Fractonic gauge theory of smectics

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Motivated by striped correlated quantum matter, and the recently developed duality between elasticity of a two-dimensional (2D) crystal and a gauge theory, we derive a dual coupled U(1) vector gauge theory for a two-dimensional (2D) quantum smectic, where the disclination is mapped onto the fractonic charge, that we demonstrate can only move transversely to smectic layers. This smectic gauge theory dual also emerges from a gauge dual of a quantum crystal through a Higgs transition corresponding to a single flavor of its dipole condensation, an anisotropic quantum melting via dislocation proliferation. A condensation of the second flavor of dislocations corresponds to another Higgs transition describing the smectic-to-nematic melting. We also utilize the electrostatic limit of this duality to formulate a melting of a 2D classical smectic in terms of a higher derivative sine-Gordon model, demonstrating its instability to a nematic at any nonzero temperature. Generalizing this classical duality to a 3D smectic, gives a formulation of a 3D nematic-to-smectic transition in terms of an anisotropic Abelian-Higgs model.

I. INTRODUCTION

A. Motivation and background

A smectic state of matter, is a liquid crystal phase that partially breaks rotational and translational symmetries spontaneously, exhibiting a periodic layered order. Classical smectics form in systems of rod-like constituents (molecules like 5CB)[1,2] and are driven by anisotropic entropic (exclusion volume) interactions. In striking contrast, quantum smectic states appear even in systems of isotropic point-like constituents as a result of frustrated competition between kinetic energy and interactions. In cold atom systems, quantum smectics may be realized in a putative Fulde-Ferrell-Larkin-Ovchinnikov paired superfluids[3,4] in imbalanced degenerate atomic gases[5,6] and in spin-orbit coupled Bose condensates[7,8]. Quantum smectics are also a natural explanation for a striking resistive anisotropy observed in quantum Hall systems at half-filled high Landau levels[9–14], and for "striped" spin and charge states of weakly doped correlated quantum magnets[15,16].

A two-dimensional (2D) smectic can emerge from partial, anisotropic melting[17] of a crystal, with only one species of dislocations unbinding, such that only one direction of translational symmetry is restored, in a Kosterlitz-Thouless (KT)-like[18] phase transition. However, a 2D smectic is unstable to thermal fluctuations, and is always driven into a nematic fluid at any nonzero temperature[17,19–21]. In contrast, a (2+1)D quantum smectic at zero temperature, is a stable state of matter, whose studies have been limited to a simplest harmonic description, with effects of topological defects and of elastic nonlinearities neglected beyond qualitative discussions (for an exception see Refs. 6,22,23). This, together with ubiquitous putative realizations provides a strong motivation for the present detailed work, a brief preview of which appeared in a recent publication.[24]

A complementary motivation for our study is its rela-

tion to a new class of topological quantum states of matter – dubbed "fractons" – discovered in theoretical exactly solvable models.[25–32] These feature a number of fascinating properties that are believed to lie beyond a conventional quantum field theoretic description.[33] The most striking of these are quasiparticles with robust (not just fine-tuned or symmetry imposed) restrictions on their mobility, such as an immobile fracton, and its subdimensional multipoles. Although experimental realizations have been sorely lacking, these theoretical models are intensely studied, motivated by their promise for a robust quantum memory[27] and fundamental interest in a new class of topological quantum liquids[34,35].

Following this gapped class of lattice qubit models, fracton-like phenomena were also uncovered in gapless symmetric tensor gauge theories, encoded in a generalized Gauss law, that conserves charge multipoles and thereby constrains mobility of charges[36–38]. Contemporaneously, a similarity of the constrained dynamics of disclination and dislocation defects in a crystal was conjectured to be dual to charges and dipoles of a gauge theory[39]. Utilizing a generalization of the familiar XY-to-gauge theory (boson-vortex) duality[40,41], Pretko and Radzihovsky[42] formalized this relation through a duality mapping (explored in other contexts by Zaanen and company[43]) between a quantum 2D crystal elasticity and a symmetric tensor gauge theory. Under this mapping the stress tensor σ_{ij} and momentum vector π_i fields map onto the electric tensor E_{ij} and magnetic vector B_i fields, respectively, with Newton's law (conservation of momentum) corresponding to Faraday's law of the tensor gauge theory. Relation of these tensor gauge theories to chiral topological elasticity was also explored in Ref.44.

This established fracton-elasticity duality allows for numerous predictions for phases and phase transitions of the fracton system, based on the extensive understanding of 2D crystal and their descendent states. For example, different phases of the scalar fracton model - fracton insulator, dipole condensate and fracton condensate, can be

regarded as gauge theory counterparts to the "commensurate" and "incommensurate" (supersolid) crystals[45–47], hexatic, and isotropic fluid phases of the elasticity theory. The associated finite-temperature dipole-unbinding transition and fracton charge unbinding transition correspond to the classical two-stage melting transitions, i.e., crystal-to-hexatic and hexatic-to-liquid transition respectively.

A complementary and physically more transparent formulation of elasticity-to-fractonic coupled *vector* gauge theory was recently presented[48]. The resulting dual coupled vector gauge theory involves three U(1) vector gauge fields $\mathbf{A}_k$ (with $k = x, y$ denoting flavors) and $\mathbf{a}$, and their canonically conjugate electric fields $\mathbf{E}_k$ and $\mathbf{e}$, that encode coupled Goldstone modes, the phonons u_k and the local bond angle θ. Building on the treatment of the quantum crystal[48] and a recent analysis of the quantum smectic[24], we derive and explore extensively the coupled vector gauge theory duality to study the (2+1)D smectic and its quantum phase transitions to a crystal and a nematic, formulated in terms of an array of Higgs transitions. We also utilize it to study 2D and 3D classical smectic and the corresponding classical nematic-to-smectic phase transitions[50–56].

B. Summary of Results

In this paper, we develop and explore in detail a dual coupled U(1) vector gauge theory for a 2D quantum smectic, building on a recent study of a 2D quantum crystal[48] and a smectic[24] by one of the authors. The dual description we derive is formulated in terms of two coupled U(1) vector gauge theories, with electric fields $\hat{\mathbf{E}}$ and $\hat{\mathbf{e}}$, and canonically conjugate vector potentials $\hat{\mathbf{A}}$ and $\hat{\mathbf{a}}$, sourced by dipole and charge current densities, $n_b, \mathbf{j}_b$ (dislocations) and $n_s, \mathbf{j}_s$ (disclinations), respectively. The corresponding dual Hamiltonian density is given by

$$\tilde{\mathcal{H}}_{\rm sm} = \frac{1}{2}\chi\hat{\mathbf{E}}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\times\hat{\mathbf{A}}\right)^2 + \frac{1}{2}K\hat{\mathbf{e}}^2 \\ + \frac{1}{2}\left(\boldsymbol{\nabla}\times\hat{\mathbf{a}} + \hat{\mathbf{x}}\times\hat{\mathbf{A}}\right)^2 - \hat{\mathbf{A}}\cdot\mathbf{j}_b - \hat{\mathbf{a}}\cdot\mathbf{j}_s, \tag{1}$$

supplemented by the generalized Gauss laws,

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{E}} = n_b + \hat{\mathbf{e}}\cdot\hat{\mathbf{x}}, \tag{2}$$
$$\boldsymbol{\nabla}\cdot\hat{\mathbf{e}} = n_s. \tag{3}$$

We demonstrate that the n_s charges (disclinations in the smectic) of this dual gauge theory are subdimensional

"lineon", mobile only transverse to smectic layers (that we take to be along $\hat{\mathbf{x}}$), enforced by generalized gauge invariance and associated continuity equation,

$$\partial_t n_b + \boldsymbol{\nabla}\cdot\mathbf{j}_b = -\hat{\mathbf{x}}\cdot\mathbf{j}. \tag{4}$$

In contrast the dipoles (dislocations) exhibit a finite but highly anisotropically mobility.

