

Synchronization dynamics in non-normal networks: the trade-off for optimality

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Synchronization is an important behavior that characterizes many natural and human made systems composed by several interacting units. It can be found in a broad spectrum of applications, ranging from neuroscience to power-grids, to mention a few. Such systems synchronize because of the complex set of coupling they exhibit, the latter being modeled by complex networks. The dynamical behavior of the system and the topology of the underlying network are strongly intertwined, raising the question of the optimal architecture that makes synchronization robust. The Master Stability Function (MSF) has been proposed and extensively studied as a generic framework to tackle synchronization problems. Using this method, it has been shown that for a class of models, synchronization in strongly directed networks is robust to external perturbations. In this paper, our approach is to transform the non-autonomous system of coupled oscillators into an autonomous one, showing that previous results are model-independent. Recent findings indicate that many real-world networks are strongly directed, being potential candidates for optimal synchronization. Inspired by the fact that highly directed networks are also strongly non-normal, in this work, we address the matter of non-normality by pointing out that standard techniques, such as the MSF, may fail in predicting the stability of synchronized behavior. These results lead to a trade-off between non-normality and directedness that should be properly considered when designing an optimal network, enhancing the robustness of synchronization.

I. INTRODUCTION

Systems in nature are often constituted by a large number of small parts that continuously interact with each other [1, 2]. Although it might be possible to accurately know the dynamics that characterize each of the individual constituents, it is, in general, nontrivial to figure out the collective behavior of the systems as a whole resulting from the individual/local interactions. A relevant example is provided by a system composed by an ensemble of coupled non-linear oscillators, that behave at unison driven by the non-local interaction, then the system is said to be synchronized [2, 3]. Synchronization has been extensively studied in network science as a paradigm of dynamical processes on a complex network, mainly due to the essential role of the coupling topology in the collective dynamics [1]. Its generic formulation allowed researchers to use it to model several applications, ranging from biology, e.g., neurons firing in synchrony, to engineering, e.g., power grids [4]. The ubiquity of synchronization in many natural or artificial systems has naturally raised questions about the stability and robustness of synchronized states [5–8]. In their seminal work, Pecora & Carroll [9] introduced a method known as Master Stability Function (MSF) to help understand the role that the topology of interactions has on system stability. Assuming a diffusive-like coupling among the oscillators, the MSF relates the stability of the synchronous state to the nontrivial spectrum of the (network) Laplace matrix; in particular, it has been

proven that the latter should lie in the region where the Lyapunov exponent that characterizes the MSF takes negative values [2, 10]. For a family of models (e.g., Rössler, Lorenz, etc.) whose stable part of the MSF has a continuous interval where the (real part of the) Laplacian eigenvalues can lie, it has been proven that they maximize their stability once the coupling network satisfies particular structural properties. Such optimal networks should be directed spanning trees and without loops [5, 6]. These networks have the peculiarity of possessing a degenerate spectrum of the Laplacian matrix and laying in the stability domain provided by the Master Stability Function. The Laplacian degeneracy is also often associated with a real spectrum or with considerably low imaginary parts compared to the real ones [11, 12].

The vast interest in complex networks in recent years has also provided an abundance of data on empirical networked systems that initiated a large study of their structural properties [1]. From this perspective, it has been recently shown that many real networks are strongly directed, namely they possess a high asymmetry adjacency matrix [13]. Most of these networks present a highly hierarchical, almost-DAG (Directed Asymmetric Graph) structure. This property potentially makes the real networks suitable candidates for optimally synchronized dynamical systems defined on top of them. Another aspect which is unavoidably associated with the high asymmetry of real networks, is their non-normality [13], namely their adjacency matrix

\mathbf{A} satisfies the condition $\mathbf{A}\mathbf{A}^T \neq \mathbf{A}^T\mathbf{A}$ [12]. The non-normality can be critical for the dynamics of networked systems [13–18]. In fact, in the non-normal dynamics regime a finite perturbation about a stable state can undergo a transient instability [12] which because of the non-linearities could never be reabsorbed [13, 14]. The effect of non-normality in dynamical systems has been studied in several contexts, such as hydrodynamics [19], ecosystems stability [20], pattern formation [21], chemical reactions [22], etc. However, it is only recently that the ubiquity of non-normal networks and the related dynamics have been put to the fore [13–18]. In this paper, we will elaborate on these lines showing the impact of non-normality on the stability of a synchronous state. We first show that a strongly non-normal network has, in general, a spectrum very close to a real one and that this in principle should imply a larger domain of parameters for which stability occurs, for systems with a generic shaped MSF. For illustration purposes, we will consider the Brusselator model [23, 24], a two-species system with a discontinuous interval of stability in the MSF representation. We will also examine the limiting cases of our analysis to two simple network models [25], namely a (normal) bidirected circulant network and a (non-normal) chain, both with tunable edge weights in such a way to allow a continuous adjustment respectively of the directedness and non-normality.

