# LOW REGULARITY SOLUTIONS OF TWO-DIMENSIONAL COMPRESSIBLE EULER EQUATIONS WITH DYNAMIC VORTICITY 

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#### Abstract

By establishing a sharp Strichartz estimate for the velocity and density, we prove the local well-posedness of solutions for the Cauchy problem of two-dimensional compressible Euler equations, where the initial velocity, density, and specific vorticity $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in H^{s}\left(\mathbb{R}^{2}\right) \times H^{s}\left(\mathbb{R}^{2}\right) \times H^{2}\left(\mathbb{R}^{2}\right), s>\frac{7}{4}$. Our strategy relies on Smith-Tataru's work [41] for quasi-linear wave equations.


## 1. Introduction

1.1. Overview. We consider the Cauchy's problem of the compressible Euler equations in $\in \mathbb{R}^{+} \times \mathbb{R}^{2}$, of the form

$$
\left\{\begin{array}{l}
\varrho_{t}+\operatorname{div}(\varrho \mathbf{v})=0  \tag{1.1}\\
\mathbf{v}_{t}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\frac{1}{\varrho} \nabla p(\varrho)=0
\end{array}\right.
$$

where the state function takes the general form

$$
\begin{equation*}
p=p(\varrho) \tag{1.2}
\end{equation*}
$$

and the initial data is

$$
\begin{equation*}
\left.(\mathbf{v}, \varrho)\right|_{t=0}=\left(\mathbf{v}_{0}, \varrho_{0}\right) \tag{1.3}
\end{equation*}
$$

Above, $\mathbf{v}=\left(v^{1}, v^{2}\right), \varrho$, and $p$ denote the fluid velocity, density, and pressure respectively, and $A$ is a constant. In the theory of partial differential equations, local well-posedness is the first question to ask. For compressible Euler equations, no matter how smooth and small the initial data is, the solution of (1.1) will blow up in finite time $[10,32,39,43]$. So we can only study the well-posedness of (1.1)(1.3) in a local sense. In many problems of this type, one is interested not only in local well-posedness in some Sobolev space $H^{s}\left(\mathbb{R}^{2}\right)$, but also in lowering the exponent $s$ as much as possible. Naturally, we ask the question: for which $s_{c}$, the Cauchy problem (1.1)-(1.3) is well-posed if $\left(\mathbf{v}_{0}, \varrho_{0}\right) \in H^{s}\left(\mathbb{R}^{2}\right)\left(s>s_{c}\right)$ and illposed if $\left(\mathbf{v}_{0}, \varrho_{0}\right) \in H^{s}\left(\mathbb{R}^{2}\right)\left(s \leq s_{c}\right)$. This question has been well studied $[6,41,31]$ for incompressible Euler equations and irrotational Euler equations. However, for (1.1)-(1.3) with non-zero vorticity, the corresponding problem remains open. Our goal is to study the local well-posedness of low regularity solutions to (1.1)-(1.3) and explore the sharp Sobolev exponent.

[^0]1.2. Background. The compressible Euler equations is a classical system in physics to describe the motion of an idea fluid. The phenomena displayed in the interior of a fluid fall into two broad classes, the phenomena of acoustics waves and the phenomena of vortex motion. The sound phenomena depend on the compressibility of a fluid, while the vortex phenomena occur even in a regime where the fluid may be considered to be incompressible.

For the Cauchy problem of $n$-D incompressible Euler equations:

$$
\left\{\begin{array}{l}
\mathbf{v}_{t}+(\mathbf{v} \cdot \nabla) \mathbf{v}+\nabla p=0, \quad(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}  \tag{1.4}\\
\operatorname{div} \mathbf{v}=0 \\
\left.\mathbf{v}\right|_{t=0}=\mathbf{v}_{0}
\end{array}\right.
$$

Kato and Ponce [23] proved the local well-posedness of (1.4) if $\mathbf{v}_{0} \in H^{s}\left(\mathbb{R}^{n}\right), s>$ $1+\frac{n}{2}$. Chae [8] proved the local existence of solutions by setting $\mathbf{v}_{0}$ in the critical Triebel-Lizorkin space. On the opposite direction, the ill-posedness of solutions of (1.1) was answered by Bourgain and $\mathrm{Li}[6,7]$, who proved that the solution will blow up instantaneously for some $\mathbf{v}_{0} \in H^{1+\frac{n}{2}}\left(\mathbb{R}^{n}\right), n=2,3$. Very recently, Guo-Li in [16] studied the continuous dependence of initial data in the critical Triebel-Lizorkin space.

In the irrotational case, the compressible Euler equations can be reduced to a special quasilinear wave equation. For general quasilinear wave equations, it can be stated as

$$
\left\{\begin{array}{l}
\square_{h(\phi)} \phi=q(d \phi, d \phi), \quad(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}  \tag{1.5}\\
\left.\phi\right|_{t=0}=\phi_{0},\left.\partial_{t} \phi\right|_{t=0}=\phi_{1}
\end{array}\right.
$$

where $\phi$ is a scalar function, $d=\left(\partial_{t}, \partial_{1}, \partial_{2}, \cdots, \partial_{n}\right)$, and $h(\phi)$ a Lorentzian metric depending on $\phi$, and $q$ a quadratic term of $d \phi$. Set the initial data $\left(\phi_{0}, \phi_{1}\right) \in$ $H^{s}\left(\mathbb{R}^{n}\right) \times H^{s-1}\left(\mathbb{R}^{n}\right)$. By using classical energy methods and Sobolev imbeddings, Hughes-Kato-Marsden [18] proved the local well-posedness of the problem (1.5) for $s>\frac{n}{2}+1$. Ifrim-Tataru [20] studied this reslut for quasilinear hyperbolic equations by using the frequency envelope approach, where the frequency envelope is introduced by Tao [48]. On the other side, Lindblad [31] constructed some counterexamples for (1.5) when $s=\frac{7}{4}, n=2$ or $s=2, n=3$. There is a gap between the result [18] and [31]. To lower the regularity of the initial data, one may seek a type of space-time estimates of $d \phi$, namely, Strichartz estimates. Of course there are several steps to obtain the sharp Strichartz estimates for (1.5). The first natural idea is to consider the the wave equation with variable coefficients

$$
\begin{equation*}
\square_{h(t, x)} \phi=0 \tag{1.6}
\end{equation*}
$$

and then exploit it to obtain the low regularity solutions of (1.5). Kapitanskij [24] and Mockenhaupt-Seeger-Sogge [38] discussed the Strichartz estimates for (1.6) with smooth coefficients $h$. With rough coefficients $h \in C^{2}$, the study of Strichartz estimates for (1.6) in two or three dimensions was begin with Smith's result [43]. At the same time, counterexamples was constructed by Smith-Sogge [44], who showed that for $\alpha<2$ there exist $h \in C^{\alpha}$ for which the Strichartz estimates fail. Later, the Strichartz estimates were established in all dimensions for $h \in C^{2}$ in Tataru [50]. The next important work was independently achieved by Bahouri-Chemin [4] and Tataru [49], who established the local well-posedness of (1.5) with $s>\frac{n}{2}+\frac{7}{8}, n=2$ or $s>\frac{n}{2}+\frac{3}{4}, n \geq 3$. Shortly afterward, Tataru [51] relaxed the Sobolev indices
$s>\frac{n+1}{2}+\frac{1}{6}, n \geq 3$. At the same time, Smith-Tataru [42] showed that the $\frac{1}{6}$ loss is sharp for general variable coefficients $h$. Thus, to improve the above results, one needs to exploit a new way or structure of Equation (1.5). Through introducing a vector-field approach and a decomposition of the Ricci curvature, the 3D result of [4, 49, 50, 51] was later improved by Klainerman-Rodnianski [26], who proved the local well-posedness of (1.5) by introducing vector-field methods for $s>2+\frac{2-\sqrt{3}}{2}$. Based on the vector-field methods, Geba [21] studied the local well-posedness of (1.5) in two dimensions for $s>\frac{7}{4}+\frac{5-\sqrt{22}}{4}$. By using wave packets of a localization to represent solutions to the linear equation, a sharp result was proved by SmithTataru [41], who established the local well-posedness of (1.5) if $s>\frac{7}{4}, n=2$ or $s>2, n=3$ or $s>\frac{n+1}{2}, 4 \leq n \leq 6$. An alternative proof of the 3 D case was also obtained through vector-field approach by Wang [57]. Besides, we should also mention substantial significant progress which has been made on low regularity solutions of Einstein vacuum equations, membrane equations, due to Andersson and Moncreif [3], Tataru [52], Ettinger and Lindblad [6], Klainerman and Rodnianski [26], Klainerman-Rodnianski-Szefel [27], Wang [54], Allen-Andersson-Restuccia [2], Speck [45] and so on.

In the general case, concerning to $n$-D compressible Euler equations, there are several aspects on studying the Cauchy problem (1.1)-(1.3), i.e. shock formation and local well-posedness. The first work on the formation of shocks was done by Riemann in [39]. Riemann considered the case of isentropic flow with plane symmetry and introduced for such systems the so-called Riemann invariants, and then proved that solutions will blow up in finite time even under smooth initial conditions. Sideris [43] considered the three dimensional compressible Euler equations and obtained the first general result on the formation of singularity. By extending the basic idea of [39], Christodoulou-Miao [10] started from geometric aspects to study the shock formation of irrotational and isentropic flow in 3D, and gave a complete description of the maximal classical development. Yin in [59] constructed a class of spherical data to discuss the formation of shock wave in three dimensions. For multi-dimensional solutions with spherical symmetry, the blow-up phenomena was obtained by Li-Wang [29]. Recently, Luk-Speck [32, 33, 46] first introduced a wave-transport structure of the flow with dynamic vorticity and entropy, and described the singularity formation in two or three dimensions. Abbrescia-Speck in [1] studied some localized integral identities for 3D compressible Euler equations. We should also mention substantial significant progress which has been made on self similar solutions due to Merle-Raphael-Rodnianski-Szeftel [36], and free boundary problems due to Coutand-Lindblad-Shkoller [11], Coutand-Shkoller [11, 13], Lei-Du-Zhang [28], Li-Wang [30], Jang-Masmoudi [17], Ifrim-Tataru [19] and so on.

To the local well-posedness problem of (1.1)-(1.3), it's well-posed if $\left(\mathbf{v}_{0}, \varrho_{0}\right) \in$ $H^{s}, s>1+\frac{n}{2}$ and the density is far away from vacuum, please refer Majda's book [35]. Very rencently, based on the wave-transport system proposed by Luk and Speck $[32,33,46]$, some researchers considered the well-posedness of rough solutions for (1.1)-(1.3) by studying Strichartz estimates, which arises from dispersive equations. We refer the reader to Strichartz's work [47]). The first work about rough solutions of three-dimensional compressible Euler equations was obtained by Disconzi-Luo-Mazzone-Speck [14] and Wang [58]. In [14], Disconzi-Luo-Mazzone-Speck proved the well-posedness of solutions with dynamic vorticity and entropy, where they assumed the initial entropy $e$, velocity $\mathbf{v}$, logarithmic density
$\rho$ and specific vorticity w(it will be defined in Definition 1.2) in $H^{3+} \times\left(H^{2+}\right)^{3}$ and curlw $\in C^{0, \delta}(0<\delta<1)$. Independently, Wang [58] proved the local well-posedness by taking the initial data of $(\mathbf{v}, \rho, \mathbf{w}) \in H^{s}\left(\mathbb{R}^{3}\right) \times H^{s}\left(\mathbb{R}^{3}\right) \times H^{s^{\prime}}\left(\mathbb{R}^{3}\right), 2<s^{\prime}<s$. The works [14] and [58] are based on vector-field approach. Later, Zhang-Andersson [61] gave an alternative proof of [58] by using Smith-Tataru's method [41]. However, there is a few result with regard to low regularity solutions of the Cauchy problem of two-dimensional compressible Euler equations. Inspired by these historical results, we wish to study the low regularity solutions of (1.1)-(1.3) by establishing sharp Strichartz estimates of the velocity and density in two dimensions.
1.3. Motivation. In view of the aforementioned results we see that most studies are focusing on the behavior of solutions of 3D compressible Euler equations. These historical results [14,58] related to rough solutions in three dimensions successfully exploited the vector-field method in the case of non-zero vorticity, where the regularity of velocity and density are optimal. One may ask that what's the sharp Sobolev regularity of the initial data for 2D compressible Euler equations if controlling it's local well-posedness, and whether the vector-field method could solve the 2 D problem. In fact, the vector-field methods may not work very well for 2 D problem, for the conformal energy in 2 D is not ideal. We persuade the readers to Geba's work [21]. But, we noticed that the sharp regularity problem of 2D quasilinear wave equations is included in Smith-Tataru [41]. Our starting point is the result of Smith-Tataru [41], which, for generic nonlinear wave equations in two dimensions, yields the sharp local well-posedness in $H^{\frac{7}{4}+}$. However, this result can not be directly applied in the case of compressible Euler equations unless the fluid is assumed to be irrotational. Instead, in the general case, the compressible Euler flow can be seen as a coupling of a wave equation and a transport equation for the vorticity, which causes many difficulties. Let us explain the difficulty of the problem and the difference between quasilinear wave equations and compressible Euler equations.

To lower the Sobolev exponent of Cauchy problem (1.1)-(1.3), the key is to prove a type of Strichartz estimates. If the vorticity is zero, one could observe that there is a type of Strichartz estimates $\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}}$ from Smith-Tataru's result [41], where the regularity $s$ should be greater than $\frac{7}{4}$. With non-zero vorticity, what's the situation of Strichartz estimate. In particular, there are no Strichartz estimates for the vorticity. Let us see the coupled system. Precisely, $\partial \varpi$ is a source term in the wave equation,

$$
\square_{g} v=\partial \varpi+\text { l.o.t. }
$$

and $\varpi$ satisfies

$$
\partial_{t} \varpi+\mathbf{v} \cdot \nabla \varpi=0 .
$$

By utilizing the method of proving Strichartz estimates for wave equations, we know that the character is crucial. Although there is independent of $\varpi$ for energy estimates of $\mathbf{v}$ and $\rho$, but $\partial \varpi$ plays essential role for character. Hence, we need some energy estimates of $\varpi$. By classical commutator estimates, the condition $\partial \varpi \in L_{x}^{\infty}$ is essential for us to get the estimate of $\|\varpi\|_{H^{a}}, a \in(1,2]$. In Zhang's first paper [61], Zhang proved that the local solution is well-posed if the initial velocity, density and specific vorticity $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in H^{s}\left(s>\frac{7}{4}\right)$ and $\partial \varpi \in L_{x}^{\infty}$. Inspired by [58], we also hope to find some good structure of the vorticity and lower the regularity of vorticity, i.e. remove the initial assumption on $\|\partial \varpi\|_{L_{x}^{\infty}}$. To be precise, by setting
$\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in H^{s} \times H^{s} \times H^{2}\left(s>\frac{7}{4}\right)$, we discuss the local existence, uniqueness and continuous dependence of solutions of the Cauchy problem (1.1)-(1.3), where $\mathbf{v}_{0}$, $\rho_{0}$, and $\varpi_{0}$ describe the initial velocity, density, and specific vorticity respectively.
1.4. Statement of the result. Before stating our result, let us introduce some following quantities and introduce a equivalent system of (1.1).
1.4.1. Some definitions. Let us first recall the classical Hadamard standard for well-posedness.

Definition 1.1. [20] The problem (1.1)-(1.3) is locally well-posed in a Sobolev space $X$ if the following properties are satisfied:
(i) For each $\left(\mathbf{v}_{0}, \varrho_{0}\right)$ there exists some time $T>0$ and a solution $(\mathbf{v}, \rho) \in$ $C([0, T] ; X)$.
(ii) The above solution is unique.
(iii) The data to solution map is continuous from $X$ into $C([0, T] ; X)$.

In the following, let us introduce the logarithmic density, specific vorticity, the speed of sound, and the acoustic metric.

Definition 1.2. [32] Let $\bar{\rho}$ be a constant background density and $\bar{\rho}>0$. We denote the logarithmic density $\rho$

$$
\begin{gather*}
\rho:=\ln \left(\bar{\rho}^{-1} \varrho\right),  \tag{1.7}\\
\varpi:=\bar{\rho}^{-1} \mathrm{e}^{-\rho} \operatorname{curlv} . \tag{1.8}
\end{gather*}
$$

and the specific vorticity $\varpi$

Definition 1.3. [32] We denote the speed of sound

$$
\begin{equation*}
c_{s}:=\sqrt{d p / d \varrho} \tag{1.9}
\end{equation*}
$$

In view of (1.7), we have

$$
\begin{equation*}
c_{s}=c_{s}(\varrho) \tag{1.10}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{s}^{\prime}=c_{s}^{\prime}(\rho):=\frac{d c_{s}}{d \rho} \tag{1.11}
\end{equation*}
$$

Definition 1.4. [32] We define the acoustical metric $g$ and the inverse acoustical metric $g^{-1}$ relative to the Cartesian coordinates as follows:

$$
\begin{align*}
& g:=-d t \otimes d t+c_{s}^{-2} \sum_{a=1}^{2}\left(d x^{a}-v^{a} d t\right) \otimes\left(d x^{a}-v^{a} d t\right)  \tag{1.12}\\
& g^{-1}:=-\left(\partial_{t}+v^{a} \partial_{a}\right) \otimes\left(\partial_{t}+v^{b} \partial_{b}\right)+c_{s}^{2} \sum_{i=1}^{2} \partial_{i} \otimes \partial_{i} \tag{1.13}
\end{align*}
$$

Based on these definitions, let us introduce the system under new variables.
Lemma 1.1. [32] For 2D compressible Euler equations (1.1), it can be reduced to the following equations:

$$
\left\{\begin{array}{l}
\mathbf{T} v^{i}=c_{s}^{2} \delta^{i a} \partial_{a} \rho  \tag{1.14}\\
\mathbf{T} \rho=-\operatorname{div} \mathbf{v}
\end{array}\right.
$$

where $\mathbf{T}=\partial_{t}+\mathbf{v} \cdot \nabla$.

To be simple, we give the notations $d=\left(\partial_{t}, \partial_{x_{1}}, \partial_{x_{2}}\right)^{\mathrm{T}}, \partial_{x_{0}}=\partial_{t}$ and $\partial=$ $\left(\partial_{x_{1}}, \partial_{x_{2}}\right)^{\mathrm{T}}$. Set

$$
\begin{equation*}
\delta \in\left(0, s-\frac{7}{4}\right) \tag{1.15}
\end{equation*}
$$

and

$$
\langle\xi\rangle=\left(1+|\xi|^{2}\right)^{\frac{1}{2}}, \xi \in \mathbb{R}^{2}
$$

Denote by $\langle\partial\rangle$ the corresponding Bessel potential multiplier. We are now ready to state the result in this paper.
1.4.2. Statement of Results.

Theorem 1.2. Let $s>\frac{7}{4}$. Consider the following Cauchy problem of two-dimensional compressible Euler equations

$$
\left\{\begin{array}{l}
\mathbf{T} v^{i}=c_{s}^{2} \delta^{i a} \partial_{a} \rho  \tag{1.16}\\
\mathbf{T} \rho=-\operatorname{div} \mathbf{v} \\
\left.(\mathbf{v}, \rho)\right|_{t=0}=\left(\mathbf{v}_{0}, \rho_{0}\right)
\end{array}\right.
$$

Assume the acoustical speed

$$
\begin{equation*}
\left.c_{s}\right|_{t=0}>c_{0}>0 \tag{1.17}
\end{equation*}
$$

where $c_{0}$ is a positive constant. Let $\varpi$ be defined in (1.8) and $M_{0}$ be any positive constant. If

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq M_{0} \tag{1.18}
\end{equation*}
$$

then there exists two positive constants $T_{*}=T\left(\left\|\mathbf{v}_{0}\right\|_{H^{s}},\left\|\rho_{0}\right\|_{H^{s}},\left\|\varpi_{0}\right\|_{H^{2}}\right)$ and $M_{1}$ such that the Cauchy problem (1.16) is locally well-posed. Precisely,
(1) there exists a unique solution $(\mathbf{v}, \rho) \in C\left(\left[0, T_{*}\right], H_{x}^{s}\right) \cap C^{1}\left(\left[0, T_{*}\right], H_{x}^{s-1}\right), \varpi \in$ $C\left(\left[0, T_{*}\right], H_{x}^{2}\right) \cap C^{1}\left(\left[0, T_{*}\right], H_{x}^{1}\right)$ and $(d \mathbf{v}, d \rho) \in L_{\left[0, T_{*}\right]}^{4} L_{x}^{\infty}$, and it satisfies the energy estimate

$$
\|\mathbf{v}, \rho\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\partial_{t} \mathbf{v}, \partial_{t} \rho\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{L_{t}^{\infty} H_{x}^{1}} \leq M_{1}
$$

(2) the solution $\mathbf{v}$ and $\rho$ satisfy the Strichartz estimate

$$
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}} \leq M_{1}
$$

(3) for any $1 \leq r \leq s+1$, and for each $t_{0} \in[0, T]$, the linear equation

$$
\left\{\begin{array}{l}
\square_{g} f=0, \quad(t, x) \in[0, T] \times \mathbb{R}^{3},  \tag{1.19}\\
f\left(t_{0}, \cdot\right)=f_{0} \in H^{r}\left(\mathbb{R}^{2}\right), \quad \partial_{t} f\left(t_{0}, \cdot\right)=f_{1} \in H^{r-1}\left(\mathbb{R}^{2}\right)
\end{array}\right.
$$

admits a solution $f \in C\left([0, T], H^{r}\right) \times C^{1}\left([0, T], H^{r-1}\right)$ and the following estimates hold:

$$
\begin{equation*}
\|f\|_{L_{t}^{\infty} H_{x}^{r}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{r-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{1.20}
\end{equation*}
$$

Additionally, the following estimates hold, provided $k<r-\frac{3}{4}$,

$$
\begin{equation*}
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{1.21}
\end{equation*}
$$

(4) the map is continued from $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in H^{s} \times H^{s} \times H^{2}$ to $(\mathbf{v}, \rho, \varpi)(t, \cdot) \in$ $C\left([0, T] ; H_{x}^{s} \times H_{x}^{s} \times H_{x}^{2}\right)$.
Remark 1.1. The condition (1.17) is used to satisfy the hyperbolicity condition of the system (1.16).

Remark 1.2. For 2D compressible Euler equations, the classical result in [35] requires $s>2$ for the regularity of velocity and density. Our result lower the regularity of the velocity and density by proving the space-time Strichartz estimates of the velocity and density. Furthermore, if the vorticity is zero(the specific vorticity is also zero), the Sobolev regularity in Theorem 1.2 is corresponding to the 2D sharp result by Smith-Tataru [41].

Remark 1.3. Compared with the prior works for 3D compressible Euler equations, i.e. $[14,58]$, the situation in $2 D$ is very different. And it's very hard to use the similar approach in $[14,58]$ to prove Theorem 1.2, even for the irrotational case in 2D. Because the conformal energy in 2D is worse than 3D, which give a sacrifice on some regularity loss on metrics, please refer Geba's result [21].

Remark 1.4. The idea of deriving a good structure of $\mathbf{T}(\Delta \varpi-\partial \rho \partial \varpi)$ in the paper is inspired by Wang [58], but our process is not trivial. Referring [58], the good structure is benefit from curl ( $\mathrm{e}^{\rho}$ curlw), not $\Delta \mathbf{w}$. For the structure of divø is very good, but $\mathbf{T}($ divw) gives us some regularity loss. Then, choosing some quantities related to $\Delta \mathbf{w}$ may not work in three dimensions. Therefore, it's not obvious to choose the quantity $\Delta \varpi-\partial \rho \partial \varpi$ in two dimensions.

