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MULTIPLE SOLUTIONS FOR SOME STRONGLY DEGENERATE SECOND ORDER ELLIPTIC EQUATIONS

JOÃO R. SANTOS JUNIOR AND GAETANO SICILIANO*

Abstract. We consider a boundary value problem in a bounded domain involving a degenerate operator of the form

$$L(u) = -\mathrm{div}(a(x)\nabla u)$$

and a suitable nonlinearity f. The function a vanishes on smooth 1-codimensional submanifolds of Ω where it is not allowed to be C^2. By using weighted Sobolev spaces we are still able to find existence of solutions which vanish, in the trace sense, on the set where a vanishes.

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1. Introduction

In this paper we are interested in the existence of "suitable" solutions for a degenerate nonlinear elliptic equation of second order in a bounded and smooth domain in $\mathbb{R}^N$ with homogeneous Dirichlet boundary condition. More specifically the equation under study is driven by the operator

$$L(u) = -\mathrm{div}(a(x)\nabla u)$$

where, $a : \overline{\Omega} \to [0, +\infty)$, among other assumptions, is a continuous function such that $a(x) > 0$ in the whole Ω except for suitable $1-$codimensional submanifolds contained in Ω where it vanishes. Hence the ellipticity of L is broken somewhere in $\overline{\Omega}$. This kind of operator is also called *degenerate* due to the fact that a^{-1} is unbounded.

Degenerate operators appear in many situations. Indeed it is known that many physical phenomena are described by degenerate evolution equations, where the degeneracy can be due to the vanishing of the time derivative coefficient or to the vanishing of the diffusion coefficient. In this context there is a strong connexion between degenerate 2nd order differential operators and Markov processes: roughly speaking these operators describe a diffusion phenomena of Markovian particle which moves until it reaches the set where the absorption takes place and here the particle "dies". Because of this fact, degenerate equations are appropriate to describe fluid diffusion in nonhomogeneous porous media taking into account saturation and porosity of the medium. For

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(∗) Corresponding author.

more applications and problems involving degenerate operators one can see e.g [1–4, 16] and the references therein.

Mathematically speaking, for degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces. A class of weights, which is particularly well understood, is the class of A_p−weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt. The importance of this class is that powers of distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [12]) and these weight have found many useful applications also in harmonic analysis (see [19]). However there are also many other interesting examples of weights (see [9] for p-admissible weights). For some references on this subject see also [5–7, 11], and for other applications of weighted Sobolev spaces see also [18].

To motivate the choice of the problem under study let us see the following example. Suppose additionally that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $f(s)=0$ if and only if $s=0$. Let $\Omega\subset\mathbb{R}^N, N\geq 2$ be a smooth and bounded domain and assume that $a\in C^1(\overline{\Omega})$ is a positive function with $a^{-1}(0)$ which is a regular connected submanifold compactly contained in Ω and such that $\nabla a(x)=0$ for any $x\in a^{-1}(0)$. Consider the problem

$$-\operatorname{div}(a(x)\nabla u)=f(u)\quad\text{in}\ \mathcal{D}'(\Omega). \tag{1.1}$$

Following [17] we say that $u_*\in\mathcal{D}'(\Omega)$ is a solution if $u_*\in C^1(\Omega)$ and the equation is satisfied in the sense of distribution, i.e.

$$\int_\Omega a(x)\nabla u_*\nabla\varphi=\int_\Omega f(u_*)\varphi\quad\forall\varphi\in C_c^\infty(\Omega).$$

But then from (1.1) it follows, that

$$-\nabla a(x)\nabla u_*-a(x)\Delta u_*=f(u_*)\quad\text{in}\ \mathcal{D}'(\Omega)$$

and since $f(u_*)$ and $\nabla a(x)\nabla u_*$ are continuous functions, so is $a(x)\Delta u(x)$ (note that a vanishes on a null set) and we obtain

$$-\nabla a(x)\nabla u_*-a(x)\Delta u_*=f(u_*(x))\quad\forall x\in\Omega.$$

From this identity we deduce

$$x\in a^{-1}(0)\Longrightarrow f(u_*(x))=0\Longrightarrow u_*(x)=0.$$

In other words, for such a problem, the solution is zero whenever a is zero.

