

Control of Unknown (Linear) Systems with Receding Horizon Learning

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Abstract

A receding horizon learning scheme is proposed to transfer the state of a discrete-time dynamical control system to zero without the need of a system model. Global state convergence to zero is proved for the class of stabilizable and detectable linear time-invariant systems, assuming that only input and output data is available and an upper bound of the state dimension is known. The proposed scheme consists of a receding horizon control scheme and a proximity-based estimation scheme to estimate and control the closed-loop trajectory. Simulations are presented for linear and nonlinear systems.

I. INTRODUCTION

Currently, a lot of research effort is centered around the interplay between control, learning, and optimization. This is driven by extensive research initiatives in artificial intelligence, by the steadily increasing online computing power, and by the wish to build autonomous and intelligent systems in all sorts of application domains. From these developments, a renewed interest in the control of systems where no system model is known, or where the model involves large uncertainties has emerged under the banner of learning-based or data-based control. Traditionally, this is a subject of adaptive control. In this vein, we address a classical problem from adaptive control, namely the stabilization of completely unknown linear time-invariant discrete-time control systems. We aim for a solution that utilizes online optimization and (past and future) receding horizon data, and that provides convergence guarantees. To this end, we propose a scheme that involves estimation, prediction, and feedback control for unknown systems, which we have subsumed under the term learning in the title of this work.

The literature on the (adaptive) stabilization of unknown systems is huge. Many different solution approaches exist in the adaptive control literature ranging from model-free approaches to model-based approaches [1], [2], [3]. The control of unknown systems has also been studied in the area of optimal control and receding horizon control for quite some time, see e.g. [4], [5], [6], [7]. Work that is related to our work in the sense that similar problems and challenges are addressed are for example [5] (Chapter 3) and [8], in which an input-output stabilizing receding horizon control was proposed and combined with a multi-step prediction method for parameter model identification. Further, in the works [9] convergence of a recursive least-squares identification algorithm was addressed with incomplete excitation and convergence of an adaptive control scheme was proven under the assumption that the system has an asymptotically stable zero dynamics. Also related is for example the publications [10], in which a framework based on a robust model predictive scheme and identification of a multi-step prediction model was proposed and convergence and feasibility for stable systems were shown. A common approach when controlling completely unknown systems is based on a combination of a control scheme (such as pole-placement) and an online estimation or identification scheme (such as recursive least-squares). Hereby, models are estimated and updated in real-time based on the measured input-output data and these models are utilized in the control scheme. A major challenge when using this so-called certainty equivalence approach to stabilize unknown (and unstructured) systems is the *loss of stabilizability problem*, i.e. how to ensure in a computationally efficient way that the estimated models are, for example, stabilizable so that adaptive pole-placement can be applied. See for example [11], [12], [13], [14], [15] and references therein on this topic. More recent related research on (partially) unknown systems and receding horizon control are for example discussed in [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], to mention only a few out of the rapidly growing literature. A complete review of the state of the art of controlling (partially) unknown systems with receding horizon schemes is out of the scope of this work.

The proposed approach in this paper is based on the classical certainty equivalence implementation. However, in contrast to the existing literature, we provide a fully online optimization-based solution with provable convergence for completely unknown linear systems. For example, we do not assume that the linear system is persistently excited, which is often assumed in order to identify (directly or indirectly) a system model. Moreover, we do not assume that the state can be measured nor that the system is stable or controllable. Under these minimal assumptions [29], [30], [31], the satisfaction of state or input constraints is not feasible and we do therefore not consider constraints as it is usually done in the receding horizon (predictive control) literature. Nevertheless, the control of unknown linear systems is an important benchmark problem, and, to the best of our knowledge, a receding horizon approach that provably ensures state convergence under these assumptions has not been reported in the literature. In particular, the contributions of this work are as follows. We propose an online optimization scheme which builds on a receding horizon control scheme and an estimation scheme. The receding horizon control scheme is based on a novel model-independent terminal state weighting in the sense that the objective function and the terminal cost can be chosen independently of a (not necessarily controllable) system model.

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The estimation scheme is based on a modified proximal minimization algorithm that guarantees convergence of the estimated quantities, and does not require that the closed loop system is persistently excited. A characteristic feature of the approach is that the estimated quantities do not correspond to a (or to the "true") system model but rather to a signal model (time series or signal predictor) of the closed loop trajectory. The overall computational online effort of the proposed scheme is rather low and requires essentially the solution of (least-squares) regression problems. Finally, the proposed scheme is also applicable to nonlinear systems as demonstrated by simulations.

II. PROBLEM STATEMENT

Consider the discrete-time linear time-invariant system

$$\begin{aligned} z(k+1) &= Fz(k) + Gv(k) \\ y(k) &= Hz(k) \end{aligned} \quad (1)$$

with state $z(k) \in \mathbb{R}^n$, input $v(k) \in \mathbb{R}^q$ and output $y(k) \in \mathbb{R}^p$ at time instant $k \in \mathbb{N}$.

Assumption 1. We assume that (F, G) is stabilizable and (F, H) detectable. Furthermore, we assume F, G, H are unknown, that an upper bound $m \geq n$ of the state dimension is known, and that only past input and output data is available.

Under Assumption 1, we aim to define a scheme which guarantees for any initial state $z(0)$ that the system state $z(k)$ of (1) converges to zero as time index k goes to infinity. We develop the scheme in three steps. In a first step, in Section III, we develop a stabilizing, model-independent receding horizon control scheme based on asymptotically accurate predictor maps for the closed loop trajectory. In a second step, in Section IV, we develop a proximity-based estimation scheme to obtain the asymptotically accurate predictor maps in terms of a so-called signal model. Section III and IV are independent of each other and also the contributions therein. In a third step, in Section V, the control scheme and the estimation scheme are combined in a proper way to solve the stated problem. In Section VI, we provide simulation results. All proofs can be found in the Appendix.

III. A MODEL-INDEPENDENT RECEDING HORIZON CONTROL SCHEME

A. Problem Setup

Consider the system

$$x(k+1) = Ax(k) + Bu(k) \quad (2)$$

with $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^q$. Further, consider the following optimization problem

$$\begin{aligned} V_1(x, p_1) &= \min \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + \frac{\Gamma(x)}{\varepsilon} x_N^\top Q_N x_N \\ \text{s.t. } & x_i = P_i(k, x_0, u_0, \dots, u_{i-1}), \quad i = 1 \dots N \\ & x_0 = x \end{aligned} \quad (3)$$

with $p_1^\top = [k, \varepsilon]$, $N, k \in \mathbb{N}$, $\varepsilon > 0$, $\Gamma: \mathbb{R}^n \rightarrow \mathbb{R}$ nonnegative. The decision variables are $u_i \in \mathbb{R}^q$, $i = 0 \dots N-1$ and $x_i \in \mathbb{R}^n$, $i = 0 \dots N$ and we refer to x and p_1 as parameters. The map $P_i: \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{iq} \rightarrow \mathbb{R}^n$ is an i -th step-ahead state (or signal) predictor. We denote the value of the objective function for some u_i , $i = 1 \dots N-1$ with $V_0(x, p_0, u_0, \dots, u_{N-1})$ (since the variables x_i are determined by u_i 's and x) and an optimal solution is denoted by $u_i(x, p_1)$, $i = 0 \dots N-1$, $x_i(x, p_1)$, $i = 1 \dots N$. If in (3) we choose x to be the state of (2) at time instant k , i.e. $x = x(k)$ and if we choose $p_1 = p_1(k)$ at time instant k for a given sequence $\{p_1(k)\}_{k \in \mathbb{N}}$, then we refer to a mapping

$$x(k) \mapsto u_0(x(k), p_1(k)) \quad (4)$$

as the *receding horizon control policy* defined by (3) and we call (3) and (2) *closed loop*, if in (2) $u(k) = u_0(x(k), p_1(k))$. We impose the following assumptions.

Assumption 2. (A, B) in (2) is stabilizable and the state can be measured.

Assumption 3. The prediction horizon satisfies $N \geq n = \dim(x(k))$ and $Q > 0, R > 0, Q_N > 0$, $\Gamma: \mathbb{R}^n \rightarrow \mathbb{R}$ positive definite.

Assumption 4. (a) For any $k \in \mathbb{N}$, we assume that the state predictor maps P_i , $i = 1 \dots N$ have the following linear structure

$$P_i(k, x_0, u_0, \dots, u_i) = A_i(k)x_0 + \sum_{l=0}^{i-1} B_{i-1-l}(k)u_l \quad (5)$$

where $\{A_i(k)\}_{k \in \mathbb{N}}$, $i = 0 \dots N$ with $A_0(k) = I$, and $\{B_i(k)\}_{k \in \mathbb{N}}$, $i = 0 \dots N-1$, $A_i(k) \in \mathbb{R}^{n \times n}$, $B_i(k) \in \mathbb{R}^{n \times q}$, are convergent matrix sequences, i.e.

$$\lim_{k \rightarrow \infty} A_i(k) = \hat{A}_i, \quad \lim_{k \rightarrow \infty} B_i(k) = \hat{B}_i. \quad (6)$$

(b) Moreover, for any $x(0) \in \mathbb{R}^n$ the state predictor maps P_i along the trajectory $x(k)$, $k \in \mathbb{N}$, of the closed loop (3) and (2) with $s_l := u_l(x(k), p_1(k))$ (or of the closed loop (11) and (2) with $s_l := v_l(x(k), p_3(k))$) satisfy for $0 \leq i+j \leq N+1$

$$\begin{aligned} P_j(k, P_i(k, x(k), s_0, \dots, s_{i-1}), s_i, \dots, s_{i+j-1}) &= \\ P_i(k, P_j(k, x(k), s_0, \dots, s_{j-1}), s_j, \dots, s_{i+j-1}) &= \\ P_{i+j}(k, x(k), s_0, \dots, s_{i+j-1}). \end{aligned} \quad (7)$$

Further, we assume that state predictor maps predict asymptotically accurate with respect to system (2) in the sense that we have for any $k \in \mathbb{N}$, $i = 0 \dots N$

$$A_i(k)x(k) + \sum_{l=0}^{i-1} B_{i-1-l}(k)s_l = A^i x(k) + \sum_{l=0}^i A^{i-1-l} B s_l + e_i(k) \quad (8)$$

where the following error bounds hold for any $i = 0 \dots N$: $\|e_i(k)\|^2 \leq \omega_1(k) + \omega_2(k)\|x(k)\|^2 + \omega_3(k)\sum_{l=0}^i \|s_l\|^2$ with $\lim_{k \rightarrow \infty} \omega_j(k) = 0$, $j = 1, 2, 3$.

(c) Finally, for any trajectory $x(k)$, $k \in \mathbb{N}$, of the closed loop (3) and (2), there exists functions $\mu_0(k), \dots, \mu_{N-1}(k)$ such that

$$\lim_{k \rightarrow \infty} \|A_N(k)x(k) + \sum_{l=0}^{N-1} B_{N-1-l}(k)\mu_l(k)\| = 0. \quad (9)$$

Remark 1. Assumption 4 postulates predictor maps for the closed loop trajectory (input and state sequence) generated by (3) and (2) (or (11) and (2)). In particular, equation (5) and (6) in Assumption 4(a) ensure that we have linear time-varying and converging state predictor maps. Equation (7) ensures a state property in the sense that the predictor maps commute like flow maps of (time-invariant) dynamical state-space models do. Equation (8) ensures that the predictor maps are able to accurately predict $x_1(x(k), p_1(k)), \dots, x_N(x(k), p_1(k))$ in (3) along the closed loop trajectory. Notice that $e(k)$ converges to zero, if, for example, the state and input stays bounded. Finally, equation (9) in Assumption 4(c) represents a stabilizability condition of the predictor maps along the closed loop trajectory. Notice that if the state predictor maps are determined (learned) online, then Assumption 4 does not imply that the knowledge of such state predictors implies a model (system) identification in the sense that neither the equation $(\hat{A}_i, \hat{B}_{i-1}) \approx (A^i, A^{i-1}B)$ must hold nor that for every initial data or every input sequence the predictions are (asymptotically) accurate.