Remark 1.5. Inspired by Andersson-Moncrief [3] and Ifrim-Tataru [20], we consider the continuous dependence of solutions for (1.16). In [3], Andersson-Moncrief studied the local well-posedness of a hyperbolic-elliptic system. In [20], Ifrim-Tataru established the local well-posedness theory for general hyperbolic equations by using the frequency envelope approach.
1.5. A sketch of the proof. In effect our discussion below is more based on the idea of Smith-Tataru's work [41]. We will adopt two classes of equivalent structure of 2 D compressible Euler equations: the hyperbolic system

$$
A_{0}(\mathbf{U}) \mathbf{U}_{t}+A_{1}(\mathbf{U}) \mathbf{U}_{x_{1}}+A_{2}(\mathbf{U}) \mathbf{U}_{x_{2}}=0, \quad \mathbf{U}=(\mathbf{v}, p(\rho))^{\mathrm{T}}
$$

and the wave-transport system

$$
\left\{\begin{array}{l}
\square_{g} \mathbf{v}=\partial \varpi+\text { quadratic terms } \\
\square_{g} \rho=\text { quadratic terms } \\
\mathbf{T} \varpi=0,
\end{array}\right.
$$

where the hyperbolic system is used to consider some energy estimates, and the wave-transport system is used to discuss the Strichartz estimate.

The first key point is how to obtain energy estimates when the Sobolev indices of $\left(\mathbf{v}_{0}, \rho\right)$ and $\varpi_{0}$ is different. We use the hyperbolic system to derive the basic energy

$$
\|\mathbf{v}, \rho\|_{H^{s}} \leq\left\|\mathbf{v}_{0}, \rho_{0}\right\|_{H^{s}} \exp \left(\|d \mathbf{v}, d \rho\|_{L_{t}^{1} L_{x}^{\infty}}\right), \quad s \geq 0
$$

Concerning to the transport equation of specific vorticity $\mathbf{T} \varpi=0$, it looks impossible for us to obtain some energy estimates if the regularity between $\mathbf{v}_{0}$ and $\varpi$ are different. By deriving the nonlinear transport equation of $\Delta \varpi-\partial \rho \partial \varpi$, we could see a hope. That is,

$$
\mathbf{T} \Delta \varpi=\Delta v \partial \varpi+\partial v \partial^{2} \varpi+\text { l.o.t, }
$$

replaced by

$$
\begin{equation*}
\mathbf{T}(\Delta \varpi-\partial \rho \partial \varpi)=\partial \mathbf{v} \partial^{2} \varpi+\partial \mathbf{v} \partial \rho \partial \varpi \tag{1.22}
\end{equation*}
$$

In the first equation, one need the norm $\|\partial \varpi\|_{L_{x}^{\infty}}$ to obtain energy estimates by utilizing standard commutator estimates, and the regularity of velocity and vorticity should be same. While, if we use the second equation (1.22), it allows us to close the basic energy estimates of $\varpi$ by using Strichartz estimates $\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}}$ and some lower-order norms of the velocity and density. Please see the proofs in Lemma 2.8, Lemma 2.28, and Theorem 2.10 for details.

The second key point is to prove the Strichartz estimate. We first reduce the problem to establish an existence result for small, supported initial data. Next, by the continuity method, we can give a bootstrap argument on the regularity of the solutions to the nonlinear equation. Then, by introducing null hypersurfaces, the key is transformed to prove characteristic energy estimates of solutions along null hypersurfaces, and the enough regularity of null hypersurfaces is crucial to prove the Strichartz estimate. To establish characteristic energy estimates, we go back to see the wave-transport system and hyperbolic system. We use the hyperbolic system to get these characteristic energy estimates for ( $\mathbf{v}, \rho$ ), which is independent with $\varpi$. As for $\varpi$, the characteristic energy estimate is very different. Let us explain it as follows. On the Cauchy slice $\{t\} \times \mathbb{R}^{2}$, we can use elliptic estimates to get the energy estimate of all derivatives of $\varpi$ only by using $\varpi$ and $\Delta \varpi$. However, on the characteristic hypersurface, these type of elliptic energy estimates don't work. We use Hodge decomposition and (1.22) to handle this difficulty. That is, operating $P_{i j}$ on (1.22) giving rise to

$$
\begin{equation*}
\mathbf{T}\left[\partial_{i j}^{2} \varpi-P_{i j}(\partial \rho \partial \varpi)\right]=P_{i j}\left(\partial \mathbf{v} \partial^{2} \varpi+\text { l.o.t }\right)+\left[P_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi) \tag{1.23}
\end{equation*}
$$

Here, the Riesz operator $P_{i j}=\partial_{i j}^{2}(-\Delta)^{-1}, i, j=1,2$. From (1.23), we can get some type of characteristic energy estimates for second derivatives of $\varpi$, where we use Sobolev imbedding to calculate the lower term

$$
\left\|P_{i j}(\partial \rho \partial \varpi)\right\|_{L_{\Sigma}^{2}} \leq\left\|P_{i j}(\partial \rho \partial \varpi)\right\|_{L_{t}^{2} H_{x}^{a}}, \quad a>\frac{1}{2}
$$

On the right hand side of (1.23), especially for the second one, we need some commutator estimates, which is introduced in Lemma 2.6. Based on these observations, we can recover some energy bounds for $\varpi$ along the characteristic hypersurface. Please refer Lemma 6.8 for detials.

After obtaining enough regularity of null hypersurfaces and coefficients from null frame, we can obtain the Strichartz estimate of a linear wave equation with the acoustical metric $g$ by using Smith-Tataru's conclusion in [41]. Through Duhamel's principle, we can prove the Strichartz estimate $\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}}$.
1.6. Notations. In the paper, the notation $X \lesssim Y$ means $X \leq C Y$, where $C$ is a universal constant. We use the notation $X \ll Y$ to mean that $X \leq C Y$ with a sufficiently large constant $C$.

We use four small parameters

$$
\begin{equation*}
\epsilon_{3} \ll \epsilon_{2} \ll \epsilon_{1} \ll \epsilon_{0} \ll 1 \tag{1.24}
\end{equation*}
$$

Let $\zeta$ be a smooth function with support in the shell $\left\{\xi: \frac{1}{2} \leq|\xi| \leq 2\right\}$. Here, $\xi$ denotes the variable of the spatial Fourier transform. Let $\zeta$ also satisfy the condition $\sum_{k \in \mathbb{Z}} \zeta\left(2^{k} \xi\right)=1$. We introduce the Littlewood-Palay operator $P_{k}$ with the frequency $2^{k}(k \in \mathbb{Z})$, which satisfies

$$
P_{k} f=\int_{\mathbb{R}^{3}} \mathrm{e}^{-\mathrm{i} x \cdot \xi} \zeta\left(2^{-k} \xi\right) \hat{f}(\xi) d \xi
$$

We also set

$$
f_{<j}=S_{j} f=\sum_{k<j} P_{k} f
$$

1.7. Outline of the paper. The organization of the remainder of this paper is as follows. In Section 2, we introduce the reductions of (1.16) and commutator estimates, and also prove the total energy estimates and stability theorem. In Section 3 , we reduce our problem to the case of smooth initial data by using compactness methods. In the subsequent Section, using a physical localized technique, we reduce the problem to the case of smooth, small, compacted supported initial data. In section 5 , we give a bootstrap argument based on continuous functional. In Section 6 we derive some characteristic energy estimates along null hypersurfaces, which is used to prove the regularity of null hypersurfaces. Finally, in section 7, we prove the Strichartz estimate and continuous dependence.

## 2. Basic energy estimates and stability theorem

In this part, our goal is to give energy estimates and stability theorem. Firstly, we introduce a hyperbolic system and a wave-transport system of (1.16). We then give some classical commutator estimates. After that, we derive new transport equations for the specific vorticity. At last, we prove the energy estimates and stability theorem.
2.1. The reduction to a hyperbolic system and a wave-transport system. In the beginning, let us introduce a hyperbolic system of 2D compressible Euler equations.
Lemma 2.1. [30] Let $\mathbf{v}$ and $\rho$ be a solution of (1.16). Then ( $\mathbf{v}, \rho$ ) satisfies the following symmetric hyperbolic system

$$
\begin{equation*}
A_{0}(\mathbf{U}) \mathbf{U}_{t}+A_{1}(\mathbf{U}) \mathbf{U}_{x_{1}}+A_{2}(\mathbf{U}) \mathbf{U}_{x_{2}}=0 \tag{2.1}
\end{equation*}
$$

where $\mathbf{U}=\left(v^{1}, v^{2}, p(\rho)\right)^{\mathrm{T}}$ and

$$
\left.\begin{array}{c}
A_{0}=\left(\begin{array}{ccc}
\bar{\rho} \mathrm{e}^{\rho} & 0 & 0 \\
0 & \bar{\rho} \mathrm{e}^{\rho} & 0 \\
0 & 0 & \bar{\rho}^{-1} \mathrm{e}^{-\rho} c_{s}^{-2}
\end{array}\right), \quad A_{1}=\left(\begin{array}{ccc}
\bar{\rho} \mathrm{e}^{\rho} v^{1} & 0 & 1 \\
0 & \bar{\rho} \mathrm{e}^{\rho} v^{1} & 0 \\
1 & 0 & v^{1} \bar{\rho}^{-1} \mathrm{e}^{-\rho} c_{s}^{-2}
\end{array}\right), \\
A_{2}=\left(\begin{array}{cc}
\bar{\rho} \mathrm{e}^{\rho} v^{2} & 0 \\
0 & \bar{\rho} \mathrm{e}^{\rho} v^{2} \\
0 & 1
\end{array} v^{2} \bar{\rho}^{-1} \mathrm{e}^{-\rho} c_{s}^{-2}\right.
\end{array}\right) .
$$

Lemma 2.2. [32] Let $(\mathbf{v}, \rho)$ be a solution of (1.16) and $\varpi$ be defined in (1.8). Then $(\mathbf{v}, \rho, \varpi)$ satisfies

$$
\left\{\begin{array}{l}
\square_{g} v^{i}=-[i a] e^{\rho} c_{s}^{2} \partial^{a} \varpi+Q^{i}+E^{i}  \tag{2.2}\\
\square_{g} \rho=\mathcal{D} \\
\mathbf{T} \varpi=0
\end{array}\right.
$$

Above, $Q^{i}, E^{i}, \mathcal{D}$ are null forms relative to $g$, which are defined by

$$
\begin{align*}
& Q^{i}:=2[i a] c_{s}^{2} \varpi \partial^{i} \rho \\
& E^{i}:=-\left(1+c_{s}^{-1} c_{s}^{\prime}\right) g^{\alpha \beta} \partial_{\alpha} \rho \partial_{\beta} v^{i}  \tag{2.3}\\
& \mathcal{D}:=-3 c_{s}^{-1} c_{s}^{\prime} g^{\alpha \beta} \partial_{\alpha} \rho \partial_{\beta} \rho+2 \sum_{1 \leq a<b \leq 2}\left\{\partial_{a} v^{a} \partial_{b} v^{b}-\partial_{a} v^{b} \partial_{b} v^{a}\right\}
\end{align*}
$$

and

$$
[i a]= \begin{cases}0, & \text { if } i=a \\ 1, & \text { if } i<a \\ -1, & \text { if } i>a\end{cases}
$$

We also define $\mathbf{Q}:=\left(Q^{1}, Q^{2}\right)^{T}, \mathbf{E}:=\left(E^{1}, E^{2}\right)^{\mathrm{T}}$.
2.2. Commutator estimates. We first introduce a classical commutator estimate.

Lemma 2.3. [23] Let $\Lambda=(-\Delta)^{\frac{1}{2}}, s \geq 0$. Then for any scalar function $h, f$, we have

$$
\begin{equation*}
\left\|\Lambda^{s}(h f)-\left(\Lambda^{s} h\right) f\right\|_{L_{x}^{2}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\Lambda^{s-1} h\right\|_{L_{x}^{2}\left(\mathbb{R}^{n}\right)}\|\partial f\|_{L_{x}^{\infty}\left(\mathbb{R}^{n}\right)}+\|h\|_{L_{x}^{p}\left(\mathbb{R}^{n}\right)}\left\|\Lambda^{s} f\right\|_{L_{x}^{q}\left(\mathbb{R}^{n}\right)} \tag{2.4}
\end{equation*}
$$ where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.

Next, let us introduce some product estimates.
Lemma 2.4. [23] Let $F(u)$ be a smooth function of $u, F(0)=0$ and $u \in L_{x}^{\infty}$. For any $s \geq 0$, we have

$$
\begin{equation*}
\|F(u)\|_{H^{s}} \lesssim\|u\|_{H^{s}}\left(1+\|u\|_{L_{x}^{\infty}}\right) \tag{2.5}
\end{equation*}
$$

Lemma 2.5. [41] Suppose that $0 \leq r, r^{\prime}<\frac{n}{2}$ and $r+r^{\prime}>\frac{n}{2}$. Then

$$
\begin{equation*}
\|h f\|_{H^{r+r^{\prime}-\frac{n}{2}\left(\mathbb{R}^{n}\right)}} \leq C_{r, r^{\prime}}\|h\|_{H^{r}\left(\mathbb{R}^{n}\right)}\|h\|_{H^{r^{\prime}}\left(\mathbb{R}^{n}\right)} \tag{2.6}
\end{equation*}
$$

Moreover, if $-r \leq r^{\prime} \leq r$ and $r>\frac{n}{2}$, then the following estimate

$$
\begin{equation*}
\|h f\|_{H^{r^{\prime}}\left(\mathbb{R}^{n}\right)} \leq C_{r, r^{\prime}}\|h\|_{H^{r}\left(\mathbb{R}^{n}\right)}\|h\|_{H^{r^{\prime}}\left(\mathbb{R}^{n}\right)} \tag{2.7}
\end{equation*}
$$

holds.
Lemma 2.6. Denote the Riesz operator $\mathbf{R}:=\partial^{2}(-\Delta)^{-1}$. For $\delta \in\left(0, s-\frac{7}{4}\right)$, then

$$
\|[\mathbf{R}, \mathbf{v} \cdot \nabla] f\|_{L_{x}^{2}\left(\mathbb{R}^{2}\right)} \lesssim\|\partial \mathbf{v}\|_{C_{x}^{\delta}}\|f\|_{L_{x}^{2}\left(\mathbb{R}^{2}\right)}
$$

Proof. By using paraproduct decomposition, we have

$$
\begin{aligned}
\Delta_{j}[\mathbf{R}, \mathbf{v} \cdot \nabla] f= & \sum_{|k-j| \leq 2} \Delta_{j}\left[\mathbf{R}\left(\Delta_{k} \mathbf{v} \cdot \nabla S_{k-1} f\right)-\Delta_{k} \mathbf{v} \cdot \nabla \mathbf{R} S_{k-1} f\right] \\
& +\sum_{|k-j| \leq 2} \Delta_{j}\left[\mathbf{R}\left(S_{k-1} \mathbf{v} \cdot \nabla \Delta_{k} f\right)-S_{k-1} \mathbf{v} \cdot \nabla \mathbf{R} \Delta_{k} f\right] \\
& +\sum_{k \geq j-1} \Delta_{j}\left[\mathbf{R}\left(\Delta_{k} \mathbf{v} \cdot \nabla \Delta_{k} f\right)-\Delta_{k} \mathbf{v} \cdot \nabla \mathbf{R} \Delta_{k} f\right] \\
= & B_{1}+B_{2}+B_{3}
\end{aligned}
$$

where

$$
\begin{aligned}
& B_{1}=\sum_{|k-j| \leq 2} \Delta_{j}\left\{\mathbf{R}\left(\Delta_{k} \mathbf{v} \cdot \nabla S_{k-1} f\right)-\Delta_{k} \mathbf{v} \cdot \nabla \mathbf{R} S_{k-1} f\right\} \\
& B_{2}=\sum_{|k-j| \leq 2} \Delta_{j}\left\{\mathbf{R}\left(S_{k-1} \mathbf{v} \cdot \nabla \Delta_{k} f\right)-S_{k-1} \mathbf{v} \cdot \nabla \mathbf{R} \Delta_{k} f\right\} \\
& B_{3}=\sum_{k \geq j-1} \Delta_{j}\left\{\mathbf{R}\left(\Delta_{k} \mathbf{v} \cdot \nabla \Delta_{k} f\right)-\Delta_{k} \mathbf{v} \cdot \nabla \mathbf{R} \Delta_{k} f\right\}
\end{aligned}
$$

By Hölder's inequality and Bernstein's inequality, we arrive at the bound

$$
\begin{align*}
\left\{\left\|B_{1}\right\|_{L_{x}^{2}}\right\}_{l_{j}^{2}} & \lesssim\left\{\sum_{|k-j| \leq 2}\left(\left\|\nabla \mathbf{R} S_{k-1} f\right\|_{L_{x}^{\infty}}+\left\|\nabla S_{k-1} f\right\|_{L_{x}^{\infty}}\right)\left\|\Delta_{j} \Delta_{k} \mathbf{v}\right\|_{L^{2}}\right\}_{l_{j}^{2}} \\
& \lesssim\left\{\sum_{|k-j| \leq 2} 2^{k}\left\|\Delta_{j} \Delta_{k} \mathbf{v}\right\|_{L_{x}^{\infty}}\left(\left\|\mathbf{R} S_{k-1} f\right\|_{L^{2}}+\left\|S_{k-1} f\right\|_{L^{2}}\right)\right\}_{l_{j}^{2}}  \tag{2.8}\\
& \lesssim\|\mathbf{v}\|_{\dot{B}_{\infty, \infty}^{1}}\left(\|\mathbf{R} f\|_{L_{x}^{2}}+\|f\|_{L_{x}^{2}} \lesssim\|\partial \mathbf{v}\|_{C_{x}^{\delta}}\|f\|_{L_{x}^{2}}\right.
\end{align*}
$$

By Hölder's inequality, we see that

$$
\begin{align*}
\left\{\left\|B_{3}\right\|_{L_{x}^{2}}\right\}_{L_{j}^{2}} & \lesssim\left\{\sum_{k \geq j-1} 2^{k}\left\|\Delta_{k} \mathbf{v}\right\|_{L_{x}^{\infty}} \cdot 2^{-k}\left(\left\|\nabla \Delta_{k} f\right\|_{L_{x}^{2}}+\left\|\nabla \mathbf{R} \Delta_{k} f\right\|_{L_{x}^{2}}\right)\right\}_{l_{j}^{2}} \\
& \lesssim\|\mathbf{v}\|_{\dot{B}_{\infty, \infty}^{1}}\|f\|_{L_{x}^{2}}  \tag{2.9}\\
& \lesssim\|\partial \mathbf{v}\|_{C_{x}^{\delta}}^{\delta}\|f\|_{L_{x}^{2}} .
\end{align*}
$$

Note

$$
B_{2}=\sum_{|k-j| \leq 2} \Delta_{j}\left[\mathbf{R}, S_{k-1} \mathbf{v} \cdot\right] \Delta_{k} \nabla f
$$

By [37](Lemma 3.2), we get

$$
\begin{align*}
\left\{\left\|B_{2}\right\|_{L_{x}^{2}}\right\}_{l_{j}^{2}} & \lesssim\left\{\sum_{|k-j| \leq 2}\|x \Phi\|_{L_{x}^{1}}\left\|\nabla S_{k-1} \mathbf{v}\right\|_{L_{x}^{\infty}}\left\|\Delta_{j} \Delta_{k} f\right\|_{L_{x}^{2}}\right\}_{l_{j}^{2}} \\
& \lesssim\left\{\sum_{|k-j| \leq 2}\left\|\nabla S_{k-1} \mathbf{v}\right\|_{L_{x}^{\infty}}\left\|\Delta_{j} \Delta_{k} f\right\|_{L_{x}^{2}}\right\}_{l_{j}^{2}}  \tag{2.10}\\
& \lesssim\|\mathbf{v}\|_{\dot{B}_{\infty, \infty}^{1}}\|f\|_{L_{x}^{2}} \\
& \lesssim\|\partial \mathbf{v}\|_{C_{x}^{\delta}}\|f\|_{L_{x}^{2}}
\end{align*}
$$

Here, we use the fact that $\Phi=\frac{x_{i} x_{j}}{|x|^{2}} 2^{2 j} \Psi\left(2^{j} x\right)$ and $\Psi$ is in Schwartz space. Gathering (2.8), (2.9) and (2.10) together, we have finished the proof of Lemma 2.6.
2.3. New transport equations. We derive a new transport equation for derivatives of $\Delta \varpi$.