Motivated by this fact we study in this paper the existence of weak solutions for a degenerate elliptic operator in a bounded domain with homogeneous Dirichlet boundary condition and with the additional condition that our solutions are zero (in the sense of trace) on the set where a vanishes. More specifically the problem under study is the following.

Let $\Omega\subset\mathbb{R}^N, N\geq 2$ be a smooth and bounded domain, $a\in C(\overline{\Omega})$, $a\geq 0$ and $f\in C(\mathbb{R})$ are functions satisfying:

- (a1) $a^{-1}(0)=\cup_{l=1}^k\Gamma_l\subset\Omega$ is the disjoint union of a finite number k of compact, connected, without boundary and 1-codimensional smooth submanifolds Γ_l of $\mathbb{R}^N$,
- (a2) $a\in A_2$ (the standard Muckenhoupt class) and $1/a\in L^t(\Omega)$, for some $t>N/2$,

and

- (f1) f has a strict local minimum in $s=0$ with $f(0)=0$, and there exists $s_*>0$ such that $f(s_*)=0$ and $f>0$ in $(0,s_*)$,
- (f2) there exists $\gamma=\lim_{t\to 0^+}f(s)/t>0$ and $a_{M_j}:=\max_{x\in\overline{D}_j}a(x)<\gamma/\lambda_1(D_j)$, where $\lambda_1(D_j)$ is the first eigenvalue of the Dirichlet Laplacian in D_j and D_j stands for any connected component of $\Omega\backslash a^{-1}\{0\}$.

Consider the problem

$$\text{(P)}\qquad \begin{cases} -\mathrm{div}(a(x)\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \cup a^{-1}(0) \end{cases}$$

The requirement that u vanishes also on the set $a^{-1}(0)$ is motivated by the previous example.

A *weak solution of* (P) is a function $u_* \in W_0^{1,1}(\Omega \setminus a^{-1}(0)) \cap L^\infty(\Omega)$ such that

$$\int_\Omega a(x)\nabla u_* \nabla\varphi = \int_\Omega f(u_*)\varphi, \quad \forall \varphi \in C_c^\infty(\Omega \backslash a^{-1}(0)).$$

Note that, since $a \in C(\overline{\Omega})$ and $f \in C(\mathbb{R})$, the above identity makes sense. The choice of the space $W_0^{1,1}(\Omega \setminus a^{-1}(0))$ in place of the more common space $H_0^1(\Omega \setminus a^{-1}(0))$ is due to the fact that we do not know if the gradient of the solution u_* we find is in $L^2(\Omega \setminus a^{-1}(0))$.

Before to continue, let us make few comments on the assumptions. First of all, note that we are just assuming the continuity of a and, in contrast to our motivating problem (1.1), the function f is also allowed to vanish in many points (assumption (f1)); however there is a relation of its first right derivative in zero with the function a (assumption (f2)).

The class A_2 which appears in assumption (a2) is the Muckenhoupt class. We prefer do not recall the right definition here (see the next Section) but roughly speaking it gives a condition on the summability of a and $1/a$ and it seems the right class to work with and define reasonable weighted Sobolev spaces for such a problem.

Finally it is worth to say that assumption (a1) appeared also in [14] where the authors study an operator of type $\mathrm{div}(A(x)\nabla u)$, for a suitable matrix A which can vanish. They are interested actually in establishing Poincaré type inequalities for such a degenerate operator.

Remark 1. *It is easy to exhibits example of functions a satisfying our assumptions. Let $\Omega = B_2(0)$ be the ball entered in 0 in $\mathbb{R}^N, N \geq 2$ of radius 2.*

Take a radial function whose profile in the radial variable has zeroes of order less then one, for example

$$a(r) = \begin{cases} \sqrt[3]{1-r^2} & \text{if } r \in [0,1], \\ \sqrt{(1-r)(r-2)} & \text{if } r \in (1,2]. \end{cases}$$

Then it is easy to check that $a \in A_2, 1/a \in L^t(\Omega)$ for any $t \in [1, 2N)$ and then (a2) holds. The function is of course not of class C^1 where it vanishes.