The MSF relies on the computation of the (real part of the maximum) Lyapunov exponent, and thus in the case of time-dependent systems, it does not possess the full predictability power it has in the autonomous case (fixed point in/stability). For this reason, we will use a homogenization method, whose validity is limited to a specific region of the model parameters, allowing us to transform the linearized periodic case problem into a time-independent one [26]. This way, we remap our problem to an identical one studied in the context of pattern formation in directed networks where spectral techniques provide significant insight [25, 27]. Such an approach allows us on one side to assess the quantitative evaluation of the role of the imaginary part of the Laplacian spectrum in the stability problem. On the other it permits the use of numerical methods, such as the pseudo-spectrum [12] in the study of the non-normal dynamics. To the best of our knowledge, this is the first attempt to use such techniques in the framework of time-varying systems, being the theory of non-normal dynamical systems limited so far to autonomous systems [12]. As expected, the non-normality plays against the stability of the synchronized ensemble of oscillators. Furthermore, a high non-normality translates to a high spectral degeneracy, which brings to a large pseudo-spectrum, indicating a high sensibility towards the instability.

Clearly, the directionality and the non-normality stand on two parallel tracks regarding the stability of

synchronized states and their robustness. As a conclusion of our work, we show that the most optimal design should be looked at as a trade-off between a high and low directionality/non-normality. Such choice should depend either on the magnitude of perturbation or the ratio directed vs. non-normal of the network structure.

II. OPTIMAL SYNCHRONIZATION: DIRECTED VS. NON-NORMAL NETWORKS

We consider a network constituted of N nodes (e.g., the idealized representation of a cell), and we assume a metapopulation framework, where the species dynamics inside each node is described by the *Brusselator* model, a portmanteau term for Brussels and oscillator. It has been initially introduced by Prigogine & Nicolis to capture the autocatalytic oscillation [23] phenomenon, resulting from a Hopf bifurcation curve in the parameter plane. This will be the framework we will consider in the following, neglecting thus the fixed point regime. Species can migrate across nodes with a diffusion-like mechanism. In formulae, this model translates to a reaction-diffusion set of equations:

$$\begin{cases} \frac{d\varphi_i}{dt} = 1 - (b+1)\varphi_i + c\varphi_i^2\psi_i + D_\varphi \sum_{j=1}^N \mathcal{L}_{ij}\varphi_j \\ \frac{d\psi_i}{dt} = b\varphi_i - c\varphi_i^2\psi_i + D_\psi \sum_{j=1}^N \mathcal{L}_{ij}\psi_j, \quad \forall i = 1, \dots, N, \end{cases} \quad (1)$$

where φ_i and ψ_i indicate the concentration of the two species per node, D_φ , D_ψ are their corresponding diffusion coefficients, and b , c are the model parameters. The coupling is represented by the matrix \mathcal{W} , whose non-negative entries \mathcal{W}_{ij} represent the strength of the edge pointing from node j to node i . The entries of the Laplacian matrix \mathcal{L} are given by $\mathcal{L}_{ij} = \mathcal{W}_{ij} - k_i^{in}\delta_{ij}$ where $k_i^{in} = \sum_j \mathcal{W}_{ij}$ stands for the incoming degree of node i , i.e. the number of all the entering edges into node i . We want to emphasize here that many other coupling operators are also possible; nevertheless, most of them will reduce at the linear level to a Laplacian involving the differences of the observable among coupled nodes [2], i.e., $\sum_{j=1}^N \mathcal{L}_{ij}x_j = \sum_{j=1}^N \mathcal{W}_{ij}(x_j - x_i)$. This form ensures that the coupling is in action only when the observable assume different values in two coupled nodes.

The reason for choosing such a model, as mentioned earlier, is mainly due to the discontinuous interval of the stability domain provided by the MSF of the problem (as it can be noticed in the inset of Fig. 3 a)). To proceed with the stability analysis, we first need to identify the homogeneous periodic solution, $\varphi^*(t)$ and $\psi^*(t)$, hereby called the *synchronized manifold* and then