Lemma 2.7. Let $(\mathbf{v}, \rho)$ be a solution of (1.16) and $\varpi$ be defined in (1.8). Then the quantities $\partial_{i} \varpi(i=1,2)$ and $\Delta \varpi$ satisfy the transport equation:

$$
\begin{equation*}
\mathbf{T}\left(\partial_{i} \varpi\right)=-\partial_{i} v^{j} \partial_{j} \varpi, \quad i=1,2 \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{T}(\Delta \varpi-\partial \rho \partial \varpi)=R \tag{2.12}
\end{equation*}
$$

where

$$
\begin{equation*}
R=-2 \sum_{i, j=1}^{2} \partial_{j} v^{i} \partial_{j} \rho \partial_{i} \varpi-\mathrm{e}^{\rho}\left(\partial^{\perp} \rho \varpi+\partial^{\perp} \varpi\right) \partial \varpi-2 \sum_{i, j=1}^{2} \partial_{i} v^{j} \partial_{i j}^{2} \varpi . \tag{2.13}
\end{equation*}
$$

Proof. Taking the spatial derivatives on $\mathbf{T} \varpi=0$, we first get

$$
\mathbf{T} \partial_{i} \varpi=-\partial_{i} v^{j} \partial_{j} \varpi, \quad i=1,2
$$

Taking the spatial derivative $\partial_{i}$ again, one has

$$
\begin{equation*}
\mathbf{T} \Delta \varpi=-\Delta v^{i} \partial_{i} \varpi-2 \sum_{i=1,2} \partial_{i} v^{k} \partial_{k i}^{2} \varpi \tag{2.14}
\end{equation*}
$$

The Hodge's decomposition implies that

$$
\begin{equation*}
\Delta \mathbf{v}=\partial \operatorname{div} \mathbf{v}+\partial^{\perp} \text { curlv}, \quad \partial^{\perp}=\left(-\partial_{2}, \partial_{1}\right)^{\mathrm{T}} \tag{2.15}
\end{equation*}
$$

Substituting $\mathbf{T} \rho=-\operatorname{divv}$ and $\varpi=\mathrm{e}^{-\rho}$ curlv to (2.15), we get

$$
\begin{aligned}
\Delta \mathbf{v} & =-\partial \mathbf{T} \rho+\partial^{\perp}\left(\mathrm{e}^{\rho} \varpi\right) \\
& =[\mathbf{T}, \partial] \rho-\mathbf{T} \partial \rho+\partial^{\perp}\left(\mathrm{e}^{\rho} \varpi\right) \\
& =-\mathbf{T} \partial \rho+\sum_{j=1,2} \partial_{j} \mathbf{v} \partial_{j} \rho+\mathrm{e}^{\rho}\left(\partial^{\perp} \rho \varpi+\partial^{\perp} \varpi\right)
\end{aligned}
$$

Putting the above formula to (2.14), we obtain the transport equation:

$$
\mathbf{T} \Delta \varpi=\mathbf{T}(\partial \rho) \partial \varpi+R_{1}
$$

where

$$
R_{1}=-\sum_{i, j=1,2} \partial_{j} v^{i} \partial_{j} \rho \partial_{i} \varpi-\mathrm{e}^{\rho}\left(\partial^{\perp} \rho \varpi+\partial^{\perp} \varpi\right) \partial \varpi-\sum_{i=1,2} \partial_{i} v^{k} \partial_{k i}^{2} \varpi .
$$

Using the fact

$$
\begin{aligned}
\mathbf{T}(\partial \rho) \partial \varpi & =\mathbf{T}(\partial \rho \partial \varpi)-\partial \rho \mathbf{T}(\partial \varpi) \\
& =\mathbf{T}(\partial \rho \partial \varpi)-\sum_{j=1,2} \partial_{j} \rho \partial_{j} v^{i} \partial_{i} \varpi
\end{aligned}
$$

we then prove

$$
\mathbf{T}(\Delta \varpi-\partial \rho \partial \varpi)=R
$$

where

$$
\begin{aligned}
R & =R_{1}-\sum_{j=1,2} \partial_{j} \rho \partial_{j} v^{i} \partial_{i} \varpi \\
& =-2 \sum_{i, j=1}^{2} \partial_{j} v^{i} \partial_{j} \rho \partial_{i} \varpi-\mathrm{e}^{\rho}\left(\partial^{\perp} \rho \varpi+\partial^{\perp} \varpi\right) \partial \varpi-2 \sum_{i, j=1}^{2} \partial_{i} v^{j} \partial_{i j}^{2} \varpi .
\end{aligned}
$$

### 2.4. Energy estimates.

Lemma 2.8. Let $(\mathbf{v}, \rho)$ be a solution of (1.16). Then for any $a \geq 0$, we have

$$
\begin{equation*}
\|\rho\|_{H_{x}^{a}}+\|\mathbf{v}\|_{H_{x}^{a}} \lesssim\left(\left\|\rho_{0}\right\|_{H^{a}}+\left\|\mathbf{v}_{0}\right\|_{H^{a}}\right) \exp \left(\int_{0}^{t}\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}} d \tau\right), \quad t \in[0, T] \tag{2.16}
\end{equation*}
$$

Proof. Let $\mathbf{U}=(\mathbf{v}, p(\rho))^{T}$. Then

$$
A_{0}(\mathbf{U}) \mathbf{U}_{t}+A_{1}(\mathbf{U}) \partial_{x_{1}} \mathbf{U}+A_{2}(\mathbf{U}) \partial_{x_{2}} \mathbf{U}=0
$$

A straightforward computation on the above equation using integration by parts and classical commutator estimates in Lemma 2.3 yields

$$
\begin{equation*}
\|\mathbf{U}(t)\|_{H_{x}^{a}} \lesssim\|\mathbf{U}(0)\|_{H^{a}} \exp \left(\int_{0}^{t}\|d \mathbf{U}\|_{L_{x}^{\infty}} d \tau\right) \tag{2.17}
\end{equation*}
$$

As a result, we obtain

$$
\|\rho\|_{H_{x}^{s}}+\|\mathbf{v}\|_{H_{x}^{s}} \lesssim\left(\left\|\rho_{0}\right\|_{H^{a}}+\left\|\mathbf{v}_{0}\right\|_{H^{a}}\right) \exp \left(\int_{0}^{t}\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}} d \tau\right), \quad t \in[0, T]
$$

Lemma 2.9. Let $\mathbf{v}$ and $\rho$ be a solution of (1.16) and $\varpi$ be defined in (1.8). Then, we have the 1 -order energy estimates for the specific vorticity

$$
\begin{equation*}
\|\varpi\|_{H_{x}^{1}}^{2} \lesssim\left\|\varpi_{0}\right\|_{H^{1}}^{2} \exp \left(\int_{0}^{t}\|\partial \mathbf{v}\|_{L_{x}^{\infty}} d \tau\right) \tag{2.18}
\end{equation*}
$$

Moreover, the following inequality

$$
\begin{align*}
& \frac{d}{d t}\left(\|\Delta \varpi\|_{L_{x}^{2}}^{2}-2 \int_{\mathbb{R}^{2}} \partial \rho \partial \varpi \Delta \varpi d x\right)  \tag{2.19}\\
\lesssim & \left(1+\|\partial \rho\|_{L_{x}^{\infty}}+\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\right)^{2}\left(\|\mathbf{v}\|_{H_{x}^{\frac{3}{2}}}^{2}+\|\rho\|_{H_{x}^{3}}^{2}+\|\varpi\|_{H_{x}^{2}}^{2}\right) .
\end{align*}
$$

holds.

Proof. By using $\mathbf{T} \varpi=0$ and Hölder's inequality, we get

$$
\begin{equation*}
\frac{d}{d t}\|\varpi\|_{L_{x}^{2}}^{2} \lesssim\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\varpi\|_{L_{x}^{2}}^{2} \tag{2.20}
\end{equation*}
$$

By using (2.11) and Hölder's inequality, we arrive at the bound

$$
\begin{equation*}
\frac{d}{d t}\|\partial \varpi\|_{L_{x}^{2}}^{2} \lesssim\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\partial \varpi\|_{L_{x}^{2}}^{2} \tag{2.21}
\end{equation*}
$$

Adding (2.20) and (2.21) together yields

$$
\frac{d}{d t}\|\varpi\|_{H_{x}^{1}}^{2} \lesssim\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\varpi\|_{H_{x}^{1}}^{2}
$$

By Gronwall's inequality, we can reach to

$$
\|\varpi\|_{H_{x}^{1}}^{2} \leq\left\|\varpi_{0}\right\|_{H^{1}}^{2} \exp \left(\int_{0}^{t}\|\partial \mathbf{v}(\tau)\|_{L_{x}^{\infty}} d \tau\right)
$$

It remains for us to prove (2.19). Multiplying $\Delta \varpi$ on (2.3) and taking inner product on $\mathbb{R}^{2}$, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left(\|\Delta \varpi\|_{L_{x}^{2}}^{2}-2 \int_{\mathbb{R}^{2}} \partial \rho \partial \varpi \Delta \varpi d x\right) \\
\leq & \int_{\mathbb{R}^{2}} \partial \rho \partial \mathbf{v} \partial \varpi \Delta \varpi d x+\int_{\mathbb{R}^{2}} R \Delta \varpi d x+\int_{\mathbb{R}^{2}} \operatorname{div} \mathbf{v}|\Delta \varpi|^{2} d x . \tag{2.22}
\end{align*}
$$

So we need to estimate the right hand side of (2.22) one by one. For the right one, by Hölder's inequality, we could derive

$$
\begin{align*}
\left|\int_{\mathbb{R}^{2}} \partial \rho \partial \mathbf{v} \partial \varpi \Delta \varpi d x\right| & \lesssim\|\partial \rho\|_{L_{x}^{\infty}}\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\Delta \varpi\|_{L_{x}^{2}}\|\partial \varpi\|_{L_{x}^{2}}  \tag{2.23}\\
& \lesssim\left(\|\partial \rho\|_{L_{x}^{\infty}}+\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\right)^{2}\|\varpi\|_{H_{x}^{2}}^{2}
\end{align*}
$$

For the second one, by using (6.43) and Hölder's inequality, we can show that

$$
\begin{align*}
\left|\int_{\mathbb{R}^{2}} R \Delta \varpi d x\right| & \lesssim\|\Delta \varpi\|_{L_{x}^{2}}\left(\|\partial \rho\|_{L_{x}^{\infty}}+\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\right)\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\partial \varpi\|_{L_{x}^{2}} \\
& +\|\Delta \varpi\|_{L_{x}^{2}}\|\partial \varpi\|_{L_{x}^{4}}^{2}+\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\Delta \varpi\|_{L_{x}^{2}}\left\|\partial^{2} \varpi\right\|_{L_{x}^{2}}  \tag{2.24}\\
& +\|\partial \rho\|_{L_{x}^{\infty}}\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\Delta \varpi\|_{L_{x}^{2}}\|\partial \varpi\|_{L_{x}^{2}} \\
& \lesssim\left(1+\|\partial \rho\|_{L_{x}^{\infty}}+\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\right)^{2}\left(\|\mathbf{v}\|_{H_{x}^{\frac{3}{2}}}^{2}+\|\rho\|_{H_{x}^{\frac{3}{2}}}^{2}+\|\varpi\|_{H_{x}^{2}}^{2}\right) .
\end{align*}
$$

For the last term, by Hölder's inequality, we deduce

$$
\begin{equation*}
\left.\left|\int_{\mathbb{R}^{2}} \operatorname{div} \mathbf{v}\right| \Delta \varpi\right|^{2} d x \mid \lesssim\|\partial \mathbf{v}\|_{L_{x}^{\infty}}\|\varpi\|_{H_{x}^{2}}^{2} \tag{2.25}
\end{equation*}
$$

Gathering (2.22), (2.23), (2.24), and (2.25), we can obtain (2.19). We have completed the proof of Theorem 2.8.

Based on the above estimate, we can get the following energy estimates.
Theorem 2.10. (Total energy estimates) Assume $s>\frac{7}{4}$. Let $(\mathbf{v}, \rho)$ be a solution of (1.16) and $\varpi$ be defined as (1.8). Set

$$
E(t)=\|(\mathbf{v}, \rho)\|_{H_{x}^{s}}+\|\varpi\|_{H_{x}^{2}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{H_{x}^{s-1}}+\left\|\partial_{t} \varpi\right\|_{H_{x}^{1}}
$$

and

$$
E_{0}=\left\|\rho_{0}\right\|_{H_{x}^{s}}+\left\|\mathbf{v}_{0}\right\|_{H_{x}^{s}}+\left\|\varpi_{0}\right\|_{H_{x}^{2}}
$$

Then the following energy estimate

$$
\begin{equation*}
E(t) \lesssim E_{0}\left(1+E_{0}^{\frac{1}{2}}\right) \exp \left(\int_{0}^{t}\left(1+\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}}\right)^{2} d \tau\right) \tag{2.26}
\end{equation*}
$$

hold.
Proof. By using (2.16), (2.18), (2.19), and Gronwall's inequality, we have

$$
\begin{align*}
&\|\rho\|_{H_{x}^{s}}^{2}+\|\mathbf{v}\|_{H_{x}^{s}}^{2}+\|\varpi\|_{H_{x}^{1}}^{2}+\|\Delta \varpi\|_{L_{x}^{2}}^{2}-2 \int_{\mathbb{R}^{2}} \partial \rho \partial \varpi \Delta \varpi d x \\
& \lesssim\left(\left\|\varpi_{0}\right\|_{H^{2}}^{2}+\left\|\rho_{0}\right\|_{H^{s}}^{2}+\left\|\mathbf{v}_{0}\right\|_{H^{s}}^{2}\right) \exp \left(\int_{0}^{t}\left(1+\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}}\right)^{2} d \tau\right) \\
& \quad+\left\|\Delta \varpi_{0}\right\|_{L^{2}}^{2}+\left|\int_{\mathbb{R}^{2}} \partial \rho_{0} \partial \varpi_{0} \Delta \varpi_{0} d x\right|  \tag{2.27}\\
& \lesssim\left(\left\|\varpi_{0}\right\|_{H^{2}}^{2}+\left\|\rho_{0}\right\|_{H^{s}}^{2}+\left\|\mathbf{v}_{0}\right\|_{H^{s}}^{2}\right) \exp \left(\int_{0}^{t}\left(1+\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}}\right)^{2} d \tau\right) \\
& \quad+\left\|\varpi_{0}\right\|_{H^{2}}^{2}+\left\|\rho_{0}\right\|_{H^{s}}\left\|\varpi_{0}\right\|_{H^{2}}^{2}
\end{align*}
$$

Note

$$
\begin{equation*}
\|\Delta \varpi\|_{L_{x}^{2}}^{2}-2 \int_{\mathbb{R}^{2}} \partial \rho \partial \varpi \Delta \varpi d x \geq\|\Delta \varpi\|_{L_{x}^{2}}^{2}-C\|\partial \rho\|_{L_{x}^{4}}\|\partial \varpi\|_{L_{x}^{4}}\|\Delta \varpi\|_{L_{x}^{2}} . \tag{2.28}
\end{equation*}
$$

By Sobolev inequality, it implies that

$$
\begin{equation*}
\|\partial \rho\|_{L_{x}^{4}}\|\partial \varpi\|_{L_{x}^{4}}\|\Delta \varpi\|_{L_{x}^{2}} \leq\|\partial \rho\|_{H_{x}^{\frac{1}{2}}}\|\partial \varpi\|_{H_{x}^{\frac{1}{2}}}\|\Delta \varpi\|_{L_{x}^{2}} \tag{2.29}
\end{equation*}
$$

By interpolation formula, we have

$$
\begin{equation*}
\|\partial f\|_{H_{x}^{\frac{1}{x}}} \lesssim\|\partial f\|_{L_{x}^{2}}^{\frac{1}{2}}\left\|\partial^{2} f\right\|_{L_{x}^{2}}^{\frac{1}{2}} \tag{2.30}
\end{equation*}
$$

Using (2.30) and Young's inequality, we can update (2.31) by

$$
\begin{equation*}
\|\partial \rho\|_{L_{x}^{4}}\|\partial \varpi\|_{L_{x}^{4}}\|\Delta \varpi\|_{L_{x}^{2}} \leq\|\rho\|_{H_{x}^{\frac{3}{2}}}^{4}\|\partial \varpi\|_{L_{x}^{2}}^{2}+\frac{1}{100}\|\varpi\|_{H_{x}^{2}}^{2} \tag{2.31}
\end{equation*}
$$

We also note that

$$
\begin{equation*}
\|\varpi\|_{H_{x}^{1}}^{2}+\|\Delta \varpi\|_{L_{x}^{2}}^{2} \geq \frac{1}{2}\|\varpi\|_{H_{x}^{2}}^{2} \tag{2.32}
\end{equation*}
$$

Gathering (2.27), (2.28), (2.31), and (2.32), we get

$$
\begin{equation*}
\|\rho\|_{H_{x}^{s}}+\|\mathbf{v}\|_{H_{x}^{s}}+\|\varpi\|_{H_{x}^{2}} \lesssim E_{0}\left(1+E_{0}^{\frac{1}{2}}\right) \exp \left(\int_{0}^{t}\left(1+\|d \mathbf{v}, d \rho\|_{L_{x}^{\infty}}\right)^{2} d \tau\right) \tag{2.33}
\end{equation*}
$$

By using (1.16) and $\mathbf{T} \varpi=0$, we can carry out

$$
\begin{equation*}
\left\|\partial_{t} \rho\right\|_{H_{x}^{s-1}}+\left\|\partial_{t} \mathbf{v}\right\|_{H_{x}^{s-1}}+\left\|\partial_{t} \varpi\right\|_{H_{x}^{1}} \lesssim\|\rho\|_{H_{x}^{s}}+\|\mathbf{v}\|_{H_{x}^{s}}+\|\varpi\|_{H_{x}^{2}} \tag{2.34}
\end{equation*}
$$

By (2.33) and (2.34), we can obtain (2.26).
We are now ready to give the stability theorem.
Theorem 2.11. (Stability theorem) Assume $\frac{7}{4}<s \leq 2$. Let $(\mathbf{v}, \rho)$ be a solution of (1.16) and $\varpi$ be defined in (1.8), where the corresponding initial data $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in$ $H^{s} \times H^{s} \times H^{2}$. Then, there exists a positive number $T_{1}=T_{1}\left(\left\|\mathbf{v}_{0}\right\|_{H^{s}},\left\|\rho_{0}\right\|_{H^{s}},\left\|\varpi_{0}\right\|_{H^{2}}\right)$ such that $(\mathbf{v}, \rho) \in C\left(\left[0, T_{1}\right], H_{x}^{s}\right) \cap C^{1}\left(\left[0, T_{1}\right], H_{x}^{s-1} \times H_{x}^{s-1}\right), \varpi \in C\left(\left[0, T_{1}\right], H_{x}^{2}\right) \cap$ $C^{1}\left(\left[0, T_{1}\right], H_{x}^{1}\right)$, and $(d \mathbf{v}, d \rho) \in L_{\left[0, T_{1}\right]}^{4} L_{x}^{\infty}$.

Let $(\mathbf{h}, \psi)$ be another solution to (1.16) and $V=\bar{\rho} \mathrm{e}^{-\psi}$ curlh. And the corresponding initial data $\left(\mathbf{h}_{0}, \psi_{0}, V_{0}\right)$ is in $H^{s} \times H^{s} \times H^{2}$. Then, there exists a positive number $T_{2}=T_{2}\left(\left\|\mathbf{h}_{0}\right\|_{H^{s}},\left\|\psi_{0}\right\|_{H^{s}},\left\|V_{0}\right\|_{H^{2}}\right)$ such that $(\mathbf{h}, \psi) \in C\left(\left[0, T_{2}\right], H_{x}^{s}\right) \cap$ $C^{1}\left(\left[0, T_{2}\right], H_{x}^{s-1}\right), V \in C\left(\left[0, T_{2}\right], H_{x}^{2}\right) \cap C^{1}\left(\left[0, T_{2}\right], H_{x}^{1}\right)$ and $d \mathbf{h}, d \psi \in L_{\left[0, T_{2}\right]}^{4} L_{x}^{\infty}$. Therefore, for $t \in\left[0, \min \left\{T_{1}, T_{2}\right\}\right]$, the following estimate
$\|(\mathbf{v}-\mathbf{h}, \rho-\psi)(t, \cdot)\|_{H_{x}^{s-1}}+\|(\varpi-V)(t, \cdot)\|_{H_{x}^{1}} \lesssim\left\|\left(\mathbf{v}_{0}-\mathbf{h}_{0}, \rho_{0}-\psi_{0}\right)\right\|_{H^{s}}+\left\|\varpi_{0}-V_{0}\right\|_{H^{2}}$
holds.
Proof. Let $\mathbf{U}=(\mathbf{v}, p(\rho))^{\mathrm{T}}$ and $\mathbf{B}=(\mathbf{h}, p(\psi))^{\mathrm{T}}$. For $t \in\left[0, \min \left\{T_{1}, T_{2}\right\}\right]$, we can derive that

$$
\begin{aligned}
& A_{0}(\mathbf{U}) \partial_{t} \mathbf{U}+\sum_{i=1}^{2} A_{i}(\mathbf{U}) \partial_{x_{i}} \mathbf{U}=0 \\
& A_{0}(\mathbf{B}) \partial_{t} \mathbf{B}+\sum_{i=1}^{2} A_{i}(\mathbf{B}) \partial_{x_{i}} \mathbf{B}=0
\end{aligned}
$$

As a result, $\mathbf{U}-\mathbf{B}$ satisfies

$$
A_{0}(\mathbf{U}) \partial_{t}(\mathbf{U}-\mathbf{B})+\sum_{i=1}^{2} A_{i}(\mathbf{U}) \partial_{x_{i}}(\mathbf{U}-\mathbf{B})=\mathbf{F}
$$

where

$$
\mathbf{F}=-\sum_{i=0}^{2}\left(A_{i}(\mathbf{U})-A_{i}(\mathbf{B})\right) \partial_{x_{i}} \mathbf{B}
$$

By using the commutator estimates in Lemma 2.3, we could show

$$
\frac{d}{d t}\|\mathbf{U}-\mathbf{B}\|_{H_{x}^{s-1}} \leq C_{\mathbf{U}, \mathbf{B}}\left(\|d \mathbf{U}, d \mathbf{B}\|_{L_{x}^{\infty}}\|\mathbf{U}-\mathbf{B}\|_{H_{x}^{s-1}}+\|\mathbf{U}-\mathbf{B}\|_{L_{x}^{\infty}}\|d \mathbf{B}\|_{H_{x}^{s-1}}\right)
$$

where $C_{\mathbf{U}, \mathbf{B}}$ depends on the $L_{x}^{\infty}$ norm of $(\mathbf{U}, \mathbf{B})$. By using $(\mathbf{v}, \rho, \mathbf{h}, \psi) \in C\left(\left[0, \min \left\{T_{1}, T_{2}\right\}\right], H_{x}^{s}\right) \cap$ $C^{1}\left(\left[0, \min \left\{T_{1}, T_{2}\right\}\right], H_{x}^{s-1}\right)$ and $d \mathbf{v}, d \rho, d \mathbf{h}, d \psi \in L_{\left[0, \min \left\{T_{1}, T_{2}\right\}\right]}^{4} L_{x}^{\infty}$, we then have

$$
\begin{aligned}
\|(\mathbf{U}-\mathbf{B})(t, \cdot)\|_{H_{x}^{s-1}} & \lesssim\|(\mathbf{U}-\mathbf{B})(0, \cdot)\|_{H^{s-1}} \\
& =\left\|\left(\mathbf{v}_{0}-\mathbf{h}_{0}, \rho_{0}-\psi_{0}\right)\right\|_{H^{s}}
\end{aligned}
$$

By Lemma 2.4, we further obtain

$$
\begin{equation*}
\|(\mathbf{v}-\mathbf{h}, \rho-\psi)(t, \cdot)\|_{H_{x}^{s-1}} \lesssim\left\|\left(\mathbf{v}_{0}-\mathbf{h}_{0}, \rho_{0}-\psi_{0}\right)\right\|_{H^{s}} \tag{2.36}
\end{equation*}
$$

On the other hand, $\varpi$ and $V$ satisfy

$$
\partial_{t} \varpi+\mathbf{v} \cdot \nabla \varpi=0
$$

and

$$
\partial_{t} V+\mathbf{h} \cdot \nabla V=0
$$

So we get

$$
\begin{equation*}
\partial_{t}(\varpi-V)+\mathbf{v} \cdot \nabla(\varpi-V)=-(\mathbf{v}-\mathbf{h}) \nabla V \tag{2.37}
\end{equation*}
$$

By using standard energy estimates for (2.37), we get

$$
\begin{align*}
\|(\varpi-V)(t, \cdot)\|_{H_{x}^{1}} & \leq\left(\|(\varpi-V)(0, \cdot)\|_{H_{x}^{1}}+\|\mathbf{v}-\varphi\|_{L_{t}^{1} H_{x}^{\frac{3}{2}}}\|V\|_{L_{t}^{\infty} H_{x}^{2}}\right) \exp \left\{\int_{0}^{t}\|\partial \mathbf{v}\|_{L_{x}^{\infty}} d \tau\right\}  \tag{2.38}\\
& \leq C_{V}\left(\|(\varpi-V)(0, \cdot)\|_{H_{x}^{1}}+\|(\mathbf{v}-\varphi)(0, \cdot)\|_{H^{s}}\right)
\end{align*}
$$

for $t \in\left[0, \min \left\{T_{1}, T_{2}\right\}\right]$. Combining (2.37) and (2.38), we complete the proof of (2.35).