Similarly, consider a function which is strictly positive in the center of the ball Ω and whose radial profile is C^1, with null derivative in the origin, and of type

$$a(r) = \begin{cases} \text{smooth and positive} & \text{if } r \in [0, 1/5], \\ \dfrac{(r-1)^2}{\sqrt{|r-1|}} & \text{if } r \in (1/5, 6/5], \\ \text{smooth and positive} & \text{if } r \in (6/5, 11/5), \\ \dfrac{(r-2)^2}{\sqrt{|r-2|}} & \text{if } r \in [11/5, 2]. \end{cases}$$

It is easy to check that $a \in A_2, 1/a \in L^t(\Omega)$ for any $t \in [1, 2N/3)$ and then (a2) holds. Such a function is C^1 in all Ω, and C^2 in Ω except where it vanishes.

Note however that functions that are C^2 where they vanish are not allowed by our hypothesis. Indeed, if a were positive and of class C^2 in a neighbourhood of $x_0 \in \Omega$ where $a(x_0) = 0$, then by *the Taylor expansion,*

$$a(x) \leq \ C|x-x_0|^2 \quad \textit{in a neighbourhood } U_{x_0} \textit{ of } x_0.$$

It follows

$$\frac{1}{a(x)^{N/2}} \geq \frac{C}{|x-x_0|^N} \qquad \textit{in } U_{x_0}$$

then $1/a \notin L^{N/2}(\Omega)$ *and hence* (a2) *cannot be satisfied.*

To state our main result let us fix some notations.

Denote $\Gamma_{k+1} = \partial\Omega$. Let $\pi_0(\Omega \setminus a^{-1}(0))$ be the usual quotient space of $\Omega \setminus a^{-1}(0)$ under the equivalence relation which identifies points that can be joint with a continuous arch. Then $\chi := \text{card } \pi_0(\Omega \setminus a^{-1}(0)) \geq 1$ gives the number of connected components of $\Omega \setminus a^{-1}(0)$. Let us write

$$\chi = \sum_{i=1}^{m} j_i, \qquad j_i \in \mathbb{N}, \ j_1 \geq 1,$$

where j_i stands for the number of subdomains of $\Omega \backslash a^{-1}(0)$ whose boundary is made exactly by i connected 1-codimensional submanifolds of $\mathbb{R}^N$. These domains are denoted with $\mathcal{A}_1^{(i)}, \mathcal{A}_2^{(i)}, \ldots, \mathcal{A}_{j_i}^{(i)}$. See the Figure 1 and Figure 2 for two examples in dimension two.

Our result states that the number of solutions of (P) is related to χ.

Theorem 1.1. *Suppose that* (a1),(a2), (f1), (f2) *hold. Then, problem* (P) *has at least* $2^\chi - 1$ *nonnegative (and nontrivial) weak solutions. More specifically, the number of positive solutions with* n *bumps,* $n \in \{1, \ldots, \chi\}$*, is given by the binomial coefficient* $\frac{\chi!}{n!(\chi-n)!}$.

We point out that we will use variational methods to prove our result and we will work in the weighted Sobolev space $H_0^1(\Omega, a)$; so the solutions we find actually will belong to this space.

The paper is organised as follows. In the next Section 2 we recall some basic facts on weighted Sobolev spaces to establish the framework of our problem. In Section 3 a suitable problem is solved which will be the main ingredient to prove our main result in the last Section 4.

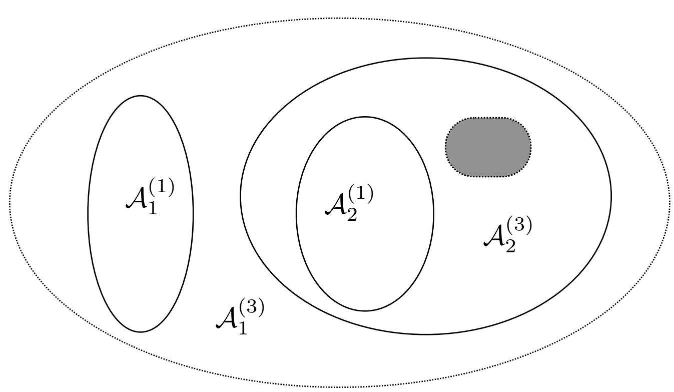

FIGURE 1. Example of a domain (with one grey hole) where $a^{-1}(0) = \sum_{i=1}^{3} \Gamma_i$. In this case $\chi = 4, j_1 = 2, j_2 = 0, j_3 = 2$.