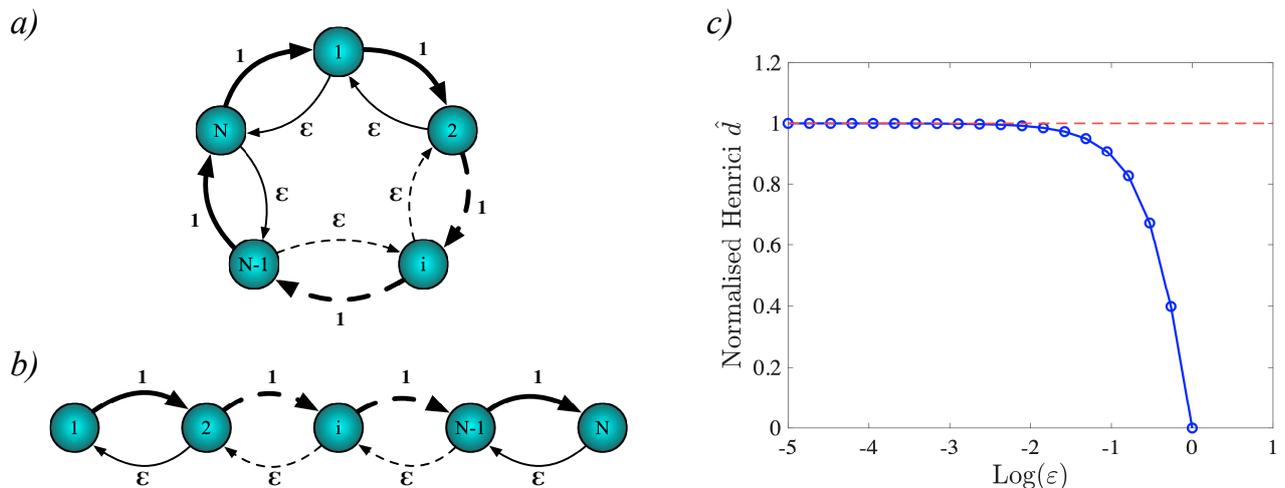


Figure 1. The network toy models for the case of a normal bidirectional circulant network, panel a), and a non-normal bidirectional chain, panel b). c) Normalized Henrici's departure from non-normality as a function of tuning parameter ϵ for the non-normal model. We observe that starting from 0, the network is symmetric, and the non-normality increases as the weight of the reciprocal edges decreases, taking the maximal value of non-normality in the limit when $\epsilon = 0$. In this case, the Laplacian spectrum is degenerate.

to linearize the system around this. Let us introduce the perturbations for the i -th node by $\delta\varphi_i$ and $\delta\psi_i$, then the linearized equations describing their evolution are given by:

$$\begin{aligned} \frac{d(\delta\varphi_i)}{dt} &= \left[f_{\varphi_i} \delta_{ij} + D_{\varphi} \sum_{j=1}^N \mathcal{L}_{ij} \right] \delta\varphi_j + f_{\psi_i} \delta\psi_i \\ \frac{d(\delta\psi_i)}{dt} &= g_{\varphi_i} \delta\varphi_i + \left[g_{\psi_i} \delta_{ij} + D_{\psi} \sum_{j=1}^N \mathcal{L}_{ij} \right] \delta\psi_j \\ \forall i &= 1, \dots, N, \end{aligned} \quad (2)$$

where the partial derivatives are given by $f_{\varphi_i} = -(b+1) + 2c\varphi^*(t)\psi^*(t)$, $f_{\psi} = c\varphi^*(t)^2$, $g_{\varphi_i} = b - 2c\varphi^*(t)\psi^*(t)$, and $g_{\psi_i} = c\varphi^*(t)^2$. Notice that the partial derivatives of the reaction part are evaluated on the synchronized manifold. This translates into a time-dependent Jacobian matrix due to the periodicity of the solutions and thus to a non-autonomous linear system. To make a step forward let us introduce the following compact notation; let $\mathbf{x} = (\delta\varphi_1, \dots, \delta\varphi_N, \delta\psi_1, \dots, \delta\psi_N)^T$ be the $2N$ -dimensional perturbations vector, \mathcal{D} the diagonal diffusion coefficients matrix and $\mathcal{J}(t)$ the time-dependent Jacobian matrix, hence Eq. (2) can be rewritten as

$$\dot{\mathbf{x}} = (\mathcal{J}(t) + \mathcal{D} \odot \mathcal{L}) \mathbf{x}, \quad (3)$$

where \odot is the coordinatewise multiplication operator. Then we proceed by diagonalizing the linearized system

using the basis of eigenvectors of the network Laplace operator \mathcal{L} . Notice that this is not always possible because the Laplacian matrix of directed networks might not have linearly independent eigenvectors. We will assume such a basis to exist for the time being, and we will consider such an issue again when discussing the non-normal case. Denoting by ξ the transformed perturbations vector, Eq. (3) becomes

$$\dot{\xi} = (\mathcal{J}(t) + \mathcal{D} \odot \Lambda) \xi, \quad (4)$$

where Λ denotes the diagonal matrix of the Laplacian eigenvalues. The (real part of the) largest Lyapunov exponent of Eq. (4), known in the literature as the Master Stability Function [1, 2, 9, 10], is thus a function of the eigenvalues Λ . Let us stress that the study of the stability of a general non-autonomous system is normally not possible through the classical spectral analysis, and one has therefore to resort to the MSF.

