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A converse to Lieb-Robinson bounds in one dimension using index theory

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Abstract

Unitary dynamics with a strict causal cone (or "light cone") have been studied extensively, under the name of quantum cellular automata (QCAs). In particular, QCAs in one dimension have been completely classified by an index theory. Physical systems often exhibit only approximate causal cones; Hamiltonian evolutions on the lattice satisfy Lieb-Robinson bounds rather than strict locality. This motivates us to study approximately locality preserving unitaries (ALPUs). We show that the index theory is robust and completely extends to one-dimensional ALPUs. As a consequence, we achieve a converse to the Lieb-Robinson bounds: any ALPU of index zero can be exactly generated by some time-dependent, quasi-local Hamiltonian in constant time. For the special case of finite chains with open boundaries, any unitary satisfying the Lieb-Robinson bound may be generated by such a Hamiltonian. We also discuss some results on the stability of operator algebras which may be of independent interest.

Contents

1 Introduction

While quantum dynamics of closed systems are always unitary, systems of interest often possess an additional property: information propagates at finite speeds. In quantum field theories or local quantum circuits, information is strictly constrained to spread within a region called the light cone, or causal cone. Systems with strict causal cones are called *quantum cellular automata* (QCA) [1, 2]; or sometimes *locality-preserving unitaries*. However, the effective theories governing laboratory systems are only constrained by an *approximate* causal cone, see Fig. 1. For instance, nontrivial time evolution by a fixed local lattice Hamiltonian never satisfies a strict causal cone,[1] but it does exhibit an approximate causal cone, given by the Lieb-Robinson bounds [3].

Evolutions with approximate causal cones constitute a wide class of natural systems. We can ask general questions about this class of dynamics, e.g. when can the evolution be generated by some local Hamiltonian, or when can one evolution be continuously deformed into another? These fundamental questions also have application in the study of topological phases in many-body physics [4, 5].

The rich theory of QCAs addresses these questions for the case of strict causal cones. In [6] (GNVW) it was shown that in one dimension, any QCA is the composition of some local circuit and translation operators. The total "flux" generated by the translation is discretized and measured by a numerical index, which completely classifies QCAs in terms of a computable invariant, yielding one of the most important tools in the study of QCAs. In Section 4 we review the GNVW index. More recently it has been shown [7, 8] that similarly in two dimensions, any QCA can be written as a composition of a circuit and a generalized shift (i.e. a permutation of nearby lattice sites). However, in higher dimensions strong evidence suggest the existence of QCAs not of this type [5]. QCAs have found many and wide-ranging applications at the interface of quantum many-body physics and quantum information theory. To mention a few recent examples, QCAs have been studied in the context of discretization of quantum field theories [9, 10], quantum hydrodynamics and operator growth [11, 12], subsystem symmetries and computational phases of matter [13], tensor

[1] In finite-dimensional lattice systems, given two operators A and B on distant sites, if $[A, B(t)]$ is exactly zero in some interval $t \in [0, t^*]$, it must be zero always by analyticity. The system therefore violates any exact causal cone unless there is zero spread of information.

networks and matrix product operators [14–17], and topological phases of many-body localized dynamics [4, 18]. See [19, 20] for recent reviews.

Nearly all rigorous results about QCAs rely heavily on their strict locality. In order to apply insights about QCAs to the "real" quantum lattice systems commonly encountered, one must first extend the theory of QCAs to the approximate case. In this work we make an important first step by developing the theory of approximately locality-preserving unitaries (ALPUs), i.e. unitary evolutions with approximate causal cones, for one-dimensional systems. In particular, we extend the topological index theory of QCAs in one dimension [6] to the case of ALPUs. The extended index theory covers local Hamiltonian evolutions and perhaps other naturally occurring evolutions with approximate locality, as in Section 6.2. We generally call a Hamiltonian local (or quasi-local, for emphasis) whenever its interactions decay sufficiently with distance, and our results refer to varying notions of decay.

Some interesting features of QCAs in one dimension are easily demonstrated by simple example. Consider an infinite spin chain. The index theory in [6] shows that it is not possible to implement a translation operator by using a finite depth circuit. Is this still true if we allow time-dependent Hamiltonian evolutions? Is this perhaps possible if we allow Hamiltonian evolutions with polynomial tails? Naively, the classification of QCAs might have been expected to "collapse" under the introduction of ALPUs (especially when allowing polynomial tails), or alternatively the classification might have become more exotic. It turns out that neither is the case; we show that almost all properties of the classification and index theory of QCAs can be generalized to ALPUs with $o(\frac{1}{r})$ tails, or even $o(1)$ tails in many cases.

More generally, while local Hamiltonians satisfy Lieb-Robinson bounds and therefore generate ALPUs, we may conversely ask the following question:

Given an automorphism α satisfying Lieb-Robinson bounds (i.e. an ALPU), can it be generated by some time-dependent local Hamiltonian? If not, what are the obstructions?

As foreshadowed by the GNVW index theory, it turns out the only obstruction to finding such a Hamiltonian in a one-dimensional system is when α has nonzero index. Thus, our classification offers a "converse" to the Lieb-Robinson bounds. For instance, we show that an ALPU with exponentially decaying tails can be generated by a time-dependent Hamiltonian with exponentially decaying interactions precisely when the ALPU has index zero; we also find related statements for tails of slower decay, as in Corollary 5.16. If α has a nonzero index it can be constructed as the composition of a generalized shift and an ALPU of index zero, which can then be generated by some time-dependent Hamiltonian evolution. Meanwhile, for a non-periodic chain of finite length, the index is always zero. In that case, we conclude that the dynamics satisfy Lieb-Robinson bounds if and only if they are generated by a local Hamiltonian with sufficiently decaying tails.

1.1 Prior work

To generalize the GNVW index to ALPUs, a natural concern is the sensitivity of the index to small perturbations of the QCA. However, the dependence of the sensitivity on the local Hilbert space dimension and the radius of the QCA is not immediately clear from the considerations in [6], which yield a relatively weak continuity estimate. Without stronger bounds, it appears possible that the homotopy classes established in [6] might collapse when considering ALPUs: two QCAs with

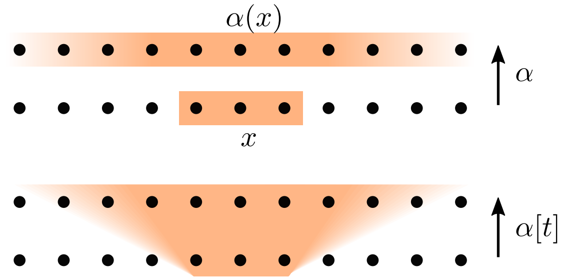

Figure 1: Illustration of an automorphism α with an approximate lightcone (an ALPU). Given such an α, does there exist a continuous dynamics $\alpha[t]$ with $\alpha[0] = \mathrm{id}$ and $\alpha[1] = \alpha$ which remains approximately local at all times?

different GNVW index might still be connected by a strongly continuous path through the space of ALPUs with some prescribed tails. Another concern is whether the generalized index would take values in the same discrete set as the GNVW index. Relatedly, we ask whether every ALPU can be approximated to arbitrary accuracy δ by a QCA whose radius does not grow too fast with δ. In the special case of Hamiltonian evolutions, Lieb-Robinson-type estimates allow one to approximate the evolution by a strictly local quantum circuit [21, 22], as in digital quantum simulation. We might hope for a similar discretization or Trotterization procedure for arbitrary ALPUs, or at least for index-0 QCAs.

These questions are recognized in existing literature. In fact, one of the main open questions in the original work [6] was how to extend the index theory to some class of automorphisms with only approximate locality. Later work asked specifically whether ALPUs could be approximated by QCAs [23]. In the review [19] such questions were raised again, highlighting their relevance for the application of index theory to actual physical systems. As an example, the GNVW index has been proposed to classify two-dimensional many-body localized Floquet phases, by computing the index of a certain dynamics that arises on the boundary of the system [4]. Such dynamics are typically not strictly local, and in Section 6.2 we comment on this specific application. Other recent work [7] also suggested the extension to ALPUs as an avenue for research, proposing that one approach might involve Ulam stability results for operator algebras. This is precisely the approach taken in this work. The stability results we use [24, 25] and augment were developed throughout the 1970s and 80s for studying how operator algebras behave under perturbation. Intriguingly, related questions about perturbations were tackled under a different guise in [26] (cf. their Theorem 3.6), in the context of quantum device certification.

Regarding the converse to the Lieb-Robinson bounds, see [27] for interesting work which develops a related converse with different assumptions. They show that if you already know α is generated by a k-local Hamiltonian satisfying a Lieb-Robinson-like condition, then the evolution can also be generated by a *geometrically* local k-local Hamiltonian. Their condition can be checked at infintesimal times. In contrast, we do not assume the ALPU is generated by any k-local Hamiltonian.

1.2 Summary of results

The theory is typically formulated in the Heisenberg picture, acting on operators rather than states. This definition is more natural for infinite systems. Unitary dynamics of the quantum system can then be described by an automorphism of an operator algebra. A *quantum cellular automaton* (QCA) with *radius* R on some lattice of spin systems is an automorphism α of the algebra $\mathcal{A}$ of operators

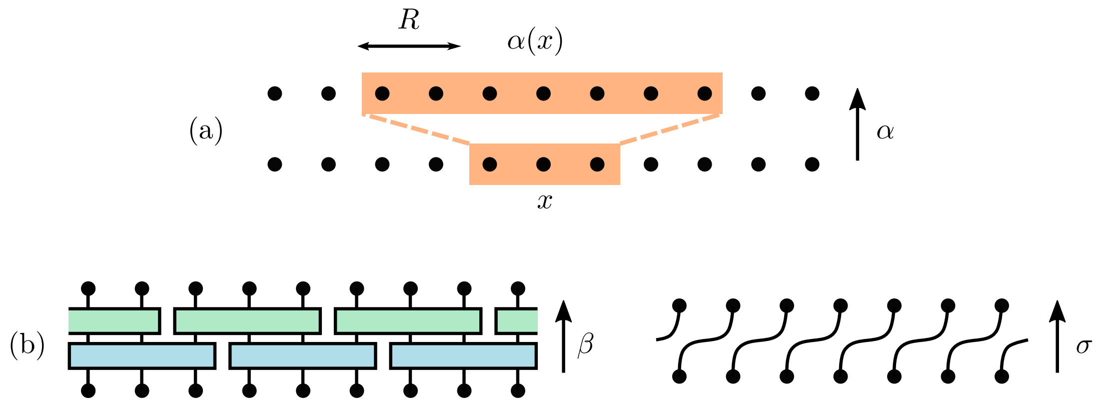

Figure 2: (a) Illustration of a QCA α with radius $R = 2$ mapping an operator x supported on three sites to an operator $\alpha(x)$ supported on seven sites. (b) A local circuit QCA β (left) and a translation QCA σ (right).

on the spin system, such that if an operator $x \in \mathcal{A}$ is supported on a set of sites X, then $\alpha(x)$ is an operator supported on $B(X, R)$, the set of sites within distance R of X. For an *approximately locality-preserving unitary* (ALPU) with *tails* $f(r)$, we only ask that $\alpha(x)$ can be approximated by operators supported on $B(X, r)$ up to an error $f(r)$ for any r, as detailed in Definition 3.5. We also require that $\lim_{r\to\infty} f(r) = 0$. In other words, the map α satisfies a Lieb-Robinson bound governed by $f(r)$. We will restrict to the situation where we have an infinite one-dimensional lattice with finite local dimension (i.e. a spin chain).

Two examples of QCAs are shown in Fig. 2,

(i) A local circuit, i.e. a composition of multiple layers of application of strictly local unitaries.

(ii) In the case where each local Hilbert space is identical, we have the translation automorphism, which simply shifts an operator by one site to the left. Notice that in the (perhaps more intuitive) Schrödinger picture this corresponds to a shift to the *right* of the state.

In [6] it was shown that these two examples generate *all* examples, in the sense that any QCA can be written as a composition of (tensor products of) translations and circuits. It is quite intuitive that in example (ii) there is a 'flux' of information to the right, whereas in example (i) the net 'flux' is zero. This suggests that computing some sort of flux allows you to extract from a QCA how many translations you need to implement it. This intuition was made precise in [6] by defining an *index* (the GNVW index) which measures the flow of quantum information, based on ideas in [28]. In this work we first observe that one can re-formulate the definition of the index as follows: one divides the chain into a left half L and right half R, and one considers the Choi state $\phi_{LR,L'R'}$ of the automorphism α. Then the mutual information difference

$$\mathrm{ind}(\alpha) = \frac{I(L' : R)_\phi - I(L : R')_\phi}{2}, \tag{1.1}$$

is precisely the index of [6], but also well-defined for ALPUs with appropriately decaying tails! In addition, the mutual information enjoys much better continuity than the related expression for the index in Eq. (45) of [6], which (in hindsight) can be understood as a difference of Renyi-2 entropies. In the context of two-dimensional Floquet phases a similar expression has been derived in [29]. We show that the expression in Eq. (1.1) generalizes to the approximately local setting.

Our first main result consists of Theorems 5.6 and 5.8, summarized as

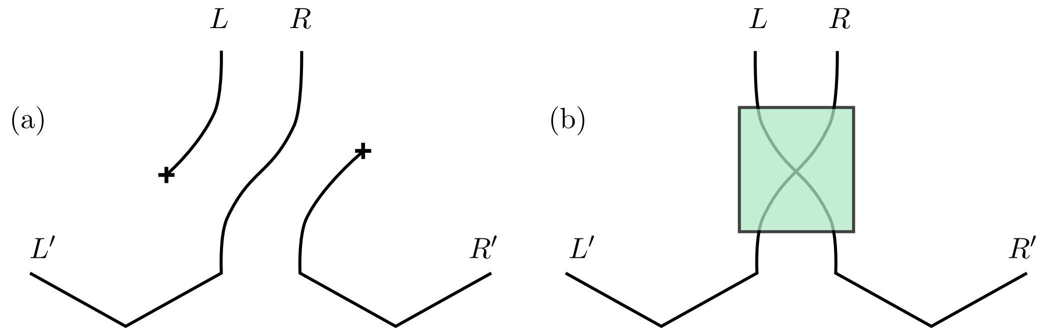

Figure 3: Illustration of (1.1). For the translation on a qudit with local dimension d in (a) we have $I(L' : R)_\phi = 2\log(d)$ and $I(L : R')_\phi = 0$, so the index equals d. For a circuit, one can show that applying a local unitary as in (b) gives $I(L' : R)_\phi = I(L : R')_\phi$, so the index is zero.

Approximation Theorem (informal)**.** *Suppose that α is an ALPU in one dimension. Then there exists a sequence of QCAs α_j of increasing radius such that $\alpha_j(x)$ converges to $\alpha(x)$ for any local operator x, and that* $\operatorname{ind}(\alpha_j)$ *stabilizes for large j. We define* $\operatorname{ind}(\alpha) = \lim_{j\to\infty} \operatorname{ind}(\alpha_j)$*. If α has $\mathcal{O}(\frac{1}{r^{1+\delta}})$-tails for $\delta > 0$, the index defined in* (1.1) *is finite and equal to* $\operatorname{ind}(\alpha)$*. The exact index may also be computed locally through a rounding procedure.*

To be precise, the error bounds are such that if α has tails like $\mathcal{O}(f(r))$, and α_j has radius j, then for an operator x supported on an interval of n sites, $\|\alpha_j(x) - \alpha(x)\| = \mathcal{O}(f(j)(\frac{n}{j} + 1)\|x\|)$. The key technical ingredient we use is a stability result for inclusions of possibly infinite algebras which we state as Theorem 2.6, an extension of results from [24, 25]. This result deals with the situation where $\mathcal{A}$ and $\mathcal{B}$ are algebras of observables and $\mathcal{A}$ is "nearly" included in $\mathcal{B}$, meaning that for each $x \in \mathcal{A}$ there is a $y \in \mathcal{B}$ such that $\|x - y\| \leq \varepsilon\|x\|$ for some small ε. Then (under some technical but very general assumptions on the algebras), there exists a unitary $u \in B(\mathcal{H})$ close to the identity, $\|u - I\| = \mathcal{O}(\varepsilon)$ with error independent of $\dim(\mathcal{H})$, such that $u\mathcal{A}u^*$ is strictly contained in $\mathcal{B}$. Loosely speaking, we construct the QCAs α_j by "localizing" the images $\alpha(\mathcal{A}_n)$ of the algebra $\mathcal{A}_n$ at each site n, by rotating $\alpha(\mathcal{A}_n)$ into an algebra supported within some radius of site n. The main technical effort in the construction is to ensure the rotations are compatible and the errors do not accumulate.

The index defines equivalence classes of ALPUs. These are characterized in our second main result, Theorem 5.15, sketched below:

Classification Theorem (informal)**.** *Suppose α and β are ALPUs with $f(r)$-tails in one dimension. Then the following are equivalent conditions:*

(i) $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$.

(ii) $\alpha = \beta\gamma$ *where* $\operatorname{ind}(\gamma) = 0$.

(iii) There exists a "blended" ALPU which (up to small error) matches α on the left of the chain and matches β on the right.

(iv) There exists a strongly continuous path from α to β through the space of ALPUs with $g(r)$-tails for some $g(r) = o(1)$. If such a path exists, then it can be generated by evolving a time-dependent quasi-local Hamiltonian for unit time.

In particular, (iv) provides a converse to the Lieb-Robinson bounds in one dimension: an automorphism can be generated by evolution along a time-dependent Hamiltonian with certain locality bounds if and only if it has index zero. Thus we see that the index theory of [6] completely generalizes to ALPUs and does not "collapse," with as only essential difference that the role of quantum circuits is replaced by time evolutions along time-dependent geometrically local Hamiltonians.

As an application, it follows immediately that the translation operator cannot be implemented by a finite time evolution of any (time-dependent) Hamiltonian satisfying Lieb-Robinson bounds. Moreover, there cannot exist a quasi-local "momentum density" that generates a lattice translation and also satisfies Lieb-Robinson bounds with $o(1)$-tails at all times. To show that it is necessary to impose some bound on the decay of the ALPU tails in our constructions, we give an example of a strongly continuous path of automorphisms generated by a Hamiltonian with $\frac{1}{r}$-decaying interactions that connects the identity map to a translation on a chain of qubits, showing that at this point the index theory does indeed collapse. As a second potential application we discuss the definition of the index for two-dimensional Floquet systems with many-body localization.

We cast our results directly in the setting of infinite one-dimensional lattices. Finite chains with non-periodic boundary conditions become a straightforward special case; see Section 5.4. There the index is always zero, and we obtain a universal converse to the Lieb-Robinson bounds.[2]

This works is organized as follows: in Section 2 we review some basic properties of operator algebras, then discuss perturbations of operator algebras, including some new tools developed for this work. (We expect that results like Lemma 2.7, Lemma B.3, and Lemma 3.4 may also find broad application, e.g. in the development of Lieb-Robinson bounds.) In Section 3 and we define ALPUs. In Section 4 we review the GNVW index theory, prove (1.1), and discuss robustness of the index. Section 5 is the technical heart of this work, where we show how to construct a sequence of approximating QCAs to any ALPU and from this result derive the index theory for ALPUs. In Section 6 we discuss the two applications: the impossibility of finding a Hamiltonian for the translation operator and the definition of an index for two-dimensional many-body localized Floquet systems. Finally, in Appendix B we provide a self-contained proof of Theorem 2.6, the stability result for algebra inclusions, which may be of independent interest.

2 Operator algebras

For infinite dimensional quantum mechanical systems it is often more convenient to work with operator algebras (algebras of observables) rather than Hilbert spaces, and use the Heisenberg rather than Schrödinger picture of quantum mechanics. A standard reference for operator algebras and their relation to quantum physics is [30]; see [31] for an accessible introduction. We review C^*-algebras and von Neumann algebras, focusing especially on facts used in subsequent proofs. Then we turn to methods for "perturbations" (e.g. small rotations) of operator algebras in Section 2.3.

2.1 C^*-algebras

The notion of an operator algebra is formalized by a *C^*-algebra*, which is a complex algebra $\mathcal{A}$ with a norm $\|\cdot\|$ and an anti-linear involution $x \mapsto x^*$, satisfying

- $\mathcal{A}$ is complete in $\|\cdot\|$,

[2]The case of finite chains with periodic boundary conditions appears more difficult. While we expect the index theory there to match that of the infinite lattice, we cannot offer rigorous results.

- $\|xy\| \leq \|x\|\|y\|$,
- $\|x^*x\| = \|x\|^2$.

We will only use algebras with an identity element I. An important example is the C^*-algebra $B(\mathcal{H})$ of operators on some Hilbert space $\mathcal{H}$, where we take the operator norm as the norm, and the adjoint as the $*$-operation. In finite dimensions this reduces to the algebra $\mathcal{M}_{d\times d}$ of complex $d \times d$ matrices with the spectral norm and Hermitian conjugate. A C^*-algebra $\mathcal{A}$ is called *approximately finite-dimensional* (AF) if it contains a directed collection of finite-dimensional subalgebras whose union is dense in $\mathcal{A}$. If $\mathcal{A} \subseteq \mathcal{B}$ are C^*-algebras we define the *commutant* of $\mathcal{A}$ in $\mathcal{B}$ as $\mathcal{A}' = \{x \in \mathcal{B}$ such that $[x, \mathcal{A}] = 0\}$, which is again a C^*-algebra. We denote by $U(\mathcal{A})$ the set of elements $u \in \mathcal{A}$ that are *unitary*, meaning that $uu^* = u^*u = I$.

A *$*$-homomorphism* $\alpha\colon \mathcal{A} \to \mathcal{B}$ between C^*-algebras is an algebra homomorphism which also preserves the $*$-operation, $\alpha(x^*) = \alpha(x)^*$. Such a $*$-homomorphism is automatically continuous and indeed contractive, i.e., $\|\alpha(x)\| \leq \|x\|$. The latter can also be written as $\|\alpha\| \leq 1$, where we define the notation $\|\beta\| = \sup_{\|x\|\leq 1}\|\beta(x)\|$ for any linear map β between C^*-algebras. A *($*$-)automorphism* is a bijective $*$-homomorphism. The inverse of an automorphism is again a $*$-homomorphism, and any automorphism is automatically unital and isometric. We write id for the identity automorphism. Finally, a *state* on a C^*-algebra $\mathcal{A}$ is given by a linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ which is positive (meaning that $\omega(x^*x) \geq 0$ for all $x \in \mathcal{A}$) and normalized (meaning that $\omega(I) = 1$).

It turns out that any C^*-algebra can be realized as a subalgebra of $B(\mathcal{H})$, the algebra of bounded operators on some Hilbert space $\mathcal{H}$. This is proven by the following result known as the *Gelfand-Naimark-Segal (GNS) construction* or *representation*:

Theorem 2.1 (Gelfand-Naimark-Segal)**.** *Given a state ω on $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}$, a $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$, and a cyclic vector ϕ (meaning $\pi(\mathcal{A})\phi$ is dense in $\mathcal{H}$) such that*

$$\omega(x) = \langle \phi, \pi(x)\phi \rangle$$

Moreover, if $(\mathcal{H}', \pi', \phi')$ is another triple as above then there exists a unique unitary $u\colon \mathcal{H} \to \mathcal{H}'$ such that $\phi' = u\phi$ and $\pi'(x) = u\pi(x)u^*$ for all $x \in \mathcal{A}$.

If ω is such that $\omega(x^*x) = 0$ implies $x = 0$, then the GNS representation is faithful (meaning that π_ω is injective). In that case, one way to construct the Hilbert space in Theorem 2.1 is by letting $\langle x, y\rangle = \omega(x^*y)$ define an inner product on $\mathcal{A}$ and letting $\mathcal{H}$ be the completion of $\mathcal{A}$ with respect to this inner product. Then $\mathcal{A}$ acts on $\mathcal{H}$ by left multiplication, which defines the $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$. The identity $I \in \mathcal{A}$ gives rise to a cyclic vector $\phi \in \mathcal{H}$.

2.2 Von Neumann algebras

A special class of C^*-algebras are von Neumann algebras. A C^*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is a *von Neumann algebra* if it is equal to its double commutant in $B(\mathcal{H})$,

$$\mathcal{A} = \mathcal{A}''.$$

In fact, for any subset $S \subseteq B(\mathcal{H})$, the double commutant S'' is always a von Neumann algebra, called the von Neumann algebra generated by S. It is the smallest von Neumann algebra that contains S. There are various relevant topologies on $B(\mathcal{H})$. The *strong operator topology* is such that a net x_i converges to some operator x if and only if $x_i v \to xv$ for each vector $v \in \mathcal{H}$. The *weak operator topology* is such that a net x_i converges to some operator x if and only if $\langle w, x_i v\rangle \to \langle w, xv\rangle$ for each pair $v, w \in \mathcal{H}$. The weak operator topology is weaker than the strong operator topology, and both are weaker than the topology induced by the norm. Sometimes also the weak-$*$ topology is relevant, induced by interpreting $B(\mathcal{H})$ as the dual space of the trace class operators on $\mathcal{H}$. The weak operator topology is weaker than the weak-$*$ topology, but the two coincide on norm-bounded subsets of $B(\mathcal{H})$. On convex subsets the weak operator closure and strong operator closure coincide. The von *Neumann bicommutant theorem* states that for unital $*$-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$, $\mathcal{A}''$ is the weak operator closure of $\mathcal{A}$, so $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}$ is weak operator closed. Any $*$-automorphism of a von Neumann algebra is continuous with respect to the weak-$*$ topology. Moreover, norm balls are compact in the weak operator topology (Theorem 5.1.3 in [32], a consequence of the Banach-Alaoglu theorem).

A useful fact in the study of von Neumann algebras is the *Kaplansky density theorem*, which states that for any self-adjoint subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ the unit ball of the strong operator closure of $\mathcal{A}$ equals the strong operator closure of the unit ball of $\mathcal{A}$. We refer to [30, §2.4] for more details. Another useful fact is that the norm is lower semi-continuous in the weak operator topology, i.e., if a net x_i converges to x in the weak operator topology then $\|x\| \leq \liminf_i \|x_i\|$.

In infinite dimensions, working with a von Neumann algebra often confers advantages over more general C^*-algebras. For instance, the output of the Borel functional calculus (taking functions of operators) on a C^*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ produces operators that sometimes lie outside $\mathcal{A}$, but they always lie in the weak operator closure $\mathcal{A}''$. A von Neumann algebra therefore allows one to use spectral projections and other technical tools.

A von Neumann algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a *factor* if it has trivial center, $\mathcal{A}' \cap \mathcal{A} = \mathbb{C}I$. In particular, $\mathcal{A} = B(\mathcal{H})$ is a factor (a so-called type I factor). Any finite dimensional factor is of this form (for a finite dimensional Hilbert space), but there also exist infinite dimensional factors *not* of the form $B(\mathcal{H})$ (so-called type II and type III factors).

A von Neumann algebra $\mathcal{A}$ is called *hyperfinite* (or approximately finite-dimensional) if it contains a directed collection of finite-dimensional subalgebras whose union is dense in the weak operator topology (equivalently, the weak-$*$ topology). Equivalently, $\mathcal{A}$ is hyperfinite when there exists an AF C^*-subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0'' = \mathcal{A}$.

Finally, if $\mathcal{M} \subseteq B(H)$ and $\mathcal{N} \subseteq B(\mathcal{K})$ are von Neumann algebras, we use $\mathcal{M} \otimes \mathcal{N} \subseteq B(\mathcal{H} \otimes \mathcal{K})$ to denote the von Neumann algebra tensor product, given by the weak operator closure of the algebraic tensor product of $\mathcal{M}$ and $\mathcal{N}$ in $B(\mathcal{H} \otimes \mathcal{K})$.

2.3 Near inclusions and stability properties

We now define our notion of near inclusions of algebras and discuss related stability properties. The notion of a near inclusion follows e.g. [25].

Definition 2.2 (Near inclusion)**.** For a C^*-algebra $\mathcal{B} \subseteq B(\mathcal{H})$ and an operator $a \in B(\mathcal{H})$, we write $a \overset{\varepsilon}{\in} \mathcal{B}$ when there exists $b \in \mathcal{B}$ such that $\|a - b\| \leq \varepsilon \|a\|$. Likewise for two C^*-algebras $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$, we write $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ and say there is a *near inclusion* whenever $a \overset{\varepsilon}{\in} \mathcal{B}$ for all $a \in \mathcal{A}$.

We note that if $\mathcal{B}$ is a von Neumann algebra, we have $a \overset{\varepsilon}{\in} \mathcal{B}$ if and only if

$$\inf_{b \in \mathcal{B}} \|a - b\| \leq \varepsilon \|a\|.$$

That is, the infimum is attained by some $b \in \mathcal{B}$. Indeed, let b_i be a sequence in $\mathcal{B}$ such that $\lim_i \|a - b_i\| \leq \varepsilon \|a\|$. In particular, $\|b_i\|$ is bounded. Since (any multiple of) the closed unit ball in $B(\mathcal{H})$ is compact in the weak operator topology (e.g., Theorem 5.1.4 in [32]), the sequence b_i has a limit point $b \in \mathcal{B}$ in the weak operator topology. Because the norm is lower semi-continuous in the weak operator topology, $\|a - b\| \leq \lim_i \|a - b_i\| \leq \varepsilon \|a\|$, which concludes the argument.

When $\mathcal{B} \subseteq B(\mathcal{H})$ is a C^*-algebra and $x \in \mathcal{B}(\mathcal{H})$ is an operator that is nearly contained in its commutant, say $x \overset{\varepsilon}{\in} \mathcal{B}'$, then it is easy to see that, for any $b \in \mathcal{B}$,

$$\|[x, b]\| \leq 2\varepsilon \|x\| \|b\|. \tag{2.1}$$

Indeed $x \overset{\varepsilon}{\in} \mathcal{B}'$ means there exists $y \in \mathcal{B}'$ such that $\|x - y\| \leq \varepsilon \|x\|$. Then we have for any $b \in \mathcal{B}$ that

$$\|[x, b]\| = \|[x - y, b]\| \leq 2\|x - y\| \|b\| \leq 2\varepsilon \|x\| \|b\|.$$

We will be interested in the converse of this statement, which is rather less clear.

To gain some intuition, we consider the finite-dimensional setting. Suppose that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ for finite-dimensional Hilbert spaces, and let $\mathcal{B} = I \otimes B(\mathcal{H}_B) \subseteq B(\mathcal{H})$ be the algebra of operators supported on the second tensor factor. Then we can define a projection onto the commutant of $\mathcal{B}$ by twirling using Haar probability measure on the group $U(\mathcal{B})$ of unitaries on $\mathcal{B}$:

$$\mathbb{E}_{\mathcal{B}'} \colon B(\mathcal{H}) \to \mathcal{B}', \quad \mathbb{E}_{\mathcal{B}'}(x) = \int_{U(\mathcal{B})} u x u^* \, \mathrm{d}u. \tag{2.2}$$

In fact, the commutant is simply $\mathcal{A} = \mathcal{B}' = B(\mathcal{H}_A) \otimes I_B$, and the projection can equivalently be written in terms of the normalized partial trace, $\mathbb{E}_{\mathcal{B}'}(x) = \frac{1}{d_B} \operatorname{tr}_{\mathcal{B}}(x)$, where $d_B = \dim \mathcal{H}_B$. The projection exhibits the desirable property that if

$$\|[x, b]\| \leq \varepsilon \|x\| \|b\| \tag{2.3}$$

for all $b \in \mathcal{B}$, then

$$\|x - \mathbb{E}_{\mathcal{B}'}(x)\| \leq \int_{U(\mathcal{B})} \|x - u x u^*\| \, \mathrm{d}u = \int_{U(\mathcal{B})} \|[x, u]\| \, \mathrm{d}u \leq \varepsilon \|x\|. \tag{2.4}$$

This shows that, in the finite-dimensional setting, the commutator bound (2.3) implies that $x \overset{\varepsilon}{\in} \mathcal{B}'$.

In infinite dimensions, where no Haar integral is available, we need a different way to define the projection. One way to do so is using the so-called "property P." If $\mathcal{B} \subseteq B(\mathcal{H})$ is a von Neumann algebra, it has *property P* if for any $x \in B(\mathcal{H})$, there exists some $y \in \mathcal{B}'$ such that y is also in the weak operator closure (equivalently, the weak-$*$ closure) of the convex hull of $\{u x u^* : u \in U(\mathcal{B})\}$. Note that in the finite-dimensional setting this is immediate from the definition in terms of the Haar integral. We can also generalize the notion of twirling by a (non-commutative) *conditional expectation* $\mathbb{E}_{\mathcal{A}}$ onto a von Neumann algebra $\mathcal{A} \subseteq B(\mathcal{H})$. This is defined to be contractive completely positive linear map $\mathbb{E}_{\mathcal{A}} \colon B(\mathcal{H}) \to \mathcal{A} \subseteq B(\mathcal{H})$ which is such that for $x \in B(\mathcal{H})$ and $a, a' \in \mathcal{A}$ we have $\mathbb{E}_{\mathcal{A}}(a) = a$ and $\mathbb{E}_{\mathcal{A}}(a x a') = a \mathbb{E}_{\mathcal{A}}(x) a'$. A von Neumann algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called *injective* if there exists such a conditional expectation [33, IV.2.1.4].

For von Neumann algebras acting on separable Hilbert spaces, these properties are equivalent to each other and to hyperfiniteness as defined earlier:

Theorem 2.3. *Let $\mathcal{B} \subseteq B(\mathcal{H})$ be a von Neumann algebra with $\mathcal{H}$ separable. Then the following are equivalent:*

(i) $\mathcal{B}$ is hyperfinite.