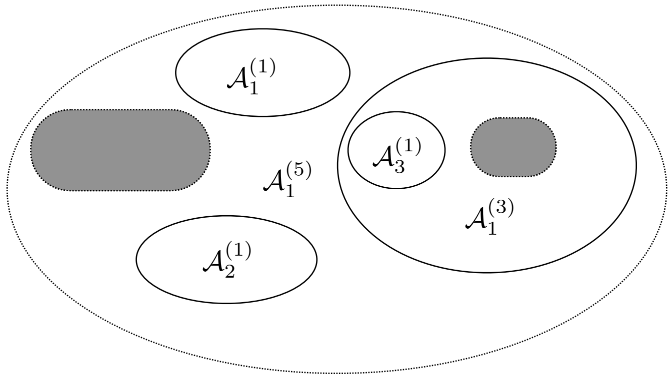

FIGURE 2. Example of a domain (with two grey holes) where $a^{-1}(0) = \sum_{i=1}^{4} \Gamma_i$. In this case $\chi = 5, j_1 = 3, j_2 = 0, j_3 = 1, j_4 = 0, j_5 = 1$.

Notations. As a matter of notations, in all the paper we denote with $W^{m,p}(\Omega)$ the usual Sobolev spaces. Whenever $p = 2$ we use the notation $H^m(\Omega)$. Finally $H_0^1(\Omega)$ is the closure of the test functions with respect to the norm in $H^1(\Omega)$. Other notations will be introduced whenever we need.

2. Some well known facts

In this section we will give some preliminary facts on suitable weighted Sobolev spaces we will use later. For more details and applications of weighted Sobolev spaces, which is the right context to study degenerate elliptic operators, we refer the reader to [6, 8, 11–13, 15], for instance.

Along this section

1. $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, and
2. $h : \Omega \to [0, +\infty)$ satisfies
$$\sup \left(\frac{1}{|B|} \int_B h(x) \right) \left(\frac{1}{|B|} \int_B h(x)^{-\frac{1}{p-1}} \right)^{p-1} \leq C, \quad p > 1,$$
where the supremum is taken over all the balls $B \subset \Omega$. In other words, h belongs to the so called Muckenhoupt class A_p (see [15]). The right hand side of the inequality above is known as the A_p-constant of h.

For each $p \geq 1$, $L^p(\Omega, h)$ is the Banach space of all measurable functions $u : \Omega \to \mathbb{R}$, for which
$$|u|_{L^p(\Omega,h)} = \left(\int_\Omega h(x)|u|^p \right)^{1/p} < \infty.$$

Whenever h is in the A_p class, $L^p(\Omega, h) \subset L^1_{loc}(\Omega)$ and then it makes sense to speak about weak derivatives and Sobolev spaces. By definition, the weighted Sobolev space $H^1(\Omega, h)$ is the set of functions $u \in L^2(\Omega, h)$ such that the (weak) derivatives of first order are all in $L^2(\Omega, h)$. The (squared) norm in $H^1(\Omega, h)$ is
$$\|u\|^2_{H^1(\Omega,h)} = \int_\Omega h(x) \left(|\nabla u|^2 + |u|^2 \right).$$

It can be proved that $H^1(\Omega, h)$ is the closure if $C^\infty(\overline{\Omega})$ with respect to the previous norm. As usual, $H^1_0(\Omega, h)$ is the closure of $C^\infty_c(\Omega)$ with respect to the norm defined by

$$\|u\|^2_{H^1_0(\Omega,h)} = \int_\Omega h(x)|\nabla u|^2. \tag{2.1}$$

Both $H^1(\Omega, h)$ and $H^1_0(\Omega, h)$ are Hilbert spaces containing the positive and negative parts of each of their elements (see [6, Corollary 2.1]). Since h may vanish somewhere on $\overline{\Omega}$, the weighted Sobolev spaces are not isomorphic the the "usual" ones.

Theorem 2.1. *(The weighted Sobolev inequality) There exists positive constants C_Ω and δ, such that for all $u \in C^\infty_c(\Omega)$ and $1 \le \theta \le N/(N-1) + \delta$,*

$$|u|_{L^{2\theta}(\Omega,h)} \le C_\Omega |\nabla u|_{L^2(\Omega,h)}.$$

See [6, Theorem 1.3] for a proof. In particular from this results it hods that the quantity defined in (2.1) gives a norm on $H^1_0(\Omega, h)$ equivalent to $\|\cdot\|_{H^1(\Omega,h)}$.

The next result is also well known (see [12, Theorem 2.8.1]).