Motivated to also understand the quantum crystal-smectic transition, we derive the smectic gauge dual and transition to it by utilizing gauge dual of the quantum crystal[48] and condensing one flavor of dipoles (dislocations). The associated Higgs transition gaps out the corresponding flavor of the gauge fields $\mathbf{A}_k$, and leads to a dual quantum smectic Lagrangian, that matches exactly the description obtained through direct duality of smectic elasticity,

$$\tilde{\mathcal{L}}_{\rm sm} = \frac{1}{2}|\left(\partial_\mu + ipA_\mu\right)\psi_x|^2 - V\left(|\psi_x|\right) + \mathcal{L}_{\rm M}^{\rm sm}, \tag{5}$$

where, $\mathcal{L}_{\rm M}^{\rm sm}$ is the Maxwell sector from Equation (1), with $V\left(|\psi_x|\right)$ a U(1)-invariant Landau potential for x-flavor dipoles, ψ_x. The flow chart in Fig. 1 summarizes these two routes to the dual gauge theory of a quantum smectic.
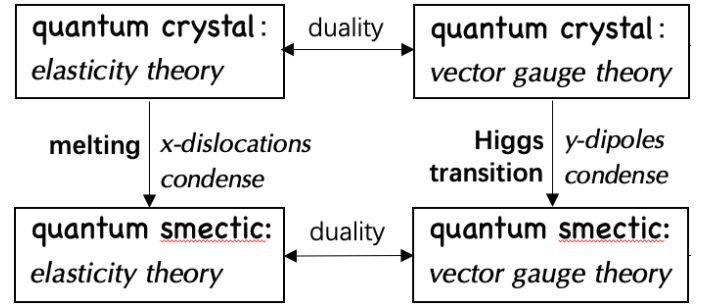

FIG. 1: Quantum crystal-smectic duality relations and the associated quantum melting transition.

In this formulation the $\psi_x = 0$ Coulomb phase corresponds to the quantum smectic phase, and the $\psi_x \neq 0$ Higgs phase gives a condensation of unbound $\hat{\mathbf{x}}$−dipoles ($\hat{\mathbf{y}}$−dislocations) that gaps out the gauge field A_μ, which drives a Higgs transition to a quantum nematic. The quantum melting transitions and the corresponding phases are illustrated in Fig. 2.

We also explore the classical limit of this duality, and formulate the 2D smectic-to-nematic melting and a subsequent nematic-to-isotropic fluid transition in terms of
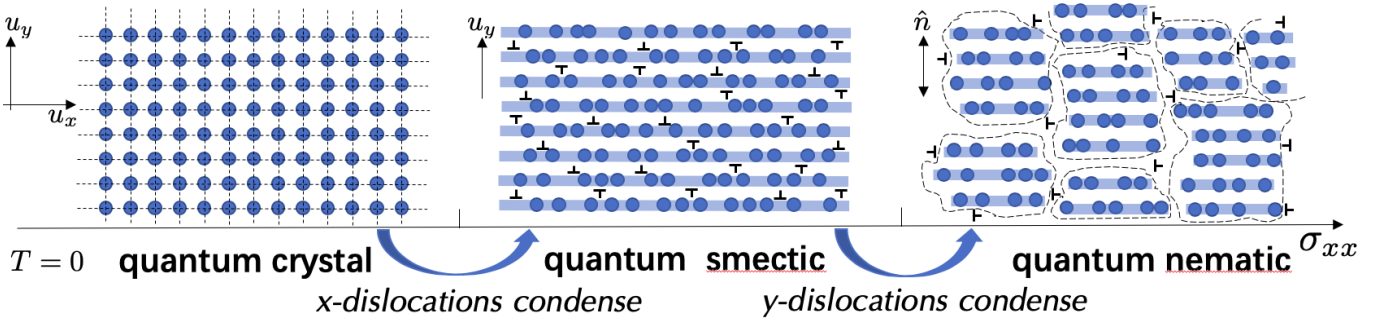
FIG. 2: Illustration of quantum melting of a 2D crystal into a smectic, followed by smectic-to-nematic melting, respectively driven by a condensation of $\hat{\mathbf{x}}$-dislocations and of $\hat{\mathbf{y}}$-dislocations, and tuned by applied shear stress, σ_{xx}.

a higher-derivative sine-Gordon model,

$$\tilde{\mathcal{H}}_{\rm sm} = \frac{1}{2}\chi^{-1}(\partial_x^2\alpha)^2 + \frac{1}{2}K^{-1}(\partial_y\alpha)^2 - g_b\cos(b\partial_x\alpha) \\ - g_s\cos(2\pi\alpha). \tag{6}$$

The first two terms capture the elasticity of a 2D smectic, and the two cosine correspond to dislocations and disclinations, tuned by the corresponding fugacities $g_{b,s}$. We demonstrate that in 2D the dislocations are always relevant, corresponding to the instability of a 2D smectic to a nematic at any nonzero temperature[17,19–21]. The resulting sine-Gordon model in α then captures the the nematic-to-isotropic fluid transition, illustrated in Fig. 3.
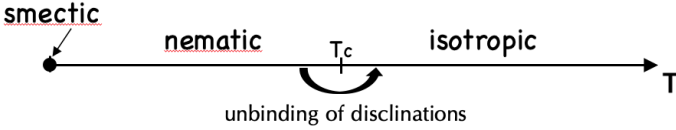

FIG. 3: A phase diagram illustrating an instability of a 2D smectic to arbitrary weak thermal fluctuations that drive it into a nematic at any nonzero temperature. The nematic phase transition into an isotropic fluid is through a KT disclination unbinding transition T_c.