Before proceeding in the quest for the optimal network topological features that minimize the MSF, we will introduce two simple network models, shown in Fig. 1, for which we can tune the directionality and the non-normality acting on a single parameter. In the first case, Fig. 1 a), we consider a bidirectional circulant network, i.e., a network whose adjacency matrix is circulant [28], made by two types of links, one of weight 1 forming a clockwise ring and the other winding a counterclockwise ring of tuneable weights ϵ . The latter can vary in the interval $\epsilon \in [0, 1]$ exploring in this way the possible topologies from a fully symmetric case when $\epsilon = 1$ to a totally mono-directed network when $\epsilon = 0$. Since such a network is circulant, the adjacency ma-

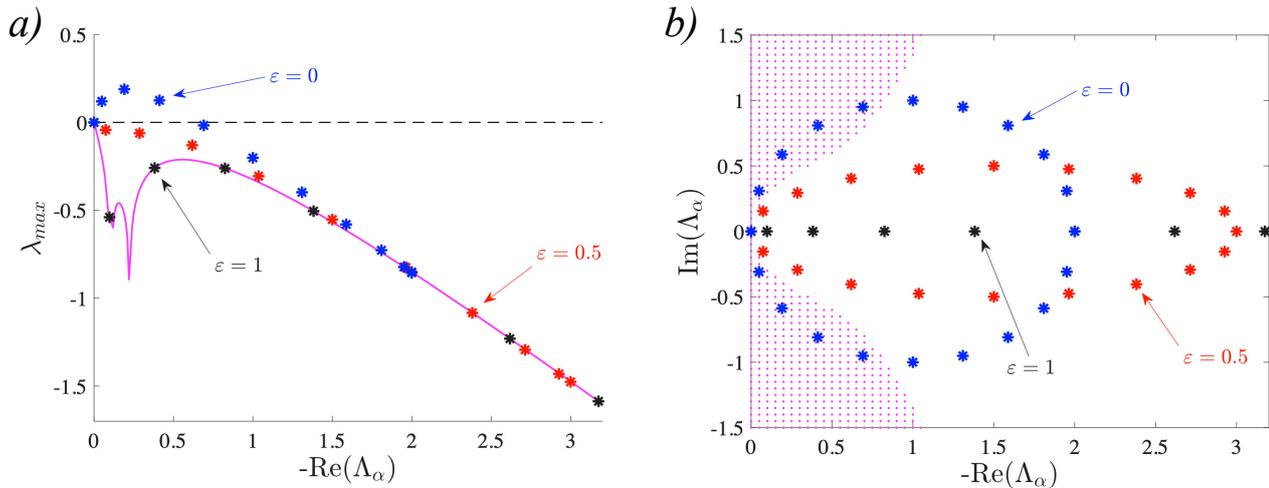


Figure 2. *a)* MSF for the Brusselator model with $b = 2.5$, $c = 1$ (limit cycle regime), $D_\varphi = 0.7$, $D_\psi = 5$ on a circulant network of 20 node; Λ_α indicates the the Laplacian's eigenvalues, of which we plot only the real part. In this setting the system should remain stable after a perturbation: in fact, when the network is symmetric ($\varepsilon = 0$), the discrete MSF (black dots) lies on the continuous one (magenta line); however, when we introduce an asymmetry in the topology as ε decreases (red and blue dots), the MSF reaches the instability region, and the system loses synchronization. *b)* The equivalent representation in the complex domain where the instability region is shaded magenta and the discrete Laplacian spectrum is denoted by the symbols. For the network topology with at least one eigenvalue that lies in the instability region, the synchronized state is lost.

trix will be normal, a property that is inherited by the Laplace operator. On the contrary, if we remove two reciprocal links, respectively, of weights 1 and ε , we obtain instead a non-normal network, as depicted in Fig. 1 *b)*. In this case, the adjacency matrix is non-normal [12], a feature also reflected on the Laplacian matrix. Even in this case, we can tune the non-normality by varying the ε parameter in the unitary interval as for the previous case, this can be appreciated from the results shown in Fig. 1 *c)* where we report the normalized Henrici index, a well-known proxy of non-normality, as a function of ε . The main advantage of using the above network models is the existence of a basis of eigenvectors for the Laplacian matrix. In the first network model, this is due to the normality of the graph Laplacian, while in the second one it is because of the tridiagonal form of the coupling operator [29]. This property is essential for the applicability of the MSF analysis, which is impossible otherwise.

A. The case of normal directed networks

We start by considering the bidirected circular network and studying the linear stability of the synchronized state using the MSF analysis. The results shown in Fig. 2 *a)*, indicate that the network topology increasingly contrasts the stability of the synchronous manifold when the directionality increases. In fact, when the

MSF computed for the directed network is compared to the symmetric case used as reference line (the continuous magenta curve), we can always observe larger values, which moreover increase as ε decreases (for the same fixed Laplacian eigenvalue). Because of the circulant property of the Laplace matrix, its spectrum can be explicitly computed [25] $\Lambda_\alpha = 1 + \varepsilon + (1 + \varepsilon) \cos(2\alpha\pi/N) + i(1 - \varepsilon) \sin(2\alpha\pi/N)$. One can easily notice that for $\varepsilon = 0$, the spectrum distributes uniformly onto the unitary circle centered at $(1, 0)$ as also shown in Fig. 2 *b)* in blue stars. On the other side, when $\varepsilon = 1$, the network turns symmetric, making the spectrum real.

