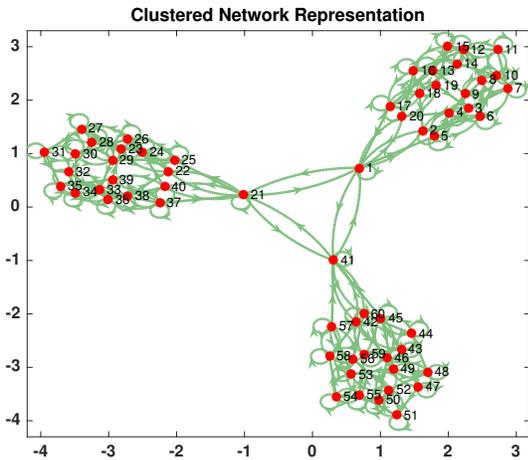


Fig. 2. A 3-cluster network consisting of 60 nodes.

The adjacency matrix \mathbf{W}^a of each cluster \mathcal{C}^a is chosen as

$$\mathbf{W}^a = [w_{ij}^a] = \begin{cases} \frac{1}{|\mathcal{N}_i^a|+1}, & \text{if } (i, j) \in \mathcal{E}^a, \\ 0, & \text{if } (i, j) \notin \mathcal{E}^a, i \neq j, \\ 1 - \sum_{j \in \mathcal{N}_i^a} w_{ij}^a, & \text{if } i = j. \end{cases}$$

Similarly, we choose the adjacency matrix $\mathbf{V}(k)$ corresponding to the sequence of graphs $\{\mathcal{G}_C(k)\}$ between the clusters as above. It is straightforward to verify that the matrices \mathbf{W}^a and $\mathbf{V}(k)$ satisfy Assumption 1 and 2. Finally, the initial conditions of all the nodes are chosen arbitrarily in $[-4, 4]$.

First, we investigate the performance of Algorithm 1 for two cases, namely $\beta = 1$ and $\beta = 0.1$ when the delays $\tau = 10$. That is, when $\beta = 1$ we simply consider a distributed consensus algorithm without handling the impact of delays τ . As shown in Fig. 3, the consensus over the clustered network

is not achieved, i.e., there is an oscillation among the nodes' iterates. This phenomenon is also observed in [2].

To handle this impact of time delays, we choose the step sizes $\beta = 0.1$ and $\gamma = 0.5$ which satisfy the condition in Lemma 2. In this case, all agents can achieve a consensus, as shown in Fig. 4. Moreover, it can be observed that the consensus within each cluster is achieved around 7s, while the one between clusters happens at 50s. It means that the information shared inside each cluster is much faster than that between clusters, which agrees with our results in Theorem 1.

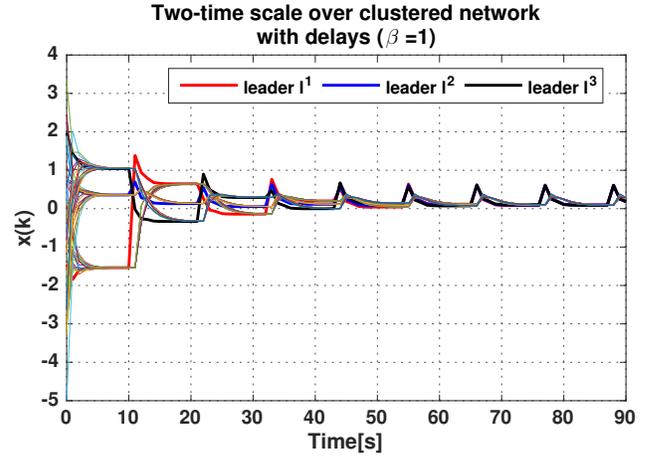


Fig. 3. States of agents with $\tau = 10s$ and $\beta = 1$.