We also derive a classical dual gauge theory for a 3D smectic, that captures its finite-temperature melting into a nematic through a dual normal-superconductor transition with an higher-derivative Maxwell sector, equivalent to Toner's original treatment of the nematic-to-smectic-A transition.[56]

C. Outline

The rest of this paper is organized as follows. In Sec. II, after briefly introducing the elasticity theory of a smectic phase and its topological defects, we map a two-dimensional quantum smectic to a dual coupled U(1) vector gauge theory, and use it to demonstrate that its charges (disclinations) exhibit subdimensional constrained mobility. In Sec. III, starting with the coupled U(1) vector gauge theory for a quantum crystal,

and "softening" it into a generalized Abelian-Higgs model, we rederive the dual gauge theory of a quantum smectic through a Higgs transition of one flavor of its dipoles. Furthermore, we derive an equivalent low-energy tensor gauge-theory description. In Sec. IV, we explore the classical analogue of these dualities and associated phase transitions, and formulate a higher derivative sine-Gordon model, capturing classical thermal smectic melting transitions. We use it to demonstrate that indeed a 2D smectic is unstable and driven into a nematic at any nonzero temperature. We also generalize this discussion to a 3D classical smectic, and reformulate the 3D nematic to smectic-A transition mediated by unbinding of dislocation loops in terms of a higher-derivative classical normal-superconductor transition. We conclude in Sec. V with a summary of our results and discussion of potential utility of our work.

II. SMECTIC AND ITS DUALITY

A. Classical smectic

Ideal smectics are equidistantly layered structures, with a well-defined interlayer spacing d, which can be determined through diffraction experiments. With the layers correlations are liquid-like and exhibit crystal-like periodic modulation transverse to the layers, with corresponding density given by,

$$\rho(\mathbf{r}) = \rho_0 + (\psi e^{\mathbf{q}_0\cdot\mathbf{r}} + h.c.), \tag{7}$$

where, $\mathbf{q}_0 = \frac{2\pi}{d}\hat{\mathbf{z}}$ is the modulation wavevector, and ψ its amplitude, that is the order parameter that distinguishes the smectic phase from the nematic phase.

The deformation of a smectic can be described by its layer displacement field $u(\mathbf{r})$. As the system is invariant under uniform translations, the elastic energy should be expressed purely in terms of derivatives of u. Furthermore, the first-order derivatives along the layer, $\boldsymbol{\nabla}_\perp u$, corresponding to merely a uniform rotation of the layers, must cost no energy. Thus, to harmonic order, only the curvature of the layers, $\boldsymbol{\nabla}_\perp^2 u$, can enter the quadratic

part of the elastic energy functional. This point can be seen more explicitly by the following argument. The order parameter $\psi(\mathbf{r})$, describing phonon fluctuations, can be represented as

$$\psi(\mathbf{r}) = |\psi| e^{-iq_0 u(\mathbf{r})}, \qquad (8)$$

and the locations of the layer planes can be determined as the constant phase of the molecular density wave,

$$\phi(\mathbf{r}) \equiv \mathbf{q}_0 \cdot \mathbf{r} - q_0 u(\mathbf{r}) = 2\pi n, n = 0, \pm 1, \pm 2, ... \qquad (9)$$

The layers local unit-normal is given by,

$$\mathbf{N} = \frac{\boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|} = \frac{(-\boldsymbol{\nabla}_\perp u, -\nabla_\parallel u, 1)}{\sqrt{1+(\boldsymbol{\nabla}u)^2}} \qquad (10)$$
$$= (-\partial_x u, -\partial_y u, 1) + O[(\boldsymbol{\nabla}u)^2].$$

The first-order derivative, $\boldsymbol{\nabla}_\perp u \approx (\partial_x u, \partial_y u, 0) \approx \hat{\mathbf{z}} - \mathbf{N}$, therefore, corresponds to a rigid rotation of the layers around an axis along the layer plane, and does not contribute to the elastic energy, and, $\nabla_\parallel u = \partial_z u$, in 3D. (See. Fig.4.)
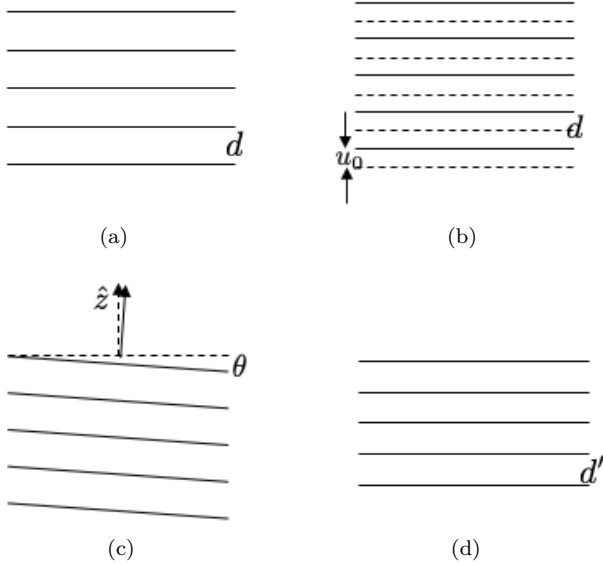

FIG. 4: Elastic deformations of a smectic. (a) Smectic with equilibrium layer spacing: $q_0 = \frac{2\pi}{d}, u = 0$. (b) A uniformly translated smectic with the same energy as that in (a): $u = u_0$. (c) A rotated smectic with the same energy as that in (a): with $u = \theta x$ to harmonic order. (d) A smectic with compressed layers with energy increased relative that in (a): $q_0' = \frac{2\pi}{d'}, u = \left(1 - \frac{d}{d'}\right) z$. [Figures adapted from Reference 2.]

Consistent with this, the continuum elastic Hamiltonian density for a D-dimensional smectic is given by a well-known expression,

$$\mathcal{H}_{\mathrm{sm}} = \frac{1}{2}\chi \left(\nabla_\parallel u\right)^2 + \frac{1}{2}K \left(\boldsymbol{\nabla}_\perp^2 u\right)^2, \qquad (11)$$

a Landau-Peierls elastic energy[19,20,23] for a one-dimensional solid, where χ is inverse of the compressional modulus, and K is the bend modulus.

Note that (10) is only correct up to $O[(\boldsymbol{\nabla}u)^2]$ for small rotations. For any finite rotations θ,

$$\nabla_\parallel u = 1 - \cos\theta, \quad \nabla_\perp u = -\sin\theta, \qquad (12)$$

and the nonlinear strain, $\nabla_\parallel u - \frac{1}{2}(\boldsymbol{\nabla}u)^2$, can be straightforwardly seen to be independent of the rotation angle θ. Thus, the rotationally invariant energy density is given by,

$$\mathcal{H}_{\mathrm{sm}} = \frac{1}{2}\chi \left[\nabla_\parallel u - \frac{1}{2}(\boldsymbol{\nabla}u)^2\right]^2 + \frac{1}{2}K \left(\boldsymbol{\nabla}_\perp^2 u\right)^2, \qquad (13)$$

which introduces non-linear elasticity into $\mathcal{H}_{\mathrm{sm}}$, that for $d \leq 3$ leads to a nontrivial anomalous smectic elasticity[22,23]. However, because the focus of our work is on a quantum smectic, these elastic nonlinearities remain irrelevant in (2+1)D and will thus be neglected in the rest of the manuscript.