Corollary 2.12. (Uniqueness of the solution) Assume $\frac{7}{4}<s \leq 2$. Suppose $(\mathbf{v}, \rho)$ and $(\mathbf{h}, \psi)$ to be solutions of (1.16) with the same initial data $\left(\mathbf{v}_{0}, \rho_{0}\right) \in H^{s} \times H^{s}$. We assume the initial specific vorticity $\varpi_{0}=\bar{\rho} \mathrm{e}^{\rho_{0}}$ curlv $\mathbf{v}_{0} \in H^{2}$. Then there exists a constant $T>0$ such that $(\mathbf{v}, \rho, \mathbf{h}, \psi) \in C\left([0, T], H_{x}^{s}\right) \cap C^{1}\left([0, T], H_{x}^{s-1}\right)$, $\varpi \in$ $C\left([0, T], H_{x}^{2}\right) \cap C^{1}\left([0, T], H_{x}^{1}\right)$ and $d \mathbf{v}, d \rho, d \mathbf{h}, d \psi \in L_{[0, T]}^{4} L_{x}^{\infty}$. Furthermore, we have

$$
\mathbf{v}=\mathbf{h}, \quad \rho=\psi
$$

## 3. Reduction to the case of smooth initial data

In this part, we will reduce Theorem 1.2 to the case of smooth initial data by compactness arguments.

Proposition 3.1. For each $R>0$, there exist constants $T, M$ and $C$ such that, for each smooth initial data $\left(\mathbf{v}_{0}, \rho_{0}\right)$ satisfies

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq R \tag{3.1}
\end{equation*}
$$

where

$$
\varpi_{0}=\bar{\rho}^{-1} \mathrm{e}^{-\rho_{0}} \operatorname{curl}_{0}
$$

Then there exists a smooth solution $(\mathbf{v}, \rho, \varpi)$ to

$$
\left\{\begin{array}{l}
\square_{g} v^{i}=-[i a] e^{\rho} c_{s}^{2} \partial^{a} \varpi+Q^{i}+E^{i}  \tag{3.2}\\
\square_{g} \rho=\mathcal{D} \\
\mathbf{T} \varpi=0 \\
\left.(\mathbf{v}, \rho, \varpi)\right|_{t=0}=\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \\
\left.\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right|_{t=0}=\left(-\mathbf{v}_{0} \cdot \nabla \mathbf{v}_{0}+c_{s}^{2} \nabla \rho_{0},-\mathbf{v}_{0} \cdot \nabla \rho_{0}-\operatorname{div} \mathbf{v}_{0}\right)
\end{array}\right.
$$

on $[-T, T] \times \mathbb{R}^{2}$, which satisfies

$$
\begin{equation*}
\|(\mathbf{v}, \rho)\|_{H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{H_{x}^{s-1}}+\|\varpi\|_{H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{H_{x}^{1}} \leq M \tag{3.3}
\end{equation*}
$$

Here, the quantities $Q^{i}, \mathcal{D}$ and $E^{i}$ are defined in Lemma 2.2. Furthermore, the solution satisfies the condition
(1) the dispersive estimate for $\mathbf{v}$ and $\rho$

$$
\begin{equation*}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} \leq M \tag{3.4}
\end{equation*}
$$

(2) for $1 \leq r \leq s+1$, the linear equation

$$
\left\{\begin{array}{l}
\square_{g} f=0  \tag{3.5}\\
\left.\left(f, \partial_{t} f\right)\right|_{t=0}=\left(f_{0}, f_{1}\right)
\end{array}\right.
$$

is well-posed in $H^{r} \times H^{r-1}$, and the following estimates

$$
\begin{equation*}
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad k<r-\frac{3}{4} \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|_{L_{t}^{\infty} H_{x}^{r}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{r-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{3.7}
\end{equation*}
$$

holds.
In the following, we will use Proposition 3.1 to prove Theorem 1.2.
proof of Theorem 1.2 by Proposition 3.1. Consider arbitrary initial data $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \in$ $H^{s} \times H^{s} \times H^{2}$ satisfying

$$
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq R
$$

Let $\left\{\left(\mathbf{v}_{0 k}, \rho_{0 k}, \varpi_{0 k}\right)\right\}_{k \in \mathbb{N}^{+}}$be a sequence of smooth data which satisfies

$$
\begin{gathered}
\lim _{k \rightarrow \infty} \mathbf{v}_{0 k}=\mathbf{v}_{0}, \quad \lim _{k \rightarrow \infty} \rho_{0 k}=\rho_{0}, \quad \text { in } H^{s}, \\
\varpi_{0 k}=\bar{\rho} \mathrm{e}^{-\rho_{0 k}} \operatorname{curlv}_{0 k}, \quad \lim _{k \rightarrow \infty} \varpi_{0 k}=\varpi_{0}, \quad \text { in } H^{2} .
\end{gathered}
$$

By Proposition 3.1, for each $k$, there exist the corresponding solution $\left(\mathbf{v}_{k}, \rho_{k}, \varpi_{k}\right)$ to (3.2). Also

$$
\left.\left(\mathbf{v}_{k}, \rho_{k}, \varpi_{k}\right)\right|_{t=0}=\left(\mathbf{v}_{0 k}, \rho_{0 k}, \varpi_{0 k}\right)^{\mathrm{T}}
$$

Notice that the solutions of (3.2) also satisfy the symmetric hyperbolic system (2.1).
Set

$$
\mathbf{U}_{k}=\left(v_{1 k}, v_{2 k}, p\left(\rho_{k}\right)\right), \quad k \in \mathbb{N}^{+}
$$

For $j, l \in \mathbb{N}^{+}$, we could derive

$$
\begin{aligned}
& A_{0}\left(\mathbf{U}_{j}\right) \partial_{t} \mathbf{U}_{j}+A_{1}\left(\mathbf{U}_{k}\right) \partial_{x_{1}} \mathbf{U}_{j}+A_{2}\left(\mathbf{U}_{j}\right) \partial_{x_{2}} \mathbf{U}_{j}=0 \\
& A_{0}\left(\mathbf{U}_{l}\right) \partial_{t} \mathbf{U}_{l}+A_{1}\left(\mathbf{U}_{l}\right) \partial_{x_{1}} \mathbf{U}_{l}+A_{2}\left(\mathbf{U}_{l}\right) \partial_{x_{2}} \mathbf{U}_{l}=0
\end{aligned}
$$

The standard energy estimates imply that
$\frac{d}{d t}\left\|\mathbf{U}_{j}-\mathbf{U}_{l}\right\|_{H_{x}^{s-1}} \leq C_{\mathbf{U}_{j}, \mathbf{U}_{l}}\left(\left\|d \mathbf{U}_{j}, d \mathbf{U}_{l}\right\|_{L_{x}^{\infty}}\left\|\mathbf{U}_{j}-\mathbf{U}_{l}\right\|_{H_{x}^{s-1}}+\left\|\mathbf{U}_{j}-\mathbf{U}_{l}\right\|_{L_{x}^{\infty}}\left\|d \mathbf{U}_{l}\right\|_{H_{x}^{s-1}}\right)$,
where $C_{\mathbf{U}_{j}, \mathbf{U}_{l}}$ depends on the $L_{x}^{\infty}$ norm of $\mathbf{U}_{j}, \mathbf{U}_{l}$. By using Strichartz estimates (3.4) for $d \mathbf{v}_{k}, d \rho_{k}, k \in \mathbb{N}^{+}$and the energy estimates (3.3) for $\mathbf{v}_{k}, \rho_{k}, k \in \mathbb{N}^{+}$, we can derive that

$$
\begin{align*}
\left\|\left(\mathbf{U}_{j}-\mathbf{U}_{l}\right)(t, \cdot)\right\|_{H_{x}^{s-1}} & \lesssim\left\|\left(\mathbf{U}_{j}-\mathbf{U}_{l}\right)(0, \cdot)\right\|_{H^{s}} \\
& \lesssim\left\|\left(\mathbf{v}_{0 j}-\mathbf{v}_{0 l}, \rho_{0 j}-\rho_{0 l}\right)\right\|_{H^{s}} . \tag{3.8}
\end{align*}
$$

Thus, the sequence $\left\{\left(\mathbf{v}_{k}, \rho_{k}\right)\right\}_{k=1}^{\infty}$ is a Cauchy sequence in $C\left([-T, T] ; H^{s-1}\right)$. Denote $(\mathbf{v}, \rho)$ to be the limit. We therefore have

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(\mathbf{v}_{k}, \rho_{k}\right)=(\mathbf{v}, \rho) \in C\left([-T, T] ; H^{s-1}\right) \tag{3.9}
\end{equation*}
$$

Consider the transport equation

$$
\partial_{t} \varpi_{k}+\mathbf{v}_{k} \cdot \nabla \varpi_{k}=0, \quad k \in \mathbb{N}^{+}
$$

It's direct to get

$$
\partial_{t}\left(\varpi_{j}-\varpi_{l}\right)+\mathbf{v}_{j} \cdot \nabla\left(\varpi_{j}-\varpi_{l}\right)=\left(\mathbf{v}_{j}-\mathbf{v}_{l}\right) \cdot \nabla \varpi_{l}, \quad j, l \in \mathbb{N}^{+}
$$

By Sobolev equality and energy estimates, we have
$\left\|\varpi_{k}-\varpi_{l}\right\|_{H^{s-1}} \lesssim\left(\left\|\varpi_{0 k}-\varpi_{0 l}\right\|_{H^{s-1}}+\left\|\mathbf{v}_{j}-\mathbf{v}_{l}\right\|_{L_{t}^{\infty} H_{x}^{s-1}}\left\|\nabla \varpi_{l}\right\|_{L_{t}^{\infty} H_{x}^{1}}\right) \exp \left\{\int_{0}^{t}\left\|\partial \mathbf{v}_{j}\right\|_{L_{x}^{\infty}} d \tau\right\}$.

For $\left\{\varpi_{0 k}\right\}_{k \in \mathbb{N}^{+}}$and $\left\{\mathbf{v}_{k}\right\}_{k \in \mathbb{N}^{+}}$being two Cauchy sequence in $H^{s}$ and $L_{t}^{\infty} H_{x}^{s-1}$ respectively, then $\left\{\varpi_{k}\right\}_{k \in \mathbb{N}^{+}}$is a Cauchy sequence in $C\left([-T, T] ; H^{s-1}\right)$. We denote the limit

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \varpi_{k}=\varpi \in C\left([-T, T] ; H^{s-1}\right) \tag{3.11}
\end{equation*}
$$

Since $\left(\mathbf{v}_{k}, \rho_{k}, \varpi^{k}\right)$ is uniformly bounded in $L_{t}^{\infty} H_{x}^{s} \times L_{t}^{\infty} H_{x}^{s} \times L_{t}^{\infty} H_{x}^{2}$. Noting the convergence (3.9) and (3.11), we can deduce that

$$
\begin{equation*}
\left(\mathbf{v}, \rho, \varpi_{k}\right) \in L_{t}^{\infty} H_{x}^{s} \times L_{t}^{\infty} H_{x}^{s} \times L_{t}^{\infty} H_{x}^{2} \tag{3.12}
\end{equation*}
$$

Also, for $(\mathbf{v}, \rho, \varpi)$ satisfying (3.2) and (3.2) being equivalent with (1.16), we get

$$
\begin{equation*}
\left(\partial_{t} \mathbf{v}, \partial_{t} \rho, \partial_{t} \varpi\right) \in L_{t}^{\infty} H_{x}^{s-1} \times L_{t}^{\infty} H_{x}^{s-1} \times L_{t}^{\infty} H_{x}^{1} \tag{3.13}
\end{equation*}
$$

On the other hand, using Proposition 3.1, $\left(d \mathbf{v}_{k}, d \rho_{k}\right)$ is uniformly bounded in $L^{4}\left([-T, T] ; C_{x}^{\delta}\right)$. As a result, we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(d \mathbf{v}_{k}, d \rho_{k}\right)=(d \mathbf{v}, d \rho), \quad \text { in } L^{4}\left([-T, T] ; L_{x}^{\infty}\right) \tag{3.14}
\end{equation*}
$$

It remains for us to prove (1.20) and (1.21) in Theorem 1.2. For $1 \leq r \leq s+1$, by Proposition 3.1, we have that there exists solutions $f_{k}$ satisfying

$$
\left\{\begin{array}{l}
\square_{g_{k}} f_{k}=0  \tag{3.15}\\
\left.\left(f_{k}, \partial_{t} f_{k}\right)\right|_{t=0}=\left(f_{0}, f_{1}\right)
\end{array}\right.
$$

Here the metric $g_{k}$ has the same formula as in Definition 1.4, and whose velocity and density should be replaced by $\mathbf{v}_{k}$ and $\rho_{k}$. Using (3.6) and (3.7), we have

$$
\begin{equation*}
\left\|\langle\partial\rangle^{a} f_{k}\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad a<r-\frac{3}{4} \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|f_{k}\right\|_{L_{t}^{\infty} H_{x}^{r}}+\left\|\partial_{t} f_{k}\right\|_{L_{t}^{\infty} H_{x}^{r-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{3.17}
\end{equation*}
$$

From (3.17), we obtain that there exists a subsequence such that there is a limit $f$ satisfying

$$
\begin{equation*}
\|f\|_{L_{t}^{\infty} H_{x}^{r}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{r-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{3.18}
\end{equation*}
$$

Utilizing (3.16), we have

$$
\begin{equation*}
\left\|\langle\partial\rangle^{a} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad a<r-\frac{3}{4} \tag{3.19}
\end{equation*}
$$

Also, taking limit to (3.15), then the limit $f$ satisfies

$$
\left\{\begin{array}{l}
\square_{g} f=0  \tag{3.20}\\
\left.\left(f, \partial_{t} f\right)\right|_{t=0}=\left(f_{0}, f_{1}\right)
\end{array}\right.
$$

Combining (3.12)-(3.14), and (3.18)-(3.20), we have finished the proof of Theorem 1.2.

## 4. Reduction to existence for small, smooth, compactly supported DATA

In this section, our goal is to give a reduction of Proposition 3.1 to the existence for small, smooth, compactly supported data by using physical localization arguments.

Proposition 4.1. Assuming $\frac{7}{4}<s \leq 2$, (1.15), and (1.24) hold. Let the initial data $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right)$ be smooth, supported in $B(0, c+2)$ such that

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq \epsilon_{3} . \tag{4.1}
\end{equation*}
$$

and

$$
\varpi_{0}=\bar{\rho} \mathrm{e}^{-\rho} \operatorname{curl}_{0} .
$$

Then the Cauchy problem (3.2) admits a unique, smooth solution ( $\mathbf{v}, \rho, \varpi$ ) on $[-1,1] \times \mathbb{R}^{2}$, which has the following properties:
(1) energy estimate

$$
\begin{equation*}
\|(\mathbf{v}, \rho)\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{L_{t}^{\infty} H_{x}^{1}} \leq \epsilon_{2} . \tag{4.2}
\end{equation*}
$$

(2) dispersive estimate for $\mathbf{v}$ and $\rho$

$$
\begin{equation*}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} \leq \epsilon_{2} \tag{4.3}
\end{equation*}
$$

(3) dispersive estimate for the linear equation for $1 \leq r \leq s+1$, the linear equation

$$
\left\{\begin{array}{l}
\square_{g} f=0  \tag{4.4}\\
\left.\left(f, \partial_{t} f\right)\right|_{t=0}=\left(f_{0}, f_{1}\right)
\end{array}\right.
$$

is well-posed in $H^{r} \times H^{r-1}$, and the following estimates

$$
\begin{equation*}
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad k<r-\frac{3}{4} \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{s-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{4.6}
\end{equation*}
$$

holds.
proof of Proposition 3.1 by Proposition 4.1. To achieve the goal, we will firstly reduce Proposition 3.1 to small data by scaling and physical localization, and then using the conclusion in Proposition 4.1 to prove Proposition 3.1.

Step 1. Scaling. The the initial data in Proposition 3.1 satisfies

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq R \tag{4.7}
\end{equation*}
$$

By scaling

$$
\tilde{\mathbf{v}}(t, x)=\mathbf{v}(T t, T x), \quad \tilde{\rho}(t, x)=\rho(T t, T x), \quad \tilde{\varpi}(t, x)=\varpi(T t, T x)
$$

we get

$$
\begin{aligned}
& \left\|\tilde{\mathbf{v}}_{0}\right\|_{\dot{H}^{s}}+\left\|\tilde{\rho}_{0}\right\|_{\dot{H}^{s}} \leq R T^{s-1} \\
& \left\|\tilde{\varpi}_{0}\right\|_{\dot{H}^{2}} \leq R T
\end{aligned}
$$

Let $\epsilon_{3}$ be stated in (1.24). Choose sufficiently small $T$ such that

$$
R T^{s-1} \ll \epsilon_{3}
$$

We then derive that

$$
\left\|\tilde{v}_{0}\right\|_{\dot{H}^{s}}+\left\|\tilde{\boldsymbol{\rho}}_{0}\right\|_{\dot{H}^{s}}+\left\|\tilde{\varpi}_{0}\right\|_{\dot{H}^{2}} \leq \epsilon_{3} .
$$

The above homogeneous norm is not enough for us to use Proposition 4.1. We then need to reduce the data in a further step.

Step 2. Localization. Let $c$ be the largest speed of propagation of (3.2). Set $\chi$ be a smooth function supported in $B(0, c+2)$, and which equals 1 in $B(0, c+1)$. For any given $y \in \mathbb{R}^{2}$, we define the localized initial data near $y$ :

$$
\begin{aligned}
& \mathbf{v}_{0}^{y}=\chi(x-y)\left(\mathbf{v}_{0}-\mathbf{v}_{0}(y)\right), \\
& \rho_{0}^{y}=\chi(x-y)\left(\rho_{0}-\rho_{0}(y)\right)
\end{aligned}
$$

Then the initial specific vorticity should be given by

$$
\varpi_{0}^{y}=\bar{\rho}^{-1} \mathrm{e}^{-\rho_{0}^{y}} \operatorname{curl}_{0}^{y}
$$

Since $s \in\left(\frac{7}{4}, 2\right]$, it is not difficult for us to verify

$$
\begin{equation*}
\left\|\left(\mathbf{v}_{0}^{y}, \rho_{0}^{y}\right)\right\|_{H_{x}^{s}}+\left\|\varpi_{0}^{y}\right\|_{H_{x}^{2}} \lesssim\left\|\mathbf{v}_{0}, \rho_{0}\right\|_{\dot{H}^{s}}+\left\|\varpi_{0}\right\|_{\dot{H}^{2}} \lesssim \epsilon_{3} \tag{4.8}
\end{equation*}
$$

Step 3. Using Proposition 4.1. By Proposition 4.1, there is a smooth solution $\left(\mathbf{v}^{y}, \rho^{y}, \varpi^{y}\right)$ on $[-1,1] \times \mathbb{R}^{2}$ satisfying the following Cauchy problem

$$
\left\{\begin{array}{l}
\square_{g} v^{i}=-[i a] e^{\rho} c_{s}^{2} \partial^{a} \varpi+Q^{i}+E^{i}  \tag{4.9}\\
\square_{g} \rho=\mathcal{D} \\
\mathbf{T} \varpi=0 \\
\left.(\mathbf{v}, \rho, \varpi)\right|_{t=0}=\left(\mathbf{v}_{0}^{y}, \rho_{0}^{y}, \varpi_{0}^{y}\right) \\
\left.\left(\partial_{t} \mathbf{v}, \partial_{t} \boldsymbol{\rho}\right)\right|_{t=0}=\left(-\mathbf{v}_{0}^{y} \cdot \nabla \mathbf{v}_{0}^{y}+c_{s}^{2} \nabla \rho_{0}^{y},-\mathbf{v}_{0}^{y} \cdot \nabla \rho_{0}^{y}-\operatorname{div} \mathbf{v}_{0}^{y}\right)
\end{array}\right.
$$

where $Q^{i}, E^{i}$ and $\mathcal{D}$ are stated as (2.3). As a result, $\mathbf{v}^{y}+\mathbf{v}_{0}(y), \rho^{y}+\rho_{0}(y), \varpi^{y}$ also solves (4.9), and its initial data coincides with $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right)$ in $B(y, c+1)$. Besides, the Strichartz estimate

$$
\begin{equation*}
\left\|d \mathbf{v}^{y}, d \rho^{y}\right\|_{L_{t}^{4} L_{x}^{\infty}} \leq \epsilon_{2} \tag{4.10}
\end{equation*}
$$

also holds. Consider the restriction, for $y \in \mathbb{R}^{2}$,

$$
\left.\left(\mathbf{v}^{y}+\mathbf{v}_{0}(y)\right)\right|_{\mathrm{K}^{y}},\left.\quad\left(\rho^{y}+\rho_{0}(y)\right)\right|_{\mathrm{K}^{y},},\left.\quad \varpi^{y}\right|_{\mathrm{K}^{y}}
$$

where

$$
\mathrm{K}^{y}:=\{(t, x): c t+|x-y| \leq c+1,|t|<1\}
$$

Then this restrictions solve (4.9) on $\mathrm{K}^{y}$. By finite speed of propagation and the uniqueness of solutions of (3.2), a smooth solution $(\mathbf{v}, \rho, \varpi)$ satisfying (3.2) in $[-1,1] \times \mathbb{R}^{2}$ could be set by

$$
\begin{aligned}
\mathbf{v}(t, x) & =\mathbf{v}^{y}(t, x)+\mathbf{v}_{0}(y), & & (t, x) \in \mathrm{K}^{y} \\
\rho(t, x) & =\rho^{y}(t, x)+\rho_{0}(y), & & (t, x) \in \mathrm{K}^{y} \\
\varpi(t, x) & =\varpi^{y}(t, x), & & (t, x) \in \mathrm{K}^{y} .
\end{aligned}
$$

For the problem (3.2) is equivalent with (1.16), using Theorem 2.10, we have for $t \in[-1,1]$

$$
\begin{align*}
& \|(\mathbf{v}, \rho)\|_{H_{x}^{s}}+\|\varpi\|_{H_{x}^{2}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{H_{x}^{s-1}}+\left\|\partial_{t} \varpi\right\|_{H_{x}^{1}} \\
= & \left\|\left(\mathbf{v}^{y}, \rho^{y}\right)\right\|_{H_{x}^{s}}+\left\|\varpi^{y}\right\|_{H_{x}^{2}}+\left\|\left(\partial_{t} \mathbf{v}^{y}, \partial_{t} \rho^{y}\right)\right\|_{H_{x}^{s-1}}+\left\|\partial_{t} \varpi^{y}\right\|_{H_{x}^{1}} \\
\leq & C\left(\left\|\left(\mathbf{v}_{0}^{y}, \rho_{0}^{y}\right)\right\|_{H_{x}^{s}}+\left\|\varpi_{0}^{y}\right\|_{H_{x}^{2}}\right) \exp \left\{\int_{0}^{t}\left(1+\left\|d \mathbf{v}^{y}, d \rho^{y}\right\|_{L_{x}^{x}}\right)^{2} d \tau\right\}  \tag{4.11}\\
\leq & M .
\end{align*}
$$

By (4.10), we can directly get

$$
\begin{equation*}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}} \leq C\left\|d \mathbf{v}^{y}, d \rho^{y}\right\|_{L_{t}^{4} L_{x}^{\infty}} \leq M \tag{4.12}
\end{equation*}
$$

It remains for us to prove (3.6) and (3.7). Let the cartesian grid $2^{-\frac{1}{2}} \mathbb{Z}^{2}$ be in $\mathbb{R}^{2}$, and a corresponding smooth partition of unity be

$$
\sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^{2}} \psi(x-y)=1
$$

such that the function $\psi$ is supported in the unit ball. Consider the solution $f^{y}$ for

$$
\left\{\begin{array}{l}
\square_{g^{y}} f^{y}=0  \tag{4.13}\\
\left.f^{y}\right|_{t=0}=\psi(x-y) f_{0},\left.\partial_{t} f^{y}\right|_{t=0}=\psi(x-y) f_{1}
\end{array}\right.
$$

where $g^{y}$ has the same formulation as in (1.4) with the velocity $\mathbf{v}^{y}$ and $\rho^{y}$. Thus,

$$
\begin{equation*}
g^{y}=g, \quad(t, x) \in \mathrm{K}^{y} \tag{4.14}
\end{equation*}
$$

By finite speed of propagation, for $(t, x) \in \mathrm{K}^{y}$, we can conclude that

$$
f^{y}=f, \quad(t, x)
$$

Write $f$ as

$$
f(t, x)=\sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^{2}} \psi(x-y) f^{y}(x, t)
$$

Using (4.5) and (4.6), for $k<r-\frac{3}{4}$, we could get

$$
\begin{align*}
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}}^{4} & \leq C \sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^{2}}\left\|\psi(x-y)\langle\partial\rangle^{k} f^{y}(x, t)\right\|_{L_{t}^{4} L_{x}^{\infty}}^{4} \\
& \leq C \sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^{2}}\left\|\psi(x-y)\left(f_{0}, f_{1}\right)\right\|_{H^{r} \times H^{r-1}}^{4} .  \tag{4.15}\\
& \lesssim\left\|\left(f_{0}, f_{1}\right)\right\|_{H^{r} \times H^{r-1}}^{4},
\end{align*}
$$

and

$$
\begin{align*}
\|f\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{s-1}} & \leq C \sum_{y \in 2^{-\frac{1}{2} \mathbb{Z}^{2}}}\left(\left\|\psi(x-y) f^{y}(t, x)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\left\|\psi(x-y) \partial_{t} f^{y}\right\|_{L_{t}^{\infty} H_{x}^{s-1}}\right)  \tag{4.16}\\
& \lesssim\left\|\left(f_{0}, f_{1}\right)\right\|_{H^{r} \times H^{r-1}}
\end{align*}
$$

Therefore, by (4.11), (4.12), (4.15), and (4.16), we have finished the proof of Proposition 4.1.