The MSF formalism ultimately relies on the maximum Lyapunov exponent, which despite having proved its validity in ruling out the chaotic behavior of dynamical system [3], remains grounded on numerical methods. To improve our analytical understanding of the problem, we proceed by transforming Eq. (4) into an autonomous one, allowing in this way to deploy the spectral analysis tools. This method is part of the broader set of homogenization methods that aim at averaging a time-dependent system to obtain a time-independent one [26]. Such methods have been found useful also for the stability analysis of synchronized states [32, 33]. The resulting autonomous version of the MSF is sometimes referred to as the dispersion relation [21]. The mathematical validity of the proposed approximation is grounded on the Magnus series expansion truncated at

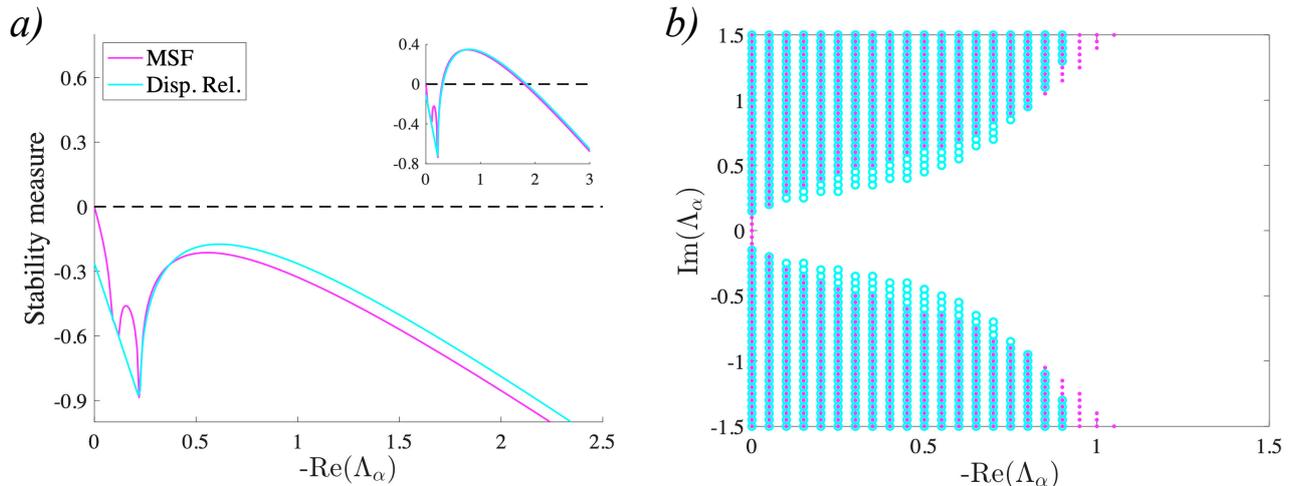


Figure 3. *a)* The comparison of the MSF and dispersion relation for the Brusselator with model parameters $b = 3$, $c = 1.8$, $D_\varphi = 0.7$, $D_\psi = 5$. We depict in magenta the MSF of the system in a limit cycle regime and cyan the dispersion relation of the averaged autonomous system. Inset: Similar comparison for a set of parameters where the instability occurs. Notice also the lack of continuity of the stability interval of eigenvalues. *b)* The same representation in the complex domain. We see that for the chosen values of the parameters, the two approaches give an excellent agreement in predicting the instability interval.

the first order [33]; hence, the set of model parameters for which we expect a good agreement with the original model corresponds to the case when higher-order terms are negligible. For more details, the interested reader should consult [33]. In formula, it translates to

$$\mathcal{J}(t) \longrightarrow \langle \mathcal{J} \rangle_T = \frac{1}{T} \int_0^T \mathcal{J}(\tau) d\tau. \quad (5)$$

Remarkably, as shown in Fig. 3, this approximation yields qualitative results in excellent agreement with the original model for a specific range of parameters. An alternative to this approach is to apply a perturbative expansion near the bifurcation point, obtaining this way the time-independent Ginzburg-Landau normal form [34]. However, the effectivity of the latter method is exclusively limited to parameters values very close to the stability threshold. In this sense, our approach is more general, both from allowing a larger set of parameters where the method remains valid, and at the same time, it is independent of the choice of the model compared to previous works [35]. The passage to an autonomous system is also essential in explaining the effect of the imaginary part of the Laplacian eigenvalues in the newly obtained stability function, the dispersion relation. It has been rigorously shown in [25, 27] that the dispersion relation increases proportional to the magnitude of the imaginary part of the spectrum. We already observed similar results for the case of the MSF presented in Fig. 2. We can in this way conclude that the averaging method sheds light on the role of the directed topology in the destabilization of a

synchronized regime.