For the smectic-A phase, the local normal field (layer orientation) $\mathbf{N}$ and the director field $\mathbf{n} = \hat{\mathbf{z}} + \delta\mathbf{n}$ are aligned in equilibrium. Thus,

$$\boldsymbol{\nabla}_\perp u = -\delta\mathbf{n}, \quad \boldsymbol{\nabla}_\perp^2 u = -\boldsymbol{\nabla}_\perp \delta\mathbf{n} = -\boldsymbol{\nabla}\mathbf{n}, \qquad (14)$$

and the elastic energy, ignoring nonlinearities, can be represented as

$$\mathcal{H}_{\mathrm{sm}} = \frac{1}{2}\chi \left(\boldsymbol{\nabla}u + \delta\mathbf{n}\right)^2 + \frac{1}{2}K \left(\boldsymbol{\nabla}\mathbf{n}\right)^2. \qquad (15)$$

For a 2D smectic, with the layers along $\hat{\mathbf{x}}$ (with layer normal along $\hat{\mathbf{y}}$), the elastic Hamiltonian density in Eq.(11) reduces to

$$\mathcal{H}_{\mathrm{sm}} = \frac{1}{2}\chi \left(\partial_y u\right)^2 + \frac{1}{2}K \left(\partial_x^2 u\right)^2, \qquad (16)$$

where the layer displacement u is along the $\hat{\mathbf{y}}$ axis, and the layer orientation (director field) is, $\mathbf{n} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta = \hat{\mathbf{y}} + \delta\mathbf{n}$.

Another way to obtain smectic elasticity is to start out with a elasticity of a 2D crystal and allow nonsingle-valued displacement field u_x, with $\boldsymbol{\nabla}u_x = \mathbf{v}_x$, accounting for a plasma of unbound dislocations with Burgers vector along the $\hat{\mathbf{x}}-$directed smectic layers, where $\mathbf{v}_x$ is an arbitrary vector with $\epsilon_{ij}\partial_i v_{jx} = b_x$. Integrating over strain tensor field $\mathbf{v}_x$, leads to the smectic harmonic elasticity,

$$\begin{aligned}
\mathcal{H}_{\mathrm{sm}} =& \mathcal{H}_{\mathrm{cr+disl.}} = \mu u_{ij}^2 + \frac{1}{2}\lambda u_{ii}^2 \\
=& \mu \left(u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2\right) + \frac{1}{2}\lambda \left(u_{xx} + u_{yy}\right)^2 + \frac{1}{2}E_c b_x^2 \\
=& \mu \left[v_{xx}^2 + \left(\partial_y u_y\right)^2 + \frac{1}{2}\left(\partial_x u_y + v_{yx}\right)^2\right] \\
& + \frac{1}{2}\lambda \left(v_{xx} + u_{yy}\right)^2 + \frac{1}{2}E_c \left(\partial_x v_{yx} - \partial_y v_{xx}\right)^2 \\
=& \frac{1}{2}\chi \left(\partial_y u_y\right)^2 + \frac{1}{2}K \left(\partial_x^2 u_y\right)^2,
\end{aligned}$$
$$(17)$$

where compressional modulus is $\chi = 4\mu(\mu+\lambda)/(2\mu+\lambda)$, bend modulus $K = E_c$, and higher derivative terms are neglected after integrating out the $\mathbf{v}_x$ field in the last step.

Equivalently, the smectic elasticity can be formulated in terms of the orientational (nematic) angle degree of freedom θ, which corresponds to the orientation of the layers, with the elastic Hamiltonian density given by,

$$\mathcal{H}_{\mathrm{sm}} = \frac{1}{2}\chi\left(\boldsymbol{\nabla}u - \theta\hat{\mathbf{x}}\right)^2 + \frac{1}{2}K\left(\boldsymbol{\nabla}\theta\right)^2. \qquad (18)$$

At low energies set by χ, this Hamiltonian reduces to the conventional form (16) after Higgs'ing out the bond angle θ, locking $\partial_x u = \theta$.

B. Two-dimensional quantum smectic

Classical elastic Hamiltonian in (18) is easily generalized to a quantum smectic by elevating u and θ to operators, and adding canonically conjugate linear and angular momenta operators, $\hat{\pi}$ and $\hat{L}$, respectively. This gives,

$$\hat{\mathcal{H}}_{\mathrm{sm}} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\hat{L}^2 + \frac{1}{2}\chi\left(\boldsymbol{\nabla}\hat{u} - \hat{\theta}\hat{\mathbf{x}}\right)^2 + \frac{1}{2}K\left(\boldsymbol{\nabla}\hat{\theta}\right)^2, \quad (19)$$

for bosonic smectic supplemented with canonical commutation relations ($\hbar = 1$),

$$[\hat{u}(\mathbf{r}), \hat{\pi}(\mathbf{r}')] = i\delta^2\left(\mathbf{r} - \mathbf{r}'\right), \qquad (20a)$$

$$\left[\hat{\theta}(\mathbf{r}), \hat{L}(\mathbf{r}')\right] = i\delta^2\left(\mathbf{r} - \mathbf{r}'\right). \qquad (20b)$$

It is convenient to work with a path-integral formulation where quantum nature of these fields is accounted for by functional integration in phase-space of these fields. We consider the evolution operator for the quantum smectic,

$$U(u', \theta', u, \theta; t) = \langle u', \theta'|e^{-i\int_{\mathbf{r}}\hat{\mathcal{H}}_{\mathrm{sm}}t}|u, \theta\rangle, \qquad (21)$$

and rewrite it in phase-space functional integral formulation as,

$$U = \int [du]\,[d\pi]\,[d\theta]\,[dL]\,e^{iS_{\mathrm{sm}}}$$
$$= \int [du]\,[d\pi]\,[d\theta]\,[dL]\,e^{i\int dt \int_{\mathbf{r}}\mathcal{L}_{\mathrm{sm}}}, \qquad (22)$$

with the corresponding Lagrangian density given by,

$$\mathcal{L}_{\mathrm{sm}} = \pi\partial_t u + L\partial_t\theta - \frac{1}{2}\pi^2 - \frac{1}{2}L^2 + \frac{1}{2}\chi^{-1}\sigma^2 + \frac{1}{2}K^{-1}\mathbf{j}^2$$
$$- \boldsymbol{\sigma}\cdot\left(\boldsymbol{\nabla}u - \theta\hat{\mathbf{x}}\right) - \mathbf{j}\cdot\boldsymbol{\nabla}\theta. \qquad (23)$$

In addition to the single-valued (smooth) Goldstone mode degrees of freedom, $\tilde{\theta}$ and $\tilde{u}$, we must also include topological defects – disclinations and dislocations, captured by including a nonsingle-valued component of the
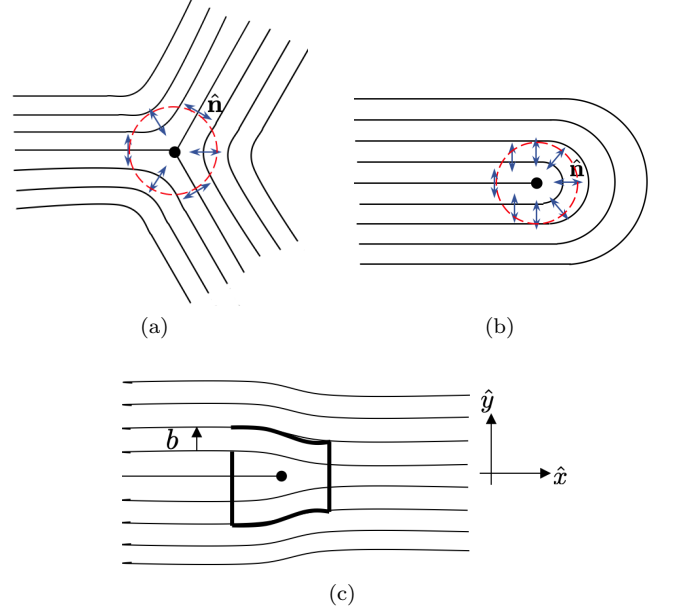

