LOW REGULARITY SOLUTIONS OF TWO-DIMENSIONAL COMPRESSIBLE EULER EQUATIONS WITH DYNAMIC VORTICITY

HUALI ZHANG

ABSTRACT. By establishing a sharp Strichartz estimate for the velocity and density, we prove the local well-posedness of solutions for the Cauchy problem of two-dimensional compressible Euler equations, where the initial velocity, density, and specific vorticity $(\mathbf{v}_0, \rho_0, \varpi_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, $s > \frac{7}{4}$. Our strategy relies on Smith-Tataru's work [41] for quasi-linear wave equations.

1. INTRODUCTION

1.1. **Overview.** We consider the Cauchy's problem of the compressible Euler equations in $\in \mathbb{R}^+ \times \mathbb{R}^2$, of the form

(1.1)
$$\begin{cases} \varrho_t + \operatorname{div}(\varrho \mathbf{v}) = 0, \\ \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\varrho} \nabla p(\varrho) = 0, \end{cases}$$

where the state function takes the general form

$$(1.2) p = p(\varrho),$$

and the initial data is

(1.3)
$$(\mathbf{v}, \varrho)|_{t=0} = (\mathbf{v}_0, \varrho_0).$$

Above, $\mathbf{v} = (v^1, v^2), \varrho$, and p denote the fluid velocity, density, and pressure respectively, and A is a constant. In the theory of partial differential equations, local well-posedness is the first question to ask. For compressible Euler equations, no matter how smooth and small the initial data is, the solution of (1.1) will blow up in finite time [10, 32, 39, 43]. So we can only study the well-posedness of (1.1)-(1.3) in a local sense. In many problems of this type, one is interested not only in local well-posedness in some Sobolev space $H^s(\mathbb{R}^2)$, but also in lowering the exponent s as much as possible. Naturally, we ask the question: for which s_c , the Cauchy problem (1.1)-(1.3) is well-posed if $(\mathbf{v}_0, \varrho_0) \in H^s(\mathbb{R}^2)(s \leq s_c)$ and ill-posed if $(\mathbf{v}_0, \varrho_0) \in H^s(\mathbb{R}^2)(s \leq s_c)$. This question has been well studied [6, 41, 31] for incompressible Euler equations and irrotational Euler equations. However, for (1.1)-(1.3) with non-zero vorticity, the corresponding problem remains open. Our goal is to study the local well-posedness of low regularity solutions to (1.1)-(1.3) and explore the sharp Sobolev exponent.

Date: January 10, 2022.

²⁰¹⁰ Mathematics Subject Classification. 76N10, 35R05, 35L60.

 $Key\ words\ and\ phrases.$ compressible Euler equations, low regularity solutions, a wave-transport system, Strichartz estimate.

1.2. **Background.** The compressible Euler equations is a classical system in physics to describe the motion of an idea fluid. The phenomena displayed in the interior of a fluid fall into two broad classes, the phenomena of acoustics waves and the phenomena of vortex motion. The sound phenomena depend on the compressibility of a fluid, while the vortex phenomena occur even in a regime where the fluid may be considered to be incompressible.

For the Cauchy problem of n-D incompressible Euler equations:

(1.4)
$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \, \mathbf{v} + \nabla p = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \operatorname{div} \mathbf{v} = 0, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0, \end{cases}$$

Kato and Ponce [23] proved the local well-posedness of (1.4) if $\mathbf{v}_0 \in H^s(\mathbb{R}^n), s > 1 + \frac{n}{2}$. Chae [8] proved the local existence of solutions by setting \mathbf{v}_0 in the critical Triebel-Lizorkin space. On the opposite direction, the ill-posedness of solutions of (1.1) was answered by Bourgain and Li [6, 7], who proved that the solution will blow up instantaneously for some $\mathbf{v}_0 \in H^{1+\frac{n}{2}}(\mathbb{R}^n), n = 2, 3$. Very recently, Guo-Li in [16] studied the continuous dependence of initial data in the critical Triebel-Lizorkin space.

In the irrotational case, the compressible Euler equations can be reduced to a special quasilinear wave equation. For general quasilinear wave equations, it can be stated as

(1.5)
$$\begin{cases} \Box_{h(\phi)}\phi = q(d\phi, d\phi), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \phi|_{t=0} = \phi_0, \partial_t \phi|_{t=0} = \phi_1, \end{cases}$$

where ϕ is a scalar function, $d = (\partial_t, \partial_1, \partial_2, \dots, \partial_n)$, and $h(\phi)$ a Lorentzian metric depending on ϕ , and q a quadratic term of $d\phi$. Set the initial data $(\phi_0, \phi_1) \in$ $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$. By using classical energy methods and Sobolev imbeddings, Hughes-Kato-Marsden [18] proved the local well-posedness of the problem (1.5) for $s > \frac{n}{2} + 1$. Ifrim-Tataru [20] studied this reslut for quasilinear hyperbolic equations by using the frequency envelope approach, where the frequency envelope is introduced by Tao [48]. On the other side, Lindblad [31] constructed some counterexamples for (1.5) when $s = \frac{7}{4}, n = 2$ or s = 2, n = 3. There is a gap between the result [18] and [31]. To lower the regularity of the initial data, one may seek a type of space-time estimates of $d\phi$, namely, Strichartz estimates. Of course there are several steps to obtain the sharp Strichartz estimates for (1.5). The first natural idea is to consider the the wave equation with variable coefficients

$$(1.6)\qquad \qquad \Box_{h(t,x)}\phi = 0,$$

and then exploit it to obtain the low regularity solutions of (1.5). Kapitanskij [24] and Mockenhaupt-Seeger-Sogge [38] discussed the Strichartz estimates for (1.6) with smooth coefficients h. With rough coefficients $h \in C^2$, the study of Strichartz estimates for (1.6) in two or three dimensions was begin with Smith's result [43]. At the same time, counterexamples was constructed by Smith-Sogge [44], who showed that for $\alpha < 2$ there exist $h \in C^{\alpha}$ for which the Strichartz estimates fail. Later, the Strichartz estimates were established in all dimensions for $h \in C^2$ in Tataru [50]. The next important work was independently achieved by Bahouri-Chemin [4] and Tataru [49], who established the local well-posedness of (1.5) with $s > \frac{n}{2} + \frac{7}{8}, n = 2$ or $s > \frac{n}{2} + \frac{3}{4}, n \geq 3$. Shortly afterward, Tataru [51] relaxed the Sobolev indices $s > \frac{n+1}{2} + \frac{1}{6}, n \ge 3$. At the same time, Smith-Tataru [42] showed that the $\frac{1}{6}$ loss is sharp for general variable coefficients h. Thus, to improve the above results, one needs to exploit a new way or structure of Equation (1.5). Through introducing a vector-field approach and a decomposition of the Ricci curvature, the 3D result of [4, 49, 50, 51] was later improved by Klainerman-Rodnianski [26], who proved the local well-posedness of (1.5) by introducing vector-field methods for $s > 2 + \frac{2-\sqrt{3}}{2}$. Based on the vector-field methods, Geba [21] studied the local well-posedness of (1.5) in two dimensions for $s > \frac{7}{4} + \frac{5-\sqrt{22}}{4}$. By using wave packets of a localization to represent solutions to the linear equation, a sharp result was proved by Smith-Tataru [41], who established the local well-posedness of (1.5) if $s > \frac{7}{4}, n = 2$ or s > 2, n = 3 or $s > \frac{n+1}{2}, 4 \le n \le 6$. An alternative proof of the 3D case was also obtained through vector-field approach by Wang [57]. Besides, we should also mention substantial significant progress which has been made on low regularity solutions of Einstein vacuum equations, membrane equations, due to Andersson and Moncreif [3], Tataru [52], Ettinger and Lindblad [6], Klainerman and Rodnianski [26], Klainerman-Rodnianski-Szefel [27], Wang [54], Allen-Andersson-Restuccia [2], Speck [45] and so on.

In the general case, concerning to n-D compressible Euler equations, there are several aspects on studying the Cauchy problem (1.1)-(1.3), i.e. shock formation and local well-posedness. The first work on the formation of shocks was done by Riemann in [39]. Riemann considered the case of isentropic flow with plane symmetry and introduced for such systems the so-called Riemann invariants, and then proved that solutions will blow up in finite time even under smooth initial conditions. Sideris [43] considered the three dimensional compressible Euler equations and obtained the first general result on the formation of singularity. By extending the basic idea of [39], Christodoulou-Miao [10] started from geometric aspects to study the shock formation of irrotational and isentropic flow in 3D, and gave a complete description of the maximal classical development. Yin in [59] constructed a class of spherical data to discuss the formation of shock wave in three dimensions. For multi-dimensional solutions with spherical symmetry, the blow-up phenomena was obtained by Li-Wang [29]. Recently, Luk-Speck [32, 33, 46] first introduced a wave-transport structure of the flow with dynamic vorticity and entropy, and described the singularity formation in two or three dimensions. Abbrescia-Speck in [1] studied some localized integral identities for 3D compressible Euler equations. We should also mention substantial significant progress which has been made on self similar solutions due to Merle-Raphael-Rodnianski-Szeftel [36], and free boundary problems due to Coutand-Lindblad-Shkoller [11], Coutand-Shkoller [11, 13], Lei-Du-Zhang [28], Li-Wang [30], Jang-Masmoudi [17], Ifrim-Tataru [19] and so on.

To the local well-posedness problem of (1.1)-(1.3), it's well-posed if $(\mathbf{v}_0, \varrho_0) \in H^s, s > 1 + \frac{n}{2}$ and the density is far away from vacuum, please refer Majda's book [35]. Very rencently, based on the wave-transport system proposed by Luk and Speck [32, 33, 46], some researchers considered the well-posedness of rough solutions for (1.1)-(1.3) by studying Strichartz estimates, which arises from dispersive equations. We refer the reader to Strichartz's work [47]). The first work about rough solutions of three-dimensional compressible Euler equations was obtained by Disconzi-Luo-Mazzone-Speck [14] and Wang [58]. In [14], Disconzi-Luo-Mazzone-Speck proved the well-posedness of solutions with dynamic vorticity and entropy, where they assumed the initial entropy e, velocity \mathbf{v} , logarithmic density

 ρ and specific vorticity $\mathbf{w}(\text{it will be defined in Definition 1.2})$ in $H^{3+} \times (H^{2+})^3$ and $\operatorname{curl} \mathbf{w} \in C^{0,\delta}(0 < \delta < 1)$. Independently, Wang [58] proved the local well-posedness by taking the initial data of $(\mathbf{v}, \rho, \mathbf{w}) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^{s'}(\mathbb{R}^3)$, 2 < s' < s. The works [14] and [58] are based on vector-field approach. Later, Zhang-Andersson [61] gave an alternative proof of [58] by using Smith-Tataru's method [41]. However, there is a few result with regard to low regularity solutions of the Cauchy problem of two-dimensional compressible Euler equations. Inspired by these historical results, we wish to study the low regularity solutions of (1.1)-(1.3) by establishing sharp Strichartz estimates of the velocity and density in two dimensions.

1.3. Motivation. In view of the aforementioned results we see that most studies are focusing on the behavior of solutions of 3D compressible Euler equations. These historical results [14, 58] related to rough solutions in three dimensions successfully exploited the vector-field method in the case of non-zero vorticity, where the regularity of velocity and density are optimal. One may ask that what's the sharp Sobolev regularity of the initial data for 2D compressible Euler equations if controlling it's local well-posedness, and whether the vector-field method could solve the 2D problem. In fact, the vector-field methods may not work very well for 2D problem, for the conformal energy in 2D is not ideal. We persuade the readers to Geba's work [21]. But, we noticed that the sharp regularity problem of 2D quasilinear wave equations is included in Smith-Tataru [41]. Our starting point is the result of Smith-Tataru [41], which, for generic nonlinear wave equations in two dimensions, yields the sharp local well-posedness in $H^{\frac{7}{4}+}$. However, this result can not be directly applied in the case of compressible Euler equations unless the fluid is assumed to be irrotational. Instead, in the general case, the compressible Euler flow can be seen as a coupling of a wave equation and a transport equation for the vorticity, which causes many difficulties. Let us explain the difficulty of the problem and the difference between quasilinear wave equations and compressible Euler equations.

To lower the Sobolev exponent of Cauchy problem (1.1)-(1.3), the key is to prove a type of Strichartz estimates. If the vorticity is zero, one could observe that there is a type of Strichartz estimates $||d\mathbf{v}, d\rho||_{L_t^4 L_x^\infty}$ from Smith-Tataru's result [41], where the regularity *s* should be greater than $\frac{7}{4}$. With non-zero vorticity, what's the situation of Strichartz estimate. In particular, there are no Strichartz estimates for the vorticity. Let us see the coupled system. Precisely, $\partial \varpi$ is a source term in the wave equation,

$$\Box_a v = \partial \varpi + \text{l.o.t.}$$

and ϖ satisfies

$$\partial_t \boldsymbol{\varpi} + \mathbf{v} \cdot \nabla \boldsymbol{\varpi} = 0$$

By utilizing the method of proving Strichartz estimates for wave equations, we know that the character is crucial. Although there is independent of ϖ for energy estimates of \mathbf{v} and ρ , but $\partial \varpi$ plays essential role for character. Hence, we need some energy estimates of ϖ . By classical commutator estimates, the condition $\partial \varpi \in L_x^{\infty}$ is essential for us to get the estimate of $\|\varpi\|_{H^a}$, $a \in (1, 2]$. In Zhang's first paper [61], Zhang proved that the local solution is well-posed if the initial velocity, density and specific vorticity $(\mathbf{v}_0, \rho_0, \varpi_0) \in H^s(s > \frac{7}{4})$ and $\partial \varpi \in L_x^{\infty}$. Inspired by [58], we also hope to find some good structure of the vorticity and lower the regularity of vorticity, i.e. remove the initial assumption on $\|\partial \varpi\|_{L_x^{\infty}}$. To be precise, by setting $(\mathbf{v}_0, \rho_0, \varpi_0) \in H^s \times H^s \times H^2(s > \frac{7}{4})$, we discuss the local existence, uniqueness and continuous dependence of solutions of the Cauchy problem (1.1)-(1.3), where \mathbf{v}_0 , ρ_0 , and ϖ_0 describe the initial velocity, density, and specific vorticity respectively.

1.4. Statement of the result. Before stating our result, let us introduce some following quantities and introduce a equivalent system of (1.1).

1.4.1. *Some definitions.* Let us first recall the classical Hadamard standard for well-posedness.

Definition 1.1. [20] The problem (1.1)-(1.3) is locally well-posed in a Sobolev space X if the following properties are satisfied:

(i) For each $(\mathbf{v}_0, \varrho_0)$ there exists some time T > 0 and a solution $(\mathbf{v}, \rho) \in C([0,T]; X)$.

(ii) The above solution is unique.

(iii) The data to solution map is continuous from X into C([0,T];X).

In the following, let us introduce the logarithmic density, specific vorticity, the speed of sound, and the acoustic metric.

Definition 1.2. [32] Let $\bar{\rho}$ be a constant background density and $\bar{\rho} > 0$. We denote the logarithmic density ρ

(1.7)
$$\rho := \ln \left(\bar{\rho}^{-1} \varrho \right),$$

and the specific vorticity ϖ

(1.8) $\varpi := \bar{\rho}^{-1} \mathrm{e}^{-\rho} \mathrm{curl} \mathbf{v}.$

Definition 1.3. [32] We denote the speed of sound

(1.9)
$$c_s := \sqrt{dp/d\varrho}$$

In view of (1.7), we have

 $(1.10) c_s = c_s(\varrho)$

(1.11)
$$c'_s = c'_s(\rho) := \frac{dc_s}{d\rho}$$

Definition 1.4. [32] We define the acoustical metric g and the inverse acoustical metric g^{-1} relative to the Cartesian coordinates as follows:

(1.12)
$$g := -dt \otimes dt + c_s^{-2} \sum_{a=1}^2 (dx^a - v^a dt) \otimes (dx^a - v^a dt),$$

(1.13)
$$g^{-1} := -(\partial_t + v^a \partial_a) \otimes (\partial_t + v^b \partial_b) + c_s^2 \sum_{i=1}^2 \partial_i \otimes \partial_i.$$

Based on these definitions, let us introduce the system under new variables.

Lemma 1.1. [32] For 2D compressible Euler equations (1.1), it can be reduced to the following equations:

(1.14)
$$\begin{cases} \mathbf{T}v^i = c_s^2 \delta^{ia} \partial_a \rho, \\ \mathbf{T}\rho = -\mathrm{div}\mathbf{v}, \end{cases}$$

where $\mathbf{T} = \partial_t + \mathbf{v} \cdot \nabla$.

To be simple, we give the notations $d = (\partial_t, \partial_{x_1}, \partial_{x_2})^{\mathrm{T}}$, $\partial_{x_0} = \partial_t$ and $\partial = (\partial_{x_1}, \partial_{x_2})^{\mathrm{T}}$. Set

(1.15)
$$\delta \in (0, s - \frac{7}{4}),$$

and

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}, \ \xi \in \mathbb{R}^2.$$

Denote by $\langle \partial \rangle$ the corresponding Bessel potential multiplier. We are now ready to state the result in this paper.

1.4.2. Statement of Results.

Theorem 1.2. Let $s > \frac{7}{4}$. Consider the following Cauchy problem of two-dimensional compressible Euler equations

(1.16)
$$\begin{cases} \mathbf{T}v^{i} = c_{s}^{2}\delta^{ia}\partial_{a}\rho, \\ \mathbf{T}\rho = -\mathrm{div}\mathbf{v}, \\ (\mathbf{v},\rho)|_{t=0} = (\mathbf{v}_{0},\rho_{0}) \end{cases}$$

Assume the acoustical speed

$$(1.17) c_s|_{t=0} > c_0 > 0,$$

where c_0 is a positive constant. Let ϖ be defined in (1.8) and M_0 be any positive constant. If

(1.18)
$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le M_0,$$

then there exists two positive constants $T_* = T(\|\mathbf{v}_0\|_{H^s}, \|\rho_0\|_{H^s}, \|\varpi_0\|_{H^2})$ and M_1 such that the Cauchy problem (1.16) is locally well-posed. Precisely,

(1) there exists a unique solution $(\mathbf{v}, \rho) \in C([0, T_*], H_x^s) \cap C^1([0, T_*], H_x^{s-1}), \varpi \in C([0, T_*], H_x^2) \cap C^1([0, T_*], H_x^1)$ and $(d\mathbf{v}, d\rho) \in L^4_{[0, T_*]}L^\infty_x$, and it satisfies the energy estimate

$$\|\mathbf{v},\rho\|_{L_{t}^{\infty}H_{x}^{s}}+\|\partial_{t}\mathbf{v},\partial_{t}\rho\|_{L_{t}^{\infty}H_{x}^{s-1}}+\|\varpi\|_{L_{t}^{\infty}H_{x}^{2}}+\|\partial_{t}\varpi\|_{L_{t}^{\infty}H_{x}^{1}}\leq M_{1},$$

(2) the solution \mathbf{v} and ρ satisfy the Strichartz estimate

$$\|d\mathbf{v}, d\rho\|_{L^4_t L^\infty_n} \le M_1.$$

(3) for any $1 \le r \le s+1$, and for each $t_0 \in [0,T]$, the linear equation

(1.19)
$$\begin{cases} \Box_g f = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^3, \\ f(t_0, \cdot) = f_0 \in H^r(\mathbb{R}^2), \quad \partial_t f(t_0, \cdot) = f_1 \in H^{r-1}(\mathbb{R}^2), \end{cases}$$

admits a solution $f \in C([0,T], H^r) \times C^1([0,T], H^{r-1})$ and the following estimates hold:

(1.20)
$$\|f\|_{L^{\infty}_{t}H^{r}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{r-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}}.$$

Additionally, the following estimates hold, provided $k < r - \frac{3}{4}$,

(1.21)
$$\|\langle \partial \rangle^k f\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}.$$

(4) the map is continued from $(\mathbf{v}_0, \rho_0, \varpi_0) \in H^s \times H^s \times H^2$ to $(\mathbf{v}, \rho, \varpi)(t, \cdot) \in C([0,T]; H^s_x \times H^s_x \times H^2_x)$.

Remark 1.1. The condition (1.17) is used to satisfy the hyperbolicity condition of the system (1.16).

 $\mathbf{6}$

Remark 1.2. For 2D compressible Euler equations, the classical result in [35] requires s > 2 for the regularity of velocity and density. Our result lower the regularity of the velocity and density by proving the space-time Strichartz estimates of the velocity and density. Furthermore, if the vorticity is zero(the specific vorticity is also zero), the Sobolev regularity in Theorem 1.2 is corresponding to the 2D sharp result by Smith-Tataru [41].

Remark 1.3. Compared with the prior works for 3D compressible Euler equations, *i.e.*[14, 58], the situation in 2D is very different. And it's very hard to use the similar approach in [14, 58] to prove Theorem 1.2, even for the irrotational case in 2D. Because the conformal energy in 2D is worse than 3D, which give a sacrifice on some regularity loss on metrics, please refer Geba's result [21].

Remark 1.4. The idea of deriving a good structure of $\mathbf{T} (\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi})$ in the paper is inspired by Wang [58], but our process is not trivial. Referring [58], the good structure is benefit from curl(e^{ρ} curl \mathbf{w}), not $\Delta \mathbf{w}$. For the structure of div $\boldsymbol{\varpi}$ is very good, but $\mathbf{T}(div\mathbf{w})$ gives us some regularity loss. Then, choosing some quantities related to $\Delta \mathbf{w}$ may not work in three dimensions. Therefore, it's not obvious to choose the quantity $\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}$ in two dimensions.

Remark 1.5. Inspired by Andersson-Moncrief [3] and Ifrim-Tataru [20], we consider the continuous dependence of solutions for (1.16). In [3], Andersson-Moncrief studied the local well-posedness of a hyperbolic-elliptic system. In [20], Ifrim-Tataru established the local well-posedness theory for general hyperbolic equations by using the frequency envelope approach.

1.5. A sketch of the proof. In effect our discussion below is more based on the idea of Smith-Tataru's work [41]. We will adopt two classes of equivalent structure of 2D compressible Euler equations: the hyperbolic system

$$A_0(\mathbf{U})\mathbf{U}_t + A_1(\mathbf{U})\mathbf{U}_{x_1} + A_2(\mathbf{U})\mathbf{U}_{x_2} = 0, \quad \mathbf{U} = (\mathbf{v}, p(\rho))^{\mathrm{T}},$$

and the wave-transport system

$$\begin{cases} \Box_g \mathbf{v} = \partial \varpi + \text{quadratic terms}, \\ \Box_g \rho = \text{quadratic terms}, \\ \mathbf{T} \varpi = 0, \end{cases}$$

where the hyperbolic system is used to consider some energy estimates, and the wave-transport system is used to discuss the Strichartz estimate.