B. The case of non-normal directed networks

The analysis performed in the previous section has been based on the study of the linearized system, in some cases, however, such analysis is not sufficient to understand the outcome of the nonlinear system. In Fig. 4 we consider again the MSF computed for the directed chain previously introduced (panel *b*) of Fig. 1). From Fig. 4 *b*) one might naively conclude that the system will synchronize, since the MSF is non-positive for all values of $\text{Re}(\Lambda_\alpha)$. Moreover, the spectrum is completely real (see panel *b*)) and thus there cannot be any contribution from the imaginary part of the spectrum. However a direct inspection of the orbit behavior (panel *c*)) clearly shows that the system does not synchronize. Once the system is defined on a symmetric support, the synchronized behavior is recovered (panel *d*)). This diversity of behavior is related to the non-normal property of the considered network, indeed it has been recently proved that such structural property can strongly alter the asymptotic behavior of networked systems [36]. A finite perturbation about a stable equilibrium goes through a transient amplification (see Fig. 4 *d*)) proportional to the level of non-normality before it is eventually reabsorbed in the linear approximation [12], while in the full non-linear system the finite perturbation could persist indefinitely. Up to now, this analysis has been limited to the case of autonomous systems; in this paper

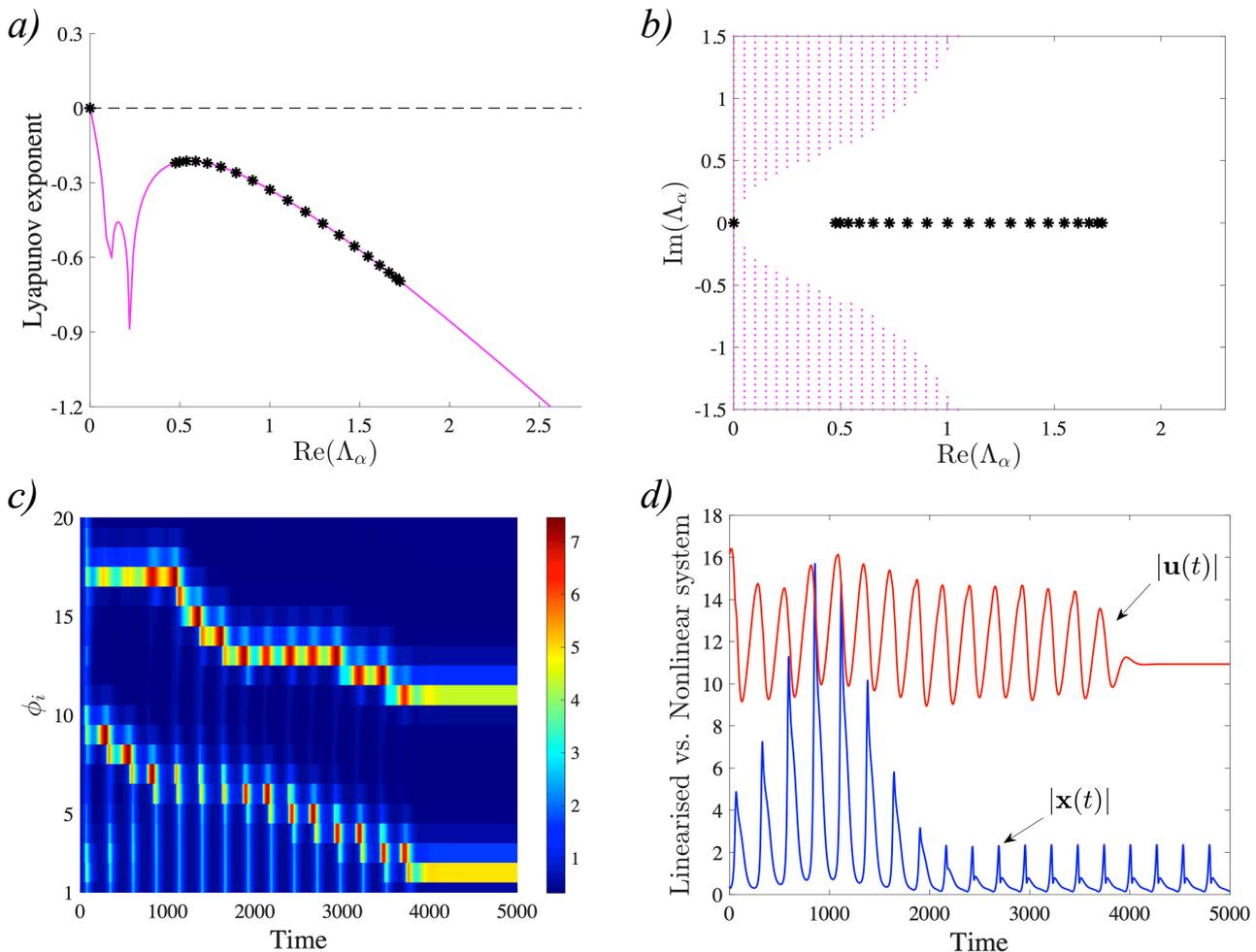


Figure 4. Desynchronization in a non-normal network. The parameters for the Brusselator model are as follows $b = 2.5$, $c = 1$, $D_\varphi = 0.7$, $D_\psi = 5$ on the (directed chain) non-normal network of 20 nodes with $\varepsilon = 0.1$ of Fig. 1 b). As it can be observed from panels a,) and b), respectively, for the MSF and the stability region, the set of parameters is such that the MSF is neatly stable. Nevertheless, the instability occurs as shown by the pattern evolution in panel c) at odd with the outcome that would have been expected from the symmetrized version. Such a result is strong evidence of the role of the network non-normality in the nonlinear dynamics of the system under investigation. The mechanism that drives the instability in the non-normal linearized regime manifests in the transition growth of the perturbations vector $\mathbf{x}(t)$ eq. (3), the blue curve in panel d), before the system relaxes to the oscillatory state of the equilibrium. Such growth might transform in a permanent instability for the nonlinear system $\mathbf{u}(t) = [\varphi(t), \psi(t)]$, red curve.

for the first time we extend it to the periodic time-dependent case making use of the homogenization process. This explains the permanent instability, shown in Fig. 4, causing the loss of stability for the synchronized state.

The non-normal dynamics study cannot be straightforwardly tackled with the analytical methods of the local stability, mostly because the instability occurs in a highly nonlinear regime. Such condition require a global analysis that can be obtained using the numerical technique based on a spectral perturbation concept known as the pseudo-spectrum. For a given matrix \mathbf{A} the lat-

ter is defined as $\sigma(\mathbf{A}_\delta) = \sigma(\mathbf{A} + \mathbf{E})$, for all $\|\mathbf{E}\| \leq \delta$ for where $\sigma(\cdot)$ represents the spectrum and $\|\cdot\|$ a given norm. The package EigTool [37] allows us to compute and draw in the complex plane the level curves of the pseudo-spectrum for a given value of ε . Although the pseudo-spectrum is not sufficient to fully explain the system behavior, it is certainly of great utility in estimating the role of non-normality in the dynamics outcomes. In particular, in panel b) of Fig. 5 we report level curves of the pseudo-spectrum for three different values of the parameter ε representing the reciprocal links of the directed chain. Notice that by increasing the non-