The main goal of the next subsection is to show that the state of the closed loop (2), (3) converges to zero under the stated assumptions. In summary, we aim for a receding horizon control scheme which guarantees that the state of a linear *stabilizable* system converges to zero, assuming that the closed loop state trajectory can be accurately predicted as time goes to infinity. A characteristic property of the proposed scheme is that the objective (and potentially constraints) can be chosen independently from the system model (predictor maps) in the sense that no terminal cost or terminal constraint needs to be computed online based on some model information or data. This is a desirable property when controlling unknown systems. Further, no stabilizing zero terminal constraint and no controllability assumption is required. Instead, a time-varying terminal state weighting scheme is introduced to ensure convergence.

B. Results

We define the following auxiliary problems

$$\begin{aligned} V_2(x, p_2) &= \min \quad \xi_N^\top Q_N \xi_N \\ \text{s.t.} \quad \xi_{i+1} &= A_{i+1}(k)\xi_0 + \sum_{l=0}^i B_{i-l}(k)v_l, \\ \xi_0 &= x, \quad i = 0 \dots N-1 \end{aligned} \quad (10)$$

with $p_2 = k$ and

$$\begin{aligned}
V_3(x, p_3) = \min & \sum_{i=0}^{N-1} \xi_i^\top Q \xi_i + v_i^\top R v_i \\
\text{s.t. } & \xi_{i+1} = A_{i+1}(k) \xi_0 + \sum_{l=0}^i B_{i-l}(k) v_l, \\
& \xi_0 = x, \quad i = 0 \dots N-1, \\
& \xi_N = r
\end{aligned} \tag{11}$$

with $p_3^\top = [k, r^\top]$. A corresponding notation as for (3) is used in (10) and (11).

Lemma 1. Suppose Assumption 3 holds true. a) Let $\{x(k)\}_{k \in \mathbb{N}}$ and $\{r(k)\}_{k \in \mathbb{N}}$ be sequences such that $\lim_{k \rightarrow \infty} r(k) = 0$ and such that problem (11) is feasible for every time instant $k \in \mathbb{N}$ with $x = x(k), r = r(k)$. Then the value $V_3(x(k), p_3(k))$ of problem (11) is given by $V_3(x(k), p_3(k)) = x(k)^\top S_3(k)x(k) + x(k)^\top S_4(k)r(k) + r(k)^\top S_5(k)r(k) \geq 0$ for some matrices $S_3(k), S_4(k), S_5(k)$ and the solution of (11) is unique and linearly parameterized in $x(k), r(k)$ in the sense of $\xi_i(x(k), p_3(k)) = K_{1,i}(k)x(k) + K_{2,i}(k)r(k)$ and $v_i(x(k), p_3(k)) = K_{3,i}(k)x(k) + K_{4,i}(k)r(k)$, for all $i = 0, \dots, N$. b) Further, the value function $V_2(x(k), p_2(k))$ of problem (10) is quadratic and positive semidefinite in x , i.e. $V_2(x(k), p_2(k)) = x(k)^\top S_6(k)x(k) \geq 0$ for some matrix $S_6(k)$, and there exists a solution $\{\xi_i(x(k), p_2(k))\}_{i=0}^N, \{v_i(x(k), p_2(k))\}_{i=0}^{N-1}$ of (11) which is linearly parameterized in $x(k)$ in the sense of a).

Assumption 5. a) Let $\{r(k)\}_{k \in \mathbb{N}}$ be a sequence which converges to zero. We assume that the solutions of (11) along the closed loop (11), (2) are uniformly bounded, i.e. $\xi_i(x(k), p_3(k)) = K_{1,i}(k)x(k) + K_{2,i}(k)r(k)$, $v_i(x(k), p_3(k)) = K_{3,i}(k)x(k) + K_{4,i}(k)r(k)$ are bounded in the sense that there exists a bound $M > 0$ such that for all $i = 0, \dots, N$, $j = 1 \dots 4$, $k \in \mathbb{N}$ it holds $\|K_{j,i}(k)\| \leq M$. b) We assume that the solutions of (3) along the closed loop (3), (2) are uniformly bounded, i.e. $x_i(x(k), p_1(k)) = K_{5,i}(k)x(k)$, $u_i(x(k), p_1(k)) = K_{6,i}(k)x(k)$ are bounded in the sense that there exists a bound $M > 0$ such that for all $i = 0, \dots, N$, $j = 5, 6$, $k \in \mathbb{N}$ it holds $\|K_{j,i}(k)\| \leq M$.

The main result of this subsection is Theorem 1, which builds on the following two lemmas.

Lemma 2. Consider the closed loop (11), (2) and suppose problem (11) is feasible for every time instant $k \in \mathbb{N}$. Let Assumption 3 and 4(a)(b) hold true and let $\{p_3(k)\}_{k \in \mathbb{N}}$ be a sequence such that $\{r(k)\}_{k \in \mathbb{N}}$ converges to zero. Then for any initial state $x(0)$, the state $x(k)$ and the input $u(k)$ of the closed loop converge to zero, i.e. $\lim_{k \rightarrow \infty} x(k) = 0$, $\lim_{k \rightarrow \infty} u(k) = 0$, assuming that Assumption 5 a) holds true.

Lemma 3. Consider (3) and suppose Assumption 4(a) holds true. Further, suppose $\Gamma: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that for all $x \in \mathbb{R}^n$, all $k \in \mathbb{N}$ and all $\varepsilon > 0$ it holds that $\Gamma(x) \geq c(\sum_{i=0}^{N-1} \|\xi_i(x, p_2)\|^2 + \|v_i(x, p_2)\|^2)$ for some $c > 0$, where $p_2 = k$ and $\{\xi_i(x, p_2)\}_{i=0}^{N-1}, \{v_i(x, p_2)\}_{i=0}^{N-1}$ is some solution of (10). Then there exists a $\rho > 0$ such that solution $x_N(x, p_1)$, $p_1^\top = [k, \varepsilon]$, of (3) satisfies for all $x \in \mathbb{R}^n$, all $k \in \mathbb{N}$ and all $\varepsilon > 0$

$$x_N(x, p_1)^\top Q_N x_N(x, p_1) \leq V_2(x, p_2) + \varepsilon \rho. \tag{12}$$

Theorem 1. Consider the closed loop (3) and (2), where $\Gamma(x) = \alpha x^\top x$ for some $\alpha > 0$ and suppose Assumption 2, 3 and 4(a)-(c) hold true. Let further $\{p_1(k)\}_{k \in \mathbb{N}}$ be a sequence such that $\{\varepsilon(k)\}_{k \in \mathbb{N}}$, $\varepsilon(k) > 0$, converges to zero. Then for any initial state $x(0)$, the state $x(k)$ of the closed loop converges to zero as k goes to infinity, assuming that Assumption 5 b) holds true.

Assumption 5 is our main technical assumption that we impose in the proposed scheme. For example, if the predictor maps P_N are controllable in the sense that $\text{rank}[B_0(k), \dots, B_{N-1}(k)] = n$ for all $k \in \mathbb{N}$ and also the limiting predictor has the same property, i.e. $\text{rank}[\hat{B}_0, \dots, \hat{B}_{N-1}] = n$, then Assumption 5 holds true (as discussed in the proof of Lemma 1). However, we admit that this assumption is in general difficult to verify.

IV. A PROXIMITY-BASED ESTIMATION SCHEME

A. Problem Setup

Consider an output sequence (or some observed signal) and an input sequence

$$\{y(k)\}_{k \in \mathbb{N}}, \{v(k)\}_{k \in \mathbb{N}} \tag{13}$$

with $y(k) \in \mathbb{R}^{\bar{p}_y}$, $v(k) \in \mathbb{R}^{\bar{q}_v}$. Let

$$\begin{aligned} x(k) &= \phi_y(y(k), \dots, y(k - \bar{N}_y + 1)) \in \mathbb{R}^{\bar{n}}, \\ u(k) &= \phi_v(v(k), \dots, v(k - \bar{N}_v + 1)) \in \mathbb{R}^{\bar{q}} \end{aligned} \quad (14)$$

and $\phi_y : \mathbb{R}^{\bar{p}_y \bar{N}_y} \rightarrow \mathbb{R}^{\bar{n}}$, $\phi_v : \mathbb{R}^{\bar{q}_v \bar{N}_v} \rightarrow \mathbb{R}^{\bar{q}}$ be some given basis (lifting) functions, e.g. $\phi_y(y_1(k), y_2(k), y_1(k-1), y_2(k-1)) = [y_1(k), y_2(k), y_1(k-1), y_2(k-1), y_1(k)y_2(k), y_1(k)^2, y_2(k)^2]^T$, $\bar{p} = 2, \bar{n} = 6, \bar{N}_y = 2$. Notice that in principle one could also consider cross-terms between input and output data, like $u(k)y(k)^3$, but such terms are not considered here for the sake of simplicity. Consider, further, at time instant k the optimization problem

$$\begin{aligned} \theta^*(k) &= \arg \min_{\theta} c(e, k) + D(\theta, \tilde{\theta}(k-1)) \\ \text{s.t.} \quad & s(k) - R(k)\theta = e \\ \tilde{\theta}(k) &= (1 - \lambda_k)\theta^*(k) + \lambda_k\tilde{\theta}(k-1) \end{aligned} \quad (15)$$

with $\lambda_k \in [0, \lambda_{\max})$, $\lambda_{\max} \in (0, 1)$, where $\bar{N} \in \mathbb{N}$, $c : \mathbb{R}^{\bar{n}\bar{N}} \times \mathbb{N} \rightarrow \mathbb{R}$ and $D(x, y) = g(x) - g(y) - (x - y)^\top \nabla_y g(y)$ defines the Bregman distance induced by a function $g : \mathbb{R}^{\bar{n}\bar{N}} \rightarrow \mathbb{R}$. The vector $s(k)$ is defined as

$$s(k) = [x(k)^\top \dots x(k - \bar{N} + 1)^\top]^\top \quad (16)$$

and the matrix $R(k)$ is defined as

$$R(k) = \begin{bmatrix} x(k-1)^\top \otimes I & u(k-1)^\top \otimes I \\ \vdots & \vdots \\ x(k-\bar{N})^\top \otimes I & u(k-\bar{N})^\top \otimes I \end{bmatrix}. \quad (17)$$

Decision variables are the parameter vector $\theta \in \mathbb{R}^{\bar{n}(\bar{n}+\bar{q})}$ and e . We refer to θ when using argmin since the (slack) variables e can be eliminated and have been introduced just for the notational convenience. Also we define in the following $y(k) = 0, v(k) = 0, v_k = 0$ etc. whenever $k < 0$. We impose now the following assumptions.