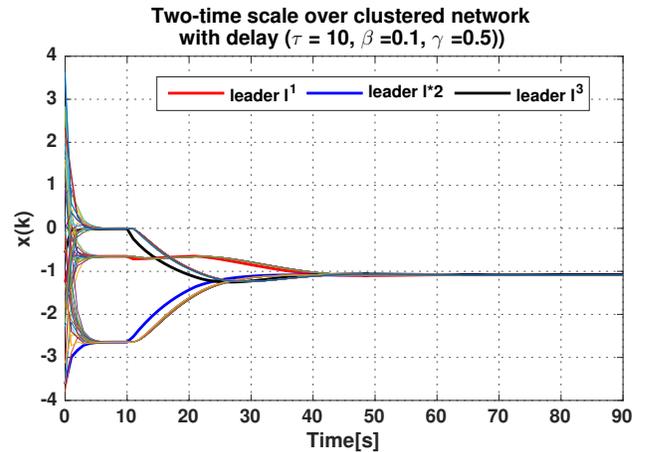


Fig. 4. States of agents with $\tau = 10s$ and $\beta = 0.1$.

Next, impact of communication delays on convergence speed of the proposed algorithm is tested, where different time delays are used, and the stopping criterion is: $\|x_i^a - x_\ell^a\| \leq 10^{-3}$. As seen in Fig. 5, the number of iterations seems to depend linearly on τ , which validates to our results shown in *Theorem 1*.

B. Larger network analysis

In this section, we validate the performance of Algorithm 1 for a larger network with much denser intra-cluster structure

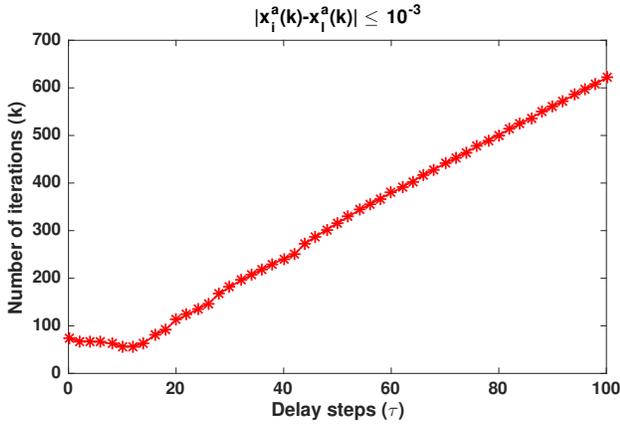


Fig. 5. Number of iterations as a function of τ

than that in Section IV-A. More specifically, we consider a network of 400 nodes partitioned into 5 clusters, as depicted in Fig. 6. Each cluster structure \mathcal{G}^a is generated by randomly initializing positions of nodes and connecting two nodes if the distance between them is less than a certain value (0.3 in this simulation). The weighted adjacency matrices $\mathbf{W}^a, \mathbf{V}(k)$ are chosen similarly to that in Section IV-A. The initial conditions of the nodes are chosen randomly within $[-4, 4]$. Moreover, we set $\beta = 0.05$ and $\gamma = 0.5$, which satisfies the condition in Lemma 2.

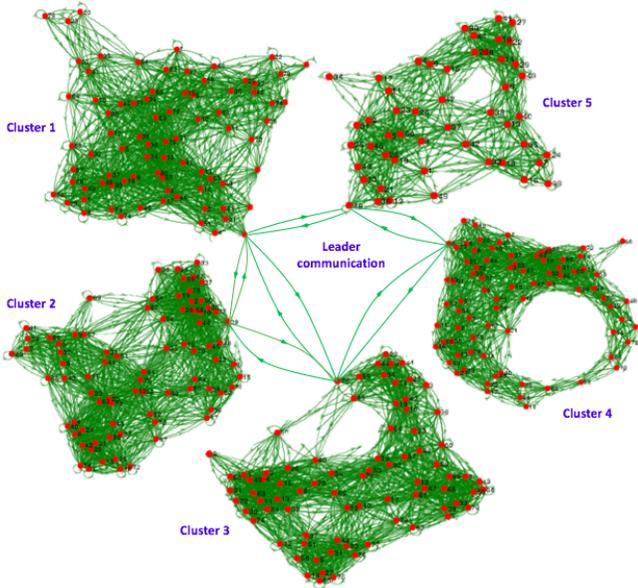


Fig. 6. A 5-cluster network of 400 nodes.