## 5. A Bootstrap argument

Let $\mathbf{m}^{\alpha \beta}$ be a standard Minkowski metric satisfying

$$
\mathbf{m}^{00}=-1, \quad \mathbf{m}^{i j}=\delta^{i j}, \quad i, j=1,2
$$

Taking $\mathbf{v}=0$ and $\rho=0$ in $g$, the inverse matrix of the metric $g$ is

$$
g^{-1}(0)=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & c_{s}^{2}(0) & 0 \\
0 & 0 & c_{s}^{2}(0)
\end{array}\right)
$$

By a linear change of coordinates which preserves $d t$, we may assume that $g^{\alpha \beta}(0)=$ $\mathbf{m}^{\alpha \beta}$. Let $\chi$ be a smooth cut-off function supported in the region $B(0,3+2 c) \times$ $\left[-\frac{3}{2}, \frac{3}{2}\right]$, which equals to 1 in the region $B(0,2+2 c) \times[-1,1]$. Set

$$
\begin{equation*}
\mathbf{g}=\chi(t, x)(g-g(0))+g(0) \tag{5.1}
\end{equation*}
$$

where $g$ is denoted in (1.4). Consider the following Cauchy problem

$$
\left\{\begin{array}{l}
\square_{\mathbf{g}} v^{i}=-[i a] e^{\boldsymbol{\rho}} c_{s}^{2} \partial^{a} \varpi+Q^{i}+E^{i}  \tag{5.2}\\
\square_{\mathbf{g}} \rho=\mathcal{D} \\
\mathbf{T} \varpi=0 \\
\left.(\mathbf{v}, \rho, \varpi)\right|_{t=0}=\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right) \\
\left.\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right|_{t=0}=\left(-\mathbf{v}_{0} \cdot \nabla \mathbf{v}_{0}+c_{s}^{2} \nabla \rho_{0},-\mathbf{v}_{0} \cdot \nabla \rho_{0}-\operatorname{div} \mathbf{v}_{0}\right)
\end{array}\right.
$$

where, $\mathbf{g}$ is defined in (5.1). We denote by $\mathcal{H}$ the family of smooth solutions $(\mathbf{v}, \rho, \varpi)$ to (5.2) for $t \in[-2,2]$, with initial data $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right)$ supported in $B(0,2+c)$, where

$$
\varpi_{0}=\bar{\rho}^{-1} \mathrm{e}^{-\rho_{0}} \operatorname{curl}_{0}
$$

and for which

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq \epsilon_{3} \tag{5.3}
\end{equation*}
$$

$\|(\mathbf{v}, \rho)\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{L_{t}^{\infty} H_{x}^{1}}+\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} \leq 2 \epsilon_{2}$.
Then, the bootstrap argument can be stated as follows:
Proposition 5.1. Let (1.24) hold. Then there is a continuous functional $G: \mathcal{H} \rightarrow$ $\mathbb{R}^{+}$, satisfying $G(0)=0$, so that for each $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ satisfying $G(\mathbf{v}, \rho) \leq 2 \epsilon_{1}$ the following hold:
(1) The function $(\mathbf{v}, \rho, \varpi)$ satisfies

$$
\begin{equation*}
G(\mathbf{v}, \rho) \leq \epsilon_{1} \tag{5.5}
\end{equation*}
$$

(2) The following estimate holds,

$$
\begin{equation*}
\|(\mathbf{v}, \rho)\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{L_{t}^{\infty} H_{x}^{1}}+\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} \leq \epsilon_{2} \tag{5.6}
\end{equation*}
$$

(3) For $1 \leq r \leq s+1$, the equation (3.5) endowed with the metric $\mathbf{g}$ is well-posed in $H^{r} \times H^{r-1}$. Moreover, the following estimates

$$
\begin{equation*}
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad k<r-\frac{3}{4} \tag{5.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{s-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}} \tag{5.8}
\end{equation*}
$$

hold.
proof of Proposition 4.1 by Proposition 5.1. The initial data in Proposition 4.1 satisfies

$$
\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}} \leq \epsilon_{3}
$$

We denote by A the subset of those $\gamma \in[0,1]$ such that the equation (5.2) admits a smooth solution $u^{\gamma}$ having the initial data

$$
\begin{aligned}
\mathbf{v}^{\gamma}(0) & =\gamma \mathbf{v}_{0} \\
\rho^{\gamma}(0) & =\gamma \rho_{0} \\
\varpi_{0}^{\gamma}(0) & =\bar{\rho} \mathrm{e}^{-\rho^{\gamma}(0)} \operatorname{curl}_{0}^{\gamma}
\end{aligned}
$$

and such that $G\left(\mathbf{v}^{\gamma}, \rho^{\gamma}\right) \leq \epsilon_{1}$ and (5.6) hold.
If $\gamma=0$, then

$$
\left(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma}\right)(t, x)=(\mathbf{0}, 0,0)
$$

is a smooth solution of 5.2 with initial data

$$
\left(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma}\right)(0, x)=(\mathbf{0}, 0,0) .
$$

Thus, the set $A$ is not empty. If we can prove that $A=[0,1]$, then $1 \in A$. As a result, the Proposition 4.1 holds. It suffices for us to prove that A is both open and closed in $[0,1]$.
(1) A is open. Let $\gamma \in \mathrm{A}$. Then $\left(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma}\right)$ is a smooth solution to (5.2), where

$$
\varpi^{\gamma}=\bar{\rho} \mathrm{e}^{-\rho^{\gamma}} \operatorname{curl}^{\gamma} .
$$

Let $\beta$ be close to $\gamma$. By the continuity of $G$, it follows that

$$
G\left(\mathbf{v}^{\beta}, \rho^{\beta}\right) \leq 2 \epsilon_{1}
$$

and also (5.4) holds. Using Proposition 5.1, we have

$$
G\left(\mathbf{v}^{\beta}, \rho^{\beta}\right) \leq \epsilon_{1}
$$

and (5.6). Thus, we have showed that $\beta \in \mathrm{A}$.
(2) A is closed. Let $\gamma_{k} \in \mathrm{~A}, k \in \mathbb{N}^{+}$and $\lim _{k \rightarrow \infty} \gamma_{k}=\gamma$. Then there exists a sequence $\left\{\left(\mathbf{v}^{\gamma_{k}}, \rho^{\gamma_{k}}, \varpi^{\gamma_{k}}\right)\right\}_{k \in \mathbb{N}^{+}}$is smooth solutions to (5.2) and

$$
\begin{aligned}
& \left\|\left(\mathbf{v}^{\gamma_{k}}, \rho^{\gamma_{k}}\right)\right\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}^{\gamma_{k}}, \partial_{t} \rho^{\gamma_{k}}\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}} \\
& +\left\|\varpi^{\gamma_{k}}\right\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi^{\gamma_{k}}\right\|_{L_{t}^{\infty} H_{x}^{1}}+\left\|d \mathbf{v}^{\gamma_{k}}, d \rho^{\gamma_{k}}\right\|_{L_{t}^{4} C_{x}^{\delta}} \leq \epsilon_{2} .
\end{aligned}
$$

Then there exists some subsequence such that there is a limit $\left(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma}\right)$ satisfying $\left\|\left(\mathbf{v}^{\gamma}, \rho^{\gamma}\right)\right\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}^{\gamma}, \partial_{t} \rho^{\gamma}\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\left\|\varpi^{\gamma}\right\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi^{\gamma}\right\|_{L_{t}^{\infty} H_{x}^{1}}+\left\|d \mathbf{v}^{\gamma}, d \rho^{\gamma}\right\|_{L_{t}^{4} C_{x}^{\delta}} \leq \epsilon_{2}$, and $G(\mathbf{v}, \rho) \leq \epsilon_{1}$. Therefore, $\gamma \in \mathrm{A}$. We could conclude that $\mathrm{A}=[0,1]$. So we complete the proof of Proposition 4.1.

## 6. Regularity of the characteristic hypersurface

Recalling Proposition 5.1, the Strichartz estimate (5.7) plays a crucial role and it is a class of Fourier restriction estimate [47]. So we need to find a background hypersurface to work. In this section, we will define the characteristic hypersurface and discuss it's regularity.

Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$, and the corresponding metric $\mathbf{g}$ which equals the Minkowski metric for $t \in\left[-1,-\frac{1}{2}\right]$. Let $\Gamma_{\theta}$ be the flowout of this section under the Hamiltonian flow of $\mathbf{g}$. For each $\theta$, the null Lagrangian manifold $\Gamma_{\theta}$ is the graph of a null covector field given by $d r_{\theta}$, where $r_{\theta}$ is a smooth extension of $\theta \cdot x-t$, and that the level sets
of $r_{\theta}$ are small perturbations of the level sets of the function $\theta \cdot x-t$ in a certain norm captured by $G$. We also let $\Sigma_{\theta, r}$ for $r \in \mathbb{R}$ denote the level sets of $r_{\theta}$. The characteristic hypersurface $\Sigma_{\theta, r}$ is thus the flow out of the set $\theta \cdot x=r-2$ along with the null geodesic flow in the direction $\theta$ at $t=-1$.

Let us introduce an orthonormal set of coordinates on $\mathbb{R}^{2}$ by setting $x_{\theta}=\theta \cdot x$. Let $x_{\theta}^{\prime}$ be given orthonormal coordinates on the hyperplane perpendicular to $\theta$, which then define coordinates on $\mathbb{R}^{2}$ by projection along $\theta$. Then $\left(t, x_{\theta}^{\prime}\right)$ induces the coordinate on $\Sigma_{\theta, r}$, where $\Sigma_{\theta, r}$ is given by

$$
\Sigma_{\theta, r}=\left\{(t, x): x_{\theta}-\phi_{\theta, r}=0\right\}
$$

for a smooth function $\phi_{\theta, r}\left(t, x_{\theta}^{\prime}\right)$. We now introduce two norms for functions defined on $[-1,1] \times \mathbb{R}^{2}$,

$$
\begin{aligned}
& \|u\|_{2, \infty}=\sup _{-1 \leq t \leq 1} \sup _{0 \leq j \leq 1}\left\|\partial_{t}^{j} u(t, \cdot)\right\|_{H^{2-j}\left(\mathbb{R}^{2}\right)} \\
& \|u\|_{2,2}=\left(\sup _{0 \leq j \leq 1} \int_{-1}^{1}\left\|\partial_{t}^{j} u(t, \cdot)\right\|_{H^{2-j}\left(\mathbb{R}^{2}\right)}^{2} d t\right)^{\frac{1}{2}}
\end{aligned}
$$

The same notation applies for functions defined on $[-1,1] \times \mathbb{R}^{2}$. Denote

$$
\left\|\|f\|_{2,2, \Sigma_{\theta, r}}=\right\||f|_{\Sigma_{\theta, r}} \|_{2,2}
$$

where the right-hand side is the norm of the restriction of $f$ to $\Sigma_{\theta, r}$, taken over the $\left(t, x_{\theta}^{\prime}\right)$ variables used to parametrise $\Sigma_{\theta, r}$. Besides, the notation

$$
\|f\|_{H^{a}\left(\Sigma_{\theta, r}\right)}
$$

denotes the $H^{a-1}(\mathbb{R})$ norm of $f$ restricted to the time $t$ slice of $\Sigma_{\theta, r}$ using the $x_{\theta}^{\prime}$ coordinates on $\Sigma_{\theta, r}^{t}$.

We now set

$$
\begin{equation*}
G(\mathbf{v}, \rho)=\sup _{\theta, r}\| \| d \phi_{\theta, r}-d t \|_{2,2, \Sigma_{\theta, r}} \tag{6.1}
\end{equation*}
$$

Proposition 6.1. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ so that $G(\mathbf{v}, \rho) \leq 2 \epsilon_{1}$. Then

$$
\begin{equation*}
\left\|\mathbf{g}^{\alpha \beta}-\mathbf{m}^{\alpha \beta}\right\|_{2,2, \Sigma_{\theta, r}}+\left\|\lambda\left(\mathbf{g}^{\alpha \beta}-\mathbf{g}_{\lambda}^{\alpha \beta}\right), d \mathbf{g}_{\lambda}^{\alpha \beta}, \lambda^{-1} \partial d \mathbf{g}_{\lambda}^{\alpha \beta}\right\|_{1,2, \Sigma_{\theta, r}} \lesssim \epsilon_{2} \tag{6.2}
\end{equation*}
$$

Proposition 6.2. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ so that $G(\mathbf{v}, \rho) \leq 2 \epsilon_{1}$. Then

$$
\begin{equation*}
G(\mathbf{v}, \rho) \lesssim \epsilon_{2} \tag{6.3}
\end{equation*}
$$

Furthermore, for each $t$, we have

$$
\begin{equation*}
\left\|d \phi_{\theta, r}(t, \cdot)-d t\right\|_{C_{x^{\prime}}^{1, \delta}} \lesssim \epsilon_{2}+\sup _{i, j}\|d \mathbf{g}(t, \cdot)\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)} \tag{6.4}
\end{equation*}
$$

6.1. Energy estimates on the characteristic hypersurface. Let $(\mathbf{v}, \rho, \varpi) \in$ $\mathcal{H}$. Then the following estimates hold:

$$
\begin{equation*}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}}+\|\mathbf{v}, \rho\|\left\|_{s, \infty}+\right\| \varpi \|_{2, \infty} \lesssim \epsilon_{2} \tag{6.5}
\end{equation*}
$$

It suffices for us to prove Proposition 6.1 and Proposition 6.2 for $\theta=(0,1)$ and $r=0$. We fix this choice, and suppress $\theta$ and $r$ in our notation. We use $\left(x_{2}, x^{\prime}\right)$ instead of $\left(x_{\theta}, x_{\theta}^{\prime}\right)$. Then $\Sigma$ is defined by

$$
\Sigma=\left\{x_{2}-\phi\left(t, x^{\prime}\right)=0\right\}
$$

The hypothesis $G \leq 2 \epsilon_{1}$ implies that

$$
\begin{equation*}
\left\|d \phi_{\theta, r}(t, \cdot)-d t\right\|_{s, 2, \Sigma} \leq 2 \epsilon_{1} \tag{6.6}
\end{equation*}
$$

According to Sobolev imbeddings, the following estimate holds:

$$
\begin{equation*}
\left\|d \phi\left(t, x^{\prime}\right)-d t\right\|_{L_{t}^{4} C_{x^{\prime}}^{1, \delta}}+\left\|\partial_{t} d \phi\left(t, x^{\prime}\right)\right\|_{L_{t}^{4} C_{x^{\prime}}^{\delta}} \lesssim \epsilon_{1} \tag{6.7}
\end{equation*}
$$

Lemma 6.3. [41] Assume $s \in\left(\frac{7}{4}, 2\right]$. Let $\tilde{h}(t, x)=h\left(t, x^{\prime}, x_{2}+\phi\left(t, x^{\prime}\right)\right)$. Then we have

$$
\|\tilde{h}\|_{s, \infty} \lesssim\| \| h\left\|_{s, \infty}, \quad\right\| d \tilde{h}\left\|_{L_{t}^{4} L^{\infty}} \lesssim\right\| d h \|_{L_{t}^{4} L^{\infty}}
$$

and

$$
\|\tilde{h}\|_{H_{x}^{a}} \lesssim\|h\|_{H_{x}^{a}}, \quad 0 \leq a \leq 2
$$

Proof.
Lemma 6.4. [41] For $r>1$ we have

$$
\|h f\|_{r, 2, \Sigma} \lesssim\|h\|_{r, 2, \Sigma}\| \| f \|_{r, 2, \Sigma}
$$

Lemma 6.5. Assume $s \in\left(\frac{7}{4}, 2\right]$. Suppose $\mathbf{U}$ to satisfy the hyperbolic system

$$
\begin{equation*}
A_{0}(\mathbf{U}) \mathbf{U}_{t}+\sum_{i=1}^{2} A_{i}(\mathbf{U}) \mathbf{U}_{x_{i}}=\mathbf{F} \tag{6.8}
\end{equation*}
$$

Then

$$
\begin{equation*}
\|\mathbf{U}\|_{s, 2, \Sigma}^{2} \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}\left(\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}}+\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}+\|\mathbf{F}\|_{L_{t}^{1} H_{x}^{s-1}}\right) \tag{6.9}
\end{equation*}
$$

Proof. Choosing the change of coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$ and setting $\tilde{\mathbf{U}}(t, x)=$ $\mathbf{U}\left(t, x^{\prime}, x_{2}+\phi\left(t, x^{\prime}\right)\right), \tilde{\mathbf{F}}(t, x)=\mathbf{F}\left(t, x^{\prime}, x_{2}+\phi\left(t, x^{\prime}\right)\right)$, the system (6.8) is transformed to

$$
\begin{equation*}
A_{0}(\mathbf{U}) \partial_{t} \tilde{\mathbf{U}}+A_{1}(\tilde{\mathbf{U}}) \partial_{x_{1}} \tilde{\mathbf{U}}+A_{2}(\tilde{\mathbf{U}}) \partial_{x_{2}} \tilde{\mathbf{U}}=-\partial_{t} \phi \partial_{2} \tilde{\mathbf{U}}-\sum_{i=0}^{2} A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{1} \tilde{\mathbf{U}}+\tilde{\mathbf{F}} \tag{6.10}
\end{equation*}
$$

Multiplying $\tilde{\mathbf{U}}$ on (6.10) and integrating it by parts on $[-1,1] \times \mathbb{R}^{2}$, we get

$$
\|\tilde{\mathbf{U}}\|_{0,2, \Sigma}^{2} \lesssim\|d \tilde{\mathbf{U}}\|_{L_{t}^{1} L_{x}^{\infty}}\|\tilde{\mathbf{U}}\|_{L_{x}^{2}}+\|\tilde{\mathbf{U}}\|_{L_{x}^{2}}\|\tilde{\mathbf{F}}\|_{L_{t}^{1} L_{x}^{2}},
$$

where we use the fact that $\phi$ is independent of $x_{2}$. Using Lemma 6.3, (6.6), and (6.7), we may bound the above expression by

$$
\begin{equation*}
\|\mathbf{U}\|_{0,2, \Sigma}^{2} \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} L_{x}^{2}}\left(\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}}+\|\mathbf{U}\|_{L_{t}^{\infty} L_{x}^{2}}+\|\mathbf{F}\|_{L_{t}^{1} L_{x}^{2}}\right) \tag{6.11}
\end{equation*}
$$

Taking the derivative of $\Lambda_{x^{\prime}}^{\beta},|\beta|=s$ on (6.10) and integrating it on $[-1,1] \times \mathbb{R}^{2}$, we could arrive at the bound

$$
\begin{equation*}
\left\|\Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}}\right\|_{L_{\Sigma}^{2}}^{2} \lesssim\|d \tilde{\mathbf{U}}\|_{L_{t}^{1} L_{x}^{\infty}}\left\|\Lambda_{x}^{\beta} \tilde{\mathbf{U}}\right\|_{L_{t}^{\infty} L_{x}^{2}}+\left\|\Lambda_{x}^{\beta} \tilde{\mathbf{U}}\right\|_{L_{t}^{\infty} L_{x}^{2}}\left\|\Lambda_{x}^{\beta} \tilde{\mathbf{F}}\right\|_{L_{t}^{1} L_{x}^{2}}+I \tag{6.12}
\end{equation*}
$$

where

$$
I=-\sum_{i=0}^{2} \int_{[-1,1] \times \mathbb{R}^{2}} \Lambda_{x^{\prime}}^{\beta}\left(A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{2} \tilde{\mathbf{U}}\right) \cdot \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}} d x d \tau
$$

Rewrite $I$ as

$$
\begin{aligned}
I= & -\sum_{i=0}^{2} \int_{[-1,1] \times \mathbb{R}^{2}}\left(\Lambda_{x^{\prime}}^{\beta}\left(A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{2} \tilde{\mathbf{U}}\right)-A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{i} \partial_{2} \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}}\right) \cdot \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}} d x d \tau \\
& +\sum_{i=0}^{2} \int_{[-1,1] \times \mathbb{R}^{2}} A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{i} \partial_{2} \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}} \cdot \Lambda_{x^{\prime}}^{\beta} \tilde{U} d x d \tau \\
& =I_{1}+I_{2}
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1} & =\sum_{i=0}^{2} \int_{[-1,1] \times \mathbb{R}^{2}}\left[\Lambda_{x^{\prime}}^{\beta}, A_{i}(\tilde{\mathbf{U}}) \partial_{i} \phi \partial_{2}\right] \tilde{\mathbf{U}} \cdot \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}} d x d \tau \\
I_{2} & =\sum_{i=0}^{2} \int_{[-1,1] \times \mathbb{R}^{2}} A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{2}\left(\Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}}\right) \cdot \Lambda_{x^{\prime}}^{\beta} \tilde{\mathbf{U}} d x d \tau
\end{aligned}
$$

By commutator estimates, we get

$$
\begin{equation*}
\left|I_{1}\right| \lesssim\left(\left\|\Lambda^{\beta} \tilde{\mathbf{U}}\right\|_{L_{t}^{\infty} L_{x}^{2}}\|\partial d \phi\|_{L_{t}^{2} L_{x}^{\infty}}+\sup _{\theta, r}\left\|\Lambda_{x^{\prime}}^{\beta} d \phi\right\|_{L^{2}\left(\Sigma_{\theta, r}\right)}\|d \tilde{\mathbf{U}}\|_{\left.L_{t}^{2} L_{x}^{\infty}\right)} \cdot\left\|\Lambda^{\beta} \tilde{\mathbf{U}}\right\|_{L_{t}^{2} L_{x}^{2}}\right. \tag{6.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|I_{2}\right| \lesssim\left(\|d \tilde{\mathbf{U}}\|_{L_{t}^{2} L_{x}^{\infty}}\|\partial \phi\|_{L_{t}^{2} L_{x}^{\infty}}+\|\tilde{\mathbf{U}}\|_{L_{t}^{2} L_{x}^{\infty}}\left\|\partial^{2} \phi\right\|_{L_{t}^{2} L_{x}^{\infty}}\right) \cdot\left\|\Lambda^{\beta} \tilde{\mathbf{U}}\right\|_{L_{t}^{\infty} L_{x}^{2}}^{2} \tag{6.14}
\end{equation*}
$$

Due to (6.13), (6.14), Lemma 6.3, (6.6) and (6.7), we obtain

$$
\begin{equation*}
\left\|\Lambda_{x^{\prime}}^{\beta} \mathbf{U}\right\|_{0,2, \Sigma}^{2} \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}\left(\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}}+\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}+\|\mathbf{F}\|_{L_{t}^{1} H_{x}^{s-1}}\right) \tag{6.15}
\end{equation*}
$$

Using $A_{0}(\mathbf{U}) \partial_{t} \mathbf{U}=-A_{1}(\mathbf{U}) \mathbf{U}_{x_{1}}-A_{2}(\mathbf{U}) \mathbf{U}_{x_{2}}$ and Lemma 6.7 , we can easily carry out

$$
\begin{align*}
\left\|\partial_{t} \mathbf{U}\right\|_{s-1,2, \Sigma}^{2} & \lesssim\|\mathbf{U}\|_{s-1,2, \Sigma}^{2}\|\partial \mathbf{U}\|_{s-1,2, \Sigma}^{2} \\
& \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}\left(\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}}+\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\mathbf{F}\|_{L_{t}^{1} H_{x}^{s-1}}\right) \tag{6.16}
\end{align*}
$$

Therefore, we can conclude the proof of Lemma 6.5 by using (6.11), (6.15), and (6.16).