The first key point is how to obtain energy estimates when the Sobolev indices of (\mathbf{v}_0, ρ) and ϖ_0 is different. We use the hyperbolic system to derive the basic energy

$$\|\mathbf{v}, \rho\|_{H^s} \le \|\mathbf{v}_0, \rho_0\|_{H^s} \exp(\|d\mathbf{v}, d\rho\|_{L^1_t L^\infty_\infty}), \quad s \ge 0.$$

Concerning to the transport equation of specific vorticity $\mathbf{T}\boldsymbol{\varpi} = 0$, it looks impossible for us to obtain some energy estimates if the regularity between \mathbf{v}_0 and $\boldsymbol{\varpi}$ are different. By deriving the nonlinear transport equation of $\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}$, we could see a hope. That is,

$$\Gamma \Delta \varpi = \Delta v \partial \varpi + \partial v \partial^2 \varpi + \text{l.o.t},$$

replaced by

(1.22)
$$\mathbf{T}(\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}) = \partial \mathbf{v} \partial^2 \boldsymbol{\varpi} + \partial \mathbf{v} \partial \rho \partial \boldsymbol{\varpi}.$$

In the first equation, one need the norm $\|\partial \varpi\|_{L^{\infty}_x}$ to obtain energy estimates by utilizing standard commutator estimates, and the regularity of velocity and vorticity should be same. While, if we use the second equation (1.22), it allows us to close the basic energy estimates of ϖ by using Strichartz estimates $\|d\mathbf{v}, d\rho\|_{L^4_t L^\infty_x}$ and some lower-order norms of the velocity and density. Please see the proofs in Lemma 2.8, Lemma 2.28, and Theorem 2.10 for details.

The second key point is to prove the Strichartz estimate. We first reduce the problem to establish an existence result for small, supported initial data. Next, by the continuity method, we can give a bootstrap argument on the regularity of the solutions to the nonlinear equation. Then, by introducing null hypersurfaces, the key is transformed to prove characteristic energy estimates of solutions along null hypersurfaces, and the enough regularity of null hypersurfaces is crucial to prove the Strichartz estimate. To establish characteristic energy estimates, we go back to see the wave-transport system and hyperbolic system. We use the hyperbolic system to get these characteristic energy estimate for (\mathbf{v}, ρ) , which is independent with ϖ . As for ϖ , the characteristic energy estimate is very different. Let us explain it as follows. On the Cauchy slice $\{t\} \times \mathbb{R}^2$, we can use elliptic estimates to get the energy estimate of all derivatives of ϖ only by using ϖ and $\Delta \varpi$. However, on the characteristic hypersurface, these type of elliptic energy estimates don't work. We use Hodge decomposition and (1.22) to handle this difficulty. That is, operating P_{ij} on (1.22) giving rise to

(1.23)
$$\mathbf{T} \left[\partial_{ij}^2 \varpi - P_{ij} (\partial \rho \partial \varpi) \right] = P_{ij} (\partial \mathbf{v} \partial^2 \varpi + \text{l.o.t}) + [P_{ij}, \mathbf{T}] (\Delta \varpi - \partial \rho \partial \varpi).$$

Here, the Riesz operator $P_{ij} = \partial_{ij}^2 (-\Delta)^{-1}$, i, j = 1, 2. From (1.23), we can get some type of characteristic energy estimates for second derivatives of ϖ , where we use Sobolev imbedding to calculate the lower term

$$\|P_{ij}(\partial\rho\partial\varpi)\|_{L^2_{\Sigma}} \le \|P_{ij}(\partial\rho\partial\varpi)\|_{L^2_t H^a_x}, \quad a > \frac{1}{2}.$$

On the right hand side of (1.23), especially for the second one, we need some commutator estimates, which is introduced in Lemma 2.6. Based on these observations, we can recover some energy bounds for ϖ along the characteristic hypersurface. Please refer Lemma 6.8 for detials.

After obtaining enough regularity of null hypersurfaces and coefficients from null frame, we can obtain the Strichartz estimate of a linear wave equation with the acoustical metric g by using Smith-Tataru's conclusion in [41]. Through Duhamel's principle, we can prove the Strichartz estimate $||d\mathbf{v}, d\rho||_{L^4_t L^\infty_\infty}$.

1.6. Notations. In the paper, the notation $X \leq Y$ means $X \leq CY$, where C is a universal constant. We use the notation $X \ll Y$ to mean that $X \leq CY$ with a sufficiently large constant C.

We use four small parameters

(1.24)
$$\epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll \epsilon_0 \ll 1.$$

Let ζ be a smooth function with support in the shell $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Here, ξ denotes the variable of the spatial Fourier transform. Let ζ also satisfy the condition $\sum_{k \in \mathbb{Z}} \zeta(2^k \xi) = 1$. We introduce the Littlewood-Palay operator P_k with the frequency $2^k (k \in \mathbb{Z})$, which satisfies

$$P_k f = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \zeta(2^{-k}\xi) \hat{f}(\xi) d\xi.$$

We also set

$$f_{$$

1.7. Outline of the paper. The organization of the remainder of this paper is as follows. In Section 2, we introduce the reductions of (1.16) and commutator estimates, and also prove the total energy estimates and stability theorem. In Section 3, we reduce our problem to the case of smooth initial data by using compactness methods. In the subsequent Section, using a physical localized technique, we reduce the problem to the case of smooth, small, compacted supported initial data. In section 5, we give a bootstrap argument based on continuous functional. In Section 6 we derive some characteristic energy estimates along null hypersurfaces, which is used to prove the regularity of null hypersurfaces. Finally, in section 7, we prove the Strichartz estimate and continuous dependence.

2. Basic energy estimates and stability theorem

In this part, our goal is to give energy estimates and stability theorem. Firstly, we introduce a hyperbolic system and a wave-transport system of (1.16). We then give some classical commutator estimates. After that, we derive new transport equations for the specific vorticity. At last, we prove the energy estimates and stability theorem.

2.1. The reduction to a hyperbolic system and a wave-transport system. In the beginning, let us introduce a hyperbolic system of 2D compressible Euler equations.

Lemma 2.1. [30] Let \mathbf{v} and ρ be a solution of (1.16). Then (\mathbf{v}, ρ) satisfies the following symmetric hyperbolic system

(2.1)
$$A_0(\mathbf{U})\mathbf{U}_t + A_1(\mathbf{U})\mathbf{U}_{x_1} + A_2(\mathbf{U})\mathbf{U}_{x_2} = 0,$$

where $\mathbf{U} = (v^1, v^2, p(q))^{\mathrm{T}}$ and

$$A_{0} = \begin{pmatrix} \bar{\rho}e^{\rho} & 0 & 0\\ 0 & \bar{\rho}e^{\rho} & 0\\ 0 & 0 & \bar{\rho}^{-1}e^{-\rho}c_{s}^{-2} \end{pmatrix}, \quad A_{1} = \begin{pmatrix} \bar{\rho}e^{\rho}v^{1} & 0 & 1\\ 0 & \bar{\rho}e^{\rho}v^{1} & 0\\ 1 & 0 & v^{1}\bar{\rho}^{-1}e^{-\rho}c_{s}^{-2} \\ A_{2} = \begin{pmatrix} \bar{\rho}e^{\rho}v^{2} & 0 & 0\\ 0 & \bar{\rho}e^{\rho}v^{2} & 1\\ 0 & 1 & v^{2}\bar{\rho}^{-1}e^{-\rho}c_{s}^{-2} \end{pmatrix}.$$

Lemma 2.2. [32] Let (\mathbf{v}, ρ) be a solution of (1.16) and ϖ be defined in (1.8). Then $(\mathbf{v}, \rho, \varpi)$ satisfies

(2.2)
$$\begin{cases} \Box_g v^i = -[ia]e^{\rho}c_s^2\partial^a \varpi + Q^i + E^i, \\ \Box_g \rho = \mathcal{D}, \\ \mathbf{T}\varpi = 0. \end{cases}$$

Above, Q^i, E^i, \mathcal{D} are null forms relative to g, which are defined by

(2.3)

$$Q^{i} := 2[ia]c_{s}^{2}\varpi\partial^{i}\rho,$$

$$E^{i} := -\left(1 + c_{s}^{-1}c_{s}'\right)g^{\alpha\beta}\partial_{\alpha}\rho\partial_{\beta}v^{i},$$

$$\mathcal{D} := -3c_{s}^{-1}c_{s}'g^{\alpha\beta}\partial_{\alpha}\rho\partial_{\beta}\rho + 2\sum_{1\leq a< b\leq 2}\left\{\partial_{a}v^{a}\partial_{b}v^{b} - \partial_{a}v^{b}\partial_{b}v^{a}\right\},$$

and

10

$$[ia] = \begin{cases} 0, & \text{if } i = a ,\\ 1, & \text{if } i < a ,\\ -1, & \text{if } i > a . \end{cases}$$

We also define $\mathbf{Q} := (Q^1, Q^2)^T, \mathbf{E} := (E^1, E^2)^T$.

2.2. Commutator estimates. We first introduce a classical commutator estimate.

Lemma 2.3. [23] Let $\Lambda = (-\Delta)^{\frac{1}{2}}$, $s \ge 0$. Then for any scalar function h, f, we have (2.4)

$$\|\Lambda^{s}(hf) - (\Lambda^{s}h)f\|_{L^{2}_{x}(\mathbb{R}^{n})} \lesssim \|\Lambda^{s-1}h\|_{L^{2}_{x}(\mathbb{R}^{n})} \|\partial f\|_{L^{\infty}_{x}(\mathbb{R}^{n})} + \|h\|_{L^{p}_{x}(\mathbb{R}^{n})} \|\Lambda^{s}f\|_{L^{q}_{x}(\mathbb{R}^{n})},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$

Next, let us introduce some product estimates.

Lemma 2.4. [23] Let F(u) be a smooth function of u, F(0) = 0 and $u \in L_x^{\infty}$. For any $s \ge 0$, we have

(2.5)
$$||F(u)||_{H^s} \lesssim ||u||_{H^s} (1 + ||u||_{L^{\infty}_x}).$$

Lemma 2.5. [41] Suppose that $0 \le r, r' < \frac{n}{2}$ and $r + r' > \frac{n}{2}$. Then

(2.6) $\|hf\|_{H^{r+r'-\frac{n}{2}}(\mathbb{R}^n)} \le C_{r,r'} \|h\|_{H^r(\mathbb{R}^n)} \|h\|_{H^{r'}(\mathbb{R}^n)}.$

Moreover, if $-r \leq r' \leq r$ and $r > \frac{n}{2}$, then the following estimate

(2.7)
$$\|hf\|_{H^{r'}(\mathbb{R}^n)} \le C_{r,r'} \|h\|_{H^r(\mathbb{R}^n)} \|h\|_{H^{r'}(\mathbb{R}^n)},$$

holds.

Lemma 2.6. Denote the Riesz operator $\mathbf{R} := \partial^2 (-\Delta)^{-1}$. For $\delta \in (0, s - \frac{7}{4})$, then $\|[\mathbf{R}, \mathbf{v} \cdot \nabla]f\|_{L^2_x(\mathbb{R}^2)} \lesssim \|\partial \mathbf{v}\|_{C^{\delta}_x} \|f\|_{L^2_x(\mathbb{R}^2)}.$

Proof. By using paraproduct decomposition, we have

$$\Delta_{j}[\mathbf{R}, \mathbf{v} \cdot \nabla] f = \sum_{|k-j| \le 2} \Delta_{j} [\mathbf{R}(\Delta_{k}\mathbf{v} \cdot \nabla S_{k-1}f) - \Delta_{k}\mathbf{v} \cdot \nabla \mathbf{R}S_{k-1}f] + \sum_{|k-j| \le 2} \Delta_{j} [\mathbf{R}(S_{k-1}\mathbf{v} \cdot \nabla \Delta_{k}f) - S_{k-1}\mathbf{v} \cdot \nabla \mathbf{R}\Delta_{k}f] + \sum_{k \ge j-1} \Delta_{j} [\mathbf{R}(\Delta_{k}\mathbf{v} \cdot \nabla \Delta_{k}f) - \Delta_{k}\mathbf{v} \cdot \nabla \mathbf{R}\Delta_{k}f] = B_{1} + B_{2} + B_{3},$$

where

$$B_{1} = \sum_{|k-j| \leq 2} \Delta_{j} \left\{ \mathbf{R}(\Delta_{k}\mathbf{v} \cdot \nabla S_{k-1}f) - \Delta_{k}\mathbf{v} \cdot \nabla \mathbf{R}S_{k-1}f \right\},\$$

$$B_{2} = \sum_{|k-j| \leq 2} \Delta_{j} \left\{ \mathbf{R}(S_{k-1}\mathbf{v} \cdot \nabla \Delta_{k}f) - S_{k-1}\mathbf{v} \cdot \nabla \mathbf{R}\Delta_{k}f \right\}\$$

$$B_{3} = \sum_{k \geq j-1} \Delta_{j} \left\{ \mathbf{R}(\Delta_{k}\mathbf{v} \cdot \nabla \Delta_{k}f) - \Delta_{k}\mathbf{v} \cdot \nabla \mathbf{R}\Delta_{k}f \right\}.$$

By Hölder's inequality and Bernstein's inequality, we arrive at the bound

By Hölder's inequality, we see that

(2.9)
$$\{ \|B_3\|_{L^2_x} \}_{l^2_j} \lesssim \{ \sum_{k \ge j-1} 2^k \|\Delta_k \mathbf{v}\|_{L^\infty_x} \cdot 2^{-k} (\|\nabla \Delta_k f\|_{L^2_x} + \|\nabla \mathbf{R} \Delta_k f\|_{L^2_x}) \}_{l^2_j}$$
$$\lesssim \|\mathbf{v}\|_{\dot{B}^1_{\infty,\infty}} \|f\|_{L^2_x}$$
$$\lesssim \|\partial \mathbf{v}\|_{C^4_x} \|f\|_{L^2_x}.$$

Note

$$B_2 = \sum_{|k-j| \le 2} \Delta_j [\mathbf{R}, S_{k-1} \mathbf{v} \cdot] \Delta_k \nabla f.$$

By [37] (Lemma 3.2), we get

$$(2.10) \begin{cases} \|B_2\|_{L^2_x}\}_{l^2_j} \lesssim \left\{ \sum_{|k-j| \le 2} \|x\Phi\|_{L^1_x} \|\nabla S_{k-1} \mathbf{v}\|_{L^\infty_x} \|\Delta_j \Delta_k f\|_{L^2_x} \right\}_{l^2_j} \\ \lesssim \left\{ \sum_{|k-j| \le 2} \|\nabla S_{k-1} \mathbf{v}\|_{L^\infty_x} \|\Delta_j \Delta_k f\|_{L^2_x} \right\}_{l^2_j} \\ \lesssim \|\mathbf{v}\|_{\dot{B}^1_{\infty,\infty}} \|f\|_{L^2_x}. \\ \lesssim \|\partial \mathbf{v}\|_{C^{\delta}_x} \|f\|_{L^2_x}. \end{cases}$$

Here, we use the fact that $\Phi = \frac{x_i x_j}{|x|^2} 2^{2j} \Psi(2^j x)$ and Ψ is in Schwartz space. Gathering (2.8), (2.9) and (2.10) together, we have finished the proof of Lemma 2.6.

2.3. New transport equations. We derive a new transport equation for derivatives of $\Delta \varpi$.

Lemma 2.7. Let (\mathbf{v}, ρ) be a solution of (1.16) and ϖ be defined in (1.8). Then the quantities $\partial_i \varpi(i = 1, 2)$ and $\Delta \varpi$ satisfy the transport equation:

(2.11)
$$\mathbf{T}(\partial_i \varpi) = -\partial_i v^j \partial_j \varpi, \quad i = 1, 2,$$

and

(2.12)
$$\mathbf{T}(\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}) = R,$$

where

(2.13)
$$R = -2\sum_{i,j=1}^{2} \partial_{j} v^{i} \partial_{j} \rho \partial_{i} \varpi - e^{\rho} (\partial^{\perp} \rho \varpi + \partial^{\perp} \varpi) \partial \varpi - 2\sum_{i,j=1}^{2} \partial_{i} v^{j} \partial_{ij}^{2} \varpi.$$

Proof. Taking the spatial derivatives on $\mathbf{T}\boldsymbol{\varpi} = 0$, we first get

 $\mathbf{T}\partial_i \boldsymbol{\varpi} = -\partial_i v^j \partial_j \boldsymbol{\varpi}, \quad i = 1, 2.$

Taking the spatial derivative ∂_i again, one has

(2.14)
$$\mathbf{T}\Delta \boldsymbol{\varpi} = -\Delta v^i \partial_i \boldsymbol{\varpi} - 2\sum_{i=1,2} \partial_i v^k \partial_{ki}^2 \boldsymbol{\varpi}.$$

The Hodge's decomposition implies that

(2.15)
$$\Delta \mathbf{v} = \partial \operatorname{div} \mathbf{v} + \partial^{\perp} \operatorname{curl} \mathbf{v}, \quad \partial^{\perp} = (-\partial_2, \partial_1)^{\mathrm{T}}$$

Substituting $\mathbf{T}\rho = -\text{div}\mathbf{v}$ and $\varpi = e^{-\rho}\text{curl}\mathbf{v}$ to (2.15), we get

$$\begin{aligned} \Delta \mathbf{v} &= -\partial \mathbf{T} \rho + \partial^{\perp} (\mathrm{e}^{\rho} \varpi) \\ &= [\mathbf{T}, \partial] \rho - \mathbf{T} \partial \rho + \partial^{\perp} (\mathrm{e}^{\rho} \varpi) \\ &= -\mathbf{T} \partial \rho + \sum_{j=1,2} \partial_j \mathbf{v} \partial_j \rho + \mathrm{e}^{\rho} (\partial^{\perp} \rho \varpi + \partial^{\perp} \varpi). \end{aligned}$$

Putting the above formula to (2.14), we obtain the transport equation:

$$\mathbf{T}\Delta\boldsymbol{\varpi} = \mathbf{T}(\partial\rho)\partial\boldsymbol{\varpi} + R_1,$$

where

12

$$R_1 = -\sum_{i,j=1,2} \partial_j v^i \partial_j \rho \partial_i \varpi - e^{\rho} (\partial^{\perp} \rho \varpi + \partial^{\perp} \varpi) \partial \varpi - \sum_{i=1,2} \partial_i v^k \partial^2_{ki} \varpi.$$

Using the fact

$$\begin{aligned} \mathbf{T}(\partial\rho)\partial\varpi &= \mathbf{T}(\partial\rho\partial\varpi) - \partial\rho\mathbf{T}(\partial\varpi) \\ &= \mathbf{T}(\partial\rho\partial\varpi) - \sum_{j=1,2}\partial_j\rho\partial_j v^i\partial_i\varpi, \end{aligned}$$

we then prove

$$\mathbf{T}(\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}) = R,$$

where

$$R = R_1 - \sum_{j=1,2} \partial_j \rho \partial_j v^i \partial_i \varpi,$$

= $-2 \sum_{i,j=1}^2 \partial_j v^i \partial_j \rho \partial_i \varpi - e^{\rho} (\partial^{\perp} \rho \varpi + \partial^{\perp} \varpi) \partial \varpi - 2 \sum_{i,j=1}^2 \partial_i v^j \partial_{ij}^2 \varpi.$

2.4. Energy estimates.

Lemma 2.8. Let (\mathbf{v}, ρ) be a solution of (1.16). Then for any $a \ge 0$, we have

(2.16)
$$\|\rho\|_{H^a_x} + \|\mathbf{v}\|_{H^a_x} \lesssim (\|\rho_0\|_{H^a} + \|\mathbf{v}_0\|_{H^a}) \exp\left(\int_0^t \|d\mathbf{v}, d\rho\|_{L^\infty_x} d\tau\right), \quad t \in [0, T].$$

Proof. Let $\mathbf{U} = (\mathbf{v}, p(\rho))^T$. Then

$$A_0(\mathbf{U})\mathbf{U}_t + A_1(\mathbf{U})\partial_{x_1}\mathbf{U} + A_2(\mathbf{U})\partial_{x_2}\mathbf{U} = 0.$$

A straightforward computation on the above equation using integration by parts and classical commutator estimates in Lemma 2.3 yields

(2.17)
$$\|\mathbf{U}(t)\|_{H^a_x} \lesssim \|\mathbf{U}(0)\|_{H^a} \exp\Big(\int_0^t \|d\mathbf{U}\|_{L^\infty_x} d\tau\Big).$$

As a result, we obtain

$$\|\rho\|_{H^s_x} + \|\mathbf{v}\|_{H^s_x} \lesssim (\|\rho_0\|_{H^a} + \|\mathbf{v}_0\|_{H^a}) \exp\left(\int_0^t \|d\mathbf{v}, d\rho\|_{L^\infty_x} d\tau\right), \quad t \in [0, T].$$

Lemma 2.9. Let \mathbf{v} and ρ be a solution of (1.16) and ϖ be defined in (1.8). Then, we have the 1-order energy estimates for the specific vorticity

(2.18)
$$\|\varpi\|_{H^{1}_{x}}^{2} \lesssim \|\varpi_{0}\|_{H^{1}}^{2} \exp\left(\int_{0}^{t} \|\partial\mathbf{v}\|_{L^{\infty}_{x}} d\tau\right).$$

Moreover, the following inequality

(2.19)
$$\frac{\frac{d}{dt} (\|\Delta \varpi\|_{L_{x}^{2}}^{2} - 2 \int_{\mathbb{R}^{2}} \partial \rho \partial \varpi \Delta \varpi dx)}{\lesssim (1 + \|\partial \rho\|_{L_{x}^{\infty}} + \|\partial \mathbf{v}\|_{L_{x}^{\infty}})^{2} (\|\mathbf{v}\|_{H_{x}^{\frac{3}{2}}}^{2} + \|\rho\|_{H_{x}^{\frac{3}{2}}}^{2} + \|\varpi\|_{H_{x}^{2}}^{2}).$$

holds.