Theorem 2.2. *If $u_n \to u$ in $L^p(\Omega, h), 1 < p < \infty$, then there exists a subsequence $\{u_{n_k}\}$ and a function $v \in L^p(\Omega, h)$ such that*

(i) $u_{n_k}(x) \to u(x),\ n_k \to \infty, h - a.e.$ on Ω;

(ii) $|u_{n_k}(x)| \le v(x), h - a.e.$ on Ω.

Finally, we will enunciate a compact embedding type result for the weighted Sobolev spaces $H^1(\Omega, h)$. See e.g. [8] for the details.

Theorem 2.3. *(Compact embeddings) Let $1 \le s \le r < Nq/(N-q)$, $q \le 2$ and*

$$K(h) = \max\left\{|h^{-\frac{1}{2}}|_{L^{\frac{2q}{2-q}}(\Omega)}, |h^{\frac{1}{s}}|_{L^{\frac{rs}{r-s}}(\Omega)}\right\} < \infty.$$

Then, the space $H^1(\Omega, h)$ is compactly embedded in $L^s(\Omega, h)$.

3. Preliminaries: a problem (possibly) degenerate on the boundary

In this section, for future reference, we consider the following elliptic problem

$$(P_D) \qquad \begin{cases} -\mathrm{div}(b(x)\nabla u) = f(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a smooth, open and bounded domain, $b \in C(\overline{D})$, $b(x) > 0$ for $x \in D$, $b \in A_2$ and $1/b \in L^t(D), t > N/2$ and f which satisfies (f1) and (f2) (with of course $\overline{\Omega}$ replaced by D and a by the function b). The operator $L(u) = -\mathrm{div}(b(x)\nabla u)$ is called also $b-$elliptic.

A weak solution for (P_D) is a function $u_* \in W^{1,1}_0(D) \cap L^\infty(D)$ such that

$$\int_D b(x)\nabla u_* \nabla v = \int_D f(u_*)v, \quad \forall v \in C^\infty_c(D). \tag{3.1}$$

Observe that b may eventually be zero somewhere on the boundary ∂D.

We will find the solution of (P_D) working in the space $H^1_0(D, b)$. Note that such a space is contained into $W^{1,2t/(1+t)}_0(D)$, where $t > N/2$ is given in (a2), and then $2t/(1+t) > 1$. Indeed, for

$u \in H_0^1(D,b)$ we have, by the Hölder inequality,

$$\begin{aligned}\int_D |\nabla u|^{2t/(t+1)} &= \int_D \left(\frac{1}{b^{t/(t+1)}}\right)(b^{t/(t+1)}|\nabla u|^{2t/(t+1)}) \\ &\le \left|\frac{1}{b}\right|_{L^t(D)}^{t/(1+t)} \|u\|_{H_0^1(D,b)}^{2t/(1+t)} < \infty\end{aligned}$$

and hence $H_0^1(D,b) \hookrightarrow W_0^{1,2t/(1+t)}(D)$. As it follows by the next proof, the solution will be also bounded.

The main result of this section is as follows.

Theorem 3.1. *Under the previous assumptions, problem (P_D) has at least a nonnegative and nontrivial weak solution.*

Proof. Due to the possibly degenerate structure of the problem, the suitable functional setting to treat (P_D) is the weighted Sobolev space $H_0^1(D,b)$. Let $J : H_0^1(D,b) \to \mathbb{R}$ be the functional

$$J(u) = \frac{1}{2}\int_D b(x)|\nabla u|^2 - \int_D F_*(u) =: \frac{1}{2}\|u\|_{H_0^1(D,b)}^2 - \psi(u),$$

where F_* is the primitive of

$$f_*(s) = \begin{cases} f(-\beta_*) & \text{if } s \in (-\infty, -\beta_*], \\ f(s) & \text{if } s \in (-\beta_*, s_*), \\ 0 & \text{if } s \in [s_*, \infty), \end{cases}$$

for some $\beta_* > 0$ such that $f > 0$ in $[-\beta_*, 0)$.

Observe that J is well defined in $H_0^1(D,b)$. In fact, since f_* is bounded, we have

$$\int_D |F_*(u)| \le C \int_D |u|, \tag{3.2}$$

for some positive constant C. On the other hand, by Hölder inequality and being $b \in A_2$

$$\int_D |u| = \int_D \frac{1}{b^{1/2}}(b^{1/2}|u|) \le \left|\frac{1}{b}\right|_{L^1(D)}^{1/2} \|u\|_{H_0^1(D,b)} < \infty, \tag{3.3}$$

for all $u \in H_0^1(D,b)$. Moreover, since f_* is continuous, $J \in C^1$.