(ii) $\mathcal{B}'$ is hyperfinite.

(iii) $\mathcal{B}$ has property P.

(iv) $\mathcal{B}'$ is injective.

If $\mathcal{H}$ is not assumed to be separable, it is still true that (i) implies (iii) and (iii) implies (iv). Moreover, $\mathcal{B}$ has property P if and only if $\mathcal{B}'$ has property P, and the same is true for injectivity.

For a comprehensive account of the theory and classification of von Neumann algebras see [33, 34]. Theorem 2.3 is proved in Proposition 4.1 of [35] in the case that $\mathcal{B}$ is a factor. The general case follows similarly by combination of well-known results, as we sketch for convenience.

Proof. The implications (i)⇒(iii) and (iii)⇒(iv) are explained in [33, IV.2.2.20], as is the fact that $\mathcal{B}$ has property P if and only if $\mathcal{B}'$ has property P. Moreover, [33, IV.2.2.7] asserts that $\mathcal{B}$ is injective if and only if $\mathcal{B}'$ is injective. Now assume that $\mathcal{H}$ is separable or, equivalently, $\mathcal{B}$ has a separable predual [36]. In this case, injectivity implies hyperfiniteness [33, IV.2.6.1], so it follows that (iv)⇒(ii). Since $\mathcal{B}'' = \mathcal{B}$, (i)⇒(ii) also yields (ii)⇒(i), so that (i)–(iv) are all equivalent. ∎

When $\mathcal{B}$ is hyperfinite and $x \in B(\mathcal{H})$ is such that $\|[x,b]\| \leq \varepsilon \|x\|\|b\|$ for all $b \in \mathcal{B}$, then $x \overset{\varepsilon}{\in} \mathcal{B}'$, providing a converse to the discussion above Eq. (2.1). Indeed, since $\mathcal{B}$ has property P by (i)⇒(iii), there exists some $y \in \mathcal{B}'$ in the weak operator closure of the convex hull of $\{uxu^* : u \in U(\mathcal{B})\}$. Using lower semicontinuity of the norm with respect to the weak operator topology, we find

$$\|x - y\| \leq \sup_{u \in U(\mathcal{B})} \|x - uxu^*\| = \sup_{u \in U(\mathcal{B})} \|[x,u]\| \leq \varepsilon \|x\|,$$

which shows that $x \overset{\varepsilon}{\in} \mathcal{B}'$. Moreover, if $x \in \mathcal{M}$, $\mathcal{B} \subseteq \mathcal{M}$ for a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, then we have that $x \overset{\varepsilon}{\in} \mathcal{B}' \cap \mathcal{M}$. Indeed, in this case $\{uxu^* : u \in U(\mathcal{B})\}$ is contained in $\mathcal{M}$, and the same is true for the weak operator closure of its convex hull. Since y was constructed as an element of the latter, it follows that $y \in \mathcal{B}' \cap \mathcal{M}$ and hence $x \overset{\varepsilon}{\in} \mathcal{B}' \cap \mathcal{M}$.

In turn, the above implies that any conditional expectation $\mathbb{E}_{\mathcal{B}'} : B(\mathcal{H}) \to \mathcal{B}'$ (and such conditional expectations exist due to (iii)⇒ (iv)) satisfies

$$\|\mathbb{E}_{\mathcal{B}'}(x) - x\| \leq \|\mathbb{E}_{\mathcal{B}'}(x) - \mathbb{E}_{\mathcal{B}'}(y)\| + \|y - x\| \leq 2\varepsilon \|x\|,$$

using that $\mathbb{E}_{\mathcal{B}'}(y) = y$ and that conditional expectations are contractions. When $\mathcal{B}$ is a factor, a different proof strategy shows that the constant 2 can be omitted; see Proposition 4.1 in [35].

As an easy consequence we obtain:

Lemma 2.4 (Near inclusions and commutators [37, Theorem 2.3])**.** *Let $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ be two C^*-algebras. If $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}'$ is a near inclusion, then*

$$\|[a,b]\| \leq 2\varepsilon \|a\|\|b\|.$$

holds for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Conversely, if $\mathcal{B}$ is a hyperfinite von Neumann algebra and

$$\|[a,b]\| \leq \varepsilon \|a\| \|b\|$$

holds for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then we have a near inclusion $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}'$. If moreover $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ for some von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, then $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}' \cap \mathcal{M}$.

Proof. The first claim follows from Eq. (2.1), since $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}'$ means that $a \overset{\varepsilon}{\in} \mathcal{B}'$ for every $a \in \mathcal{A}$. For the converse claim, the discussion above the lemma shows that for every $a \in \mathcal{A}$ we have $a \overset{\varepsilon}{\in} \mathcal{B}'$, hence $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}'$; moreover, if $a \in \mathcal{M}$, $\mathcal{B} \subseteq \mathcal{M}$ then $a \overset{\varepsilon}{\in} \mathcal{B}' \cap \mathcal{M}$, and hence $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}' \cap \mathcal{M}$. ■

As a straightforward consequence of Lemma 2.4, we in turn obtain the following:

Lemma 2.5 (Near inclusion of commutants)**.** *Let $\mathcal{C}, \mathcal{D} \subseteq B(\mathcal{H})$ be von Neumann algebras with $\mathcal{C}$ hyperfinite. If $\mathcal{C} \overset{\varepsilon}{\subseteq} \mathcal{D}$, then $\mathcal{D}' \overset{2\varepsilon}{\subseteq} \mathcal{C}'$. If moreover $\mathcal{C} \subseteq \mathcal{M}$ for a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, then $\mathcal{D}' \cap \mathcal{M} \overset{2\varepsilon}{\subseteq} \mathcal{C}' \cap \mathcal{M}$.*

Proof. It suffices to prove the second statement, since it reduces to the first if we choose $\mathcal{M} = B(\mathcal{H})$. Since $\mathcal{C} \overset{\varepsilon}{\subseteq} \mathcal{D} = (\mathcal{D}')'$, the first claim in Lemma 2.4 (with $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{D}'$) shows that

$$\|[a,b]\| \leq 2\varepsilon \|a\| \|b\| = 2\varepsilon \|a\| \|b\|$$

for all $a \in \mathcal{C}$ and $b \in \mathcal{D}'$. Since $\mathcal{C}$ is hyperfinite, we can now use the converse in Lemma 2.4 (with $\mathcal{A} = \mathcal{D}' \cap \mathcal{M}$ and $\mathcal{B} = \mathcal{C} \subseteq \mathcal{M}$) to conclude that $\mathcal{D}' \cap \mathcal{M} \overset{2\varepsilon}{\subseteq} \mathcal{C}' \cap \mathcal{M}$. ■

We now come to a central and nontrivial result. For hyperfinite von Neumann algebras, if $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ for sufficiently small ε, there exists a unitary close to the identity that rotates $\mathcal{A}$ into $\mathcal{B}$.

Theorem 2.6 (Near inclusions of subalgebras)**.** *For hyperfinite von Neumann algebras $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ with $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ for $\varepsilon \leq \frac{1}{64}$, there exists a unitary $u \in (\mathcal{A} \cup \mathcal{B})''$ such that $u^* \mathcal{A} u \subseteq \mathcal{B}$ and we have:*

(i) $\|I - u\| \leq 12\varepsilon$.

(ii) If $z \in B(\mathcal{H})$ satisfies $\|[z,c]\| \leq \delta \|z\| \|c\|$ for all $c \in \mathcal{A} \cup \mathcal{B}$, then $\|u^* z u - z\| \leq 10\delta \|z\|$.

Moreover, if $\mathcal{A}_0 \subseteq \mathcal{A}$ is an AF C^*-algebra such that $\mathcal{A}_0'' = \mathcal{A}$, then u can be chosen such that also:

(iii) If $z \in B(\mathcal{H})$ satisfies $z \overset{\delta}{\in} \mathcal{A}_0$ and $z \overset{\delta}{\in} \mathcal{B}$, then $\|u^* z u - z\| \leq 16\delta \|z\|$.

This theorem extends Theorem 4.1 of Christensen [25]. The first item re-states his result, and we develop the remaining claims. A self-contained proof appears in Appendix B. Similar stability theorems exist for various other classes of C^*-algebras [25, 38]. The stability of subalgebra inclusions is closely related to what is often (especially in the context of groups) referred to as *Ulam stability* [39, 40]. There, one is given a map that "almost" satisfies the homomorphism properties, and one asks whether the map can be slightly deformed into a true homomorphism. See for instance [41, 42] for Ulam stability results on C^*-algebras. The proof of Theorem 2.6 implicitly involves one such Ulam stability property: a completely positive map on a hyperfinite von Neumann algebra that is almost a homomorphism is then deformed to a true homomorphism; see e.g. [41] more generally.

Using related methods, we also obtain the following useful lemma. Here, we control the global error between two homomorphisms using the sum of errors on their local restrictions.

Lemma 2.7. *Consider two injective weak-$*$ continuous unital $*$-homomorphisms $\alpha_1, \alpha_2 \colon \mathcal{A} \to \mathcal{B}$ between von Neumann algebras $\mathcal{A} \subseteq B(\mathcal{H})$ and $\mathcal{B} \subseteq B(\mathcal{K})$, with hyperfinite von Neumann subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ that pairwise commute, i.e., $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$, and generate $\mathcal{A}$ in the sense that $(\cup_{i=1}^n \mathcal{A}_i)'' = \mathcal{A}$. Define*

$$\varepsilon = \sum_{i=1}^{n} \|(\alpha_1 - \alpha_2)|_{\mathcal{A}_i}\|.$$

Then if $\varepsilon < 1$,

$$\|\alpha_1 - \alpha_2\| \leq 2\sqrt{2}\varepsilon \left(1 + (1 - \varepsilon^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \leq 2\sqrt{2}\varepsilon,$$

where we note that the expression in the middle is $2\varepsilon + \mathcal{O}(\varepsilon^2)$.

The proof appears in Appendix B. We find the difference between the homomorphisms α_1 and α_2 is controlled by the sum of their local differences. It appears possible that in general, a tighter bound $\|\alpha_1 - \alpha_2\| \leq \varepsilon + \mathcal{O}(\varepsilon^2)$ may be correct. An easy example demonstrates the bound is optimal to within a constant factor. Let $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes A_n$ for matrix algebras A_i, and let $\alpha_1(x) = x$ and $\alpha_2(x) = u^* x u$ for $u = u_1 \otimes \cdots \otimes u_n$, choosing any $u_i \in U(\mathcal{A}_i)$ with spectrum $\{1, e^{i\frac{\varepsilon}{n}}\}$, so that $\|u_i - I\| = \frac{\varepsilon}{n} + \mathcal{O}(\varepsilon^2)$ and hence $\|u - I\| = \varepsilon + \mathcal{O}(\varepsilon^2)$. Note that by e.g. Theorem 26 of [43], the map on operators $x \mapsto vxv^* - x$ for unitary v has norm given by the diameter of the smallest closed disk containing the spectrum of v. For u_i and u, that diameter is given by $\frac{\varepsilon}{n} + \mathcal{O}(\varepsilon^2)$ and $\varepsilon + \mathcal{O}(\varepsilon^2)$, respectively. Then $\|(\alpha_1 - \alpha_2)|_{\mathcal{A}_i}\| = \frac{\varepsilon}{n} + \mathcal{O}((\frac{\varepsilon}{n})^2)$, while $\|\alpha_1 - \alpha_2\| = \varepsilon + \mathcal{O}(\varepsilon^2)$.

3 Dynamics on spin systems

We introduce in Section 3.1 the quasi-local algebra, which is the appropriate C^*-algebra to describe a lattice of quantum spin systems. Next we discuss the celebrated Lieb-Robinson bounds in Section 3.2, give a definition of approximately locality-preserving unitaries, and prove some of their basic properties.

3.1 The quasi-local algebra

If we have a system of a finite number of spins $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, the corresponding operator algebra is simply the full matrix algebra $\mathcal{M}_{d_1 \times d_1} \otimes \cdots \otimes \mathcal{M}_{d_n \times d_n}$. However, for infinitely many spins the tensor product structure becomes ambiguous. If the spins form a lattice the physically appropriate choice of C^*-algebra is the *quasi-local algebra*. Consider a lattice Γ, and associate a finite-dimensional matrix algebra $\mathcal{A}_n = \mathcal{M}_{d_n \times d_n}$ to each element n of the lattice. We assume that there is a uniform upper bound on the dimensions d_n. For any *finite* subset $X \subseteq \Gamma$ we can define the algebra $\mathcal{A}_X = \bigotimes_{n \in X} \mathcal{A}_n$. These algebras naturally form a local net, meaning that for any two subsets $X \subseteq X'$ we have a natural inclusion $\mathcal{A}_X \subseteq \mathcal{A}_{X'}$ (by tensoring with the identity on $X' \setminus X$), and for any two disjoint subsets $X \cap X' = \emptyset$ we have that $[\mathcal{A}_X, \mathcal{A}_{X'}] = 0$ (we embed the two algebras into any $\mathcal{A}_{X''}$ such that $X \cup X' \subseteq X''$). This allows us to define the algebra of all *strictly local* operators as

$$\mathcal{A}_\Gamma^{\text{strict}} = \bigcup_{X \subseteq \Gamma \text{ finite}} \mathcal{A}_X.$$

This is a $*$-algebra which inherits a norm from the $\mathcal{A}_X$, but it is not complete for this norm. We define the *quasi-local algebra* $\mathcal{A}_\Gamma$ to be the norm completion of $\mathcal{A}_\Gamma^{\text{strict}}$. Thus, $\mathcal{A}_\Gamma$ is a C^*-algebra. For infinite subsets $X \subseteq \Gamma$, we define $\mathcal{A}_X$ correspondingly as a norm-complete C^*-subalgebra of $\mathcal{A}_\Gamma$. Then we have inclusions $\mathcal{A}_X \subseteq \mathcal{A}_\Gamma$ for any subset $X \subseteq \Gamma$. If $x \in \mathcal{A}_X$, we say x is *supported* on X.

The quasi-local algebra has a natural state τ, called the *tracial state*, which can be thought of as the generalization of the maximally mixed state to an infinite lattice. It is defined on $x \in \mathcal{A}_X$ for finite $X \subseteq \Lambda$ by

$$\tau(x) = \frac{1}{d_X} \operatorname{tr}(x),$$

where $d_X = \prod_{n \in X} d_n$, and can be extended to the full algebra.

We consider the GNS representation $\pi \colon \mathcal{A}_\Gamma \to B(\mathcal{H})$ from Theorem 2.1 of the quasi-local algebra using the tracial state τ, and we let

$$\mathcal{A}_\Gamma^{\text{vN}} = \pi(\mathcal{A}_\Gamma)'' \subseteq B(\mathcal{H}), \tag{3.1}$$

denote the von Neumann algebra generated by the GNS representation of the quasi-local algebra. The right-hand side is also the weak operator closure of the image $\pi(\mathcal{A}_\Gamma)$. It turns out $\mathcal{A}_\Gamma^{\text{vN}}$ is a proper subalgebra of $B(\mathcal{H})$, which remains true even for Γ finite; in our case that Γ is infinite, $\mathcal{A}_\Gamma^{\text{vN}}$ is the (unique up to unique isomorphism) *hyperfinite type II_1 factor*. This algebra is extensively studied, but for our purpose we will only need to observe that this factor is hyperfinite (as follows directly from its construction). If $X \subseteq \Gamma$ we denote $\mathcal{A}_X^{\text{vN}} = \pi(\mathcal{A}_X)''$. Each $\mathcal{A}_X^{\text{vN}}$ is hyperfinite and has the property that $(\mathcal{A}_X^{\text{vN}})' \cap \mathcal{A}_\Gamma^{\text{vN}} = \mathcal{A}_{\Gamma \setminus X}^{\text{vN}}$. Since the tracial state is faithful, we may think of each $\mathcal{A}_X$ as a subalgebra of $\mathcal{A}_X^{\text{vN}}$. Moreover, it holds that $\mathcal{A}_X^{\text{vN}} \cap \mathcal{A}_\mathbb{Z} = \mathcal{A}_X$ for any $X \subseteq \mathbb{Z}$.[3]

The reason for introducing $\mathcal{A}_\Gamma^{\text{vN}}$ is purely to be able to use technical tools, especially Theorem 2.6, from the study of von Neumann algebras. Our main results are all formulated in terms of the quasi-local algebra.

We observe that an automorphism α of $\mathcal{A}_\Gamma$ extends naturally to the associated von Neumann algebra in (3.1), as follows. If τ is the tracial state on the quasi-local algebra $\mathcal{A}_\Gamma$, then for any

[3] Since $\mathcal{A}_X^{\text{vN}} = (\mathcal{A}_{\mathbb{Z}\setminus X}^{\text{vN}})' \cap \mathcal{A}_\mathbb{Z}^{\text{vN}}$, it suffices to show that $(\mathcal{A}_{\mathbb{Z}\setminus X}^{\text{vN}})' \cap \mathcal{A}_\mathbb{Z} = \mathcal{A}_X$. We will argue that $(\mathcal{A}_{\mathbb{Z}\setminus X}^{\text{vN}})' \cap \mathcal{A}_\mathbb{Z} \subseteq \mathcal{A}_X$, since the other inclusion is immediate. To this end, let $x \in (\mathcal{A}_{\mathbb{Z}\setminus X}^{\text{vN}})' \cap \mathcal{A}_\mathbb{Z}$. Since $x \in \mathcal{A}_\mathbb{Z}$, we can choose a sequence $x_i \in \mathcal{A}_{X_i}$ converging to x in norm and such that each X_i is a finite set. On the other hand, $x \in (\mathcal{A}_{\mathbb{Z}\setminus X}^{\text{vN}})'$ implies that, for any $y \in \mathcal{A}_{\mathbb{Z}\setminus X}$ it holds that

$$\|[y, x_i]\| = \|[y, x_i - x]\| \leq 2\|y\|\|x - x_i\|. \tag{3.2}$$

Now let $\tilde{x_i} = \int_{U(\mathcal{A}_{X_i \cap (\mathbb{Z}\setminus X)})} u x_i u^* \,\mathrm{d}u$, similarly as in Eq. (2.2). Then $\tilde{x_i} \in \mathcal{A}_{X_i \cap X} \subseteq \mathcal{A}_X$, since it is an element of $\mathcal{A}_{X_i}$ that commutes with $\mathcal{A}_{X_i \cap (\mathbb{Z}\setminus X)} = \mathcal{A}_{X_i \setminus X}$. On the other hand, Eq. (3.2) implies (cf. Eqs. (2.3) and (2.4)) that

$$\|\tilde{x_i} - x_i\| \leq \int_{U(\mathcal{A}_{X_i \cap (\mathbb{Z}\setminus X)})} \|u x_i u^* - x_i\| \,\mathrm{d}u \leq \int_{U(\mathcal{A}_{X_i \cap (\mathbb{Z}\setminus X)})} \|[u, x_i]\| \,\mathrm{d}u \leq 2\|x - x_i\|.$$

Therefore $\|\tilde{x_i} - x\| \leq 3\|x - x_i\| \to 0$ and since each $\tilde{x_i} \in \mathcal{A}_X$, we conclude that $x \in \mathcal{A}_X$.

automorphism of $\mathcal{A}_\Gamma$ this state is left invariant, i.e., $\tau \circ \alpha = \tau$. (One way to see this is by using that τ is the unique state for which $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}_\Gamma$.) By the uniqueness of the GNS construction this implies that α can be implemented by a unitary u on $\mathcal{H}$, in the sense that $\pi(\alpha(x)) = u\pi(x)u^*$. Therefore, α extends to an automorphism of the hyperfinite von Neumann algebra $\mathcal{A}_\Gamma^{\mathrm{vN}}$, which we denote by the same symbol α if there is no danger of confusion. Note that this extension is necessarily unique.

From Section 4 onwards we will only consider the situation where $\Gamma = \mathbb{Z}$ is the discrete line. If $X = \{m \in \mathbb{Z} \text{ such that } m \leq n\}$ we will write $\mathcal{A}_{\leq n} := \mathcal{A}_X$ and similarly if $X = \{m \in \mathbb{Z} \text{ such that } m \geq n\}$ we will write $\mathcal{A}_{\geq n} := \mathcal{A}_X$. We use the same notation to describe subalgebras $\mathcal{A}_{\leq n}^{\mathrm{vN}}, \mathcal{A}_{\geq n}^{\mathrm{vN}}$ of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$.

3.2 QCAs and approximately locality-preserving unitaries

Consider a spin system on a lattice Γ with some metric d and the associated quasi-local C^*-algebra $\mathcal{A}_\Gamma$. If $X \subseteq \Gamma$ we will denote

$$B(X, r) = \{n \in \Gamma \text{ such that } d(n, X) \leq r\}.$$

Definition 3.1 (QCA)**.** A *quantum cellular automaton* (QCA) *with radius* R is an automorphism $\alpha \colon \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma$ such that if x is an operator supported on a finite subset $X \subseteq \Gamma$, then $\alpha(x)$ is supported on $B(X, R)$. We call R a *radius* of the QCA.

One of the reasons to study QCAs is that many physical quantum dynamics preserve locality in some form. However, the locality in Definition 3.1 is very stringent, and one the most important classes of automorphisms violates strict locality, while preserving a form of approximate locality: evolution by a geometrically local Hamiltonian. The locality of these evolutions is expressed by so-called *Lieb-Robinson bounds* [3].

We will state a fairly general form of the Lieb-Robinson bounds which also holds for Hamiltonians which are not strictly local, but have a sufficiently fast decay, following e.g. [44] or [45]. Suppose that Γ is a lattice with a metric d. Then a monotonically decreasing function $F \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *reproducing* (implying fast decay) if there exists a constant $C > 0$ such that for all $n, m \in \Gamma$,

$$\sum_l F(d(n,l))F(d(l,m)) \leq CF(d(n,m)),$$

$$\sup_y \sum_x F(d(x,y)) < \infty.$$

These conditions are related to a convolution and integral, respectively. For $\Gamma = \mathbb{Z}^D$ with the Euclidean distance, the function $F(r) = (1+r)^{-(D+\varepsilon)}$ is reproducing for any $\varepsilon > 0$. Note the reproducing property is not strictly a measure of fast decay: an exponential decay alone is not reproducing, despite having faster decay than the previous power law, because it fails the first inequality. Meanwhile, $F(r) = (1+r)^{-(D+\varepsilon)}e^{-ar}$ for any $a > 0$ is again reproducing ([44], Appendix 8.2).

Now we consider the automorphism α on the quasi-local algebra $\mathcal{A}_\Gamma$ which is generated by time evolution for some fixed time T by a Hamiltonian

$$H = \sum_{n \in \Gamma} H_n + \sum_{X \subseteq \Gamma} H_X.$$

The terms H_n act only on site n, and the terms H_X act on the sites in X. Then, if the interaction terms H_X have sufficient decay, we have the following bounds on $\alpha(x) = e^{iHt}xe^{-iHt}$.[4] We state them without the dependence on the time t, which only affects the constant C below, and which is irrelevant for our purposes:

Theorem 3.2 (Lieb-Robinson [44])**.** *For $\alpha(x) = e^{iHt}xe^{-iHt}$ as above, if F is reproducing and*

$$\sup_{n,m\in\Gamma} \sum_{\substack{X\subseteq\Gamma \\ s.t.\ n,m\in X}} \frac{\|H_X\|}{F(d(n,m))} \leq \infty \tag{3.3}$$

then there exists a constant $C > 0$ such that for all $X, Y \subseteq \Gamma$ and for all $x \in \mathcal{A}_X$, $y \in \mathcal{A}_Y$ we have

$$\|[\alpha(x), y]\| \leq C\|x\|\|y\| \sum_{n\in X}\sum_{m\in Y} F(d(n,m)). \tag{3.4}$$

Here, the Hamiltonian is also allowed to be time-dependent, as long as (3.3) holds uniformly. See [44] for a proof and extensive discussion.

We are particularly interested in the one-dimensional case, where $\Gamma = \mathbb{Z}$ and $d(x,y) = |x-y|$ for $x, y \in \mathbb{Z}$. In that setting we consider the case where X is an *interval* (a finite or infinite sequence of consecutive sites) and Y has bounded distance away from X. A consequence of the Lieb-Robinson bounds in (3.4) is that certain algebras form near inclusions.

Lemma 3.3. *Suppose α is an automorphism of $\mathcal{A}_{\mathbb{Z}}$ and suppose there exists a monotonically decreasing function $F\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $X, Y \subseteq \Gamma$,*

$$\|[\alpha(x), y]\| \leq \|x\|\|y\| \sum_{n\in X}\sum_{m\in Y} F(|n-m|). \tag{3.5}$$

and suppose $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F(n+m) < \infty$. Then for any (finite or infinite) interval $X \subseteq \mathbb{Z}$, we have

$$\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)},$$

where

$$f(r) = 4\sum_{n,m=0}^{\infty} F(n+m+r+1).$$

Proof. We first prove the near inclusion for finite X. By Lemma 2.4, it suffices to show that for any $x \in \mathcal{A}_X$ and any $y \in \mathcal{A}_{B(X,r)^c}$ we have

$$\|[\alpha(x), y]\| \leq f(r)\|x\|\|y\|$$

as in that case $\mathcal{A}^{\mathrm{vN}}_{B(X,r)} = \mathcal{A}_{B(X,r)}$. By Eq. (3.5), we know that

$$\|[\alpha(x), y]\| \leq \|x\|\|y\| \sum_{n\in X}\sum_{m\in B(X,r)^c} F(|n-m|)$$

[4] In fact, one generally needs these bounds to prove that the time evolution defines a dynamics on the quasi-local algebra, i.e. that time-evolved quasi-local operators are still quasi-local [44].

Let

$$X_k = \{n \in X \text{ such that } d(n, X^c) = k\},$$
$$Y_l = \{m \in X^c \text{ such that } d(m, X) = r + l\},$$

using the notation $d(n, X) = \min_{x \in X} |n - x|$. Since X is an interval the size of each of these sets is upper bounded by 2. We can therefore estimate

$$\begin{aligned}\sum_{n\in X}\sum_{m\in B(X,r)^c} F(|n-m|) &\leq \sum_{k\geq 1}\sum_{l\geq 1}\sum_{n\in X_k}\sum_{m\in Y_l} F(k+l+r-1)\\ &\leq 4\sum_{\geq 1}\sum_{l\geq 1} F(k+l+r-1)\\ &= f(r).\end{aligned}$$

We conclude that $\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}$ for any finite interval X.

If X is infinite and $x \in \mathcal{A}_X$, we can take a sequence x_i such that $\lim_i x_i = x$ in norm and each x_i is supported on a finite interval inside X. By what we showed above, for each i there exists some $y_i \in \mathcal{A}_{B(X,r)}$ such that $\|\alpha(x_i) - y_i\| \leq f(r)\|x\|$. Then

$$\begin{aligned}\inf_{y\in\mathcal{A}_{B(X,r)}} \|\alpha(x) - y\| &\leq \liminf_i \|\alpha(x) - y_i\|\\ &\leq \liminf_i \left(\|\alpha(x) - \alpha(x_i)\| + \|\alpha(x_i) - y_i\|\right)\\ &\leq f(r)\|x\|.\end{aligned}$$

■

For instance, if $F(r) = \frac{1}{r^4}$, then $f(r) = \mathcal{O}(\frac{1}{r^2})$; if $F(r) = e^{-ar}\frac{1}{r^2}$ for $a > 0$, then $f(r) = \mathcal{O}(e^{-ar})$. As a side note, we observe that one can use Lemma B.3 on simultaneous near inclusions to show that (in any dimension) Lieb-Robinson type bounds for single-site operators imply bounds for operators on arbitrary sets (which has already been remarked upon in a more restricted setting in [27]):

Lemma 3.4. *Suppose α is an automorphism of the quasi-local algebra $\mathcal{A}_\Gamma$ and suppose there exists a function $G\colon \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ such that for any $n, m \in \Gamma$, $x \in \mathcal{A}_n$ and $y \in \mathcal{A}_m$*

$$\|[\alpha(x), y]\| \leq \|x\|\|y\| G(n, m).$$

Then for any finite sets $X, Y \subseteq \Gamma$ and $x \in \mathcal{A}_X$, $y \in \mathcal{A}_Y$,

$$\|[\alpha(x), y]\| \leq 128\|x\|\|y\| \sum_{n\in X}\sum_{m\in Y} G(n, m).$$

Proof. By assumption and Lemma 2.4 with $\mathcal{M} = \mathcal{A}_\Gamma^{\mathrm{vN}}$ we have

$$\alpha(\mathcal{A}_n) \overset{G(n,m)}{\subseteq} \mathcal{A}_m' \cap \mathcal{A}_\Gamma^{\mathrm{vN}} = \mathcal{A}_{\Gamma\setminus\{m\}}^{\mathrm{vN}}$$

for all $m, n \in \Gamma$. Applying Lemma B.3 we find that

$$\alpha(\mathcal{A}_X) \overset{4\sum_{n\in X} G(n,m)}{\subseteq} \mathcal{A}_{\Gamma\setminus\{m\}}^{\mathrm{vN}}$$

for all $m \in \Gamma$. Lemma 2.5 with $\mathcal{M} = \mathcal{A}_\Gamma^{\mathrm{vN}}$ shows that

$$\mathcal{A}_m \overset{8\sum_{n\in X} G(n,m)}{\subseteq} \alpha(\mathcal{A}_X)' \cap \mathcal{A}_\Gamma^{\mathrm{vN}} = \alpha(\mathcal{A}_X' \cap \mathcal{A}_\Gamma^{\mathrm{vN}}) = \alpha(\mathcal{A}_{\Gamma\setminus X}^{\mathrm{vN}}).$$

Again applying Lemma B.3 and Lemma 2.5 as above yields

$$\alpha(\mathcal{A}_X) \overset{64\sum_{n\in X, m\in Y} G(n,m)}{\subseteq} \mathcal{A}_{\Gamma\setminus Y}^{\mathrm{vN}}$$

which implies the desired commutator bound by Lemma 2.4. ∎

Following [23] we would like to generalize the notion of a QCA to the case where the automorphism does not preserve strict locality, but only approximate locality. Such an automorphism is often called quasi-local. There are various choices of definition that require different decays or dependence on support size; see for instance [44]. For our purpose, the definition should at least include Hamiltonian evolutions satisfying Lieb-Robinson bounds. We will restrict to the one-dimensional case, where Theorem 3.2 and Lemma 3.3 inspire the following definition:

Definition 3.5 (ALPU in one dimension)**.** An automorphism α of the quasi-local algebra $\mathcal{A}_\mathbb{Z}$ is called an *approximately locality-preserving unitary* (ALPU) if for all (possibly infinite) intervals $X \subseteq \mathbb{Z}$ and for all $r \geq 0$ we have

$$\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}$$

for some positive function $f(r)$ with $\lim_{r\to\infty} f(r) = 0$. Here we use the notation in Definition 2.2.

We say α has $f(r)$-tails when it satisfies the above, or $\mathcal{O}(g(r))$-tails if $f(r) = \mathcal{O}(g(r))$. We will always assume, without loss of generality, that $f(r)$ is non-increasing.

Note that by definition, if α has $f(r)$-tails, it also has $h(r)$-tails for any function $h(r)$ with $h(r) \geq f(r)$ for all r, i.e., $f(r)$ only serves as an upper bound on the spread of α. Furthermore, note that any ALPU has $o(1)$-tails, by definition.

It suffices to check the conditions in Definition 3.5 either for all finite or for all infinite intervals (see Lemmas 3.10 and 3.11 below). If the above conditions on α are satisfied for all intervals X of some fixed size (and arbitrary $r \geq 0$), but $f(r)$ decays exponentially, then in fact α is an ALPU with $\mathcal{O}(f(r))$-tails by Lemma 3.4. We note an equivalent definition of ALPUs when passing to von Neumann algebras in Remark 3.9.

Remark 3.6. In [23] what we call an ALPU is simply called a locality-preserving unitary (LPU). Moreover, there it is said that an automorphism is a *locally generated unitary* (LGU) if it arises from time evolution by some time-dependent Hamiltonian. We have chosen the more explicit term ALPU instead of LPU, since in the literature the latter has also been used as a synonym for QCA (e.g. [15]).

We note that to call such automorphisms "unitary" is perhaps slightly misleading: there need not be a unitary $u \in \mathcal{A}_\mathbb{Z}$ such that $\alpha(x) = u^*xu$ (but there will be a unique unitary implementing α on the GNS Hilbert space with respect to the tracial state, as discussed in Section 3.1).

Example 3.7. Lemma 3.3 states that for the class of local Hamiltonians in Theorem 3.2 (Lieb-Robinson), the automorphism $\alpha(x) = e^{iHt}xe^{-iHt}$ is an ALPU at fixed t. It turns out that if the Hamiltonian has *exponentially decaying tails* in the sense that $\|H_X\| = \mathcal{O}(e^{-k|X|})$ decays exponentially with the size of the support X, then for any $k' < k$ we may take $f(r) = \mathcal{O}(e^{-k'r})$ and α has $\mathcal{O}(e^{-k'r})$-tails [44, 45]. Such evolutions composed with translations are also ALPUs.

To use Theorem 2.6, we would like to work in the von Neumann algebra $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$. However, in the definition of an ALPU we consider an automorphism of $\mathcal{A}_{\mathbb{Z}}$, not $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$. We therefore prove some results allowing us to translate between the tails for automorphisms of $\mathcal{A}_{\mathbb{Z}}$ versus $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$.