In this simulation, our goal is to investigate the impact of intra-cluster time delays to the proposed algorithm performance. To do so, we set the inter-cluster time delay to be $\tau = 20s$, and vary the intra-cluster time delay τ_a to be 0, 2, and 15s.

Simulation results are then displayed in Fig. 7. As anticipated, when the intra-cluster delay is small compared to the inter-cluster delay, i.e., as $\tau_a = 0s$ or $\tau_a = 2s$, the

convergence inside each cluster is reached much faster than that across clusters. Moreover, network convergence, both intra-cluster and inter-cluster, is faster as the intra-cluster delay is smaller. On the other hand, when τ_a is large and comparable with τ (here $\tau_a = 15s$ and $\tau = 20s$), it greatly affects to the algorithm performance, where the convergence inside clusters cannot be distinguished from that between clusters.

Note that most of the existing literature consider a uniform delay for all nodes, hence cannot capture the true behavior of clustered networks, where the delays within each cluster are much smaller as compared to the ones across the clusters. However, the simulations for our proposed distributed two-time-scale method here clearly show such behavior, and help provide a better understanding about the dynamical evolution in clustered networks.

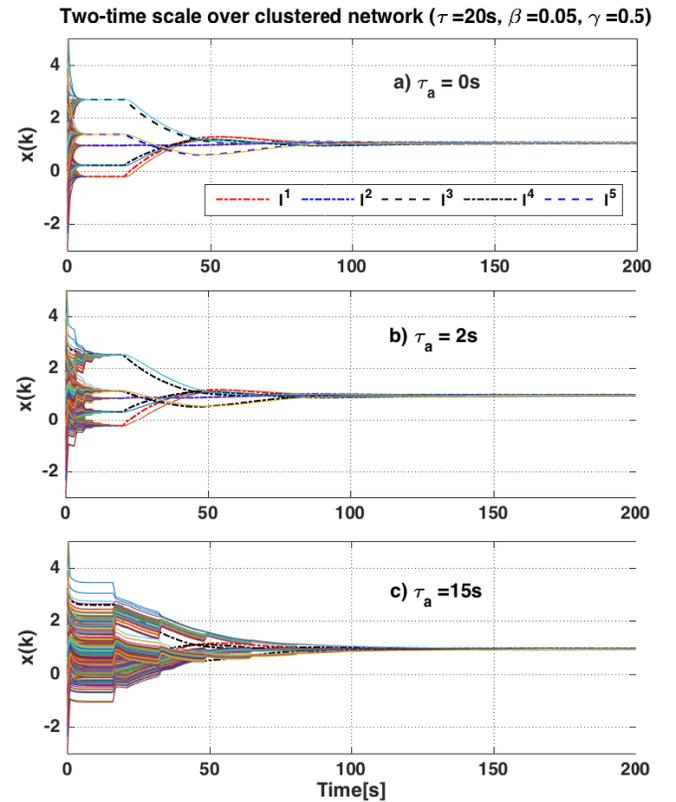


Fig. 7. States of agents with $\beta = 0.05, \gamma = 0.5$, inter-cluster delay $\tau = 20s$, and varied intra-cluster delay τ_a .

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we consider a distributed two-time-scale consensus algorithm for clustered networks with inter-cluster time delays, where a faster consensus protocol is employed for each cluster while a slower consensus update is accounted for inter-cluster time delays. We proved the convergence of the two-time scale consensus algorithm in the presence of uniform, but possibly arbitrarily large, communication delays between the leaders. In addition, we provided an explicit formula for the convergence rate of such algorithms,

which characterizes the impact of delays and the network topology. Our theoretical results are validated by a number of numerical simulations. A few more interesting questions left from this work including handling quantized communication and directed graphs, which we leave for our future studies.

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