Observe that J is coercive. Indeed, from (3.2) and (3.3) we deduce

$$J(u) \ge \frac{1}{2}\|u\|_{H_0^1(D,b)}^2 - C\left|\frac{1}{b}\right|_{L^1(\Omega)}^{1/2} \|u\|_{H_0^1(\Omega,b)}.$$

To prove that J is weakly lower semicontinuous, it is enough to note that if $u_n \rightharpoonup u$ in $H_0^1(D,b)$, then, by (a2) and being $b \in C(\overline{D})$ the number $K(b)$ in Theorem 2.3 is finite if we choose

$$q = \frac{2t}{t+1}, \quad s = 2, \quad r \in \Big(2, \frac{2Nt}{N(t+1) - 2t}\Big).$$

Therefore, by the compact embedding, we get

$$u_n \to u \ \text{ in } L^2(D,b).$$

From Theorem 2.2, up to a subsequence, there exists $g \in L^2(D,b)$ such that

$$u_n(x) \to u(x) \ \text{ and } \ |u_n(x)| \le g(x), \quad b - \text{a.e. in } D.$$

Since b is positive in D, we obtain

$$u_n(x) \to u(x) \ \text{ and } \ |u_n(x)| \le g(x), \quad \text{a.e. in } D.$$

Consequently,

$$F_*(u_n(x)) \to F_*(u(x)) \text{ a.e. in } D$$

and

$$|F_*(u_n(x))| \leq C|u_n(x)| \leq Cg(x) \text{ a.e. in } D.$$

On the other hand, from (a2)

$$\int_D |g(x)| = \int_D \frac{1}{b(x)^{1/2}}(b(x)^{1/2}|g(x)|) \leq \left|\frac{1}{b}\right|^{1/2}_{L^1(D)} |g|_{L^2(D,b)} < \infty,$$

showing that $g \in L^1(D)$. Then by using the Lebesgue dominated convergence theorem, we conclude that

$$\psi(u_n) = \int_D F_*(u_n) \to \int_D F_*(u) = \psi(u).$$

Thus, ψ is weakly continuous and, consequently, J is weakly lower semicontinuous in the Hilbert space $H^1_0(D,b)$. Let $u_* : \Omega \to \mathbb{R}$ a minimum point of J. Since J is C^1,

$$\int_D b(x)\nabla u_* \nabla v = \int_D f_*(u_*)v, \quad \forall v \in H^1_0(D,b),$$

showing that u_* is a weak solution of (P_D).

Now we are going to prove that u_* is nontrivial. For that, it is enough to realise that J takes negatives values. Indeed let e_1 be a positive eigenfunction associated to the first eigenvalue $\lambda_1(D)$ of Laplacian operator in D with homogeneous Dirichlet boundary condition and consider

$$\frac{1}{s^2}J(se_1) = \frac{1}{2}\|e_1\|^2_{H^1_0(D,b)} - \int_D \frac{F_*(se_1)}{(se_1)^2}e_1^2,$$

for each $s > 0$. By (f2), de L'Hospital rule and Lebesgue dominated convergence theorem, by passing to the limit as $s \to 0^+$, we obtain

$$\lim_{s\to 0^+} \frac{1}{s^2}J(se_1) = \frac{1}{2}\int_D \left(b(x) - \frac{\gamma}{\lambda_1(D)}\right)|\nabla u|^2 < 0.$$

Thus, for $s > 0$ small enough, we have $J(u_*) \leq J(se_1) < 0$, showing that u_* is nontrivial.

It follows from (f1) and the definition of f_* that by choosing $v = u_*^- := \min\{u_*, 0\}$ in (3.1), we have

$$\int_D b(x)|\nabla u_*^-|^2 = \int_D f_*(u_*)u_*^- \leq 0.$$

Since $b > 0$, we conclude that $\nabla u_*^- = 0$ a.e. in D. Therefore $u_*^- = c$ a.e. in D, for some $c \in \mathbb{R}$. Finally, from $u_* \in H^1_0(D,b)$, we have that $u_*^- = 0$ and $u_* = u_*^+ := \max\{u_*, 0\} \geq 0$. To conclude that $u_* \leq s_*$, it is enough to choose $v = (u_* - s_*)^+$ in (3.1) and reasoning in a similar way. Therefore $f_*(u_*) = f(u_*)$, concluding the proof. □

4. Proof of Theorem 1.1

Finally we are ready to treat the problem

$$\text{(P)} \qquad \begin{cases} -\text{div}(a(x)\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \cup a^{-1}\{0\} \end{cases}$$

and prove Theorem 1.1.