Based on Lemma 6.5, we get
Corollary 6.6. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$. Then

$$
\begin{equation*}
\|\mathbf{v}, \rho\|_{s, 2, \Sigma} \lesssim \epsilon_{2} \tag{6.17}
\end{equation*}
$$

Lemma 6.7. Suppose $f$ to satisfy the linear equation

$$
\begin{equation*}
\mathbf{T} f=G \tag{6.18}
\end{equation*}
$$

Then

$$
\begin{equation*}
\|f\|_{0,2, \Sigma}^{2} \lesssim\|G\|_{L_{t}^{1} L_{x}^{2}}\|f\|_{L_{x}^{2}}+\|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\|f\|_{L_{t}^{\infty} L_{x}^{2}} \tag{6.19}
\end{equation*}
$$

If $\varpi \in \mathcal{H}$ satisfies

$$
\begin{equation*}
\mathbf{T} \varpi=0 \tag{6.20}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\|\varpi\|_{0,2, \Sigma} \lesssim \epsilon_{2} . \tag{6.21}
\end{equation*}
$$

Proof. Choosing the change of coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$, then the equation (6.18) is transformed to

$$
\partial_{t} \tilde{f}+\tilde{\mathbf{v}} \cdot \nabla \tilde{f}=\tilde{G}-\partial_{t} \phi \cdot \partial_{2} \tilde{f}-\tilde{v}^{i} \partial_{i} \phi \partial_{2} \tilde{f}
$$

Taking the inner product with $\tilde{f}$ on $[-1,1] \times \mathbb{R}^{2}$, it gives

$$
\begin{equation*}
\|f\|_{L_{\Sigma}^{2}}^{2} \lesssim\|G\|_{L_{t}^{1} H_{x}^{s}}\|f\|_{L_{x}^{2}}+\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|f\|_{L_{x}^{2}}+I_{1}+I_{2} \tag{6.22}
\end{equation*}
$$

where

$$
\begin{aligned}
& I_{1}=-\int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{t} \phi \cdot \partial_{2} \tilde{f} \cdot \tilde{f} d x d \tau \\
& I_{2}=-\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{v}^{i} \partial_{i} \phi \partial_{2} \tilde{f} \cdot \tilde{f} d x d \tau
\end{aligned}
$$

For $\phi$ is independent of $x_{2}$, we have

$$
\begin{equation*}
I_{1}=\frac{1}{2} \int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{2} \partial_{t} \phi \cdot|\tilde{f}|^{2} d x d \tau=0 \tag{6.23}
\end{equation*}
$$

and

$$
\begin{aligned}
\left|I_{2}\right| & \left.=\left.\frac{1}{2}\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{2} \tilde{v}^{i} \partial_{i} \phi \cdot\right| \tilde{f}\right|^{2} d x d \tau \right\rvert\, \\
& \lesssim\|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\|f\|_{L_{x}^{2}}^{2}\|\partial \phi\|_{L_{t}^{4} L_{x}^{\infty}} .
\end{aligned}
$$

Using (6.33), we get

$$
\begin{equation*}
\left|I_{2}\right| \lesssim \epsilon_{1}\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|f\|_{L^{2}}^{2} \leq\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|f\|_{L_{x}^{2}}^{2} \tag{6.24}
\end{equation*}
$$

By (6.22), (6.23), and (6.24), we can obtain (6.19). If $G=0$, using (6.19), we can conclude (6.21).

Lemma 6.8. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$. Let $s \in\left(\frac{7}{4}, 2\right]$. Then we have

$$
\begin{equation*}
\|\varpi\|_{s, 2, \Sigma}+\| \|\left\|_{2,2, \Sigma}+\right\|\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}+\| \| \varpi \|_{1,2, \Sigma} \lesssim \epsilon_{2} . \tag{6.25}
\end{equation*}
$$

Proof. The proof is separated into several steps.
Step 1: $\left\|\|\partial \varpi\|_{0,2, \Sigma}\right.$. Recall

$$
\mathbf{T} \partial \varpi=\partial v \partial \varpi
$$

By changing coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$, we have

$$
\left(\partial_{t}+\partial_{t} \phi \partial_{2}\right) \widetilde{\partial \varpi}+\tilde{v}^{i} \cdot\left(\partial_{i}+\partial_{i} \phi \partial_{2}\right) \widetilde{\partial \varpi}=\left(\partial+\partial \phi \partial_{2}\right) \tilde{\mathbf{v}} \cdot\left(\partial+\partial \phi \partial_{2}\right) \tilde{\varpi}
$$

where $\tilde{\sim}$ denotes the function under new coordinates. Multiplying $\widetilde{\partial \varpi}$ on the above equation, we derive that

$$
\|\partial \varpi\|_{0,2, \Sigma}^{2} \lesssim\|d \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\left(1+\|d \phi\|_{L_{t, x}^{\infty}}\right)^{2}\|\partial \varpi\|_{L_{x}^{2}} \lesssim \epsilon_{2}^{2}
$$

Taking square of the above expression, we conclude that

$$
\begin{equation*}
\|\partial \varpi\|_{0,2, \Sigma} \lesssim \epsilon_{2} \tag{6.26}
\end{equation*}
$$

Step 2: $\left\|\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}\right.$. We find $\Delta \varpi$ satisfying

$$
\begin{equation*}
\mathbf{T}(\Delta \varpi-\partial \rho \partial \varpi)=R \tag{6.27}
\end{equation*}
$$

where $R$ is defined in (6.43). Denote the operator $\mathrm{P}_{i j}$ by

$$
\begin{equation*}
\mathrm{P}_{i j}=\partial_{i j}^{2}(-\Delta)^{-1} \tag{6.28}
\end{equation*}
$$

Then

$$
\begin{equation*}
\partial_{i j}^{2} \varpi=\mathrm{P}_{i j} \Delta \varpi, \quad i, j=1,2 . \tag{6.29}
\end{equation*}
$$

Operating $\mathrm{P}_{i j}$ on (6.27), we then get

$$
\begin{equation*}
\mathbf{T}\left\{\mathrm{P}_{i j}(\Delta \varpi-\partial \rho \partial \varpi)\right\}=\mathrm{P}_{i j} R+\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi) \tag{6.30}
\end{equation*}
$$

Inserting (6.29) into (6.30), we have

$$
\begin{equation*}
\mathbf{T}\left(\partial_{i j}^{2} \varpi-\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right)=K . \tag{6.31}
\end{equation*}
$$

Above, we define

$$
\begin{equation*}
K=\mathrm{P}_{i j} R+\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi) \tag{6.32}
\end{equation*}
$$

Choosing the change of coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$ and setting $\tilde{\varpi}\left(t, x^{\prime}, x_{2}\right)=$ $\varpi\left(t, x_{1}, x_{2}-\phi\left(t, x^{\prime}\right)\right)$, then the term $\partial_{i j}^{2} \varpi$ is transformed to

$$
\partial_{i j}^{2} \tilde{\varpi}-\partial_{i j}^{2} \phi \partial_{2} \tilde{\varpi}-\partial_{j} \phi \partial_{2 i}^{2} \tilde{\varpi}-\partial_{i} \phi \partial_{2 j}^{2} \tilde{\varpi}+\partial_{i} \phi \partial_{j} \phi \partial_{22}^{2} \tilde{\varpi}+\partial_{i} \phi \partial_{2 j}^{2} \phi \partial_{2} \tilde{\varpi} .
$$

Under the change of coordinates, we see the term $\mathrm{P}_{i j}(\partial \rho \partial \varpi)$ and $K$ as a whole part,i.e,

$$
\begin{aligned}
{\left.\left[\mathrm{P}_{i j} \widetilde{(\partial \rho \partial} \varpi\right)\right] } & =\left[\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right]\left(t, x_{1}, x_{2}-\phi\left(t, x^{\prime}\right)\right) \\
\tilde{K} & =K\left(t, x_{1}, x_{2}-\phi\left(t, x^{\prime}\right)\right)
\end{aligned}
$$

As a result, the left side of (6.31) becomes
$\left.\widetilde{\mathbf{T}}\left(\partial_{i j}^{2} \tilde{\varpi}-\partial_{i j}^{2} \phi \partial_{2} \tilde{\varpi}-\partial_{j} \phi \partial_{2 i}^{2} \tilde{\varpi}-\partial_{i} \phi \partial_{2 j}^{2} \tilde{\varpi}+\partial_{i} \phi \partial_{j} \phi \partial_{22}^{2} \tilde{\varpi}+\partial_{i} \phi \partial_{2 j}^{2} \phi \partial_{2} \tilde{\varpi}-\left[\mathrm{P}_{i j} \widetilde{(\partial \rho \partial} \varpi\right)\right]\right)$,
where

$$
\widetilde{\mathbf{T}}=\left(\partial_{t}+\partial_{t} \phi \partial_{2}\right)+\tilde{v}^{i}\left(\partial_{i}+\partial_{i} \phi \partial_{2}\right) .
$$

Organizing it in order, the expression of (6.27) could be

$$
\begin{equation*}
\left(\partial_{t}+\tilde{v}^{i} \partial_{i}\right) \tilde{B}+\left(\partial_{t} \phi+\tilde{v}^{i} \partial_{i} \phi\right) \partial_{2} \tilde{B}=\tilde{K} \tag{6.33}
\end{equation*}
$$

where

$$
\begin{align*}
\tilde{B}:= & \partial_{i j}^{2} \tilde{\varpi}-\partial_{i j}^{2} \phi \partial_{2} \tilde{\varpi}-\partial_{j} \phi \partial_{2 i}^{2} \tilde{\varpi}-\partial_{i} \phi \partial_{2 j}^{2} \tilde{\varpi} \\
& \left.+\partial_{i} \phi \partial_{j} \partial_{22}^{2} \tilde{\varpi}+\partial_{i} \phi \partial_{2 j}^{2} \phi \partial_{2} \tilde{\varpi}-\left[\mathrm{P}_{i j} \widetilde{(\partial \rho \partial} \varpi\right)\right] . \tag{6.34}
\end{align*}
$$

If we set

$$
\begin{equation*}
B=\partial_{i j}^{2} \varpi-\mathrm{P}_{i j}(\partial \rho \partial \varpi), \tag{6.35}
\end{equation*}
$$

then $B$ is transformed to $\tilde{B}$ under changing of coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$. Multiplying $\tilde{B}$ on (6.33) and integrating it on $[-1,1] \times \mathbb{R}^{2}$, one has

$$
\begin{align*}
\|\tilde{B}\|_{L^{2}(\Sigma)}^{2} \leq & \left|\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{K} \cdot \tilde{B} d x d \tau\right|+\|d \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|\tilde{B}\|_{L_{x}^{2}}^{2} \\
& +\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}}\left(\partial_{t} \phi+\tilde{v}^{i} \partial_{i} \phi\right) \partial_{2} \tilde{B} \cdot \tilde{B} d x d \tau\right| \tag{6.36}
\end{align*}
$$

On the left side, we see that

$$
\begin{equation*}
\|\tilde{B}\|_{L^{2}(\Sigma)}^{2}=\left\|\left.B\right|_{\Sigma}\right\|_{L_{t}^{2} L_{x^{\prime}}^{2}(\Sigma)}=\|B\|_{0,2, \Sigma}^{2} \tag{6.37}
\end{equation*}
$$

Let us estimate the right hand of (6.36) as follows. By using Lemma 6.3, we have

$$
\begin{equation*}
\|\tilde{B}\|_{L_{x}^{2}}^{2} \leq\|B\|_{L_{x}^{2}}^{2} \tag{6.38}
\end{equation*}
$$

By Hölder's inequality and $\phi$ independent with $x_{2}$, we arrive at the bound

$$
\begin{align*}
\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}}\left(\partial_{t} \phi+\tilde{v}^{i} \partial_{i} \phi\right) \partial_{2} \tilde{B} \cdot \tilde{B} d x d \tau\right| & =\left.\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{2}\left(\partial_{t} \phi+\tilde{v}^{i} \partial_{i} \phi\right)\right| \tilde{B}\right|^{2} d x d \tau \mid \\
& =\left.\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}}\right| \partial_{2} \tilde{v}^{i}|\cdot| \partial_{i} \phi| | \tilde{B}\right|^{2} d x d \tau \mid  \tag{6.39}\\
& \leq\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|\partial \phi\|_{L_{t}^{\infty} L_{x}^{\infty}}\|\tilde{B}\|_{L_{t}^{\infty} L_{x}^{2}} \\
& \leq\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|\partial \phi\|_{L_{t}^{\infty} L_{x}^{\infty}}\|B\|_{L_{t}^{\infty} L_{x}^{2}}
\end{align*}
$$

We note that there is a Riesz operator in $K$, we then pull the coordinate back by the transform $x_{2}-\phi\left(t, x^{\prime}\right) \rightarrow x_{2}$. Then, we have

$$
\begin{align*}
\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{K} \cdot \tilde{B} d x d \tau\right| & =\left|\int_{-1}^{1} \int_{\mathbb{R}^{2}} K \cdot B d x d \tau\right|  \tag{6.40}\\
& \leq\|K\|_{L_{t}^{1} L_{x}^{2}}\|B\|_{L_{t}^{\infty} L_{x}^{2}}
\end{align*}
$$

Combining (6.36)-(6.39) yields
$\|B\|_{0,2, \Sigma}^{2} \lesssim\|\partial \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|\partial \phi\|_{L_{t}^{\infty} L_{x}^{\infty}}\|B\|_{L_{t}^{\infty} L_{x}^{2}}+\|d \mathbf{v}\|_{L_{t}^{1} L_{x}^{\infty}}\|B\|_{L_{x}^{2}}^{2}+\|K\|_{L_{t}^{1} L_{x}^{2}}\|B\|_{L_{t}^{\infty} L_{x}^{2}}$.
By (5.4) and (6.7), we have

$$
\begin{equation*}
\|B\|_{0,2, \Sigma}^{2} \lesssim \epsilon_{2}^{2}+\|K\|_{L_{t}^{1} L_{x}^{2}}\|B\|_{L_{t}^{\infty} L_{x}^{2}} \tag{6.41}
\end{equation*}
$$

It remains for us to bound $\|K\|_{L_{t}^{1} L_{x}^{2}}$. Recalling (6.32), we can obtain

$$
\begin{equation*}
\|K\|_{L_{t}^{1} L_{x}^{2}} \leq\left\|\mathrm{P}_{i j} R\right\|_{L_{t}^{1} L_{x}^{2}}+\left\|\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi)\right\|_{L_{t}^{1} L_{x}^{2}} \tag{6.42}
\end{equation*}
$$

Using $\mathrm{P}_{i j}$, a Riesz operator, we can show that by Hölder's inequality

$$
\begin{align*}
\left\|\mathrm{P}_{i j} R\right\|_{L_{t}^{1} L_{x}^{2}} \lesssim & \|R\|_{L_{t}^{1} L_{x}^{2}}  \tag{6.43}\\
\lesssim & \|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\|\partial \rho\|_{L_{t}^{\infty} L_{x}^{2}}+\left\|\mathrm{e}^{\rho}\right\|_{L_{t}^{\infty} L_{x}^{\infty}}\left(\|\partial \rho\|_{L_{t}^{4} L_{x}^{\infty}}\|\varpi\|_{L_{t}^{\infty} L_{x}^{2}}+\|\partial \varpi\|_{L_{t}^{1} L_{x}^{2}}\right) \\
& +\|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\left\|\partial^{2} \varpi\right\|_{L_{t}^{\infty} L_{x}^{2}}+\|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\|\partial \rho\|_{L_{t}^{\infty} L_{x}^{4}}\|\partial \varpi\|_{L_{t}^{\infty} L_{x}^{4}} \\
\lesssim & \left.\|\partial \mathbf{v}, \partial \rho\|_{L_{t}^{4} L_{x}^{\infty}}+\|\partial \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\|\partial \rho\|_{L_{t}^{4} L_{x}^{\infty}}\right)\left(\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\|\rho\|_{L_{t}^{\infty} H_{x}^{s}}\right) \\
\lesssim & \epsilon_{2}^{2} .
\end{align*}
$$

By Lemma 2.6, we see that

$$
\begin{equation*}
\left\|\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi)\right\|_{L_{t}^{1} L_{x}^{2}} \leq\|\partial \mathbf{v}\|_{L_{t}^{4} C_{x}^{\delta}}\|\Delta \varpi-\partial \rho \partial \varpi\|_{L_{t}^{\infty} L_{x}^{2}} \tag{6.44}
\end{equation*}
$$

On the other hand, by using (5.4), we have

$$
\begin{align*}
\|\Delta \varpi-\partial \rho \partial \varpi\|_{L_{t}^{2} L_{x}^{2}} & \leq\|\Delta \varpi\|_{L_{t}^{2} L_{x}^{2}}+\|\partial \boldsymbol{\rho} \partial \varpi\|_{L_{t}^{2} L_{x}^{2}} \\
& \leq\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\|\partial \boldsymbol{\rho}\|_{L_{t}^{4} L_{x}^{\infty}}\|\partial \varpi\|_{L_{t}^{\infty} L_{x}^{2}}  \tag{6.45}\\
& \lesssim \epsilon_{2}+\epsilon_{2}^{2} \lesssim \epsilon_{2} .
\end{align*}
$$

Substituting (6.45) to (6.44) and using (5.4), we could get the bound

$$
\begin{equation*}
\left\|\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi)\right\|_{L_{t}^{1} L_{x}^{2}} \lesssim \epsilon_{2} . \tag{6.46}
\end{equation*}
$$

Adding (6.46) and (6.43), one has

$$
\begin{equation*}
\|K\|_{L_{t}^{1} L_{x}^{2}} \leq\left\|\mathrm{P}_{i j} R\right\|_{L_{t}^{1} L_{x}^{2}}+\left\|\left[\mathrm{P}_{i j}, \mathbf{T}\right](\Delta \varpi-\partial \rho \partial \varpi)\right\|_{L_{t}^{1} L_{x}^{2}} \lesssim \epsilon_{2} \tag{6.47}
\end{equation*}
$$

which when inserted into (6.41) yields the inequality

$$
\begin{equation*}
\|B\|_{0,2, \Sigma}^{2} \lesssim \epsilon_{2}^{2}+\epsilon_{2}\|B\|_{L_{t}^{\infty} L_{x}^{2}} . \tag{6.48}
\end{equation*}
$$

Recalling (6.35) and using (5.4), we have

$$
\begin{aligned}
\|B\|_{L_{t}^{\infty} L_{x}^{2}} & \leq\left\|\partial^{2} \varpi-\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{L_{t}^{\infty} L_{x}^{2}} \\
& \leq\left\|\partial^{2} \tilde{\varpi}\right\|_{L_{t}^{\infty} L_{x}^{2}}+\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{L_{t}^{\infty} L_{x}^{2}} \\
& \leq\left\|\partial^{2} \varpi\right\|_{L_{t}^{\infty} L_{x}^{2}}+\|\partial \rho \partial \varpi\|_{L_{t}^{\infty} L_{x}^{2}} \\
& \lesssim\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}\left(1+\|\partial \rho\|_{L_{t}^{\infty} H_{x}^{s-1}}\right) \lesssim \epsilon_{2}
\end{aligned}
$$

which combing with (6.48) give us

$$
\begin{equation*}
\|B\|_{0,2, \Sigma} \lesssim \epsilon_{2} \tag{6.49}
\end{equation*}
$$

Using (6.35) again, we derive that

$$
\begin{align*}
\|B\|_{0,2, \Sigma} & =\left\|\partial^{2} \tilde{\varpi}-\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{0,2, \Sigma} \\
& \geq\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}-\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{0,2, \Sigma} \tag{6.50}
\end{align*}
$$

It remains for us to estimate $\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{0,2, \Sigma}$. For the 1 -codimension of $\Sigma$ in $\mathbb{R}^{+} \times \mathbb{R}^{2}$, by Sobolev imbedding, we have

$$
\begin{align*}
\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{0,2, \Sigma} & =\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{L_{t}^{2} L_{x^{\prime}}^{2}} \\
& \leq\left\|\mathrm{P}_{i j}(\partial \rho \partial \varpi)\right\|_{L_{t}^{2} H_{x}^{a}}, \quad a>\frac{1}{2}  \tag{6.51}\\
& \leq\|\partial \rho \partial \varpi\|_{L_{t}^{2} H_{x}^{a}} \\
& \leq\|\partial \rho\|_{L_{t}^{2} H_{x}^{s-1}}\|\partial \varpi\|_{L_{t}^{2} H_{x}^{1}} \lesssim \epsilon_{2}^{2} .
\end{align*}
$$

Combining (6.50) and (6.51), we derive

$$
\begin{equation*}
\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma} \leq\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}+\left\|P_{i j}(\partial \rho \partial \varpi)\right\|_{0,2, \Sigma} \lesssim \epsilon_{2} \tag{6.52}
\end{equation*}
$$

Step 3: $\left\|\|\partial\|_{1,2, \Sigma}\right.$. We also note

$$
\partial_{t} \partial \varpi+\mathbf{v} \cdot \nabla \partial \varpi=\partial \mathbf{v} \cdot \partial \varpi
$$

By (6.54) and Sobolev imbedding, we see that

$$
\begin{aligned}
\left\|\partial_{t} \partial \varpi\right\|_{0,2, \Sigma} & \leq\|\mathbf{v} \cdot \nabla \partial \varpi\|_{0,2, \Sigma}+\|\partial \mathbf{v} \cdot \partial \varpi\|_{0,2, \Sigma} \\
& \leq\|\mathbf{v}\|_{L_{t, x}^{\infty}}\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}+\|\partial \mathbf{v} \cdot \partial \varpi\|_{L_{t}^{2} H_{x}^{a}}, \quad a>\frac{1}{2} \\
& \leq\|\mathbf{v}\|_{L_{t}^{\infty} H_{x}^{s}}\left\|\partial^{2} \varpi\right\|_{0,2, \Sigma}+\|\partial \mathbf{v}\|_{L_{t}^{\infty} H_{x}^{s-1}}\|\partial \varpi\|_{L_{t}^{\infty} H_{x}^{1}} \\
& \lesssim \epsilon_{2}
\end{aligned}
$$