Proof. By using $\mathbf{T}\boldsymbol{\varpi} = 0$ and Hölder's inequality, we get

(2.20)
$$\frac{d}{dt} \|\varpi\|_{L^2_x}^2 \lesssim \|\partial \mathbf{v}\|_{L^\infty_x} \|\varpi\|_{L^2_x}^2$$

By using (2.11) and Hölder's inequality, we arrive at the bound

(2.21)
$$\frac{d}{dt} \|\partial \varpi\|_{L^2_x}^2 \lesssim \|\partial \mathbf{v}\|_{L^\infty_x} \|\partial \varpi\|_{L^2_x}^2.$$

Adding (2.20) and (2.21) together yields

$$\frac{d}{dt} \|\varpi\|_{H^1_x}^2 \lesssim \|\partial \mathbf{v}\|_{L^\infty_x} \|\varpi\|_{H^1_x}^2.$$

By Gronwall's inequality, we can reach to

$$\|\varpi\|_{H^1_x}^2 \le \|\varpi_0\|_{H^1}^2 \exp\left(\int_0^t \|\partial \mathbf{v}(\tau)\|_{L^\infty_x} d\tau\right).$$

It remains for us to prove (2.19). Multiplying $\Delta \varpi$ on (2.3) and taking inner product on \mathbb{R}^2 , we have

(2.22)
$$\frac{\frac{1}{2} \frac{d}{dt} (\|\Delta \varpi\|_{L_x^2}^2 - 2 \int_{\mathbb{R}^2} \partial \rho \partial \varpi \Delta \varpi dx)}{\leq \int_{\mathbb{R}^2} \partial \rho \partial \mathbf{v} \partial \varpi \Delta \varpi dx + \int_{\mathbb{R}^2} R \Delta \varpi dx + \int_{\mathbb{R}^2} \operatorname{div} \mathbf{v} |\Delta \varpi|^2 dx.}$$

So we need to estimate the right hand side of (2.22) one by one. For the right one, by Hölder's inequality, we could derive

(2.23)
$$\left| \int_{\mathbb{R}^2} \partial \rho \partial \mathbf{v} \partial \varpi \Delta \varpi dx \right| \lesssim \|\partial \rho\|_{L^{\infty}_x} \|\partial \mathbf{v}\|_{L^{\infty}_x} \|\Delta \varpi\|_{L^2_x} \|\partial \varpi\|_{L^2_x} \\ \lesssim (\|\partial \rho\|_{L^{\infty}_x} + \|\partial \mathbf{v}\|_{L^{\infty}_x})^2 \|\varpi\|_{H^2_x}^2.$$

For the second one, by using (6.43) and Hölder's inequality, we can show that

$$(2.24) \qquad \left| \int_{\mathbb{R}^2} R\Delta \varpi dx \right| \lesssim \|\Delta \varpi\|_{L^2_x} \left(\|\partial \rho\|_{L^\infty_x} + \|\partial \mathbf{v}\|_{L^\infty_x} \right) \|\partial \mathbf{v}\|_{L^\infty_x} \|\partial \varpi\|_{L^2_x} \\ + \|\Delta \varpi\|_{L^2_x} \|\partial \varpi\|_{L^4_x}^2 + \|\partial \mathbf{v}\|_{L^\infty_x} \|\Delta \varpi\|_{L^2_x} \|\partial^2 \varpi\|_{L^2_x} \\ + \|\partial \rho\|_{L^\infty_x} \|\partial \mathbf{v}\|_{L^\infty_x} \|\Delta \varpi\|_{L^2_x} \|\partial \varpi\|_{L^2_x} \\ \lesssim (1 + \|\partial \rho\|_{L^\infty_x} + \|\partial \mathbf{v}\|_{L^\infty_x})^2 (\|\mathbf{v}\|_{H^2_x}^2 + \|\rho\|_{H^2_x}^2 + \|\varpi\|_{H^2_x}^2)$$

For the last term, by Hölder's inequality, we deduce

(2.25)
$$\left| \int_{\mathbb{R}^2} \operatorname{div} \mathbf{v} |\Delta \varpi|^2 dx \right| \lesssim \|\partial \mathbf{v}\|_{L^\infty_x} \|\varpi\|_{H^2_x}^2$$

Gathering (2.22), (2.23), (2.24), and (2.25), we can obtain (2.19). We have completed the proof of Theorem 2.8. $\hfill \Box$

Based on the above estimate, we can get the following energy estimates.

Theorem 2.10. (Total energy estimates) Assume $s > \frac{7}{4}$. Let (\mathbf{v}, ρ) be a solution of (1.16) and ϖ be defined as (1.8). Set

$$E(t) = \|(\mathbf{v}, \rho)\|_{H^s_x} + \|\varpi\|_{H^2_x} + \|(\partial_t \mathbf{v}, \partial_t \rho)\|_{H^{s-1}_x} + \|\partial_t \varpi\|_{H^1_x}$$

and

$$E_0 = \|\rho_0\|_{H^s_x} + \|\mathbf{v}_0\|_{H^s_x} + \|\varpi_0\|_{H^2_x}.$$

Then the following energy estimate

(2.26)
$$E(t) \lesssim E_0(1 + E_0^{\frac{1}{2}}) \exp\left(\int_0^t (1 + \|d\mathbf{v}, d\rho\|_{L^{\infty}_x})^2 d\tau\right),$$

hold.

Proof. By using (2.16), (2.18), (2.19), and Gronwall's inequality, we have

$$\begin{aligned} \|\rho\|_{H_x^s}^2 + \|\mathbf{v}\|_{H_x^s}^2 + \|\varpi\|_{H_x^1}^2 + \|\Delta\varpi\|_{L_x^2}^2 - 2\int_{\mathbb{R}^2} \partial\rho\partial\varpi\Delta\varpi dx \\ \lesssim (\|\varpi_0\|_{H^2}^2 + \|\rho_0\|_{H^s}^2 + \|\mathbf{v}_0\|_{H^s}^2) \exp\left(\int_0^t (1 + \|d\mathbf{v}, d\rho\|_{L_x^\infty})^2 d\tau\right) \\ + \|\Delta\varpi_0\|_{L^2}^2 + \left|\int_{\mathbb{R}^2} \partial\rho_0\partial\varpi_0\Delta\varpi_0 dx\right| \\ \lesssim (\|\varpi_0\|_{H^2}^2 + \|\rho_0\|_{H^s}^2 + \|\mathbf{v}_0\|_{H^s}^2) \exp\left(\int_0^t (1 + \|d\mathbf{v}, d\rho\|_{L_x^\infty})^2 d\tau\right) \\ + \|\varpi_0\|_{H^2}^2 + \|\rho_0\|_{H^s}\|\varpi_0\|_{H^2}^2. \end{aligned}$$

Note

$$(2.28) \qquad \|\Delta\varpi\|_{L^2_x}^2 - 2\int_{\mathbb{R}^2} \partial\rho\partial\varpi\Delta\varpi dx \ge \|\Delta\varpi\|_{L^2_x}^2 - C\|\partial\rho\|_{L^4_x}\|\partial\varpi\|_{L^4_x}\|\Delta\varpi\|_{L^2_x}.$$

By Sobolev inequality, it implies that

(2.29)
$$\|\partial\rho\|_{L^4_x}\|\partial\varpi\|_{L^4_x}\|\Delta\varpi\|_{L^2_x} \le \|\partial\rho\|_{H^{\frac{1}{2}}_x}\|\partial\varpi\|_{H^{\frac{1}{2}}_x}\|\Delta\varpi\|_{L^2_x}$$

By interpolation formula, we have

(2.30)
$$\|\partial f\|_{H^{\frac{1}{2}}_{x}} \lesssim \|\partial f\|_{L^{2}_{x}}^{\frac{1}{2}} \|\partial^{2} f\|_{L^{2}_{x}}^{\frac{1}{2}}$$

Using (2.30) and Young's inequality, we can update (2.31) by

(2.31)
$$\|\partial\rho\|_{L^4_x} \|\partial\varpi\|_{L^4_x} \|\Delta\varpi\|_{L^2_x} \le \|\rho\|_{H^{\frac{3}{2}}_x}^4 \|\partial\varpi\|_{L^2_x}^2 + \frac{1}{100} \|\varpi\|_{H^2_x}^2$$

We also note that

(2.32)
$$\|\varpi\|_{H^1_x}^2 + \|\Delta\varpi\|_{L^2_x}^2 \ge \frac{1}{2}\|\varpi\|_{H^2_x}^2$$

Gathering (2.27), (2.28), (2.31), and (2.32), we get

(2.33)
$$\|\rho\|_{H^s_x} + \|\mathbf{v}\|_{H^s_x} + \|\varpi\|_{H^2_x} \lesssim E_0(1+E_0^{\frac{1}{2}})\exp\left(\int_0^t (1+\|d\mathbf{v},d\rho\|_{L^\infty_x})^2 d\tau\right).$$

By using (1.16) and $\mathbf{T}\boldsymbol{\varpi} = 0$, we can carry out

$$(2.34) \|\partial_t \rho\|_{H^{s-1}_x} + \|\partial_t \mathbf{v}\|_{H^{s-1}_x} + \|\partial_t \varpi\|_{H^1_x} \lesssim \|\rho\|_{H^s_x} + \|\mathbf{v}\|_{H^s_x} + \|\varpi\|_{H^2_x}.$$

By (2.33) and (2.34), we can obtain (2.26).

We are now ready to give the stability theorem.

Theorem 2.11. (Stability theorem) Assume $\frac{7}{4} < s \leq 2$. Let (\mathbf{v}, ρ) be a solution of (1.16) and ϖ be defined in (1.8), where the corresponding initial data $(\mathbf{v}_0, \rho_0, \varpi_0) \in H^s \times H^s \times H^2$. Then, there exists a positive number $T_1 = T_1(||\mathbf{v}_0||_{H^s}, ||\rho_0||_{H^s}, ||\varpi_0||_{H^2})$ such that $(\mathbf{v}, \rho) \in C([0, T_1], H^s_x) \cap C^1([0, T_1], H^{s-1}_x \times H^{s-1}_x), \ \varpi \in C([0, T_1], H^2_x) \cap C^1([0, T_1], H^{s-1}_x), \ \varpi \in C([0, T_1], H^2_x) \cap C^1([0, T_1], H^1_x), \ and \ (d\mathbf{v}, d\rho) \in L^4_{[0, T_1]}L^{\infty}_x.$

Let (\mathbf{h}, ψ) be another solution to (1.16) and $V = \bar{\rho} e^{-\psi} \text{curl} \mathbf{h}$. And the corresponding initial data $(\mathbf{h}_0, \psi_0, V_0)$ is in $H^s \times H^s \times H^2$. Then, there exists a positive number $T_2 = T_2(||\mathbf{h}_0||_{H^s}, ||\psi_0||_{H^s}, ||V_0||_{H^2})$ such that $(\mathbf{h}, \psi) \in C([0, T_2], H^s_x) \cap C^1([0, T_2], H^{s-1}_x)$, $V \in C([0, T_2], H^2_x) \cap C^1([0, T_2], H^1_x)$ and $d\mathbf{h}, d\psi \in L^4_{[0, T_2]}L^\infty_x$. Therefore, for $t \in [0, \min\{T_1, T_2\}]$, the following estimate

(2.35)

$$\|(\mathbf{v} - \mathbf{h}, \rho - \psi)(t, \cdot)\|_{H^{s-1}_x} + \|(\varpi - V)(t, \cdot)\|_{H^1_x} \lesssim \|(\mathbf{v}_0 - \mathbf{h}_0, \rho_0 - \psi_0)\|_{H^s} + \|\varpi_0 - V_0\|_{H^2}$$

holds.

Proof. Let $\mathbf{U} = (\mathbf{v}, p(\rho))^{\mathrm{T}}$ and $\mathbf{B} = (\mathbf{h}, p(\psi))^{\mathrm{T}}$. For $t \in [0, \min\{T_1, T_2\}]$, we can derive that

$$A_0(\mathbf{U})\partial_t \mathbf{U} + \sum_{i=1}^2 A_i(\mathbf{U})\partial_{x_i} \mathbf{U} = 0,$$
$$A_0(\mathbf{B})\partial_t \mathbf{B} + \sum_{i=1}^2 A_i(\mathbf{B})\partial_{x_i} \mathbf{B} = 0.$$

As a result, $\mathbf{U} - \mathbf{B}$ satisfies

$$A_0(\mathbf{U})\partial_t(\mathbf{U}-\mathbf{B}) + \sum_{i=1}^2 A_i(\mathbf{U})\partial_{x_i}(\mathbf{U}-\mathbf{B}) = \mathbf{F},$$

where

$$\mathbf{F} = -\sum_{i=0}^{2} (A_i(\mathbf{U}) - A_i(\mathbf{B})) \partial_{x_i} \mathbf{B}.$$

By using the commutator estimates in Lemma 2.3, we could show

$$\frac{d}{dt} \|\mathbf{U} - \mathbf{B}\|_{H^{s-1}_x} \le C_{\mathbf{U},\mathbf{B}} \left(\|d\mathbf{U}, d\mathbf{B}\|_{L^{\infty}_x} \|\mathbf{U} - \mathbf{B}\|_{H^{s-1}_x} + \|\mathbf{U} - \mathbf{B}\|_{L^{\infty}_x} \|d\mathbf{B}\|_{H^{s-1}_x} \right),$$

where $C_{\mathbf{U},\mathbf{B}}$ depends on the L_x^{∞} norm of (\mathbf{U},\mathbf{B}) . By using $(\mathbf{v},\rho,\mathbf{h},\psi) \in C([0,\min\{T_1,T_2\}],H_x^s) \cap C^1([0,\min\{T_1,T_2\}],H_x^{s-1})$ and $d\mathbf{v},d\rho,d\mathbf{h},d\psi \in L^4_{[0,\min\{T_1,T_2\}]}L_x^{\infty}$, we then have

$$\begin{aligned} \| (\mathbf{U} - \mathbf{B})(t, \cdot) \|_{H^{s-1}_x} &\lesssim \| (\mathbf{U} - \mathbf{B})(0, \cdot) \|_{H^{s-1}} \\ &= \| (\mathbf{v}_0 - \mathbf{h}_0, \rho_0 - \psi_0) \|_{H^s}. \end{aligned}$$

By Lemma 2.4, we further obtain

(2.36)
$$\| (\mathbf{v} - \mathbf{h}, \rho - \psi)(t, \cdot) \|_{H^{s-1}_x} \lesssim \| (\mathbf{v}_0 - \mathbf{h}_0, \rho_0 - \psi_0) \|_{H^s}$$

On the other hand, ϖ and V satisfy

$$\partial_t \boldsymbol{\varpi} + \mathbf{v} \cdot \nabla \boldsymbol{\varpi} = 0,$$

and

$$\partial_t V + \mathbf{h} \cdot \nabla V = 0.$$

So we get

(2.37)
$$\partial_t(\varpi - V) + \mathbf{v} \cdot \nabla(\varpi - V) = -(\mathbf{v} - \mathbf{h})\nabla V.$$

By using standard energy estimates for (2.37), we get (2.38)

$$\begin{aligned} \|(\varpi - V)(t, \cdot)\|_{H^{1}_{x}} &\leq (\|(\varpi - V)(0, \cdot)\|_{H^{1}_{x}} + \|\mathbf{v} - \varphi\|_{L^{1}_{t}H^{\frac{3}{2}}_{x}} \|V\|_{L^{\infty}_{t}H^{2}_{x}}) \exp\{\int_{0}^{t} \|\partial \mathbf{v}\|_{L^{\infty}_{x}} d\tau\} \\ &\leq C_{V}\left(\|(\varpi - V)(0, \cdot)\|_{H^{1}_{x}} + \|(\mathbf{v} - \varphi)(0, \cdot)\|_{H^{s}}\right), \end{aligned}$$

for $t \in [0, \min\{T_1, T_2\}]$. Combining (2.37) and (2.38), we complete the proof of (2.35).

Corollary 2.12. (Uniqueness of the solution) Assume $\frac{7}{4} < s \leq 2$. Suppose (\mathbf{v}, ρ) and (\mathbf{h}, ψ) to be solutions of (1.16) with the same initial data $(\mathbf{v}_0, \rho_0) \in H^s \times H^s$. We assume the initial specific vorticity $\varpi_0 = \bar{\rho} e^{\rho_0} \operatorname{curl} \mathbf{v}_0 \in H^2$. Then there exists a constant T > 0 such that $(\mathbf{v}, \rho, \mathbf{h}, \psi) \in C([0, T], H^s_x) \cap C^1([0, T], H^{s-1}_x), \ \varpi \in C([0, T], H^2_x) \cap C^1([0, T], H^1_x)$ and $d\mathbf{v}, d\rho, d\mathbf{h}, d\psi \in L^4_{[0,T]} L^\infty_x$. Furthermore, we have

 $\mathbf{v}=\mathbf{h},\quad \rho=\psi.$

3. Reduction to the case of smooth initial data

In this part, we will reduce Theorem 1.2 to the case of smooth initial data by compactness arguments.

Proposition 3.1. For each R > 0, there exist constants T, M and C such that, for each smooth initial data (\mathbf{v}_0, ρ_0) satisfies

(3.1)
$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le R,$$

where

$$\varpi_0 = \bar{\rho}^{-1} \mathrm{e}^{-\rho_0} \mathrm{curl} \mathbf{v}_0.$$

Then there exists a smooth solution $(\mathbf{v}, \rho, \varpi)$ to

(3.2)
$$\begin{cases} \Box_g v^i = -[ia]e^{\rho}c_s^2\partial^a \varpi + Q^i + E^i, \\ \Box_g \rho = \mathcal{D}, \\ \mathbf{T}\varpi = 0. \\ (\mathbf{v}, \rho, \varpi)|_{t=0} = (\mathbf{v}_0, \rho_0, \varpi_0), \\ (\partial_t \mathbf{v}, \partial_t \rho)|_{t=0} = (-\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + c_s^2 \nabla \rho_0, -\mathbf{v}_0 \cdot \nabla \rho_0 - \operatorname{div} \mathbf{v}_0) \end{cases}$$

on $[-T,T] \times \mathbb{R}^2$, which satisfies

(3.3)
$$\|(\mathbf{v},\rho)\|_{H^s_x} + \|(\partial_t \mathbf{v},\partial_t \rho)\|_{H^{s-1}_x} + \|\varpi\|_{H^2_x} + \|\partial_t \varpi\|_{H^1_x} \le M$$

Here, the quantities Q^i, \mathcal{D} and E^i are defined in Lemma 2.2. Furthermore, the solution satisfies the condition

(1) the dispersive estimate for \mathbf{v} and ρ

$$(3.4) \|d\mathbf{v}, d\rho\|_{L^4_t C^\delta_x} \le M,$$

(2) for $1 \le r \le s+1$, the linear equation

(3.5)
$$\begin{cases} \Box_g f = 0\\ (f, \partial_t f)|_{t=0} = (f_0, f_1) \end{cases}$$

is well-posed in $H^r \times H^{r-1}$, and the following estimates

(3.6)
$$\|\langle \partial \rangle^k f\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}, \quad k < r - \frac{3}{4},$$

and

(3.7)
$$\|f\|_{L^{\infty}_{t}H^{r}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{r-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}},$$

holds.