Assumption 6. *The objective function c in (15) is continuously differentiable and strictly convex in the first argument and it satisfies for all k and $e \neq 0$: $c(e, k) > c(0, k)$. Further, the function g , which defines the Bregman distance D , is continuously differentiable and strictly convex.*

Assumption 7. *For the given sequences in (13) and given $x(k) = \phi_y(y(k), \dots, y(k - \bar{N}_y + 1))$ and $u(k) = \phi_v(v(k), \dots, v(k - \bar{N}_v + 1))$ in (14), there exist matrices A, B and $x_0 \in \mathbb{R}^{\bar{n}}$ that satisfy*

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0. \quad (18)$$

Remark 2. *a) Notice that $s(k) = R(k)\theta$ with $\theta^\top = [\text{vec}(A)^\top, \text{vec}(B)^\top]$, where $\text{vec}(A)$ corresponds to the (column-wise) vectorization of a matrix A , is the linear system of equations $x(j) = Ax(j-1) + Bu(j-1)$, $j = k \dots k - \bar{N} + 1$. b) If $c(e, k) = \|e\|^2, g(x) = \|x\|^2$, then (17) reduces to a least squares parameter estimation problem, where a closed form solution to it is known. The motivation for a general convex cost is its flexibility in tuning the estimator. Similarly as in a recently proposed state estimation scheme based on proximal minimization [32], specifying different c, D allows to take into account various aspects like outliers in the data, sparsity in the parameters or cost-biased objectives [11].*

Remark 3. *Assumption 7 imposes that the given (lifted) signal $\{x(k)\}_{k \in \mathbb{N}}$ can be reproduced by some linear time-invariant system that is driven by the given (lifted) input sequence $\{u(k)\}_{k \in \mathbb{N}}$. Notice that reproducing a given signal by (18) does not imply that the signal $x(k)$ itself originates from (18) nor by a linear time-invariant system at all. For example, a given (single) trajectory of a nonlinear system or even all trajectories of a large class of nonlinear systems can be reproduced by or embedded into high dimensional linear (not necessarily controllable) systems using for example Carleman or Koopman lifting techniques. Hence, (18) represents a signal model of the actual closed-loop trajectory, rather than a system model of all possible trajectories of the plant. A signal model is therefore a parsimonious modeling approach in the sense that it aims to predict nothing more than the closed-loop trajectory.*

The main goal of the next subsection is to show that the (parameter) estimates $\tilde{\theta}(k)$ ($\theta^*(k)$) obtained from (15) converge and that the estimates can be used to define i th step-ahead signal predictor maps $v_i = P_i(k, v_0, u_0, \dots, u_{i-1})$ which have the properties as described in Assumption 4 for the signal model (18) and for the given data (14). The convex combination in (15) is introduced to resolve the loss of stabilizability problem (see next subsection). In summary, we therefore aim for an estimation scheme to obtain asymptotically accurate predictor maps for a given input and output sequence that can be embedded in a potentially high-dimensional linear signal model. An important property of the proposed scheme is that no system identification is carried out and no persistency of excitation condition is needed. The online computational burden is again rather low since, in its simplest form, the problem boils down to a least-squares regression problem.

B. Results

We first prove a lemma which is a key step for the convergence of the estimation scheme. It provides a convergence result for a proximal minimization scheme with a time-varying objective function which has at least one common (time-invariant) minimizer.

Lemma 4. *Let $f : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}$ be convex and continuous differentiable in the first argument. Suppose the set of minimizers $\mathcal{X}_k = \{x_k^* \in \mathbb{R}^n : f(x, k) \geq f(x_k^*, k) := 0 \forall x \in \mathbb{R}^n\}$ of f at any time instant k is nonempty and also their intersections*

$$\mathcal{X} = \bigcap_{k=0}^{\infty} \mathcal{X}_k \neq \emptyset, \quad (19)$$

i.e. there exists a common (time-invariant) minimizer $x^ \in \mathcal{X}$ which minimizes f for any k with a common minimum value zero. Let further $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be strictly convex, continuous differentiable, $\lambda_{\max} \in [0, 1)$, and let D denotes the Bregman distance induced by g . Assume that D is convex in the second argument, then the proximal minimization iterations x_k^*, \tilde{x}_k given by*

$$\begin{aligned} x_{k+1}^* &= \arg \min_x f(x, k) + D(x, \tilde{x}_k) \\ \tilde{x}_{k+1} &= (1 - \lambda_{k+1})x_{k+1}^* + \lambda_{k+1}\tilde{x}_k \end{aligned} \quad (20)$$

with $\lambda_{k+1} \in [0, \lambda_{\max}]$, converge to a point in \mathcal{X} , i.e. $\lim_{k \rightarrow \infty} x_k^ = \lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}^0 \in \mathcal{X}$.*

Notice that the Bregman distance is in general not convex in the second argument, but there are important cases, such as $g(x) = x^T Q x$, $Q > 0$, where this holds [33]. Notice further that in a classical proximal minimization scheme $\tilde{x}_{k+1} = x_{k+1}^*$ ($\lambda_{k+1} = 0$). Here, we pick $\tilde{x}_{k+1} = (1 - \lambda_{k+1})x_{k+1}^* + \lambda_{k+1}\tilde{x}_k$ instead of x_{k+1}^* as the next iterate, because with appropriately chosen λ_{k+1} 's, the loss of stabilizability problem in the estimation scheme (15) can be avoided and it can be guaranteed that every estimated signal model is controllable, if the initial model is controllable, as shown next.

Lemma 5. *Consider $A_c, A_u \in \mathbb{R}^{n \times n}$, $B_c, B_u \in \mathbb{R}^{n \times q}$, $q \leq n$, and let (A_c, B_c) be controllable and (A_u, B_u) be not controllable. Then for any $\lambda_{\max} \in (0, 1)$, there exists a $\lambda \in (0, \lambda_{\max})$ such that $(A(\lambda), B(\lambda))$ with $A(\lambda) = (1 - \lambda)A_u + \lambda A_c$, $B(\lambda) = (1 - \lambda)B_u + \lambda B_c$ is controllable. In particular, take some λ_j 's with $0 < \lambda_1 < \dots < \lambda_{2n^2+1} < \lambda_{\max}$, then there exists an $i \in \{1, \dots, 2n^2 + 1\}$ such that $(A(\lambda_i), B(\lambda_i))$ is controllable.*

The next theorem is the main result of this subsection.

Theorem 2. *Consider the sequences (13) with some given basis functions (14) and consider the optimization problem (15). Suppose Assumption 6 and 7 hold true and assume that D is convex in the second argument. Then the following statements hold true.*

- (i) *The solution sequence $\{\tilde{\theta}(k)\}_{k \in \mathbb{N}}$ converges, i.e. $\lim_{k \rightarrow \infty} \tilde{\theta}(k) = \theta^*$.*
- (ii) *If one defines $\tilde{\theta}(k)^\top = [\text{vec}(A(k))^\top, \text{vec}(B(k))^\top]$ and predictor maps (5) according to*

$$\begin{aligned} A_i(k) &= A(k)^i \\ B_i(k) &= A(k)^i B(k), \end{aligned} \quad (21)$$

$i = 0 \dots \tilde{N}$, then the predictor fulfills the properties (6), (7) and (8) in Assumption 4(a)(b) with respect to the signal model (18) and the sequences (14).

- (iii) *In addition, if the initialization $(A(0), B(0))$, $\tilde{\theta}(0)^\top = [\text{vec}(A(0))^\top, \text{vec}(B(0))^\top]$, of (15) is controllable, then there exists a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$, $\lambda_{\max} \in (0, 1)$, (constructed for example according to Lemma 5) such that for any $k \in \mathbb{N}$ the pair $(A(k), B(k))$ is controllable and hence also Assumption 4(c) is fulfilled.*

V. THE OVERALL SCHEME

In Section III, Theorem 1, we have established a control scheme which drives the state of the linear system (2) to zero assuming that the system is stabilizable and that state measurements as well as asymptotically accurate predictor maps are available. In Section IV, Theorem 2, we have established an estimation scheme which delivers asymptotically accurate predictor maps for any lifted signals (14) assuming that these signals can be embedded in a linear signal model of the form (18). Utilizing the estimation scheme (15) in the control scheme (3) means now that the signal model (18) replaces the system model (2) and the predictor maps in (21) are used to define the predictor (5). However, it needs to be clarified how the output and input sequence $\{y(k)\}_{k \in \mathbb{N}}$, $\{v(k)\}_{k \in \mathbb{N}}$ of (1) under Assumption 1 can be related to a signal model of the form (18) such that (A, B) is stabilizable and such that the (observable part of the) state $x(k)$ is available. This issue is addressed in the next lemma.

Lemma 6. Consider an arbitrary output and input sequence $\{y(k)\}_{k \in \mathbb{N}}$, $\{v(k)\}_{k \in \mathbb{N}}$ of system (1) and suppose Assumption 1 holds true. Let

$$\begin{aligned} x(k) &= \phi_y(y(k), \dots, y(k-m+1)) = [y(k)^\top, \dots, y(k-m+1)^\top]^\top, \\ u(k) &= \phi_v(v(k), \dots, v(k-m+1)) = [v(k)^\top, \dots, v(k-m+1)^\top]^\top \end{aligned} \quad (22)$$

with $m \geq n$. Then the sequences $\{u(k)\}_{k \in \mathbb{N}}$, $\{x(k)\}_{k \in \mathbb{N}}$ satisfy Assumption 7 with a stabilizable pair of matrices (A, B) . In addition, if the sequences $\{u(k)\}_{k \in \mathbb{N}}$, $\{x(k)\}_{k \in \mathbb{N}}$ converge to zero when k goes to infinity, then so do the sequences $\{v(k)\}_{k \in \mathbb{N}}$, $\{y(k)\}_{k \in \mathbb{N}}$.

Remark 4. Notice that the lifted input vector $u(k)$ in (22) contain past values of the actual input vector $v(k)$. In order to obtain a state space model with input $v(k)$, one can just add state variables to the signal model. In more detail, define an integrator chain dynamics of the form $\zeta_1(k+1) = \zeta_2(k), \dots, \zeta_{m-2}(k+1) = \zeta_{m-1}(k), \zeta_{m-1}(k+1) = v(k)$, hence $\zeta_1(k)$ corresponds to $v(k-m)$ etc. This augmentation does not effect the stabilizability property, since the states of the integrator chain converge to zero, if $v(k)$ converges to zero. This state augmentation in the signal model leads to matrices with at least the size $A \in \mathbb{R}^{mp+(m-1)q \times mp+(m-1)q}$, $B \in \mathbb{R}^{mp+(m-1)q \times q}$ and needs to be taken into account when implementing the receding horizon scheme.

We are now ready to close the loop. By Lemma 6, we known that the output and input sequences $\{y(k)\}_{k \in \mathbb{N}}$, $\{v(k)\}_{k \in \mathbb{N}}$ of system (1) satisfy Assumption 7 and Assumption 2 w.r.t. the signal model (18) (equation (55)). Assumption 3 and Assumption 6 can be satisfied by setting up the optimization problem accordingly. By Theorem 2, Assumption 4 holds. Hence all assumptions are satisfied and thus the receding horizon scheme guarantees, by Theorem 1 with $\Gamma(x) = \alpha x^T x$, $\alpha > 0$, a sequence $\varepsilon(k) \rightarrow 0$, and Assumption 5 b), together with Lemma 6 that the state and the input of (1) converges to zero. These arguments lead to the next theorem.

Theorem 3. Consider the closed loop system consisting of the system (1), the receding horizon control scheme (3) and the proximity-based estimation scheme (15). Define $x(k)$, $u(k)$ (ϕ_y, ϕ_v) according to equation (22) and set up the predictor scheme (15) according to Assumption 6. Further, set up the receding horizon scheme (3) according to Assumption 3 with $n = m$, $\alpha > 0$, $\Gamma(x) = \alpha x^T x$ and a sequence $\{\varepsilon(k)\}_{k \in \mathbb{N}}$, $\varepsilon(k) > 0$ that converges to zero. Then, under Assumption 1, for any initial state $z(0)$, the state $z(k)$ of the closed loop and the input $v(k)$ of the closed loop converges to zero as k goes to infinity, assuming that Assumption 5 b) holds true.