FIG. 5: Topological defects in a 2D smectic. $\hat{\mathbf{n}}$ is the Frank director, and $\mathbf{b}$ is the Burger's vector. (a) A $-\frac{1}{2}$ disclination, i.e., the director $\hat{\mathbf{n}}$ changes $-\pi$ around a closed loop counterclockwise. (b) A $+\frac{1}{2}$ disclination, i.e., the director $\hat{\mathbf{n}}$ changes $+\pi$ around a closed loop counterclockwise. (c) A single positive dislocation, i.e., the layer displacement u changes $+d$ around a closed circuit counterclockwise, which can be regarded as a tightly bound dipole of two, opposite disclinations with charges $\pm\frac{1}{2}$.

bond angle field θ^s and of the phonon distortion field u^s, respectively. In the smectic, a disclination at a point $\mathbf{r}_0$, is defined by a nonzero closed line-integral of the gradient of the bond angle around $\mathbf{r}_0$, $\oint_{\mathbf{r}_0} d\theta = 2\pi n_s$, or equivalently in a differential form,

$$\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\theta^s = 2\pi n_s\delta^2(\mathbf{r} - \mathbf{r}_0) \equiv n_s(\mathbf{r}), \qquad (24)$$

measuring the deficit/surplus bond angle. $n_s(\mathbf{r})$ is the disclination charge density. A disclination with charge $n_s = -\frac{1}{2}$, and a disclination with charge $n_s = +\frac{1}{2}$ are illustrated in Fig. 5(a, b).

A dislocation at $\mathbf{r}_0$ with a Burgers charge n_b (that is an integer multiples of the elementary layer spacing), is defined by a closed line-integral, $\oint_{\mathbf{r}_0} du = n_b$, or equivalently in the differential form,

$$\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}u^s = n_b\delta^2(\mathbf{r} - \mathbf{r}_0) \equiv n_b(\mathbf{r}), \qquad (25)$$

where $n_b(\mathbf{r})$ is the Burgers charge density. A dislocation in the smectic is shown in Fig. 5(c), which can be regarded as a tightly bound pair of $+\frac{1}{2}$ and $-\frac{1}{2}$ disclinations.

In anticipation of our more rigorous duality derivation, already here we can argue for the subdimensional nature of the disclination dynamics in a smectic. Consider a pair of oppositely charged $\pm\frac{1}{2}$ disclinations separated along $\hat{x}$ axis, as shown in Fig. 6. The separation between the pair

is large such that we can regard them as isolated disclinations. Moving the '+' disclination along the layers (i.e., $\hat{\mathbf{x}}$ axis) by two layer spacings, requires the introduction of four extra half-layers of molecules, which is a highly nonlocal process in terms of atoms quantum dynamics, and is therefore not allowed. While moving the '+' disclination transversely (i.e., in $\hat{y}$ direction) preserves the strength of the dislocation, and thus, is allowed dynamically, even though there is an energy cost for separating the '+' and '−' disclination pair that make up the dislocation. Similar analysis applies to the '−' disclination. Thus, we conclude that disclinations can only move transversely to the layers, i.e., they manifests the subdimensional lineon dynamics.

To include the topological defects in the complete description of the smectic, we decompose the distortion field u and the bond angle θ into the smooth elastic and nonsingle-valued components,

$$u = \tilde{u} + u^s, \quad \theta = \tilde{\theta} + \theta^s. \tag{26}$$

Integrating out the single-valued parts $\tilde{u}$ and $\tilde{\theta}$ out of the total generating function,

$$U = \int [d\tilde{u}]\,[d\tilde{\theta}]\,[du^s]\,[d\pi]\,[d\theta^s]\,[dL]\,e^{i\int dt\int_{\mathbf{r}}\mathcal{L}_{\mathrm{sm}}}, \tag{27}$$

leads to,

$$U = \int [du^s]\,[d\pi]\,[d\theta^s]\,[dL]\,e^{i\int dt\int_{\mathbf{r}}\mathcal{L}_{\mathrm{sm}}}, \tag{28}$$

where the new Lagrangian density is given in terms of only nonsingular components,

$$\begin{aligned}\mathcal{L}_{\mathrm{sm}} =&\pi\partial_t u^s + L\partial_t\theta^s - \frac{1}{2}\pi^2 - \frac{1}{2}L^2 + \frac{1}{2}\chi^{-1}\sigma^2 \\ &- \sigma\cdot(\boldsymbol{\nabla}u^s - \theta^s\hat{\mathbf{x}}) + \frac{1}{2}K^{-1}\mathbf{j}^2 - \mathbf{j}\cdot\boldsymbol{\nabla}\theta^s,\end{aligned} \tag{29}$$

with two enforced constraints,

$$\partial_t\pi - \boldsymbol{\nabla}\cdot\sigma \equiv \partial_\mu J_\mu = 0, \tag{30}$$
$$\partial_t L - \boldsymbol{\nabla}\cdot\mathbf{j} - \sigma_x = 0, \tag{31}$$

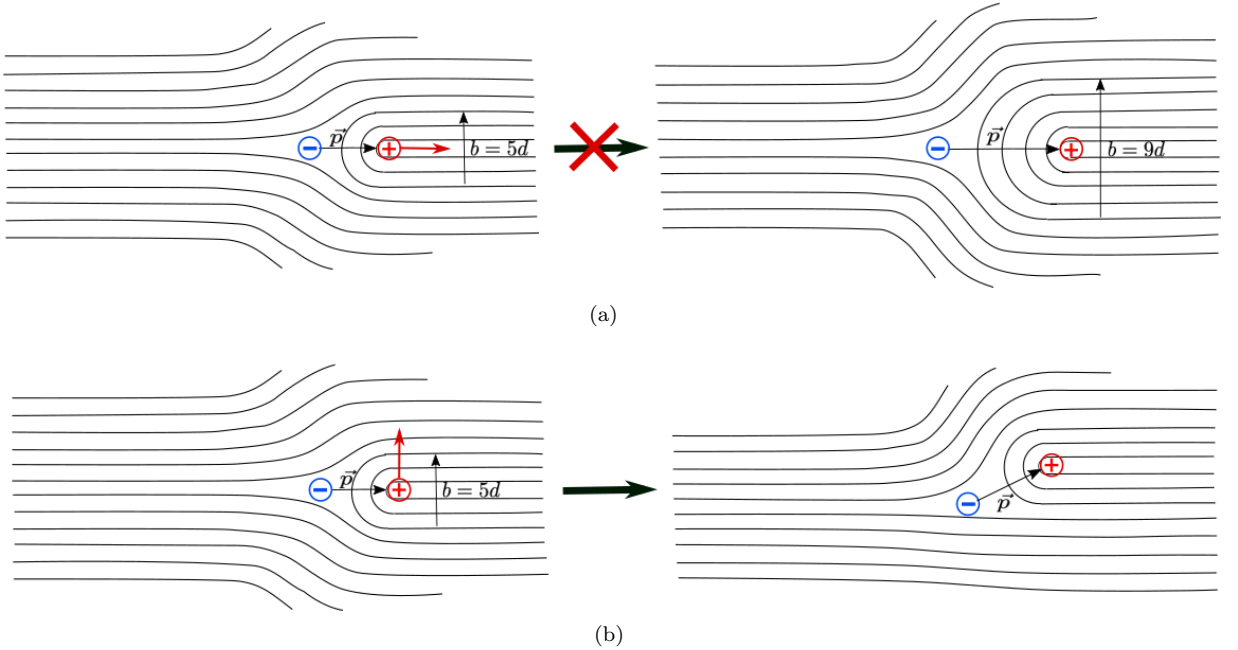where we implicitly introduced currents $J_\mu = (\pi, -\sigma_i)$.