Lemma 3.8. *(i) Suppose α is an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ and $\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_Y^{\mathrm{vN}}$ for some $X, Y \subseteq \mathbb{Z}$ and $\varepsilon \geq 0$. Then $\alpha(\mathcal{A}_X^{\mathrm{vN}}) \overset{\varepsilon}{\subseteq} \mathcal{A}_Y^{\mathrm{vN}}$. In particular, any ALPU with $f(r)$-tails extends to an automorphism α of $\mathcal{A}_X^{\mathrm{vN}}$ such that $\alpha(\mathcal{A}_X^{\mathrm{vN}}) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}^{\mathrm{vN}}$ for any interval X and $r \geq 0$.*

(ii) Suppose α is an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ such that $\alpha(\mathcal{A}_X^{\mathrm{vN}}) \overset{\varepsilon}{\subseteq} \mathcal{A}_{B(X,r)}^{\mathrm{vN}}$ for any interval X and some fixed $r, \varepsilon \geq 0$. Then $\alpha^{-1}(\mathcal{A}_X^{\mathrm{vN}}) \overset{4\varepsilon}{\subseteq} \mathcal{A}_{B(X,r)}^{\mathrm{vN}}$ for any interval X.

(iii) Suppose α is an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ such that $\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}^{\mathrm{vN}}$ for any interval X and any $r \geq 0$, where $f(r)$ is a function with $\lim_{r\to\infty} f(r) = 0$. Then α restricts to an ALPU with $f(r)$-tails.

(iv) If α is an ALPU with $f(r)$-tails, then α^{-1} is an ALPU with $4f(r)$-tails.

Proof. (i) Let $x \in \mathcal{A}_X^{\mathrm{vN}}$. Using the Kaplansky density theorem, choose a net $x_i \in \mathcal{A}_X$ with $\|x_i\| \leq \|x\|$, converging to x in the weak operator topology, hence also in the weak-$*$ topology (since these topologies are the same on bounded subsets). Because α is weak-$*$ continuous, $\alpha(x_i)$ converges to $\alpha(x)$ in the weak-$*$ and hence also in the weak operator topology. Meanwhile, by our assumption that $\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_Y^{\mathrm{vN}}$, there exist $y_i \in \mathcal{A}_Y^{\mathrm{vN}}$ such that $\|\alpha(x_i) - y_i\| \leq \varepsilon\|x\|$. In particular, $\|y_i\| \leq (1+\varepsilon)\|x\|$. Hence the net y_i is bounded in norm. Since norm balls are compact in the weak operator topology, this implies there must be a converging subnet, which we also denote by y_i. Denoting the limit of y_i by y, then $y \in \mathcal{A}_Y^{\mathrm{vN}}$, and by lower semi-continuity of the norm in the weak operator topology,

$$\|y - \alpha(x)\| \leq \liminf_i \|y_i - \alpha(x_i)\| \leq \varepsilon\|x\|.$$

This shows that $\alpha(x) \overset{\varepsilon}{\in} \mathcal{A}_Y^{\mathrm{vN}}$, proving (i).

(ii) Note that $\mathbb{Z}\backslash B(X,r)$ is a disjoint union of at most two intervals Y_1 and Y_2, and $B(Y_i, r) \subseteq \mathbb{Z}\backslash X$ for $i = 1, 2$, so $\alpha(\mathcal{A}_{Y_i}^{\mathrm{vN}}) \overset{\varepsilon}{\subseteq} \mathcal{A}_{\mathbb{Z}\backslash X}^{\mathrm{vN}}$. Then applying (B.11) in Lemma B.3 to these two near inclusions with $\mathcal{M} = \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$,

$$\begin{aligned}
\mathcal{A}_X^{\mathrm{vN}} &= (\mathcal{A}_{\mathbb{Z}\backslash X}^{\mathrm{vN}})' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}} \overset{4\varepsilon}{\subseteq} \left(\alpha(\mathcal{A}_{Y_1}^{\mathrm{vN}}) \cup \alpha(\mathcal{A}_{Y_2}^{\mathrm{vN}})\right)' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}} = (\alpha(\mathcal{A}_{Y_1}^{\mathrm{vN}}))' \cap (\alpha(\mathcal{A}_{Y_2}^{\mathrm{vN}}))' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}} \\
&= \alpha((\mathcal{A}_{Y_1}^{\mathrm{vN}})' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}) \cap \alpha((\mathcal{A}_{Y_2}^{\mathrm{vN}})' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}) = \alpha((\mathcal{A}_{Y_1}^{\mathrm{vN}})' \cap (\mathcal{A}_{Y_2}^{\mathrm{vN}})' \cap \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}) \\
&= \alpha(\mathcal{A}_{\mathbb{Z}\backslash Y_1}^{\mathrm{vN}} \cap \mathcal{A}_{\mathbb{Z}\backslash Y_2}^{\mathrm{vN}}) = \alpha(\mathcal{A}_{B(X,r)}^{\mathrm{vN}})
\end{aligned}$$

and the conclusion follows by applying α^{-1}.

(iii) We need to show that if $x \in \mathcal{A}_\mathbb{Z}$, then $\alpha(x) \in \mathcal{A}_\mathbb{Z}$. First consider x strictly local, on some finite interval X. Then by assumption there is a sequence $y_r \in \mathcal{A}^{\mathrm{vN}}_{B(X,r)} = \mathcal{A}_{B(X,r)}$ such that $\|\alpha(x) - y_r\| \leq f(r)\|x\|$. Hence, y_r is a sequence of strictly local operators converging in norm to $\alpha(x)$ and hence $\alpha(x) \in \mathcal{A}_\mathbb{Z}$. If $x \in \mathcal{A}_\mathbb{Z}$ is not strictly local, let x_i be a sequence of strictly local operators converging in norm to x. Then $\alpha(x_i) \in \mathcal{A}_\mathbb{Z}$ and $\alpha(x_i)$ converges in norm to $\alpha(x)$. Similarly, α^{-1} maps $\mathcal{A}_\mathbb{Z}$ into $\mathcal{A}_\mathbb{Z}$ (using that (ii) implies locality bounds for α^{-1}) and hence we conclude that α restricts to an automorphism of $\mathcal{A}_\mathbb{Z}$.

This implies the desired result, as then

$$\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{B(X,r)} \cap \mathcal{A}_\mathbb{Z} = \mathcal{A}_{B(X,r)}$$

using the general fact that $\mathcal{A}^{\mathrm{vN}}_Y \cap \mathcal{A}_\mathbb{Z} = \mathcal{A}_Y$ for any $Y \subseteq \mathbb{Z}$.

(iv) By (i), α extends to an automorphism of $\mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$ with $f(r)$-tails, by (ii) the inverse of this extension has $4f(r)$-tails, and by (iii) the restriction of the latter is an ALPU with $4f(r)$-tails. ∎

Recall that any automorphism α of the quasi-local algebra $\mathcal{A}_\mathbb{Z}$ extends uniquely to an automorphism of the von Neumann algebra $\mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$, which we denote by the same symbol α. Then Lemma 3.8(i) and (iii) together allow an equivalent definition of ALPUs with $f(r)$-tails using the von Neumann algebra, rather than using the quasi-local algebra as in Definition 3.5. We summarize below.

Remark 3.9. Any automorphism $\alpha : \mathcal{A}_\mathbb{Z} \to \mathcal{A}_\mathbb{Z}$ with $f(r)$-tails, i.e. that satisfies $\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}$ for all intervals X and $r \geq 0$, uniquely extends to an automorphism $\alpha : \mathcal{A}^{\mathrm{vN}}_\mathbb{Z} \to \mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$ that has the same tails, i.e. that satisfies $\alpha(\mathcal{A}^{\mathrm{vN}}_X) \overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{B(X,r)}$. Conversely, any $\alpha : \mathcal{A}^{\mathrm{vN}}_\mathbb{Z} \to \mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$ satisfying the latter (for all X and r) restricts to an ALPU $\alpha : \mathcal{A}_\mathbb{Z} \to \mathcal{A}_\mathbb{Z}$ with $f(r)$-tails. Hence we may identify an ALPU α with its extension to $\mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$ and refer to the latter also as an "ALPU with $f(r)$-tails."

We can use Lemma 3.8 to show that in Definition 3.5 we may in fact restrict to either only finite intervals or only half-infinite intervals, as shown by the following lemmas.

Lemma 3.10. *Suppose α is an automorphism of $\mathcal{A}_\mathbb{Z}$ such that $\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}_{B(X,r)}$ for any* finite *interval $X \subseteq \mathbb{Z}$ and any $r \geq 0$, where $f(r)$ is a positive function with $\lim_{r\to\infty} f(r) = 0$. Then α is an ALPU with $f(r)$-tails.*

Proof. As explained earlier, we can extend α to an automorphism of $\mathcal{A}^{\mathrm{vN}}_\mathbb{Z}$ (denoted again by α). Let $X \subseteq \mathbb{Z}$ be an infinite interval, $r > 0$, and $0 \neq x \in \mathcal{A}_X$. We first show that for any $\delta > 0$,

$$\alpha(x) \overset{(1+\delta)f(r)}{\in} \mathcal{A}_{B(X,r)}. \tag{3.6}$$

By definition of $\mathcal{A}_\mathbb{Z}$, we can approximate x with a sequence $x_i \to x$ converging in norm, with $x_i \in \mathcal{A}_{X_i}$, where each $X_i \subseteq X$ is a finite interval. Then $\alpha(x_i) \to \alpha(x)$ converges in norm as well, and

$$\begin{aligned}\inf_{y\in\mathcal{A}_{B(X,r)}} \|y - \alpha(x)\| &\leq \liminf_i \Big(\inf_{y\in\mathcal{A}_{B(X,r)}} \|y - \alpha(x_i)\| + \|\alpha(x) - \alpha(x_i)\| \Big) \\ &\leq \liminf_i \Big(f(r)\|x_i\| + \|\alpha(x) - \alpha(x_i)\| \Big) = f(r)\|x\|,\end{aligned}$$

so for any $\delta' > 0$ there exists some $y \in \mathcal{A}_{B(X,r)}$ such that $\|y - \alpha(x)\| \leq f(x)\|x\| + \delta'$. If we apply this with $\delta' = f(x)\delta\|x\|$ then Eq. (3.6) follows. Next, we claim that

$$\alpha(x) \overset{f(r)}{\in} \mathcal{A}^{\mathrm{vN}}_{B(X,r)}. \tag{3.7}$$

Indeed, by Eq. (3.6) we can for any $n > 0$ take some $y_n \in \mathcal{A}_{B(X,r)}$ such that $\|\alpha(x) - y_n\| \leq (1+\frac{1}{n})f(r)$. In particular, y_n is a bounded sequence in $\mathcal{A}^{\mathrm{vN}}_{B(X,r)}$. Since norm balls are compact in the weak operator topology, there is a subsequence y_{n_i} converging to some $y \in \mathcal{A}^{\mathrm{vN}}_{B(X,r)}$, and

$$\|\alpha(x) - y\| \leq \liminf_i \|\alpha(x) - y_{n_i}\| \leq \liminf_i \left(1 + \tfrac{1}{n_i}\right) f(r) = f(r).$$

Thus we have proved Eq. (3.7). As a consequence, we have for any interval $X \subseteq \mathbb{Z}$ and any $r \geq 0$,

$$\alpha(\mathcal{A}_X) \overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{B(X,r)}.$$

Now the lemma follows from Lemma 3.8(iii). ■

Lemma 3.11. *Suppose α is an automorphism of $\mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$ such that $\alpha(\mathcal{A}_{\leq n}) \overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\leq n+r}$ and $\alpha(\mathcal{A}_{\geq n}) \overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\geq n-r}$ for any $n \in \mathbb{Z}$ and $r \geq 0$, where $f(r)$ is a positive function with $\lim_{r\to\infty} f(r) = 0$. Then α restricts to an ALPU with $8f(r)$-tails.*

Proof. By (iii) of Lemma 3.8 we only need to show that for any finite interval $X = \{n, n+1, \ldots, n+m\}$ it holds that

$$\alpha(\mathcal{A}_X) \overset{8f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{B(X,r)}.$$

Now, by (i) of Lemma 3.8 we have

$$\begin{aligned} \alpha(\mathcal{A}^{\mathrm{vN}}_{\leq n+m}) &\overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\leq n+m+r}, \\ \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}) &\overset{f(r)}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\geq n-r}, \end{aligned}$$

hence we obtain by taking commutants and applying Lemma 2.5 with $\mathcal{M} = \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$ that

$$\begin{aligned} \mathcal{A}^{\mathrm{vN}}_{\geq n+m+r+1} &\overset{2f(r)}{\subseteq} \alpha(\mathcal{A}^{\mathrm{vN}}_{\leq n+m})' \cap \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}} = \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n+m+1}) \subseteq \alpha(\mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}\setminus X}), \\ \mathcal{A}^{\mathrm{vN}}_{\leq n-r-1} &\overset{2f(r)}{\subseteq} \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})' \cap \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}} = \alpha(\mathcal{A}^{\mathrm{vN}}_{\leq n-1}) \subseteq \alpha(\mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}\setminus X}). \end{aligned}$$

By Eq. (B.11) of Lemma B.3 it follows that

$$\alpha(\mathcal{A}_X) = \alpha(\mathcal{A}^{\mathrm{vN}}_X) = \alpha(\mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}\setminus X})' \cap \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}} \overset{8f(r)}{\subseteq} \left(\mathcal{A}^{\mathrm{vN}}_{\geq n+m+r+1} \cup \mathcal{A}^{\mathrm{vN}}_{\leq n-r-1}\right)' \cap \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}} = \mathcal{A}^{\mathrm{vN}}_{B(X,r)}.$$

■

If we consider an ALPU, we may coarse-grain the lattice by grouping together (or 'blocking') sites. This yields again an ALPU, but with faster decaying tails. In particular, for any fixed $\varepsilon > 0$, we can always coarse-grain by sufficiently large blocks of sites so that on the coarse-grained lattice, $\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_{B(X,1)}$ for any interval X. This motivates the following definition:

Definition 3.12 (ε-nearest neighbor automorphism in one dimension)**.** An automorphism α of $\mathcal{A}_{\mathbb{Z}}$ is called *ε-nearest neighbor* for some $\varepsilon \geq 0$ if for any (finite or infinite) interval $X \subseteq \mathbb{Z}$ we have

$$\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_{B(X,1)}. \tag{3.8}$$

If α is an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ we instead require the weaker condition that

$$\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_{B(X,1)}^{\mathrm{vN}} \tag{3.9}$$

for all intervals $X \subseteq \mathbb{Z}$. Note that Eq. (3.9) is equivalent to $\alpha(\mathcal{A}_X^{\mathrm{vN}}) \overset{\varepsilon}{\subseteq} \mathcal{A}_{B(X,1)}^{\mathrm{vN}}$ by Lemma 3.8(i).

If an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ extends an automorphism of $\mathcal{A}_{\mathbb{Z}}$, as will usually be the case for us, then Eqs. (3.8) and (3.9) are equivalent, since $\mathcal{A}_{B(X,1)}^{\mathrm{vN}} \cap \mathcal{A}_{\mathbb{Z}} = \mathcal{A}_{B(X,1)}$ for any $X \subseteq \mathbb{Z}$. As such, Definition 3.12 is unambiguous.

4 Index theory of one-dimensional QCAs revisited

In this section we discuss the index theory of QCAs in one dimension. First, in Section 4.1 we review the definition and some of the most important properties of the GNVW index as proven in [6]. In Section 4.2 we provide an alternative formula for the index in terms of a difference of mutual informations. In Section 4.3 we prove some results about QCAs in one dimension which are locally close to each other. These results are interesting in their own right, but will also be crucial when extending the index to ALPUs.

4.1 The structure of one-dimensional QCAs and the GNVW index

One-dimensional QCAs have a beautifully simple structure theory, which we will now review. The material in this section is based on [6], which we recommend for a more extensive discussion. The same material is also covered in the review [19]. The discussion below refers only to QCAs, serving as a warm-up for the case of ALPUs.

Suppose that α is a nearest-neighbor QCA, which we may assume without loss of generality after blocking sites. Let

$$\begin{aligned} \mathcal{B}_n &= \mathcal{A}_{\{2n,2n+1\}} \\ \mathcal{C}_n &= \mathcal{A}_{\{2n-1,2n\}} \end{aligned} \tag{4.1}$$

be algebras on pairs of adjacent sites; with $\mathcal{B}_n$ and $\mathcal{C}_n$ corresponding to pairs staggered by one. In particular, $\alpha(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$. Define

$$\begin{aligned} \mathcal{L}_n &= \alpha(\mathcal{B}_n) \cap \mathcal{C}_n \\ \mathcal{R}_n &= \alpha(\mathcal{B}_n) \cap \mathcal{C}_{n+1}. \end{aligned} \tag{4.2}$$

See Fig. 4 as a mnemonic. These are manifestly algebras, but naively they might be trivial. Instead, it turns out that they provide factorizations of $\mathcal{C}_n$ and $\mathcal{B}_n$. Using the notation $\mathcal{M} \otimes \mathcal{N} := (\mathcal{M} \cup \mathcal{N})''$ for finite-dimensional mutually commuting subalgebras $\mathcal{M}, \mathcal{N} \subset \mathcal{A}_{\mathbb{Z}}$, one has the following result.

Theorem 4.1 (Factorization [6])**.**

$$\mathcal{C}_n := \mathcal{A}_{\{2n-1,2n\}} = \mathcal{L}_n \otimes \mathcal{R}_{n-1} \tag{4.3}$$
$$\mathcal{B}_n := \mathcal{A}_{\{2n,2n+1\}} = \alpha^{-1}(\mathcal{L}_n) \otimes \alpha^{-1}(\mathcal{R}_n). \tag{4.4}$$

Thus $\alpha^{-1}(\mathcal{L}_n)$ is the part of $\mathcal{B}_n$ that α sends to the left, and $\alpha^{-1}(\mathcal{R}_n)$ is the part of $\mathcal{B}_n$ that α sends to the right. Likewise, $\mathcal{C}_n$ is composed of a part $\mathcal{L}_n$ that was sent leftward from $\mathcal{B}_n$, and a part $\mathcal{R}_{n-1}$ that was sent rightward from $\mathcal{B}_{n-1}$.

Proof. Recall from Eq. (2.2) that in general, for a finite-dimensional subalgebra $\mathcal{M} \subset \mathcal{A}_{\mathbb{Z}}$, we have the conditional expectation $\mathbb{E}_{\mathcal{M}'}(x) = \int_{U(\mathcal{M})} uxu^* \,\mathrm{d}u$. We first show

$$\mathcal{L}_n := \alpha(\mathcal{B}_n) \cap \mathcal{C}_n = \mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(\mathcal{B}_n)) \tag{4.5}$$
$$\mathcal{R}_{n-1} := \alpha(\mathcal{B}_{n-1}) \cap \mathcal{C}_n = \mathbb{E}_{\mathcal{C}'_{n-1}}(\alpha(\mathcal{B}_{n-1})). \tag{4.6}$$

Clearly, $\mathcal{L}_n \subseteq \mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(\mathcal{B}_n))$. To show the reverse inclusion, let $y = \mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(x))$ for some $x \in \mathcal{B}_n$, i.e.

$$y = \mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(x)) = \int_{U(\mathcal{C}_{n+1})} u\alpha(x)u^* \,\mathrm{d}u.$$

From this expression, we see $[y, \alpha(\mathcal{B}_{n-1})] = 0$ because $[\alpha(x), \alpha(\mathcal{B}_{n-1})] = 0$ and $[\mathcal{C}_{n+1}, \alpha(\mathcal{B}_{n-1})] = 0$ (the latter because $\alpha(\mathcal{B}_{n-1}) \subseteq \mathcal{C}_{n-1} \otimes \mathcal{C}_n$). On the other hand, it follows from $\alpha(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$ that $y \in \mathcal{C}_n$. Moreover, α^{-1} is again a nearest neighbor QCA, so we have $\alpha^{-1}(\mathcal{C}_n) \subset \mathcal{B}_{n-1} \otimes \mathcal{B}_n$, so we find that $y \in \alpha(\mathcal{B}_{n-1} \otimes \mathcal{B}_n)$. Then $[y, \alpha(\mathcal{B}_{n-1})] = 0$ implies $y \in \alpha(\mathcal{B}_n)$. We conclude Eq. (4.5) holds; a similar argument shows Eq. (4.6).

Finally we demonstrate $\mathcal{C}_n \subseteq \mathcal{L}_n \otimes \mathcal{R}_{n-1}$, which then becomes an equality. For any $c \in \mathcal{C}_n$, we can express $\alpha^{-1}(c) \in \mathcal{B}_{n-1} \otimes \mathcal{B}_n$ as $\alpha^{-1}(c) = \sum_i a_i b_i$ for some elements $a_i \in \mathcal{B}_{n-1}, b_i \in \mathcal{B}_n$. Then

$$c = \mathbb{E}_{\mathcal{C}'_{n-1}}\mathbb{E}_{\mathcal{C}'_{n+1}}(c) = \sum_i \mathbb{E}_{\mathcal{C}'_{n-1}}\mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(a_i)\alpha(b_i)) = \sum_i \mathbb{E}_{\mathcal{C}'_{n-1}}(\alpha(a_i))\mathbb{E}_{\mathcal{C}'_{n+1}}(\alpha(b_i)) \in \mathcal{L}_n \otimes \mathcal{R}_{n-1},$$

as desired. The final equality follows from $\alpha(a_i) \in \mathcal{C}_{n-1} \otimes \mathcal{C}_n$ and $\alpha(b_i) \in \mathcal{C}_n \otimes \mathcal{C}_{n+1}$, and the final inclusion is manifest from Eqs. (4.5) and (4.6). Thus we have proved Eq. (4.3).

Noting again that α^{-1} is a nearest neighbor QCA, similar logic applied to α^{-1} yields Eq. (4.4). Specifically, Eqs. (4.5) and (4.6) are replaced by

$$\alpha^{-1}(\mathcal{L}_n) = \mathcal{B}_n \cap \alpha^{-1}(\mathcal{C}_n) = \mathbb{E}_{\mathcal{B}'_{n-1}}(\alpha^{-1}(\mathcal{C}_n))),$$
$$\alpha^{-1}(\mathcal{R}_n) = \mathcal{B}_n \cap \alpha^{-1}(\mathcal{C}_{n+1}) = \mathbb{E}_{\mathcal{B}'_{n+1}}(\alpha^{-1}(\mathcal{C}_{n+1}))),$$

which follow using $\alpha^{-1}(\mathcal{C}_n) \subseteq \mathcal{B}_{n-1} \otimes \mathcal{B}_n$, $\alpha^{-1}(\mathcal{C}_{n+1}) \subseteq \mathcal{B}_n \otimes \mathcal{B}_{n+1}$, and $\alpha(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$, and one uses this to prove the nontrivial inclusion $\mathcal{B}_n \subseteq \alpha^{-1}(\mathcal{L}_n)) \otimes \alpha^{-1}(\mathcal{R}_n)$. ■

For later use in Section 5, below we note Theorem 4.1 also holds for weaker assumptions, by an identical argument.

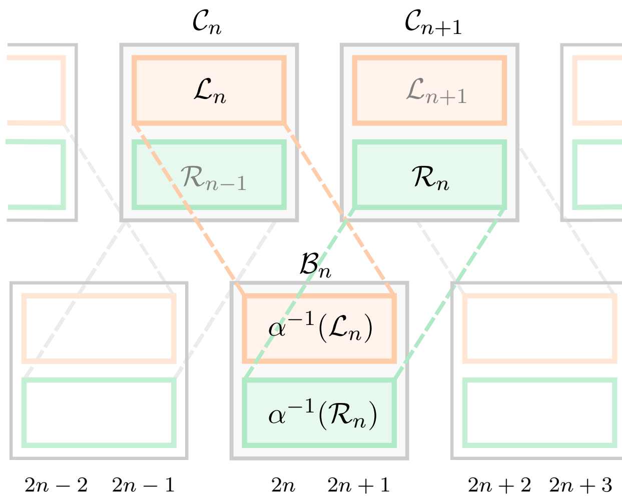

Figure 4: The factorization Theorem 4.1 decomposes every two-site algebra into a left-moving and right-moving part.

Remark 4.2. Although in Theorem 4.1 we assumed the automorphism α was a QCA, the only locality properties of α required to achieve $\mathcal{C}_n = \mathcal{L}_n \otimes \mathcal{R}_{n-1}$ were $\alpha(\mathcal{B}_{n-1}) \subseteq \mathcal{C}_{n-1} \otimes \mathcal{C}_n$, $\alpha(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$, and $\alpha^{-1}(\mathcal{C}_n) \subseteq \mathcal{B}_{n-1} \otimes \mathcal{B}_n$. Similarly, to achieve $\mathcal{B}_n = \alpha^{-1}(\mathcal{L}_n) \otimes \alpha^{-1}(\mathcal{R}_n)$ we need only $\alpha^{-1}(\mathcal{C}_n) \subseteq \mathcal{B}_{n-1} \otimes \mathcal{B}_n$, $\alpha^{-1}(\mathcal{C}_{n+1}) \subseteq \mathcal{B}_n \otimes \mathcal{B}_{n+1}$, and $\alpha(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$.

Based on Theorem 4.1 we can show that the ratio of $\dim(\mathcal{L}_n)$ and $\dim(\mathcal{A}_{2n})$ is independent of n, motivating the following definition:

Definition 4.3 (Index of QCA)**.** Suppose α is a one-dimensional nearest neighbor QCA. Let $\mathcal{L}_n$ and $\mathcal{R}_n$ be defined as in (4.2), then the *index* of α is given by

$$\begin{aligned} \operatorname{ind}(\alpha) &:= \frac{1}{2}(\log(\dim(\mathcal{L}_n)) - \log(\dim(\mathcal{A}_{2n}))) \\ &= \frac{1}{2}(\log(\dim(\mathcal{A}_{2n+1})) - \log(\dim(\mathcal{R}_n))). \end{aligned} \tag{4.7}$$

The value of $\operatorname{ind}(\alpha)$ is independent of the choice of n [6]. We choose to take the logarithm of the original definition. The index of a QCA with radius $R > 1$ may be defined by blocking sites such that the resulting QCA is nearest neighbor, and one can show that the index is independent of the choice of blocking. This index can be thought of as a 'flux', measuring the difference between how much quantum information is flowing to the right vs. left. From the definition it is clear it cannot take arbitrary values, but is restricted to integer linear combinations $\mathbb{Z}[\{\log(p_i)\}]$ where the p_i are all prime factors of local Hilbert space dimensions d_n.

The index can be used to characterize all one-dimensional QCAs. In order to do so, we introduce two types of QCAs: circuits and shifts. We will say a QCA α is a *block partitioned unitary* if it can be written as

$$\alpha(x) = \left(\prod_j u_j^*\right) x \left(\prod_j u_j\right)$$

where the u_j are a family of local unitaries, the u_j having disjoint and finite support. We will say α is a *circuit* (in [6] a similar notion is called *locally implementable*) if it can be written as a composition of block partitioned unitaries where each local unitary is supported on a uniformly bounded finite set. In one dimension any circuit QCA of radius R can be written as a composition of at most two block partitioned unitaries where each local unitary is supported on at most $2R$ contiguous sites. We denote by σ_d^k the *translation* QCA which has local Hilbert space dimension d and which translates any operator by k sites, mapping $\sigma_d^k(\mathcal{A}_n) = \mathcal{A}_{n-k}$. Here k can be negative. We will say a QCA is a *shift* if it is a tensor product of translations of the form σ_d^k.

Theorem 4.4 (Properties of GNVW index [6])**.** *Let α, β be one-dimensional QCAs. Then:*

(i) $\mathrm{ind}(\alpha \otimes \beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$

(ii) If α and β are defined on the same quasi-local algebra (i.e., with the same local dimensions), $\mathrm{ind}(\alpha\beta) = \mathrm{ind}(\alpha) + \mathrm{ind}(\beta)$.

(iii) α is a circuit if and only if $\mathrm{ind}(\alpha) = 0$.

(iv) $\mathrm{ind}(\sigma_d^k) = k \log(d)$.

(v) Every one-dimensional QCA is a composition of a shift and a circuit.[5]

(vi) If α and β are defined on the same quasi-local algebra the following are equivalent:

(a) $\mathrm{ind}(\alpha) = \mathrm{ind}(\beta)$.

(b) There exists a circuit γ such that $\alpha = \beta\gamma$.

(c) There exists a strongly continuous path from α to β through the space of QCAs with a uniform bound on the radius.

(d) There exists a blending *of α and β, meaning a QCA γ which is identical to α on a region extending to left infinity and equal to β on a region extending to right infinity.*

The "classification" of one-dimensional QCAs refers to the set of QCAs modulo an equivalence relation, given either by circuits (b), continuous deformations (c), or blending (d). These equivalence classes are identical and characterized by the index, as expressed in (vi). If α and β are not defined on the same quasi-local algebra (i.e. have different local dimensions), analogous statements to (b), (c), (d) hold after separately tensoring α and β with appropriate identity automorphisms, i.e. adding extra tensor factors on which they act trivially, such that α and β then have the same local dimensions. The notion of equivalence between QCAs that further allows extra tensor factors is called "stable equivalence," discussed in [7]. We will prove generalizations of all these properties for ALPUs.

As observed in [6], the tensor product property together with the normalization on shifts and circuits completely determines the index.

Lemma 4.5. *Suppose I assigns a real number $I(\alpha)$ to any one-dimensional QCA α such that*

(i) $I(\alpha \otimes \beta) = I(\alpha) + I(\beta)$ for all one-dimensional QCAs α and β.

[5] Strictly speaking this only makes sense if all the local dimensions are equal. We can always achieve this by taking a tensor product with the identity automorphism on a quasi-local algebra with appropriate local dimensions.

(ii) I takes the same values as ind *on circuits and on σ_d^k.*

Then $I(\alpha) = \mathrm{ind}(\alpha)$ for any one-dimensional QCA α.

Proof. Let α be any one-dimensional QCA and let β be a shift with $I(\beta) = \mathrm{ind}(\beta) = -\mathrm{ind}(\alpha)$, using (ii). Then $I(\alpha \otimes \beta) = I(\alpha) + I(\beta) = I(\alpha) - \mathrm{ind}(\alpha)$ by (i). On the other hand, $\mathrm{ind}(\alpha \otimes \beta) = 0$ so it is a circuit. Again by property (ii) this implies that $I(\alpha \otimes \beta) = 0$, showing that $I(\alpha) = \mathrm{ind}(\alpha)$. ■

4.2 An entropic definition of the GNVW index

Here we provide a new formula for the index in terms of the mutual information, which can also be defined for infinite C^*-algebras. This reformulation is not strictly necessary to develop an index theory for ALPUs, but it does allow us to give a clean expression for the index of an ALPU.

We consider two copies of the quasi-local algebra $\mathcal{A}_{\mathbb{Z}}$. Then the tensor product $\mathcal{A}_{\mathbb{Z}} \otimes \mathcal{A}_{\mathbb{Z}}$ is uniquely defined as a C^*-algebra since $\mathcal{A}_{\mathbb{Z}}$ is nuclear (so there is no ambiguity in the norm completion of the tensor product). We choose a transposition on each local algebra, which gives rise to a transposition $x \mapsto x^T$ on $\mathcal{A}_{\mathbb{Z}}$. Let τ be the tracial state on $\mathcal{A}_{\mathbb{Z}}$. Then we define the *maximally entangled state* ω by

$$\omega(x \otimes y) = \tau(xy^T) \tag{4.8}$$

for $x \otimes y \in \mathcal{A}_{\mathbb{Z}} \otimes \mathcal{A}_{\mathbb{Z}}$. It is not hard to see that if we restrict to a finite number of sites, ω indeed restricts to the usual maximally entangled state. Then we define

$$\phi = (\alpha^\dagger \otimes \mathrm{id})(\omega).$$

where id is the identity channel, and $\alpha^\dagger$ is the adjoint channel. In other words, ϕ is the *Choi state* of α.

Split the algebra $\mathcal{A}_{\mathbb{Z}}$ at any point n in the chain, letting

$$\begin{aligned} \mathcal{A}_L &:= \mathcal{A}_{\leq n} \\ \mathcal{A}_R &:= \mathcal{A}_{>n}. \end{aligned} \tag{4.9}$$

and similarly split the copy as $\mathcal{A}_{L'}$ and $\mathcal{A}_{R'}$. For a QCA with radius r, we will also consider

$$\begin{aligned} \mathcal{A}_{L_1} &= \mathcal{A}_{n-r+1,\ldots,n}, \qquad & \mathcal{A}_{L_2} &= \mathcal{A}_{\leq n-r}, \\ \mathcal{A}_{R_1} &= \mathcal{A}_{n+1,\ldots,n+r}, \qquad & \mathcal{A}_{R_2} &= \mathcal{A}_{\geq n+r+1}. \end{aligned} \tag{4.10}$$

We will define the index in terms of a difference of mutual informations of the Choi state. If ϕ, ψ are states on a C^*-algebra we may define the relative entropy $S(\phi, \psi)$ [46]. The mutual information of a state ϕ on $\mathcal{A}_A \otimes \mathcal{A}_B$ is then defined as $I(A : B)_\phi = S(\phi, \phi|_{\mathcal{A}_A} \otimes \phi|_{\mathcal{A}_B})$. On finite dimensional subsystems this definition coincides with the usual one. The only property we need is that relative entropies, and hence mutual informations, on the full algebra can be computed as limits:

Proposition 4.6 (Proposition 5.23 in [46])**.** *Let $\mathcal{A}$ be a C^*-algebra and let $\{\mathcal{A}_i\}_i$ be an increasing net of C^*-subalgebras so that $\cup_i \mathcal{A}_i$ is dense in $\mathcal{A}$. Then for any two states ϕ, ψ on $\mathcal{A}$ the net $S(\phi_i, \psi_i)$ converges to $S(\phi, \psi)$ where $\phi_i = \phi|_{\mathcal{A}_i}$, $\psi_i = \psi|_{\mathcal{A}_i}$.*

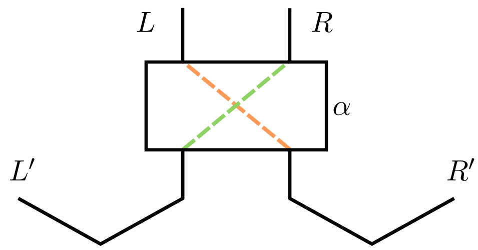

Figure 5: Illustration of (4.11). The index measures the difference in information flows, left to right minus right to left, as $\mathrm{ind}(\alpha) = \frac{1}{2}\left(I(L' : R)_\phi - I(L : R')_\phi\right)$.