For any function $f$, the term $\partial_{x^{\prime}} \tilde{f}$ can be calculated by

$$
\partial_{x^{\prime}} \tilde{f}=\nabla f \cdot(1, d \phi)^{\mathrm{T}}
$$

 $\left.\phi\left(t, x^{\prime}\right)\right)$. We then have

$$
\left\|\partial_{x^{\prime}} f\right\|_{0,2, \Sigma} \leq\left(1+\|d \phi\|_{L_{t, x^{\prime}}^{\infty}}\right)\|\partial f\|_{0,2, \Sigma}
$$

Based on this fact, we can deduce

$$
\begin{equation*}
\left\|\partial_{x^{\prime}} \partial \varpi\right\|_{0,2, \Sigma} \leq\left(1+\|d \phi\|_{L_{t, x^{\prime}}^{\infty}}\right)\|\partial(\partial \varpi)\|_{0,2, \Sigma} \leq\left(1+\epsilon_{1}\right) \epsilon_{2} \lesssim \epsilon_{2} \tag{6.54}
\end{equation*}
$$

Gathering (6.26), (6.54), and (6.53), we get

$$
\begin{equation*}
\|\partial \varpi\|_{1,2, \Sigma} \lesssim \epsilon_{2} \tag{6.55}
\end{equation*}
$$

Step 4: $\left\|\|\varpi\|_{2,2, \Sigma}\right.$. Note

$$
\mathbf{T} \varpi=0
$$

By changing of coordinates $x_{2} \rightarrow x_{2}-\phi\left(t, x^{\prime}\right)$, we can get

$$
\left(\partial_{t}+\partial_{t} \phi \partial_{2}\right) \tilde{\varpi}+\tilde{v}^{i}\left(\partial_{i}+\partial_{i} \phi \partial_{2}\right) \tilde{\varpi}=0
$$

Multiplying $\tilde{W}$ and integrating it on the whole space-time, we have

$$
\|\varpi\|_{0,2, \Sigma}^{2} \lesssim\|d \mathbf{v}\|_{L_{t}^{4} L_{x}^{\infty}}\left(1+\|d \phi\|_{L_{t, x^{\prime}}^{\infty}}\right)\|\varpi\|_{L_{t}^{\infty} L_{x}^{2}}^{2} \lesssim \epsilon_{2}^{2}
$$

which when taken square yields

$$
\begin{equation*}
\|\varpi\|_{0,2, \Sigma} \lesssim \epsilon_{2} . \tag{6.56}
\end{equation*}
$$

By using

$$
\begin{equation*}
\partial_{x^{\prime}} \tilde{\varpi}=\nabla \varpi \cdot(1, d \phi)^{\mathrm{T}} \tag{6.57}
\end{equation*}
$$

we thus have

$$
\partial_{x^{\prime}}^{2} \tilde{\varpi}=\partial_{x^{\prime}}(\nabla \varpi) \cdot(1, d \phi)^{\mathrm{T}}+\nabla \varpi \cdot\left(0, \partial_{x^{\prime}} d \phi\right)^{\mathrm{T}}
$$

Combining (6.7), (6.26), (6.54), and (6.59), we see that

$$
\begin{align*}
\left\|\partial_{x^{\prime}}^{2} \tilde{\varpi}\right\|_{0,2, \Sigma} \leq & \left\|\partial_{x^{\prime}}(\nabla \varpi)\right\|_{0,2, \Sigma}\left\|(1, d \phi)^{\mathrm{T}}\right\|_{L_{t, x^{\prime}}^{\infty}} \\
& +\|\nabla \varpi\|_{L_{t}^{\infty} L_{x^{\prime}}^{2}(\Sigma)}\left\|\left(0, \partial_{x^{\prime}} d \phi\right)^{\mathrm{T}}\right\|_{L_{t}^{2} L_{x^{\prime}}^{\infty}(\Sigma)} \\
\lesssim & \epsilon_{2} \epsilon_{1}+\|\varpi\|_{L_{t}^{\infty} H_{x^{\prime}}^{\frac{3}{2}}} \|(0)  \tag{6.58}\\
\lesssim & \left.\epsilon_{2} \epsilon_{1}+\| \varpi \partial_{x^{\prime}} d \phi\right)^{\mathrm{T}} \|_{L_{t}^{2} L_{x^{\prime}}^{\infty}(\Sigma)} \\
\lesssim & \epsilon_{2} \epsilon_{1}+\epsilon_{1}\|\varpi\|_{2,2, \Sigma}\|d \phi-d t\|_{L_{t}^{2} H_{x^{\prime}}^{s}(\Sigma)}
\end{align*}
$$

Above, we use the trace theorem

$$
\begin{equation*}
\|\varpi\|_{L_{t}^{\infty} H_{x^{\prime}}^{\frac{3}{2}}(\Sigma)} \leq\|\varpi\|_{2,2, \Sigma} \tag{6.59}
\end{equation*}
$$

Operating $\partial_{t}$ on (6.57), we get

$$
\begin{align*}
\left\|\partial_{t} \partial_{x^{\prime}} \varpi\right\|_{0,2, \Sigma} \leq & \left\|\partial_{t}(\nabla \varpi)\right\|_{0,2, \Sigma} \cdot\left\|(1, d \phi)^{\mathrm{T}}\right\|_{L_{t, x}^{\infty}} \\
& +\|\nabla \varpi\|_{L_{t}^{\infty} L_{x^{\prime}}^{2}(\Sigma)} \cdot\left\|\left(0, \partial_{t} d \phi\right)^{\mathrm{T}}\right\|_{L_{t}^{2} L_{x^{\prime}}^{\infty}(\Sigma)} \\
\lesssim & \epsilon_{2} \epsilon_{1}+\|\varpi\|_{L_{t}^{\infty} H_{x^{\prime}}^{\frac{3}{2}}(\Sigma)} \cdot\left\|\left(0, \partial_{x^{\prime}} d \phi\right)^{\mathrm{T}}\right\|_{L_{t}^{4} L_{x^{\prime}}^{\infty}(\Sigma)}  \tag{6.60}\\
\lesssim & \epsilon_{2} \epsilon_{1}+\|\varpi\|_{2,2, \Sigma} \cdot\left\|\left(0, \partial_{x^{\prime}} d \phi\right)^{\mathrm{T}}\right\|_{L_{t}^{4} L_{x^{\prime}}^{\infty}(\Sigma)} \\
\lesssim & \epsilon_{2} \epsilon_{1}+\epsilon_{1}\|\varpi\|_{2,2, \Sigma} .
\end{align*}
$$

Adding (6.58), (6.56), and (6.60) can give us

$$
\|\varpi\|_{2,2, \Sigma} \lesssim \epsilon_{2} \epsilon_{1}+\epsilon_{1}\|\varpi\|_{2,2, \Sigma}
$$

For $\epsilon_{1}$ is sufficiently small, we can see

$$
\begin{equation*}
\|\varpi\|_{2,2, \Sigma} \lesssim \epsilon_{2} \epsilon_{1} \lesssim \epsilon_{2} \tag{6.61}
\end{equation*}
$$

By using $s \in\left(\frac{7}{4}, 2\right]$, we have

$$
\begin{equation*}
\|\varpi\|_{s, 2, \Sigma} \leq\|\varpi\|_{2,2, \Sigma} \lesssim \epsilon_{2} \tag{6.62}
\end{equation*}
$$

Combining (6.54), (6.55), (6.62), and (6.62), we complete the proof of Lemma 6.8 .

Lemma 6.9. Let $\mathbf{U}$ be stated in Lemma 6.5. Then

$$
\begin{equation*}
\left\|2^{j}\left(\mathbf{U}-P_{j} \mathbf{U}\right), d S_{k} \mathbf{U}, 2^{-j} d \partial S_{j} \mathbf{U}\right\|_{s-1,2, \Sigma} \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}+\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}} \tag{6.63}
\end{equation*}
$$

Proof. Let $P$ be a standard multiplier of order 0 on $\mathbb{R}^{2}$, such that $P$ is additionally bounded on $L_{x}^{\infty}\left(\mathbb{R}^{2}\right)$. Clearly,

$$
A_{0}(\mathbf{U})(P \mathbf{U})_{t}+A_{1}(\mathbf{U})(P \mathbf{U})_{x_{1}}+A_{2}(\mathbf{U})(P \mathbf{U})_{x_{2}}=-\sum_{i=0}^{2}\left[P, A_{i}(\mathbf{U})\right] \partial_{x_{i}} \mathbf{U}
$$

By Lemma 6.5, this implies that

$$
\begin{equation*}
\|P \mathbf{U}\|\left\|_{s, 2, \Sigma} \lesssim\right\| d \mathbf{U}\left\|_{L_{t}^{4} L_{x}^{\infty}}+\right\| \mathbf{U}\left\|_{L_{t}^{\infty} H_{x}^{s}}+\right\| \mathbf{F} \|_{L_{t}^{1} H_{x}^{s-1}} \tag{6.64}
\end{equation*}
$$

To control the norm of $2^{j}\left(\mathbf{U}-P_{j} \mathbf{U}\right)$, we write

$$
2^{j}\left(\mathbf{U}-P_{j} \mathbf{U}\right)=\sum_{k=1}^{2} \partial_{k} P_{k} \mathbf{U}
$$

where $P_{k}$ satisfies the above conditions for $P$. Using (6.64), we get

$$
\left\|2^{j}\left(\mathbf{U}-P_{j} \mathbf{U}\right)\right\|_{s-1,2, \Sigma} \lesssim\|\mathbf{U}\|_{L_{t}^{\infty} H_{x}^{s}}+\|d \mathbf{U}\|_{L_{t}^{4} L_{x}^{\infty}}
$$

Finally, applying (6.64) to $P=S_{j}$ and $P=2^{-j} \partial S_{j}$ can give us

$$
\left\|d S_{j} \mathbf{U}\right\|_{s-1,2, \Sigma}+\left\|2^{-j} d \partial S_{j} \mathbf{U}\right\|\left\|_{s-1,2, \Sigma} \lesssim\right\| \mathbf{U}\left\|_{L_{t}^{\infty} H_{x}^{s}}+\right\| d \mathbf{U} \|_{L_{t}^{4} L_{x}^{\infty}}
$$

Therefore, the proof of Lemma 6.9 is completed.
As a direct corollary, we can see
Lemma 6.10. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ and $\mathbf{J}=(\mathbf{v}, \rho)^{\mathrm{T}}$. Then

$$
\begin{equation*}
\left\|2^{j}\left(\mathbf{J}-P_{j} \mathbf{J}\right), d S_{j} \mathbf{J}, 2^{-j} d \partial S_{j} \mathbf{J}\right\|_{s-1,2, \Sigma} \lesssim\|\mathbf{v}, \rho\|_{L_{t}^{\infty} H_{x}^{s}}+\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim \epsilon_{2} \tag{6.65}
\end{equation*}
$$

We are now ready to give a proof of Proposition 6.1.
proof of Proposition 6.1. For $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$, then $(\mathbf{v}, \rho, \varpi)$ is the solution of (5.2). Using Lemma 6.10, it suffices for us to verify that

$$
\left\|\mathbf{g}^{\alpha \beta}-\mathbf{m}^{\alpha \beta}\right\|_{s, 2, \Sigma_{\theta, r}} \lesssim \epsilon_{2} .
$$

By Corollary 6.6, one has

$$
\sup _{\theta, r}\| \| \mathbf{v}\left\|_{s, 2, \Sigma_{\theta, r}}+\sup _{\theta, r}\right\| \rho \|_{s, 2, \Sigma_{\theta, r}} \lesssim \epsilon_{2} .
$$

Using the expression of $\mathbf{g}$, and using Lemma 6.7, we arrive at the bound

$$
\begin{aligned}
\left\|\mathbf{g}^{\alpha \beta}-\mathbf{m}^{\alpha \beta}\right\| \|_{s, 2, \Sigma_{\theta, r}} & \lesssim\|\mathbf{v}\|_{s, 2, \Sigma_{\theta, r}}+\|\mathbf{v} \cdot \mathbf{v}\|_{s, 2, \Sigma_{\theta, r}}+\left\|c_{s}^{2}(\rho)-c_{s}^{2}(0)\right\|_{s, 2, \Sigma_{\theta, r}} \\
& \lesssim \epsilon_{2}
\end{aligned}
$$

Consequently, the conclusion of Proposition 6.1 holds.
6.2. The null frame. We introduce a null frame along $\Sigma$ as follows. Let

$$
V=(d r)^{*}
$$

where $r$ is the defining function of the foliation $\Sigma$, and where $*$ denotes the identification of covectors and vectors induced by $\mathbf{g}$. Then $V$ is the null geodesic flow field tangent to $\Sigma$. Let

$$
\begin{equation*}
\sigma=d t(V), \quad l=\sigma^{-1} V \tag{6.66}
\end{equation*}
$$

Thus $l$ is the g-normal field to $\Sigma$ normalized so that $d t(l)=1$, hence

$$
\begin{equation*}
l=\left\langle d t, d x_{2}-d \phi\right\rangle_{\mathbf{g}}^{-1}\left(d x_{2}-d \phi\right)^{*} \tag{6.67}
\end{equation*}
$$

So the coefficients $l^{j}$ are smooth functions of $\mathbf{v}, \rho$ and $d \phi$. Conversely,

$$
\begin{equation*}
d x_{2}-d \phi=\left\langle l, \partial_{x_{2}}\right\rangle_{\mathbf{g}}^{-1} l^{*} \tag{6.68}
\end{equation*}
$$

so that $d \phi$ is a smooth function of $\mathbf{v}, \rho$ and the coefficients of $l$.
Next, we introduce the vector fields $e_{1}$ tangent to the fixed-time slice $\Sigma^{t}$ of $\Sigma$. We do this by applying Grahm-Schmidt orthogonalization in the metric $\mathbf{g}$ to the $\Sigma^{t}$-tangent vector fields $\partial_{x_{1}}+\partial_{x_{1}} \phi \partial_{x_{2}}$.

Finally, we let

$$
\underline{l}=l+2 \partial_{t} .
$$

It follows that $\left\{l, \underline{l}, e_{1}\right\}$ form a null frame in the sense that

$$
\begin{array}{ll}
\langle l, \underline{l}\rangle_{\mathbf{g}}=2, & \left\langle e_{1}, e_{1}\right\rangle_{\mathbf{g}}=1, \\
\langle l, l\rangle_{\mathbf{g}}=\langle\underline{l}, \underline{l}\rangle_{\mathbf{g}}=0, & \left\langle l, e_{1}\right\rangle_{\mathbf{g}}=\left\langle\underline{l}, e_{1}\right\rangle_{\mathbf{g}}=0 .
\end{array}
$$

The coefficient of each of the fields is a smooth function of $(\mathbf{v}, \rho)$ and $d \phi$, and by assumption, we also have the pointwise bound

$$
\left|e_{1}-\partial_{x_{1}}\right|+\left|l-\left(\partial_{t}+\partial_{x_{2}}\right)\right|+\left|\underline{l}-\left(-\partial_{t}+\partial_{x_{2}}\right)\right| \lesssim \epsilon_{1} .
$$

After that, we can state the following lemma concerning the decomposition of the Ricci curvature tensor.

Corollary 6.11. Let $R$ be the Riemann curvature tensor of the metric $\mathbf{g}$. Let $e_{0}=l$. Then

$$
\begin{equation*}
R_{l l}=l\left(f_{2}\right)+f_{1}, \tag{6.69}
\end{equation*}
$$

where $\left|f_{1}\right| \lesssim|\partial \varpi|+|d \mathbf{g}|^{2},\left|f_{2}\right| \lesssim|d \mathbf{g}|$,

$$
\begin{equation*}
\left\|f_{2}\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)}+\left\|f_{1}\right\|_{L_{t}^{1} H_{x^{\prime}}^{s-1}(\Sigma)} \lesssim \epsilon_{2} \tag{6.70}
\end{equation*}
$$

and for any $t \in[0, T]$,

$$
\begin{equation*}
\left\|f_{2}(t, \cdot)\right\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim\|d \mathbf{g}\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)} \tag{6.71}
\end{equation*}
$$

Proof. By using the remarkable decomposition in Klainerman-Rodianiski [26], we have

$$
R_{l l}=l\left(f_{2}\right)-\frac{1}{2} l^{\alpha} l^{\beta} \square_{\mathbf{g}} \mathbf{g}_{\alpha \beta}+H
$$

where $|H| \lesssim|d \mathbf{g}|^{2}$ and

$$
z=l^{\gamma} \mathbf{g}^{\alpha \beta} \partial_{\beta} \mathbf{g}_{\alpha \gamma}-\frac{1}{2} \mathbf{g}^{\alpha \beta} l\left(\mathbf{g}_{\alpha \beta}\right)
$$

According to (2.2), we derive that

$$
\left|f_{1}\right| \lesssim|\partial \varpi|+|d \mathbf{g}|^{2},\left|f_{2}\right| \lesssim|d \mathbf{g}| .
$$

Due to Lemma 6.5 and Lemma 6.7, we get

$$
\left\|f_{2}\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)}+\left\|f_{1}\right\|_{L_{t}^{1} H_{x^{\prime}}^{s-1}(\Sigma)} \lesssim \epsilon_{2}
$$

It's clear that the estimate (6.71) can be obtained directly from Sobolev embeddings. Thus, the proof is completed.
6.3. The estimate of connection coefficients. Define

$$
\chi=\left\langle D_{e_{1}} l, e_{1}\right\rangle_{\mathbf{g}}, \quad l(\ln \sigma)=\frac{1}{2}\left\langle D_{l} l, l\right\rangle_{\mathbf{g}}
$$

For $\sigma$, we set the initial data $\sigma=1$ at the time -2 . Thanks to Proposition 6.1, we have

$$
\begin{equation*}
\|\chi\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)}+\|l(\ln \sigma)\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)} \lesssim \epsilon_{1} . \tag{6.72}
\end{equation*}
$$

In a similar way, if we expand $l=l^{\alpha} \not \partial_{\alpha}$ in the tangent frame $\partial_{t}, \partial_{x^{\prime}}$ on $\Sigma$, then

$$
\begin{equation*}
l^{0}=1, \quad\left\|l^{1}\right\|_{s-1,2, \Sigma} \lesssim \epsilon_{1} . \tag{6.73}
\end{equation*}
$$

Lemma 6.12. Let $\chi$ be defined as before. Then

$$
\begin{equation*}
\|\chi\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)} \lesssim \epsilon_{2} \tag{6.74}
\end{equation*}
$$

Furthermore, for any $t \in[0, T]$,

$$
\begin{equation*}
\|\chi\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}+\|d \mathbf{g}\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)} \tag{6.75}
\end{equation*}
$$

Proof. The famous transport equation for $\chi$ along null hypersurfaces (see references [25] and [41]) can be described as

$$
l(\chi)=\left\langle R\left(l, e_{1}\right) l, e_{1}\right\rangle_{\mathbf{g}}-\chi^{2}-l(\ln \sigma) \chi
$$

Due to Corollary 6.11, we write the above equation as

$$
\begin{equation*}
l\left(\chi-f_{2}\right)=f_{1}-\chi^{2}-l(\ln \sigma) \chi \tag{6.76}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\|f_{2}\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)}+\left\|f_{1}\right\|_{L_{t}^{1} H_{x^{\prime}}^{s-1}(\Sigma)} \lesssim \epsilon_{2} \tag{6.77}
\end{equation*}
$$

and for any $t \in[0, T]$,

$$
\begin{equation*}
\left\|f_{2}(t, \cdot)\right\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim\|d \mathbf{g}\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)} \tag{6.78}
\end{equation*}
$$

Let $\Lambda$ be the fractional derivative operator in the $x^{\prime}$ variables. We thus have

$$
\begin{align*}
\left\|\Lambda^{s-1}\left(\chi-f_{2}\right)(t, \cdot)\right\|_{L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)} \lesssim & \left\|\left[\Lambda^{s-1}, l\right]\left(\chi-f_{2}\right)\right\|_{L_{t}^{1} L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)}  \tag{6.79}\\
& +\left\|\Lambda^{s-1}\left(f_{1}-\chi^{2}-l(\ln \sigma) \chi\right)\right\|_{L_{t}^{1} L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)}
\end{align*}
$$

A direct calculation shows that

$$
\begin{align*}
\left\|\Lambda^{s-1}\left(f_{1}-\chi^{2}-l(\ln \sigma) \chi\right)\right\|_{L_{t}^{1} L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)} \lesssim & \left\|f_{1}\right\|_{L_{t}^{1} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}+\|\chi\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}^{2}  \tag{6.80}\\
& \left.+\|\chi\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \cdot\|l(\ln \sigma)\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}\right)
\end{align*}
$$

where we use the fact that $H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)$ is an algebra.
We next bound

$$
\begin{aligned}
\left\|\left[\Lambda^{s-1}, l\right]\left(\chi-f_{2}\right)\right\|_{L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)} \leq & \left\|\not \partial_{\alpha} l^{\alpha}\left(\chi-f_{2}\right)(t, \cdot)\right\|_{H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \\
& +\left\|\left[\Lambda^{s-1} \not \partial_{\alpha}, l^{\alpha}\right]\left(\chi-f_{2}\right)(t, \cdot)\right\|_{L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)}
\end{aligned}
$$

By Kato-Ponce commutator estimate and Sobolev embeddings, the above could be bounded by

$$
\begin{equation*}
\left\|l^{1}(t, \cdot)\right\|_{H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}\left\|\Lambda^{s-1}\left(\chi-f_{2}\right)(t, \cdot)\right\|_{L_{x^{\prime}}^{2}\left(\Sigma^{t}\right)} \tag{6.81}
\end{equation*}
$$

Gathering (6.72), (6.73), (6.77), (6.79), (6.80), and (6.81) together, we thus prove that

$$
\sup _{t}\left\|\left(\chi-f_{2}\right)(t, \cdot)\right\|_{H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}
$$

From (6.76), we can see

$$
\begin{equation*}
\left\|\chi-f_{2}\right\|_{C_{x^{\prime}}^{\delta}} \lesssim\left\|f_{1}\right\|_{L_{t}^{1} C_{x^{\prime}}^{\delta}}+\left\|\chi^{2}\right\|_{L_{t}^{1} C_{x^{\prime}}^{\delta}}+\|l(\ln \sigma) \chi\|_{L_{t}^{1} C_{x^{\prime}}^{\delta}} \tag{6.82}
\end{equation*}
$$

Using the Sobolev imbedding $H^{1}(\mathbb{R}) \hookrightarrow C^{\delta}(\mathbb{R})$ and Gronwall's inequality, we can derive that

$$
\|\chi\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}+\|d \mathbf{g}\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)}
$$

6.4. The proof of Proposition 6.2. We first recall that

$$
G(\mathbf{v}, \rho)=\left\|d \phi\left(t, x^{\prime}\right)-d t\right\|_{s, 2, \Sigma}
$$