To take advantage of the degeneracy of a in order to prove existence of multiple solutions to problem (P), we will divide the proof in two steps. In the first one will be considered a suitable class of problems ($P_{i,l}$) with diffusion operator involving coefficients degenerating on the boundary of the domain where the problem is settled, that is, for each $i \in \{1, \dots, m\}$ and $l \in \{1, \dots, j_i\}$, we will look for weak solutions of the problem

$$(P_{i,l}) \qquad \begin{cases} -\mathrm{div}(a(x)\nabla u) = f(u) & \text{in } \mathcal{A}_l^{(i)}, \\ u = 0 & \text{on } \partial\mathcal{A}_l^{(i)}. \end{cases}$$

In the second one, the solutions obtained for ($P_{i,l}$) will be used to construct solutions to (P), which have different numbers of positive bumps.

Step I: Existence of χ one-bump weak solutions to ($P_{i,l}$).

It follows from (a1) that each set $\mathcal{A}_l^{(i)}$ is a bounded domain of $\mathbb{R}^N$ with a smooth boundary, on which function a can be zero. Consequently, Step I is a straightforward consequence of hypotheses (a2),(f1),(f2) and Theorem 3.1 in previous Section. Let us call $u_{i,l}$ the one-bump weak solution obtained to ($P_{i,l}$).

Step II: Existence of $2^\chi - 1$ nonnegative (and nontrivial) weak solutions to (P).

Let us consider the extensions $\widetilde{u}_{i,l}$ of $u_{i,l}$ to Ω, that is,

$$\widetilde{u}_{i,l}(x) = \begin{cases} u_{i,l} & \text{in } \mathcal{A}_l^{(i)}, \\ 0 & \text{in } \Omega \backslash \mathcal{A}_l^{(i)}. \end{cases}$$

Since $0 \le u_{i,l} \le s_*$, $u_{i,l} \in H_0^1(\mathcal{A}_l^{(i)}, a_{|_{\mathcal{A}_l^{(i)}}})$ and

$$\int_{\mathcal{A}_l^{(i)}} |\nabla u_{i,l}| = \int_{\mathcal{A}_l^{(i)}} \left(\frac{1}{a(x)^{1/2}}\right)(a(x)^{1/2}|\nabla u_{i,l}|) \le \left|\frac{1}{a}\right|^{1/2}_{L^1(\mathcal{A}_l^{(i)})} \|u_{i,l}\|^2_{H_0^1(\mathcal{A}_l^{(i)}, a_{|_{\mathcal{A}_l^{(i)}}})} < \infty,$$

where in the last inequality we have used the Holder inequality. It is clear that $\widetilde{u}_i \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$. Moreover, since $a \in C(\overline{\Omega})$ and $\mathcal{A}_l^{(i)} \subset \Omega \backslash a^{-1}\{0\}$, if $v \in C_c^\infty(\Omega \backslash a^{-1}\{0\})$ then $v_{|_{\mathcal{A}_l^{(i)}}} \in H_0^1(\mathcal{A}_l^{(i)}, a_{|_{\mathcal{A}_l^{(i)}}})$. Thus, since $u_{i,l}$ is a weak solution of ($P_{i,l}$), for all $v \in C_c^\infty(\Omega \backslash a^{-1}\{0\})$:

$$\int_\Omega a(x)\nabla\left(\sum_{i,l} \widetilde{u}_{i,l}\right)\nabla v = \sum_{i,l}\int_{\mathcal{A}_l^{(i)}} a(x)\nabla u_{i,l}\nabla(v_{|_{\mathcal{A}_l^{(i)}}}) = \sum_{i,l}\int_{\mathcal{A}_l^{(i)}} f(u_{i,l})v_{|_{\mathcal{A}_l^{(i)}}} = \int_\Omega f\left(\sum_{i,l}\widetilde{u}_{i,l}\right)v,$$

where the summation $\sum_{i,l}$ runs over all the possible combinations of indexes i, l, so as to include all the connected components of $\Omega \backslash a^{-1}\{0\}$, showing that $\widetilde{u}_{i,l}$ is a nonnegative and nontrivial weak solution of (P) for each $i \in \{1, \dots, m\}$ and $l \in \{1, \dots, j_i\}$. Since the sum of n of the previous weak solutions $\widetilde{u}_{i,l}$ ($2 \le n \le \chi$) is still a solution of (P) (by (a1)), the result follows. Observe finally that, arguing as in Section 3, the solutions found are in $H_0^1(\Omega, a)$.