Proposition 4.7. *For any choice of n in Eq.* (4.9) *the index of a one-dimensional QCA* α *is given by*

$$\mathrm{ind}(\alpha) = \frac{1}{2}\left(I(L' : R)_\phi - I(L : R')_\phi\right). \tag{4.11}$$

For a QCA with radius r*, this can also be computed locally as*

$$\mathrm{ind}(\alpha) = \frac{1}{2}\left(I(L_1' : R_1)_\phi - I(L_1 : R_1')_\phi\right). \tag{4.12}$$

Here, the mutual information terms are computed with respect to the corresponding subalgebras of $\mathcal{A}_\mathbb{Z} \otimes \mathcal{A}_\mathbb{Z}$ *(with primed systems corresponding to subalgebras of the second factor).*

Proof. Denote by $I(\alpha)$ the expression in (4.11). First we will argue that $I(L' : R)_\phi = I(L_1' : R_1)$ and $I(L : R')_\phi = I(L_1 : R_1')$. One sees this by verifying that

$$\begin{aligned}\phi_{L'R} &= \phi_{L_1'R_1} \otimes \tau_{L_2'R_2}\\ \phi_{LR'} &= \phi_{L_1R_1} \otimes \tau_{L_2R_2'}\end{aligned}$$

where the τ denote tracial (i.e maximally mixed) states. Next, to see that $I(\alpha) = \mathrm{ind}(\alpha)$ we will apply Lemma 4.5. From the definition it is clear that $I(\alpha \otimes \beta) = I(\alpha) + I(\beta)$, so it suffices to compute $I(\alpha)$ for a circuit and a shift. For a shift $\alpha = \sigma_d^k$ it is clear from the definition that for positive k

$$\begin{aligned}I(L' : R)_\phi &= 2k\log(d)\\ I(L : R')_\phi &= 0\end{aligned}$$

and for negative k

$$\begin{aligned}I(L' : R)_\phi &= 0\\ I(L : R')_\phi &= 2k\log(d).\end{aligned}$$

Finally, for a circuit α, notice that we can ignore any unitaries that act only on L or R as they keep the mutual information invariant. In this way, we may also reduce to the finite subsystem $L_1R_1L_1'R_1'$. In order to see that $I(\alpha) = 0$ we thus only need to check that

$$I(L_1' : R_1)_\phi = I(L_1 : R_1')_\phi$$

where $|\phi\rangle = U \otimes I\,|\omega\rangle$ for some unitary U acting on L_1R_1 and where $|\omega\rangle$ is a maximally entangled state between L_1R_1 and $L'_1R'_1$. In that case $|\phi\rangle$ is a maximally entangled state between L_1R_1 and L_2R_2 and

$$\begin{aligned} S(L'_1)_\phi &= S(L_1)_\phi \\ S(R'_1)_\phi &= S(R_1)_\phi \\ S(L'_1R_1)_\phi &= S(L_1R'_1)_\phi. \end{aligned}$$

The first two equalities hold because ϕ is maximally entangled, and the third equality holds because ϕ is pure. Thus we see that

$$\begin{aligned} I(L'_1 : R_1)_\phi &= S(L'_1)_\phi + S(R_1)_\phi - S(L'_1R_1)_\phi \\ &= S(L_1)_\phi + S(R'_1)_\phi - S(L_1R'_1)_\phi \\ &= I(L_1 : R'_1)_\phi. \end{aligned}$$

■

The expression of the index in (4.11) is intuitive: $I(L' : R)_\phi$ and $I(L : R')_\phi$ measure the flow of information to the right and left respectively. Notice that depending on the choice of cut $I(L' : R)_\phi$ and $I(L : R')_\phi$ can vary individually, but the total *flux* as defined by (4.11) is invariant. One reason this expression for the index is useful is that, contrary to the original definition, it is plausibly well-defined for automorphisms which are not strictly local (or for channels which are not automorphisms). In Theorem 5.8 we will show that taking the limit of the finite subalgebras in (4.10) with increasing radius gives a well-defined and finite limit for any ALPU with appropriately decaying tails, and hence using Proposition 4.6 we conclude that both mutual information terms in (4.11) are finite and (4.11) gives a finite, quantized answer also for an ALPU.

In [6], a similar numerical expression for the index is provided in terms of overlaps of algebras (their Eq. 45). In fact, their formula (or rather its logarithm) can be interpreted as (4.11) but with the entropies replaced by Renyi-2 entropies,

$$\operatorname{ind}(\alpha) = \frac{1}{2}\left(I_2(L' : R)_\phi - I_2(L : R')_\phi\right),$$

where $I_2(A : B)_\rho := S_2(A)_\rho + S_2(B)_\rho - S_2(AB)_\rho$. While the values of the individual mutual information terms depend on the choice of Renyi-2 or von Neumann entropy, for QCAs, the difference of mutual informations used to define the index does *not* depend on this choice, and in the proof of Proposition 4.7 one can simply replace the entropies S by Renyi entropies S_2. However, the mutual information has better continuity properties with respect to the dimension of the local Hilbert spaces compared to the Renyi-2 mutual information (compare the following with the continuity bound in Lemma 12 of [6]):

Theorem 4.8 (Continuity of mutual information [47–49])**.** *Suppose ρ, σ are states on $\mathcal{H}_A \otimes \mathcal{H}_B$, and $\frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon < 1$. Then*

$$\left|I(A : B)_\rho - I(A : B)_\sigma\right| \leq 3\varepsilon \log(d_A) + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right) \leq 3\varepsilon \log(d_A) + \varepsilon \log \tfrac{1}{\varepsilon}$$

where $d_A = \dim(\mathcal{H}_A)$ and $h(x) = -x\log(x) - (1-x)\log(1-x)$ is the binary entropy.

This continuity is important for the extension to ALPUs, where we need to compute the approximation to the index on a sequence of increasing finite subalgebras. In that case, the indices defined using the Renyi-2 and von Neumann entropies give different answers when restricted to the finite subalgebras. A final remark is that (4.12) can also be rewritten as an entropy difference

$$\begin{aligned}\operatorname{ind}(\alpha) &= \frac{1}{2}\left(I(L_1' : R_1)_\phi - I(L_1 : R_1')_\phi\right) \\ &= \frac{1}{2}\left(S(L_1 R_1')_\phi - S(L_1' R_1)_\phi\right). \end{aligned} \tag{4.13}$$

However, the extension of this expression to infinite-dimensional setting is less clear, because both terms diverge.

4.3 Robustness of the GNVW index

Because the index can be computed locally, it appears that two QCAs with different index should be easy to distinguish locally. We make this quantitative in Proposition 4.10: two QCAs which look locally similar must have equal index. We begin with a cruder but more general estimate, describing how the mutual information of the Choi state varies continuously with respect to the automorphism that defines it. This estimate applies to general automorphisms which may not be QCAs, proving useful in the argument for Theorem 5.8.

Let α be an automorphism of $\mathcal{A}_\mathbb{Z}$. Even when α is not a QCA, we can mimic the local definition of the index in (4.12) using finite disjoint regions L, R. We denote this quantity $\widetilde{\operatorname{ind}}_{L,R}(\alpha)$ to emphasize α may not be a QCA nor even an ALPU,

$$\widetilde{\operatorname{ind}}_{L,R}(\alpha) = \frac{1}{2}\left(I(L' : R)_\phi - I(L : R')_\phi\right), \tag{4.14}$$

where the mutual information terms are computed with respect to the corresponding subalgebras of $\mathcal{A}_\mathbb{Z} \otimes \mathcal{A}_\mathbb{Z}$. (As above, primed systems refer to the second copy of $\mathcal{A}_\mathbb{Z}$.) Clearly, Eq. (4.14) only depends on the restriction of the Choi state to $\mathcal{A}_X \otimes \mathcal{A}_{X'}$, where $X = L \cup R$, i.e. on the state $\widetilde{\phi}_{XX'} := \phi|_{\mathcal{A}_X \otimes \mathcal{A}_{X'}}$, which is given by

$$\widetilde{\phi}_{XX'}(x) = \omega\left((\alpha \otimes \mathrm{id})(x)\right)$$

for all $x \in \mathcal{A}_X \otimes \mathcal{A}_{X'}$. Then we have the following continuity estimate.

Lemma 4.9. *For two automorphisms α_1 and α_2 of $\mathcal{A}_\mathbb{Z}$ with maximum local dimension d, the quantity $\widetilde{\operatorname{ind}}_{L,R}$ in (4.14) obeys*

$$\left|\widetilde{\operatorname{ind}}_{L,R}(\alpha_1) - \widetilde{\operatorname{ind}}_{L,R}(\alpha_2)\right| = \mathcal{O}(\varepsilon |X| \log(d) + \varepsilon \log \tfrac{1}{\varepsilon}),$$

where $\varepsilon = \|(\alpha_1 - \alpha_2)|_{\mathcal{A}_X}\|$. The same continuity estimate with respect to α_1 and α_2 holds for the individual terms in (4.14).

Proof. First we compare the restricted Choi states $\widetilde{\phi}_{XX',1}$ and $\widetilde{\phi}_{XX',2}$ of α_1 and α_2, respectively. For any $x \in \mathcal{A}_X \otimes \mathcal{A}_{X'}$ with $\|x\| = 1$,

$$\left|\widetilde{\phi}_{XX',1}(x) - \widetilde{\phi}_{XX',2}(x)\right|_1 = \left|\omega((\alpha_1 \otimes \mathrm{id} - \alpha_2 \otimes \mathrm{id})(x))\right| \leq \|(\alpha_1 \otimes \mathrm{id} - \alpha_2 \otimes \mathrm{id})|_{\mathcal{A}_X \otimes \mathcal{A}_{X'}}\|.$$

Thus the trace distance between the two Choi states is bounded by

$$\|\widetilde{\phi}_{XX',1} - \widetilde{\phi}_{XX',2}\|_1 \leq \|(\alpha_1 \otimes \mathrm{id} - \alpha_2 \otimes \mathrm{id})|_{\mathcal{A}_X \otimes \mathcal{A}_{X'}}\| \leq 2\varepsilon + \mathcal{O}(\varepsilon^2)$$

using Lemma 2.7 for the last inequality (with $\mathcal{A}_1 = \mathcal{A}_X \otimes I$, $\mathcal{A}_2 = I \otimes \mathcal{A}_{X'}$ and $\mathcal{A} = \mathcal{A}_X \otimes \mathcal{A}_{X'}$ finite-dimensional and $\mathcal{B} = \mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$). The conclusion follows from the continuity of mutual information in Theorem 4.8 with respect to the state, noting the region X has associated Hilbert space of dimension at most $d^{|X|}$. ■

If α_1 and α_2 are one-dimensional QCAs of radius r, then because the index takes discrete values, there exists ε_0 such that if $\varepsilon \leq \frac{\varepsilon_0}{r\log(d)}$ then $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$. However, we can do better and eliminate the dependence on the local dimension, as a simple application of Theorem 2.6. By blocking sites, we may assume without loss of generality that the QCA is nearest neighbor.

Proposition 4.10 (Robustness of GNVW index for QCAs)**.** *Suppose α_1 and α_2 are two nearest-neighbor QCAs defined on the same quasi-local algebra $\mathcal{A}_{\mathbb{Z}}$ such that $\|(\alpha_1 - \alpha_2)|_{\mathcal{A}_{\{2n,2n+1\}}}\| \leq \varepsilon$ for some n with $\varepsilon \leq \frac{1}{192}$. Then* $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$.

Moreover, the algebras $\mathcal{L}_n^{(1)}$ and $\mathcal{L}_n^{(2)}$ defined by (4.2) *using α_1 and α_2 respectively are isomorphic, with isomorphism implemented by a unitary $u \in \mathcal{A}_{\{2n-1,2n\}}$ satisfying $\|u - I\| \leq 36\varepsilon$.*

Note that when working with a coarse-grained QCA, where each site is composed of many smaller sites, the hypotheses like $\|(\alpha_1 - \alpha_2)|_{\mathcal{A}_{\{2n,2n+1\}}}\| \leq \varepsilon$ constraining error on coarse-grained sites may always be replaced by hypotheses constraining the sum of errors on fine-grained sites, using Lemma 2.7. (In other words, upper bounds for errors on small regions control errors on larger regions.)

Proof. By the structure theory for QCAs in Theorem 4.1 there exist algebras $\mathcal{L}_n^{(i)}$, $\mathcal{R}_{n-1}^{(i)}$ for $i = 1, 2$ defined as in (4.2) that satisfy

$$\mathcal{A}_{\{2n-1,2n\}} = \mathcal{L}_n^{(i)} \otimes \mathcal{R}_{n-1}^{(i)}$$

To prove that $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$, by (4.7) it suffices to show that $\mathcal{L}_n^{(1)}$ and $\mathcal{L}_n^{(2)}$ are isomorphic. To see the isomorphism, take $x \in \mathcal{L}_n^{(1)}$ with $\|x\| = 1$ and let $y = \alpha_2(\alpha_1^{-1}(x))$. Then $\|x - y\| = \|\alpha_1(\alpha_1^{-1}(x)) - \alpha_2(\alpha_1^{-1}(x))\| \leq \varepsilon$ using the assumption $\|(\alpha_1 - \alpha_2)|_{\mathcal{A}_{\{2n,2n+1\}}}\| \leq \varepsilon$ and noting $\alpha_1^{-1}(x) \in \mathcal{A}_{\{2n,2n+1\}}$ since $x \in \mathcal{L}_n^{(1)}$. Using the conditional expectation from (2.2), define

$$z = \mathbb{E}_{\mathcal{A}'_{\{2n+1,2n+2\}}}(y) = \int_{U(\mathcal{A}_{\{2n+1,2n+2\}})} uyu^* \,\mathrm{d}u$$

such that $z \in \mathcal{L}_n^{(2)}$ by the characterization of $\mathcal{L}_n$ in (4.5). Note $\|[a, y]\| = \|[a, y - x]\| \leq 2\varepsilon\|a\|$ for all $a \in \mathcal{A}_{\{2n+1,2n+2\}}$, so by its definition z satisfies $\|y - z\| \leq 2\varepsilon$, and $\|x - z\| \leq \|x - y\| + \|y - z\| \leq 3\varepsilon$. We conclude $\mathcal{L}_n^{(1)} \overset{3\varepsilon}{\subseteq} \mathcal{L}_n^{(2)}$, and by a symmetric argument we see $\mathcal{L}_n^{(2)} \overset{3\varepsilon}{\subseteq} \mathcal{L}_n^{(1)}$. By Theorem 2.6, noting that $3\varepsilon \leq \frac{1}{64}$, we obtain that $\mathcal{L}_n^{(1)}$ and $\mathcal{L}_n^{(2)}$ are isomorphic, and the isomorphism is implemented by a unitary $u \in \mathcal{A}_{\{2n-1,2n\}}$ with $\|u - I\| \leq 36\varepsilon$. ■

For later use in Section 5, below we build on Remark 4.2 to note that Proposition 4.10 also holds for weaker assumptions, by an identical argument.

Remark 4.11. Although in Proposition 4.10 we assumed the automorphisms α_1 and α_2 were QCAs, the only locality properties required to achieve the isomorphism between $\mathcal{L}_n^{(1)}$ and $\mathcal{L}_n^{(2)}$ are the properties listed in Remark 4.2 as those required to achieve $\mathcal{A}_{\{2n-1,2n\}} = \mathcal{L}_n^{(i)} \otimes \mathcal{R}_{n-1}^{(i)}$ for $i = 1, 2$. More explicitly, we only require that $\alpha_i(\mathcal{A}_{\{2n-2,2n-1\}}) \subseteq \mathcal{A}_{\{2n-3,\ldots,2n\}}$, $\alpha_i(\mathcal{A}_{\{2n,2n+1\}}) \subseteq \mathcal{A}_{\{2n-1,\ldots,2n+2\}}$, and $\alpha_i^{-1}(\mathcal{A}_{\{2n-1,2n\}}) \subseteq \mathcal{A}_{\{2n-2,\ldots,2n+1\}}$ for $i = 1, 2$.

This also allows us to confirm the intuition that a one-dimensional QCA which is locally close to the identity can be implemented locally with unitaries close to the identity.

Proposition 4.12. *Suppose α is a one-dimensional QCA with radius R and suppose that for $\varepsilon \leq \frac{1}{192}$ we have $\|\alpha(x) - x\| \leq \varepsilon \|x\|$ for any $x \in \mathcal{A}_{\mathbb{Z}}$ supported on at most $2R$ sites. Then α can be implemented as a composition of two block partitioned unitaries $u = \prod_n u_n$ and $v = \prod_n v_n$, i.e.*

$$\alpha(x) = v^* u^* x u v$$

with each of the unitaries u_n, v_n acting on $2R$ adjacent sites and satisfying

$$\|u_n - I\| = \mathcal{O}(\varepsilon), \quad \|v_n - I\| = \mathcal{O}(\varepsilon).$$

Proof. By blocking sites in groups of R sites, we may assume without loss of generality that α is nearest neighbor. Let $\alpha_1 = \mathrm{id}$ and $\alpha_2 = \alpha$ in Proposition 4.10. Clearly, $\mathcal{L}_n^{(1)} = \mathcal{A}_{2n}$ and $\mathcal{R}_{n-1}^{(1)} = \mathcal{A}_{2n-1}$. By Proposition 4.10 there exists some $v_n \in \mathcal{A}_{\{2n-1,2n\}}$ such that $v_n \mathcal{L}_n^{(2)} v_n^* = \mathcal{A}_{2n}$ with $\|v_n - I\| = \mathcal{O}(\varepsilon)$. It follows that $v_n \mathcal{R}_{n-1}^{(2)} v_n^* = \mathcal{A}_{2n-1}$. Let $v = \prod v_n$ and let $\tilde{\alpha} = v\alpha(x)v^*$. Then $\tilde{\alpha}(\mathcal{A}_{\{2n,2n+1\}}) = \mathcal{A}_{\{2n,2n+1\}}$. Moreover, for all $x \in \mathcal{A}_{\{2n,2n+1\}}$, we estimate

$$\begin{aligned}
\|\tilde{\alpha}(x) - x\| &\leq \|v\alpha(x)v^* - \alpha(x)\| + \|\alpha(x) - x\| \\
&\leq 2\|v_n \otimes v_{n+1} - I\|\|x\| + \varepsilon\|x\| \\
&\leq 2(\|v_n - I\| + \|v_{n+1} - I\|)\|x\| + \varepsilon\|x\| \\
&= \mathcal{O}(\varepsilon)\|x\|.
\end{aligned}$$

Then Proposition B.1 shows that $\tilde{\alpha}|_{\mathcal{A}_{\{2n,2n+1\}}}$ can be implemented by a unitary u_n on $\mathcal{A}_{\{2n,2n+1\}}$ with $\|u_n - I\| = \mathcal{O}(\varepsilon)$. ∎

5 Index theory of approximately locality-preserving unitaries in one dimension

In this section we develop the index theory of ALPUs in one dimension. Just like in the rest of the paper, all ALPUs will be one-dimensional.

For a general ALPU α, we show in Theorems 5.6 and 5.8 that there always exist an approximation of α by a sequence of QCAs β_j. We can use the limit of the indices of the latter (which become stationary for large j) as the definition of the index of α. If α has $\mathcal{O}(r^{-(1+\delta)})$-tails for some $\delta > 0$, we further show that this index can be computed as a difference of mutual informations,

$$\mathrm{ind}(\alpha) = \frac{1}{2}\left(I(L' : R)_\phi - I(L : R')_\phi\right), \tag{5.1}$$

with both terms being finite, just like we saw in Eq. (4.11) for QCAs. The local computation of the index in (4.12) does not yield the exact index for ALPUs. However, the exact index can still be computed locally; we show that on sufficiently large regions, the local index computation gives the exact answer when rounded to the nearest value in the fixed set of discrete index values.

In the remainder of the section, we discuss the properties of this index. We find that once circuits are replaced by evolutions by time-dependent Hamiltonians, the results of [6] stated in Theorem 4.4 generalize in a natural way. Our results are summarized in Theorem 5.15.

5.1 Approximating an ALPU by a QCA

We sketch the general strategy for approximating an ALPU α by a QCA. We first develop a method for deforming α into an ALPU α_n that behaves as a QCA with a strict causal cone in the proximity of the site n, exhibited by Proposition 5.4 and Fig. 7. In Proposition 5.5 we then we stitch the different α_n together into a QCA β using the structure theory for one-dimensional QCAs, obtaining a QCA approximation to α. If we apply this result to increasingly coarse-grained lattices, in Theorem 5.6 we obtain a sequence of QCAs of increasing radius that approximate α with increasing accuracy.

To achieve Proposition 5.4 localizing an ALPU α on a local patch, we compose α with a sequence of unitary rotations. Some individual rotation steps are described by Lemma 5.1 and Lemma 5.3, with proof illustrated in Fig. 6. Each step uses Theorem 2.6 to rotate nearby subalgebras, e.g. rotating an algebra $\alpha(\mathcal{A}_X) \overset{\varepsilon}{\subseteq} \mathcal{A}_Y$ to obtain an exact inclusion. We start with these two lemmas. Lemma 5.1, Lemma 5.3 and Proposition 5.4 are each divided into two parts, (i) and (ii). In each case, part (i) is valid for ε-nearest neighbour automorphisms (which need not be ALPUs), while part (ii) gives a more refined statement when assuming an ALPU as input. For the majority of the further development in this paper, in fact only the parts (i) will be necessary, and so the first-time reader may wish to skip part (ii) of these results, as well as the supporting Lemma 5.2. Those parts will only be necessary for later results about blending, following Definition 5.13.

Lemma 5.1. *(i) There exist universal constants $C_0, \varepsilon_0 > 0$ such that if α is an ε-nearest neighbor automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ with $\varepsilon \le \varepsilon_0$ and*

$$\alpha(\mathcal{A}_{\ge n}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\ge n-1}^{\mathrm{vN}}$$

for some site $n \in \mathbb{Z}$, then there exists an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ of the form $\tilde{\alpha}(x) = u^* \alpha(x) u$ for some unitary $u \in \mathcal{A}_{\ge n-1}^{\mathrm{vN}}$ with $\|u - I\| \le C_0 \varepsilon$ and

$$\tilde{\alpha}(\mathcal{A}_{\le n-1}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\le n}^{\mathrm{vN}}, \tag{5.2}$$

$$\tilde{\alpha}(\mathcal{A}_{\ge n}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\ge n-1}^{\mathrm{vN}}. \tag{5.3}$$

(ii) If additionally α is an ALPU with $f(r)$-tails, we can take u such that $\tilde{\alpha}$ is an ALPU with $\mathcal{O}(f(r-1))$-tails and such that we have, for $r \to \infty$,

$$\|(\alpha - \tilde{\alpha})|_{\mathcal{A}_{\le n-r-1}^{\mathrm{vN}}}\| = \mathcal{O}(f(r)), \tag{5.4}$$

$$\|(\alpha - \tilde{\alpha})|_{\mathcal{A}_{\ge n+r}^{\mathrm{vN}}}\| = \mathcal{O}(f(r-1)) \tag{5.5}$$

and, for all $x \in \mathcal{A}_{\ge n+r+1}$,

$$\|u^* x u - x\| = \mathcal{O}(f(r)\|x\|). \tag{5.6}$$

Proof. (i) Note α^{-1} is 4ε-nearest neighbor by (ii) of Lemma 3.8. Thus $\alpha^{-1}(\mathcal{A}^{\mathrm{vN}}_{\geq n+1}) \overset{4\varepsilon}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\geq n}$ and then $\mathcal{A}^{\mathrm{vN}}_{\geq n+1} \overset{4\varepsilon}{\subseteq} \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})$. By Theorem 2.6 with $\mathcal{A}_0 = \mathcal{A}_{\geq n+1}$, $\mathcal{A} = \mathcal{A}_0'' = \mathcal{A}^{\mathrm{vN}}_{\geq n+1}$ and $\mathcal{B} = \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})$, provided that $\varepsilon \leq \frac{1}{256}$, there exists a unitary $u \in (\mathcal{A}^{\mathrm{vN}}_{\geq n+1} \cup \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}))''$ such that

$$u\mathcal{A}^{\mathrm{vN}}_{\geq n+1}u^* \subseteq \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})$$

and $\|u - I\| \leq 48\varepsilon$. Because $\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}) \subseteq \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$, we also have $u \in \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$. We define a new automorphism $\tilde{\alpha}(x) = u^*\alpha(x)u$ that satisfies $\tilde{\alpha}(\mathcal{A}^{\mathrm{vN}}_{\geq n}) \supseteq \mathcal{A}^{\mathrm{vN}}_{\geq n+1}$, and then satisfies (5.2) by taking commutants. Moreover,

$$\tilde{\alpha}(\mathcal{A}^{\mathrm{vN}}_{\geq n}) = u^*\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})u \subseteq u^*\mathcal{A}^{\mathrm{vN}}_{\geq n-1}u = \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$$

using the assumption $\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}) \subseteq \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$ and the fact $u \in \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$. Then $\tilde{\alpha}$ also satisfies (5.3).

(ii) Now we further assume α is an ALPU with $f(r)$-tails and show (5.4). By our use of Theorem 2.6 to construct $u \in (\mathcal{A}^{\mathrm{vN}}_{\geq n+1} \cup \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}))''$, we know that for $x \in \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$,

$$\|[x,y]\| \leq \delta\|x\|\|y\| \;\; \forall y \in \mathcal{A}^{\mathrm{vN}}_{\geq n+1} \cup \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}) \implies \|u^*xu - x\| = \mathcal{O}(\delta\|x\|).$$

For $r \geq 0$ and $x \in \alpha(\mathcal{A}^{\mathrm{vN}}_{\leq n-r-1})$, the above condition is satisfied for $\delta = 2f(r+1)$ because $x \overset{f(r+1)}{\in} \mathcal{A}^{\mathrm{vN}}_{\leq n}$ (using Lemma 2.4) and $x \in \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})'$, so $\|(\alpha - \tilde{\alpha})|_{\mathcal{A}^{\mathrm{vN}}_{\leq n-r-1}}\| = \mathcal{O}(f(r+1))$ and Eq. (5.4) follows.

Our application of Theorem 2.6 also implies that for $x \in \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$,

$$x \overset{\delta}{\in} \mathcal{A}_{\geq n+1} \;\text{ and }\; x \overset{\delta}{\in} \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n}) \implies \|u^*xu - x\| = \mathcal{O}(\delta\|x\|). \tag{5.7}$$

For $r \geq 1$ and $x \in \alpha(\mathcal{A}_{\geq n+r})$, those conditions are satisfied for $\delta = f(r-1)$ because $x \overset{f(r-1)}{\in} \mathcal{A}_{\geq n+1}$ and $x \in \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})$, so $\|(\alpha - \tilde{\alpha})|_{\mathcal{A}_{\geq n+r}}\| = \mathcal{O}(f(r-1))$ and hence Eq. (5.5) follows.

Next, we prove Eq. (5.6). Recall that by Lemma 3.8(iv), α^{-1} is also an ALPU with $\mathcal{O}(f(r))$-tails. Therefore, for any $r \geq 0$ and $x \in \mathcal{A}_{\geq n+r+1}$ we have $\alpha^{-1}(x) \overset{f(r+1)}{\in} \mathcal{A}_{\geq n}$, hence $x \overset{f(r+1)}{\in} \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq n})$, and now Eq. (5.7) shows that $\|u^*xu - x\| = \mathcal{O}(f(r+1)\|x\|)$ and Eq. (5.6) follows.

Finally we show that $\tilde{\alpha}$ is an ALPU with $\mathcal{O}(f(r-1))$-tails. This follows from Lemma 5.2 below and the fact that

$$\|uxu^* - x\| = \mathcal{O}(f(r-1)\|x\|)$$

holds for the following x and all $r \geq 0$: for $x \in \mathcal{A}_{\leq n-r-2}$ since $u \in \mathcal{A}^{\mathrm{vN}}_{\geq n-1}$, for $x \in \mathcal{A}_{\geq n+r+1}$ by Eq. (5.6), for $x \in \alpha(\mathcal{A}_{\leq n-r-1})$ by Eq. (5.4) and for $x \in \alpha(\mathcal{A}_{\geq n+r})$ by Eq. (5.5). ■

The following lemma is used in the proof above (and in similar proofs below) that the construction gives rise to an ALPU when the input is an ALPU.

Lemma 5.2. *Suppose that α is an ALPU with $f(r)$-tails and $\tilde{\alpha}$ is an automorphism of $\mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$ of the form $\tilde{\alpha}(x) = u^*\alpha(x)u$ for some $u \in \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$ and all $x \in \mathcal{A}^{\mathrm{vN}}_{\mathbb{Z}}$, which satisfies*

$$\begin{aligned} \tilde{\alpha}(\mathcal{A}_{\leq n-1}) &\subseteq \mathcal{A}^{\mathrm{vN}}_{\leq n}, \\ \tilde{\alpha}(\mathcal{A}_{\geq n}) &\subseteq \mathcal{A}^{\mathrm{vN}}_{\geq n-1} \end{aligned} \tag{5.8}$$

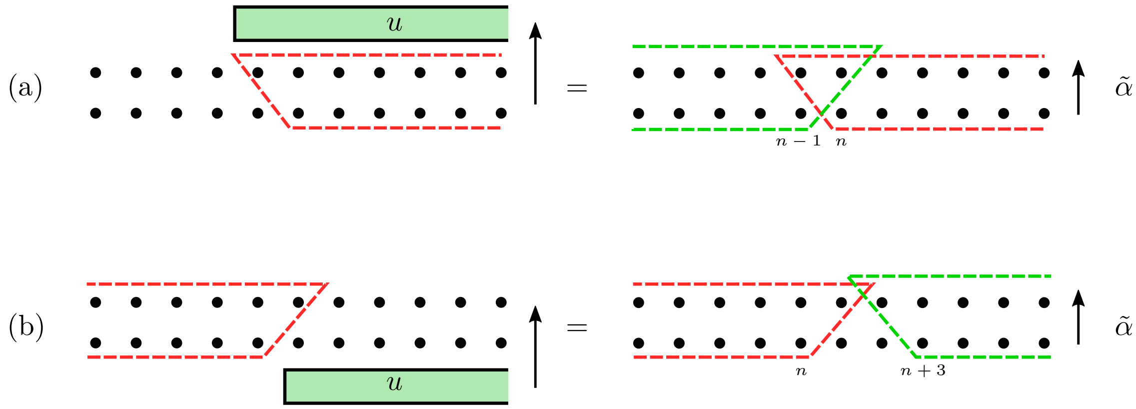

Figure 6: (a) Illustration of the construction in Lemma 5.1. The dashed lines indicate causal cones. (b) Analogous illustration of Lemma 5.3.

for some site $n \in \mathbb{Z}$. If for any $r \geq 0$ and $x \in \mathcal{A}_{\leq n-r-2} \cup \mathcal{A}_{\geq n+r+1} \cup \alpha(\mathcal{A}_{\leq n-r-1}) \cup \alpha(\mathcal{A}_{\geq n+r})$ we have

$$\|u^* x u - x\| \leq g(r)\|x\|,$$

where $g(r)$ is non-increasing with $\lim_{r\to\infty} g(r) = 0$, then $\tilde{\alpha}$ is an ALPU with $\mathcal{O}(f(r)+g(r))$-tails.

Proof. We abbreviate $h(r) = f(r) + g(r)$. By Lemma 3.11, it suffices to show

$$\tilde{\alpha}(\mathcal{A}_{\leq m}) \overset{\mathcal{O}(h(r))}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\leq m+r} \quad \text{and} \quad \tilde{\alpha}(\mathcal{A}_{\geq m}) \overset{\mathcal{O}(h(r))}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\geq m-r}$$

for all $m \in \mathbb{Z}$ and $r \geq 0$. We only prove the former, since the proof of the latter proceeds analogously. We distinguish two cases:

- $m < n$: Then $m = n - k - 1$ for some $k \geq 0$. By assumption, $\tilde{\alpha}(\mathcal{A}_{\leq n-k-1}) \subseteq \mathcal{A}^{\mathrm{vN}}_{\leq n}$, so it remains to show

  $$\tilde{\alpha}(\mathcal{A}_{\leq n-k-1}) \overset{\mathcal{O}(h(r))}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\leq n-k-1+r}.$$

  for $0 \leq r \leq k$. This holds since, by assumption, $\|u^* x u - x\| \leq g(k)\|x\|$ for all $x \in \alpha(\mathcal{A}_{\leq n-k-1})$, and hence $\|(\tilde{\alpha} - \alpha)|_{\mathcal{A}_{\leq n-k-1}}\| \leq g(k) \leq g(r)$ for any $0 \leq r \leq k$.