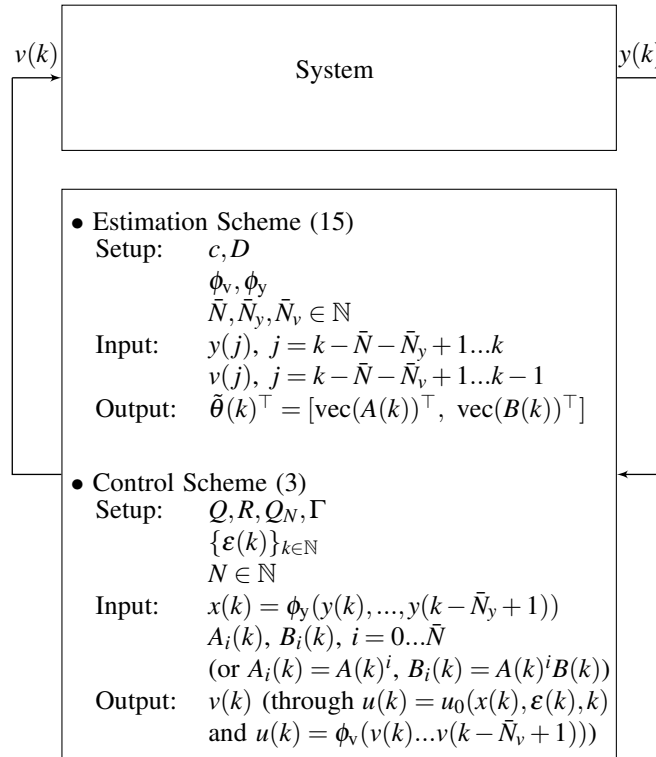


Fig. 1: The overall scheme

We conclude with some remarks. Firstly, if we postulate that the limiting signal model \hat{A}, \hat{B} is controllable, then we can drop Assumption 5 b) in Theorem 3. This is for example the case, when the signal model is controllable and when the estimates are unique (persistently excitation). Notice further that the state extension in Remark 4 has to be taken into account when implementing the overall scheme. Moreover, if ϕ_v, ϕ_y are nonlinear functions (notice that in (22) these are linear functions), then this requires further considerations (e.g. one has to be able to extract the input v from ϕ_v), which is out of the scope of this work. Secondly, notice that the convergence result of Theorem 3 also holds for certain classes of nonlinear systems. As already mention in Remark 3, certain (trajectories of) nonlinear systems can be embedded into high-dimensional linear (signal) models, e.g. $z_1(k+1) = z_1(k) + z_2(k)^2 + u(k), z_2(k+1) = 0.5z_2(k)$, which can be written as a linear system with an additional state variable $z_3(k) = z_2(k)^2$, see [34], [35], [20], [21]. Thirdly, if a model of (1) is known, then it follows from the proofs of Theorem 1 and 3 that global asymptotic stability (instead of global convergence) can be guaranteed. Moreover, in applications often models are available for at least some parts of the system. The proposed scheme can eventually be applied to such situations, for example, if we have two subsystem $x_1(k+1) = F_1x_1(k) + F_2x_2(k) + G_1u(k), x_2(k+1) = F_3x_1(k) + F_4x_2(k) + G_2u(k), y(k) = x_1(k)$, where F_3, F_4, G_2 is unknown, than one can consider the second subsystem as the unknown system with input $u(k), x_1(k)$ and adjust the receding horizon control scheme accordingly. It is also possible to model unknown disturbances or reference signals by a so-called unknown exosystem and to treat this system as a signal model [21]. Finally, a priori knowledge about the system model can be taken into account by initializing the estimator appropriately or by including parameter constraints in the estimator.

VI. SIMULATION RESULTS

In the following, we show simulation results of the proposed overall scheme for a linear and a nonlinear system.

A. Linear system

We consider the system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 1 & 0.1 \\ 0 & 1.02 & 0 \\ 0 & 0 & 0.92 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= [1 \quad 0 \quad 1] x(k). \end{aligned} \quad (23)$$

The system is unstable, stabilizable and observable. For the estimation scheme, we choose $c(e) = \|e\|^2$, $g(x) = \|x\|^2$ ($D(x, y) = \|x - y\|^2$), $\bar{N} = 8$, $\bar{N}_y = 4$, $\bar{N}_v = 4$, $\phi_v(v(k), \dots, v(k-3)) = [v(k), \dots, v(k-3)]^\top$, $\phi_y(y(k), \dots, y(k-3)) = [y(k), \dots, y(k-3)]^\top$. Further, we initialize the estimator with a controllable signal model. In this example, we assumed that we know that the system order is at most four, i.e. $m = 4$, hence we have chosen $\bar{N}_y = 4 = \bar{N}_v$. For the control scheme, we choose $Q = 100I$, $R = 10000I$, $Q_N = 100I$, $\Gamma(x) = x^\top x$, $\varepsilon(k) = \frac{1}{1+1000k}$, $N = 20$. Some simulation results for the initial condition to be $x(0) = [0.1 \quad 0.1 \quad -10]^\top$ are shown in the Figures 2 to 6. It can be verified that the state converges to zero (see Figure 5 for evolution of the state $x_1(k)$).

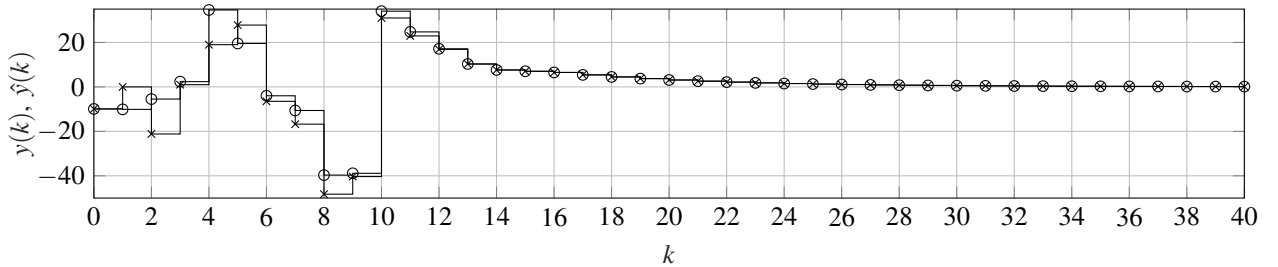


Fig. 2: Evolution of the true system output $y(k)$ ($-o-$) and the estimated system output $\hat{y}(k)$ ($-x-$)

B. Nonlinear system

In the following, we show the performance of the proposed algorithm for a single-link robot arm with a DC motor [36]. After an Euler-forward discretization of the model equations with step-size $h = 0.01$, we obtain the system

$$\begin{aligned} x_1(k+1) &= x_1(k) + hx_2(k) \\ x_2(k+1) &= x_2(k) + h(36.4x_3(k) - 1.7x_2(k) - 1309\sin(x_1(k))) \\ x_3(k+1) &= x_3(k) + h(-1000x_3(k) - 3.6x_2(k) + 100u(k)) \end{aligned} \quad (24)$$

where the system state $x = [x_1, x_2, x_3]^\top$ characterizes the position, velocity of the robot and the current of the DC motor. We assume that the state can be measured, i.e. $y(k) = x(k)$. The control input u correspond to the input voltage

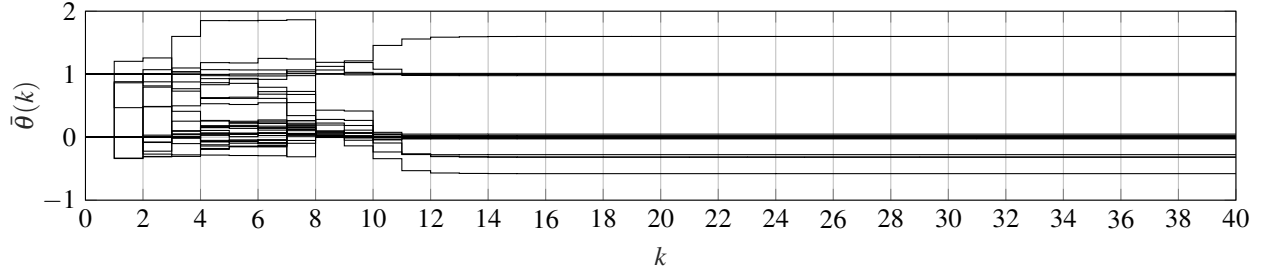


Fig. 3: Evolution of the estimates $\bar{\theta}(k)$

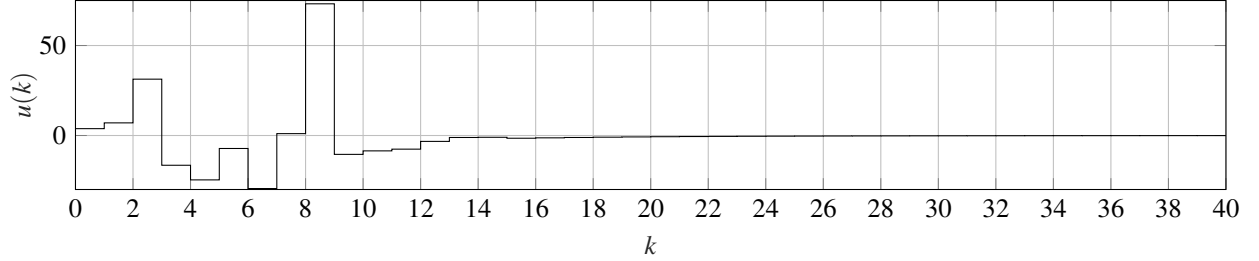


Fig. 4: Evolution of the control input $u(k)$

of the DC motor. For the estimation scheme, we choose $c(e) = \|e\|^2$, $g(x) = \|x\|^2$ ($D(x, y) = \|x - y\|^2$), $\bar{N} = 10$, $\bar{N}_y = 2$, $\bar{N}_v = 2$, $\phi_v(v(k), v(k-1)) = [v(k), v(k-1)]^\top$, $\phi_y(y(k), y(k-1)) = [y(k)^\top, y(k-1)^\top]^\top$. Further, we initialize the estimator with a controllable signal model. For the control scheme, we choose $Q = 10I$, $R = 100I$, $Q_N = 10I$, $\Gamma(x) = x^T x$, $\varepsilon(k) = \frac{1}{1+10k}$, $N = 15$. The state evolution for the initial condition $x_0 = [5, -5, 1]^\top$ are depicted in Figure 7 to 9.

Overall, our simulation experiments on various examples show that the proposed approach performs well in many cases. However, as also known from adaptive control [37], the state trajectories of unstable systems often show quite a strong peaking behavior. This is to some extent a fundamental limitation when controlling unknown unstable systems but more research on this issue is necessary.

VII. CONCLUSION AND OUTLOOK

Motivation of this research was to develop a basic online optimization-based approach that guarantees convergence for a prototypical problem from adaptive control and that may serve as a basis for other online optimization-based (model-free) learning schemes. To this end, a receding horizon learning scheme consisting of a receding horizon control scheme and a proximity-based estimation scheme was proposed. For unknown linear time-invariant systems, zero state convergence was proven under rather minimal assumptions. The motivation to consider linear time-invariant systems stems not only from the fact that they define an important and tractable benchmark class but that they also serve as a measure for the *local* stabilization of unknown nonlinear systems. Since the proposed approach relies on predictor maps, it can be considered as an indirect adaptive optimal control method and thus stands in contrast to direct adaptive optimal control methods such as reinforcement learning. From a conceptual point of view, the main ideas and results of this work were a time-varying model-independent terminal state weighting in the receding horizon control scheme which does not rely on a controllability assumption (Section III), a convergent proximal estimation scheme that estimates controllable signal models for predicting the closed loop trajectory (Section IV) as well as a proper combination of the control and estimation scheme to achieve guaranteed zero state convergence for completely unknown linear system (Section V).