In the following, we will use Proposition 3.1 to prove Theorem 1.2.

proof of Theorem 1.2 by Proposition 3.1. Consider arbitrary initial data $(\mathbf{v}_0, \rho_0, \varpi_0) \in$ $H^s \times H^s \times H^2$ satisfying

$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le R.$$

Let $\{(\mathbf{v}_{0k}, \rho_{0k}, \varpi_{0k})\}_{k \in \mathbb{N}^+}$ be a sequence of smooth data which satisfies

$$\lim_{k \to \infty} \mathbf{v}_{0k} = \mathbf{v}_0, \quad \lim_{k \to \infty} \rho_{0k} = \rho_0, \quad \text{in } H^s,$$
$$\varpi_{0k} = \bar{\rho} e^{-\rho_{0k}} \operatorname{curl} \mathbf{v}_{0k}, \quad \lim_{k \to \infty} \varpi_{0k} = \varpi_0, \quad \text{in } H^2$$

By Proposition 3.1, for each k, there exist the corresponding solution $(\mathbf{v}_k, \rho_k, \varpi_k)$ to (3.2). Also

$$(\mathbf{v}_k, \rho_k, \varpi_k)|_{t=0} = (\mathbf{v}_{0k}, \rho_{0k}, \varpi_{0k})^{\mathrm{T}}$$

Notice that the solutions of (3.2) also satisfy the symmetric hyperbolic system (2.1). Set

$$\mathbf{U}_k = (v_{1k}, v_{2k}, p(\rho_k)), \quad k \in \mathbb{N}^+$$

For $j, l \in \mathbb{N}^+$, we could derive

$$\begin{aligned} A_0(\mathbf{U}_j)\partial_t\mathbf{U}_j + A_1(\mathbf{U}_k)\partial_{x_1}\mathbf{U}_j + A_2(\mathbf{U}_j)\partial_{x_2}\mathbf{U}_j &= 0, \\ A_0(\mathbf{U}_l)\partial_t\mathbf{U}_l + A_1(\mathbf{U}_l)\partial_{x_1}\mathbf{U}_l + A_2(\mathbf{U}_l)\partial_{x_2}\mathbf{U}_l &= 0. \end{aligned}$$

The standard energy estimates imply that

$$\frac{d}{dt} \|\mathbf{U}_{j} - \mathbf{U}_{l}\|_{H_{x}^{s-1}} \le C_{\mathbf{U}_{j},\mathbf{U}_{l}} \left(\|d\mathbf{U}_{j}, d\mathbf{U}_{l}\|_{L_{x}^{\infty}} \|\mathbf{U}_{j} - \mathbf{U}_{l}\|_{H_{x}^{s-1}} + \|\mathbf{U}_{j} - \mathbf{U}_{l}\|_{L_{x}^{\infty}} \|d\mathbf{U}_{l}\|_{H_{x}^{s-1}} \right),$$

where $C_{\mathbf{U}_j,\mathbf{U}_l}$ depends on the L_x^{∞} norm of $\mathbf{U}_j,\mathbf{U}_l$. By using Strichartz estimates (3.4) for $d\mathbf{v}_k, d\rho_k, k \in \mathbb{N}^+$ and the energy estimates (3.3) for $\mathbf{v}_k, \rho_k, k \in \mathbb{N}^+$, we can derive that

(3.8)
$$\| (\mathbf{U}_j - \mathbf{U}_l)(t, \cdot) \|_{H^{s-1}_x} \lesssim \| (\mathbf{U}_j - \mathbf{U}_l)(0, \cdot) \|_{H^s} \\ \lesssim \| (\mathbf{v}_{0j} - \mathbf{v}_{0l}, \rho_{0j} - \rho_{0l}) \|_{H^s}$$

Thus, the sequence $\{(\mathbf{v}_k, \rho_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in $C([-T, T]; H^{s-1})$. Denote (\mathbf{v}, ρ) to be the limit. We therefore have

(3.9)
$$\lim_{k \to \infty} (\mathbf{v}_k, \rho_k) = (\mathbf{v}, \rho) \in C([-T, T]; H^{s-1}).$$

Consider the transport equation

$$\partial_t \varpi_k + \mathbf{v}_k \cdot \nabla \varpi_k = 0, \quad k \in \mathbb{N}^+.$$

It's direct to get

$$\partial_t(\varpi_j - \varpi_l) + \mathbf{v}_j \cdot \nabla(\varpi_j - \varpi_l) = (\mathbf{v}_j - \mathbf{v}_l) \cdot \nabla \varpi_l, \quad j, l \in \mathbb{N}^+.$$

By Sobolev equality and energy estimates, we have (3.10)

$$\|\varpi_k - \varpi_l\|_{H^{s-1}} \lesssim (\|\varpi_{0k} - \varpi_{0l}\|_{H^{s-1}} + \|\mathbf{v}_j - \mathbf{v}_l\|_{L^{\infty}_t H^{s-1}_x} \|\nabla \varpi_l\|_{L^{\infty}_t H^1_x}) \exp\{\int_0^t \|\partial \mathbf{v}_j\|_{L^{\infty}_x} d\tau\}.$$

For $\{\varpi_{0k}\}_{k\in\mathbb{N}^+}$ and $\{\mathbf{v}_k\}_{k\in\mathbb{N}^+}$ being two Cauchy sequence in H^s and $L_t^{\infty}H_x^{s-1}$ respectively, then $\{\varpi_k\}_{k\in\mathbb{N}^+}$ is a Cauchy sequence in $C([-T,T]; H^{s-1})$. We denote the limit

(3.11)
$$\lim_{k \to \infty} \varpi_k = \varpi \in C([-T,T]; H^{s-1}).$$

Since $(\mathbf{v}_k, \rho_k, \varpi^k)$ is uniformly bounded in $L_t^{\infty} H_x^s \times L_t^{\infty} H_x^s \times L_t^{\infty} H_x^2$. Noting the convergence (3.9) and (3.11), we can deduce that

(3.12)
$$(\mathbf{v},\rho,\varpi_k) \in L_t^{\infty} H_x^s \times L_t^{\infty} H_x^s \times L_t^{\infty} H_x^2$$

Also, for $(\mathbf{v}, \rho, \varpi)$ satisfying (3.2) and (3.2) being equivalent with (1.16), we get

$$(3.13) \qquad (\partial_t \mathbf{v}, \partial_t \rho, \partial_t \varpi) \in L^{\infty}_t H^{s-1}_x \times L^{\infty}_t H^{s-1}_x \times L^{\infty}_t H^1_x.$$

On the other hand, using Proposition 3.1, $(d\mathbf{v}_k, d\rho_k)$ is uniformly bounded in $L^4([-T, T]; C_x^{\delta})$. As a result, we have

(3.14)
$$\lim_{k \to \infty} (d\mathbf{v}_k, d\rho_k) = (d\mathbf{v}, d\rho), \quad \text{in } L^4([-T, T]; L^{\infty}_x).$$

It remains for us to prove (1.20) and (1.21) in Theorem 1.2. For $1 \le r \le s+1$, by Proposition 3.1, we have that there exists solutions f_k satisfying

(3.15)
$$\begin{cases} \Box_{g_k} f_k = 0\\ (f_k, \partial_t f_k)|_{t=0} = (f_0, f_1). \end{cases}$$

Here the metric g_k has the same formula as in Definition 1.4, and whose velocity and density should be replaced by \mathbf{v}_k and ρ_k . Using (3.6) and (3.7), we have

(3.16)
$$\|\langle \partial \rangle^a f_k\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}, \quad a < r - \frac{3}{4},$$

and

$$(3.17) ||f_k||_{L^{\infty}_t H^r_x} + ||\partial_t f_k||_{L^{\infty}_t H^{r-1}_x} \lesssim ||f_0||_{H^r} + ||f_1||_{H^{r-1}}.$$

From (3.17), we obtain that there exists a subsequence such that there is a limit f satisfying

(3.18)
$$\|f\|_{L^{\infty}_{t}H^{r}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{r-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}}.$$

Utilizing (3.16), we have

(3.19)
$$\| \langle \partial \rangle^a f \|_{L^4_t L^\infty_x} \lesssim \| f_0 \|_{H^r} + \| f_1 \|_{H^{r-1}}, \quad a < r - \frac{3}{4}.$$

Also, taking limit to (3.15), then the limit f satisfies

(3.20)
$$\begin{cases} \Box_g f = 0\\ (f, \partial_t f)|_{t=0} = (f_0, f_1) \end{cases}$$

Combining (3.12)-(3.14), and (3.18)-(3.20), we have finished the proof of Theorem 1.2.

4. Reduction to existence for small, smooth, compactly supported data

In this section, our goal is to give a reduction of Proposition 3.1 to the existence for small, smooth, compactly supported data by using physical localization arguments.

Proposition 4.1. Assuming $\frac{7}{4} < s \leq 2$, (1.15), and (1.24) hold. Let the initial data $(\mathbf{v}_0, \rho_0, \varpi_0)$ be smooth, supported in B(0, c+2) such that

(4.1)
$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le \epsilon_3.$$

and

$$\varpi_0 = \bar{\rho} \mathrm{e}^{-\rho} \mathrm{curl} \mathbf{v}_0.$$

Then the Cauchy problem (3.2) admits a unique, smooth solution $(\mathbf{v}, \rho, \varpi)$ on $[-1, 1] \times \mathbb{R}^2$, which has the following properties:

(1) energy estimate

(4.2)
$$\| (\mathbf{v}, \rho) \|_{L_t^{\infty} H_x^s} + \| (\partial_t \mathbf{v}, \partial_t \rho) \|_{L_t^{\infty} H_x^{s-1}} + \| \varpi \|_{L_t^{\infty} H_x^2} + \| \partial_t \varpi \|_{L_t^{\infty} H_x^1} \le \epsilon_2.$$

(2) dispersive estimate for \mathbf{v} and ρ

(4.3)
$$\|d\mathbf{v}, d\rho\|_{L^4_t C^\delta_x} \le \epsilon_2,$$

(3) dispersive estimate for the linear equation

for $1 \leq r \leq s+1$, the linear equation

(4.4)
$$\begin{cases} \Box_g f = 0\\ (f, \partial_t f)|_{t=0} = (f_0, f_1) \end{cases}$$

is well-posed in $H^r \times H^{r-1}$, and the following estimates

(4.5)
$$\|\langle \partial \rangle^k f\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}, \quad k < r - \frac{3}{4},$$

and

(4.6)
$$\|f\|_{L^{\infty}_{t}H^{s}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{s-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}}$$

holds.

proof of Proposition 3.1 by Proposition 4.1. To achieve the goal, we will firstly reduce Proposition 3.1 to small data by scaling and physical localization, and then using the conclusion in Proposition 4.1 to prove Proposition 3.1.

Step 1. Scaling. The the initial data in Proposition 3.1 satisfies

(4.7)
$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le R.$$

By scaling

$$\tilde{\mathbf{v}}(t,x) = \mathbf{v}(Tt,Tx), \quad \tilde{\rho}(t,x) = \rho(Tt,Tx), \quad \tilde{\varpi}(t,x) = \varpi(Tt,Tx),$$

we get

$$\|\tilde{\mathbf{v}}_{0}\|_{\dot{H}^{s}} + \|\tilde{\rho}_{0}\|_{\dot{H}^{s}} \le RT^{s-1}, \\ \|\tilde{\varpi}_{0}\|_{\dot{H}^{2}} \le RT.$$

Let ϵ_3 be stated in (1.24). Choose sufficiently small T such that

$$RT^{s-1} \ll \epsilon_3.$$

We then derive that

$$\|\tilde{v}_0\|_{\dot{H}^s} + \|\tilde{\rho}_0\|_{\dot{H}^s} + \|\tilde{\varpi}_0\|_{\dot{H}^2} \le \epsilon_3$$

The above homogeneous norm is not enough for us to use Proposition 4.1. We then need to reduce the data in a further step.

Step 2. Localization. Let c be the largest speed of propagation of (3.2). Set χ be a smooth function supported in B(0, c+2), and which equals 1 in B(0, c+1). For any given $y \in \mathbb{R}^2$, we define the localized initial data near y:

$$\mathbf{v}_{0}^{y} = \chi(x - y) \left(\mathbf{v}_{0} - \mathbf{v}_{0}(y) \right), \rho_{0}^{y} = \chi(x - y) \left(\rho_{0} - \rho_{0}(y) \right).$$

Then the initial specific vorticity should be given by

$$\varpi_0^y = \bar{\rho}^{-1} \mathrm{e}^{-\rho_0^y} \mathrm{curl} \mathbf{v}_0^y$$

Since $s \in (\frac{7}{4}, 2]$, it is not difficult for us to verify

(4.8)
$$\|(\mathbf{v}_0^y, \rho_0^y)\|_{H^s_x} + \|\varpi_0^y\|_{H^2_x} \lesssim \|\mathbf{v}_0, \rho_0\|_{\dot{H}^s} + \|\varpi_0\|_{\dot{H}^2} \lesssim \epsilon_3$$

Step 3. Using Proposition 4.1. By Proposition 4.1, there is a smooth solution $(\mathbf{v}^y, \rho^y, \varpi^y)$ on $[-1, 1] \times \mathbb{R}^2$ satisfying the following Cauchy problem

(4.9)
$$\begin{cases} \Box_g v^i = -[ia]e^{\rho}c_s^2\partial^a \varpi + Q^i + E^i, \\ \Box_g \rho = \mathcal{D}, \\ \mathbf{T}\varpi = 0. \\ (\mathbf{v}, \rho, \varpi)|_{t=0} = (\mathbf{v}_0^y, \rho_0^y, \varpi_0^y), \\ (\partial_t \mathbf{v}, \partial_t \rho)|_{t=0} = (-\mathbf{v}_0^y \cdot \nabla \mathbf{v}_0^y + c_s^2 \nabla \rho_0^y, -\mathbf{v}_0^y \cdot \nabla \rho_0^y - \operatorname{div} \mathbf{v}_0^y), \end{cases}$$

where Q^i, E^i and \mathcal{D} are stated as (2.3). As a result, $\mathbf{v}^y + \mathbf{v}_0(y), \rho^y + \rho_0(y), \varpi^y$ also solves (4.9), and its initial data coincides with $(\mathbf{v}_0, \rho_0, \varpi_0)$ in B(y, c+1). Besides, the Strichartz estimate

(4.10)
$$\|d\mathbf{v}^y, d\rho^y\|_{L^4_t L^\infty_x} \le \epsilon_2.$$

also holds. Consider the restriction, for $y \in \mathbb{R}^2$,

$$(\mathbf{v}^{y} + \mathbf{v}_{0}(y))|_{\mathbf{K}^{y}}, \quad (\rho^{y} + \rho_{0}(y))|_{\mathbf{K}^{y}}, \quad \varpi^{y}|_{\mathbf{K}^{y}},$$

where

$$\mathbf{K}^{y} := \{(t, x) : ct + |x - y| \le c + 1, |t| < 1\}.$$

Then this restrictions solve (4.9) on K^y . By finite speed of propagation and the uniqueness of solutions of (3.2), a smooth solution $(\mathbf{v}, \rho, \varpi)$ satisfying (3.2) in $[-1, 1] \times \mathbb{R}^2$ could be set by

$$\begin{aligned} \mathbf{v}(t,x) &= \mathbf{v}^y(t,x) + \mathbf{v}_0(y), \quad (t,x) \in \mathbf{K}^y, \\ \rho(t,x) &= \rho^y(t,x) + \rho_0(y), \quad (t,x) \in \mathbf{K}^y, \\ \varpi(t,x) &= \varpi^y(t,x), \qquad (t,x) \in \mathbf{K}^y. \end{aligned}$$

20

For the problem (3.2) is equivalent with (1.16), using Theorem 2.10, we have for $t \in [-1, 1]$

$$(4.11) \begin{aligned} \|(\mathbf{v},\rho)\|_{H^s_x} + \|\varpi\|_{H^2_x} + \|(\partial_t \mathbf{v},\partial_t \rho)\|_{H^{s-1}_x} + \|\partial_t \varpi\|_{H^1_x} \\ = \|(\mathbf{v}^y,\rho^y)\|_{H^s_x} + \|\varpi^y\|_{H^2_x} + \|(\partial_t \mathbf{v}^y,\partial_t \rho^y)\|_{H^{s-1}_x} + \|\partial_t \varpi^y\|_{H^1_x} \\ \leq C\left(\|(\mathbf{v}^y_0,\rho^y_0)\|_{H^s_x} + \|\varpi^y_0\|_{H^2_x}\right) \exp\{\int_0^t (1+\|d\mathbf{v}^y,d\rho^y\|_{L^\infty_x})^2 d\tau\} \\ \leq M. \end{aligned}$$

By (4.10), we can directly get

(4.12)
$$\|d\mathbf{v}, d\rho\|_{L^4_t L^\infty_x} \le C \|d\mathbf{v}^y, d\rho^y\|_{L^4_t L^\infty_x} \le M.$$

It remains for us to prove (3.6) and (3.7). Let the cartesian grid $2^{-\frac{1}{2}}\mathbb{Z}^2$ be in \mathbb{R}^2 , and a corresponding smooth partition of unity be

$$\sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^2} \psi(x - y) = 1,$$

such that the function ψ is supported in the unit ball. Consider the solution f^y for

(4.13)
$$\begin{cases} \Box_{g^y} f^y = 0, \\ f^y|_{t=0} = \psi(x-y)f_0, \ \partial_t f^y|_{t=0} = \psi(x-y)f_1, \end{cases}$$

where g^y has the same formulation as in (1.4) with the velocity \mathbf{v}^y and ρ^y . Thus,

$$(4.14) g^y = g, \quad (t,x) \in \mathbf{K}^y$$

By finite speed of propagation, for $(t, x) \in \mathbf{K}^y$, we can conclude that

$$f^y = f, \quad (t, x).$$

Write f as

$$f(t,x) = \sum_{y \in 2^{-\frac{1}{2}} \mathbb{Z}^2} \psi(x-y) f^y(x,t),$$

Using (4.5) and (4.6), for $k < r - \frac{3}{4}$, we could get

(4.15)
$$\begin{aligned} \|\langle \partial \rangle^{k} f\|_{L^{4}_{t}L^{\infty}_{x}}^{4} \leq C \sum_{y \in 2^{-\frac{1}{2}}\mathbb{Z}^{2}} \|\psi(x-y) \langle \partial \rangle^{k} f^{y}(x,t)\|_{L^{4}_{t}L^{\infty}_{x}}^{4} \\ \leq C \sum_{y \in 2^{-\frac{1}{2}}\mathbb{Z}^{2}} \|\psi(x-y)(f_{0},f_{1})\|_{H^{r} \times H^{r-1}}^{4}. \\ \leq \|(f_{0},f_{1})\|_{H^{r} \times H^{r-1}}^{4}, \end{aligned}$$

and

$$\begin{aligned} (4.16) \\ \|f\|_{L^{\infty}_{t}H^{s}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{s-1}_{x}} \leq & C \sum_{y \in 2^{-\frac{1}{2}}\mathbb{Z}^{2}} (\|\psi(x-y)f^{y}(t,x)\|_{L^{\infty}_{t}H^{s-1}_{x}} + \|\psi(x-y)\partial_{t}f^{y}\|_{L^{\infty}_{t}H^{s-1}_{x}}) \\ \lesssim & \|(f_{0},f_{1})\|_{H^{r} \times H^{r-1}}, \end{aligned}$$

Therefore, by (4.11), (4.12), (4.15), and (4.16), we have finished the proof of Proposition 4.1.

5. A BOOTSTRAP ARGUMENT

Let $\mathbf{m}^{\alpha\beta}$ be a standard Minkowski metric satisfying

$$\mathbf{m}^{00} = -1, \quad \mathbf{m}^{ij} = \delta^{ij}, \quad i, j = 1, 2$$

Taking $\mathbf{v} = 0$ and $\rho = 0$ in g, the inverse matrix of the metric g is

$$g^{-1}(0) = \begin{pmatrix} -1 & 0 & 0\\ 0 & c_s^2(0) & 0\\ 0 & 0 & c_s^2(0) \end{pmatrix}.$$

By a linear change of coordinates which preserves dt, we may assume that $g^{\alpha\beta}(0) = \mathbf{m}^{\alpha\beta}$. Let χ be a smooth cut-off function supported in the region $B(0, 3 + 2c) \times [-\frac{3}{2}, \frac{3}{2}]$, which equals to 1 in the region $B(0, 2 + 2c) \times [-1, 1]$. Set

(5.1)
$$\mathbf{g} = \chi(t, x)(g - g(0)) + g(0)$$

where g is denoted in (1.4). Consider the following Cauchy problem

(5.2)
$$\begin{cases} \Box_{\mathbf{g}} v^{i} = -[ia] e^{\boldsymbol{\rho}} c_{s}^{2} \partial^{a} \varpi + Q^{i} + E^{i}, \\ \Box_{\mathbf{g}} \rho = \mathcal{D}, \\ \mathbf{T} \varpi = 0. \\ (\mathbf{v}, \rho, \varpi)|_{t=0} = (\mathbf{v}_{0}, \rho_{0}, \varpi_{0}), \\ (\partial_{t} \mathbf{v}, \partial_{t} \rho)|_{t=0} = (-\mathbf{v}_{0} \cdot \nabla \mathbf{v}_{0} + c_{s}^{2} \nabla \rho_{0}, -\mathbf{v}_{0} \cdot \nabla \rho_{0} - \operatorname{div} \mathbf{v}_{0}), \end{cases}$$

where, **g** is defined in (5.1). We denote by \mathcal{H} the family of smooth solutions $(\mathbf{v}, \rho, \varpi)$ to (5.2) for $t \in [-2, 2]$, with initial data $(\mathbf{v}_0, \rho_0, \varpi_0)$ supported in B(0, 2+c), where

$$\varpi_0 = \bar{\rho}^{-1} \mathrm{e}^{-\rho_0} \mathrm{curl} \mathbf{v}_0$$

and for which

(5.3)
$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le \epsilon_3,$$

(5.4)

 $\|(\mathbf{v},\rho)\|_{L^{\infty}_{t}H^{s}_{x}}+\|(\partial_{t}\mathbf{v},\partial_{t}\rho)\|_{L^{\infty}_{t}H^{s-1}_{x}}+\|\varpi\|_{L^{\infty}_{t}H^{2}_{x}}+\|\partial_{t}\varpi\|_{L^{\infty}_{t}H^{1}_{x}}+\|d\mathbf{v},d\rho\|_{L^{4}_{t}C^{\delta}_{x}}\leq 2\epsilon_{2}.$ Then, the bootstrap argument can be stated as follows:

Proposition 5.1. Let (1.24) hold. Then there is a continuous functional $G : \mathcal{H} \to \mathbb{R}^+$, satisfying G(0) = 0, so that for each $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ satisfying $G(\mathbf{v}, \rho) \leq 2\epsilon_1$ the following hold:

(1) The function $(\mathbf{v}, \rho, \varpi)$ satisfies

(5.5)
$$G(\mathbf{v},\rho) \le \epsilon_1.$$

(2) The following estimate holds,

(5.6)

$$\|(\mathbf{v},\rho)\|_{L^{\infty}_{t}H^{s}_{x}} + \|(\partial_{t}\mathbf{v},\partial_{t}\rho)\|_{L^{\infty}_{t}H^{s-1}_{x}} + \|\varpi\|_{L^{\infty}_{t}H^{2}_{x}} + \|\partial_{t}\varpi\|_{L^{\infty}_{t}H^{1}_{x}} + \|d\mathbf{v},d\rho\|_{L^{4}_{t}C^{\delta}_{x}} \le \epsilon_{2}$$
(2) For 1 < n < s + 1, the equation (2.5) and equal with the metric π is called a set L^{∞}_{t}

(3) For $1 \le r \le s+1$, the equation (3.5) endowed with the metric **g** is well-posed in $H^r \times H^{r-1}$. Moreover, the following estimates

(5.7)
$$\|\langle \partial \rangle^k f\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}, \quad k < r - \frac{3}{4},$$

and

(5.8)
$$\|f\|_{L^{\infty}_{t}H^{s}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{s-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}},$$

hold.

proof of Proposition 4.1 by Proposition 5.1. The initial data in Proposition 4.1 satisfies

$$\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2} \le \epsilon_3.$$

We denote by A the subset of those $\gamma \in [0, 1]$ such that the equation (5.2) admits a smooth solution u^{γ} having the initial data

$$\begin{aligned} \mathbf{v}^{\gamma}(0) &= \gamma \mathbf{v}_{0}, \\ \rho^{\gamma}(0) &= \gamma \rho_{0}, \\ \varpi^{\gamma}_{0}(0) &= \bar{\rho} \mathrm{e}^{-\rho^{\gamma}(0)} \mathrm{curl} \mathbf{v}_{0}^{\gamma}, \end{aligned}$$

and such that $G(\mathbf{v}^{\gamma}, \rho^{\gamma}) \leq \epsilon_1$ and (5.6) hold.

If $\gamma = 0$, then

$$(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma})(t, x) = (\mathbf{0}, 0, 0).$$

is a smooth solution of 5.2 with initial data

$$(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma})(0, x) = (\mathbf{0}, 0, 0).$$

Thus, the set A is not empty. If we can prove that A = [0, 1], then $1 \in A$. As a result, the Proposition 4.1 holds. It suffices for us to prove that A is both open and closed in [0, 1].

(1) A is open. Let $\gamma \in A$. Then $(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma})$ is a smooth solution to (5.2), where $\varpi^{\gamma} = \bar{\rho} e^{-\rho^{\gamma}} \operatorname{curl} \mathbf{v}^{\gamma}.$

Let β be close to γ . By the continuity of G, it follows that

$$G(\mathbf{v}^{\beta}, \rho^{\beta}) \leq 2\epsilon_1,$$

and also (5.4) holds. Using Proposition 5.1, we have

$$G(\mathbf{v}^{\beta}, \rho^{\beta}) \leq \epsilon_1,$$

and (5.6). Thus, we have showed that $\beta \in A$.