- $m \geq n$: Then $m = n + k$ for some $k \geq 0$. To prove that

  $$\tilde{\alpha}(\mathcal{A}_{\leq n+k}) \overset{\mathcal{O}(h(r))}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\leq n+k+r}$$

  by Lemma 2.4 it suffices to show that for all $x \in \mathcal{A}_{\leq n+k}$ and for all $y \in \mathcal{A}^{\mathrm{vN}}_{\geq n+k+r+1}$,

  $$\|[\tilde{\alpha}(x), y]\| = \mathcal{O}(h(r))\|x\|\|y\|.$$

  By assumption $\|uyu^* - y\| = \|u^* y u - y\| \leq g(k+r)\|y\| \leq g(r)\|y\|$ for all $y \in \mathcal{A}_{\geq n+k+r+1}$ and thus for all $\mathcal{A}^{\mathrm{vN}}_{\geq n+k+r+1}$, and hence we indeed have that

  $$\|[\tilde{\alpha}(x), y]\| = \|[\alpha(x), uyu^*]\| \leq \|[\alpha(x), y]\| + \|[\alpha(x), uyu^* - y]\| = O(h(r))\|x\|\|y\|,$$

  using that $\|[\alpha(x), y]\| \leq 2f(r)\|x\|\|y\|$ by Lemma 2.4, since α is an ALPU with $f(r)$-tails. ∎

Lemma 5.3. *(i) There exist universal constants $C_0', \varepsilon_0' > 0$ such that if α is an ε-nearest neighbor automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ with $\varepsilon \leq \varepsilon_0'$ and*

$$\alpha(\mathcal{A}_{\leq n}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\leq n+1}^{\mathrm{vN}}$$

for some site $n \in \mathbb{Z}$, then there exists an automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ of the form $\tilde{\alpha}(x) = \alpha(uxu^*)$ for some unitary $u \in \mathcal{A}_{\geq n+1}^{\mathrm{vN}}$ with $\|u - I\| \leq C_0'\varepsilon$ and

$$\tilde{\alpha}(\mathcal{A}_{\geq n+3}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\geq n+2}^{\mathrm{vN}}, \tag{5.9}$$

$$\tilde{\alpha}(\mathcal{A}_{\leq n}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\leq n+1}^{\mathrm{vN}}. \tag{5.10}$$

In fact, it holds that $\alpha|_{\mathcal{A}_{\leq n}^{\mathrm{vN}}} = \tilde{\alpha}|_{\mathcal{A}_{\leq n}^{\mathrm{vN}}}$.

(ii) If additionally α is an ALPU with $f(r)$-tails, we can take u such that $\tilde{\alpha}$ is an ALPU with $\mathcal{O}(f(r-1))$-tails and such that for $r \to \infty$,

$$\|(\alpha - \tilde{\alpha})|_{\mathcal{A}_{\geq n+r+3}^{\mathrm{vN}}}\| = \mathcal{O}(f(r)). \tag{5.11}$$

Proof. (i) This follows by application of Lemma 5.1 to $\beta = \alpha^{-1}$. Here we use that if α is an ε-nearest neighbour automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$, then β is a 4ε-nearest neighbour automorphism by Lemma 3.8(ii). Now, $\alpha(\mathcal{A}_{\leq n}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\leq n+1}^{\mathrm{vN}}$ implies that $\mathcal{A}_{\leq n}^{\mathrm{vN}} \subseteq \beta(\mathcal{A}_{\leq n+1}^{\mathrm{vN}})$ and hence $\beta(\mathcal{A}_{\geq n+2}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\geq n+1}^{\mathrm{vN}}$. Let $\varepsilon_0' := \varepsilon_0/4$ and $C_0' = 4C_0$ for $C_0, \varepsilon_0 > 0$ the constants from Lemma 5.1. Thus we may apply Lemma 5.1 to β (with $n+2$ in place of n) to find an automorphism $\tilde{\beta}$ of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ that is of the form $\tilde{\beta}(x) = u^*\beta(x)u$ for some unitary $u \in \mathcal{A}_{\geq n+1}^{\mathrm{vN}}$ with $\|u - I\| \leq 4C_0\varepsilon = C_0'\varepsilon$ and which satisfies

$$\tilde{\beta}(\mathcal{A}_{\leq n+1}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\leq n+2}^{\mathrm{vN}},$$
$$\tilde{\beta}(\mathcal{A}_{\geq n+2}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\geq n+1}^{\mathrm{vN}}.$$

We then see that $\tilde{\alpha} = \tilde{\beta}^{-1}$ is given by $\tilde{\alpha}(x) = \alpha(uxu^*)$ and satisfies the desired properties in (i). In particular, note that $u \in \mathcal{A}_{\geq n+1}^{\mathrm{vN}}$ immediately implies that $\alpha|_{\mathcal{A}_{\leq n}^{\mathrm{vN}}} = \tilde{\alpha}|_{\mathcal{A}_{\leq n}^{\mathrm{vN}}}$.

(ii) If α is an ALPU with $f(r)$-tails, then by Lemma 3.8(iv) β is an ALPU with $\mathcal{O}(f(r))$-tails. Hence by part (ii) of Lemma 5.1, $\tilde{\beta}$ is an ALPU with $\mathcal{O}(f(r-1))$-tails and thus the same is true for $\tilde{\alpha}$, again by Lemma 3.8(iv). Eq. (5.11) follows since by Eq. (5.6) we have

$$\|uxu^* - x\| = \|u^*xu - x\| = \mathcal{O}(f(r)\|x\|)$$

for all $x \in \mathcal{A}_{\geq n+r+3}$. ∎

We iteratively apply Lemma 5.1 and Lemma 5.3 to show that for an ε-nearest neighbor automorphism, for any small patch, one can find a nearby $\mathcal{O}(\varepsilon)$-nearest neighbor automorphism that is strictly local on that patch. Below we work with a patch near site $2n$, and the modified automorphism is denoted α_n.

Proposition 5.4. *(i) There exist universal constants $C_1, \varepsilon_1 > 0$ such that for any ε-nearest neighbor automorphism α of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ with $\varepsilon \leq \varepsilon_1$ and for any site $n \in \mathbb{Z}$, there exists an automorphism α_n of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ such that for $k \in \{0, 1, 2, 3\}$,*

$$\alpha_n(\mathcal{A}_{\leq 2n+2k-1}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\leq 2n+2k}^{\mathrm{vN}},$$
$$\alpha_n(\mathcal{A}_{\geq 2n+2k}^{\mathrm{vN}}) \subseteq \mathcal{A}_{\geq 2n+2k-1}^{\mathrm{vN}},$$
$$\|\alpha_n - \alpha\| \leq C_1\varepsilon.$$

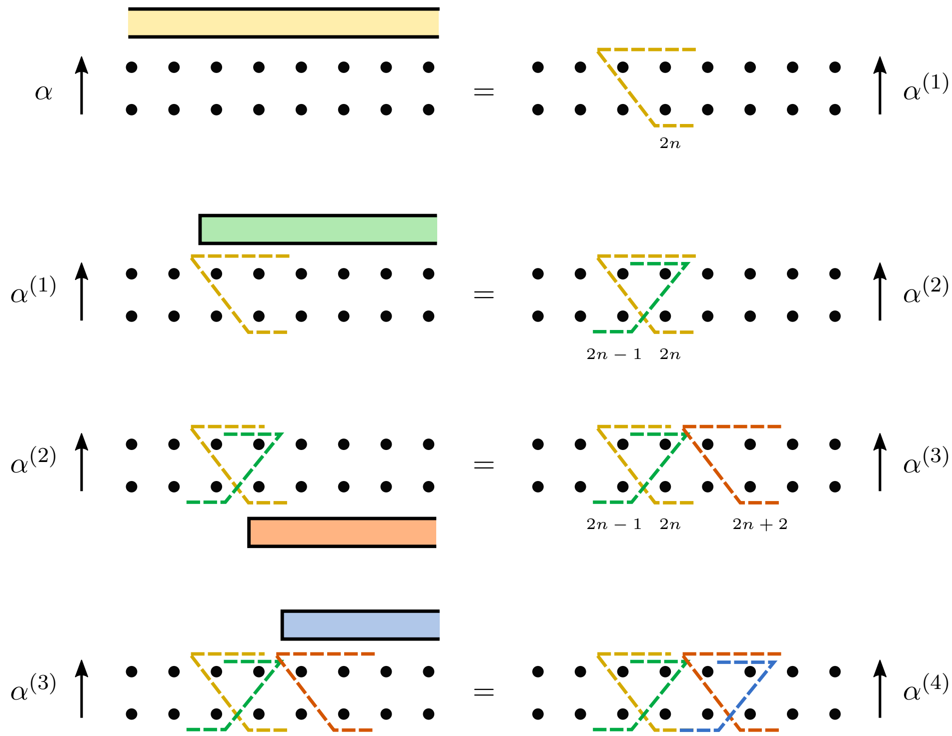

Figure 7: Illustration of the construction of $\alpha^{(i)}$ for $i = 1, 2, 3, 4$ in the proof of Proposition 5.4. Each row depicts an equation, and the solid strips on the left depict applications of unitary operators supported on those regions. The second, third, and fourth row use Lemma 5.1, Lemma 5.3, and Lemma 5.1 again respectively. Dashed lines indicate causal cones.

In particular, denoting $\mathcal{B}_m = \mathcal{A}_{\{2m,2m+1\}}$ *and* $\mathcal{C}_m = \mathcal{A}_{\{2m-1,2m\}}$ *as in Eq.* (4.1)*, we have*

$$\alpha_n(\mathcal{B}_m) \subseteq \mathcal{C}_m \otimes \mathcal{C}_{m+1} \quad \text{for } m \in \{n, n+1, n+2\},$$
$$\alpha_n^{-1}(\mathcal{C}_m) \subseteq \mathcal{B}_{m-1} \otimes \mathcal{B}_m \quad \text{for } m \in \{n+1, n+2\}.$$

(ii) Moreover, if α *is an ALPU with* $f(r)$*-tails, we may take* α_n *to be an ALPU with* $\mathcal{O}(f(r-7))$*-tails and such that, for* $r \to \infty$*,*

$$\|(\alpha - \alpha_n)|_{\mathcal{A}^{\mathrm{vN}}_{\leq 2n-r-1}}\| = \mathcal{O}(f(r-1)), \tag{5.12}$$
$$\|(\alpha - \alpha_n)|_{\mathcal{A}^{\mathrm{vN}}_{\geq 2n+r+6}}\| = \mathcal{O}(f(r-7)). \tag{5.13}$$

Proof. (i) We define a sequence of automorphisms $\alpha_n^{(i)}$, $i = 1, \dots, 8$ to obtain $\alpha_n := \alpha_n^{(8)}$ with the desired properties. To begin,

$$\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq 2n}) \overset{\varepsilon}{\subseteq} \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$$

so by Theorem 2.6, with $\mathcal{A}_0 = \alpha(\mathcal{A}_{\geq 2n})$, $\mathcal{A} = \mathcal{A}_0'' = \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq 2n})$ and $\mathcal{B} = \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$, there exists $u_1 \in (\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq 2n}) \cup \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1})''$ such that

$$u_1^* \alpha(\mathcal{A}^{\mathrm{vN}}_{\geq 2n}) u_1 \subseteq \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$$

with $\|u_1^* - I\| \le 12\varepsilon$. We define $\alpha_n^{(1)}(x) = u_1^*\alpha(x)u_1$, which by construction satisfies

$$\begin{aligned}\alpha_n^{(1)}(\mathcal{A}_{\ge 2n}^{\mathrm{vN}}) &\subseteq \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}},\\ \|\alpha - \alpha_n^{(1)}\| &= \mathcal{O}(\varepsilon).\end{aligned}$$

Then $\alpha_n^{(1)}$ is an $\mathcal{O}(\varepsilon)$-nearest neighbor automorphism by the above, and we are in a situation where we can apply Lemma 5.1 (but replacing n with $2n$) to obtain an automorphism $\alpha_n^{(2)}(x) = u_2^*\alpha_n^{(1)}(x)u_2$ for unitary $u_2 \in \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}}$, such that

$$\begin{aligned}\alpha_n^{(2)}(\mathcal{A}_{\le 2n-1}^{\mathrm{vN}}) &\subseteq \mathcal{A}_{\le 2n}^{\mathrm{vN}},\\ \alpha_n^{(2)}(\mathcal{A}_{\ge 2n}^{\mathrm{vN}}) &\subseteq \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}},\\ \|\alpha_n^{(1)} - \alpha_n^{(2)}\| &= \mathcal{O}(\varepsilon).\end{aligned}$$

Then $\alpha_n^{(2)}$ is again an $\mathcal{O}(\varepsilon)$-nearest neighbor automorphism, and we can apply Lemma 5.3 (but replacing n with $2n-1$) to obtain an automorphism $\alpha_n^{(3)}(x) = \alpha_n^{(2)}(u_3xu_3^*)$ for unitary $u_3 \in \mathcal{A}_{\ge 2n}^{\mathrm{vN}}$, such that

$$\begin{aligned}\alpha_n^{(3)}(\mathcal{A}_{\ge 2n+2}^{\mathrm{vN}}) &\subseteq \mathcal{A}_{\ge 2n+1}^{\mathrm{vN}},\\ \|\alpha_n^{(3)} - \alpha_n^{(2)}\| &= \mathcal{O}(\varepsilon).\end{aligned}$$

Since $u_3 \in \mathcal{A}_{\ge 2n}^{\mathrm{vN}}$, $\alpha_n^{(3)}$ also satisfies the locality properties listed for $\alpha_n^{(2)}$ above. See Fig. 7 for an illustration of the construction.

We continue to apply Lemma 5.1 and Lemma 5.3 alternatingly. Explicitly, we apply Lemma 5.1 (with $n \to 2n+2$) to define $\alpha_n^{(4)}$, as illustrated in the figure, and then Lemma 5.3 (with $n \to 2n+1$) to define $\alpha_n^{(5)}$, followed by Lemma 5.1 (with $n \to 2n+4$) to define $\alpha_n^{(6)}$ and Lemma 5.3 (with $n \to 2n+3$) to define $\alpha_n^{(7)}$. Finally we use Lemma 5.1 (with $n \to 2n+6$) to obtain $\alpha_n^{(8)}$. We take $\alpha_n := \alpha_n^{(8)}$; then α_n has the desired locality properties in the proposition statement. We must assume ε is sufficiently small to meet the conditions of these lemmas at each step, determining the universal constant ε_1 in the proposition statement.

(ii) Now we further assume α is an ALPU with $f(r)$-tails, to demonstrate Eqs. (5.12) and (5.13) and prove that α_n is an ALPU.

We first show that $\alpha_n^{(2)}$ is an ALPU with $\mathcal{O}(f(r-1))$ tails, using Lemma 5.2 (with $n \mapsto 2n$). Note that $\alpha_n^{(2)}(x) = v^*\alpha(x)v$ for $v = u_1u_2$, and $\alpha_n^{(2)}$ satisfies the necessary locality properties in Eq. (5.8) (unlike $\alpha_n^{(1)}$!), so in order to apply the lemma we only need to show that

$$\|v^*xv - x\| = \mathcal{O}(f(r-1))\|x\|, \tag{5.14}$$

for all $r \ge 0$ and $x \in \mathcal{A}_{\le 2n-r-2} \cup \mathcal{A}_{\ge 2n+r+1} \cup \alpha(\mathcal{A}_{\le 2n-r-1}) \cup \alpha(\mathcal{A}_{\ge 2n+r})$. To this end, recall that for u_1 we applied Theorem 2.6 with $\mathcal{A}_0 = \alpha(\mathcal{A}_{\ge 2n})$, $\mathcal{A} = \mathcal{A}_0'' = \alpha(\mathcal{A}_{\ge 2n}^{\mathrm{vN}})$ and $\mathcal{B} = \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}}$, and in the construction of $u_2 \in \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}}$ in Lemma 5.1 (with $n \mapsto 2n$) we applied Theorem 2.6 with $\mathcal{A}_0 = \mathcal{A}_{\ge 2n+1}$, $\mathcal{A} = \mathcal{A}_0'' = \mathcal{A}_{\ge 2n+1}^{\mathrm{vN}}$ and $\mathcal{B} = \alpha_n^{(1)}(\mathcal{A}_{\ge 2n}^{\mathrm{vN}})$.

First consider $x \in \mathcal{A}_{\le 2n-r-2}$. As $\alpha(\mathcal{A}_{\ge 2n}^{\mathrm{vN}}) \overset{f(r+1)}{\subseteq} \mathcal{A}_{\ge 2n-r-1}^{\mathrm{vN}}$, we have $\|[x,y]\| = 2f(r+1)\|x\|\|y\|$ for all $y \in \alpha(\mathcal{A}_{\ge 2n}^{\mathrm{vN}})$ by Lemma 2.4. Since moreover $[x,y] = 0$ for all $y \in \mathcal{A}_{\ge 2n-1}^{\mathrm{vN}}$, Theorem 2.6(ii) shows that $\|u_1^* x u_1 - x\| = \mathcal{O}(f(r+1)\|x\|)$. In addition, we have $u_2^* x u_2 = x$ since $u_2 \in \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$. Together we find that $\|v^* x v - x\| = \mathcal{O}(f(r+1)\|x\|)$.

Next consider $x \in \mathcal{A}_{\geq 2n+r+1}$. By Lemma 3.8, α^{-1} is an ALPU with $\mathcal{O}(f(r))$-tails, so we have

$$\alpha^{-1}(x) \overset{\mathcal{O}(f(r+1))}{\in} \mathcal{A}_{\geq 2n} \tag{5.15}$$

and hence $x \overset{\mathcal{O}(f(r+1))}{\in} \alpha(\mathcal{A}_{\geq 2n})$. Since moreover $x \in \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$, Theorem 2.6(iii) shows that $\|u_1^* x u_1 - x\| = \mathcal{O}(f(r+1)\|x\|)$. Since $(\alpha_n^{(1)})^{-1}(x) = \alpha^{-1}(u_1 x u_1^*)$, the latter along with Eq. (5.15) in turn implies that $x \overset{\mathcal{O}(f(r+1))}{\in} \alpha_n^{(1)}(\mathcal{A}_{\geq 2n})$. Also, $x \in \mathcal{A}_{\geq 2n+1}$, hence we obtain $\|u_2^* x u_2 - x\| = \mathcal{O}(f(r+1))$, again by Theorem 2.6(iii). Together we find that $\|v^* x v - x\| = \mathcal{O}(f(r+1)\|x\|)$.

Now consider $x \in \alpha(\mathcal{A}_{\leq 2n-r-1})$, i.e., $x = \alpha(z)$ for some $z \in \mathcal{A}_{\leq 2n-r-1}$. Then x commutes with $\alpha(\mathcal{A}^{\mathrm{vN}}_{\geq 2n})$. Moreover, $x \overset{f(r-1)}{\in} \mathcal{A}_{\leq 2n-2}$, hence $\|[x,y]\| \leq 2f(r-1)\|x\|\|y\|$ for all $y \in \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$. Thus we obtain $\|u_1^* x u_1 - x\| = \mathcal{O}(f(r-1)\|x\|)$ by Theorem 2.6(ii). The preceding in turn implies that for all $y \in \mathcal{A}^{\mathrm{vN}}_{\geq 2n+1}$,

$$\|[\alpha_n^{(1)}(z), y]\| = \|[u_1^* x u_1, y]\| \leq 2\|u_1^* x u_1 - x\|\|y\| + \|[x,y]\| = \mathcal{O}(f(r-1)\|x\|\|y\|).$$

Also, $\alpha_n^{(1)}(z)$ commutes with $\alpha_n^{(1)}(\mathcal{A}^{\mathrm{vN}}_{\geq 2n})$. Therefore, again by Theorem 2.6(ii) we see that

$$\|v^* x v - u_1^* x u_1\| = \|u_2^* \alpha_n^{(1)}(z) u_2 - \alpha_n^{(1)}(z)\| = \mathcal{O}(f(r-1)\|x\|).$$

We conclude that $\|v^* x v - x\| = \mathcal{O}(f(r-1)\|x\|)$.

Finally, let $x \in \alpha(\mathcal{A}_{\geq 2n+r})$, i.e., $x = \alpha(z)$ for some $z \in \mathcal{A}_{\geq 2n+r}$. Then $x \in \alpha(\mathcal{A}_{\geq 2n})$ and $x \overset{f(r+1)}{\in} \mathcal{A}^{\mathrm{vN}}_{\geq 2n-1}$. So, by Theorem 2.6(iii) we find that $\|u_1^* x u_1 - x\| = \mathcal{O}(f(r+1)\|x\|)$. Using the latter, as well as $\|\alpha_n^{(1)}(z) - \alpha(z)\| = \|u_1^* x u_1 - x\|$ and $\alpha(z) \overset{f(r-1)}{\in} \mathcal{A}_{\geq 2n+1}$, we find $\alpha_n^{(1)}(z) \overset{\mathcal{O}(f(r-1))}{\in} \mathcal{A}_{\geq 2n+1}$. Moreover, $\alpha_n^{(1)}(z) \in \alpha_n^{(1)}(\mathcal{A}^{\mathrm{vN}}_{\geq 2n})$, so by Theorem 2.6(iii) we obtain that

$$\|v^* x v - u_1^* x u_1\| = \|u_2^* \alpha_n^{(1)}(z) u_2 - \alpha_n^{(1)}(z)\| = \mathcal{O}(f(r-1)\|x\|),$$

and hence $\|v^* x v - x\| = \mathcal{O}(f(r-1)\|x\|)$. Altogether we have verified that Eq. (5.14) holds for all $r \geq 0$ and $x \in \mathcal{A}_{\leq 2n-r-2} \cup \mathcal{A}_{\geq 2n+r+1} \cup \alpha(\mathcal{A}_{\leq 2n-r-1}) \cup \alpha(\mathcal{A}_{\geq 2n+r})$. We may therefore apply Lemma 5.2 and conclude that $\alpha_n^{(2)}$ is an ALPU with $\mathcal{O}(f(r-1))$-tails.

For $i = 3, \ldots, 8$, we simply observe that by our applications of Lemma 5.1 and Lemma 5.3, the automorphisms $\alpha_n^{(i)}$ are guaranteed to be APLUs with $\mathcal{O}(f(r+1-i))$-tails.

To see that Eq. (5.12) holds, note that Eq. (5.14) implies that $\|\alpha_n^{(2)}(x) - \alpha(x)\| = \mathcal{O}(f(r-1)\|x\|)$ for all $x \in \mathcal{A}_{\leq 2n-r-1}$. Moreover, $\alpha_n(x) = \alpha_n^{(2)}(x)$ for such x, since $\alpha_n = \alpha_n^{(8)}$ is obtained from $\alpha_n^{(2)}$ by conjugating the input with unitaries in $\mathcal{A}^{\mathrm{vN}}_{\geq 2n}$ (leaving x unchanged) and the output by unitaries in $\mathcal{A}^{\mathrm{vN}}_{\geq 2n+1}$ (leaving $\alpha_n^{(2)}(x) \in \mathcal{A}^{\mathrm{vN}}_{\leq 2n}$ unchanged). Thus Eq. (5.12) follows.

Finally, Eq. (5.13) follows since the $\alpha_n^{(i)}$ for $i = 3, \ldots, 8$ satisfy analogs of Eqs. (5.5) and (5.11) and we have $\|\alpha_n^{(2)}(x) - \alpha(x)\| \leq \mathcal{O}(f(r-1)\|x\|)$ for all $x \in \mathcal{A}_{\geq 2n+r}$, again by Eq. (5.14). ■

Proposition 5.5 (QCA approximation of ε-nearest neighbor automorphism)**.** *There exists a universal constant $\varepsilon_2 > 0$ such that if α is an ε-nearest neighbor automorphism of $\mathcal{A}_{\mathbb{Z}}$ with $\varepsilon \leq \varepsilon_2$, then there exists a QCA β with radius 2 such that*

$$\|(\alpha - \beta)|_{\mathcal{A}_X}\| = \mathcal{O}(\varepsilon|X|)$$

for all regions X with $|X|$ sites.

Proof. Recall that α extends to a ε-nearest neighbor automorphism of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ by Lemma 3.8, which we will denote by the same symbol. Let C_1 and ε_1 be the constants from Proposition 5.4, and take $\varepsilon_2 := \min\{\frac{\varepsilon_1}{2}, \frac{1}{384C_1}\}$. As usual, we write $\mathcal{B}_n = \mathcal{A}_{\{2n,2n+1\}}$ and $\mathcal{C}_n = \mathcal{A}_{\{2n-1,2n\}}$. Now apply part (i) of Proposition 5.4 to find automorphisms α_m, one for each $m \in \mathbb{Z}$, which satisfy the locality properties $\alpha_m(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$ for $n \in \{m, m+1, m+2\}$ as well as $\alpha_m^{-1}(\mathcal{C}_n) \subseteq \mathcal{B}_{n-1} \otimes \mathcal{B}_n$ for $n \in \{m+1, m+2\}$. Then by Theorem 4.1 and the subsequent Remark 4.2, we can define

$$\mathcal{L}_n^{(m)} = \alpha_m(\mathcal{B}_n) \cap \mathcal{C}_n,$$
$$\mathcal{R}_{n-1}^{(m)} = \alpha_m(\mathcal{B}_{n-1}) \cap \mathcal{C}_n$$

such that, for $m \in \{n-1, n-2\}$,

$$\mathcal{C}_n = \mathcal{L}_n^{(m)} \otimes \mathcal{R}_{n-1}^{(m)}. \tag{5.16}$$

Moreover, again by Theorem 4.1 and Remark 4.2, we have

$$\mathcal{B}_n = \alpha_{n-1}^{-1}(\mathcal{L}_n^{(n-1)}) \otimes \alpha_{n-1}^{-1}(\mathcal{R}_n^{(n-1)}), \tag{5.17}$$

which we will use below.

Note that $\|\alpha_{n-1} - \alpha_{n-2}\| \leq \|\alpha_{n-1} - \alpha\| + \|\alpha_{n-2} - \alpha\| \leq 2C_1\varepsilon \leq \frac{1}{192}$. Because α_{n-1} and α_{n-2} are nearby ALPUs with locality properties satisfying Remark 4.11, we can apply the argument from Proposition 4.10 to α_{n-1} and α_{n-2}, finding that $\mathcal{L}_n^{(n-2)}$ and $\mathcal{L}_n^{(n-1)}$ are related by a unitary $u_n \in \mathcal{C}_n$, i.e. $u_n \mathcal{L}_n^{(n-1)} u_n^* = \mathcal{L}_n^{(n-2)}$, with $\|u_n - I\| = \mathcal{O}(\varepsilon)$. Finally we define

$$\beta_n \colon \mathcal{B}_n \to \mathcal{C}_n \otimes \mathcal{C}_{n+1}, \quad \beta_n(x) = u_n \alpha_{n-1}(x) u_n^*.$$

Each β_n is an injective homomorphism and by (5.17) we obtain

$$\beta_n(\mathcal{B}_n) = u_n \left(\mathcal{L}_n^{(n-1)} \otimes \mathcal{R}_n^{(n-1)}\right) u_n^* = \mathcal{L}_n^{(n-2)} \otimes \mathcal{R}_n^{(n-1)}, \tag{5.18}$$

where the second equality holds because $u_n \in \mathcal{C}_n$ and $\mathcal{R}_n^{(n-1)} \subseteq \mathcal{C}_{n+1}$ commute. From Eq. (5.18) we conclude that $\beta_n(\mathcal{B}_n)$ and $\beta_m(\mathcal{B}_m)$ commute for $n \neq m$. Hence we can define a global injective homomorphism β that acts as β_n on each $\mathcal{B}_n$. By (5.16) and (5.18), this homomorphism is surjective. Indeed, $\beta_{n-1}(\mathcal{B}_{n-1}) \otimes \beta_n(\mathcal{B}_n) \supseteq \mathcal{R}_{n-1}^{(n-2)} \otimes \mathcal{L}_n^{(n-2)} = \mathcal{C}_n$ for all n. Thus the map β is an automorphism. By construction it is clear that this automorphism is a QCA with radius 2. For any single site operator x we have that $x \in \mathcal{B}_n$ for some n, so

$$\begin{aligned}\|\beta(x) - \alpha(x)\| &\leq 2\|u_n - I\| + \|\alpha - \alpha_{n-1}\| \\ &\leq \mathcal{O}(\varepsilon).\end{aligned}$$

We showed $\|(\beta - \alpha)|_{\mathcal{A}_n}\| = \mathcal{O}(\varepsilon)$ for all single sites n, and the desired result holds by Lemma 2.7. ■

By Proposition 5.5 and coarse-graining, we obtain the main result of this section, which shows that any ALPU in one dimensions can be approximated by a sequence of QCAs.

Theorem 5.6 (QCA approximations)**.** *If α is a one-dimensional ALPU with $f(r)$-tails, then there exists a sequence of QCAs $\{\beta_j\}_{j=1}^{\infty}$, of radius $2j$ such that for any finite subset $X \subset \mathbb{Z}$,*[6]

$$\|(\alpha - \beta_j)|_{\mathcal{A}_X}\| = \mathcal{O}\big(f(j) \min\Big\{|X|, \Big\lceil \tfrac{\operatorname{diam}(X)}{j} \Big\rceil\Big\}\big). \tag{5.19}$$

Moreover, there is a constant $C_f > 0$, depending only on $f(r)$, such that the following holds for all j and finite $X \subset \mathbb{Z}$:

$$\|(\alpha - \beta_j)|_{\mathcal{A}_X}\| \leq C_f\, f(j) \min\Big\{|X|, \Big\lceil \tfrac{\operatorname{diam}(X)}{j} \Big\rceil\Big\}. \tag{5.20}$$

In particular, the β_j converge strongly to α, meaning that $\lim_{j\to\infty}\|\alpha(x) - \beta_j(x)\| = 0$ for all $x \in \mathcal{A}_{\mathbb{Z}}$.

Proof. By blocking j sites we obtain an ε_j-nearest neighbor QCA on the coarse-grained lattice where $\varepsilon_j = f(j)$. For $j > j_0$ sufficiently large, we can apply Proposition 5.5 to obtain a QCA β_j of radius 2 on the coarse-grained lattice satisfying $\|(\alpha-\beta)|_{\mathcal{A}_X}\| = \mathcal{O}(f(j)m)$ for all regions X composed of m coarse-grained sites. If we now consider β_j as a QCA of radius $2j$ on the original lattice (before coarse-graining), we arrive at (5.19). To obtain (5.20), for smaller $j \leq j_0$, we may choose some arbitrary QCA β_j and use that $\|\alpha - \beta_j\| \leq 2$ at the expense of incurring a tails-dependent constant $C_f > 0$.

We now show that the sequence of QCAs β_j converges strongly to α. For $x \in \mathcal{A}_{\mathbb{Z}}$ arbitrary, let x_n be a sequence of strictly local operators, where x_n is supported on n contiguous sites, such that $\lim_{n\to\infty} x_n = x$ converges in norm. Then,

$$\begin{aligned}\limsup_{j\to\infty} \|\alpha(x) - \beta_j(x)\| &\leq \limsup_{j\to\infty} \Big(\|\alpha(x) - \alpha(x_n)\| + \|\alpha(x_n) - \beta_j(x_n)\| + \|\beta_j(x_n) - \beta_j(x)\|\Big) \\ &\leq 2\|x - x_n\| + \limsup_{j\to\infty} \|\alpha(x_n) - \beta_j(x_n)\| = 2\|x - x_n\|.\end{aligned}$$

The second inequality holds since α and the β_j are $*$-homomorphisms; the final equality follows by (5.20). Since the above holds for all n, we conclude that $\lim_{j\to\infty}\|\alpha(x) - \beta_j(x)\| = 0$. ■

5.2 Definition of the index for ALPUs

We now use the QCA approximations developed in the preceding to define an index for general ALPUs. In addition, we give two alternative ways of computing the index for ALPUs with appropriately decaying tails, and we prove that the index is stable also for ALPUs.

Definition 5.7 (Index for ALPUs)**.** Let α be a one-dimensional ALPU with $f(r)$-tails and let β_j be a sequence of QCAs of radius at most $2j$ such that for any finite subset $X \subset \mathbb{Z}$,

$$\|(\alpha - \beta_j)|_{\mathcal{A}_X}\| \leq C_f\, f(j) \Big\lceil \tfrac{\operatorname{diam}(X)}{j} \Big\rceil, \tag{5.21}$$

where $C_f > 0$ is the constant from Theorem 5.6. We define the *index* of α by

$$\operatorname{ind}(\alpha) := \lim_{j\to\infty} \operatorname{ind}(\beta_j). \tag{5.22}$$

[6]Below $|X|$ denotes the number of sites in the subset X, while $\operatorname{diam}(X) = \max(X) - \min(X) + 1$ denotes the diameter of the subset, and $\lceil\cdot\rceil$ is the integer ceiling.

Note that by Theorem 5.6 such a sequence β_j always exists. The following theorem shows that the index is a well-defined, finite quantity.

Theorem 5.8 (Index for ALPUs)**.** *Let α be a one-dimensional ALPU with $f(r)$-tails and let β_j be a sequence of QCAs as in Definition 5.7. Then the following hold:*

(i) There exists j_0, depending only on $f(r)$, such that $\operatorname{ind}(\beta_j)$ is constant for $j \geq j_0$. Accordingly, the limit (5.22) exists and is in $\mathbb{Z}[\{\log(p_i)\}]$, where the p_i are the finitely many prime factors of the local Hilbert space dimensions d_n, and $\mathbb{Z}[\cdot]$ denotes integer linear combinations. Moreover, this limit does not depend on the choice of sequence β_j. Thus, $\operatorname{ind}(\alpha)$ is well-defined by (5.22).