References

[1] J. Chabrowski *Degenerate elliptic equation involving a subcritical Sobolev exponent* Portugaliae Mathematica Vol. 53 Fasc. 2 – 1996 2

[2] F.S Cîrstea, V. Rădulescu, *Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent,* Topological Methods in Nonlinear Analysis **15** (2000), 283–300. 2

[3] C.L. Epstein, R. Mazzeo. *Degenerate Diffusion Operators Arising in Population Biology*. Annals of Mathematics Studies 185. Princeton, NJ: Princeton University Press, 2013. 2

[4] C.L. Epstein, R. Mazzeo, *The Geometric Microlocal Analysis of Generalized Kimura and Heston Diffusions,* in Analysis and Topology in Nonlinear Differential Equations, edited by D. G. de Figueiredo, J. M. do Ó , and C. Tomei, 241–66. Progress in Nonlinear Differential Equations and Their Applications 85. New York, NY: Springer International Publishing AG, 2014. 2

[5] E. Fabes, D. Jerison, and C. Kenig, *The Wiener test for degenerate elliptic equations,* Ann. Inst. Fourier (Grenoble) **32** (1982), No. 3, vi, 151–182. 2

[6] E. G. Fabes, C. E. Kenig and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Commun. Part. Diff. Eq. **7**, n. 1, (1982), 77-116. 2, 5, 6

[7] B. Franchi and R. Serapioni, *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), No. 4, 527–568. 2

[8] V. Gol'dshtein and A. Ukhlov, *Weighted Sobolev spaces and embedding theorems,* Trans. Amer. Math. Soc. **361**, n. 7, (2009), 3829-3850. 5, 6

[9] J. Heinonen, T. Kilpelaïnen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. 2

[10] M. Kimura, *Diffusion models in population genetics,* Journal of Applied Probability, **1** (1964), 177–232. 2, 5

[11] A. Kufner, Weighted Sobolev spaces, Teubner-Texte zur Math., Bd. 31, Teubner Verlagsge- sellschaft, Leipzig (1980). 2, 5

[12] A. Kufner, O. John, and S. Fučik, Function spaces. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977. 2, 5, 6

[13] A. Kufner, B. Opic, *How to define reasonably weighted Sobolev spaces,* Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 3, 537–554. 5

[14] D. Monticelli, K.R. Payne, and F. Punzo, *Poincaré inequalities for Sobolev spaces with matrix-valued weights and applications to degenerate partial differential equations* Proc. Roy. Soc. Edinburgh Sect. A **149** (2019), no. 1, 61–100. 3

[15] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. Soc. **165** (1972), 207–226. 5

[16] M.K.V. Murthy, G. Stampacchia, *Boundary problems for some degenerate elliptic operators,* Ann. Mat. Pura Appl (4), 1968, 80: 1–122. 2

[17] P. Pucci and J. Serrin, *Dead cores and bursts for quasilinear singular elliptic equations,* Siam J. Math. Anal. **38** No. 1, pp. 259–278. 2

[18] A-M. Sändig, A. Kufner, Some applications of weighted Sobolev spaces, Teubner-Texte zur Mathematik 100 Vieweg Teubner Verlag (1987). 2

[19] A. Torchinsky, Real-variable methods in harmonic analysis. Pure and Applied Mathematics, 123. Academic Press, Inc., Orlando, FL, 1986. 2

(J. R. Santos Jr.)
Faculdade de Matemática
Instituto de Ciências Exatas e Naturais
Universidade Federal do Pará
Avenida Augusto corrêa 01, 66075-110, Belém, PA, Brazil
Email address: joaojunior@ufpa.br

(G. Siciliano)
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, 05508-090, São Paulo, SP, Brazil
Email address: sicilian@ime.usp.br