(2) A is closed. Let $\gamma_k \in A, k \in \mathbb{N}^+$ and $\lim_{k\to\infty} \gamma_k = \gamma$. Then there exists a sequence $\{(\mathbf{v}^{\gamma_k}, \rho^{\gamma_k}, \varpi^{\gamma_k})\}_{k\in\mathbb{N}^+}$ is smooth solutions to (5.2) and

$$\begin{aligned} \| (\mathbf{v}^{\gamma_k}, \rho^{\gamma_k}) \|_{L_t^\infty H_x^s} + \| (\partial_t \mathbf{v}^{\gamma_k}, \partial_t \rho^{\gamma_k}) \|_{L_t^\infty H_x^{s-1}} \\ + \| \varpi^{\gamma_k} \|_{L_t^\infty H_x^2} + \| \partial_t \varpi^{\gamma_k} \|_{L_t^\infty H_x^1} + \| d\mathbf{v}^{\gamma_k}, d\rho^{\gamma_k} \|_{L_t^4 C_x^\delta} \le \epsilon_2 \end{aligned}$$

Then there exists some subsequence such that there is a limit $(\mathbf{v}^{\gamma}, \rho^{\gamma}, \varpi^{\gamma})$ satisfying $\|(\mathbf{v}^{\gamma}, \rho^{\gamma})\|_{L_{t}^{\infty}H_{x}^{s}} + \|(\partial_{t}\mathbf{v}^{\gamma}, \partial_{t}\rho^{\gamma})\|_{L_{t}^{\infty}H_{x}^{s-1}} + \|\varpi^{\gamma}\|_{L_{t}^{\infty}H_{x}^{2}} + \|\partial_{t}\varpi^{\gamma}\|_{L_{t}^{\infty}H_{x}^{1}} + \|d\mathbf{v}^{\gamma}, d\rho^{\gamma}\|_{L_{t}^{4}C_{x}^{\delta}} \leq \epsilon_{2},$ and $G(\mathbf{v}, \rho) \leq \epsilon_{1}$. Therefore, $\gamma \in A$. We could conclude that A = [0, 1]. So we complete the proof of Proposition 4.1.

6. Regularity of the characteristic hypersurface

Recalling Proposition 5.1, the Strichartz estimate (5.7) plays a crucial role and it is a class of Fourier restriction estimate [47]. So we need to find a background hypersurface to work. In this section, we will define the characteristic hypersurface and discuss it's regularity.

Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$, and the corresponding metric \mathbf{g} which equals the Minkowski metric for $t \in [-1, -\frac{1}{2}]$. Let Γ_{θ} be the flowout of this section under the Hamiltonian flow of \mathbf{g} . For each θ , the null Lagrangian manifold Γ_{θ} is the graph of a null covector field given by dr_{θ} , where r_{θ} is a smooth extension of $\theta \cdot x - t$, and that the level sets

of r_{θ} are small perturbations of the level sets of the function $\theta \cdot x - t$ in a certain norm captured by G. We also let $\Sigma_{\theta,r}$ for $r \in \mathbb{R}$ denote the level sets of r_{θ} . The characteristic hypersurface $\Sigma_{\theta,r}$ is thus the flow out of the set $\theta \cdot x = r - 2$ along with the null geodesic flow in the direction θ at t = -1.

Let us introduce an orthonormal set of coordinates on \mathbb{R}^2 by setting $x_{\theta} = \theta \cdot x$. Let x'_{θ} be given orthonormal coordinates on the hyperplane perpendicular to θ , which then define coordinates on \mathbb{R}^2 by projection along θ . Then (t, x'_{θ}) induces the coordinate on $\Sigma_{\theta,r}$, where $\Sigma_{\theta,r}$ is given by

$$\Sigma_{\theta,r} = \{(t,x) : x_{\theta} - \phi_{\theta,r} = 0\}$$

for a smooth function $\phi_{\theta,r}(t, x'_{\theta})$. We now introduce two norms for functions defined on $[-1, 1] \times \mathbb{R}^2$,

$$\begin{split} \|\|u\|\|_{2,\infty} &= \sup_{-1 \le t \le 1} \sup_{0 \le j \le 1} \|\partial_t^j u(t, \cdot)\|_{H^{2-j}(\mathbb{R}^2)}, \\ \|\|u\|\|_{2,2} &= \big(\sup_{0 \le j \le 1} \int_{-1}^1 \|\partial_t^j u(t, \cdot)\|_{H^{2-j}(\mathbb{R}^2)}^2 dt \big)^{\frac{1}{2}}. \end{split}$$

The same notation applies for functions defined on $[-1, 1] \times \mathbb{R}^2$. Denote

$$||f||_{2,2,\Sigma_{\theta,r}} = |||f|_{\Sigma_{\theta,r}}||_{2,2}$$

where the right-hand side is the norm of the restriction of f to $\Sigma_{\theta,r}$, taken over the (t, x'_{θ}) variables used to parametrise $\Sigma_{\theta,r}$. Besides, the notation

$$\|f\|_{H^a(\Sigma_{\theta,r})}$$

denotes the $H^{a-1}(\mathbb{R})$ norm of f restricted to the time t slice of $\Sigma_{\theta,r}$ using the x'_{θ} coordinates on $\Sigma^t_{\theta,r}$.

We now set

(6.1)
$$G(\mathbf{v},\rho) = \sup_{\theta,r} |||d\phi_{\theta,r} - dt|||_{2,2,\Sigma_{\theta,r}}.$$

Proposition 6.1. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ so that $G(\mathbf{v}, \rho) \leq 2\epsilon_1$. Then

(6.2)
$$\|\|\mathbf{g}^{\alpha\beta} - \mathbf{m}^{\alpha\beta}\|\|_{2,2,\Sigma_{\theta,r}} + \|\lambda(\mathbf{g}^{\alpha\beta} - \mathbf{g}^{\alpha\beta}_{\lambda}), d\mathbf{g}^{\alpha\beta}_{\lambda}, \lambda^{-1}\partial d\mathbf{g}^{\alpha\beta}_{\lambda}\|\|_{1,2,\Sigma_{\theta,r}} \lesssim \epsilon_2.$$

Proposition 6.2. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ so that $G(\mathbf{v}, \rho) \leq 2\epsilon_1$. Then

(6.3)
$$G(\mathbf{v},\rho) \lesssim \epsilon_2.$$

Furthermore, for each t, we have

(6.4)
$$\|d\phi_{\theta,r}(t,\cdot) - dt\|_{C^{1,\delta}_{x'}} \lesssim \epsilon_2 + \sup_{i,j} \|d\mathbf{g}(t,\cdot)\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

6.1. Energy estimates on the characteristic hypersurface. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$. Then the following estimates hold:

(6.5)
$$\|d\mathbf{v}, d\rho\|_{L^4_t C^\delta_x} + \||\mathbf{v}, \rho\||_{s,\infty} + \||\varpi\||_{2,\infty} \lesssim \epsilon_2.$$

It suffices for us to prove Proposition 6.1 and Proposition 6.2 for $\theta = (0, 1)$ and r = 0. We fix this choice, and suppress θ and r in our notation. We use (x_2, x') instead of $(x_{\theta}, x'_{\theta})$. Then Σ is defined by

$$\Sigma = \{x_2 - \phi(t, x') = 0\}.$$

The hypothesis $G \leq 2\epsilon_1$ implies that

(6.6)
$$|||d\phi_{\theta,r}(t,\cdot) - dt|||_{s,2,\Sigma} \le 2\epsilon_1.$$

According to Sobolev imbeddings, the following estimate holds:

(6.7)
$$\|d\phi(t,x') - dt\|_{L^4_t C^{1,\delta}_{x'}} + \|\partial_t d\phi(t,x')\|_{L^4_t C^{\delta}_{x'}} \lesssim \epsilon_1$$

Lemma 6.3. [41] Assume $s \in (\frac{7}{4}, 2]$. Let $\tilde{h}(t, x) = h(t, x', x_2 + \phi(t, x'))$. Then we have

$$|||h|||_{s,\infty} \lesssim |||h|||_{s,\infty}, \quad ||dh||_{L^4_t L^\infty} \lesssim ||dh||_{L^4_t L^\infty},$$

and

$$|||h|||_{H^a_x} \lesssim |||h|||_{H^a_x}, \quad 0 \le a \le 2.$$

Proof.

Lemma 6.4. [41] For r > 1 we have

$$|||hf|||_{r,2,\Sigma} \lesssim |||h|||_{r,2,\Sigma} |||f|||_{r,2,\Sigma}.$$

Lemma 6.5. Assume $s \in (\frac{7}{4}, 2]$. Suppose U to satisfy the hyperbolic system

(6.8)
$$A_0(\mathbf{U})\mathbf{U}_t + \sum_{i=1}^2 A_i(\mathbf{U})\mathbf{U}_{x_i} = \mathbf{F}.$$

Then

(6.9)
$$\|\|\mathbf{U}\|_{s,2,\Sigma}^2 \lesssim \|\mathbf{U}\|_{L_t^{\infty} H_x^s} \left(\|d\mathbf{U}\|_{L_t^4 L_x^{\infty}} + \|\mathbf{U}\|_{L_t^{\infty} H_x^s} + \|\mathbf{F}\|_{L_t^1 H_x^{s-1}} \right).$$

Proof. Choosing the change of coordinates $x_2 \to x_2 - \phi(t, x')$ and setting $\tilde{\mathbf{U}}(t, x) = \mathbf{U}(t, x', x_2 + \phi(t, x'))$, $\tilde{\mathbf{F}}(t, x) = \mathbf{F}(t, x', x_2 + \phi(t, x'))$, the system (6.8) is transformed to (6.10)

$$A_0(\mathbf{U})\partial_t\tilde{\mathbf{U}} + A_1(\tilde{\mathbf{U}})\partial_{x_1}\tilde{\mathbf{U}} + A_2(\tilde{\mathbf{U}})\partial_{x_2}\tilde{\mathbf{U}} = -\partial_t\phi\partial_2\tilde{\mathbf{U}} - \sum_{i=0}^2 A_i(\tilde{\mathbf{U}})\partial_{x_i}\phi\partial_1\tilde{\mathbf{U}} + \tilde{\mathbf{F}}.$$

Multiplying $\tilde{\mathbf{U}}$ on (6.10) and integrating it by parts on $[-1, 1] \times \mathbb{R}^2$, we get

$$\|\|\tilde{\mathbf{U}}\|_{0,2,\Sigma}^{2} \lesssim \|d\tilde{\mathbf{U}}\|_{L_{t}^{1}L_{x}^{\infty}}\|\tilde{\mathbf{U}}\|_{L_{x}^{2}} + \|\tilde{\mathbf{U}}\|_{L_{x}^{2}}\|\tilde{\mathbf{F}}\|_{L_{t}^{1}L_{x}^{2}},$$

where we use the fact that ϕ is independent of x_2 . Using Lemma 6.3, (6.6), and (6.7), we may bound the above expression by

(6.11)
$$\| \mathbf{U} \|_{0,2,\Sigma}^2 \lesssim \| \mathbf{U} \|_{L_t^\infty L_x^2} \left(\| d\mathbf{U} \|_{L_t^4 L_x^\infty} + \| \mathbf{U} \|_{L_t^\infty L_x^2} + \| \mathbf{F} \|_{L_t^1 L_x^2} \right).$$

Taking the derivative of $\Lambda_{x'}^{\beta}$, $|\beta| = s$ on (6.10) and integrating it on $[-1, 1] \times \mathbb{R}^2$, we could arrive at the bound

(6.12)
$$\|\Lambda_{x'}^{\beta} \tilde{\mathbf{U}}\|_{L_{\Sigma}^{2}}^{2} \lesssim \|d\tilde{\mathbf{U}}\|_{L_{t}^{1}L_{x}^{\infty}} \|\Lambda_{x}^{\beta} \tilde{\mathbf{U}}\|_{L_{t}^{\infty}L_{x}^{2}} + \|\Lambda_{x}^{\beta} \tilde{\mathbf{U}}\|_{L_{t}^{\infty}L_{x}^{2}} \|\Lambda_{x}^{\beta} \tilde{\mathbf{F}}\|_{L_{t}^{1}L_{x}^{2}} + I,$$

where

$$I = -\sum_{i=0}^{2} \int_{[-1,1]\times\mathbb{R}^{2}} \Lambda_{x'}^{\beta} \left(A_{i}(\tilde{\mathbf{U}})\partial_{x_{i}}\phi\partial_{2}\tilde{\mathbf{U}} \right) \cdot \Lambda_{x'}^{\beta} \tilde{\mathbf{U}} dx d\tau.$$

Rewrite I as

$$\begin{split} I &= -\sum_{i=0}^{2} \int_{[-1,1]\times\mathbb{R}^{2}} \left(\Lambda_{x'}^{\beta} \left(A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{2} \tilde{\mathbf{U}} \right) - A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{i} \partial_{2} \Lambda_{x'}^{\beta} \tilde{\mathbf{U}} \right) \cdot \Lambda_{x'}^{\beta} \tilde{\mathbf{U}} dx d\tau \\ &+ \sum_{i=0}^{2} \int_{[-1,1]\times\mathbb{R}^{2}} A_{i}(\tilde{\mathbf{U}}) \partial_{x_{i}} \phi \partial_{i} \partial_{2} \Lambda_{x'}^{\beta} \tilde{\mathbf{U}} \cdot \Lambda_{x'}^{\beta} \tilde{\mathbf{U}} dx d\tau \\ &= I_{1} + I_{2}, \end{split}$$

where

$$I_{1} = \sum_{i=0}^{2} \int_{[-1,1]\times\mathbb{R}^{2}} [\Lambda_{x'}^{\beta}, A_{i}(\tilde{\mathbf{U}})\partial_{i}\phi\partial_{2}]\tilde{\mathbf{U}}\cdot\Lambda_{x'}^{\beta}\tilde{\mathbf{U}}dxd\tau,$$
$$I_{2} = \sum_{i=0}^{2} \int_{[-1,1]\times\mathbb{R}^{2}} A_{i}(\tilde{\mathbf{U}})\partial_{x_{i}}\phi\partial_{2}(\Lambda_{x'}^{\beta}\tilde{\mathbf{U}})\cdot\Lambda_{x'}^{\beta}\tilde{\mathbf{U}}dxd\tau.$$

By commutator estimates, we get

(6.13)

$$|I_1| \lesssim \left(\|\Lambda^{\beta} \tilde{\mathbf{U}}\|_{L^{\infty}_t L^2_x} \|\partial d\phi\|_{L^2_t L^{\infty}_x} + \sup_{\theta, r} \|\Lambda^{\beta}_{x'} d\phi\|_{L^2(\Sigma_{\theta, r})} \|d\tilde{\mathbf{U}}\|_{L^2_t L^{\infty}_x} \right) \cdot \|\Lambda^{\beta} \tilde{\mathbf{U}}\|_{L^2_t L^2_x}$$

and

(6.14)
$$|I_2| \lesssim \left(\|d\tilde{\mathbf{U}}\|_{L^2_t L^\infty_x} \|\partial\phi\|_{L^2_t L^\infty_x} + \|\tilde{\mathbf{U}}\|_{L^2_t L^\infty_x} \|\partial^2\phi\|_{L^2_t L^\infty_x} \right) \cdot \|\Lambda^\beta \tilde{\mathbf{U}}\|_{L^\infty_t L^2_x}^2.$$

Due to (6.13) (6.14) Lemma 6.3 (6.6) and (6.7) we obtain

Due to (6.13), (6.14), Lemma 6.3, (6.6) and (6.7), we obtain

(6.15)
$$\| \Lambda_{x'}^{\beta} \mathbf{U} \|_{0,2,\Sigma}^{2} \lesssim \| \mathbf{U} \|_{L_{t}^{\infty} H_{x}^{s}} \left(\| d\mathbf{U} \|_{L_{t}^{4} L_{x}^{\infty}} + \| \mathbf{U} \|_{L_{t}^{\infty} H_{x}^{s}} + \| \mathbf{F} \|_{L_{t}^{1} H_{x}^{s-1}} \right).$$

Using $A_0(\mathbf{U})\partial_t \mathbf{U} = -A_1(\mathbf{U})\mathbf{U}_{x_1} - A_2(\mathbf{U})\mathbf{U}_{x_2}$ and Lemma 6.7, we can easily carry out

(6.16)
$$\| \partial_t \mathbf{U} \|_{s-1,2,\Sigma}^{z} \lesssim \| \mathbf{U} \|_{s-1,2,\Sigma}^{z} \| \| \partial \mathbf{U} \|_{s-1,2,\Sigma}^{z} \\ \lesssim \| \mathbf{U} \|_{L_t^{\infty} H_x^s} \left(\| d \mathbf{U} \|_{L_t^4 L_x^{\infty}} + \| \mathbf{U} \|_{L_t^{\infty} H_x^{s-1}} + \| \mathbf{F} \|_{L_t^1 H_x^{s-1}} \right).$$

Therefore, we can conclude the proof of Lemma 6.5 by using (6.11), (6.15), and (6.16). $\hfill \Box$

Based on Lemma 6.5, we get

Corollary 6.6. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$. Then (6.17) $\|\|\mathbf{v}, \rho\|\|_{s,2,\Sigma} \lesssim \epsilon_2$. Lemma 6.7. Suppose f to satisfy the linear equation (6.18) $\mathbf{T}f = G$. Then (6.19) $\|\|f\|\|_{0,2,\Sigma}^2 \lesssim \|G\|_{L_t^1 L_x^2} \|f\|_{L_x^2} + \|\partial \mathbf{v}\|_{L_t^4 L_x^\infty} \|f\|_{L_t^\infty L_x^2}.$ If $\varpi \in \mathcal{H}$ satisfies (6.20) $\mathbf{T}\varpi = 0$, then we have (6.21) $\|\|\varpi\|\|_{0,2,\Sigma} \lesssim \epsilon_2.$

26

Proof. Choosing the change of coordinates $x_2 \to x_2 - \phi(t, x')$, then the equation (6.18) is transformed to

$$\partial_t \tilde{f} + \tilde{\mathbf{v}} \cdot \nabla \tilde{f} = \tilde{G} - \partial_t \phi \cdot \partial_2 \tilde{f} - \tilde{v}^i \partial_i \phi \partial_2 \tilde{f}.$$

Taking the inner product with f on $[-1,1] \times \mathbb{R}^2$, it gives

(6.22)
$$\|f\|_{L_{\Sigma}^{2}}^{2} \lesssim \|G\|_{L_{t}^{1}H_{x}^{s}}\|f\|_{L_{x}^{2}} + \|\partial \mathbf{v}\|_{L_{t}^{1}L_{x}^{\infty}}\|f\|_{L_{x}^{2}} + I_{1} + I_{2},$$

where

$$I_{1} = -\int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{t} \phi \cdot \partial_{2} \tilde{f} \cdot \tilde{f} dx d\tau,$$

$$I_{2} = -\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{v}^{i} \partial_{i} \phi \partial_{2} \tilde{f} \cdot \tilde{f} dx d\tau.$$

For ϕ is independent of x_2 , we have

(6.23)
$$I_1 = \frac{1}{2} \int_{-1}^{1} \int_{\mathbb{R}^2} \partial_2 \partial_t \phi \cdot |\tilde{f}|^2 dx d\tau = 0$$

and

$$\begin{split} I_2| &= \frac{1}{2} \Big| \int_{-1}^1 \int_{\mathbb{R}^2} \partial_2 \tilde{v}^i \partial_i \phi \cdot |\tilde{f}|^2 dx d\tau \Big| \\ &\lesssim \|\partial \mathbf{v}\|_{L^4_t L^\infty_x} \|f\|^2_{L^2_x} \|\partial \phi\|_{L^4_t L^\infty_x}. \end{split}$$

Using (6.33), we get

(6.24)
$$|I_2| \lesssim \epsilon_1 \|\partial \mathbf{v}\|_{L^1_t L^\infty_x} \|f\|_{L^2}^2 \le \|\partial \mathbf{v}\|_{L^1_t L^\infty_x} \|f\|_{L^2_x}^2$$

By (6.22), (6.23), and (6.24), we can obtain (6.19). If G = 0, using (6.19), we can conclude (6.21).

Lemma 6.8. Let $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$. Let $s \in (\frac{7}{4}, 2]$. Then we have

(6.25)
$$\| \varpi \|_{s,2,\Sigma} + \| \varpi \|_{2,2,\Sigma} + \| \partial^2 \varpi \|_{0,2,\Sigma} + \| \partial \varpi \|_{1,2,\Sigma} \lesssim \epsilon_2.$$

Proof. The proof is separated into several steps.