(a)

(b)

FIG. 6: Restricted along-layers mobility of disclinations in a 2D quantum smectic. Consider a pair of disclinations separated by five layer spacings. (a) Moving the '+' disclination along the layers by two layer spacings requires an introduction of four half-layers of underlying bosons, which is a non-local process, and therefore is not allowed. (b) Moving the '+' disclination transverse to the layers by two layer spacings preserves the strength of the dislocation, and thus, is dynamically allowed.

C. Quantum smectic-gauge theory duality

The momentum continuity equation (30) can be solved in terms of gauge potential fields, A_μ, with

$$J_\mu = \epsilon_{\mu\nu\gamma}\partial_\nu A_\gamma = (\epsilon_{ij}\partial_i A_j, \epsilon_{i\nu\gamma}\partial_\nu A_\gamma), \tag{32}$$

such that

$$\pi = \epsilon_{ij}\partial_i A_j = \hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}\times\mathbf{A}), \tag{33}$$
$$\sigma_i = -\epsilon_{i\nu\gamma}\partial_\nu A_\gamma = \epsilon_{ij}(\partial_t A_j - \partial_j A_0), \tag{34}$$

and Eq.(31) transforms into

$$\partial_t (L - \epsilon_{xj} A_j) - \partial_i (j_i - \epsilon_{xi} A_0) \equiv \partial_\mu \tilde{j}_\mu = 0, \qquad (35)$$

with $\tilde{j}_\mu = (L - \epsilon_{xj} A_j, -j_i + \epsilon_{xi} A_0)$, which is then solved by introducing another vector gauge potential a_μ,

$$\tilde{j}_\mu = \epsilon_{\mu\nu\gamma} \partial_\nu a_\gamma, \qquad (36)$$

such that

$$L = \epsilon_{ij} \partial_i a_j + \epsilon_{xj} A_j = \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla} \times \mathbf{a} + \hat{\mathbf{x}} \times \mathbf{A}), \quad (37)$$
$$j_i = \epsilon_{ij} (\partial_t a_j - \partial_j a_0) + \epsilon_{xi} A_0. \qquad (38)$$

Substituting the solutions of $\pi, \sigma_i, L,$ and j_i in terms of the gauge fields into the Lagrangian density, leads to the dual Lagrangian density

$$\tilde{\mathcal{L}}_{\text{sm}} = \mathcal{L}_{\text{M}}^{\text{sm}} + \mathbf{A} \cdot \mathbf{j}^b + A_0 n_b + \mathbf{a} \cdot \mathbf{j}^s + a_0 n_s, \qquad (39)$$

where $\mathcal{L}_{\text{M}}$ is the Maxwell part, given by

$$\mathcal{L}_{\text{M}}^{\text{sm}} = \frac{1}{2} \chi^{-1} (\partial_t \mathbf{A} - \boldsymbol{\nabla} A_0)^2 - \frac{1}{2} (\boldsymbol{\nabla} \times \mathbf{A})^2$$
$$+ \frac{1}{2} K^{-1} (\partial_t \mathbf{a} - \boldsymbol{\nabla} a_0 - A_0 \hat{\mathbf{x}})^2 - \frac{1}{2} (\boldsymbol{\nabla} \times \mathbf{a} + \hat{\mathbf{x}} \times \mathbf{A})^2,$$
$$(40)$$

and the charge contributions are obtained by integrating by parts and defining the dislocation and disclination charge and current densities as,

$$n_b = \epsilon_{ij} \partial_i \partial_j u, \qquad (41a)$$
$$n_s = \epsilon_{ij} \partial_i \partial_j \theta, \qquad (41b)$$
$$j_i^b = \epsilon_{ij} (\partial_j \partial_t u - \partial_t \partial_j u), \qquad (41c)$$
$$j_i^s = \epsilon_{ij} (\partial_j \partial_t \theta - \partial_t \partial_j \theta). \qquad (41d)$$

Introducing Hubbard-Stratonovich fields $\mathbf{E}$ and $\mathbf{e}$, the Lagrangian density transforms into

$$\tilde{\mathcal{L}}_{\text{sm}} = -\mathbf{E} \cdot (\partial_t \mathbf{A} - \boldsymbol{\nabla} A_0) - \frac{1}{2} \chi \mathbf{E}^2 - \frac{1}{2} (\boldsymbol{\nabla} \times \mathbf{A})^2 - \frac{1}{2} K \mathbf{e}^2$$
$$- \mathbf{e} \cdot (\partial_t \mathbf{a} - \boldsymbol{\nabla} a_0 - A_0 \hat{\mathbf{x}}) - \frac{1}{2} (\boldsymbol{\nabla} \times \mathbf{a} + \hat{\mathbf{x}} \times \mathbf{A})^2$$
$$+ \mathbf{A} \cdot \mathbf{j}^b + A_0 n_b + \mathbf{a} \cdot \mathbf{j}^s + a_0 n_s.$$
$$(42)$$

Integrating over A_0 and a_0 gives the Gauss law, leaving the standard Lagrangian form, $\tilde{\mathcal{L}}_{\text{sm}} = -\mathbf{E} \cdot \partial_t \mathbf{A} - \mathbf{e} \cdot \partial_t \mathbf{a} - \tilde{\mathcal{H}}_{\text{sm}}$, from which we can read off the dual Hamiltonian density, that is given by,

$$\tilde{\mathcal{H}}_{\text{sm}} = \frac{1}{2} \chi \hat{\mathbf{E}}^2 + \frac{1}{2} (\boldsymbol{\nabla} \times \hat{\mathbf{A}})^2 + \frac{1}{2} K \hat{\mathbf{e}}^2$$
$$+ \frac{1}{2} (\boldsymbol{\nabla} \times \hat{\mathbf{a}} + \hat{\mathbf{x}} \times \hat{\mathbf{A}})^2 - \hat{\mathbf{A}} \cdot \mathbf{j}^b - \hat{\mathbf{a}} \cdot \mathbf{j}^s,$$
$$(43)$$

supplemented by the generalized Gauss laws

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{E}} = n_b + \hat{\mathbf{e}} \cdot \hat{\mathbf{x}}, \qquad (44)$$
$$\boldsymbol{\nabla} \cdot \hat{\mathbf{e}} = n_s, \qquad (45)$$

where, $\mathbf{E}$ and $\mathbf{e}$, are independent electric fields, canonically conjugate to the corresponding vector potentials, $\mathbf{A}$ and $\mathbf{a}$, respectively.