Using (6.68) and the estimate of $\left\|\|\mathbf{g}-\mathbf{m}\|_{s, 2, \Sigma}\right.$ in Proposition 6.1, then the estimate (6.3) follows from the bound

$$
\left\|l-\left(\partial_{t}-\partial_{x_{2}}\right)\right\|_{s, 2, \Sigma} \lesssim \epsilon_{2},
$$

where it is understood that one takes the norm of the coefficients of $l-\left(\partial_{t}-\partial_{x_{2}}\right)$ in the standard frame on $\mathbb{R}^{2+1}$. The geodesic equation, together with the bound for Christoffel symbols $\left\|\Gamma_{\beta \gamma}^{\alpha}\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\|d \mathbf{g}\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim \epsilon_{2}$, imply that

$$
\left\|l-\left(\partial_{t}-\partial_{x_{2}}\right)\right\|_{L_{t, x}^{\infty}} \lesssim \epsilon_{2}
$$

so it suffices to bound the tangential derivatives of the coefficients of $l-\left(\partial_{t}-\partial_{x_{2}}\right)$ in the norm $L_{t}^{2} H_{x^{\prime}}^{s-1}(\Sigma)$. By Proposition 6.1, we can estimate Christoffel symbols

$$
\left\|\Gamma_{\beta \gamma}^{\alpha}\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}
$$

Note that $H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)$ is a algebra. We then have

$$
\left\|\Gamma_{\beta \gamma}^{\alpha} e_{1}^{\beta} l^{\gamma}\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}
$$

We are now in a position to establish the following bound,

$$
\left\|\left\langle D_{e_{1}} l, e_{1}\right\rangle\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}+\left\|\left\langle D_{e_{1}} l, \underline{l}\right\rangle\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)}+\left\|\left\langle D_{l} l, \underline{l}\right\rangle\right\|_{L_{t}^{2} H_{x^{\prime}}^{s-1}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}
$$

The first term is $\chi$, which has estimated in Lemma 6.12. For the second term, noting

$$
\left\langle D_{e_{1}} l, \underline{l}\right\rangle=\left\langle D_{e_{1}} l, 2 \partial_{t}\right\rangle=-2\left\langle D_{e_{1}} \partial_{t}, l\right\rangle,
$$

then it can be bounded by using Proposition 6.1. Similarly, we can control the last term by proposition 6.1. It remains for us to show that

$$
\left\|d \phi\left(t, x^{\prime}\right)-d t\right\|_{C_{x^{\prime}}^{1, \delta}(\mathbb{R})} \lesssim \epsilon_{2}+\|d \mathbf{g}(t, \cdot)\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)}
$$

To do that, it suffices to establish

$$
\left\|l(t, \cdot)-\left(\partial_{t}-\partial_{x_{2}}\right)\right\|_{C_{x^{\prime}}^{1, \delta}(\mathbb{R})} \lesssim \epsilon_{2}+\|d \mathbf{g}(t, \cdot)\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)}
$$

The coefficients of $e_{1}$ are small in $C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)$ perturbations of their constant-coefficient analogs, so it suffices to show that

$$
\left\|\left\langle D_{e_{1}} l, e_{1}\right\rangle(t, \cdot)\right\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)}+\left\|\left\langle D_{e_{1}} l, \underline{l}\right\rangle(t, \cdot)\right\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim \epsilon_{2}+\|d \mathbf{g}(t, \cdot)\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)}
$$

Above, the first term is bounded by Lemma 6.12, and the second by using

$$
\left\|\left\langle D_{e_{1}} \partial_{t}, l\right\rangle(t, \cdot)\right\|_{C_{x^{\prime}}^{\delta}\left(\Sigma^{t}\right)} \lesssim\|d \mathbf{g}(t, \cdot)\|_{C_{x}^{\delta}\left(\mathbb{R}^{2}\right)}
$$

Consequently, we complete the proof of Proposition 6.2.

## 7. proof of Proposition 5.1 and continuous dependence

7.1. Proof of Proposition 5.1. To prove Proposition 5.1, let us first give a type of Stricharz estimates. In the above sections, we obtain characteristic energy estimates of solutions and get enough regularity of hypersurfaces. By using the result of Smith and Tataru([41], Proposition 7.1, page 36), we can directly obtain the following

Proposition 7.1. Suppose that $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ and $G(\mathbf{v}, \rho) \leq 2 \epsilon_{1}$. For $1 \leq r \leq s+1$, then the linear equation $\square_{g} f=0$ is well-posed with the initial data in $H^{r} \times H^{r-1}$. Moreover, the following estimates

$$
\left\|\langle\partial\rangle^{k} f\right\|_{L_{t}^{4} L_{x}^{\infty}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}}, \quad k<r-\frac{3}{4}
$$

and

$$
\|f\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\partial_{t} f\right\|_{L_{t}^{\infty} H_{x}^{s-1}} \lesssim\left\|f_{0}\right\|_{H^{r}}+\left\|f_{1}\right\|_{H^{r-1}},
$$

hold.
Proposition 7.2. Suppose that $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ and $G(\mathbf{v}, \rho) \leq 2 \epsilon_{1}$. Then $(\mathbf{v}, \rho)$ of (5.2) satisfies the Strichartz estimate

$$
\begin{equation*}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} \leq \epsilon_{2} \tag{7.1}
\end{equation*}
$$

Proof. Note (5.2). Using Duhamel's principle, we can get

$$
\begin{aligned}
\|d \mathbf{v}, d \rho\|_{L_{t}^{4} C_{x}^{\delta}} & \leq C\left(\|\partial \varpi\|_{L_{t}^{1} H_{x}^{s-1}}+\|\mathbf{Q}\|_{L_{t}^{1} H_{x}^{s-1}}+\|\mathbf{E}\|_{L_{t}^{1} H_{x}^{s-1}}+\|\left.\mathcal{D}\right|_{L_{t}^{1} H_{x}^{s-1}}\right) \\
& \leq 4 C\|\partial \varpi\|_{L_{t}^{\infty} H_{x}^{1}}+C[1-(-1)]^{\frac{3}{4}}\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}}\|d \mathbf{v}, d \rho\|_{L_{t}^{\infty} H_{x}^{s-1}} \\
& \leq C\left(\left\|\rho_{0}\right\|_{H^{s}}+\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\partial \varpi_{0}\right\|_{H^{2}}\right) \exp \left(1+\|d \mathbf{v}, d \rho\|_{L_{t}^{4} L_{x}^{\infty}}\right)^{2} \\
& \leq C \epsilon_{3} \leq \epsilon_{2},
\end{aligned}
$$

where we use (5.3) and Lemma 6.5.
Proof of Proposition 5.1. By using Proposition 6.2, we know that (5.5) holds. By using Proposition 7.1, we obtain (5.7) and (5.8). Using (2.10) and (7.1), we have

$$
\begin{align*}
& \|(\mathbf{v}, \rho)\|_{L_{t}^{\infty} H_{x}^{s}}+\left\|\left(\partial_{t} \mathbf{v}, \partial_{t} \rho\right)\right\|_{L_{t}^{\infty} H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty} H_{x}^{2}}+\left\|\partial_{t} \varpi\right\|_{L_{t}^{\infty} H_{x}^{1}} \\
\lesssim & \epsilon_{3}\left(1+\epsilon_{3}^{\frac{1}{2}}\right) \exp \left(\int_{-1}^{1}\left[1+\epsilon_{2}\right]^{2}\right)  \tag{7.2}\\
\leq & \epsilon_{2} .
\end{align*}
$$

The estimate (7.2) combining (7.1) can yield (5.6). Therefore, we complete the proof of Proposition 5.1.
7.2. Continuous dependence. We will discuss the continuous dependence by referring Ifrim-Tataru's paper [20].

Corollary 7.3 (Continuous dependence on data). If $\left(\mathbf{v}_{0 j}, \rho_{0 j}, \varpi_{0 j}\right)$ is a sequence of initial data converging to $\left(\mathbf{v}_{0}, \rho_{0}, \varpi_{0}\right)$ in space $H^{s} \times H^{s} \times H^{2}$, then the associated solutions $\left(\mathbf{v}_{j}, \rho_{j}, \varpi_{j}\right)$ of (5.2) converge uniformly to $(\mathbf{v}, \rho, \varpi)$ on $[0, T]$ in $\in H_{x}^{s} \times$ $H_{x}^{s} \times H_{x}^{2}$. Moreover,

$$
\begin{align*}
&\left\|\left(\mathbf{v}_{j}-\mathbf{v}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(\rho_{j}-\rho\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(\varpi_{j}-\varpi\right)(t)\right\|_{H_{x}^{2}} \\
& \lesssim\left\|\mathbf{v}_{0 j}-\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0 j}-\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0 j}-\varpi_{0}\right\|_{H^{2}} . \tag{7.3}
\end{align*}
$$

Before we prove it, let us introduce a frequency envelope in [20, 48].
Definition 7.1. We say that $\left\{c_{k}\right\}_{k \in \mathbb{N}^{+}} \in \ell^{2}$ is a frequency envelope for a function $f$ in $H^{s}$ if we have the following two properties:
(1) Energy bound:

$$
\begin{equation*}
\left\|P_{k} f\right\|_{H^{s}} \lesssim c_{k} \tag{7.4}
\end{equation*}
$$

(2) Slowly varying:

$$
\begin{equation*}
\frac{c_{k}}{c_{j}} \lesssim 2^{\delta|j-k|}, \quad j, k \in \mathbb{N}^{+} \tag{7.5}
\end{equation*}
$$

We call such envelopes sharp, if

$$
\|f\|_{H^{s}}^{2} \approx \sum_{k \geq 0} c_{k}^{2}
$$

proof of Corollary 7.3. We divide the proof into three steps.
Step 1: the convergence in a weaker space. By using Theorem 2.11, it yields

$$
\left\|\left(\mathbf{v}_{j}-\mathbf{v}, \rho_{j}-\rho\right)(t, \cdot)\right\|_{H_{x}^{s-1}}+\left\|\left(\varpi_{j}-\varpi\right)(t, \cdot)\right\|_{H_{x}^{1}} \lesssim\left\|\left(\mathbf{v}_{0 j}-\mathbf{v}_{0}, \rho_{0 j}-\rho_{0}\right)\right\|_{H^{s}}+\left\|\varpi_{0 j}-\varpi_{0}\right\|_{H^{2}}
$$

Taking $j \rightarrow \infty$, we obtain

$$
\lim _{j \rightarrow \infty}\left(\mathbf{v}_{j}, \rho_{j}\right) \rightarrow(\mathbf{v}, \rho) \text { in } H_{x}^{s-1}, \quad \lim _{j \rightarrow \infty} \varpi_{j} \rightarrow \varpi \text { in } H_{x}^{1}
$$

By interpolation formula, we then have
$\left\|\left(\mathbf{v}_{j}-\mathbf{v}, \rho_{j}-\rho\right)\right\|_{H_{x}^{\sigma}} \lesssim\left\|\left(\mathbf{v}_{j}-\mathbf{v}, \rho_{j}-\rho\right)\right\|_{H_{x}^{s-1}}^{s-\sigma}\left\|\left(\mathbf{v}_{j}-\mathbf{v}, \rho_{j}-\rho\right)\right\|_{H_{x}^{s}}^{1+\sigma-s}, \quad s-1 \leq \sigma<s$.
and

$$
\left\|\varpi_{j}-\varpi\right\|_{H_{x}^{\gamma}} \lesssim\left\|\varpi_{j}-\varpi\right\|_{H_{x}^{1}}^{2-\gamma}\left\|\varpi_{j}-\varpi\right\|_{H_{x}^{2}}^{\gamma-1}, \quad 1 \leq \gamma<2
$$

As a result, we get

$$
\lim _{j \rightarrow \infty}\left(\mathbf{v}_{j}, \rho_{j}\right) \rightarrow(\mathbf{v}, \rho) \text { in } H_{x}^{\sigma}, \quad 0 \leq \sigma<s
$$

and

$$
\lim _{j \rightarrow \infty} \varpi_{j} \rightarrow \varpi \text { in } H_{x}^{\gamma}, \quad 0 \leq \gamma<2
$$

Step 2: the construction of smooth solutions. Consider the initial data $\mathbf{v}_{0}=$ $\left(v_{0}^{1}, v_{0}^{2}\right)$ and $\rho_{0} \in H^{s}$. We set $\mathbf{U}_{0}=\left(v_{0}^{1}, v_{0}^{2}, \rho_{0}\right)$. By [20], there exists a sharp frequency envelope for $v_{0}^{1}, v_{0}^{2}$, and $\rho$ respectively. Let $\left\{c_{k}^{(i)}\right\}_{k \geq 0}(i=1,2,3)$ be a sharp frequency envelope for $v_{0}^{1}, v_{0}^{2}, \rho_{0}$ in $H^{s}$. Set $\mathbf{C}_{k}=\left(c_{k}^{(1)}, c_{k}^{(2)}, c_{k}^{(3)}\right)$. We choose
a family of regularizations $\mathbf{U}_{0}^{h}=\left(v_{0}^{1 h}, v_{0}^{2 h}, \rho_{0}^{h}\right) \in H^{\infty}:=\cap_{s=0}^{\infty} H^{s}$ at frequencies $\lesssim 2^{h}$ where $h$ is a dyadic frequency parameter. Denote

$$
\varpi_{0}^{h}=\bar{\rho}^{-1} \mathrm{e}^{-\rho_{0}^{h}} \operatorname{curlv}_{0}^{h} .
$$

Then we have $\varpi_{0}^{h} \in H^{\infty}$. At the same time, there exists a sharp frequency for $\varpi_{0}^{h}$, we record it $\left\{c_{k}^{(4)}\right\}_{k \geq 0}$. Also, the function $v_{0}^{1 h}, v_{0}^{2 h}, \rho_{0}^{h}, \varpi_{0}^{h}$, and $\varpi_{0}^{h}$ has the following properties:
(i) uniform bounds

$$
\left\|P_{k} v_{0}^{i h}\right\|_{H^{s}} \lesssim c_{k}^{(i)}, i=1,2, \quad\left\|P_{k} \rho_{0}^{h}\right\|_{H^{s}} \lesssim c_{k}^{(3)}, \quad\left\|P_{k} \varpi_{0}^{h}\right\|_{H^{2}} \lesssim c_{k}^{(4)}
$$

(ii) high frequency bounds

$$
\left\|v_{0}^{i h}\right\|_{H^{s+j}} \lesssim 2^{j h} c_{h}^{(i)}, i=1,2, \quad\left\|\rho_{0}^{h}\right\|_{H^{s+j}} \lesssim 2^{j h} c_{h}^{(3)}, \quad\left\|\varpi_{0}^{h}\right\|_{H^{2+j}} \lesssim 2^{j h} c_{h}^{(4)}
$$

(iii) difference bounds

$$
\begin{aligned}
& \left\|v_{0}^{i(h+1)}-v_{0}^{i h}\right\|_{L^{2}} \lesssim 2^{-s h} c_{h}^{(i)}, \quad i=1,2, \\
& \left\|\rho_{0}^{h+1}-\rho_{0}^{h}\right\|_{L^{2}} \lesssim 2^{-s h} c_{h}^{(3)}, \quad\left\|\varpi_{0}^{h+1}-\varpi_{0}^{h}\right\|_{L^{2}} \lesssim 2^{-2 h} c_{h}^{(4)},
\end{aligned}
$$

(iv) limit

$$
\begin{array}{ll}
\mathbf{U}_{0}=\lim _{h \rightarrow \infty} \mathbf{U}_{0}^{h} & \text { in } H^{s},  \tag{7.6}\\
\varpi_{0}=\lim _{h \rightarrow \infty} \varpi_{0}^{h} & \text { in } H^{2} .
\end{array}
$$

Taking the smooth initial data $\left(v_{0}^{1 h}, v_{0}^{2 h}, \rho_{0}^{h}, \varpi_{0}^{h}\right)$, we obtain a family of smooth solutions ( $v^{1 h}, v^{2 h}, \rho^{h}, \varpi^{h}$ ) satisfying (5.2). Based on the existence of (5.2), this yields a time interval $[0, T]$ where all these solutions $\left(v^{1 h}, v^{2 h}, \rho^{h}, \varpi^{h}\right)$ exists, and $T$ depends only on the size of $\left\|\mathbf{v}_{0}\right\|_{H^{s}}+\left\|\rho_{0}\right\|_{H^{s}}+\left\|\varpi_{0}\right\|_{H^{2}}$. Furthermore, we have:
(i) high frequency bounds

$$
\begin{equation*}
\left\|v^{i h}\right\|_{H_{x}^{s+j}} \lesssim 2^{j h} c_{h}^{(i)}, \quad\left\|\rho^{h}\right\|_{H_{x}^{s+j}} \lesssim 2^{j h} c_{h}^{(3)}, \quad\left\|\varpi^{h}\right\|_{H_{x}^{2+j}} \lesssim 2^{j h} c_{h}^{(4)} \tag{7.7}
\end{equation*}
$$

(ii) difference bounds

$$
\begin{equation*}
\left\|v^{i(h+1)}-v^{i h}\right\|_{L_{x}^{2}} \lesssim 2^{-s h} c_{h}^{(i)}, \quad\left\|\rho^{h+1}-\rho^{h}\right\|_{L_{x}^{2}} \lesssim 2^{-s h} c_{h}^{(3)}, \quad\left\|\varpi^{h+1}-\varpi^{h}\right\|_{L_{x}^{2}} \lesssim 2^{-2 h} c_{h}^{(4)} \tag{7.8}
\end{equation*}
$$

Taking the convergence $h \rightarrow \infty$ on (7.8), we get

$$
\left\|\mathbf{v}-\mathbf{v}^{h}\right\|_{L_{x}^{2}} \lesssim 2^{-s h}, \quad\left\|\rho-\rho^{h}\right\|_{L_{x}^{2}} \lesssim 2^{-s h}, \quad\left\|\varpi-\varpi^{h}\right\|_{L_{x}^{2}} \lesssim 2^{-2 h}
$$

where $\mathbf{v}^{h}=\left(v^{1 h}, v^{2 h}\right)$. By using

$$
\rho-\rho^{h}=\sum_{m=h}^{\infty} \rho^{m+1}-\rho^{m},
$$

we obtain

$$
\left\|\rho-\rho^{h}\right\|_{H_{x}^{s}} \lesssim c_{\geq h}^{(3)}:=\left(\sum_{m \geq h}\left[c_{m}^{(3)}\right]^{2}\right)^{\frac{1}{2}}
$$

Similarly, we also have

$$
\left\|v^{i}-v^{i h}\right\|_{H_{x}^{s}} \lesssim c_{\geq h}^{(i)}, \quad\left\|\rho-\rho^{h}\right\|_{H_{x}^{s}} \lesssim c_{\geq h}^{(3)}, \quad\left\|\varpi-\varpi^{h}\right\|_{H_{x}^{2}} \lesssim c_{\geq h}^{(4)}
$$

Step 3: the convergence. Based on these facts above, we deduce that

$$
\begin{align*}
\left\|\left(\mathbf{v}_{j}-\mathbf{v}\right)(t)\right\|_{H_{x}^{s}} \lesssim & \left\|\left(v_{j}^{1 h}-v^{1 h}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(v^{1 h}-v^{1}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(v_{j}^{1 h}-v_{j}^{1}\right)(t)\right\|_{H_{x}^{s}}  \tag{7.9}\\
& +\left\|\left(v_{j}^{2 h}-v^{2 h}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(v^{2 h}-v^{2}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(v_{j}^{2 h}-v_{j}^{2}\right)(t)\right\|_{H_{x}^{s}}
\end{align*}
$$

and

$$
\begin{equation*}
\left\|\left(\rho_{j}-\rho\right)(t)\right\|_{H_{x}^{s}} \lesssim\left\|\left(\rho_{j}^{h}-\rho^{h}\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(\rho^{h}-\rho\right)(t)\right\|_{H_{x}^{s}}+\left\|\left(\rho_{j}^{h}-\rho_{j}\right)(t)\right\|_{H_{x}^{s}} \tag{7.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(\varpi_{j}-\varpi\right)(t)\right\|_{H_{x}^{2}} \lesssim\left\|\left(\varpi_{j}^{h}-\varpi^{h}\right)(t)\right\|_{H_{x}^{2}}+\left\|\left(\varpi^{h}-\varpi\right)(t)\right\|_{H_{x}^{2}}+\left\|\left(\varpi_{j}^{h}-\varpi_{j}\right)(t)\right\|_{H_{x}^{2}} \tag{7.11}
\end{equation*}
$$

Now, let us first estimate (7.10). Taking the limit for $j \rightarrow \infty$, this leads to

$$
\begin{equation*}
\left\|\left(\rho^{h}-\rho\right)(t)\right\|_{H_{x}^{s}} \rightarrow 0, \quad j \rightarrow \infty, \quad \text { for fixed } h \tag{7.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(\rho_{j}^{h}-\rho_{j}\right)(t)\right\|_{H_{x}^{2}} \rightarrow 0, \quad j \rightarrow \infty \tag{7.13}
\end{equation*}
$$

Let $\left\{c_{k}^{(i) j}\right\}_{k \geq 0}$ be frequency envelopes for the initial data $v_{0 j}^{i}$ in $H^{s}, i=1,2$. Let $\left\{c_{k}^{(3) j}\right\}_{k \geq 0}$ be frequency envelopes for the initial data $\rho_{0 j}$ in $H^{s}$. Let $\left\{c_{k}^{(4) j}\right\}_{k \geq 0}$ be frequency envelopes for the initial data $\varpi_{0 j}$ in $H^{2}$. By (7.12) and (7.13), we can update (7.10) by

$$
\begin{equation*}
\left\|\left(\rho_{j}-\rho\right)(t)\right\|_{H_{x}^{s}} \lesssim\left\|\left(\rho_{j}^{h}-\rho^{h}\right)(t)\right\|_{H_{x}^{s}}+c_{\geq h}^{(3)}+c_{\geq h}^{(3) j} \tag{7.14}
\end{equation*}
$$

On the other hand, we know

$$
\begin{equation*}
\rho_{0 j}^{h} \rightarrow \rho_{0}^{h} \quad \text { in } H_{x}^{\sigma}, 0 \leq \sigma<\infty \tag{7.15}
\end{equation*}
$$

By using a similar way in Step 1, we can derive that

$$
\begin{equation*}
\rho_{j}^{h} \rightarrow \rho^{h} \quad \text { in } H_{x}^{\sigma}, 0 \leq \sigma<\infty \tag{7.16}
\end{equation*}
$$

From (7.15), it yields

$$
\begin{equation*}
c_{k}^{(3) j} \rightarrow c_{k}^{(3)}, j \rightarrow \infty \tag{7.17}
\end{equation*}
$$

Therefore, using (7.15), (7.16), (7.17), and passing to the limit $j \rightarrow \infty$ for (7.14), we have

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left\|\left(\rho_{j}-\rho\right)(t)\right\|_{H_{x}^{s}} \lesssim c_{\geq h}^{(3)} \tag{7.18}
\end{equation*}
$$

Taking $h \rightarrow \infty$ in (7.18), we derive

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left\|\left(\rho_{j}-\rho\right)(t)\right\|_{H_{x}^{s}}=0 \tag{7.19}
\end{equation*}
$$

In a similar process, we can also obtain

$$
\lim _{j \rightarrow \infty}\left\|\left(\mathbf{v}_{j}-\mathbf{v}\right)(t)\right\|_{H_{x}^{s}}=0, \quad \lim _{j \rightarrow \infty}\left\|\left(\varpi_{j}-\varpi\right)(t)\right\|_{H_{x}^{2}}=0
$$

At this stage, we have finished the proof of Theorem 1.2.

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