Step 1: $\|\partial \varpi\|_{0,2,\Sigma}$. Recall

$$\mathbf{T}\partial \boldsymbol{\varpi} = \partial \boldsymbol{v}\partial \boldsymbol{\varpi}.$$

By changing coordinates $x_2 \to x_2 - \phi(t, x')$, we have

$$(\partial_t + \partial_t \phi \partial_2) \partial \overline{\omega} + \tilde{v}^i \cdot (\partial_i + \partial_i \phi \partial_2) \partial \overline{\omega} = (\partial + \partial \phi \partial_2) \tilde{\mathbf{v}} \cdot (\partial + \partial \phi \partial_2) \tilde{\omega},$$

where $\tilde{\cdot}$ denotes the function under new coordinates. Multiplying $\partial \overline{\omega}$ on the above equation, we derive that

$$\|\partial \varpi\|_{0,2,\Sigma}^2 \lesssim \|d\mathbf{v}\|_{L^4_t L^\infty_x} (1 + \|d\phi\|_{L^\infty_{t,x}})^2 \|\partial \varpi\|_{L^2_x} \lesssim \epsilon_2^2.$$

Taking square of the above expression, we conclude that

(6.26) $\|\partial w\|_{0,2,\Sigma} \lesssim \epsilon_2.$

Step 2:
$$\|\partial^2 \varpi\|_{0,2,\Sigma}$$
. We find $\Delta \varpi$ satisfying

(6.27)
$$\mathbf{T}(\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}) = R,$$

where R is defined in (6.43). Denote the operator P_{ij} by

(6.28)
$$\mathbf{P}_{ij} = \partial_{ij}^2 (-\Delta)^{-1}$$

Then

(6.29) $\partial_{ij}^2 \varpi = \mathcal{P}_{ij} \Delta \varpi, \quad i, j = 1, 2.$

Operating P_{ij} on (6.27), we then get

(6.30)
$$\mathbf{T} \{ \mathbf{P}_{ij} (\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}) \} = \mathbf{P}_{ij} R + [\mathbf{P}_{ij}, \mathbf{T}] (\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi}).$$

Inserting (6.29) into (6.30), we have

(6.31) $\mathbf{T}\left(\partial_{ij}^2 \varpi - \mathbf{P}_{ij}(\partial \rho \partial \varpi)\right) = K.$

Above, we define

(6.32)
$$K = P_{ij}R + [P_{ij}, \mathbf{T}](\Delta \varpi - \partial \rho \partial \varpi).$$

Choosing the change of coordinates $x_2 \to x_2 - \phi(t, x')$ and setting $\tilde{\varpi}(t, x', x_2) = \varpi(t, x_1, x_2 - \phi(t, x'))$, then the term $\partial_{ij}^2 \varpi$ is transformed to

$$\partial_{ij}^2 \tilde{\omega} - \partial_{ij}^2 \phi \partial_2 \tilde{\omega} - \partial_j \phi \partial_{2i}^2 \tilde{\omega} - \partial_i \phi \partial_{2j}^2 \tilde{\omega} + \partial_i \phi \partial_j \phi \partial_{22}^2 \tilde{\omega} + \partial_i \phi \partial_{2j}^2 \phi \partial_2 \tilde{\omega}.$$

Under the change of coordinates, we see the term $P_{ij}(\partial \rho \partial \varpi)$ and K as a whole part, i.e,

$$\begin{split} [\mathbf{P}_{ij}(\partial\rho\partial\varpi)] &= [\mathbf{P}_{ij}(\partial\rho\partial\varpi)] \left(t, x_1, x_2 - \phi(t, x')\right), \\ \tilde{K} &= K(t, x_1, x_2 - \phi(t, x')), \end{split}$$

As a result, the left side of (6.31) becomes

$$\widetilde{\mathbf{T}}\left(\partial_{ij}^{2}\tilde{\boldsymbol{\omega}}-\partial_{ij}^{2}\phi\partial_{2}\tilde{\boldsymbol{\omega}}-\partial_{j}\phi\partial_{2i}^{2}\tilde{\boldsymbol{\omega}}-\partial_{i}\phi\partial_{2j}^{2}\tilde{\boldsymbol{\omega}}+\partial_{i}\phi\partial_{j}\phi\partial_{22}^{2}\tilde{\boldsymbol{\omega}}+\partial_{i}\phi\partial_{2j}^{2}\phi\partial_{2}\tilde{\boldsymbol{\omega}}-[\mathbf{P}_{ij}(\widetilde{\partial\rho\partial\boldsymbol{\omega}})]\right),$$

where

$$\widetilde{\mathbf{T}} = (\partial_t + \partial_t \phi \partial_2) + \widetilde{v}^i (\partial_i + \partial_i \phi \partial_2),$$

Organizing it in order, the expression of (6.27) could be

(6.33)
$$(\partial_t + \tilde{v}^i \partial_i)\tilde{B} + (\partial_t \phi + \tilde{v}^i \partial_i \phi)\partial_2 \tilde{B} = \tilde{K},$$

where

(6.34)
$$\begin{split} \tilde{B} := \partial_{ij}^2 \tilde{\omega} - \partial_{ij}^2 \phi \partial_2 \tilde{\omega} - \partial_j \phi \partial_{2i}^2 \tilde{\omega} - \partial_i \phi \partial_{2j}^2 \tilde{\omega} \\ + \partial_i \phi \partial_j \partial_{22}^2 \tilde{\omega} + \partial_i \phi \partial_{2j}^2 \phi \partial_2 \tilde{\omega} - [\mathrm{P}_{ij} \widetilde{(\partial \rho \partial \omega)}] \end{split}$$

If we set

(6.35)
$$B = \partial_{ij}^2 \varpi - \mathcal{P}_{ij}(\partial \rho \partial \varpi),$$

then B is transformed to \tilde{B} under changing of coordinates $x_2 \to x_2 - \phi(t, x')$. Multiplying \tilde{B} on (6.33) and integrating it on $[-1, 1] \times \mathbb{R}^2$, one has

(6.36)
$$\begin{aligned} \|\tilde{B}\|_{L^{2}(\Sigma)}^{2} \leq |\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{K} \cdot \tilde{B} dx d\tau| + \|d\mathbf{v}\|_{L^{1}_{t}L^{\infty}_{x}} \|\tilde{B}\|_{L^{2}_{x}}^{2} \\ + |\int_{-1}^{1} \int_{\mathbb{R}^{2}} (\partial_{t}\phi + \tilde{v}^{i}\partial_{i}\phi) \partial_{2}\tilde{B} \cdot \tilde{B} dx d\tau|. \end{aligned}$$

On the left side, we see that

(6.37)
$$\|\tilde{B}\|_{L^{2}(\Sigma)}^{2} = \|B|_{\Sigma}\|_{L^{2}_{t}L^{2}_{x'}(\Sigma)} = \|B\|_{0,2,\Sigma}^{2}$$

Let us estimate the right hand of (6.36) as follows. By using Lemma 6.3, we have (6.38) $\|\tilde{B}\|_{L_x^2}^2 \leq \|B\|_{L_x^2}^2.$

28

By Hölder's inequality and ϕ independent with x_2 , we arrive at the bound

$$(6.39) \qquad |\int_{-1}^{1} \int_{\mathbb{R}^{2}} (\partial_{t}\phi + \tilde{v}^{i}\partial_{i}\phi)\partial_{2}\tilde{B} \cdot \tilde{B}dxd\tau| = |\int_{-1}^{1} \int_{\mathbb{R}^{2}} \partial_{2}(\partial_{t}\phi + \tilde{v}^{i}\partial_{i}\phi)|\tilde{B}|^{2}dxd\tau$$
$$= |\int_{-1}^{1} \int_{\mathbb{R}^{2}} |\partial_{2}\tilde{v}^{i}| \cdot |\partial_{i}\phi||\tilde{B}|^{2}dxd\tau|$$
$$\leq ||\partial\mathbf{v}||_{L^{1}_{t}L^{\infty}_{x}} ||\partial\phi||_{L^{\infty}_{t}L^{\infty}_{x}} ||\tilde{B}||_{L^{\infty}_{t}L^{2}_{x}}.$$

We note that there is a Riesz operator in K, we then pull the coordinate back by the transform $x_2 - \phi(t, x') \to x_2$. Then, we have

(6.40)
$$|\int_{-1}^{1} \int_{\mathbb{R}^{2}} \tilde{K} \cdot \tilde{B} dx d\tau | = |\int_{-1}^{1} \int_{\mathbb{R}^{2}} K \cdot B dx d\tau |$$
$$\leq ||K||_{L_{t}^{1} L_{x}^{2}} ||B||_{L_{t}^{\infty} L_{x}^{2}}.$$

Combining (6.36)-(6.39) yields

$$\begin{aligned} \|B\|_{0,2,\Sigma}^2 \lesssim \|\partial \mathbf{v}\|_{L_t^1 L_x^\infty} \|\partial \phi\|_{L_t^\infty L_x^\infty} \|B\|_{L_t^\infty L_x^2} + \|d\mathbf{v}\|_{L_t^1 L_x^\infty} \|B\|_{L_x^2}^2 + \|K\|_{L_t^1 L_x^2} \|B\|_{L_t^\infty L_x^2} \\ \text{By (5.4) and (6.7), we have} \end{aligned}$$

(6.41)
$$|||B|||_{0,2,\Sigma}^2 \lesssim \epsilon_2^2 + ||K||_{L_t^1 L_x^2} ||B||_{L_t^\infty L_x^2}$$

It remains for us to bound $||K||_{L_t^1 L_x^2}$. Recalling (6.32), we can obtain

(6.42)
$$\|K\|_{L^{1}_{t}L^{2}_{x}} \leq \|\mathbf{P}_{ij}R\|_{L^{1}_{t}L^{2}_{x}} + \|[\mathbf{P}_{ij},\mathbf{T}](\Delta \varpi - \partial \rho \partial \varpi)\|_{L^{1}_{t}L^{2}_{x}}$$

Using P_{ij} , a Riesz operator, we can show that by Hölder's inequality (6.43)

$$\begin{split} \| \mathbf{P}_{ij} R \|_{L_{t}^{1} L_{x}^{2}} &\lesssim \| R \|_{L_{t}^{1} L_{x}^{2}} \\ &\lesssim \| \partial \mathbf{v} \|_{L_{t}^{4} L_{x}^{\infty}} \| \partial \rho \|_{L_{t}^{\infty} L_{x}^{2}} + \| \mathbf{e}^{\rho} \|_{L_{t}^{\infty} L_{x}^{\infty}} (\| \partial \rho \|_{L_{t}^{4} L_{x}^{\infty}} \| \varpi \|_{L_{t}^{\infty} L_{x}^{2}} + \| \partial \varpi \|_{L_{t}^{1} L_{x}^{2}}) \\ &+ \| \partial \mathbf{v} \|_{L_{t}^{4} L_{x}^{\infty}} \| \partial^{2} \varpi \|_{L_{t}^{\infty} L_{x}^{2}} + \| \partial \mathbf{v} \|_{L_{t}^{4} L_{x}^{\infty}} \| \partial \rho \|_{L_{t}^{\infty} L_{x}^{4}} \| \partial \varpi \|_{L_{t}^{\infty} L_{x}^{4}} \\ &\lesssim (\| \partial \mathbf{v}, \partial \rho \|_{L_{t}^{4} L_{x}^{\infty}} + \| \partial \mathbf{v} \|_{L_{t}^{4} L_{x}^{\infty}} \| \partial \rho \|_{L_{t}^{4} L_{x}^{\infty}}) (\| \varpi \|_{L_{t}^{\infty} H_{x}^{2}} + \| \rho \|_{L_{t}^{\infty} H_{x}^{8}}) \\ &\lesssim \epsilon_{2}^{2}. \end{split}$$

By Lemma 2.6, we see that

(6.44)
$$\|[\mathbf{P}_{ij},\mathbf{T}](\Delta \varpi - \partial \rho \partial \varpi)\|_{L^1_t L^2_x} \le \|\partial \mathbf{v}\|_{L^4_t C^\delta_x} \|\Delta \varpi - \partial \rho \partial \varpi\|_{L^\infty_t L^2_x}$$

On the other hand, by using (5.4), we have

(6.45)
$$\begin{aligned} \|\Delta \varpi - \partial \rho \partial \varpi\|_{L^2_t L^2_x} &\leq \|\Delta \varpi\|_{L^2_t L^2_x} + \|\partial \rho \partial \varpi\|_{L^2_t L^2_x} \\ &\leq \|\varpi\|_{L^\infty_t H^2_x} + \|\partial \rho\|_{L^4_t L^\infty_x} \|\partial \varpi\|_{L^\infty_t L^2_x} \\ &\lesssim \epsilon_2 + \epsilon_2^2 \lesssim \epsilon_2. \end{aligned}$$

Substituting (6.45) to (6.44) and using (5.4), we could get the bound

(6.46)
$$\|[\mathbf{P}_{ij},\mathbf{T}](\Delta \boldsymbol{\varpi} - \partial \rho \partial \boldsymbol{\varpi})\|_{L^{1}_{t}L^{2}_{x}} \lesssim \epsilon_{2}.$$

Adding (6.46) and (6.43), one has

(6.47)
$$\|K\|_{L^{1}_{t}L^{2}_{x}} \leq \|\mathbf{P}_{ij}R\|_{L^{1}_{t}L^{2}_{x}} + \|[\mathbf{P}_{ij},\mathbf{T}](\Delta \varpi - \partial \rho \partial \varpi)\|_{L^{1}_{t}L^{2}_{x}} \lesssim \epsilon_{2},$$

which when inserted into (6.41) yields the inequality

(6.48)
$$|||B|||_{0,2,\Sigma}^2 \lesssim \epsilon_2^2 + \epsilon_2 ||B||_{L_t^\infty L_x^2}$$

Recalling (6.35) and using (5.4), we have

$$\begin{split} \|B\|_{L^{\infty}_{t}L^{2}_{x}} &\leq \|\partial^{2}\varpi - \mathcal{P}_{ij}(\partial\rho\partial\varpi)\|_{L^{\infty}_{t}L^{2}_{x}} \\ &\leq \|\partial^{2}\tilde{\varpi}\|_{L^{\infty}_{t}L^{2}_{x}} + \|\mathcal{P}_{ij}(\partial\rho\partial\varpi)\|_{L^{\infty}_{t}L^{2}_{x}} \\ &\leq \|\partial^{2}\varpi\|_{L^{\infty}_{t}L^{2}_{x}} + \|\partial\rho\partial\varpi\|_{L^{\infty}_{t}L^{2}_{x}} \\ &\lesssim \|\varpi\|_{L^{\infty}_{t}H^{2}_{x}}(1 + \|\partial\rho\|_{L^{\infty}_{t}H^{2}_{x}-1}) \lesssim \epsilon_{2}, \end{split}$$

which combing with (6.48) give us

$$(6.49) |||B|||_{0,2,\Sigma} \lesssim \epsilon_2.$$

Using (6.35) again, we derive that

(6.50)
$$\|\|B\|\|_{0,2,\Sigma} = \||\partial^2 \tilde{\omega} - \mathcal{P}_{ij}(\partial \rho \partial \varpi)\||_{0,2,\Sigma} \\ \geq \||\partial^2 \varpi\||_{0,2,\Sigma} - \|\mathcal{P}_{ij}(\partial \rho \partial \varpi)\||_{0,2,\Sigma}.$$

It remains for us to estimate $|||P_{ij}(\partial \rho \partial \varpi)|||_{0,2,\Sigma}$. For the 1-codimension of Σ in $\mathbb{R}^+ \times \mathbb{R}^2$, by Sobolev imbedding, we have

(6.51)
$$\begin{split} \||\mathbf{P}_{ij}(\partial\rho\partial\varpi)||_{0,2,\Sigma} &= \|\mathbf{P}_{ij}(\partial\rho\partial\varpi)\|_{L^{2}_{t}L^{2}_{x'}} \\ &\leq \|\mathbf{P}_{ij}(\partial\rho\partial\varpi)\|_{L^{2}_{t}H^{a}_{x}}, \ a > \frac{1}{2}, \\ &\leq \|\partial\rho\partial\varpi\|_{L^{2}_{t}H^{a}_{x}} \\ &\leq \|\partial\rho\|_{L^{2}_{t}H^{s-1}_{x}} \|\partial\varpi\|_{L^{2}_{t}H^{1}_{x}} \lesssim \epsilon^{2}_{2} \end{split}$$

Combining (6.50) and (6.51), we derive

(6.52)
$$\| \partial^2 \varpi \|_{0,2,\Sigma} \le \| \partial^2 \varpi \|_{0,2,\Sigma} + \| \mathbf{P}_{ij}(\partial \rho \partial \varpi) \|_{0,2,\Sigma} \lesssim \epsilon_2.$$

Step 3: $\||\partial \varpi\||_{1,2,\Sigma}$. We also note

$$\partial_t \partial \varpi + \mathbf{v} \cdot \nabla \partial \varpi = \partial \mathbf{v} \cdot \partial \varpi.$$

By (6.54) and Sobolev imbedding, we see that

 $\|\!|\partial_t \partial \varpi\|\!|_{0,2,\Sigma} \le \|\!|\mathbf{v} \cdot \nabla \partial \varpi\|\!|_{0,2,\Sigma} + \|\!|\partial \mathbf{v} \cdot \partial \varpi\|\!|_{0,2,\Sigma}$

(6.53)
$$\leq \|\mathbf{v}\|_{L^{\infty}_{t,x}} \|\partial^{2}\varpi\|_{0,2,\Sigma} + \|\partial\mathbf{v}\cdot\partial\varpi\|_{L^{2}_{t}H^{a}_{x}}, \quad a > \frac{1}{2},$$
$$\leq \|\mathbf{v}\|_{L^{\infty}_{t}H^{s}_{x}} \|\partial^{2}\varpi\|_{0,2,\Sigma} + \|\partial\mathbf{v}\|_{L^{\infty}_{t}H^{s-1}_{x}} \|\partial\varpi\|_{L^{\infty}_{t}H^{1}_{x}}$$
$$\lesssim \epsilon_{2}.$$

For any function f, the term $\partial_{x'}\tilde{f}$ can be calculated by

$$\partial_{x'}\tilde{f} = \nabla f \cdot (1, d\phi)^{\mathrm{T}},$$

where $\tilde{\cdot}$ denotes the function expressed in the new coordinates and $\tilde{f}(t,x) = f(t,x',x_2 + \phi(t,x'))$. We then have

$$\|\|\partial_{x'}f\|\|_{0,2,\Sigma} \le (1+\|d\phi\|_{L^{\infty}_{t,r'}})\|\|\partial f\|\|_{0,2,\Sigma}.$$

Based on this fact, we can deduce

$$(6.54) |||\partial_{x'}\partial\varpi|||_{0,2,\Sigma} \le (1+||d\phi||_{L^{\infty}_{t,r'}})|||\partial(\partial\varpi)|||_{0,2,\Sigma} \le (1+\epsilon_1)\epsilon_2 \le \epsilon_2.$$

Gathering (6.26), (6.54), and (6.53), we get

$$(6.55) ||| \partial \varpi |||_{1,2,\Sigma} \lesssim \epsilon_2.$$

Step 4: $\|\varpi\|_{2,2,\Sigma}$. Note

 $\mathbf{T}\boldsymbol{\varpi}=0.$

By changing of coordinates $x_2 \rightarrow x_2 - \phi(t, x')$, we can get

$$(\partial_t + \partial_t \phi \partial_2)\tilde{\varpi} + \tilde{v}^i (\partial_i + \partial_i \phi \partial_2)\tilde{\varpi} = 0.$$

Multiplying \tilde{W} and integrating it on the whole space-time, we have

$$|||\varpi|||_{0,2,\Sigma}^2 \lesssim ||d\mathbf{v}||_{L_t^4 L_x^\infty} (1 + ||d\phi||_{L_{t,x'}^\infty}) ||\varpi||_{L_t^\infty L_x^2}^2 \lesssim \epsilon_2^2,$$

which when taken square yields

(6.56)

 $\|\!|\!| \varpi \|\!|_{0,2,\Sigma} \lesssim \epsilon_2.$

,

By using

(6.57)
$$\partial_{x'}\tilde{\varpi} = \nabla \varpi \cdot (1, d\phi)^{\mathrm{T}}$$

we thus have

$$\partial_{x'}^2 \tilde{\varpi} = \partial_{x'} (\nabla \varpi) \cdot (1, d\phi)^{\mathrm{T}} + \nabla \varpi \cdot (0, \partial_{x'} d\phi)^{\mathrm{T}}.$$

Combining (6.7), (6.26), (6.54), and (6.59), we see that

$$\|\partial_{x'}^{2}\tilde{\varpi}\|_{0,2,\Sigma} \leq \|\partial_{x'}(\nabla \varpi)\|_{0,2,\Sigma} \|(1,d\phi)^{\mathrm{T}}\|_{L_{t,x'}^{\infty}} + \|\nabla \varpi\|_{L_{t}^{\infty}L_{x'}^{2}(\Sigma)}\|\|(0,\partial_{x'}d\phi)^{\mathrm{T}}\|_{L_{t}^{2}L_{x'}^{\infty}(\Sigma)} \leq \epsilon_{2}\epsilon_{1} + \|\varpi\|_{L_{t}^{\infty}H_{x'}^{\frac{3}{2}}(\Sigma)}\|\|(0,\partial_{x'}d\phi)^{\mathrm{T}}\|_{L_{t}^{2}L_{x'}^{\infty}(\Sigma)} \leq \epsilon_{2}\epsilon_{1} + \|\varpi\|_{2,2,\Sigma}\|d\phi - dt\|_{L_{t}^{2}H_{x'}^{s}(\Sigma)} \leq \epsilon_{2}\epsilon_{1} + \epsilon_{1}\|\varpi\|_{2,2,\Sigma}.$$

Above, we use the trace theorem

(6.59)
$$\|\varpi\|_{L^{\infty}_{*}H^{\frac{3}{2}}(\Sigma)} \le \|\|\varpi\|_{2,2,\Sigma}.$$

Operating ∂_t on (6.57), we get

$$\begin{aligned} \|\partial_t \partial_{x'} \varpi\|_{0,2,\Sigma} &\leq \|\partial_t (\nabla \varpi)\|_{0,2,\Sigma} \cdot \|(1,d\phi)^{\mathrm{T}}\|_{L^{\infty}_{t,x}} \\ &+ \|\nabla \varpi\|_{L^{\infty}_t L^2_{x'}(\Sigma)} \cdot \|(0,\partial_t d\phi)^{\mathrm{T}}\|_{L^2_t L^{\infty}_{x'}(\Sigma)} \\ &\lesssim \epsilon_2 \epsilon_1 + \|\varpi\|_{L^{\infty}_t H^{\frac{3}{2}}_{x'}(\Sigma)} \cdot \|(0,\partial_{x'} d\phi)^{\mathrm{T}}\|_{L^4_t L^{\infty}_{x'}(\Sigma)} \\ &\lesssim \epsilon_2 \epsilon_1 + \|\varpi\|_{2,2,\Sigma} \cdot \|(0,\partial_{x'} d\phi)^{\mathrm{T}}\|_{L^4_t L^{\infty}_{x'}(\Sigma)} \\ &\lesssim \epsilon_2 \epsilon_1 + \epsilon_1 \|\varpi\|_{2,2,\Sigma}. \end{aligned}$$

Adding (6.58), (6.56), and (6.60) can give us

 $\|\|\varpi\|\|_{2,2,\Sigma} \lesssim \epsilon_2 \epsilon_1 + \epsilon_1 \|\|\varpi\|\|_{2,2,\Sigma}.$

For ϵ_1 is sufficiently small, we can see

(6.61) $\|\!|\!| \varpi \|\!|_{2,2,\Sigma} \lesssim \epsilon_2 \epsilon_1 \lesssim \epsilon_2.$

By using $s \in (\frac{7}{4}, 2]$, we have

(6.62) $\||\varpi\||_{s,2,\Sigma} \le \||\varpi\||_{2,2,\Sigma} \lesssim \epsilon_2.$

Combining (6.54), (6.55), (6.62), and (6.62), we complete the proof of Lemma 6.8.