There are several interesting points of future research. One important question is to investigate under which conditions Assumption 5 b) holds true. Related to this question is the issue whether or not the limiting estimates correctly identify the excited controllable modes of the unknown system and how fast $\varepsilon(k)$ converges to zero relative to the uncontrollable modes of the system. Another interesting issue are constraints. Indeed, constraints satisfaction is impossible without additional assumptions, but the satisfaction of polytopic input and output constraints, as time goes to infinity (under some constraint tightening or the use (relaxed-)barrier functions similarly to [38]), should be feasible. In particular, such constraints would lead to piecewise quadratic respectively strongly convex value functions of the underlying optimization problems (see Lemma 1) and hence it seems reasonable that similar proof arguments as in this work are applicable. As mentioned above, shaping the transient behavior is a key challenge and a well known problem from adaptive control. The proposed optimization-based formulation allows in principle to specify objectives, constraints (e.g. saturation functions to reduce peaking) and possibly a prior information about the system model in order to improve the transient behavior [37]. Other important research directions

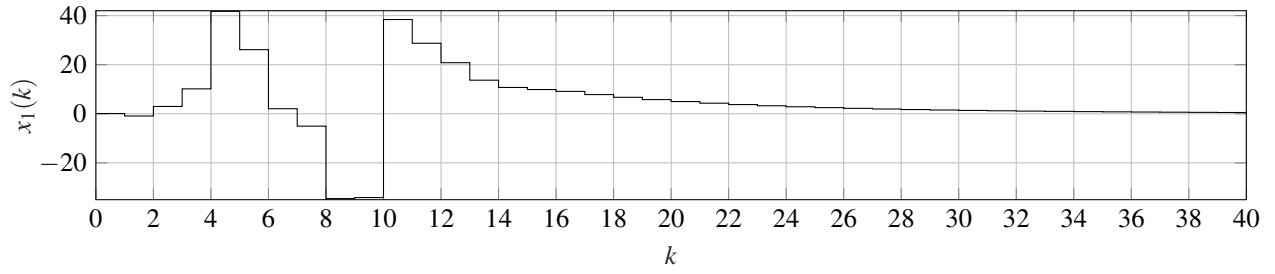


Fig. 5: Evolution of the system state $x_1(k)$

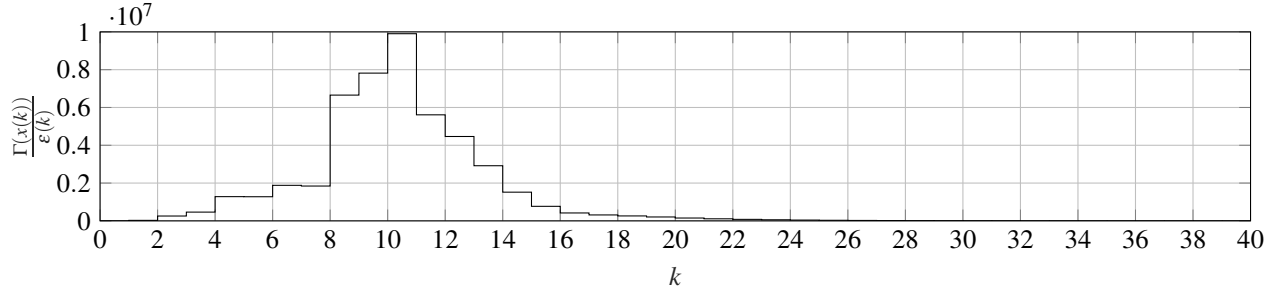


Fig. 6: Evolution of $\Gamma(x(k))/\varepsilon(k)$

are i) to address robustness and uncertainties, for example by using tube techniques [39] or techniques from robust control [40] or ii) to fit the input-output data not to (lifted) linear signal models but to nonlinear models such as neural networks or iii) to study the approach (i.e. the regression) in a Bayesian context. Moreover, it would be interesting to replace in the receding horizon control scheme the time-varying terminal state weighting with an adaptive one, in the sense that, for instance, the weight increases only if there is no decrease in the value function after two consecutive time steps. Finally, despite the involved online optimization is computationally not very demanding, it is important to take real-time aspects into account and to develop anytime iteration schemes or to exploit dual (kernel) formulations of the underlying regression problems.

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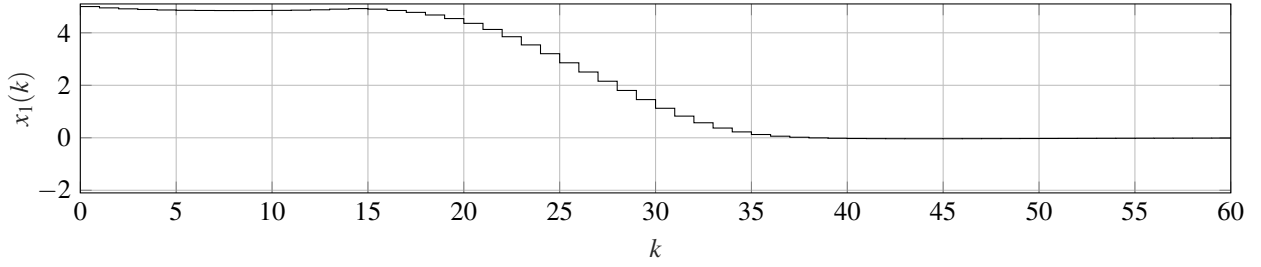


Fig. 7: Evolution of the system state $x_1(k)$

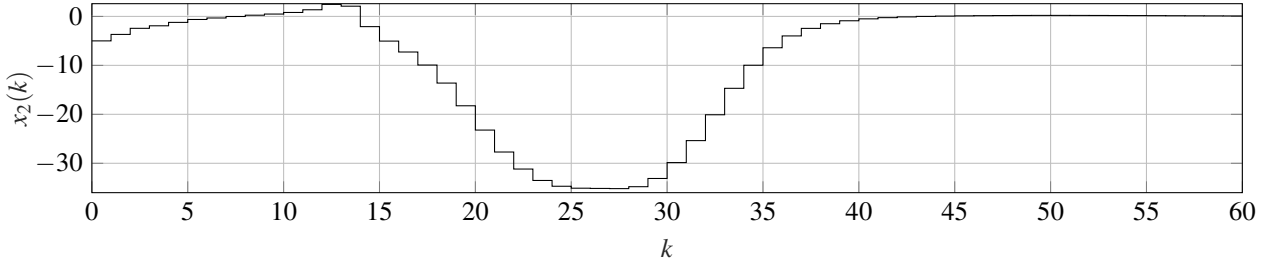


Fig. 8: Evolution of the system state $x_2(k)$

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APPENDIX

A. Proof of Lemma 1 and Remark to Assumption 5

Case a) Notice that (11) can be written as minimize $w^\top H w$ subject to $C(k)w = B(k)q(k)$ where $q(k)^\top = [x(k)^\top \ r(k)^\top]$, $w^\top = [v_0^\top, \xi_0^\top, \dots, v_{N-1}^\top, \xi_N^\top]$. Since ξ_N is given by $r(k)$, it is easy to see from (11) and Assumption 3 that the solution

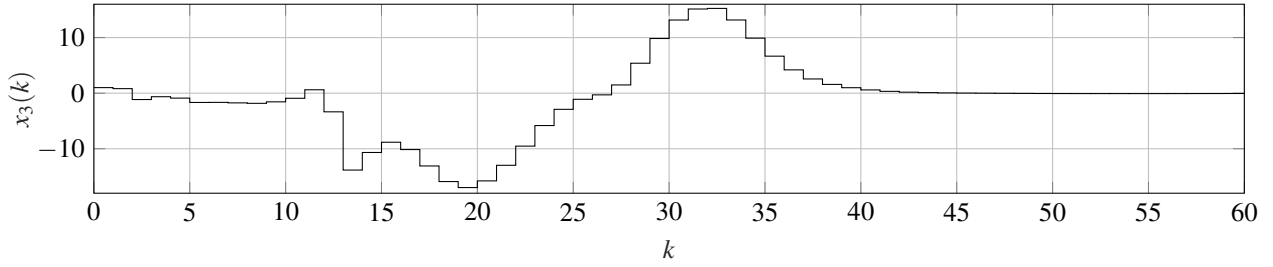


Fig. 9: Evolution of the system state $x_3(k)$

of (11) is unique. Further, since the solution is zero for $x(k) = 0, r(k) = 0$, we can write the solution as $\xi_i(x(k), p_3(k)) = K_{1,i}(k)x(k) + K_{2,i}(k)r(k)$ and $v_i(x(k), p_3(k)) = K_{3,i}(k)x(k) + K_{4,i}(k)r(k)$ and the value as $V_3(x(k), p_3(k)) = x(k)^\top S_3(k)x(k) + x(k)^\top S_4(k)r(k) + r(k)^\top S_5(k)r(k) \geq 0$ for all k because $Q > 0, R > 0$.

Remark to Assumption 5. Notice that if $\text{rank}[B_0(k), \dots, B_{N-1}(k)] = n$ for all $k \in \mathbb{N}$ and also $\text{rank}[\hat{B}_0, \dots, \hat{B}_{N-1}] = n$, then $C(k)$ and $\lim_{k \rightarrow \infty} C(k)$ have full column rank, then problem (11) is feasible for any $x(k), r(k) \in \mathbb{R}^n$. Hence, $\tilde{w}(k) = (C(k)^\top C(k))^{-1} C(k)^\top B(k)q(k) = \tilde{K}(k)q(k)$ is a suboptimal (feasible) solution of problem (11) (i.e. minimize $w^\top H w$ subject to $C(k)w = B(k)q(k)$). Since $\tilde{w}(k)$ is suboptimal, $W_3(x(k), p_3(k), \tilde{w}(k)) := x(k)^\top \tilde{S}_3(k)x(k) + x(k)^\top \tilde{S}_4(k)r(k) + r(k)^\top \tilde{S}_5(k)r(k)$ is an upper bound of the value function. Further, the limits of $\{B(k)\}_{k \in \mathbb{N}}$ and $\{(C(k)^\top C(k))^{-1}\}_{k \in \mathbb{N}}$ exist, hence also the limit of $\lim_{k \rightarrow \infty} \tilde{K}(k)$ exists. Thus the sequence $\{\tilde{K}(k)\}_{k \in \mathbb{N}}$ is bounded. Hence there exists a $M > 0$ such that $\|\tilde{S}_i(k)\| \leq M$, $i = 3, 4, 5$, $\|\tilde{K}_{j,i}(k)\| \leq M$, $i = 0 \dots N, j = 1 \dots 4, k \in \mathbb{N}$. Since $W_3(x(k), p_3(k), \tilde{w}(k)) \geq V_3(x(k), p_3(k))$ for all $k, x(k), r(k)$, it follows that $\|S_i(k)\|, i = 3, 4, 5, \|K_{j,i}(k)\|, i = 0 \dots N, j = 1 \dots 4$ are uniformly bounded and thus Assumption 5 a) holds true.

Case b) Notice that the solution of (10) is not unique. Thus we consider the least norm solution of (10), i.e. let $\{\tilde{v}_i(x(k), p_2(k))\}_{i=0}^{N-1}, \{\tilde{\xi}_i(x(k), p_2(k))\}_{i=0}^{N-1}$ be any solution of (10). Then the unique solution $\{v_i(x(k), p_3(k))\}_{i=0}^{N-1}, \{\xi_i(x(k), p_3(k))\}_{i=0}^{N-1}$ of (11) with the constraint $r(k) = \xi_N = \tilde{\xi}_N(x(k), p_2(k))$ and $R = Q = I$ is a solution (i.e. the least norm solution) of (10). Hence the arguments in Case a) can be applied. \square

B. Proof of Lemma 2

Since $Q > 0$, there exists a $\alpha > 0$ such that $V_3(x, p_3) \geq \alpha x^\top x$ for any p_3 , i.e. V_3 is positive definite and radially unbounded. For the sake of convenience, we use the notation $x := x(k), x^+ := x(k+1), p_3 := p_3(k), p_3^+ := p_3(k+1), r = r(k), r^+ := r(k+1)$. In the following, we consider the Lyapunov increment

$$\begin{aligned} V_3(x^+, p_3^+) - V_3(x, p_3) &= V_3(x^+, p_3^+) - V_3(x^+, \tilde{p}_3^+) \\ &\quad + V_3(x^+, \tilde{p}_3^+) - V_3(x, p_3), \end{aligned} \quad (25)$$

where $\tilde{p}_3^+ = [k+1, \tilde{r}^+]$ is defined below.