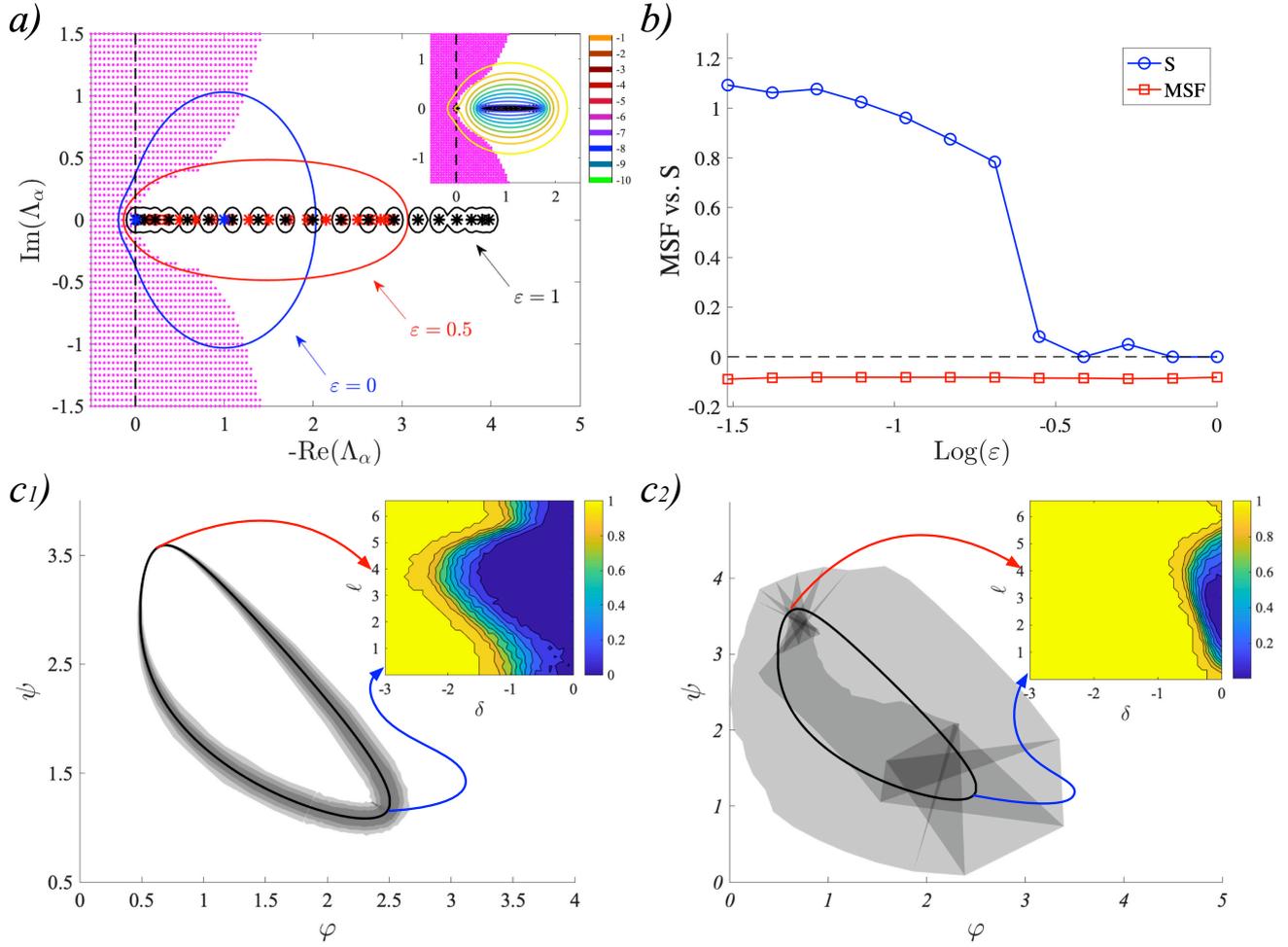


Figure 5. *a)* The pseudo-spectral description of the stability of the directed chain of 20 nodes for the Brusselator model with $b = 2.5$, $c = 1.12$, $D_\varphi = 0.7$, $D_\psi = 5$, and an initial condition perturbation of the average magnitude $\delta = 0.1$. We show the pseudo-spectra for three different values of the control parameter ϵ for the chain network, emphasizing the considerably large difference between the pseudo-spectra regions and the spectrum of the Laplacian matrix. Inset: the pseudo-spectra for many other values of the perturbation magnitude δ for the chain with $\epsilon = 0.1$. Notice that although the eigenvalues do not lie inside the instability region due to the lack of an imaginary part, the pseudo-spectra might do. *b)* The comparison between the expected outcome as predicted from the MSF and the actual outcome as measures by the standard deviation of the desynchronized pattern. The stability basin (shaded grey) projected onto the limit cycle plane for the non-normal case, panel *c1)* and the symmetrized (normal) one, panel *c2)*, calculated over 300 different initial conditions (of the same averaged magnitude) and a perturbation whose maximum magnitude varies from 10^{-3} to 1. Inset: In the y -axis we plot the points of limit cycle we perturb and in the x -axis the magnitude of the perturbation; the colormap gives the fraction of orbits that conserve the synchronized regime. It can be clearly noticed that the attraction basin for the non-normal network is strongly reduced, though not at the same amount compared to where the perturbation occurs.

normality of the toy network, the pseudo-spectrum will also increase the chances of intersection with the instability region. In panel *d)* of Fig. 5, we have shown a comparison between a proxy of the presence of a synchronized state, i.e. the standard deviation S [38] of the asymptotic orbit behavior and the MSF demonstrating a clear different behavior. For all the considered values of ϵ the MSF is always negative suggesting a stable synchronized state, on the other hand S becomes positive

and large for small enough ϵ , testifying a loss of synchronization. The dependence on the different values of the initial conditions is further shown in panels *c1)* and *c2)*. As expected, the instability is more probable for both larger values of non-normality and magnitude of the initial conditions. In particular, it can be observed that the synchronization basin of attraction is strongly reduced for the non-normal network compared to the normal one, and moreover its width varies along