Lemma 6.9. Let U be stated in Lemma 6.5. Then

(6.63) $||| 2^{j} (\mathbf{U} - P_{j} \mathbf{U}), dS_{k} \mathbf{U}, 2^{-j} d\partial S_{j} \mathbf{U} |||_{s-1,2,\Sigma} \lesssim ||\mathbf{U}||_{L_{t}^{\infty} H_{x}^{s}} + || d\mathbf{U} ||_{L_{t}^{4} L_{x}^{\infty}}.$

Proof. Let P be a standard multiplier of order 0 on \mathbb{R}^2 , such that P is additionally bounded on $L^{\infty}_x(\mathbb{R}^2)$. Clearly,

$$A_0(\mathbf{U})(P\mathbf{U})_t + A_1(\mathbf{U})(P\mathbf{U})_{x_1} + A_2(\mathbf{U})(P\mathbf{U})_{x_2} = -\sum_{i=0}^2 [P, A_i(\mathbf{U})]\partial_{x_i}\mathbf{U}.$$

By Lemma 6.5, this implies that

(6.64)
$$|||P\mathbf{U}|||_{s,2,\Sigma} \lesssim ||d\mathbf{U}||_{L_t^4 L_x^\infty} + ||\mathbf{U}||_{L_t^\infty H_x^s} + ||\mathbf{F}||_{L_t^1 H_x^{s-1}}.$$

To control the norm of $2^{j}(\mathbf{U} - P_{j}\mathbf{U})$, we write

$$2^{j}(\mathbf{U}-P_{j}\mathbf{U})=\sum_{k=1}^{2}\partial_{k}P_{k}\mathbf{U},$$

where P_k satisfies the above conditions for P. Using (6.64), we get

$$||| 2^{j} (\mathbf{U} - P_{j} \mathbf{U}) |||_{s-1,2,\Sigma} \lesssim ||\mathbf{U}||_{L_{t}^{\infty} H_{x}^{s}} + || d\mathbf{U} ||_{L_{t}^{4} L_{x}^{\infty}}.$$

Finally, applying (6.64) to $P = S_j$ and $P = 2^{-j} \partial S_j$ can give us

$$|||dS_{j}\mathbf{U}|||_{s-1,2,\Sigma} + |||2^{-j}d\partial S_{j}\mathbf{U}|||_{s-1,2,\Sigma} \lesssim ||\mathbf{U}||_{L_{t}^{\infty}H_{x}^{s}} + ||d\mathbf{U}||_{L_{t}^{4}L_{x}^{\infty}}$$

Therefore, the proof of Lemma 6.9 is completed.

As a direct corollary, we can see

Lemma 6.10. Let
$$(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$$
 and $\mathbf{J} = (\mathbf{v}, \rho)^{\mathrm{T}}$. Then

$$(6.65) \quad ||| 2^{j} (\mathbf{J} - P_{j} \mathbf{J}), dS_{j} \mathbf{J}, 2^{-j} d\partial S_{j} \mathbf{J} |||_{s-1,2,\Sigma} \lesssim ||\mathbf{v}, \rho||_{L_{t}^{\infty} H_{x}^{s}} + ||d\mathbf{v}, d\rho||_{L_{t}^{4} L_{x}^{\infty}} \lesssim \epsilon_{2}.$$

We are now ready to give a proof of Proposition 6.1.

proof of Proposition 6.1. For $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$, then $(\mathbf{v}, \rho, \varpi)$ is the solution of (5.2). Using Lemma 6.10, it suffices for us to verify that

$$\|\|\mathbf{g}^{\alpha\beta}-\mathbf{m}^{\alpha\beta}\|\|_{s,2,\Sigma_{\theta,r}}\lesssim\epsilon_2.$$

By Corollary 6.6, one has

$$\sup_{\theta,r} \| \| \mathbf{v} \| \|_{s,2,\Sigma_{\theta,r}} + \sup_{\theta,r} \| \| \rho \| \|_{s,2,\Sigma_{\theta,r}} \lesssim \epsilon_2$$

Using the expression of \mathbf{g} , and using Lemma 6.7, we arrive at the bound

$$\begin{aligned} \|\mathbf{g}^{\alpha\beta} - \mathbf{m}^{\alpha\beta}\|_{s,2,\Sigma_{\theta,r}} &\lesssim \|\|\mathbf{v}\|_{s,2,\Sigma_{\theta,r}} + \|\|\mathbf{v}\cdot\mathbf{v}\|_{s,2,\Sigma_{\theta,r}} + \||c_s^2(\rho) - c_s^2(0)\|_{s,2,\Sigma_{\theta,r}} \\ &\lesssim \epsilon_2. \end{aligned}$$

Consequently, the conclusion of Proposition 6.1 holds.

6.2. The null frame. We introduce a null frame along Σ as follows. Let

$$V = (dr)^*,$$

where r is the defining function of the foliation Σ , and where * denotes the identification of covectors and vectors induced by **g**. Then V is the null geodesic flow field tangent to Σ . Let

(6.66)
$$\sigma = dt(V), \qquad l = \sigma^{-1}V.$$

Thus l is the g-normal field to Σ normalized so that dt(l) = 1, hence

(6.67)
$$l = \langle dt, dx_2 - d\phi \rangle_{\mathbf{g}}^{-1} (dx_2 - d\phi)^*.$$

So the coefficients l^j are smooth functions of \mathbf{v}, ρ and $d\phi$. Conversely,

(6.68)
$$dx_2 - d\phi = \langle l, \partial_{x_2} \rangle_{\mathbf{g}}^{-1} l^*$$

so that $d\phi$ is a smooth function of \mathbf{v}, ρ and the coefficients of l.

Next, we introduce the vector fields e_1 tangent to the fixed-time slice Σ^t of Σ . We do this by applying Grahm-Schmidt orthogonalization in the metric **g** to the Σ^t -tangent vector fields $\partial_{x_1} + \partial_{x_1}\phi \partial_{x_2}$.

Finally, we let

$$\underline{l} = l + 2\partial_t$$

It follows that $\{l, \underline{l}, e_1\}$ form a null frame in the sense that

$$\begin{split} \langle l, \underline{l} \rangle_{\mathbf{g}} &= 2, \qquad \langle e_1, e_1 \rangle_{\mathbf{g}} = 1, \\ \langle l, l \rangle_{\mathbf{g}} &= \langle \underline{l}, \underline{l} \rangle_{\mathbf{g}} = 0, \quad \langle l, e_1 \rangle_{\mathbf{g}} = \langle \underline{l}, e_1 \rangle_{\mathbf{g}} = 0. \end{split}$$

The coefficient of each of the fields is a smooth function of (\mathbf{v}, ρ) and $d\phi$, and by assumption, we also have the pointwise bound

$$e_1 - \partial_{x_1}| + |l - (\partial_t + \partial_{x_2})| + |\underline{l} - (-\partial_t + \partial_{x_2})| \lesssim \epsilon_1.$$

After that, we can state the following lemma concerning the decomposition of the Ricci curvature tensor.

Corollary 6.11. Let R be the Riemann curvature tensor of the metric g. Let $e_0 = l$. Then

(6.69)
$$R_{ll} = l(f_2) + f_1$$

where $|f_1| \lesssim |\partial \varpi| + |d\mathbf{g}|^2$, $|f_2| \lesssim |d\mathbf{g}|$,

(6.70)
$$\|f_2\|_{L^2_t H^{s-1}_{x'}(\Sigma)} + \|f_1\|_{L^1_t H^{s-1}_{x'}(\Sigma)} \lesssim \epsilon_2,$$

and for any $t \in [0, T]$,

(6.71)
$$\|f_2(t,\cdot)\|_{C^{\delta}_{x'}(\Sigma^t)} \lesssim \|d\mathbf{g}\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

Proof. By using the remarkable decomposition in Klainerman-Rodianiski [26], we have

$$R_{ll} = l(f_2) - \frac{1}{2} l^{\alpha} l^{\beta} \Box_{\mathbf{g}} \mathbf{g}_{\alpha\beta} + H,$$

where $|H| \lesssim |d\mathbf{g}|^2$ and

$$z = l^{\gamma} \mathbf{g}^{\alpha\beta} \partial_{\beta} \mathbf{g}_{\alpha\gamma} - \frac{1}{2} \mathbf{g}^{\alpha\beta} l(\mathbf{g}_{\alpha\beta}).$$

According to (2.2), we derive that

$$|f_1| \lesssim |\partial \varpi| + |d\mathbf{g}|^2, |f_2| \lesssim |d\mathbf{g}|.$$

Due to Lemma 6.5 and Lemma 6.7, we get

$$\|f_2\|_{L^2_t H^{s-1}_{x'}(\Sigma)} + \|f_1\|_{L^1_t H^{s-1}_{x'}(\Sigma)} \lesssim \epsilon_2.$$

It's clear that the estimate (6.71) can be obtained directly from Sobolev embeddings. Thus, the proof is completed.

6.3. The estimate of connection coefficients. Define

$$\chi = \langle D_{e_1}l, e_1 \rangle_{\mathbf{g}}, \qquad l(\ln \sigma) = \frac{1}{2} \langle D_l \underline{l}, l \rangle_{\mathbf{g}}.$$

For σ , we set the initial data $\sigma = 1$ at the time -2. Thanks to Proposition 6.1, we have

(6.72)
$$\|\chi\|_{L^2_t H^{s-1}_{x'}(\Sigma)} + \|l(\ln \sigma)\|_{L^2_t H^{s-1}_{x'}(\Sigma)} \lesssim \epsilon_1.$$

In a similar way, if we expand $l = l^{\alpha} \partial_{\alpha}$ in the tangent frame $\partial_t, \partial_{x'}$ on Σ , then

(6.73)
$$l^0 = 1, \quad ||l^1||_{s-1,2,\Sigma} \lesssim \epsilon_1$$

Lemma 6.12. Let χ be defined as before. Then

$$\|\chi\|_{L^2_t H^{s-1}(\Sigma)} \lesssim \epsilon_2$$

Furthermore, for any $t \in [0, T]$,

(6.75)
$$\|\chi\|_{C^{\delta}_{x'}(\Sigma^t)} \lesssim \epsilon_2 + \|d\mathbf{g}\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

Proof. The famous transport equation for χ along null hypersurfaces (see references [25] and [41]) can be described as

$$l(\chi) = \langle R(l, e_1)l, e_1 \rangle_{\mathbf{g}} - \chi^2 - l(\ln \sigma)\chi.$$

Due to Corollary 6.11, we write the above equation as

(6.76)
$$l(\chi - f_2) = f_1 - \chi^2 - l(\ln \sigma)\chi,$$

where

(6.77)
$$\|f_2\|_{L^2_t H^{s-1}_{x'}(\Sigma)} + \|f_1\|_{L^1_t H^{s-1}_{x'}(\Sigma)} \lesssim \epsilon_2,$$

and for any $t \in [0, T]$,

(6.78)
$$\|f_2(t,\cdot)\|_{C^{\delta}_{x'}(\Sigma^t)} \lesssim \|d\mathbf{g}\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

Let Λ be the fractional derivative operator in the x' variables. We thus have

(6.79)
$$\|\Lambda^{s-1}(\chi - f_2)(t, \cdot)\|_{L^2_{x'}(\Sigma^t)} \lesssim \|[\Lambda^{s-1}, l](\chi - f_2)\|_{L^1_t L^2_{x'}(\Sigma^t)} + \|\Lambda^{s-1} \left(f_1 - \chi^2 - l(\ln \sigma)\chi\right)\|_{L^1_t L^2_{x'}(\Sigma^t)}.$$

A direct calculation shows that

(6.80)

$$\begin{split} \|\Lambda^{s-1} \left(f_1 - \chi^2 - l(\ln \sigma) \chi \right) \|_{L^1_t L^2_{x'}(\Sigma^t)} &\lesssim \|f_1\|_{L^1_t H^{s-1}_{x'}(\Sigma^t)} + \|\chi\|^2_{L^2_t H^{s-1}_{x'}(\Sigma^t)} \\ &+ \|\chi\|_{L^2_t H^{s-1}_{x'}(\Sigma^t)} \cdot \|l(\ln \sigma)\|_{L^2_t H^{s-1}_{x'}(\Sigma^t)}, \end{split}$$

where we use the fact that $H^{s-1}_{x'}(\Sigma^t)$ is an algebra.

We next bound

$$\begin{split} \| [\Lambda^{s-1}, l](\chi - f_2) \|_{L^2_{x'}(\Sigma^t)} &\leq \| \partial_{\alpha} l^{\alpha} (\chi - f_2)(t, \cdot) \|_{H^{s-1}_{x'}(\Sigma^t)} \\ &+ \| [\Lambda^{s-1} \partial_{\alpha}, l^{\alpha}](\chi - f_2)(t, \cdot) \|_{L^2_{x'}(\Sigma^t)}. \end{split}$$

34

By Kato-Ponce commutator estimate and Sobolev embeddings, the above could be bounded by

(6.81)
$$\|l^1(t,\cdot)\|_{H^{s-1}_{x'}(\Sigma^t)} \|\Lambda^{s-1}(\chi-f_2)(t,\cdot)\|_{L^2_{x'}(\Sigma^t)}.$$

Gathering (6.72), (6.73), (6.77), (6.79), (6.80), and (6.81) together, we thus prove that

$$\sup_{t} \|(\chi - f_2)(t, \cdot)\|_{H^{s-1}_{x'}(\Sigma^t)} \lesssim \epsilon_2.$$

From (6.76), we can see

(6.82)
$$\|\chi - f_2\|_{C^{\delta}_{x'}} \lesssim \|f_1\|_{L^1_t C^{\delta}_{x'}} + \|\chi^2\|_{L^1_t C^{\delta}_{x'}} + \|l(\ln \sigma)\chi\|_{L^1_t C^{\delta}_{x'}}.$$

Using the Sobolev imbedding $H^1(\mathbb{R}) \hookrightarrow C^\delta(\mathbb{R})$ and Gronwall's inequality, we can derive that

$$\|\chi\|_{C^{\delta}_{-\prime}(\Sigma^t)} \lesssim \epsilon_2 + \|d\mathbf{g}\|_{C^{\delta}_x(\mathbb{R}^2)}$$

6.4. The proof of Proposition 6.2. We first recall that

$$G(\mathbf{v},\rho) = |||d\phi(t,x') - dt|||_{s,2,\Sigma}.$$

Using (6.68) and the estimate of $|||\mathbf{g} - \mathbf{m}|||_{s,2,\Sigma}$ in Proposition 6.1, then the estimate (6.3) follows from the bound

$$|||l - (\partial_t - \partial_{x_2})|||_{s,2,\Sigma} \lesssim \epsilon_2,$$

where it is understood that one takes the norm of the coefficients of $l - (\partial_t - \partial_{x_2})$ in the standard frame on \mathbb{R}^{2+1} . The geodesic equation, together with the bound for Christoffel symbols $\|\Gamma^{\alpha}_{\beta\gamma}\|_{L^4_t L^\infty_x} \lesssim \|d\mathbf{g}\|_{L^4_t L^\infty_x} \lesssim \epsilon_2$, imply that

$$\|l - (\partial_t - \partial_{x_2})\|_{L^{\infty}_{t,x}} \lesssim \epsilon_2,$$

so it suffices to bound the tangential derivatives of the coefficients of $l - (\partial_t - \partial_{x_2})$ in the norm $L_t^2 H_{x'}^{s-1}(\Sigma)$. By Proposition 6.1, we can estimate Christoffel symbols

$$\|\Gamma^{\alpha}_{\beta\gamma}\|_{L^{2}_{t}H^{s-1}_{\pi'}(\Sigma^{t})} \lesssim \epsilon_{2}$$

Note that $H^{s-1}_{x'}(\Sigma^t)$ is a algebra. We then have

$$\|\Gamma^{\alpha}_{\beta\gamma}e_1^{\beta}l^{\gamma}\|_{L^2_tH^{s-1}_{x'}(\Sigma^t)} \lesssim \epsilon_2.$$

We are now in a position to establish the following bound,

$$\| \langle D_{e_1}l, e_1 \rangle \|_{L^2_t H^{s-1}_{x'}(\Sigma^t)} + \| \langle D_{e_1}l, \underline{l} \rangle \|_{L^2_t H^{s-1}_{x'}(\Sigma^t)} + \| \langle D_ll, \underline{l} \rangle \|_{L^2_t H^{s-1}_{x'}(\Sigma^t)} \lesssim \epsilon_2.$$

The first term is χ , which has estimated in Lemma 6.12. For the second term, noting

$$\langle D_{e_1}l, \underline{l} \rangle = \langle D_{e_1}l, 2\partial_t \rangle = -2 \langle D_{e_1}\partial_t, l \rangle,$$

then it can be bounded by using Proposition 6.1. Similarly, we can control the last term by proposition 6.1. It remains for us to show that

$$\|d\phi(t,x') - dt\|_{C^{1,\delta}_{-\epsilon'}(\mathbb{R})} \lesssim \epsilon_2 + \|d\mathbf{g}(t,\cdot)\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

To do that, it suffices to establish

$$\|l(t,\cdot) - (\partial_t - \partial_{x_2})\|_{C^{1,\delta}_{x'}(\mathbb{R})} \lesssim \epsilon_2 + \|d\mathbf{g}(t,\cdot)\|_{C^{\delta}_x(\mathbb{R}^2)}.$$

The coefficients of e_1 are small in $C_{x'}^{\delta}(\Sigma^t)$ perturbations of their constant-coefficient analogs, so it suffices to show that

 $\|\langle D_{e_1}l, e_1\rangle(t, \cdot)\|_{C^{\delta}_{-\prime}(\Sigma^t)} + \|\langle D_{e_1}l, \underline{l}\rangle(t, \cdot)\|_{C^{\delta}_{-\prime}(\Sigma^t)} \lesssim \epsilon_2 + \|d\mathbf{g}(t, \cdot)\|_{C^{\delta}_x(\mathbb{R}^2)}.$

Above, the first term is bounded by Lemma 6.12, and the second by using

 $\| \langle D_{e_1} \partial_t, l \rangle (t, \cdot) \|_{C^{\delta}_{-\prime}(\Sigma^t)} \lesssim \| d\mathbf{g}(t, \cdot) \|_{C^{\delta}_x(\mathbb{R}^2)}.$

Consequently, we complete the proof of Proposition 6.2.

7. PROOF OF PROPOSITION 5.1 AND CONTINUOUS DEPENDENCE

7.1. **Proof of Proposition 5.1.** To prove Proposition 5.1, let us first give a type of Stricharz estimates. In the above sections, we obtain characteristic energy estimates of solutions and get enough regularity of hypersurfaces. By using the result of Smith and Tataru([41], Proposition 7.1, page 36), we can directly obtain the following

Proposition 7.1. Suppose that $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ and $G(\mathbf{v}, \rho) \leq 2\epsilon_1$. For $1 \leq r \leq s+1$, then the linear equation $\Box_{\mathbf{g}} f = 0$ is well-posed with the initial data in $H^r \times H^{r-1}$. Moreover, the following estimates

$$\|\langle \partial \rangle^k f\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{H^r} + \|f_1\|_{H^{r-1}}, \quad k < r - \frac{3}{4}$$

and

$$\|f\|_{L^{\infty}_{t}H^{s}_{x}} + \|\partial_{t}f\|_{L^{\infty}_{t}H^{s-1}_{x}} \lesssim \|f_{0}\|_{H^{r}} + \|f_{1}\|_{H^{r-1}},$$

hold.

Proposition 7.2. Suppose that $(\mathbf{v}, \rho, \varpi) \in \mathcal{H}$ and $G(\mathbf{v}, \rho) \leq 2\epsilon_1$. Then (\mathbf{v}, ρ) of (5.2) satisfies the Strichartz estimate

(7.1)
$$\|d\mathbf{v}, d\rho\|_{L^4_t C^\delta_{\pi}} \le \epsilon_2.$$

Proof. Note (5.2). Using Duhamel's principle, we can get

$$\begin{aligned} \|d\mathbf{v}, d\rho\|_{L_{t}^{4}C_{x}^{\delta}} &\leq C(\|\partial\varpi\|_{L_{t}^{1}H_{x}^{s-1}} + \|\mathbf{Q}\|_{L_{t}^{1}H_{x}^{s-1}} + \|\mathbf{E}\|_{L_{t}^{1}H_{x}^{s-1}} + \|\mathcal{D}|_{L_{t}^{1}H_{x}^{s-1}}) \\ &\leq 4C\|\partial\varpi\|_{L_{t}^{\infty}H_{x}^{1}} + C[1 - (-1)]^{\frac{3}{4}} \|d\mathbf{v}, d\rho\|_{L_{t}^{4}L_{x}^{\infty}} \|d\mathbf{v}, d\rho\|_{L_{t}^{\infty}H_{x}^{s-1}} \\ &\leq C(\|\rho_{0}\|_{H^{s}} + \|\mathbf{v}_{0}\|_{H^{s}} + \|\partial\varpi_{0}\|_{H^{2}}) \exp\left(1 + \|d\mathbf{v}, d\rho\|_{L_{t}^{4}L_{x}^{\infty}}\right)^{2} \\ &\leq C\epsilon_{3} \leq \epsilon_{2}, \end{aligned}$$

where we use (5.3) and Lemma 6.5.

Proof of Proposition 5.1. By using Proposition 6.2, we know that (5.5) holds. By using Proposition 7.1, we obtain (5.7) and (5.8). Using (2.10) and (7.1), we have

(7.2)
$$\| (\mathbf{v}, \rho) \|_{L^{\infty}_{t}H^{s}_{x}} + \| (\partial_{t}\mathbf{v}, \partial_{t}\rho) \|_{L^{\infty}_{t}H^{s-1}_{x}} + \| \varpi \|_{L^{\infty}_{t}H^{2}_{x}} + \| \partial_{t}\varpi \|_{L^{\infty}_{t}H^{1}_{x}}$$
$$\lesssim \epsilon_{3}(1 + \epsilon_{3}^{\frac{1}{2}}) \exp(\int_{-1}^{1} [1 + \epsilon_{2}]^{2})$$
$$\le \epsilon_{2}.$$

The estimate (7.2) combining (7.1) can yield (5.6). Therefore, we complete the proof of Proposition 5.1.