Step 1. We first consider the term $V_3(x^+, \tilde{p}_3^+) - V_3(x, p_3)$. Let $\{v_i(x, p_3)\}_{i=0}^{N-1}$ be the solution of (11) which we denote in the following by $\{v_i\}_{i=0}^{N-1}$ and let $\{\xi_i(x, p_3)\}_{i=0}^N$ be the corresponding predicted states. Define

$$\begin{aligned} \tilde{r}^+ &= A_N(k+1)x^+ + \sum_{l=0}^{N-1} B_{N-1-l}(k+1)v_{l+1} \\ &= P_N(k+1, x^+, v_1, \dots, v_N), \end{aligned} \quad (26)$$

hence \tilde{r}^+ is the predicted terminal state at time instant $k+1$ using the state predictor matrices at time $k+1$ and the input sequence $v_1, \dots, v_{N-1}, v_N := v_N(x, p_3) := 0$ obtained at time k . Hence, $v_1, \dots, v_{N-1}, 0$ is, by construction, feasible for the problem (11) at time $k+1$ with $(\tilde{p}_3^+)^\top = [k+1, (\tilde{r}^+)^\top]$. Notice that by (8) it holds $x^+ = Ax + Bv_0 = A_1(k)x + B_0(k)v_0 - e_0(k)$. Thus, we have for $i \leq N-1$

$$\begin{aligned} \tilde{\xi}_i &:= \tilde{\xi}_i(x^+, \tilde{p}_3^+) = P_i(k+1, P_1(k, x, v_0) - e_0(k), v_1, \dots, v_i) \\ &= P_i(k, P_1(k, x, v_0), v_1, \dots, v_i) + P_i(k+1, -e_0(k), 0, \dots, 0) \\ &\quad + P_i(k+1, P_1(k, x, v_0), v_1, \dots, v_i) - P_i(k, P_1(k, x, v_0), v_1, \dots, v_i) \\ &= \xi_{i+1}(x, p_3) + w_i(x, p_3) \end{aligned} \quad (27)$$

with $w_i(x, p_3) = P_i(k+1, -e_0(k), 0, \dots, 0) + P_i(k+1, P_1(k, x, v_0), v_1, \dots, v_i) - P_i(k, P_1(k, x, v_0), v_1, \dots, v_i)$ and where we used

linearity and commutativity of P_i (Assumption 4 a) and b)). Specifically, for w_i we have

$$\begin{aligned} w_i(x, p_3) &= -A_i(k+1)e_0(k) \\ &+ (A_i(k+1) - A_i(k))(A_1(k)x + B_0(k)v_0) \\ &+ \sum_{l=1}^i (B_{i-l}(k+1) - B_{i-l}(k))v_l. \end{aligned} \quad (28)$$

By Assumption 4 we can bound $w_i(x, p_3)$ for $k \rightarrow \infty$ as follows. Step i): Let $Q(k)$ be positive definite and uniformly bounded in k by $x^\top Q(k)x \leq \lambda x^\top x$ for all $x, k \in \mathbb{N}$, then it follows from the Cauchy-Schwarz's and Young's inequality, that for any $\rho > 0$ we have

$$x^\top Q(k)y \leq \lambda \left(\frac{\rho}{2} x^\top x + \frac{1}{2\rho} y^\top y \right). \quad (29)$$

Step ii) Using the fact that $v_i(x, p_3) = K_{3,i}(k)x + K_{4,i}(k)r$ with bounded $A_i(k), B_i(k)$ and $K_{3,i}(k), K_{4,i}(k)$ (see Lemma 1 and Assumption 5 a)) and using that $\|e_0(k)\|^2 \leq \omega_1(k) + \omega_2(k)\|x(k)\|^2 + \omega_3(k)\|v_0\|^2$ we obtain (after some elementary calculations) the bound

$$\|w_i(x, p_3)\|^2 \leq \alpha_{i,1}(k)x^\top x + \alpha_{i,2}(k)r^\top r + \alpha_{i,3}(k) \quad (30)$$

with $\lim_{k \rightarrow \infty} \alpha_{i,j}(k) = 0$ for $j = 1, 2, 3$, $i = 0 \dots N-1$. Consider now

$$\begin{aligned} V_3(x, p_3) &= \sum_{i=0}^{N-1} \xi_i(x, p_3)^\top Q \xi_i(x, p_3) + v_i(x, p_3)^\top R v_i(x, p_3) \\ &V_3(x^+, \tilde{p}_3^+, v_1(x, r), \dots, v_{N-1}(x, p_3), 0) = \\ &\sum_{i=0}^{N-1} \tilde{\xi}_i^\top Q \tilde{\xi}_i + \sum_{i=1}^{N-1} v_i(x, p_3)^\top R v_i(x, p_3) \\ &= \sum_{i=1}^N (\xi_i(x, p_3) + w_{i-1}(x, p_3))^\top Q (\xi_i(x, p_3) + w_{i-1}(x, p_3)) \\ &\quad + \sum_{i=1}^{N-1} v_i(x, p_3)^\top R v_i(x, p_3). \end{aligned} \quad (31)$$

Due to optimality, we have the inequality

$$\begin{aligned} &V_3(x^+, \tilde{p}_3^+) - V_3(x, p_3) \\ &\leq V_3(x^+, \tilde{p}_3^+, v_1(x, p_3), \dots, v_{N-1}(x, p_3), 0) - V_3(x, p_3) \\ &= -\xi_0(x, p_3)^\top Q \xi_0(x, p_3) - v_0(x, p_3)^\top R v_0(x, p_3) \\ &\quad + (\xi_N(x, p_3) + w_{N-1}(x, p_3))^\top Q (\xi_N(x, p_3) + w_{N-1}(x, p_3)) \\ &\quad + \sum_{i=1}^{N-1} w_{i-1}(x, p_3)^\top Q w_{i-1}(x, p_3) + 2\xi_i(x, p_3)^\top Q w_{i-1}(x, p_3). \end{aligned} \quad (32)$$

Notice that $\xi_0(x, p_3) = x$ and $\xi_N(x, p_3) = r$. Similarly as above, i.e. using (30) and (29) and exploiting the fact that $\xi_i(x, p_3)$ is linear in x, r and with bounded gains $K_{1,i}(k), K_{2,i}(k)$ (Assumption 5, a)), we can upper bound the expressions in the last two lines of equation (32) by $(\beta_1(k) + \beta_4)x^\top x + (\beta_2(k) + \beta_5)r^\top r + \beta_3(k)$ with $\lim_{k \rightarrow \infty} \beta_j(k) = 0$ for $j = 1, 2, 3$ and where $\beta_4 > 0, \beta_5 > 0$ can be chosen arbitrarily small. In more detail, we have for example for the expression $\xi_i Q w_{i-1}$ the bound $\xi_i Q w_{i-1} \leq \frac{\tilde{\beta}_4}{2} \xi_i Q \xi_i + \frac{1}{2\tilde{\beta}_4} w_{i-1} Q w_{i-1}$ with $\tilde{\beta}_4$ arbitrarily small. Further $\xi_i(x, p_3)$ is linear in x, r and w_{i-1} obeys (30), which yields a bound of type $(\beta_1(k) + \beta_4)x^\top x + (\beta_2(k) + \beta_5)r^\top r + \beta_3(k)$ for this expression. Applying these arguments to each expression and summing up the obtained bounds we obtain

$$\begin{aligned} V_3(x^+, \tilde{p}_3^+) - V_3(x, p_3) &\leq -x^\top Q x - v_0(x, p_3)^\top R v_0(x, p_3) \\ &\quad + (\beta_1(k) + \beta_4)x^\top x + (\beta_2(k) + \beta_5)r^\top r + \beta_3(k), \end{aligned} \quad (33)$$

where β_4, β_5 can be chosen arbitrarily small.

Step 2. We next consider $V_3(x^+, \tilde{p}_3^+) - V_3(x^+, \tilde{p}_3^+)$. Firstly, notice that from (26) and (27), we have $\tilde{r}^+ = \tilde{\xi}_N = P_N(k, P_1(k, x, v_0), v_1, \dots, v_N) + w_N(x, p_3) = P_1(k, P_N(k, x, v_0, \dots, v_{N-1}), v_N) + w_N(x, p_3) = P_1(k, \xi_N, v_N) + w_N(x, p_3) = P_1(k, r, 0) + w_N(x, p_3) = A_1(k)r + w_N(x, p_3)$ hence \tilde{r}^+ depends on x, p_3 and converges to zero, i.e. analogous to (30), we

have the bound $\|w_N(x, p_3)\|^2 \leq \alpha_{N,1}(k)x^\top x + \alpha_{N,2}(k)r^\top r + \alpha_{N,3}(k)$ with $\lim_{k \rightarrow \infty} \alpha_{N,j}(k) = 0$ for $j = 1, 2, 3$ and because r converges to zero, we also have

$$\|\tilde{r}^+\|^2 \leq \alpha_{N,1}(k)x^\top x + \alpha_{N,2}(k)r^\top r + \tilde{\alpha}_{N,3}(k) \quad (34)$$

with $\lim_{k \rightarrow \infty} \tilde{\alpha}_{N,3}(k) = 0$. Secondly, by Lemma 1 and Assumption 5 a), it follows $V_3(x, p_3) = x^\top S_3(k)x + x^\top S_4(k)r + r^\top S_5(k)r$ where $S_i(k)$, $i = 3, 4, 5$ are uniformly bounded for all $k \in \mathbb{N}$. Hence,

$$\begin{aligned} V_3(x^+, p_3^+) - V_3(x^+, \tilde{p}_3^+) &= (x^+)^{\top} S_4(k+1)(r^+ - \tilde{r}^+) \\ &\quad + (r^+)^{\top} S_5(k+1)r^+ - (\tilde{r}^+)^{\top} S_5(k+1)\tilde{r}^+. \end{aligned} \quad (35)$$

Since $x^+ = Ax + Bv_0$ and using again Young's inequality (29) with an arbitrary $\bar{\rho} > 0$, we get

$$\begin{aligned} &V_3(x^+, p_3^+) - V_3(x^+, \tilde{p}_3^+) \\ &\leq \frac{\bar{\rho}}{2}(Ax + Bv_0)^{\top}(Ax + Bv_0) + \gamma(x, p_3) \end{aligned} \quad (36)$$

with $\gamma(x, p_3) = (r^+)^{\top} S_5(k+1)r^+ - (\tilde{r}^+)^{\top} S_5(k+1)\tilde{r}^+ + \frac{1}{2\bar{\rho}}(r^+ - \tilde{r}^+)^{\top} S_4(k+1)^{\top} S_4(k+1)(r^+ - \tilde{r}^+)$ that is converging to zero for k to infinity. In particular, it can be bounded by $\gamma(x, p_3) \leq \gamma_1(k)x^\top x + \gamma_2(k)r^\top r$ with $\lim_{k \rightarrow \infty} \gamma_i(k) = 0$, $i = 1, 2$, which follows from $\tilde{r}^+ = r + w_N(x, p_3)$, Lemma 1, Assumption 5 a) and once more by application of (29).