The above Hamiltonian must be invariant under the gauge transformations:

$$(A_0, A_i) \rightarrow A_\mu' = (A_0 + \partial_t \chi, (A_i + \partial_i \chi)), \qquad (46a)$$
$$(a_0, a_i) \rightarrow a_\mu' = (a_0 + \partial_t \phi, (a_i + \partial_i \phi + \hat{\mathbf{x}}_i \chi)). \quad (46b)$$

Requiring the source term to preserve this gauge invariance, we obtain coupled continuity equations for charges (disclinations) and dipoles (dislocations), satisfying,

$$\partial_t n_s + \boldsymbol{\nabla} \cdot \mathbf{j}_s = 0, \qquad (47)$$
$$\partial_t n_b + \boldsymbol{\nabla} \cdot \mathbf{j}_b = -j_s^x. \qquad (48)$$

We observe that the dipole (dislocation) continuity equation is violated by a nonzero charge (disclination) current j_s^x in the $\hat{\mathbf{x}}$ (along the layers) direction. Thus, in the absence of gapped $\hat{\mathbf{x}}$−dipoles ($\hat{\mathbf{y}}$-dislocations), we find that $j_s^x = 0$, i.e., motion of isolated fracton charges (disclinations) is restricted to be transverse to the smectic layers, as moving along the layers requires $\hat{\mathbf{x}}$-dipoles ($\hat{\mathbf{y}}$-dislocations) that are gapped in the smectic. Therefore, the fractons (disclinations) exhibit subdimensional lineon dynamics, as argued in Section II B.

III. HIGGS TRANSITION OF QUANTUM CRYSTAL-TO-SMECTIC MELTING

As discussed in the Introduction, a smectic can emerge from anisotropic melting[17] of a crystal, understood in terms of a Kosterlitz-Thouless (KT)-like,[18] single-species dislocation unbinding transition. This classical partial melting transition of an anisotropic solid was studied by Halperin and Ostlund[17], and we have used this fact in Eq.(17) to derive the smectic harmonic elasticity by including $\hat{\mathbf{x}}$−orientated dislocations in a classical crystal. Although such a 2D classical smectic is unstable to thermal fluctuations, driven into a nematic fluid at any nonzero temperature[17,19–21] (see Section IV), a (2+1)D quantum smectic at zero temperature is a stable state of matter. In this section, we demonstrate that similarly, a quantum smectic can also emerge from partial, anisotropic quantum melting of a crystal.

The Hamiltonian density of a 2D quantum crystal is,

$$\mathcal{H}_{\text{cr}} = \frac{1}{2} C_{ij,kl} (\partial_i \hat{u}_j - \hat{\theta} \epsilon_{ij})(\partial_k \hat{u}_l - \hat{\theta} \epsilon_{kl}) + \frac{1}{2} K (\boldsymbol{\nabla} \hat{\theta})^2$$
$$+ \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} \hat{L}^2,$$
$$(49)$$

where, $\hat{\mathbf{u}}$ is the phonon field operator, $\hat{\theta}$ is the orientational bond-angle field operator, and, $\hat{\pi}$ and $\hat{L}$ are their corresponding canonically conjugate momentum respectively. $C_{ij,kl}$ is the elastic constant tensor, which takes the form, $C_{ij,kl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}$, for an

isotropic hexagonal lattice, characterized by two independent Lamé coefficients, λ and μ. Working in the path-integral formulation with the field operators replaced by corresponding classical fields, the action is

$$S_{\mathrm{cr}} = \int dt \int_{\mathbf{r}} \frac{1}{2} \left[(\partial_t \mathbf{u})^2 + (\partial_t \theta)^2 - K(\boldsymbol{\nabla}\theta)^2 \right. $$
$$\left. -C_{ij,kl}(\partial_i u_j - \theta\epsilon_{ij})(\partial_k u_\ell - \theta\epsilon_{kl}) \right]. \quad (50)$$

Now, as the $\hat{\mathbf{x}}$−dislocations condensed, we include dislocations with Burgers vector along the layers ($\hat{\mathbf{x}}$) by replacing $\partial_i u_x = v_{ix}$, where v_i is an arbitrary vector field with $\epsilon_{ij}\partial_i v_{jx} = b_x$. Then, the Lagrangian density

$$\mathcal{L}_{\mathrm{cr+disl.}} = \frac{1}{2} \left[(\partial_t \mathbf{u})^2 + (\partial_t \theta)^2 - K(\boldsymbol{\nabla}\theta)^2 - \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})^2 \right.$$
$$\left. -2\mu(\partial_i u_j - \theta\epsilon_{ij})^2 \right]$$
$$= \frac{1}{2} \left[(\partial_t \mathbf{u})^2 + (\partial_t \theta)^2 - K(\boldsymbol{\nabla}\theta)^2 - \lambda(v_{xx} + \partial_y u_y)^2 \right.$$
$$-2\mu\left(v_{xx}^2 + (\partial_y u_y)^2 + (v_{yx} - \theta)^2 + (\partial_x u_y - \theta)^2\right)$$
$$\left. -\frac{1}{2}E_c(\partial_x v_{yx} - \partial_y v_{xx})^2 \right]$$
$$= \frac{1}{2} \left[(\partial_t \mathbf{u})^2 + (\partial_t \theta)^2 - K(\boldsymbol{\nabla}\theta)^2 - \chi_1(\partial_y u_y)^2 \right.$$
$$\left. -\chi_2(\partial_x u_y - \theta)^2 \right]$$
$$= \frac{1}{2} \left[(\partial_t u_y)^2 + (\partial_t \theta)^2 - \chi(\boldsymbol{\nabla}u_y - \theta\hat{\mathbf{x}})^2 - K(\boldsymbol{\nabla}\theta)^2 \right], \quad (51)$$

where, in the third line, we have integrated out v_{ix} and u_x, and have defined $\chi_1 = 4\mu(\mu + \lambda)/(2\mu + \lambda)$ and $\chi_2 = 2\mu$, which are taken to be equal in the last line, $\chi_1 = \chi_2 = \chi$, for simplicity, corresponding to the case when $\lambda \to 0$. Then, we arrive at the Lagrangian density of a smectic, starting with that of a quantum crystal,

$$\mathcal{L}_{\mathrm{sm}} = \mathcal{L}_{\mathrm{cr+disl.}}, \quad (52)$$

as summarized by the flow chart in Fig. 1.