36

7.2. Continuous dependence. We will discuss the continuous dependence by referring Ifrim-Tataru's paper [20].

Corollary 7.3 (Continuous dependence on data). If $(\mathbf{v}_{0j}, \rho_{0j}, \varpi_{0j})$ is a sequence of initial data converging to $(\mathbf{v}_0, \rho_0, \varpi_0)$ in space $H^s \times H^s \times H^2$, then the associated solutions $(\mathbf{v}_j, \rho_j, \varpi_j)$ of (5.2) converge uniformly to $(\mathbf{v}, \rho, \varpi)$ on [0, T] in $\in H^s_x \times H^s_x \times H^s_x \times H^s_x$. Moreover,

(7.3)
$$\begin{aligned} \|(\mathbf{v}_j - \mathbf{v})(t)\|_{H^s_x} + \|(\rho_j - \rho)(t)\|_{H^s_x} + \|(\varpi_j - \varpi)(t)\|_{H^2_x} \\ \lesssim \|\mathbf{v}_{0j} - \mathbf{v}_0\|_{H^s} + \|\rho_{0j} - \rho_0\|_{H^s} + \|\varpi_{0j} - \varpi_0\|_{H^2}. \end{aligned}$$

Before we prove it, let us introduce a frequency envelope in [20, 48].

Definition 7.1. We say that $\{c_k\}_{k \in \mathbb{N}^+} \in \ell^2$ is a frequency envelope for a function f in H^s if we have the following two properties:

(1) Energy bound:

$$(7.4) ||P_k f||_{H^s} \lesssim c_k$$

(2) Slowly varying:

(7.5)
$$\frac{c_k}{c_j} \lesssim 2^{\delta|j-k|}, \quad j,k \in \mathbb{N}^+.$$

We call such envelopes sharp, if

$$\|f\|_{H^s}^2 \approx \sum_{k \ge 0} c_k^2.$$

proof of Corollary 7.3. We divide the proof into three steps.

Step 1: the convergence in a weaker space. By using Theorem 2.11, it yields

$$\|(\mathbf{v}_{j} - \mathbf{v}, \rho_{j} - \rho)(t, \cdot)\|_{H^{s-1}_{x}} + \|(\varpi_{j} - \varpi)(t, \cdot)\|_{H^{1}_{x}} \lesssim \|(\mathbf{v}_{0j} - \mathbf{v}_{0}, \rho_{0j} - \rho_{0})\|_{H^{s}} + \|\varpi_{0j} - \varpi_{0}\|_{H^{2}}$$

Taking $j \to \infty$, we obtain

$$\lim_{j \to \infty} (\mathbf{v}_j, \rho_j) \to (\mathbf{v}, \rho) \text{ in } H^{s-1}_x, \qquad \lim_{j \to \infty} \varpi_j \to \varpi \text{ in } H^1_x.$$

By interpolation formula, we then have

$$\|(\mathbf{v}_{j} - \mathbf{v}, \rho_{j} - \rho)\|_{H_{x}^{\sigma}} \lesssim \|(\mathbf{v}_{j} - \mathbf{v}, \rho_{j} - \rho)\|_{H_{x}^{s-1}}^{s-\sigma} \|(\mathbf{v}_{j} - \mathbf{v}, \rho_{j} - \rho)\|_{H_{x}^{s}}^{1+\sigma-s}, \quad s-1 \le \sigma < s.$$

and

$$\|\varpi_j - \varpi\|_{H^{\gamma}_x} \lesssim \|\varpi_j - \varpi\|_{H^1_x}^{2-\gamma}\|\varpi_j - \varpi\|_{H^2_x}^{\gamma-1}, \quad 1 \le \gamma < 2.$$

As a result, we get

$$\lim_{j \to \infty} (\mathbf{v}_j, \rho_j) \to (\mathbf{v}, \rho) \text{ in } H_x^{\sigma}, \quad 0 \le \sigma < s,$$

and

$$\lim_{j \to \infty} \varpi_j \to \varpi \quad \text{in} \quad H_x^{\gamma}, \quad 0 \le \gamma < 2.$$

Step 2: the construction of smooth solutions. Consider the initial data $\mathbf{v}_0 = (v_0^1, v_0^2)$ and $\rho_0 \in H^s$. We set $\mathbf{U}_0 = (v_0^1, v_0^2, \rho_0)$. By [20], there exists a sharp frequency envelope for v_0^1, v_0^2 , and ρ respectively. Let $\{c_k^{(i)}\}_{k\geq 0} (i = 1, 2, 3)$ be a sharp frequency envelope for v_0^1, v_0^2, ρ_0 in H^s . Set $\mathbf{C}_k = (c_k^{(1)}, c_k^{(2)}, c_k^{(3)})$. We choose

a family of regularizations $\mathbf{U}_0^h = (v_0^{1h}, v_0^{2h}, \rho_0^h) \in H^\infty := \bigcap_{s=0}^\infty H^s$ at frequencies $\leq 2^h$ where h is a dyadic frequency parameter. Denote

$$\varpi_0^h = \bar{\rho}^{-1} \mathrm{e}^{-\rho_0^h} \mathrm{curl} \mathbf{v}_0^h$$

Then we have $\varpi_0^h \in H^\infty$. At the same time, there exists a sharp frequency for ϖ_0^h , we record it $\{c_k^{(4)}\}_{k\geq 0}$. Also, the function $v_0^{1h}, v_0^{2h}, \rho_0^h, \varpi_0^h$, and ϖ_0^h has the following properties:

(i) uniform bounds

$$||P_k v_0^{ih}||_{H^s} \lesssim c_k^{(i)}, i = 1, 2, \quad ||P_k \rho_0^h||_{H^s} \lesssim c_k^{(3)}, \quad ||P_k \varpi_0^h||_{H^2} \lesssim c_k^{(4)},$$

(ii) high frequency bounds

$$\|v_0^{ih}\|_{H^{s+j}} \lesssim 2^{jh} c_h^{(i)}, i = 1, 2, \quad \|\rho_0^h\|_{H^{s+j}} \lesssim 2^{jh} c_h^{(3)}, \quad \|\varpi_0^h\|_{H^{2+j}} \lesssim 2^{jh} c_h^{(4)},$$

(iii) difference bounds

$$\begin{aligned} \|v_0^{i(h+1)} - v_0^{ih}\|_{L^2} &\lesssim 2^{-sh} c_h^{(i)}, \quad i = 1, 2, \\ \|\rho_0^{h+1} - \rho_0^h\|_{L^2} &\lesssim 2^{-sh} c_h^{(3)}, \quad \|\varpi_0^{h+1} - \varpi_0^h\|_{L^2} &\lesssim 2^{-2h} c_h^{(4)}, \end{aligned}$$

(iv) limit

(7.6)
$$\begin{aligned} \mathbf{U}_0 &= \lim_{h \to \infty} \mathbf{U}_0^h \qquad \text{in } H^s, \\ \boldsymbol{\varpi}_0 &= \lim_{h \to \infty} \boldsymbol{\varpi}_0^h \qquad \text{in } H^2. \end{aligned}$$

Taking the smooth initial data $(v_0^{1h}, v_0^{2h}, \rho_0^h, \varpi_0^h)$, we obtain a family of smooth solutions $(v^{1h}, v^{2h}, \rho^h, \varpi^h)$ satisfying (5.2). Based on the existence of (5.2), this yields a time interval [0, T] where all these solutions $(v^{1h}, v^{2h}, \rho^h, \varpi^h)$ exists, and T depends only on the size of $\|\mathbf{v}_0\|_{H^s} + \|\rho_0\|_{H^s} + \|\varpi_0\|_{H^2}$. Furthermore, we have:

(i) high frequency bounds

(7.7)
$$\|v^{ih}\|_{H^{s+j}_x} \lesssim 2^{jh} c_h^{(i)}, \quad \|\rho^h\|_{H^{s+j}_x} \lesssim 2^{jh} c_h^{(3)}, \quad \|\varpi^h\|_{H^{2+j}_x} \lesssim 2^{jh} c_h^{(4)},$$

(ii) difference bounds

(7.8) $\|v^{i(h+1)} - v^{ih}\|_{L^2_x} \lesssim 2^{-sh} c_h^{(i)}, \quad \|\rho^{h+1} - \rho^h\|_{L^2_x} \lesssim 2^{-sh} c_h^{(3)}, \quad \|\varpi^{h+1} - \varpi^h\|_{L^2_x} \lesssim 2^{-2h} c_h^{(4)}.$ Taking the convergence $h \to \infty$ on (7.8), we get

$$\|\mathbf{v} - \mathbf{v}^h\|_{L^2_x} \lesssim 2^{-sh}, \quad \|\rho - \rho^h\|_{L^2_x} \lesssim 2^{-sh}, \quad \|\varpi - \varpi^h\|_{L^2_x} \lesssim 2^{-2h},$$

where $\mathbf{v}^h = (v^{1h}, v^{2h})$. By using

$$\rho - \rho^h = \sum_{m=h}^{\infty} \rho^{m+1} - \rho^m,$$

we obtain

$$\|\rho - \rho^h\|_{H^s_x} \lesssim c^{(3)}_{\geq h} := \left(\sum_{m \geq h} [c^{(3)}_m]^2\right)^{\frac{1}{2}}$$

Similarly, we also have

$$\|v^i - v^{ih}\|_{H^s_x} \lesssim c^{(i)}_{\ge h}, \quad \|\rho - \rho^h\|_{H^s_x} \lesssim c^{(3)}_{\ge h}, \quad \|\varpi - \varpi^h\|_{H^2_x} \lesssim c^{(4)}_{\ge h}.$$

$$\begin{aligned} \| (\mathbf{v}_{j} - \mathbf{v})(t) \|_{H_{x}^{s}} \lesssim \| (v_{j}^{1h} - v^{1h})(t) \|_{H_{x}^{s}} + \| (v^{1h} - v^{1})(t) \|_{H_{x}^{s}} + \| (v_{j}^{1h} - v_{j}^{1})(t) \|_{H_{x}^{s}} \\ &+ \| (v_{j}^{2h} - v^{2h})(t) \|_{H_{x}^{s}} + \| (v^{2h} - v^{2})(t) \|_{H_{x}^{s}} + \| (v_{j}^{2h} - v_{j}^{2})(t) \|_{H_{x}^{s}}, \end{aligned}$$

and

(7.10)
$$\|(\rho_j - \rho)(t)\|_{H^s_x} \lesssim \|(\rho_j^h - \rho^h)(t)\|_{H^s_x} + \|(\rho^h - \rho)(t)\|_{H^s_x} + \|(\rho_j^h - \rho_j)(t)\|_{H^s_x},$$

and

(7.11)

$$\|(\varpi_j - \varpi)(t)\|_{H^2_x} \lesssim \|(\varpi_j^h - \varpi^h)(t)\|_{H^2_x} + \|(\varpi^h - \varpi)(t)\|_{H^2_x} + \|(\varpi_j^h - \varpi_j)(t)\|_{H^2_x}.$$

Now, let us first estimate (7.10). Taking the limit for $j \to \infty$, this leads to

(7.12)
$$\|(\rho^h - \rho)(t)\|_{H^s_x} \to 0, \quad j \to \infty, \quad \text{for fixed } h,$$

and

(7.13)
$$\|(\rho_j^h - \rho_j)(t)\|_{H^2_x} \to 0, \quad j \to \infty.$$

Let $\{c_k^{(i)j}\}_{k\geq 0}$ be frequency envelopes for the initial data v_{0j}^i in H^s , i = 1, 2. Let $\{c_k^{(3)j}\}_{k\geq 0}$ be frequency envelopes for the initial data ρ_{0j} in H^s . Let $\{c_k^{(4)j}\}_{k\geq 0}$ be frequency envelopes for the initial data ϖ_{0j} in H^2 . By (7.12) and (7.13), we can update (7.10) by

(7.14)
$$\|(\rho_j - \rho)(t)\|_{H^s_x} \lesssim \|(\rho_j^h - \rho^h)(t)\|_{H^s_x} + c^{(3)}_{\geq h} + c^{(3)j}_{\geq h},$$

On the other hand, we know

(7.15)
$$\rho_{0j}^h \to \rho_0^h \quad \text{in } H_x^\sigma, \ 0 \le \sigma < \infty.$$

By using a similar way in Step 1, we can derive that

(7.16)
$$\rho_j^h \to \rho^h \quad \text{in } H_x^\sigma, \ 0 \le \sigma < \infty.$$

From (7.15), it yields

(7.17)
$$c_k^{(3)j} \to c_k^{(3)}, \ j \to \infty$$

Therefore, using (7.15), (7.16), (7.17), and passing to the limit $j \to \infty$ for (7.14), we have

(7.18)
$$\lim_{j \to \infty} \|(\rho_j - \rho)(t)\|_{H^s_x} \lesssim c^{(3)}_{\ge h}.$$

Taking $h \to \infty$ in (7.18), we derive

(7.19)
$$\lim_{j \to \infty} \|(\rho_j - \rho)(t)\|_{H^s_x} = 0.$$

In a similar process, we can also obtain

$$\lim_{j \to \infty} \| (\mathbf{v}_j - \mathbf{v})(t) \|_{H^s_x} = 0, \quad \lim_{j \to \infty} \| (\varpi_j - \varpi)(t) \|_{H^2_x} = 0.$$

At this stage, we have finished the proof of Theorem 1.2.

39

Acknowledgments

The author would like to express the great gratitude to the reviewers, who give us much helpful advice. The author also express great thanks Professor Lars Andersson for hours of discussions throughout the preparation of this work. The author is also supported by National Natural Science Foundation of China(Grant No. 12101079) and Natural Science Foundation of Hunan Province, China(Grant No. 2021JJ40561).

References

- L.Abbrescia, J. Speck. Remarkable localized integral identities for 3D compressible Euler flow and the double-null framework, arXiv:2003.02815
- [2] P.T. Allen, L. Andersson, A. Restuccia. Local well-posedness for membranes in the light cone gauge, Comm. Math. Phys., 301, 383-410 (2011).
- [3] L. Andersson, V. Moncrief. Elliptic-hyperbolic systems and the Einstein equations, Ann. Henri Poincaré, 4, 1-34(2003).
- [4] H. Bahouri and J.Y. Chemin. Equations dóndes quasilineaires et estimation de Strichartz, Amer. J. Math., 121, 1337-1377 (1999).
- [5] H. Bahouri, J. Y. Chemin, and R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, vol. 343, Springer, Heidelberg, 2011.
- [6] J. Bourgain, D. Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces, Invent. Math., 201:97-157 (2015).
- [7] J. Bourgain, L. Dong. Strong ill-posedness of the 3D incompressible Euler equation in borderline spaces, Int. Math. Res. Notices, 00(0), 1-110 (2019).
- [8] D. Chae. On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. Comm. Pure Appl. Math. 55(5), 654-678.
- [9] D. Chae. On the Euler equations in the critical Triebel-Lizorkin spaces. Arch. Ration. Mech. Anal. 170(3), 185-210 (2003).
- [10] D. Christodoulou and S. Miao. Compressible flow and Euler's equations, Surveys of Modern Mathematics, vol. 9, International Press, Somerville, MA; Higher Education Press, Beijing, 2014.
- [11] D. Coutand, H. Lindblad, and S. Shkoller. A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum, Comm. Math. Phys. 296(2), 559-587 (2010).
- [12] D. Coutand and S. Shkoller. Well-posedness in smooth function spaces for moving-boundary 1-D compressible Euler equations in physical vacuum, Comm. Pure Appl. Math. 64(3), 328-366 (2011).
- [13] D. Coutand and S. Shkoller. Well-posedness in smooth function spaces for the movingboundary three-dimensional compressible Euler equations in physical vacuum, Arch. Ration. Mech. Anal. 206(2), 515-616 (2012).
- [14] M. Disconzi, C. Luo, G. Mazzone and J. Speck. Rough sound waves in 3D compressible Euler flow with vorticity, arXiv:1909.02550v1, 100 pages.
- [15] B. Ettinger, H. Lindblad. A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates, Ann. Math., 185, 311-330 (2017).
- [16] Z.H. Guo, K.L. Li. Remarks on the well-posedness of the Euler equations in the Triebel-Lizorkin spaces, arXiv:1903.09437
- [17] J. Jang and N. Masmoudi. Well-posedness for compressible Euler equations with physical vacuum singularity, Comm. Pure Appl. Math. 62(10), 1327-1385 (2009).
- [18] T. Hughes, T. Kato and J.E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear electrodynamics and general relativity, Arch. Rat. Mech. Anal., 63, 273- 294 (1977).
- [19] M. Ifrim, D. Tataru. The compressible Euler equations in a physical vacuum: a comprehensive Eulerian approach, arXiv:2007.05668, 79pages.
- [20] M. Ifrim, D. Tataru. Local well-posedness for quasilinear problems: a primer, arXiv:2008.05684
- [21] D.G. Geba. A local well-posedness result for the quasilinear wave equation in ℝ²⁺¹, Comm. Part. Diff. Equ, 29, 323-360 (2004).

- [22] L. V. Kapitanskij. Norm estimates in Besov and Lizorkin-Triebel spaces for the solutions of second-order linear hyperbolic equations, J. Soviet Math. 56, 2348-2389 (1991).
- [23] T. Kato, G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41(7), 891-907 (1988).
- [24] S. Klainerman. A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations, Int. Math. Res. Notices, 5, 221-274 (2001).
- [25] S. Klainerman, I. Rodnianski. Improved local well-posedness for quasilinear wave equations in dimension three, Duke Math. J., 117, 1-124(2003).
- [26] S. Klainerman, I. Rodnianski. Rough solutions of the Einstein vacuum equations, Ann. Math., 161, 1143-1193 (2005).
- [27] S. Klainerman, I. Rodnianski and J. Szeftel. The bounded L² curvature conjecture, Invent. Math., 202(1), 91216 (2015).
- [28] Z. Lei, Y. Du, Q.T. Zhang. Singularities of solutions to compressible Euler equations with vacuum, Math. Res. Lett., 20, 41-50 (2013).
- [29] T. Li. Global classical solutions for quasilinear hyperbolic systems, Research in Applied Mathematics 32, Wiley/Masson, Paris, 1994.
- [30] T. Li, D. Wang. Blowup phenomena of solutions to the Euler equations for compressible fluid flow, J. Differential Equations, 221, 91-101 (2006).
- [31] H. Lindblad. Counterexamples to local existence for quasilinear wave equations, Math. Res. Letters, 5(5), 605-622 (1998).
- [32] J. Luk, J. Speck. Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity, Invent. Math., 214, 1-169 (2018).
- [33] J. Luk, J. Speck. The hidden null structure of the compressible Euler equations and a prelude to applications, to appear in Journal of Hyperbolic Differential Equations.
- [34] J. Luk, J. Speck. The stability of simple plane-symmetric shock formation for 3D compressible Euler flow with vorticity and entropy, arXiv:2107.03426
- [35] A. Majda. Compressible fluid flow and systems of conservation laws in several space variables, Applied Mathematical Sciences, 53. Springer, New York, 1984.
- [36] F. Merle, P. Raphael, I. Rodnianski, J. Szeftel. On smooth self similar solutions to the compressible Euler equations, arXiv:1912.10998
- [37] C. Miao, L. Xue, On the global well-posedness of a class of Boussinesq-Navier-Stokes systems, NoDEA Nonlinear Differential Equations Appl. 18 (6), 707-735 (2011).
- [38] G. Mockenhaupt, A. Seeger, and C. D. Sogge, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc. 6, 65-130 (1993).
- [39] B. Riemann. Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, Mathematisch-physikalische 8, 43 (1858).
- [40] T.C. Sideris. Formation of singularities in three dimensional compressible fluids, Commun. Math. Phys. 101, 475-485 (1985).
- [41] H.F. Smith, D. Tataru. Sharp local well-posedness results for the nonlinear wave equation, Ann. Math., 162, 291-366 (2005).
- [42] H.F. Smith and D. Tataru. Sharp counterexamples for Strichartz estimates for low regularity metrics, Mathematical Research Letters, 199-204 (2002).
- [43] H. F. Smith, A parametrix construction for wave equations with $C^{1,1}$ coefficients, Ann. Inst. Fourier (Grenoble) 48, 797-835 (1998).
- [44] H. F. Smith and C. D. Sogge, On Strichartz and eigenfunction estimates for low regularity metrics, Math. Res. Lett. 1, 729-737 (1994).
- [45] J. Speck. Shock formation for 2D quasilinear wave systems featuring multiple speeds:Blowup for the fastest wave, with non-trivial interactions up to the singularity, arXiv:1701.06728
- [46] J. Speck. A New Formulation of the 3D Compressible Euler Equations With Dynamic Entropy:Remarkable Null Structures and Regularity Properties, arXiv:1701.06626
- [47] R.S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke. Math. 44(3), 705-714 (1977).
- [48] T. Tao. Global regularity of wave maps. II. Small energy in two dimensions, Comm. Math. Phys., 224(2), 443-544 (2001).
- [49] D. Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation, Am. J. Math., 122, 349-376 (2000).
- [50] D. Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients II, Am. J. Math. 123, 385-423 (2001).

- [51] D. Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III. J. Am. Math. Soc. 15, 419-442 (2002).
- [52] D. Tataru. Rough solutions for the wave maps equation. Amer. J. Math., 127(2), 293-377 (2005).
- [53] C.B. Wang. Sharp local well-posedness for quasilinear wave equations with spherical symmetry, appeared in Journal of the European Mathematical Society.
- [54] Q. Wang. Rough Solutions of Einstein vacuum equations in CMCSH gauge, Comm. Math. Phys. 328, 1275-1340 (2014).
- [55] Q. Wang. Causal geometry of rough Einstein CMCSH spacetime, J. Hyperbolic Differ. Equ. 11(3), 563-601 (2014).
- [56] Q. Wang. Improved breakdown criterion for Einstein vacuum equations in CMC gauge, Commun. Pure Appl. Math. 65(1), 21-76 (2012).
- [57] Q. Wang. A geometric approach for sharp local well-posedness of quasilinear wave equations, Ann. PDE, 3:12 (2017).
- [58] Q. Wang. Rough solutions of the 3-D compressible Euler equations, to appear in Ann. Math., (2021).
- [59] H.C. Yin. Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data, Nagoya Math. J. 175, 125-164 (2004).
- [60] H.L. Zhang, L. Andersson. On the rough solutions of 3D compressible Euler equations: alternative proof, arXiv:2104.12299
- [61] H.L. Zhang. Local existence theory for 2D compressible Euler equations with low regularity, to appear in Journal of Hyperbolic Differential Equations.

School of Mathematics and Statistics,, Changsha University of Science and Technology, Changsha, 410114, P. R. China

Email address: zhlmath@yahoo.com