Finally (again (29)), there exists $c > 0$ such that $(Ax + Bv_0)^{\top}(Ax + Bv_0) \leq 2c(x^\top x + v_0^\top v_0)$, hence for any $\bar{\rho} > 0$ we have

$$\begin{aligned} &V_3(x^+, p_3^+) - V_3(x^+, \tilde{p}_3^+) \\ &\leq \bar{\rho}c(x^\top x + v_0^\top v_0) + \gamma_1(k)x^\top x + \gamma_2(k)r^\top r. \end{aligned} \quad (37)$$

Step 3. Plugging in (37) and (33) in (32) with $\bar{\rho}$ such that $2\bar{\rho} < \mu := \min\{\lambda_{\min}(Q), \lambda_{\min}(R)\}$ and β_4 such that $\mu > 2\beta_4$ we get

$$\begin{aligned} &V_3(x^+, p_3^+) - V_3(x, p_3) \\ &\leq -\frac{\mu}{2}(x^\top x + v_0(x, p_3)^\top v_0(x, p_3)) \\ &\quad + (\beta_1(k) + \beta_4)x^\top x + (\beta_2(k) + \beta_5)r^\top r \\ &\quad + \beta_3(k) + \gamma_1(k)x^\top x + \gamma_2(k)r^\top r. \end{aligned} \quad (38)$$

Since $r = r(k)$, $\gamma_i(k)$, $\beta_i(k)$ converge to zero (and thus are bounded sequences) and since $x(k)$ is defined for all $k \in \mathbb{N}$, it follows that $x(k)$ converges to zero, i.e there exists a time \tilde{k} , such that for all $k \geq \tilde{k}$, the right hand side of (38) is negative for $x(k) \neq 0$. Finally, due to Lemma 1, $V_3(0, p_3) = 0$, hence if $x(k)$ converges to zero then also $u(k)$. \square

C. Proof of Lemma 3

Proof by contradiction. Suppose for all $\rho > 0$ there exists an $x \in \mathbb{R}^N$, $k \in \mathbb{N}$ and $\varepsilon > 0$ such that $x_N^\top(x, p_1)^\top Q_N x_N(x, p_1) > V_2(x, p_2) + \varepsilon\rho$. Then by (3) and Assumption 4(a), we have $V_1(x, p_1) > \frac{\Gamma(x)V_2(x, p_2)}{\varepsilon} + \rho\Gamma(x)$. On the other hand we have $V_1(x, p_1, v_0(x, p_2), \dots, v_{N-1}(x, p_2)) = \sum_{i=0}^{N-1} \xi_i(x, p_2)^\top Q \xi_i(x, p_2) + v_i(x, p_2)^\top R v_i(x, p_2) + \frac{\Gamma(x)V_2(x, p_2)}{\varepsilon}$, where $\{v_i(x, p_2)\}_{i=0}^{N-1}$, $\{\xi_i(x, p_2)\}_{i=0}^N$ is a solution of (10) and it is feasible for (3). By optimality, we must have $V_1(x, p_1) \leq V_1(x, p_1, v_0(x, p_2), \dots, v_{N-1}(x, p_2))$, hence

$$\begin{aligned} &\frac{\Gamma(x)V_2(x, p_2)}{\varepsilon} + \rho\Gamma(x) < \frac{\Gamma(x)V_2(x, p_2)}{\varepsilon} \\ &+ \sum_{i=0}^{N-1} \xi_i(x, p_2)^\top Q \xi_i(x, p_2) + v_i(x, p_2)^\top R v_i(x, p_2) \end{aligned} \quad (39)$$

must hold. However, (39) cannot be true for all $\rho > 0$ since $\Gamma(x) \geq c(\sum_{i=0}^{N-1} \|\xi_i(x, p_2)\|^2 + \|v_i(x, p_2)\|^2)$ and therefore there exists a $\rho > 0$ (independent of x, p_2) such that $\rho\Gamma(x) \geq \rho c(\sum_{i=0}^{N-1} \|\xi_i(x, p_2)\|^2 + \|v_i(x, p_2)\|^2) \geq \sum_{i=0}^{N-1} \xi_i(x, p_2)^\top Q \xi_i(x, p_2) + v_i(x, p_2)^\top R v_i(x, p_2)$. \square

D. Proof of Theorem 1

To prove the result, we use Lemma 2. We first show the following: *There exists an (appropriately constructed) sequence $\{p_3(k) := \tilde{p}_1(k)^\top = [k, r(k)^\top]\}_{k \in \mathbb{N}}$, such that the solution of (3) at time instant k with $p_1 = p_1(k)$ and $x = x(k)$, where $x(k)$ is the state of the closed loop (3),(2), is equivalent to the solution of (11) for $x = x(k)$, $p_3 = p_3(k)$ and thus the state sequence of the closed loop (3),(2) coincides with that of (11),(2). Moreover we show that the constructed sequence $\{r(k)\}_{k \in \mathbb{N}}$ converges to zero.*

Step 1. Let $\{u_i(x, p_1)\}_{i=0}^{N-1}$, $\{x_i(x, p_1)\}_{i=0}^N$ be the unique solution of (3). Then due to (5), it is also the unique solution of

$$\begin{aligned} \tilde{V}_1(x, \tilde{p}_1) &= \min \sum_{i=0}^{N-1} \xi_i^\top Q \xi_i + v_i^\top R v_i \\ \text{s.t. } \quad &\xi_{i+1} = A_{i+1}(k)x + \sum_{l=0}^i B_{i-l}(k)v_l, \\ &\xi_0 = x, \quad i = 0 \dots N-1, \\ &\xi_N = x_N(x, p_1) \end{aligned} \quad (40)$$

with $\tilde{p}_1^\top = [k, x_N(x, p_1)^\top]$. Thus, any closed loop trajectory of (2) and (3) coincides with a closed loop trajectory of (2) and (40) with the same initial data. Since (40) is an instance of (11) with $r = x_N(x, p_1)$ and thus Assumption 5 b) implies Assumption 5 a), it remains to show (in Step 2a and 2b below) that $r(k) := x_N(x(k), p_1(k))$ converges to zero as k goes to infinity in order to apply Lemma 2.

Step 2. We next apply Lemma 3. Therefore, we first show that there exists a $\tilde{c} > 0$ such that for any $k \in \mathbb{N}$ ($p_2(k) = k$) and any $x(k)$ we have $\tilde{c}\alpha x(k)^\top x(k) \geq \sum_{i=0}^{N-1} \|\xi_i(x(k), p_2(k))\|^2 + \|v_i(x(k), p_2(k))\|^2$, where $\{v_i(x(k), p_2(k))\}_{i=0}^{N-1}$, $\{\xi_i(x(k), p_2(k))\}_{i=0}^{N-1}$ is some solution of (10) at a time step k .

Notice that (10) has not necessarily a unique solution and Lemma 3 refers to *some* solution. Therefore, according to Lemma 1, there is a solution $\{\xi_i(x(k), p_2(k))\}_{i=0}^{N-1}$, $\{v_i(x(k), p_2(k))\}_{i=0}^{N-1}$ of (10) (i.e. the least norm solution) which is linear in x and uniformly bounded in k , i.e. $\xi_i(x(k), p_2(k)) = K_{1,i}(k)x(k)$, $v_i(x(k), p_2(k)) = K_{2,i}(k)x(k)$.

Consequently, there exist $c_1 > 0$ such that for $i = 0, \dots, N$ $\|\xi_i(x(k), p_2(k))\|^2 \leq c_1 x(k)^\top x(k)$, $\|v_i(x(k), p_2(k))\|^2 \leq c_1 x(k)^\top x(k)$.

Hence, we have $\sum_{i=0}^{N-1} \|\xi_i(x(k), p_2(k))\|^2 + \|v_i(x(k), p_2(k))\|^2 \leq \tilde{c}\alpha x(k)^\top x(k)$ for some $\tilde{c} > 0$ and we can finally apply Lemma 3 to get $x_N(x(k), p_1(k))^\top Q_N x_N(x(k), p_1(k)) \leq V_2(x(k), p_2(k)) + \varepsilon(k)\rho$.

Step 3. Since $Q_N > 0$ and $\{\varepsilon(k)\}_{k \in \mathbb{N}}$ converges to zero, it remains to show that $V_2(x(k), p_2(k))$ defined in (10) converges to zero, which however is a direct consequence of Assumption 4(c). Hence, $r(k) := x_N(x(k), p_1(k))$ converges to zero.

Summarizing, we have shown that there exists a sequence $\{p_3(k) := \tilde{p}_1(k) = [k, x_N(x(k), p_1(k))^\top]^\top\}_{k \in \mathbb{N}}$ such that the solution of the closed loop (2),(3) is equivalent to the solution of the closed loop (2), (11) and $\{\tilde{p}_1(k)\}_{k \in \mathbb{N}}$ converges to zero. Thus, by Lemma 2, the state and input sequence of the closed loop system (2) and (3) converges to zero. \square

E. Proof of Lemma 4

From the first order optimality condition for the right hand side of (20), we get

$$\nabla_x f(x_{k+1}^*, k) = -\nabla_x D(x_{k+1}^*, \tilde{x}_k). \quad (41)$$

Moreover, due to the convexity of f , it holds for all $x \in \mathbb{R}^n$

$$f(x, k) \geq f(x_{k+1}^*, k) + \nabla_x f(x_{k+1}^*, k)^\top (x - x_{k+1}^*). \quad (42)$$

Evaluating (42) at a time invariant minimizer $x^* \in X$ and inserting (41) into (42), we get

$$f(x^*, k) - f(x_{k+1}^*, k) \geq -\nabla_x D(x_{k+1}^*, \tilde{x}_k)(x^* - x_{k+1}^*). \quad (43)$$

Inserting the definition of the Bregman distance and the gradient of the Bregman distance with $\nabla_x D(x, y) = \nabla g(x) - \nabla g(y)$ into (43) yields

$$\begin{aligned} f(x^*, k) - f(x_{k+1}^*, k) &\geq [\nabla g(x_k^*) - \nabla g(x_{k+1}^*)]^\top (x^* - x_{k+1}^*) \\ &= D(x^*, x_{k+1}^*) + D(x_{k+1}^*, \tilde{x}_k) - D(x^*, \tilde{x}_k), \end{aligned} \quad (44)$$

where the last equality follows from the three-point identity of the Bregman distance (see also [41, Proof of Proposition 3.6]). Since D is convex in the second argument, we have

$$\begin{aligned} D(x^*, \tilde{x}_{k+1}) &\leq (1 - \lambda_{k+1})D(x^*, x_{k+1}^*) + \lambda_{k+1}D(x^*, \tilde{x}_k) \\ &\leq (1 - \lambda_{k+1})[D(x^*, \tilde{x}_k) - D(x_{k+1}^*, \tilde{x}_k)] \\ &\quad + f(x^*, k) - f(x_{k+1}^*, k) + \lambda_{k+1}D(x^*, \tilde{x}_k), \end{aligned} \quad (45)$$

and hence

$$\begin{aligned} & D(x^*, \tilde{x}_{k+1}) - D(x^*, \tilde{x}_k) \\ & \leq (1 - \lambda_{\max})[-D(x_{k+1}^*, \tilde{x}_k) + f(x^*, k) - f(x_{k+1}^*, k)]. \end{aligned} \quad (46)$$

Based on (46), we now utilize $V(x) = D(x^*, x)$ as a Lyapunov-like function. In particular, V is nonnegative, zero if and only if $x = x^*$ and strictly convex and hence radially unbounded. Since the right hand side of (46) is non-positive, \tilde{x}_k is a bounded sequence. Moreover, V is strictly monotonically decreasing as long as i) $D(x_{k+1}^*, \tilde{x}_k) > 0$ and ii) $f(x^*, k) - f(x_{k+1}^*, k) < 0$. Hence, we infer that $D(x_{k+1}^*, \tilde{x}_k)$ must converge to zero, and we have $\lim_{k \rightarrow \infty} \|x_{k+1}^* - \tilde{x}_k\| = 0$. By $\tilde{x}_{k+1} = (1 - \lambda_{k+1})x_{k+1}^* + \lambda_{k+1}\tilde{x}_k$, we also have $\lim_{k \rightarrow \infty} \|\tilde{x}_{k+1} - \tilde{x}_k\| = 0$. Consequently \tilde{x}_k converges to a point, say \tilde{x}^0 . In addition, due to $\lim_{k \rightarrow \infty} \|\tilde{x}_{k+1} - \tilde{x}_k\| = 0$ and $\lambda_k \leq \lambda_{\max} < 1$, it must also hold that x_k^* converges to \tilde{x}^0 . Finally, also $f(x^*, k) - f(x_{k+1}^*, k)$ must converge to zero, which implies that $f(\tilde{x}^0, k) = f(x^*, k) = 0$. \square

F. Proof of Lemma 5

Let $C(\lambda) = [B(\lambda), A(\lambda)B(\lambda), \dots, A^{n-1}(\lambda)B(\lambda)]$ be the controllability matrix and let $p(\lambda) = \det(C(\lambda)C(\lambda)^\top)$. Then $p(\lambda)$ is a polynomial of at most degree $2n^2$ with $p(0) = 0$ and $p(1) > 0$. Hence, p is not identical zero and there exist at most $2n^2$ real zeros of p over any interval. Therefore, if p is evaluated at $2n^2 + 1$ distinct points, then there is at least one point where p is nonzero. The claim follows from this basic observation. \square

G. Proof of Theorem 2

(i) By Assumption 6 and 7 and by eliminating the variable e in (15), we can apply Lemma 4. Hence, we have $\lim_{k \rightarrow \infty} \theta(k) = \theta^*$.