(ii) There is a constant r_1, depending only on $f(r)$, with the following property: Let α' be another one-dimensional ALPU with $f(r)$-tails. Then, for any interval X with $|X| = \operatorname{diam}(X) \geq r_1$,

$$\|(\alpha - \alpha')|_{\mathcal{A}_X}\| \leq \frac{1}{384} \implies \operatorname{ind}(\alpha) = \operatorname{ind}(\alpha').$$

In particular, the index is completely determined by $\alpha|_{\mathcal{A}_X}$ for any such X.

(iii) If $f(r) = o(\frac{1}{r})$ then there exist a constant r_2, depending only on $f(r)$ and the local Hilbert space dimensions d_n, such that the index may also be computed locally as in (4.12),

$$\operatorname{ind}(\alpha) = \operatorname{round}_{\mathbb{Z}[\{\log(p_i)\}]} \frac{1}{2} \left(I(L_1' : R_1)_\phi - I(L_1 : R_1')_\phi \right),$$

where ϕ denotes the Choi state, the intervals L_1, R_1, L_1', R_1' must be of size at least r_2, and the notation means that we round to the nearest value in $\mathbb{Z}[\{\log(p_i)\}]$.

(iv) If $f(r) = \mathcal{O}(\frac{1}{r^{1+\delta}})$ for some $\delta > 0$, then the index can also be computed as in (5.1), by

$$\operatorname{ind}(\alpha) = \frac{1}{2} \left(I(L' : R)_\phi - I(L : R')_\phi \right),$$

where both $I(L' : R)_\phi$ and $I(L : R')_\phi$ are finite.

In both calculations (iii) and (iv) of the index, the cut defining the regions L, R may be chosen anywhere on the chain.

Proof. Throughout this proof, the implicit constants in the $\mathcal{O}$ notation are allowed to depend on the tails $f(r)$.

(i) and (ii): To see that $\operatorname{ind}(\beta_j)$ stabilizes at large j and hence the limit (5.22) exists, consider β_j and β_{j+1}. After coarse-graining by blocking $2(j+1)$ sites, both β_j and β_{j+1} are nearest neighbor. Moreover, on any subset X_j that consists of two neighboring coarse-grained sites,

$$\|(\beta_j - \beta_{j+1})|_{\mathcal{A}_{X_j}}\| \leq \|(\beta_j - \alpha)|_{\mathcal{A}_{X_j}}\| + \|(\alpha - \beta_{j+1})|_{\mathcal{A}_{X_j}}\| = \mathcal{O}(f(j)) \tag{5.23}$$

by Eq. (5.21). Since $f(r) = o(1)$ this implies that $\|(\beta_j - \beta_{j+1})|_{\mathcal{A}_{X_j}}\|$ approaches zero as $j \to \infty$. By Proposition 4.10 this implies that $\operatorname{ind}(\beta_j) = \operatorname{ind}(\beta_{j+1})$ for sufficiently large $j \geq j_0$, where the constant j_0 can be taken as the minimum j such that the right-hand side of Eq. (5.23) remains below $\frac{1}{192}$. Thus we conclude that the limit (5.22) exists and equals $\operatorname{ind}(\beta_j)$ for $j \geq j_0$. Moreover, $\operatorname{ind}(\alpha) \in \mathbb{Z}[\{\log(p_i)\}]$, since the same is true for the index of the QCAs β_j.

To conclude the proof of (i), we still need to argue that the index is well-defined. We will demonstrate this together with (ii). Consider an ALPU α' that also has $f(r)$-tails, and let β'_j be a corresponding sequence of QCAs as in Definition 5.7. Note that $\mathrm{ind}(\beta_j)$ and $\mathrm{ind}(\beta'_j)$ stabilize for $j \geq j_0$, with the same constant j_0. We claim that $\mathrm{ind}(\beta_j) = \mathrm{ind}(\beta'_j)$ for some (and hence for all) $j \geq j_0$. To see this, we consider β_j and β'_j as nearest-neighbor QCAs on a coarse-grained lattice obtained by blocking $2j$ sites. Then by Proposition 4.10, it is sufficient to show $\|(\beta_j - \beta'_j)|_{\mathcal{A}_Y}\| \leq \frac{1}{192}$ for a region Y consisting of two neighboring coarse-grained sites. Note Y then consists of $4j$ sites on the original lattice. Now,

$$\begin{aligned}\|(\beta_j - \beta'_j)|_{\mathcal{A}_Y}\| &\leq \|(\beta_j - \alpha)|_{\mathcal{A}_Y}\| + \|(\alpha - \alpha')|_{\mathcal{A}_Y}\| + \|(\alpha' - \beta'_j)|_{\mathcal{A}_Y}\| \\ &\leq \mathcal{O}(f(j)) + \|(\alpha - \alpha')|_{\mathcal{A}_Y}\|.\end{aligned}$$

Since $f(r) = o(1)$, we can find $j_1 \geq j_0$ large enough such that the $\mathcal{O}(f(j))$ term is smaller than $\frac{1}{384}$. Take $r_1 := 8j_1$ to ensure that any interval X with r_1 sites contains two neighboring sites of the coarse-grained lattice, so that $\|(\alpha - \alpha')|_{\mathcal{A}_Y}\| \leq \frac{1}{384}$ by assumption. Then, $\|(\beta_{j_1} - \beta'_{j_1})|_{\mathcal{A}_Y}\| \leq \frac{1}{192}$, and Proposition 4.10 implies that $\mathrm{ind}(\beta_j) = \mathrm{ind}(\beta'_j)$ for $j = j_1$ and hence for all $j \geq j_0$. This implies that the index is well-defined (take $\alpha = \alpha'$), concluding the proof of (i), and it also establishes (ii).

(iii) Let $L_j = \{-2j+1, \ldots, 0\}$ and $R_j = \{1, \ldots, 2j\}$. Since β_j is a QCA of radius $2j$, by Proposition 4.7 we can compute

$$\mathrm{ind}(\beta_j) = \frac{1}{2}\Big(I(L'_j : R_j)_{\phi_j} - I(L_j : R'_j)_{\phi_j}\Big) \tag{5.24}$$

where $\phi_j = (\beta_j^\dagger \otimes \mathrm{id})(\omega)$, with ω a maximally entangled state on $\mathcal{A}_\mathbb{Z} \otimes \mathcal{A}_\mathbb{Z}$. We let

$$\widetilde{\mathrm{ind}}_j(\alpha) := \widetilde{\mathrm{ind}}_{L_j,R_j}(\alpha) = \frac{1}{2}\Big(I(L'_j : R_j)_{\phi} - I(L_j : R'_j)_{\phi}\Big) \tag{5.25}$$

as in (4.14), where $\phi = (\alpha^\dagger \otimes \mathrm{id})(\omega)$. By Eq. (5.21), $\|(\alpha - \beta_j)|_{A_{X_j}}\| = \mathcal{O}(f(j))$, where $X_j = L_j \cup R_j$. Thus Lemma 4.9 shows that

$$\big|\mathrm{ind}(\beta_j) - \widetilde{\mathrm{ind}}_j(\alpha)\big| = \mathcal{O}\big(jf(j)\log(d) + f(j)\log\tfrac{1}{f(j)}\big) \tag{5.26}$$

where $d = \max_n d_n$ is the maximum of the local Hilbert space dimensions associated to $\mathcal{A}_\mathbb{Z}$. Assuming that $f(j) = o(\frac{1}{j})$ the above approaches zero as $j \to \infty$. Because the sequence $\mathrm{ind}(\beta_j)$ stabilizes to $\mathrm{ind}(\alpha)$ by definition in (5.22), this implies that

$$\lim_{j\to\infty} \widetilde{\mathrm{ind}}_j(\alpha) = \mathrm{ind}(\alpha). \tag{5.27}$$

Since $\mathrm{ind}(\alpha)$ takes values in the nowhere dense set $\mathbb{Z}[\{\log(p_i)\}]$, rounding $\widetilde{\mathrm{ind}}_j(\alpha)$ must yield $\mathrm{ind}(\alpha)$ for sufficiently large j, proving (iii).

(iv) Even though the quantities in Eqs. (5.24) and (5.25) converge with j, we have not yet shown that the individual mutual information terms converge. We will show this next, assuming that $f(r) = \mathcal{O}(\frac{1}{r^{1+\delta}})$ for some $\delta > 0$. We consider the subsequence $\{\beta_{2^k}\}$. Then, by Eq. (5.21)

$$\|(\beta_{2^k} - \alpha)|_{\mathcal{A}_{X_{2^{k+1}}}}\| = \mathcal{O}(f(2^k)), \tag{5.28}$$

and thus

$$\|(\beta_{2^k} - \beta_{2^{k+1}})|_{\mathcal{A}_{X_{2^{k+1}}}}\| = \mathcal{O}(f(2^k)).$$

Hence by Lemma 4.9, noting that $I(L'_{2^{k+1}} : R_{2^{k+1}})_{\phi_{2^k}} = I(L'_{2^k} : R_{2^k})_{\phi_{2^k}}$ since β_{2^k} has radius 2^{k+1}, as similarly observed in the proof of Proposition 4.7, this implies

$$\begin{aligned}|I(L'_{2^k} : R_{2^k})_{\phi_{2^k}} - I(L'_{2^{k+1}} : R_{2^{k+1}})_{\phi_{2^{k+1}}}| &= \mathcal{O}(2^k f(2^k)\log(d) + f(2^k)\log\tfrac{1}{f(2^k)}) \\ &= \mathcal{O}(2^{-\delta k}).\end{aligned}$$

Thus $I(L'_{2^k} : R_{2^k})_{\phi_{2^k}}$ is a Cauchy sequence and hence converges. Moreover, by Lemma 4.9, Eq. (5.28) also implies that

$$\begin{aligned}|I(L'_{2^k} : R_{2^k})_{\phi} - I(L'_{2^k} : R_{2^k})_{\phi_{2^k}}| &= \mathcal{O}(2^k f(2^k)\log(d) + f(2^k)\log\tfrac{1}{f(2^k)}) \\ &= \mathcal{O}(2^{-\delta k}).\end{aligned}$$

Thus $I(L'_{2^k} : R_{2^k})_{\phi}$ also converges, with the same limit as $I(L'_{2^k} : R_{2^k})_{\phi_{2^k}}$. Then using Proposition 4.6, this implies that

$$I(L' : R)_\phi = \lim_{k\to\infty} I(L'_{2^k} : R_{2^k})_\phi = \lim_{k\to\infty} I(L'_{2^k} : R_{2^k})_{\phi_{2^k}}$$

is finite. A similar argument shows that $I(L : R')_\phi$ is finite and can be computed as

$$I(L' : R)_\phi = \lim_{k\to\infty} I(L_{2^k} : R'_{2^k})_\phi = \lim_{k\to\infty} I(L_{2^k} : R'_{2^k})_{\phi_{2^k}}.$$

It follows that

$$\operatorname{ind}(\alpha) = \frac{1}{2}\left(I(L' : R)_\phi - I(L : R')_\phi\right),$$

as a consequence either of Eq. (5.27) or of Eq. (5.22).

In parts (iii) and (iv) we took the cut between L and R to be at $n = 0$, but the index may be calculated using regions translated anywhere along the chain, which follows from the same fact for the QCAs β_j. ■

The proof of Theorem 5.8 also shows that in part (iv), the two mutual information quantities can be computed as limits of corresponding mutual information quantities for finite intervals.

5.3 Properties of the index for ALPUs

In this section we will show that the index for ALPUs defined in Theorem 5.8 inherits essentially all properties of the GNVW index for QCAs stated in Theorem 4.4.

We first use Theorem 5.6 to construct a path between any ALPU α with $\operatorname{ind}(\alpha) = 0$ and the identity automorphism id, using a one-parameter family of ALPUs $\beta[t]$ for $t \in [0, 1]$, with $\beta[0] = \mathrm{id}$ and $\beta[1] = \alpha$. The path will be *strongly continuous*, in the sense that for all $x \in \mathcal{A}_{\mathbb{Z}}, t_0 \in [0, 1]$,

$$\lim_{t\to t_0} \|\alpha[t](x) - \alpha[t_0](x)\| = 0. \tag{5.29}$$

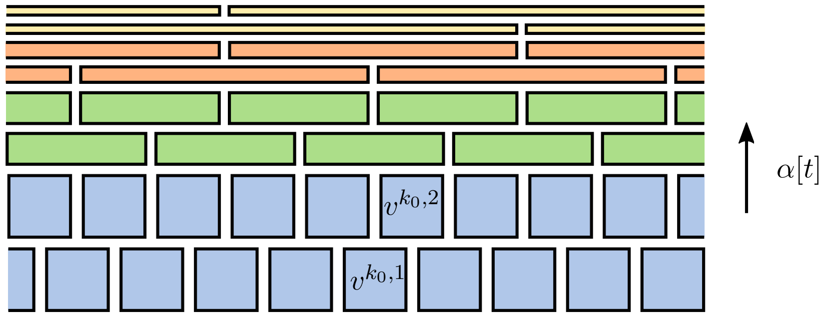

Figure 8: Illustration of the construction of $\alpha[t]$ in Theorem 5.9. The evolution consists of successive evolutions by different time-independent Hamiltonians, depicted as successive layers, with interaction terms increasing in diameter but decreasing in strength.

Theorem 5.9 (Continuous deformations)**.** *If α is a one-dimensional ALPU with $f(r)$-tails with $\operatorname{ind}(\alpha) = 0$, then there exists a strongly continuous path $\alpha(t)$ with $\alpha[0] = \mathrm{id}$ and $\alpha[1] = \alpha$ such that $\alpha[t]$ has $g(r)$-tails for all t, for some $g(r) = \mathcal{O}(f(Cr))$ and some universal constant $C > 0$. Moreover, this path may be given by a time evolution using a time-dependent Hamiltonian $H(t)$ evolving for unit time. For every $t < 1$ there exists l such that the Hamiltonian $H(t)$ has only terms H_X on (nonoverlapping) sets X of diameter at most $16l$, and it holds that $\|H_X(t)\| = \mathcal{O}(f(l)\log(l))$.*

The above $H(t)$ is constructed as piecewise-constant for $t \in [0, 1)$. The idea of the proof is that we continuously interpolate between consecutive QCAS β_{2^j} as constructed in Theorem 5.6. For large j we need to use a Hamiltonian with a correspondingly large support to interpolate between β_{2^j} and $\beta_{2^{j+1}}$, but on the other hand β_{2^j} and $\beta_{2^{j+1}}$ are locally close, so the interaction strength is small. As we increase j, we "speed up" the interpolation, so we get to α in unit time. In particular, the Hamiltonian is piecewise constant on time intervals that decrease in size as t goes to 1, and the support of the Hamiltonian increases as t goes to 1. This procedure is illustrated in Fig. 8 and leads to the given bound on the terms $H_X(t)$ of the Hamiltonian. For $f(r) = \mathcal{O}((\log r)^{-1})$ the norms $\|H_X(t)\|$ are uniformly bounded as $t \to 1$; more generally for $f(r) = o(1)$ the terms may diverge in norm as $t \to 1$, but nonetheless the path $H(t)$ is strongly continuous on $[0, 1]$. Of course, the path $\alpha(t)$ and associated Hamiltonian $H(t)$ are not unique; we just provide one particular construction.

Proof. We apply Theorem 5.6 to obtain a sequence of QCA approximations β_j of radius $2j$ with error $\|(\alpha - \beta_j)|_{\mathcal{A}_X}\| = \mathcal{O}(f(j)\lceil \frac{\operatorname{diam}(X)}{j} \rceil)$ as $j \to \infty$. Therefore

$$\|(\beta_{2j} - \beta_j)|_{\mathcal{A}_X}\| \leq \|(\alpha - \beta_j)|_{\mathcal{A}_X}\| + \|(\alpha - \beta_{2j})|_{\mathcal{A}_X}\| = \mathcal{O}(f(j)\lceil \tfrac{\operatorname{diam}(X)}{j} \rceil),$$

having used that f is non-increasing, and hence

$$\|(\beta_{2j}\beta_j^{-1} - \mathrm{id})|_{\mathcal{A}_X}\| \leq \|(\beta_{2j} - \beta_j)|_{\mathcal{A}_{B(X,2j)}}\| = \mathcal{O}(f(j)\lceil \tfrac{\operatorname{diam}(X)}{j} \rceil)$$

We can therefore define QCAs

$$\gamma_k = \beta_{2^{k+1}}\beta_{2^k}^{-1}$$

which have at most radius $R_k = 2^{k+3}$ and satisfy $\|(\gamma_k - \mathrm{id})|_{\mathcal{A}_X}\| = \mathcal{O}(f(2^k))$ for $\mathrm{diam}(X) \leq 2R_k = 2^{k+4}$. For sufficiently large $k \geq k_0$, $\mathrm{ind}(\beta_{2^k}) = \mathrm{ind}(\alpha) = 0$, and hence $\mathrm{ind}(\gamma_k) = 0$.

By Theorem 4.1 and Theorem 4.4, any index-0 QCA of radius R can be decomposed as a two-layer circuit with unitaries on blocks of diameter $2R$. If the QCA is ε-near the identity when restricted to intervals of size $2R$, the individual unitaries in the circuit are $\mathcal{O}(\varepsilon)$-near the identity by Proposition 4.12. Therefore γ_k may be implemented by a two-layer unitary circuit for $k \geq k_0$. We proceed to describe this circuit as a Hamiltonian evolution, with a different time-independent Hamiltonian generating each layer, in the straightforward way. To be precise, from Proposition 4.12 we obtain that

$$\gamma_k(x) = (v^{(k,2)})^*(v^{(k,1)})^* x v^{(k,1)} v^{(k,2)}$$

where for each layer $a \in \{1, 2\}$

$$v^{(k,a)} = \prod_n v_n^{(k,a)}$$

and where the $\{v_n^{(k,a)}\}_n$ are unitary gates acting on disjoint regions of diameter $2R_k = 2^{k+4}$. Moreover, each gate satisfies $\|v_n^{(k,a)} - I\| = \mathcal{O}(f(2^k))$. Each gate is generated by a Hamiltonian $H_n^{(k,a)} = -i\log(v_n^{(k,a)})$, defined using the principal logarithm, with $\|H_n^{(k,a)}\| = \mathcal{O}(f(2^k))$. Let $H^{(k,a)} = \sum_n H_n^{(k,a)}$ denote the total Hamiltonian generating the a-th layer. Then we can define a Hamiltonian evolution $\gamma_k[t]$ for $t \in [0, 1]$ with $\gamma_k[0] = I$, $\gamma_k[1] = \gamma_k$:

$$\gamma_k[t](x) = e^{2iH^{(k,1)}t}(x)e^{-2iH^{(k,1)}t}$$

for $t \in [0, \frac{1}{2}]$ and

$$\gamma_k[t](x) = e^{2iH^{(k,2)}(t-\frac{1}{2})}e^{iH^{(k,1)}}(x)e^{-iH^{(k,1)}}e^{-2iH^{(k,2)}(t-\frac{1}{2})}$$

for $t \in (\frac{1}{2}, 1]$. Note that the gates implementing $\gamma_k[t]$ are all of the form $(v_n^{(k,a)})^s$ for some $s \in [0, 1]$. From this it is clear that $\gamma_k[t]$ defines a strongly continuous path and the evolution is gentle in the sense that $\gamma_k[t]$ never strays far from id:

$$\|(\gamma_k[t] - \mathrm{id})|_{\mathcal{A}_X}\| = \mathcal{O}(f(2^k)).$$

for $\mathrm{diam}(X) \leq 2R_k$. By construction, $\gamma_k[t]$ is a QCA with radius at most $3R_k$ for every $t \in [0, 1]$.

We let $\alpha_{k+1}[t] := \gamma_k[t]\beta_{2^k}$, which is a strongly continuous path with $\alpha_{k+1}[0] = \beta_{2^k}$ and $\alpha_{k+1}[1] = \beta_{2^{k+1}}$. For all $t \in [0, 1]$

$$\|(\alpha_{k+1}[t] - \alpha)|_{\mathcal{A}_X}\| \leq \|(\gamma_k - \mathrm{id})|_{\mathcal{A}_{B(X,2^{k+1})}}\| + \|(\alpha - \beta_{2^k})|_{\mathcal{A}_X}\| = \mathcal{O}(f(2^k)) \tag{5.30}$$

for $\mathrm{diam}(X) \leq R_k$. Moreover $\alpha_{k+1}[t]$ is a QCA with radius $3R_k + 2^{k+1} \leq 4R_k$.

We defined $\alpha_k[t]$ only for $k > k_0$. Let $\alpha_{k_0}[t]$ be the Hamiltonian evolution implementing the index-0 QCA $\beta_{2^{k_0}}$ for $t \in [0, 1]$, in the same way we defined $\gamma_k[t]$. Let

$$T = \sum_{k=0}^{\infty} \frac{1}{1 + k^2}, \quad t_k = \sum_{l=0}^{k-1} \frac{1}{T(1 + l^2)}.$$

We define $\alpha[t]$ by gluing together the $\alpha_k[t]$, "speeding up" α_{k_0+k} by a factor $T(k^2+1)$ in order to make this a unit time evolution:

$$\alpha[t] = \alpha_{k_0+k}\left[\frac{t-t_k}{T(k^2+1)}\right] \text{ if } t \in (t_k, t_{k+1})$$

for $t \in [0,1)$ and $\alpha[1] = \alpha$. The construction of the path $\alpha[t]$ is illustrated in Fig. 8. Going through γ_{k_0+k} faster by a factor $T(k^2+1)$ is equivalent to rescaling the Hamiltonian by $T(k^2+1)$, and is still strongly continuous. Hence $\alpha[t]$ is strongly continuous for $t \in [0,1)$. The strong continuity at $t = 1$ follows from the fact that the sequence β_{2^k} converges strongly to α. Indeed, let $x \in \mathcal{A}_X$ for finite X. Then consider k such that $\operatorname{diam}(X) \leq R_{k_0+k}$, then we see that for $l \geq k_0 + k$ $\|\alpha_{l+1}[s](x) - \alpha(x)\| = \mathcal{O}(f(2^l))$ for $s \in [0,1]$. Hence, $\|\alpha[t](x) - \alpha(x)\|$ goes to zero as $t \to 1$. As in the proof of Theorem 5.6 we see that the same holds for general $x \in \mathcal{A}_{\mathbb{Z}}$. Moreover, we see that at each point in time the Hamiltonian will have terms H_X with support of diameter $16l = 2^{k+4}$ for some k with $\|H_X\| = \mathcal{O}(f(2^k)k^2) = \mathcal{O}(f(l)\log(l))$.

Finally, we need to show that $\alpha[t]$ has uniform tail bounds for $t \in (0,1)$. (We already have tail bounds at the initial and final time.) Let $X \subseteq \mathbb{Z}$ be an arbitrary (finite or infinite) interval. Take some $r > 4R_{k_0} = r_0$. There will be some k such that $4R_k \leq r < 4R_{k+1}$, and there will be some l and $s \in [0,1]$ such that $\alpha[t] = \alpha_{l+1}[s]$. If $k \geq l$, by construction $\alpha[t](\mathcal{A}_X) \subseteq \mathcal{A}_{B(X,4R_l)} \subseteq \mathcal{A}_{B(X,r)}$. On the other hand, suppose that $k < l$. Write $X = X_1 \cup X_2$ where X_1 is the (possibly empty set) of elements with distance from the boundary larger than $4R_l$. Then $\alpha_{l+1}[s](\mathcal{A}_{X_1}) \subset \mathcal{A}_X$. Moreover, since X_2 consists of at most two intervals of size $4R_l$ we have, using Lemma 2.7 and (5.30) that $\|(\alpha - \alpha_{l+1}[s])|_{\mathcal{A}_{X_2}}\| = \mathcal{O}(f(2^l))$. Since α has $f(r)$-tails,

$$\alpha(\mathcal{A}_{X_2}) \overset{\mathcal{O}(f(r))}{\subseteq} \mathcal{A}_{B(X,r)},$$

and since $r < 4R_l = 2^{l+5}$ we see that

$$\begin{aligned} \alpha_{l+1}[s](\mathcal{A}_{X_2}) &\overset{\mathcal{O}(f(r)+f(2^l))}{\subseteq} \mathcal{A}_{B(X,r)} \\ \alpha_{l+1}[s](\mathcal{A}_{X_2}) &\overset{\mathcal{O}(f(\frac{r}{32}))}{\subseteq} \mathcal{A}_{B(X,r)}. \end{aligned}$$

Lemma B.3 allows us to conclude that

$$\alpha[t](\mathcal{A}_X) = \alpha_{l+1}[s](\mathcal{A}_X) \overset{\mathcal{O}(f(\frac{r}{32}))}{\subseteq} \mathcal{A}_{B(X,r)}.$$

■

Remark 5.10. If α has $\mathcal{O}(\frac{1}{r^{1+a}})$-tails for $a > 0$, then for $0 < b < a$ and reproducing function $F(r) = \frac{1}{(1+r)^{1+b}}$ the Hamiltonian constructed in Theorem 5.9 satisfies the hypotheses in Theorem 3.2 (Lieb-Robinson). However, notice that the locality estimates you get from applying the Lieb-Robinson bounds to these bounds are weaker than the original locality bounds on $\alpha[t]$.

Remark 5.11. The Hamiltonian evolution constructed in Theorem 5.9 cannot always be approximated by a 2-local quantum circuit of constant depth. Likewise, even QCAs of radius r may have circuit complexity exponential in r when using 2-local gates.

Remark 5.12. Finally, we observe that in Theorem 5.9 if we have exponential tails, with $f(r) = \mathcal{O}(e^{-ar})$, one obtains that $\alpha[t]$ has $\mathcal{O}(e^{-aCr})$-tails. This is not entirely optimal, and for exponential tails one can slightly change the proof, by considering the sequence β_k rather than β_{2^k} and correspondingly $\gamma_k = \beta_{k+1}\beta_k^{-1}$ instead of $\gamma_k = \beta_{2^{k+1}}\beta_{2^k}^{-1}$. The same arguments as in the proof of Theorem 5.9 then lead to a path $\alpha[t]$ with $\mathcal{O}(f(r+C)) = \mathcal{O}(e^{-ar})$-tails, which is implemented by a Hamiltonian $H(t)$. In this case the Hamiltonian is such that for every t, there exists k such that $H(t)$ has only terms H_X on (nonoverlapping) sets X of diameter at most k, with $\|H_X(t)\| = \mathcal{O}(k^2 e^{-ak})$.

Next we discuss blending. We need a slightly weaker notion than for QCAs.

Definition 5.13. Two ALPUs α_1 and α_2 in one dimension can be *blended* (at the origin) if there exists an ALPU β on some $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ such that

$$\lim_{r\to\infty} \|(\beta - \alpha_1)|_{\mathcal{A}_{\leq -r}}\| = 0,$$
$$\lim_{r\to\infty} \|(\beta - \alpha_2)|_{\mathcal{A}_{\geq r}}\| = 0.$$

Proposition 5.14. *Two ALPUs α_1, α_2 can be blended if and only if* $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$.

When $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$ and both ALPUs have $f(r)$-tails, the approximation requirement of the blending as defined in Definition 5.13 can be refined as in (5.33) as discussed in the proof. The blending proceeds similarly to the construction in Proposition 5.5.

Proof. First we assume α_1 and α_2 can be blended and show $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$. Consider a blended ALPU β as in Definition 5.13. By Theorem 5.8(ii), one may compute $\mathrm{ind}(\beta)$ locally on either half of the blended chain. Both calculations must yield the same index, which does not depend on where it is locally calculated. By (ii) of Theorem 5.8, the index computed locally at the sufficiently far left must be $\mathrm{ind}(\alpha_1)$, and the index computed at the far right must be $\mathrm{ind}(\alpha_2)$. Thus, $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$.

Next we show that if $\mathrm{ind}(\alpha_1) = \mathrm{ind}(\alpha_2)$, then α_1 and α_2 can be blended. We assume both ALPUs are defined on the same $\mathcal{A}_{\mathbb{Z}}$ (i.e., the local dimensions are the same) and address the general case afterward. Both ALPUs extend to automorphisms of $\mathcal{A}_{\mathbb{Z}}^{\mathrm{vN}}$ as in Remark 3.9. Coarse-grain the lattice until both α_1 and α_2 are ε-nearest neighbor ALPUs, with ε smaller than a universal constant determined by the remainder of the proof. Then we can apply Proposition 5.4 (if $\varepsilon \leq \varepsilon_1$) separately to α_1 and α_2 at site $n = 0$. Denote the ALPUs resulting from Proposition 5.4 as $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively. Then by construction $\|\tilde{\alpha}_i - \alpha_i\| \leq C_1\varepsilon$ for $i = 1, 2$. Moreover by Theorem 5.8(ii), we can take ε small enough that $\mathrm{ind}(\alpha_i) = \mathrm{ind}(\tilde{\alpha}_i)$ for $i = 1, 2$, hence $\mathrm{ind}(\tilde{\alpha}_1) = \mathrm{ind}(\tilde{\alpha}_2)$.

As usual, we write $\mathcal{B}_n = \mathcal{A}_{\{2n,2n+1\}}$ and $\mathcal{C}_n = \mathcal{A}_{\{2n-1,2n\}}$. Then by their construction, $\tilde{\alpha}_i$ for $i = 1, 2$ both satisfy the locality properties $\tilde{\alpha}_i(\mathcal{B}_n) \subseteq \mathcal{C}_n \otimes \mathcal{C}_{n+1}$ for $n = 0, 1, 2$, as well as $\tilde{\alpha}_i^{-1}(\mathcal{C}_n) \subseteq \mathcal{B}_{n-1} \otimes \mathcal{B}_n$ for $n = 1, 2$. Then by Theorem 4.1 and subsequent Remark 4.2, for each $i = 0, 1$ and $n = 1, 2$ we can define

$$\mathcal{L}_n^{(i)} = \tilde{\alpha}_i(\mathcal{B}_n) \cap \mathcal{C}_n,$$
$$\mathcal{R}_{n-1}^{(i)} = \tilde{\alpha}_i(\mathcal{B}_{n-1}) \cap \mathcal{C}_n$$

such that

$$\mathcal{C}_n = \mathcal{L}_n^{(i)} \otimes \mathcal{R}_{n-1}^{(i)} \tag{5.31}$$

and, for $n = 1$,

$$\mathcal{B}_n = \tilde{\alpha}_i^{-1}(\mathcal{L}_n^{(i)}) \otimes \tilde{\alpha}_i^{-1}(\mathcal{R}_n^{(i)}). \tag{5.32}$$

Following the structure theory of QCAs in Theorem 4.1, one can for each $i = 1, 2$ find a QCA β_i of radius 2 such that $\beta_i|_{\mathcal{A}_{\{0,\ldots,5\}}} = \tilde{\alpha}_i|_{\mathcal{A}_{\{0,\ldots,5\}}}$.[7] By the latter condition, Theorem 5.8, (ii) implies (if we have sufficiently coarse-grained in our initial step) that $\operatorname{ind}(\tilde{\alpha}_i) = \operatorname{ind}(\beta_i)$. Recalling that $\operatorname{ind}(\tilde{\alpha}_1) = \operatorname{ind}(\tilde{\alpha}_2)$, we then have $\operatorname{ind}(\beta_1) = \operatorname{ind}(\beta_2)$, so from Eq. (4.7) we conclude that $\mathcal{R}_0^{(1)}$ and $\mathcal{R}_0^{(2)}$ have the same dimension and hence are isomorphic finite-dimensional subalgebras of $\mathcal{C}_1$. Hence there exists a unitary $u \in \mathcal{C}_1$ such that $u\mathcal{R}_0^{(1)}u^* = \mathcal{R}_0^{(2)}$.

Now we are in position to define the blended ALPU β. Let $\beta(x) = u\tilde{\alpha}_1(x)u^*$ for $x \in \mathcal{A}_{\leq 1}$, and let $\beta|_{\mathcal{A}_{\geq 2}} = \tilde{\alpha}_2|_{\mathcal{A}_{\geq 2}}$. Then $\beta(\mathcal{A}_{\leq 1}) = (\mathcal{A}_{\leq 0} \cup \mathcal{R}_0^{(2)})''$ and $\beta(\mathcal{A}_{\geq 2}) = (\mathcal{L}_1^{(2)} \cup \mathcal{A}_{\geq 3})''$ commute by construction, so β is a well-defined injective unital $*$-homomorphism. Moreover $\mathcal{C}_1 = \mathcal{L}_1^{(2)} \otimes \mathcal{R}_0^{(2)}$ from (5.31), so β is surjective, hence a well-defined ALPU. By construction of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ using Proposition 5.4, for $r \to \infty$,

$$\begin{aligned} \|(\beta - \alpha_1)|_{\mathcal{A}_{\leq -r-1}}\| &= \mathcal{O}(f(r-1)), \\ \|(\beta - \alpha_2)|_{\mathcal{A}_{\geq r+6}}\| &= \mathcal{O}(f(r-7)). \end{aligned} \tag{5.33}$$

Above we assumed both α_1 and α_2 were defined on the same $\mathcal{A}_{\mathbb{Z}}$ (i.e., that both chains use algebras $\mathcal{A}_n$ of the same dimensions). If α_1 and α_2 have different local dimensions, then in the region where we blend them above, we can first pad them with extra tensor factors so that they have identical local dimensions within that region. ■

The following theorem extends all properties in Theorem 4.4 for QCAs to ALPUs, replacing the role of circuits by Hamiltonian evolutions, and allowing strongly continuous paths through the space of ALPUs with uniform tail bounds.

Theorem 5.15 (Properties of index for ALPUs)**.** *Suppose α and β are ALPUs in one dimension. Then:*

(i) $\operatorname{ind}(\alpha \otimes \beta) = \operatorname{ind}(\alpha) + \operatorname{ind}(\beta)$.