the limit cycle, implying that desynchronization will depend also on the point at which the perturbation starts.

III. CONCLUSIONS

In this paper, we have studied the quest for the optimal conditions ensuring the stability of synchronization dynamics in directed networks. Such conditions determine the design of a networked system that makes the synchronization regime as robust as possible. Previous results have proven that a strictly directed topology is necessary for the synchronized state's robustness. Based on the well-known Master Stability Function, it has been shown that directed tree-like networks are optimal for models with a discontinuous interval of the Laplacian spectrum in the stability range of MSF. Here, we have extended such results proving that they are generally independent of the dynamic model. Using an averaging procedure, we transformed the problem from a time-dependent (non-autonomous) to a time-invariant (autonomous) one. This method allows to prove that networks whose Laplacian matrix exhibits a spectrum that lacks an imaginary part are the most optimal. In general, the loss of synchronization increases with the magnitude of the imaginary part of the spectrum. Secondly, recent findings have shown that real-world networks present strong directed traits, resulting in a strong non-normality. This latter feature can play a very important role in the linear dynamics influencing the local stability of the synchronized state through a strong transient amplification of the perturbations. We have extended the idea of non-normal dynamics to

the case of non-autonomous synchronization dynamics, revealing how network non-normality can drive the system to instability, thus increasing the understanding of synchronization in complex networks. We have also numerically quantified the effect of non-normality in driving the instability through the pseudo-spectrum technique. In conclusion, we have analytically and numerically demonstrated that there is no compelling recipe for optimal network architecture in order to conserve the synchronized state, but rather a trade-off between the network directedness and its non-normality. We are aware that the interesting outcomes of the interaction of structural non-normality networks with the fascinating synchronization phenomenon require deeper and further investigation (e.g. synchronization basin). In this sense, with this work we aim to initiate a new direction of research of the synchronization problem.

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