(ii) From the construction (21) of the predictor maps and the fact that $s(k) - R(k)\theta^*(k) = e^*(k)$ corresponds to the linear system of equations

$$x(j) = A(k)x(j-1) + B(k)u(j-1) + e_{j-k+\bar{N}}^*(k), \quad (47)$$

$j = k \dots k - \bar{N} + 1$, $e^*(k) = [e_N^*(k)^\top \dots e_1^*(k)^\top]^\top$, as well as from the convergence property (i), i.e. by $\lim_{k \rightarrow \infty} A(k) = \hat{A}$, $\lim_{k \rightarrow \infty} B(k) = \hat{B}$, property (6), (7) in Assumption 4 is satisfied. It remains to show property (8). Notice that by (47), we have for any $k \in \mathbb{N}$ and $j = 0 \dots \bar{N} - 1$

$$\begin{aligned} x(j+k-\bar{N}+1) &= A_j(k)x(k-\bar{N}+1) \\ &+ \sum_{l=0}^{j-1} B_{j-1-l}(k)u(l+k-\bar{N}+1) + \sum_{l=0}^{j-1} A(k)^{j-1-l}e_{l+1}^*(k) \end{aligned} \quad (48)$$

Since $\lim_{k \rightarrow \infty} A_i(k) = \hat{A}_i$, $\lim_{k \rightarrow \infty} B_i(k) = \hat{B}_i$, we have

$$\begin{aligned} x(j+k-\bar{N}+1) &= \hat{A}_j x(k-\bar{N}+1) \\ &+ \sum_{l=0}^{j-1} \hat{B}_{j-1-l} u(l+k-\bar{N}+1) + \sum_{l=0}^{j-1} A(k)^{j-1-l} e_{l+1}^*(k) \\ &\quad + (A_j(k) - \hat{A}_j) x(k-\bar{N}+1) \\ &\quad + \sum_{l=0}^{j-1} (B_{j-1-l}(k) - \hat{B}_{j-1-l}) u(l+k-\bar{N}+1). \end{aligned} \quad (49)$$

By Lemma 4, $\lim_{k \rightarrow \infty} e_{l+1}^*(k) = 0$, $l = 0 \dots \bar{N} - 1$, hence there exists a sequence $\{\omega_l(k)\}_{k \in \mathbb{N}}$ that converges to zero and such that for all $j = 0 \dots \bar{N} - 1$ it holds: $\|\sum_{l=0}^{j-1} A(k)^{j-1-l} e_{l+1}^*(k)\|^2 \leq \omega_1(k)$.

By Cauchy-Schwarz's and Young's inequality, there exist sequences $\{\omega_2(k)\}_{k \in \mathbb{N}}$, $\{\omega_3(k)\}_{k \in \mathbb{N}}$ that both converge to zero and such that $\|(A_j(k) - \hat{A}_j)x(k-\bar{N}+1)\|^2 \leq \omega_2(k)\|x(k-\bar{N}+1)\|^2$ and such that for all $j = 0 \dots \bar{N} - 1$ $\|\sum_{l=0}^{j-1} (B_{j-1-l}(k) - \hat{B}_{j-1-l})u(l+k-\bar{N}+1)\|^2 \leq \omega_2(k)\sum_{l=0}^{j-1} \|u(l+k-\bar{N}+1)\|^2$. Hence

$$\begin{aligned} x(j+k-\bar{N}+1) &= \hat{A}_j x(k-\bar{N}+1) + e_j(k-\bar{N}+1) \\ &\quad + \sum_{l=0}^{j-1} \hat{B}_{j-1-l} u(l+k-\bar{N}+1) \end{aligned} \quad (50)$$

where $e_j(k - \bar{N} + 1) = \sum_{l=0}^{j-1} A(k)^{j-1-l} e_{l+1}^*(k) + (A_j(k) - \hat{A}_j)x(k - \bar{N} + 1) + \sum_{l=0}^{j-1} (B_{j-1-l}(k) - \hat{B}_{j-1-l})u(l + k - \bar{N} + 1)$, $j = 0 \dots \bar{N} - 1$, satisfies the error bounds in Assumption 4(b) as shown above. In more detail, since (50) holds for any $k \in \mathbb{N}$, let $k = \tilde{k} + \bar{N} - 1$, then we get the desired asymptotically correct prediction property (8) with respect to (18) with $N = \bar{N} - 1$:

$$\begin{aligned} & \hat{A}_j x(\tilde{k}) + \sum_{l=0}^{j-1} \hat{B}_{j-1-l} u(l + \tilde{k}) + e_j(\tilde{k}) \\ &= x(j + \tilde{k}) = A^j x(\tilde{k}) + \sum_{l=0}^{j-1} A^{j-1-l} B u(l + \tilde{k}). \end{aligned} \quad (51)$$

(iii) Equation (9) follows directly from Lemma 5. In particular, suppose $\tilde{\theta}(k-1)^\top = [\text{vec}(A(k-1))^\top, \text{vec}(B(k-1))^\top]^\top$ is controllable (which holds for $k=1$) and suppose $(A_u(k), B_u(k))$ obtained from $\theta^*(k)^\top = [\text{vec}(A_u(k))^\top, \text{vec}(B_u(k))^\top]^\top$ is not controllable. Then λ_k and thus $\tilde{\theta}(k)^\top = [\text{vec}(A(k))^\top, \text{vec}(B(k))^\top]^\top$ such that $(A(k), B(k))$ is controllable can be constructed according to Lemma 5. (If $\theta^*(k)^\top = [\text{vec}(A_u(k))^\top, \text{vec}(B_u(k))^\top]^\top$ is controllable, then simply choose $\lambda_k = 0$). \square

H. Proof of Lemma 6

We utilize the Kalman decomposition. Since we consider only input and output data and since Assumption 1 holds, we can assume without loss of generality that (F, G, H) in (1) is structured as follows

$$\begin{aligned} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \end{bmatrix} &= \begin{bmatrix} F_1 & F_2 & 0 \\ 0 & F_3 & 0 \\ F_4 & F_5 & F_6 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \\ G_2 \end{bmatrix} v(k) \\ y(k) &= \begin{bmatrix} H_1 & H_2 & 0 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \end{bmatrix}, \end{aligned} \quad (52)$$

where F_3, F_6 are stable (eigenvalues are in the interior of the complex unit disc) and the subsystem

$$\begin{aligned} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} &= \begin{bmatrix} F_1 & F_2 \\ 0 & F_3 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} v(k) \\ &:= F_s z_s(k) + G_s v(k) \\ y(k) &= \begin{bmatrix} H_1 & H_2 \end{bmatrix} z_s(k) := H_s z_s(k) \end{aligned} \quad (53)$$

is observable and stabilizable. Due to the cascaded structure, it follows that if the state and input of the subsystem (53) goes to zero, then also the state (and the input) of the overall system (52), since F_6 is stable. Notice further that the output sequences of the subsystem are equivalent to the output sequences of the overall system (for the same input sequences). Since the subsystem is observable and $m \geq n$ is known, it follows that if $x(k) = \phi_y(y(k), \dots, y(k-m+1)) = [y(k)^\top, \dots, y(k-m+1)^\top]^\top$ and $u(k) = \phi_v(v(k), \dots, v(k-m+1)) = [v(k)^\top, \dots, v(k-m+1)^\top]^\top$ goes to zero, then also $\{v(k)\}_{k \in \mathbb{N}}, \{y(k)\}_{k \in \mathbb{N}}$ of the subsystem (53) (as well as of the overall system (52)). As a final step, consider $x(k) = \phi_y(y(k), \dots, y(k-m+1)) = [y(k)^\top, \dots, y(k-m+1)^\top]^\top$ which is given by

$$\begin{bmatrix} H_s F_s^{m-1} z_s(k-m+1) + H_s \sum_{l=0}^{m-2} F_s^{m-2-l} G_s v(k-m+1+l) \\ \vdots \\ H_s F_s z_s(k-m+1) + H_s G_s v(k-m+1) \\ H_s z_s(k-m+1) \end{bmatrix}. \quad (54)$$

Due to (22), (54) can be compactly written as $x(k) = O z_s(k-m+1) + R u(k-1)$. Since (53) is observable, the (observability) matrix O has full column rank, hence we have $z_s(k-m+1) = (O^\top O)^{-1} O^\top x(k) - (O^\top O)^{-1} O^\top R u(k-1)$. Thus

$$\begin{aligned} x(k+1) &= O z_s(k-m+2) + R u(k) \\ &= O F_s z_s(k-m+1) + O G_s v(k-m+1) + R u(k) \\ &= O F_s ((O^\top O)^{-1} O^\top x(k) - (O^\top O)^{-1} O^\top R u(k-1)) \\ &\quad + O G_s v(k-m+1) + R u(k) \end{aligned} \quad (55)$$

which can be written as $x(k+1) = A x(k) + B u(k)$. Consequently $\{u(k)\}_{k \in \mathbb{N}}, \{x(k)\}_{k \in \mathbb{N}}$ satisfy Assumption 7. To show that (A, B) is stabilizable, choose $v(k) = K_s z_s(k)$ such that $F_s + G_s K_s$ is stable, which is possible since (53) is stabilizable. Then this choice implies that (55) is stable, which follows directly from (54). In more detail, from (55) we have $w_{k+1} = (F_s + G_s K_s) w_k$ with $w_k := O^\top x_k$. Therefore w_k goes to zero and thus x_k , hence we have shown that the obtained pair (A, B) is stabilizable. Concerning Remark 4. Notice that the input vector $u(k)$ contain past values of the actual input $v(k)$. These past input values, however, can be eliminated by augmenting additional state variables to (55) and by defining an integrator chain dynamics

of the form $\zeta_1(k+1) = \zeta_2(k), \dots, \zeta_{m-1}(k+1) = \zeta_m(k), \zeta_m(k+1) = v(k)$, hence we have $\zeta_1(k) = v(k-m)$ etc. Stabilizability of the augmented system follows from the stabilizability of (A, B) , since the states of the integrator chain converge to zero, if $v(k)$ converges to zero, i.e. a stabilizing feedback is again $v(k) = K_s z_s(k)$. \square