(ii) If α and β are defined on the same algebra, $\operatorname{ind}(\alpha\beta) = \operatorname{ind}(\alpha) + \operatorname{ind}(\beta)$*.*

(iii) The following are equivalent:

  (a) $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$.

  (b) α and β may be blended.

[7] Indeed, β_i can be constructed as follows. First we define $\beta_i|_{\mathcal{A}_{\{0,\ldots,5\}}} = \tilde{\alpha}_i|_{\mathcal{A}_{\{0,\ldots,5\}}}$ and then we define the action of β_i on the remainder of $\mathcal{A}_{\mathbb{Z}}$ as follows. We focus on defining β_i for $\mathcal{A}_{\leq -1}$; the definition for $\mathcal{A}_{\geq 6}$ is directly analogous. An easy argument shows that $\mathcal{L}_0^{(i)} := \tilde{\alpha}_i(\mathcal{B}_0) \cap \mathcal{C}_0$ is a factor and moreover Eq. (5.32) also holds for $n = 0$. Then Eq. (5.31) will also hold for $n = 0$ if we define $\mathcal{R}_{-1}^{(i)}$ alternatively as the complementary factor to $\mathcal{L}_0^{(i)} \subset \mathcal{C}_0$. For each $n \leq -1$, choose an arbitrary factorization $\mathcal{C}_n = \mathcal{L}_n^{(i)} \otimes \mathcal{R}_{n-1}^{(i)}$ with $\mathcal{L}_n^{(i)} \cong \mathcal{L}_1^{(i)}$ and $\mathcal{R}_{n-1}^{(i)} \cong \mathcal{R}_{-1}^{(i)} \cong \mathcal{R}_0^{(i)} \cong \mathcal{R}_1^{(i)}$ (using that, by assumption, all local dimensions are the same). For each $n \leq -1$, choose an arbitrary factorization $\mathcal{B}_n = \tilde{\mathcal{L}}_n^{(i)} \otimes \tilde{\mathcal{R}}_n^{(i)}$ into factors isomorphic to those used for $\mathcal{B}_1$ in Eq. (5.32) for $n = 1$. Then we have $\tilde{\mathcal{L}}_n^{(i)} \cong \mathcal{L}_n^{(i)}$ and $\tilde{\mathcal{R}}_n^{(i)} \cong \mathcal{R}_n^{(i)}$, so we can define β_i to act as $\beta_i(\tilde{\mathcal{L}}_n^{(i)}) = \mathcal{L}_n^{(i)}$ and $\beta_i(\tilde{\mathcal{R}}_n^{(i)}) = \mathcal{R}_n^{(i)}$ for $n \leq -1$. This completes the definition of β_i for $\mathcal{A}_{\leq -1}$.

(c) There exists an index-0 ALPU γ such that $\alpha = \beta\gamma$.

(d) There exists $g(r) = o(1)$ and a strongly continuous path $\alpha[t]$ through the space of ALPUs with $g(r)$-tails such that $\alpha[0] = \alpha$ and $\alpha[1] = \beta$.

In (d), if α and β have $f(r)$-tails, we may take $g(r) = \mathcal{O}(f(Cr))$ for a universal constant C. If they have $\mathcal{O}(e^{-ar})$-tails, we may take $g(r) = \mathcal{O}(e^{-ar})$. In general, the path in (d) may be implemented by composing α (or β) with a Hamiltonian evolution with time-dependent Hamiltonian $H(t)$, with interactions bounded as in Theorem 5.9 and Remark 5.12.

In (c), if α and β do not have the same local dimensions, the statement only holds after separately tensoring α and β with appropriate identity automorphisms, such that α and β then have the same local dimensions. The analogous modification is needed for (d).

Proof. If α and β are ALPUs with approximating sequences α_n and β_n as in Theorems 5.6 and 5.8, then $\alpha_n \otimes \beta_n$ and $\alpha_{2n}\beta_n$ approximate $\alpha \otimes \beta$ and $\alpha\beta$ respectively. Then (i) and (ii) follow from the corresponding property for QCAs (Theorem 4.4). For (iii) the equivalence $(a) \Leftrightarrow (b)$ is stated by Proposition 5.14. The equivalence $(a) \Leftrightarrow (c)$ follows from $\mathrm{ind}(\beta\alpha^{-1}) = \mathrm{ind}(\beta) - \mathrm{ind}(\alpha)$, using property (ii). The implication $(a) \Rightarrow (d)$ follows from Theorem 5.9 applied to $\beta\alpha^{-1}$. The comment about exponential tails follows from the remark after Theorem 5.9. Next we show $(d) \Rightarrow (a)$, i.e. that the index must remain constant along a strongly continuous path. Because all ALPUs in the path are assumed to have $g(r)$-tails for some fixed $g(r)$, by Theorem 5.8(ii) there exists a finite interval X such that any two ALPUs γ and γ' with $\|(\gamma - \gamma')|_{\mathcal{A}_X}\|$ sufficiently small must have $\mathrm{ind}(\gamma) = \mathrm{ind}(\gamma')$. By the strong continuity (5.29) of the path, the index must then be constant along the path. ■

In the terminology of [23], Theorem 5.15 shows that an (A)LPU is an LGU (locally generated unitary) if and only if it has index zero. We can also interpret Theorem 5.15 as a converse to the Lieb-Robinson bounds in one dimension. Again, recall that Lieb-Robinson bounds demonstrate that local Hamiltonian evolution exhibits an approximate causal cone, quantified by the bound. Conversely, we ask whether evolutions that satisfy Lieb-Robinson-type bounds (i.e. ALPUs) can be generated by some time-dependent Hamiltonian. We find the following converse, emphasized below.

Corollary 5.16 (Converse to Lieb-Robinson bounds)**.** *Suppose α is an ALPU in one dimension with $f(r)$-tails. If (and only if) $\mathrm{ind}(\alpha) = 0$, α can be implemented by a strongly continuous path $\alpha[t]$ generated by some time-dependent Hamiltonian $H(t)$, such that $\alpha[0] = \mathrm{id}$, $\alpha[1] = \alpha$, and $\alpha[t]$ has $g(r)$-tails for all t, for some $g(r) = o(1)$. If α has $f(r)$-tails, we may take $g(r) = \mathcal{O}(f(Cr))$ for a universal constant C. If it has $\mathcal{O}(e^{-ar})$-tails, we may take $g(r) = \mathcal{O}(e^{-ar})$. The Hamiltonian $H(t)$ can be taken to have interactions bounded as in Theorem 5.9 and Remark 5.12.*

More generally, every ALPU in one dimension is a composition of a shift and a Hamiltonian evolution as above.

Proof. The equivalence follows immediately from (iii) in Theorem 5.15. The final statement follows by letting σ be a shift with $\mathrm{ind}(\sigma) = \mathrm{ind}(\alpha)$, then $\mathrm{ind}(\alpha\sigma^{-1}) = 0$ by (ii) in Theorem 5.15 so there exists a Hamiltonian evolution γ such that $\gamma = \alpha\sigma^{-1}$ and hence $\alpha = \gamma\sigma$. ■

The use of time-dependent rather than time-independent Hamiltonians is necessary: [50] shows that there exist QCAs with index 0 that cannot be implemented by any time-independent local Hamiltonian.

5.4 Finite chains

We developed the above structure theory of ALPUs on the infinite one-dimensional lattice. The statements are easily be adapted to the case of a finite one-dimensional chain with non-periodic ("open") boundary conditions. The statements as well as the proofs essentially hold unchanged, but we make some clarifying remarks. In summary, the theorems only become nontrivial when the length $|\Gamma|$ of the chain is taken larger than some finite threshold, but this threshold depends only on the tails and local dimensions of the ALPU. Meanwhile, the index is always zero.

We work with the algebra $\mathcal{A}_\Gamma$, where Γ is now a finite interval $\Gamma \subset \mathbb{Z}$. By non-periodic boundary conditions, we mean that Γ is considered as an interval rather than a circle, i.e. Γ inherits the metric from $\mathbb{Z}$, and the sites at either end of the interval are not considered neighbors. We again consider ALPUs on $\mathcal{A}_\Gamma$ with $f(r)$ tails, where $f(r)$ is only meaningful for $r < |\Gamma|$. In our arguments, $\mathcal{A}_{\leq n}$ becomes the finite-dimensional algebra corresponding to all sites left of $n+1$, and so on.

With this modification, Lemma 5.1 holds as stated, and the proof is identical. Importantly, all unspecified constants appearing as $\mathcal{O}(\cdot)$ in e.g. (5.4) are independent of the chain length $|\Gamma|$.

We then arrive at Theorem 5.6 for finite one-dimensional lattices, describing QCA approximations to ALPUs. Given ALPU α with $f(r)$ tails, the theorem describes an increasing sequence of QCA approximations β_j of radius j. For finite Γ, we restrict attention to $j \leq |\Gamma|$, so that the notion of a QCA of radius j remains meaningful. Recall the QCA approximations β_j were only guaranteed to have the listed properties in Theorem 5.8 for $j > j_0$, with j_0 chosen such that $f(j_0)$ is smaller than some universal constant independent of $|\Gamma|$. Then we only need $|\Gamma| > j_0$ for Theorem 5.8 to yield nontrivial QCA approximations, and this threshold size is determined only by the tails $f(r)$. Finally, the assumption $f(r) = o(\frac{1}{r})$ used for the latter claims of Theorem 5.8 may expressed more explicitly as the assumption that $f(j_0)j_0$ is smaller than some constant depending only on the local dimensions d_n of $\mathcal{A}_\Gamma$. This assumption then increases the minimum length $|\Gamma|$ for the theorem to become nontrivial, but with the minimum depending only on the tails and local dimensions, rather the details of α.

While Theorem 5.8 holds as written for finite Γ, it also reduces to a special case: the index is always zero. Calculating the index as the entropy difference in (4.13), we see the entropies correspond to complementary regions of a pure state, yielding zero. In fact, the trivial index was inevitable. On the infinite lattice, ALPUs with nonzero index implement shifts, and these shifts have no analog on the finite interval with non-periodic boundary conditions.

We can therefore apply Theorem 5.9 about Hamiltonian evolutions to every ALPU on finite $\Gamma \subset \mathbb{Z}$. As above, the theorem becomes nontrivial lattices of a certain size, using the same threshold discussed above. We then obtain a local Hamiltonian evolution generating the ALPU, with locality as specified by Theorem 5.9.

While finite chains with non-periodic boundary conditions descend as a special case from the infinite lattice, the case of periodic boundary conditions (i.e. Γ inherits the metric of a circle) appears more difficult. Many of the tools we develop appear useful there, but the key Lemma 5.1 has no obvious analog. Therefore we cannot offer a rigorous index theory of ALPUs on finite chains with periodic boundary conditions. The question is nonetheless important, and perhaps crucial for a generalization to higher dimensions. We leave the question to future work.

6 Many-body physics applications

In this section, we discuss two specific ways in which our results answer natural questions about quantum many-body systems. In Section 6.1, we show there cannot be a local "momentum density" on the one-dimensional lattice and discuss examples. In Section 6.2 we discuss an application of the ALPU index theory to the classification of topological phases in many-body localized Floquet systems, though this application requires further analysis and poses an interesting question for future work.

6.1 Translations cannot be implemented by local Hamiltonians

In quantum many-body systems, local conserved quantities dramatically influence dynamics. For instance, under local Hamiltonian evolution, energy itself is a local conserved quantity, and after the system has locally equilibrated, the dynamics are often governed by energy diffusion. More generally, when a system admits more local conserved quantities in addition to energy, the near-equilibrium dynamics are often governed by the hydrodynamics of these quantities [51–53]. For translation-invariant systems, one expects momentum is also a local conserved quantity. For instance, in scalar quantum field theory, the i'th component of the total momentum operator may be expressed as $P^i = \int dx\, \pi(x)\partial_i\phi(x)$ which is manifestly local, with local momentum density $\pi(x)\partial_i\phi(x)$.

The long-wavelength, low-energy regime of a lattice system like a spin chain is often described by a field theory, and a local momentum density is well-defined under this approximation. However, we might also ask for a local momentum operator $P = \sum_x p_x$ on the spin chain that generates translations, yielding $U = e^{iP}$ as the one-site translation operator. If P were constructed with local terms p_x, and if P commuted with some translation-invariant Hamiltonian, this exactly conserved momentum density might play an important role in dynamics.

The existence of such a local P is precisely the question of whether the shift QCA can be generated by a local "Hamiltonian," referring now to P as a Hamiltonian. We show such a local Hamiltonian cannot exist. In particular, on the infinite one-dimensional chain, it is impossible to implement the translation operator by time evolution using any time-dependent Hamiltonian satisfying Lieb-Robinson bounds, if the Lieb-Robinson bounds lead to an ALPU with $o(1)$-tails. This follows immediately from Theorem 5.15(iii)(d). For instance, we have:

Corollary 6.1 (No-go for local momentum densities). *If a local Hamiltonian $P = \sum_{X\subseteq\mathbb{Z}} P_X$ on an infinite one-dimensional spin chain has decaying interactions such that for all $n \in \mathbb{Z}$,*

$$\sum_{\substack{X\subseteq\mathbb{Z} \\ s.t.\ n\in X}} \|P_X\| = \mathcal{O}\left(\mathrm{diam}(X)^{-(2+\varepsilon)}\right)$$

for some $\varepsilon > 0$, then e^{iP} cannot be the unitary lattice translation operator that translates by one site.

Recall our notation that e.g. P_X is a term local to region $X \subset \mathbb{Z}$ on the lattice $\mathbb{Z}$. The no-go result is also robust: Theorem 5.8 constrains how well e^{iP} can approximate the translation operator locally. (Note that while [6] already demonstrated that finite-depth circuits cannot achieve translations, their statements about circuits cannot be easily re-cast as claims about Hamiltonian evolution, at least not without further robustness results such as those developed here.)

Given that the translation operator cannot be generated by a finite-depth circuits, our analogous for claim for sufficiently local Hamiltonians might seem in intuitive. However the claim is not obvious, as demonstrated by the following example: if we allow evolution generated by Hamiltonians with $\frac{1}{r}$-decaying interaction terms (which then violate Lieb-Robinson bounds), we *can* implement a translation. The example involves a chain of qubits; we only sketch the construction but the details are easily verified. A Jordan-Wigner transformation maps the chain of qubits to a chain of fermions (or formally, it maps the quasi-local algebra to the CAR-algebra). Let $c_n^\dagger$ and c_n be the fermionic creation and annihilation operators at site $n \in \mathbb{Z}$. The Jordan-Wigner transform of the translation automorphism T is again the translation automorphism, $T(c_n) = c_{n-1}$. Taking a Fourier transform we see that

$$T(\hat{c}_k) = e^{ik}\hat{c}_k$$

Hence time evolution for time $t = 1$ using Hamiltonian

$$H = \int_{-\pi}^{\pi} \mathrm{d}k\, k\hat{c}_k^\dagger \hat{c}_k$$

implements T. In real space

$$H = \sum_{n,m\in\mathbb{Z}} h_{n-m} c_n^\dagger c_m$$

where the coefficients h_r (of which the precise form is not important) have magnitude $\frac{1}{r}$, . Of course, we can also take the inverse Jordan-Wigner transform of this Hamiltonian to obtain a Hamiltonian on the spin chain

$$\tilde{H} = \sum_{n,m} h_{n-m}\sigma_{n,m}$$

where $\sigma_{n,m}$ is a Pauli operator supported on sites $\min\{n, m\}, \ldots, \max\{n, m\}$. In this way we can construct a Hamiltonian *not* satisfying Lieb-Robinson bounds which does implement T. This shows that our demand that the ALPUs have $o(1)$-tails in our construction of the index is not arbitrary; the classification by index collapses once we allow evolutions such as those generated by $\tilde{H}$ above with $\frac{1}{r}$-decaying interactions. In fact, by Theorem 5.15 we conclude that $e^{-i\tilde{H}t}$ cannot have $o(1)$-tails.

For the case of a single-particle Hamiltonian (i.e. a quantum walk), the obstruction to generating the translation operator with a local Hamiltonian hinges on the non-trivial winding of the dispersion relation [6]. It has been observed that for quadratic fermion Hamiltonians, every such Hamiltonian that implements the translation operator will need to have a discontinuity in its dispersion relation (in our example at $k = \pm\pi$) and hence at least $\frac{1}{r}$-tails in real space [27, 50]. These single-particle and free fermion results do not permit obvious generalization to the broader many-body case; our results allow us to draw conclusions for all local many-body Hamiltonians satisfying Lieb-Robinson bounds.

6.2 Floquet phases

The GNVW index has been used to define a topological invariant that classifies phases of systems with *dynamical many-body localization* for Floquet systems in two dimensions [4]. The intuitive idea is that on the two-dimensional lattice, under certain localization assumptions, time evolution of a subsystem with boundary defines an associated evolution on the one-dimensional boundary.

The GNVW index of this boundary automorphism then captures whether the boundary has chiral transport, and relatedly whether the two-dimensional system has vortex-like behavior.

Below we sketch the setup described by [4], pointing to where our results could make the former rigorous. However, the first step of reducing the two-dimensional dynamics to a one-dimensional boundary dynamics in a rigorous fashion presents an interesting problem of its own, which we leave to future work.

We consider a (time-dependent) local Hamiltonian H on a two-dimensional lattice, and we let U be the unitary obtained by time evolution for some fixed time T. The system exhibits many-body localization (MBL) if U can be written as a product of commuting unitaries which are all approximately local, i.e. when there exists a complete set of approximately local integrals of motion. More precisely, one says U is MBL in the sense of [4] when it can be written

$$U = \prod_X u_X \tag{6.1}$$

where u_X is approximately supported in a set X of some bounded size and $[u_X, u_{X'}] = 0$ for all X, X'. What "approximately supported" means here depends on one's definition of many-body localization. A reasonable definition may be

$$\|u_X - E_{B(X,r)(u_x)}\| \leq Ce^{-\gamma}$$

for some positive C and γ, where $E_{B(X,r)}$ denotes the projection onto the algebras supported on $B(X,r)$ as defined in (2.2). To define the invariant we let D denote the upper half plane (or in fact any simply connected infinite subset of the lattice) and we let U_D denote time evolution for time T using only the terms in the Hamiltonian strictly supported inside D. Also, we use (6.1) to define

$$U'_D = \prod_{X \subseteq D} U_X.$$

Then we let $V = U_D^{-1} U'_D$, so that the map $a \mapsto VaV^{-1}$ approximately preserves the algebra supported on a thick boundary strip ∂D of D. More precisely, the Lieb-Robinson bounds and (6.1) together show that for an operator a on a single site in ∂D, $VaV^\dagger$ is approximately supported within ∂D.

In [4], one implicitly assumes that V (or some deformation thereof) actually defines an ALPU on ∂D. Given this assumption that MBL dynamics define some APLU on ∂D, one could then apply the index theory of ALPUs to obtain a rigorous classification of MBL Floquet evolutions in 2D. Rigorously justifying that assumption presents an interesting future direction.

7 Discussion

We have defined and studied the index for approximately locality-preserving unitaries (ALPUs) on spin chains. Various open questions remain:

(i) Our results are restricted to the infinitely extended chain, or an open finite chain as in Section 5.4. One could also investigate what happens with a finite *periodic* chain with an ε-nearest neighbor automorphism for small ε. It appears that our proof technique relies on the fact that the chain is infinite (or open), so probably a different strategy is needed for finite periodic chains.

(ii) An obvious question of interest is the generalization to higher dimensions. In that case there is no immediate index theory, but one could still hope that for any ALPU α there exists a sequence of QCAs α_j approximating α as in Theorem 5.6. Our constructions of approximating QCAs for an ALPU rely rather heavily on the structure theory (i.e., the GNVW index theory) of one-dimensional QCAs. Hence, it is not immediately clear how to generalize to higher dimensions. In fact, we have not even given a definition of what an ALPU is in higher dimensions, where some choices exist. For two-dimensional QCAs there is also a complete classification (in which any QCA is a composition of a circuit and a generalized shift). Potentially, this structure theory, as developed in [7, 8] can be used in a similar fashion to construct the α_j. This could involve proving stability results for the notion of a "visibly simple algebra" as introduced in [7]. However, in higher dimensions there is strong evidence for the existence of "nontrivial" QCAs (meaning that they cannot be written as a composition of a circuit and a shift) [5], so this would require a different approach. Perhaps more generic, topological arguments (e.g. using fixed point theorems) are possible. A direct physical application would be a rigorous understanding of the index discussed in Section 6.2.

(iii) Another direction to generalize in is to channels which preserve locality but which are not unitary (i.e. an automorphism), see [17] for definitions and a recent discussion. In other words, what happens if the dynamics is slightly noisy? Is the index robust under small amounts of noise? Perhaps the type of algebraic stability results we used can also be applied to prove that any locality preserving channel which is almost unitary can be approximated by a QCA.

(iv) There is also a notion of *fermionic* QCAs, with a corresponding GNVW index. It should be possible to use similar arguments to extend the index to fermionic ALPUs.

(v) Finally, the algebra stability results of Appendix B or the related results Lemma 2.7, Lemma B.3, and Lemma 3.4 could find application in different aspects of quantum information theory. An example of recent work using similar techniques for very different purposes is [26]. Other potential applications could include approximate error correction.

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A Commutator lemmas

In this appendix we bound commutators $[x, f(y)]$ in terms of commutators $[x, y]$, assuming that y is near the identity.

Lemma A.1 (Commutators with powers)**.** *Let $\mathcal{A}$ be a C^*-algebra and let $y \in \mathcal{A}$ be a normal element with $\|I - y\| \leq \varepsilon < 1$. Then, for any $s \in [-1, 1]$ and $x, y \in \mathcal{A}$, we have*

$$\|[x, y^s]\| \leq \frac{|s|}{(1-\varepsilon)^{1-s}} \|[x, y]\|.$$

For fractional powers, y^s is defined using the functional calculus, with branch cut on the negative imaginary axis (away from the spectrum because $\|y - I\| < 1$).

Proof. We assume that $s \notin \{0, 1\}$ since otherwise the claim holds trivially. Let $z = I - y$. The function $t \mapsto (1-t)^s$ is holomorphic on the open unit disk, so we may expand

$$y^s = (I - z)^s = \sum_{n=0}^{\infty} c_n z^n.$$

The exact form of the coefficients c_n here is unimportant, but note $\operatorname{sgn}(c_n) = -\operatorname{sgn}(s)$ for $n \geq 1$ by our assumption that $s \notin \{0, 1\}$.

$$\begin{aligned}
\|[x, y^s]\| = \|[x, (I-z)^s]\| &\leq \sum_{n=1}^{\infty} |c_n| \, \|[x, z^n]\| \leq -\operatorname{sgn}(s) \sum_{n=1}^{\infty} c_n \, n \, \|z\|^{n-1} \|[x, z]\| \\
&= -\operatorname{sgn}(s) \frac{d}{dw} (1-w)^s \Big|_{w = \|z\|} \|[x, y]\| = \frac{|s|}{(1 - \|z\|)^{1-s}} \|[x, y]\| \leq \frac{|s|}{(1-\varepsilon)^{1-s}} \|[x, y]\|
\end{aligned}$$

as desired. ∎

Lemma A.2 (Commutators with polar decompositions)**.** *Let $\mathcal{A}$ be a C^*-algebra and $y \in \mathcal{A}$ an element with $\|y - I\| \leq \varepsilon \leq \frac{1}{8}$. Let $y = u|y|$ be its polar decomposition, with $|y| = (y^* y)^{\frac{1}{2}}$. Then, for any $x \in \mathcal{A}$,*

$$\|[x, u]\| < 3\|[x, y]\| + 2\|[x, y^*]\|.$$

More generally for any $\varepsilon < \sqrt{2} - 1$, an estimate of the above form holds for some choice of constants on the right-hand side depending only on ε.

Proof. Note $\|y - I\| \leq \varepsilon < 1$ implies that y is invertible, hence the unitary u in the polar decomposition is uniquely given by $u = y|y|^{-1} = y(y^* y)^{-\frac{1}{2}}$. Moreover, we have $\|y\| \leq 1 + \varepsilon$ and $\|y^* y - I\| \leq (2 + \varepsilon)\varepsilon$, which also implies that $\|(y^* y)^{-\frac{1}{2}}\| \leq (1 - (2+\varepsilon)\varepsilon)^{-\frac{1}{2}}$, since $\varepsilon < \sqrt{2} - 1$. We obtain

$$\begin{aligned}
\|[x,u]\| &= \|[x, y(y^*y)^{-\frac{1}{2}}]\| \le \|y\|\|[x,(y^*y)^{-\frac{1}{2}}\| + \|(y^*y)^{-\frac{1}{2}}\|\|[x,y]\| \\
&\le (1+\varepsilon)\|[x,(y^*y)^{-\frac{1}{2}}\| + \frac{1}{(1-2\varepsilon-\varepsilon^2)^{\frac{1}{2}}}\|[x,y]\| \\
&\le \frac{1+\varepsilon}{2(1-2\varepsilon-\varepsilon^2)^{\frac{3}{2}}}\|[x,y^*y]\| + \frac{1}{(1-2\varepsilon-\varepsilon^2)^{\frac{1}{2}}}\|[x,y]\| \\
&\le \frac{(1+\varepsilon)^2}{2(1-2\varepsilon-\varepsilon^2)^{\frac{3}{2}}}\left(\|[x,y^*]\| + \|[x,y]\|\right) + \frac{1}{(1-2\varepsilon-\varepsilon^2)^{\frac{1}{2}}}\|[x,y]\| \\
&= \frac{(1+\varepsilon)^2 + 2(1-2\varepsilon-\varepsilon^2)}{2(1-2\varepsilon-\varepsilon^2)^{\frac{3}{2}}}\|[x,y]\| + \frac{(1+\varepsilon)^2}{2(1-2\varepsilon-\varepsilon^2)^{\frac{3}{2}}}\|[x,y^*]\|.
\end{aligned}$$

Here we use the above comments to bound the relevant norms, as well as Lemma A.1 for $s = -\frac{1}{2}$. Using $\varepsilon \le \frac{1}{8}$ this implies the desired bounds. ■

B Near inclusions of algebras

In this appendix, we prove Theorem 2.6 about near inclusions of von Neumann algebras. The result is an extension of Theorem 4.1 of Christensen [25], but we give a self-contained proof. We follow closely the exposition in [24, 25]. Note that in [25] it is assumed that injective von Neumann algebras have a property called D_1. However, whether this is true is unknown, see comments in [54]. We slightly adapt the arguments of [25] to avoid this issue.

We begin with Proposition B.1, which generalizes Proposition 4.2 of Christensen [24]. There Christensen considers two subalgebras $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ that are isomorphic via an isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$. Note that Φ is defined only on $\mathcal{A}$, not $B(\mathcal{H})$. Roughly speaking, the theorem says that if the isomorphism nearly fixes $\mathcal{A}$, it is inner and implemented by a unitary near the identity. Our Proposition B.1 below extends this result to the case of multiple commuting subalgebras $\mathcal{A}_i$. Our generalization will be useful for Lemma 2.7. We also extend Christensen's result with the following observation: for elements of $B(\mathcal{H})$ that nearly commute with $\mathcal{A}$ and $\mathcal{B}$, the distance these elements are moved by the inner automorphism is controlled by the size of their commutator with $\mathcal{A}$ and $\mathcal{B}$.

Proposition B.1 (Making homomorphisms inner)**.** *Consider C^*-algebras $\mathcal{A}_i, \mathcal{B}_i \subseteq B(\mathcal{H})$ for $i = 1, \dots, n$, such that each $\mathcal{A}_i''$ is hyperfinite and $[\mathcal{A}_i, \mathcal{A}_j] = [\mathcal{B}_i, \mathcal{B}_j] = 0$ for $i \neq j$. Consider unital $*$-homomorphisms $\Phi_i\colon \mathcal{A}_i \to \mathcal{B}_i$, with $\|\Phi_i(a_i) - a_i\| \le \gamma_i \|a_i\|$ for all $a_i \in \mathcal{A}_i$ and $i = 1, \dots, n$. Denote $\mathcal{A} = (\cup_{i=1}^n \mathcal{A}_i)''$, $\mathcal{B} = (\cup_{i=1}^n \mathcal{B}_i)''$, and $\varepsilon = \sum_{i=1}^n \gamma_i$. If $\varepsilon < 1$, then there exists a unitary $u \in (\mathcal{A} \cup \mathcal{B})''$ such that $\Phi_i(a_i) = u^* a_i u$ for all i and $a_i \in \mathcal{A}_i$, with*

$$\|I - u\| \le \sqrt{2}\varepsilon(1 + (1-\varepsilon^2)^{\frac{1}{2}})^{-\frac{1}{2}} \le \sqrt{2}\varepsilon,$$

where we note that the expression in the middle is in fact $\varepsilon + \mathcal{O}(\varepsilon^2)$.

Moreover, for $\varepsilon \le \frac{1}{8}$, u can be chosen such that for any $z \in B(\mathcal{H})$, if $\|[z,c]\| \le \delta \|z\|\|c\|$ for all $c \in \mathcal{A} \cup \mathcal{B}$, then $\|u^* z u - z\| \le 10\,\delta \|z\|$.

Note $c \in \mathcal{A} \cup \mathcal{B}$ refers to the union of sets, i.e. $c \in \mathcal{A}$ or $c \in \mathcal{B}$. The proof extends the proof of Proposition 4.2 in [24].

Proof of Proposition B.1. We will define an element $y \in (\mathcal{A} \cup \mathcal{B})''$ whose polar decomposition yields the desired unitary u. We construct the element y to satisfy the properties $\|I - y\| \le \sum_{i=1}^{n} \gamma_i$ and $y\Phi_i(u_i) = u_i y$ for all $u_i \in U(\mathcal{A}_i)$ and $i = 1, \ldots, n$.[8]

By Proposition B.2 further below, each homomorphism $\Phi_i \colon \mathcal{A}_i \to \mathcal{B}_i$ can be extended to a $*$-isomorphism $\Phi_i' \colon \mathcal{A}_i'' \to \Phi_i(\mathcal{A}_i)'' \subseteq \mathcal{B}_i''$. Moreover, we obtain $\|\Phi_i'(a_i) - a_i\| \le \gamma_i \|a_i\|$. Without loss of generality, we may assume $\mathcal{A}_i$ is a hyperfinite von Neumann algebra and Φ_i is a weak-$*$ continuous unital homomorphism (this can always be achieved by replacing $\mathcal{A}_i$ by $\mathcal{A}_i''$, $\mathcal{B}_i$ by $\mathcal{B}_i''$, Φ_i by Φ_i'; the latter is weak-$*$ continuous because it is a $*$-isomorphism of von Neumann algebras).

Consider $B(\mathcal{H} \oplus \mathcal{H})$ with pairwise commuting subalgebras

$$\mathcal{C}_i = \left\{ \begin{pmatrix} a_i & 0 \\ 0 & \Phi_i(a_i) \end{pmatrix} : a_i \in \mathcal{A}_i \right\} \subseteq B(\mathcal{H} \oplus \mathcal{H}).$$

Since Φ_i is weak-$*$ continuous, the map $a_i \mapsto a_i \oplus \Phi_i(a_i)$ is a weak-$*$-continuous unital $*$-homomorphism. Therefore, $\mathcal{C}_i$, which is its image, is a von Neumann algebra is isomorphic to $\mathcal{A}_i$ and hence hyperfinite.

Therefore, by Theorem 2.3 $\mathcal{C}_i$ has property P and for

$$x_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$$

there exists an element $x_1 \in \mathcal{C}_i'$ that is also in the weak operator closure of the convex hull of $\{c_1^* x_0 c_1 \;:\; c_1 \in U(\mathcal{C}_1)\}$. Note that unitaries $c_1 \in U(\mathcal{C}_1)$ are of the form

$$\begin{pmatrix} u_1 & 0 \\ 0 & \Phi_1(u_1) \end{pmatrix}$$

for $u_1 \in U(\mathcal{A}_1)$, so elements $c_1^* x_0 c_1$ are of the form

$$\begin{pmatrix} u_1^* & 0 \\ 0 & \Phi_1(u_1^*) \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & \Phi_1(u_1) \end{pmatrix} = \begin{pmatrix} 0 & u_1^* \Phi_1(u_1) \\ 0 & 0 \end{pmatrix}.$$

Hence x_1 is of the form

$$x_1 = \begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix} \tag{B.1}$$

for some $y_1 \in (\mathcal{A}_1 \cup \mathcal{B}_1)''$. By direct calculation, $x_1 \in \mathcal{C}_1'$ implies $y_1 \Phi(u_1) = u_1 y_1$ for any unitary $u_1 \in \mathcal{A}_1$, and hence

$$y_1 \Phi_1(a_1) = a_1 y_1$$

for any $a_1 \in \mathcal{A}_1$.

If $n = 1$, we take $y_1 = y$. Otherwise, we repeat the above construction but with x_1 taking the place of x_0, and applying property P of $\mathcal{C}_2$. We obtain $x_2 \in \mathcal{C}_2'$ and associated y_2, with

[8] In finite dimension, one can define the element y using $y = \int_{U(\mathcal{A}_1)} \mathrm{d}u_1 \cdots \int_{U(\mathcal{A}_n)} \mathrm{d}u_n \; u_n^* \cdots u_1^* \Phi_1(u_1) \cdots \Phi_n(u_n)$, using the Haar measure of the unitary groups $U(\mathcal{A}_i)$. This y is easily seen to satisfy the abovementioned properties.

$y_2\Phi_2(a_2) = a_2y_2$ for all $a_2 \in \mathcal{A}_2$. Also note $x_2 \in \mathcal{C}_1'$, so $y_2\Phi_1(a_1) = a_1y_2$ for all $a_1 \in U(\mathcal{A}_1)$. We continue in this way, until we obtain $y := y_n$, with the property

$$y\Phi_i(a_i) = a_iy \tag{B.2}$$

for all $a_i \in \mathcal{A}_i$ and $i = 1, \dots, n$.

By construction, y_1 is in the weak operator closure of the convex hull of $\{u_1^*\Phi_1(u_1) : u_1 \in U(\mathcal{A}_1)\}$, and likewise y_2 is in the weak operator closure of the convex hull of $\{u_2^*y_1\Phi_2(u_2) : u_2 \in U(\mathcal{A}_2)\}$, and so on. Then y is in the weak operator closure of the convex hull of

$$S := \{u_n^* \dots u_1^*\Phi_1(u_1) \dots \Phi_n(u_n) : u_1 \in U(\mathcal{A}_1), \dots, u_n \in U(\mathcal{A}_n)\}. \tag{B.3}$$

Elements of this form are near the identity,

$$\begin{aligned}\|I - u_n^* \dots u_1^*\Phi_1(u_1) \dots \Phi_n(u_n)\| &\leq \|I - u_n^* \dots u_2^*\Phi_2(u_2) \dots \Phi_n(u_n)\| + \|\Phi_1(u_1) - u_1\| \\ &\leq \sum_{i=1}^{n} \|\Phi_i(u_i) - u_i\|,\end{aligned}$$

and thus, by convexity and lower semicontinuity of the norm in the weak operator topology,

$$\|I - y\| \leq \sum_{i=1}^{n} \gamma_i = \varepsilon.$$

Define $u = y|y|^{-1}$ as the unitary in the polar decomposition of y. By the above estimate, it generally follows (Lemma 2.7 of [55]) that

$$\|u - I\| \leq \sqrt{2}\varepsilon(1 + (1 - \varepsilon^2)^{\frac{1}{2}})^{-\frac{1}{2}} \leq \sqrt{2}\varepsilon.$$

We now show

$$u^*a_iu = \Phi_i(a_i)$$

for all $a_i \in \mathcal{A}_i$ and $i = 1, \dots, n$. To see this, first note that (B.2) implies $y^*y = \Phi_i(u_i)^*y^*y\Phi(u_i)$ for any $u_i \in U(\mathcal{A}_i)$, so that $[\Phi_i(u_i), y^*y] = 0$. Then, since any $a_i \in \mathcal{A}_i$ can be written as a linear combination of unitary elements, $[\Phi_i(a_i), y^*y] = 0$, hence $[\Phi_i(a_i), |y|^{-1}] = 0$ and

$$u^*a_iu = |y|^{-1}y^*a_iy|y|^{-1} = |y|^{-1}y^*y\Phi_i(a_i)|y|^{-1} = |y|^{-1}y^*y|y|^{-1}\Phi_i(a_i) = \Phi(a_i)$$

where we first used (B.2) and then that $[\Phi_i(a_i), |y|^{-1}] = 0$.

Finally, we show the last claim of the theorem. Consider any $z \in B(\mathcal{H})$ with the property that $\|[z, c]\| \leq \delta\|z\|\|c\|$ for all $c \in \mathcal{A} \cup \mathcal{B}$. Then, $\|[z, s]\| \leq 2\delta\|z\|$ for any $s \in S$, since any element of S is a product of a unitary in $U(\mathcal{A})$ and a unitary in $U(\mathcal{B})$, and likewise $\|[z, s^*]\| \leq 2\delta\|z\|$. We find that

$$\|[z, y]\| \leq 2\delta\|z\|,$$

using that y is in the weak operator closure of the convex hull of S as defined in (B.3). To see this, let y_i be a net of elements in the convex hull of S that converges to y in the weak operator topology. Since the elements in S have norm at most one, by convexity it holds that $\|y_i\| \leq 1$ as well and hence $\|[z, y_i]\| \leq 2\delta\|z\|$. The norm is lower semicontinuous in the weak operator topology which implies that $\|[z,y]\| \leq \liminf_i \|[z,y_i]\| \leq 2\delta\|z\|$. The above reasoning holds for y^* as well. Then we can apply Lemma A.2, using that $\|I - y\| \leq \varepsilon \leq \frac{1}{8}$. We find

$$\|u^* z u - z\| = \|[z,u]\| \leq 3\|[z,y]\| + 2\|[z,y^*]\| \leq 10\,\delta\|z\|$$

as desired. ∎

The above proof is completed by the technical proposition below. The proof follows from the proof of Theorem 5.4 in [24].

Proposition B.2. *Given a C^*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ with unital $*$-homomorphism $\Phi\colon \mathcal{A} \to B(\mathcal{H})$ and $\|\Phi(a) - a\| \leq \varepsilon\|a\|$ for all $a \in \mathcal{A}$ and some $\varepsilon < 1$, then Φ can be extended to a $*$-isomorphism $\Phi'\colon \mathcal{A}'' \to \Phi(\mathcal{A})''$ with $\|\Phi(a) - a\| \leq \varepsilon\|a\|$ for all $a \in \mathcal{A}''$.*

Proof. To extend Φ, consider $B(\mathcal{H} \oplus \mathcal{H})$ with subalgebra

$$\mathcal{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \Phi(a) \end{pmatrix} : a \in \mathcal{A} \right\} \subseteq B(\mathcal{H} \oplus \mathcal{H}).$$

We first show that for any $a \in \mathcal{A}''$, there exists unique $b \in \Phi(A)''$ such that

$$c = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{C}''.$$

For $a \in \mathcal{A}''$, by Kaplansky's density theorem, there exists a net $\{a_i\}$ in $\mathcal{A}$ converging in the strong and hence in the weak operator topology to a, with $\|a_i\| \leq \|a\|$. Then $\|\Phi(a_i) - a_i\| \leq \varepsilon\|a\|$, and $\|\Phi(a_i)\| \leq (1+\varepsilon)\|a\|$, so we can define a net

$$c_i = \begin{pmatrix} a_i & 0 \\ 0 & \Phi(a_i) \end{pmatrix}$$

within a ball of finite radius in $B(\mathcal{H})$. Since such balls are compact in the weak operator topology, c_i must have a convergent subnet, which then converges to some

$$c = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{C}'',$$

as claimed. To see the uniqueness of b given a, suppose otherwise that there exist corresponding $b_1, b_2 \in \Phi(A)''$ with $(a, b_1), (a, b_2) \in \mathcal{C}''$, so that $z = b_1 - b_2 \in \Phi(\mathcal{A})''$ with

$$c = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in \mathcal{C}''.$$

By Kaplansky's density theorem, there exists a net $\{c_i\}$ in $\mathcal{C}$ converging strongly to c with $\|c_i\| \leq \|c\| = \|z\|$. Write $c_i = (a_i, \Phi(a_i))$ for $a_i \in \mathcal{A}$. Then $\|a_i\| \leq \|z\|$, $\{a_i\}$ converges strongly to zero, and $\{\Phi(a_i)\}$ converges strongly to z, so $\Phi(a_i) - a_i$ converges strongly to z and hence also weakly. By the lower semicontinuity of the norm for the weak operator topology, $\|z\| \leq \liminf_i \|\Phi(a_i) - a_i\| \leq \varepsilon\|z\|$, so that $\|z\| = 0$ and $b_1 = b_2$, demonstrating uniqueness.

A similar argument shows that for any $b \in \Phi(\mathcal{A})''$, there exists unique $a \in \mathcal{A}''$ such that

$$c = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{C}''.$$

The above maps $a \mapsto b$ and $b \mapsto a$ define a bijection $\Phi' \colon \mathcal{A}'' \to \Phi(\mathcal{A})''$. The linearity, multiplicativity, and $*$-property of Φ' follow from the above uniqueness. Thus Φ' is a $*$-isomorphism. Finally, we show $\|\Phi'(a) - a\| \leq \|a\|\varepsilon$ for all $a \in \mathcal{A}''$. By Kaplansky's density theorem, there exists a net $\{a_i\}$ strongly converging to a for $a_i \in \mathcal{A}$ with $\|a_i\| \leq \|a\|$. By the above constructions, there exists a subnet such that $\Phi(a_i)$ converges in the weak operator topology to $\Phi'(a)$. Then, again by the lower semicontinuity of the norm, $\|\Phi'(a) - a\| \leq \liminf_i \|\Phi(a_i) - a_i\| \leq \varepsilon\|a\|$, as desired. ■

Now we turn to Theorem 2.6. In Theorem 4.1 of [25], Christensen proves that if a subalgebra $\mathcal{A}$ is approximately contained in another subalgebra $\mathcal{B}$ then there exists a unitary near the identity that rotates $\mathcal{A}$ into $\mathcal{B}$. Our Theorem 2.6 extends his result with the following observations. First, elements of $B(\mathcal{H})$ already close to both $\mathcal{A}$ and $\mathcal{B}$ are not moved much by the automorphism. Second, elements that nearly commute with both $\mathcal{A}$ and $\mathcal{B}$ are are not moved much either. Thus the automorphism "does no more than it needs."

For convenience, we recall the notion of near inclusions in Definition 2.2. We write $a \overset{\varepsilon}{\in} \mathcal{B}$ when there exists $b \in \mathcal{B}$ such that

$$\|a - b\| \leq \varepsilon\|a\|,$$

and we write $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ when $a \overset{\varepsilon}{\in} \mathcal{B}$ for all $a \in \mathcal{A}$. Also recall the notion of hyperfinite von Neumann algebras, reviewed in Section 2.2. Then we are equipped to state Theorem 2.6, repeated below.

Theorem 2.6 (Near inclusions of subalgebras)**.** *For hyperfinite von Neumann algebras $\mathcal{A}, \mathcal{B} \subseteq B(\mathcal{H})$ with $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$ for $\varepsilon \leq \frac{1}{64}$, there exists a unitary $u \in (\mathcal{A} \cup \mathcal{B})''$ such that $u^*\mathcal{A}u \subseteq \mathcal{B}$ and we have:*

(i) $\|I - u\| \leq 12\varepsilon$.

(ii) If $z \in B(\mathcal{H})$ satisfies $\|[z, c]\| \leq \delta\|z\|\|c\|$ for all $c \in \mathcal{A} \cup \mathcal{B}$, then $\|u^*zu - z\| \leq 10\delta\|z\|$.

Moreover, if $\mathcal{A}_0 \subseteq \mathcal{A}$ is an AF C^*-algebra such that $\mathcal{A}_0'' = \mathcal{A}$, then u can be chosen such that also:

(iii) If $z \in B(\mathcal{H})$ satisfies $z \overset{\delta}{\in} \mathcal{A}_0$ and $z \overset{\delta}{\in} \mathcal{B}$, then $\|u^*zu - z\| \leq 16\delta\|z\|$.

The first item re-states Theorem 4.1 of Christensen [25], or specifically part (b) of his Corollary 4.2 (noting that hyperfinite algebras are injective). The remaining items constitute our extension.

Now we proceed with the proof of Theorem 2.6, closely following and elaborating on some technical details and then extending the proof of Theorem 4.1 in [25].

Proof of Theorem 2.6. By Theorem 2.3, since $\mathcal{B}$ is hyperfinite, it is injective and hence there exists a conditional expectation

$$\mathbb{E}_{\mathcal{B}} \colon B(\mathcal{H}) \to \mathcal{B} \subseteq B(\mathcal{H}).$$

This map is completely positive and unital, and thus it has a Stinespring dilation [56]. That is, there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi\colon B(\mathcal{H}) \to B(\mathcal{K})$, and an isometry $v\colon \mathcal{H} \to \mathcal{K}$ such that

$$\mathbb{E}_{\mathcal{B}}(x) = v^* \pi(x) v \quad \forall x \in B(\mathcal{H}). \tag{B.4}$$

Let $p = vv^* \in B(\mathcal{K})$ be the projection onto the image of v. Then $p \in \pi(\mathcal{B})'$, since $\mathbb{E}_{\mathcal{B}}$ restricted to $\mathcal{B}$ is an isomorphism.[9][10]

Next we show that p nearly commutes with $\pi(\mathcal{A})$ as well. For any $a \in \mathcal{A}$, choose $b \in \mathcal{B}$ with $\|a - b\| \le \varepsilon \|a\|$, using $\mathcal{A} \overset{\varepsilon}{\subseteq} \mathcal{B}$. Then,

$$\|[\pi(a), p]\| = \frac{1}{2} \|[\pi(a-b), 2p - I]\| \le \|\pi(a-b)\| \, \|2p - I\| \le \varepsilon \|a\|, \tag{B.5}$$

noting that for any projection, $\|2p - I\| = 1$.

Note that although $\mathcal{A}$ itself is hyperfinite, $\pi(\mathcal{A})''$ is not immediately guaranteed hyperfinite, because π is not guaranteed weak-$*$ continuous. On the other hand, since $\mathcal{A}$ is hyperfinite, there exists an AF C^*-algebra $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{A}_0'' = \mathcal{A}$, as in the theorem statement. Then $\pi(\mathcal{A}_0)$ is also AF, and $\pi(\mathcal{A}_0)''$ is hyperfinite.

Because $\pi(\mathcal{A}_0)''$ hyperfinite, it satisfies property P, so there exists $\tilde{p}$ in the weak operator closure of the convex hull of $\{upu^* : u \in U(\pi(\mathcal{A}_0)'')\}$ such that $\tilde{p} \in \pi(\mathcal{A}_0)'$. By convexity of the norm and lower semicontinuity of the norm with respect to the weak operator topology,

$$\|\tilde{p} - p\| \le \sup_{u \in U(\pi(\mathcal{A}_0)'')} \|\tilde{p} - upu^*\| = \sup_{u \in U(\pi(\mathcal{A}_0)'')} \|[u, p]\| \le \varepsilon.$$

The final inequality follows from Eq. (B.5) in the following way. First note that $\|[u,p]\| \le \varepsilon$ for any $u \in \pi(U(\mathcal{A}))$. By the lower semicontinuity of the norm in the weak operator topology, this estimate extends directly to the weak operator closure of $\pi(U(\mathcal{A}))$. Accordingly, it suffices to show that any $u \in U(\pi(\mathcal{A}_0)'')$ is contained in the weak operator closure of $\pi(U(\mathcal{A}))$. This can be seen as follows. By Theorem 5.2.5 in [32] there exists self-adjoint $y \in \pi(\mathcal{A}_0)''$ such that $u = e^{iy}$. By a version of the Kaplansky density theorem, there exists a net $\{y_n\}$ of self-adjoint elements $y_n \in \pi(\mathcal{A}_0)$ converging strongly to y, with $\|y_n\| \le \|y\|$. Then $\{e^{iy_n}\}$ is a net of elements in $\pi(U(\mathcal{A}_0))$, since we can always write $y_n = \pi(x_n)$ with self-adjoint $x_n \in \mathcal{A}_0$, hence $e^{iy_n} = e^{i\pi(x_n)} = \pi(e^{ix_n})$ and e^{ix_n} is unitary. On the other hand, by Proposition 5.3.2 in [32], $\{e^{iy_n}\}$ converges strongly to $e^{iy} = u$. We conclude that u is in the strong (and hence in the weak) operator closure of $\pi(U(\mathcal{A}_0))$, hence in particular of $\pi(U(\mathcal{A}))$. Note that, by construction, $\|\tilde{p}\| \le \|p\| = 1$.

[9]In more detail, to see $p \in \pi(\mathcal{B})'$, first note $\pi(\mathcal{B}) \to B(\mathcal{K})$, $\pi(b) \mapsto p\pi(b)p$ is a $*$-homomorphism. Then note the following general fact: for any algebra $\mathcal{A} \subset B(\mathcal{H})$ and projection $p \in B(\mathcal{H})$, if the map $f(a) = pap$ is a $*$-homomorphism, then $p \in \mathcal{A}'$. To see this, note for any $a \in \mathcal{A}$, $f(a^*a) = f(a^*)f(a) = pa^*pap$ and $f(a^*a) = pa^*ap = pa^*(p + p^\perp)ap = pa^*pap + pa^*p^\perp ap$, so the difference yields $0 = pa^*p^\perp ap = (p^\perp ap)^*(p^\perp ap)$, so $p^\perp ap = 0$. The same is true for a^*, so $pap^\perp = 0$ also. Then $[p, a] = (p + p^\perp)[p, a](p + p^\perp) = pap^\perp - p^\perp ap = 0$.

[10]It may be helpful to understand the Stinespring dilation explicitly in finite dimensions where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{B} = I_A \otimes B(\mathcal{H}_B)$, with commutant $\mathcal{B}' = \mathcal{A} = B(\mathcal{H}_A) \otimes I_B$. Then the conditional expectation is the normalized partial trace $\mathbb{E}_{\mathcal{B}}(x) = \frac{1}{d_A} \operatorname{tr}_{\mathcal{A}}(x)$. For a minimal Stinespring dilation we can take the Hilbert space $\mathcal{K} = \mathcal{H}_A^1 \otimes H_A^2 \otimes \mathcal{H}_A^3 \otimes \mathcal{H}_B$, where the $\mathcal{H}_A^i$ are three copies of the Hilbert space $\mathcal{H}_A$. We define $\pi\colon B(\mathcal{H}) \to B(\mathcal{K})$ by identifying operators on $\mathcal{H}$ with operators on $\mathcal{H}_A^1 \otimes \mathcal{H}_B$. Finally, we take the isometry v as adding a maximally entangled state on $\mathcal{H}_A^1 \otimes \mathcal{H}_A^2$. Note that the projection p onto the image of v commutes with $\pi(\mathcal{B}) = I_{A^1A^2A^3} \otimes B(\mathcal{H}_B)$.

Next we would like to project $\tilde{p}$ onto $(\pi(B(\mathcal{H})) \cup \{p\})''$, the von Neumann algebra generated by $\pi(B(\mathcal{H}))$ and the projection p inside $B(\mathcal{K})$.[11] By Corollary 1.3.2 in [57], $(\pi(B(\mathcal{H})) \cup \{p\})'$ is isomorphic to $\mathcal{B}'$. Because the commutant of an injective von Neumann algebra is injective, and $\mathcal{B}$ is injective, $\mathcal{B}'$ is also injective, hence also $(\pi(B(\mathcal{H})) \cup \{p\})'$ and $(\pi(B(\mathcal{H})) \cup \{p\})''$. Thus we can use a conditional expectation to define

$$x = \mathbb{E}_{(\pi(B(\mathcal{H}))\cup\{p\})''}(\tilde{p}) \in (\pi(B(\mathcal{H})) \cup \{p\})'', \quad \|x - p\| \leq \varepsilon,$$

where the norm bound follows from $\|\tilde{p} - p\| \leq \varepsilon$, because the conditional expectation is a contraction. Moreover, it holds that $x \in \pi(\mathcal{A}_0)'$. To see this, compute $[x, z] = 0$ for $z \in \pi(\mathcal{A}_0)$, using that $\tilde{p} \in \pi(\mathcal{A}_0)'$ and $\pi(\mathcal{A}_0) \subseteq (\pi(B(\mathcal{H})) \cup \{p\})''$, and the general property of conditional expectations that $\mathbb{E}_{\mathcal{Z}}(z_1 y z_2) = z_1 \mathbb{E}_{\mathcal{Z}}(y) z_2$ for $z_1, z_2 \in \mathcal{Z}$.

The next steps follow Lemma 3.3 of [24]. Note that because x is self-adjoint, $\|x - p\| \leq \varepsilon$, and $\|x\| \leq 1$ (as conditional expectations are contractions), its spectrum is in $[-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1]$. Define the projection $q \in \pi(\mathcal{A}_0)'$ as the spectral projection of x corresponding to the part of the spectrum in $[1-\varepsilon, 1]$. Then, $\|q - x\| \leq \varepsilon$ and $\|q - p\| \leq 2\varepsilon$.

Using the projection $p \in \pi(\mathcal{B})'$ and the nearby projection $q \in \pi(\mathcal{A}_0)'$, define

$$y = qp + q^{\perp} p^{\perp},$$

where $p^{\perp} = (I - p)$ denotes the projection onto the orthogonal complement. Then

$$\|y - I\| = \|(2q - I)(p - q)\| \leq \|p - q\| \leq 2\varepsilon. \tag{B.6}$$

In particular, y is invertible. Now consider the unitary $w = y|y|^{-1}$ from the polar decomposition $y = w|y|$. Because y is near the identity, w must be as well. Namely, by Lemma 2.7 of [55], we find

$$\|w - I\| \leq 2\sqrt{2}\,\varepsilon. \tag{B.7}$$

Since $y^* y = pqp + p^{\perp} q^{\perp} p^{\perp}$, we have $[p, y^* y] = 0$ and hence $[p, |y|^{-1}] = 0$. Moreover, $yp = qy$, so

$$wpw^* = y|y|^{-1} p |y|^{-1} y^* = yp|y|^{-1}|y|^{-1} y^* = qy|y|^{-1}|y|^{-1} y^* = q. \tag{B.8}$$

With the unitary $w \in (p \cup \pi(B(\mathcal{H})))''$, we can finally define the homomorphism to which we soon apply Proposition B.1. Let

$$\Phi \colon \mathcal{A}_0 \to \mathcal{B} \subseteq B(\mathcal{H}), \quad \Phi(a) = v^* w^* \pi(a) w v.$$

This is a unital $*$-homomorphism, since it clearly preserves the $*$-operation and we have for all $a_1, a_2 \in \mathcal{A}_0$ that

$$\begin{aligned} \Phi(a_1)\Phi(a_2) &= v^* w^* \pi(a_1) w p w^* \pi(a_2) w v = v^* w^* \pi(a_1) q \pi(a_2) w v \\ &= v^* w^* \pi(a_1 a_2) q w v = v^* w^* \pi(a_1 a_2) w p v = \Phi(a_1 a_2), \end{aligned}$$

[11] In the finite-dimensional setting of Footnote 10, where again $\mathcal{K} = \mathcal{H}_A^1 \otimes \mathcal{H}_A^2 \otimes \mathcal{H}_A^3 \otimes \mathcal{H}_B$, we have $(\pi(B(\mathcal{H})) \cup \{p\})' = B(\mathcal{H}_A^3)$ and hence $(\pi(B(\mathcal{H})) \cup \{p\})'' = B(\mathcal{H}_A^1 \otimes \mathcal{H}_A^2 \otimes \mathcal{H}_B)$.

using (B.8), $q \in \pi(\mathcal{A}_0)'$, $p = vv^*$, and that v is an isometry. To see that its image lies in $\mathcal{B}$, note that $w^*\pi(a)w \in (p \cup \pi(B(\mathcal{H})))''$ and recall the original construction of the Stinespring dilation in (B.4). Moreover, for any $a \in \mathcal{A}_0$, there exists $b \in \mathcal{B}$ with $\|b - a\| \leq \varepsilon\|a\|$, so that

$$\begin{aligned}\|\Phi(a) - a\| &\leq \|\Phi(a) - b\| + \|b - a\| \\ &= \|v^*(w^*\pi(a)w - \pi(b))v\| + \|b - a\| \\ &\leq \|w^*\pi(a)w - \pi(b)\| + \|b - a\| \\ &\leq \|w^*\pi(a)w - \pi(a)\| + 2\|b - a\| \\ &= \|[\pi(a), w]\| + 2\|b - a\| \\ &= \|[\pi(a), w - I]\| + 2\|b - a\| \\ &\leq 2\|w - I\|\|a\| + 2\|b - a\| \leq 8\varepsilon\|a\|.\end{aligned} \tag{B.9}$$

using $b = \mathbb{E}_{\mathcal{B}}(b) = v^*\pi(b)v$ in the second step and (B.7) in the last step.

We can thus apply Proposition B.1 (for $n = 1$) to obtain a unitary $u \in (\mathcal{A} \cup \mathcal{B})''$ with $\|u - I\| \leq \sqrt{2} \cdot 8\varepsilon \leq 12\varepsilon$ such that $u^*au = \Phi(a) \in \mathcal{B}$ for all $a \in \mathcal{A}_0$, and we extend $\Phi\colon \mathcal{A} \to \mathcal{B}$ by $\Phi(a) = u^*au$ for $a \in \mathcal{A} = \mathcal{A}_0''$. Moreover, by Proposition B.1, we are already ensured the desired property of Theorem 2.6 that if $z \in B(\mathcal{H})$ satisfies $\|[z, c]\| \leq \delta\|z\|\|c\|$ for all $c \in \mathcal{A} \cup \mathcal{B}$, then $\|uzu^* - z\| \leq 10\delta\|z\|$.

Finally, we need to show the additional property that for any $z \in B(\mathcal{H})$ with $z \overset{\delta}{\in} \mathcal{A}_0$ and $z \overset{\delta}{\in} \mathcal{B}$, we have $\|u^*zu - z\| \leq 16\delta\|z\|$. First take $a \in \mathcal{A}_0$ with $\|z - a\| \leq \delta\|z\|$. Then,

$$\|u^*zu - z\| = \|u^*(z - a)u - (z - a) + u^*au - a\| \leq 2\delta\|z\| + \|\Phi(a) - a\|.$$

Now take $b \in \mathcal{B}$ with $\|z - b\| \leq \delta\|z\|$, hence also $\|a - b\| \leq 2\delta\|z\|$. Then we can bound just like in (B.9) to obtain (note that $a \in \mathcal{A}_0$)

$$\begin{aligned}\|\Phi(a) - a\| &\leq \|[\pi(a), w]\| + 2\|a - b\| \\ &\leq \|[\pi(a), w]\| + 4\delta\|z\| \\ &\leq 3\|[\pi(a), y]\| + 2\|[\pi(a), y^*]\| + 4\delta\|z\|,\end{aligned}$$

where we used Lemma A.2 in the last line, noting that $\|y - I\| \leq 2\varepsilon \leq \frac{1}{8}$ by Eq. (B.6) and our assumption on ε. To bound the commutators' norms, recall that $p \in \pi(\mathcal{B})'$ and $q \in \pi(\mathcal{A}_0)'$. Hence,

$$\begin{aligned}\|[\pi(a), y]\| &= \|[\pi(a), qp + q^\perp p^\perp]\| = \|(2q - I)[\pi(a), p]\| \\ &\leq \|[\pi(a), p]\| = \|[\pi(a - b), p]\| = \frac{1}{2}\|[\pi(a - b), 2p - I]\| \leq \|a - b\| \leq 2\delta\|z\|,\end{aligned}$$

and likewise for $[\pi(b), y^*]$. Therefore, $\|\Phi(a) - a\| \leq 14\delta\|z\|$, and hence

$$\|u^*zu - z\| \leq 16\delta\|z\|$$

as desired. ■

As another application of Proposition B.1, Lemma 2.7 controls the distance between homomorphisms using the distance between their local restrictions. We repeat the statement for convenience.

Lemma 2.7. *Consider two injective weak-$*$ continuous unital $*$-homomorphisms $\alpha_1, \alpha_2 \colon \mathcal{A} \to \mathcal{B}$ between von Neumann algebras $\mathcal{A} \subseteq B(\mathcal{H})$ and $\mathcal{B} \subseteq B(\mathcal{K})$, with hyperfinite von Neumann subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ that pairwise commute, i.e., $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$, and generate $\mathcal{A}$ in the sense that $(\cup_{i=1}^n \mathcal{A}_i)'' = \mathcal{A}$. Define*

$$\varepsilon = \sum_{i=1}^{n} \|(\alpha_1 - \alpha_2)|_{\mathcal{A}_i}\|.$$

Then if $\varepsilon < 1$,

$$\|\alpha_1 - \alpha_2\| \leq 2\sqrt{2}\varepsilon \left(1 + (1 - \varepsilon^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \leq 2\sqrt{2}\varepsilon,$$

where we note that the expression in the middle is $2\varepsilon + \mathcal{O}(\varepsilon^2)$.

Proof. Since we assume the α_i to be weak-$*$ continuous, $\alpha_1(\mathcal{A}_i)$ and $\alpha_2(\mathcal{A}_i)$ are von Neumann algebras which are isomorphic to $\mathcal{A}_i$ (and in particular are hyperfinite). Define $*$-isomorphisms Φ_i between $\alpha_1(\mathcal{A}_i)$ and $\alpha_2(\mathcal{A}_i)$, given by $\alpha_1(a_i) \mapsto \alpha_2(a_i)$ for $a_i \in \mathcal{A}_i$. Then we apply Proposition B.1 (with $\alpha_1(\mathcal{A}_i)$ as $\mathcal{A}_i$, $\alpha_2(\mathcal{A}_i)$ as $\mathcal{B}_i$, $\gamma_i = \|(\alpha_1 - \alpha_2)|_{\mathcal{A}_i}\|$) to find a unitary $u \in \mathcal{B}$ such that $\alpha_2(a_i) = u^*\alpha_1(a_i)u$ for all $a_i \in \mathcal{A}_i$ for $i = 1, \ldots n$ with

$$\|I - u\| \leq \sqrt{2}\varepsilon \left(1 + (1 - \varepsilon^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

This implies that $\alpha_2(a) = u^*\alpha_1(a)u$ for all $a \in \mathcal{A}$, and hence

$$\|\alpha_1 - \alpha_2\| \leq 2\sqrt{2}\varepsilon \left(1 + (1 - \varepsilon^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

∎

Finally, we mention another result about simultaneous near inclusions. If several mutually commuting subalgebras $\mathcal{A}_i$ each nearly include into $\mathcal{B}$, then so does the algebra they generate. We use this lemma to prove Lemma 3.4 which shows that Lieb-Robinson type bounds for single site operators imply Lieb-Robinson bounds for operators supported on arbitrary sets.

Lemma B.3 (Simultaneous near inclusions)**.** *Let $\mathcal{A}_i \subseteq B(\mathcal{H})$ for $i = 1, ..., n$ and $\mathcal{B} \subseteq B(\mathcal{H})$ be von Neumann algebras, where the $\mathcal{A}_i$ are hyperfinite and $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$. If $\mathcal{A}_i \overset{\varepsilon_i}{\subseteq} \mathcal{B}$ for each i, then for $\varepsilon := \sum_{i=1}^n \varepsilon_i$ we have*

$$\mathcal{B}' \overset{2\varepsilon}{\subseteq} (\cup_i \mathcal{A}_i)'. \tag{B.10}$$

If additionally $\mathcal{A}_i \subseteq \mathcal{M}$ for some von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ for $i = 1, \ldots, n$, then

$$\mathcal{B}' \cap \mathcal{M} \overset{2\varepsilon}{\subseteq} (\cup_i \mathcal{A}_i)' \cap \mathcal{M}. \tag{B.11}$$

Finally, if $\mathcal{B}'$ is hyperfinite then

$$(\cup_i \mathcal{A}_i)'' \overset{4\varepsilon}{\subseteq} \mathcal{B}.$$

Proof. First we show $\mathcal{B}'$ nearly includes into $(\cup_i \mathcal{A}_i)'$. By hyperfiniteness (and therefore property P) of $\mathcal{A}_1$, for each $b_0' \in \mathcal{B}'$ there exists $b_1' \in \mathcal{A}_1'$ in the weak operator closure of the convex hull of $\{u_1^* b_0' u_1 \ : \ u_1 \in U(\mathcal{A}_1)\}$. Then by property P of $\mathcal{A}_2$, there exists $b_2' \in \mathcal{A}_2'$ in the weak operator closure of the convex hull of $\{u_2^* b_1' u_2 \ : \ u \in U(\mathcal{A}_2)\}$. Note that $b_2' \in \mathcal{A}_1'$ still, using $[\mathcal{A}_1, \mathcal{A}_2] = 0$. We continue in this way until we find b_n' in the weak operator closure of the convex hull of[12]

$$\{u_n^* \cdots u_1^* b_0' u_1 \cdots u_n \ : \ u_1 \in U(\mathcal{A}_1), \ldots, u_n \in U(\mathcal{A}_n)\}.$$

Note $\|[u_i, b_0']\| \leq 2\varepsilon_i \|b_0'\|$, by $\mathcal{A}_i \overset{\varepsilon_i}{\subseteq} \mathcal{B}$ and Lemma 2.4. Thus, elements in the above set are near b_0', since by a telescoping sum

$$\begin{aligned}
\|b_0' - u_n^* \cdots u_1^* b_0' u_1 \cdots u_n\| &\leq \sum_{i=1}^{n} \|u_n^* \cdots u_{i+1}^* b_0' u_{i+1} \cdots u_n - u_n^* \cdots u_i^* b_0' u_i \cdots u_n\| \\
&= \sum_{i=1}^{n} \|b_0' - u_i^* b_0' u_i\| = \sum_{i=1}^{n} \|[u_i, b_0']\| \leq 2\varepsilon \|b_0'\|
\end{aligned}$$

and hence, using the convexity of the norm and its lower semicontinuity with respect to the weak operator topology,

$$\|b_0' - b_n'\| \leq 2\varepsilon \|b_0'\|.$$

By construction, $b_n' \in \mathcal{A}_i'$ for each i, so $b_n' \in (\cup_i \mathcal{A}_i)'$. The above construction held for any $b_0' \in \mathcal{B}'$, so Eq. (B.10) follows. Note that if we assume that each $\mathcal{A}_i \subseteq \mathcal{M}$ and we take $b_0' \in \mathcal{B}' \cap \mathcal{M}$, then also $b_n' \in \mathcal{M}$, which shows Eq. (B.11). Finally, by Lemma 2.5 and the assumption that $\mathcal{B}'$ is hyperfinite we conclude that $(\cup_i \mathcal{A}_i)'' \overset{4\varepsilon}{\subseteq} \mathcal{B}$. ■

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