Motivated by this possibility of partial quantum melting of a crystal into a smectic and the subsequent melting into a quantum nematic, we will explore its dual in this section. Below, we will also derive a dual gauge theory of a quantum smectic, through a Higgs transition from a dual gauge theory of an incommensurate quantum crystal (supersolid) by condensing one flavor of dipoles, and thereby Higgs'ing out a flavor component of the gauge fields. As required by consistency, we indeed find that the resulting quantum smectic dual is in full agreement with a direct duality derived in Section II.

A. Soft-spin descriptions of quantum crystal and quantum smectic

1. Quantum crystal

The dual coupled U(1) vector gauge theory for a quantum crystal was first derived in Ref. [48], characterized

by the Lagrangian density

$$\tilde{\mathcal{L}}_{\mathrm{cr}} = \mathcal{L}_{\mathrm{M}}^{\mathrm{cr}} + \mathbf{A}_k \cdot \mathbf{J}_k + A_{0k} n_k^b + \mathbf{a}\cdot\mathbf{j}^s + a_0 n_s, \quad (53)$$

where $k = (x, y)$ indexes different flavors, $\mathbf{A}_k$, $\mathbf{a}$ gauge fields capture the $k = x, y$ phonons and bond orientational order respectively. The Maxwell part, $\mathcal{L}_{\mathrm{M}}^{\mathrm{cr}}$, is given by

$$\mathcal{L}_{\mathrm{M}}^{\mathrm{cr}} = \frac{1}{2}\chi^{-1}(\partial_t \mathbf{A}_k - \boldsymbol{\nabla}A_{0k})^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}_k)^2$$
$$+\frac{1}{2}K^{-1}(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0 - A_{0k}\hat{\mathbf{e}}_k)^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a} - \hat{\mathbf{z}}\times\mathbf{A}_k)^2. \quad (54)$$

To access descendant phases and corresponding quantum phase transitions, we need to treat dislocation and disclination defects as dynamical charges. Following a standard analysis and focusing on dislocations (dipoles) for the moment, we introduce the dynamical field $\psi_k(\mathbf{r}, t) = \sqrt{\rho_k}e^{i\varphi_k}$ for each gauge-charged dipole species $\mathbf{p}_k$, and add corresponding kinetic energies, $\frac{\rho_k}{2}(\partial_t\varphi_k + pA_{0k})^2$, where we have integrated out the massive magnitude fluctuations to focus on the low-energy phase fluctuations only.
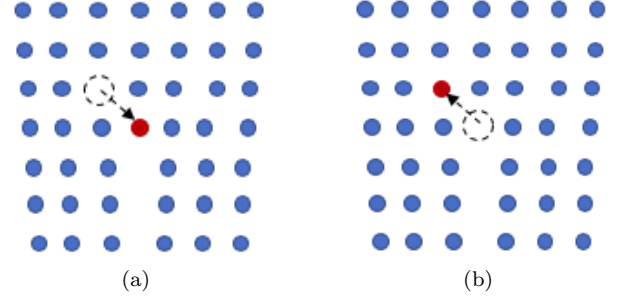

FIG. 7: Dislocation climb via vacancies diffusion. An edge dislocation moves out of the slip plane onto a parallel plane directly above or below the slip plane. This movement (climb) is termed nonconservative, as compared with conservative movement (glide). (a) Diffusion of vacancy to edge dislocation. (b) Dislocation climbs up one lattice spacing.

As discussed in the Introduction, in the Mott-insulating commensurate crystal phase, the dipole $\mathbf{p}_k$ can only move in the direction perpendicular to $\mathbf{p}_k$ while the along-dipole climb is forbidden due to the U(1) particle-number conservation symmetry ("glide-constraint"). Thus, we have only the glide motion, $\Pi_{ij}^{\perp\mathbf{p}_k}D_j\psi_{\mathbf{p}_k}$, with, $\mathbf{D} = \boldsymbol{\nabla} + ip_k\mathbf{A}_k$, the covariant spatial derivative, and $\Pi_{ij}^{\perp\mathbf{p}_k} = \delta_{ij} - \frac{p_{i,k}p_{j,k}}{p_k^2}$, the transverse projection operator. However, the crystal also exhibits scalar non-topological point defects, corresponding to deficiency and excess in atom density, which permits the climb process of the dipoles (dislocations), as shown in Fig. 7. Combined with the bosonic statistics of the underlying particles, the quantum crystal can first develop

into a super-solid phase ("incommensurate" crystal), featuring both the crystalline and the superfluid orders. The condensation of vacancies or interstitials in the supersolid phase (as illustrated in Fig.8 is a bound state of opposite charge dislocatons - a quadrupole), therefore, frees these symmetry-forbidden climb constraints[45–47,57], $\Pi_{ij}^{\|\mathbf{P}_k} D_j \psi_{\mathbf{P}_k}$, where $\Pi_{ij}^{\|\mathbf{P}_k} = \frac{p_{i,k}p_{j,k}}{p_k^2}$ is the longitudinal projection operator. In Appendix A, we show in detail how the dislocation-superfluid coupling alleviates the glide constraint and converts the dislocations (dipoles) from subdimensional quasi-particles to ordinary mobile defects, acted upon by the full spatial derivative, $\mathbf{D}\psi_{\mathbf{P}_k}$.
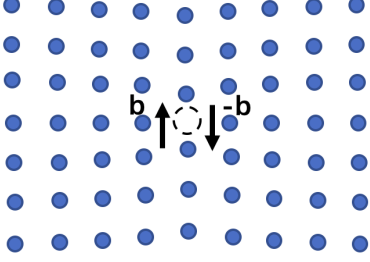

FIG. 8: A disclination quadrupole, constructed as a bound pair of two equal and opposite dislocations with Burgers vectors $\mathbf{b}$ and $-\mathbf{b}$, carries a unit of atom number, as can be seen by the deficiency of a single atom in the middle of the configuration. As the dislocations proliferate, the condensation of pairs of opposite dislocations $(\mathbf{b}, -\mathbf{b})$ must always accompany the condensation of vacancies/interstitials, which correspond to terms of $\mathcal{L}_{\text{sf-dis}}$, the coupling between dislocation climb operators and vacancy or interstitial condensate (see Appendix A).

Introducing defect's core energy, $E_{j_\mu^b}$, to account for lattice-scale physics, and the dipole (dislocation) charge density and current density on a discrete lattice is given as a sum of their discrete charges

$$n_k^b(\mathbf{r}) = \sum_{\mathbf{r}_n} d \cdot n_k^b(\mathbf{r}_n)\delta^2(\mathbf{r} - \mathbf{r}_n), \qquad (55)$$

$$\mathbf{j}_k^b(\mathbf{r}) = \sum_{\mathbf{r}_n} d \cdot \mathbf{j}_k^b(\mathbf{r}_n)\delta^2(\mathbf{r} - \mathbf{r}_n), \qquad (56)$$

where d is the lattice spacing for a 2D crystal, which is also the elementary charge of the gauge dipoles (unit of the dislocation charge), i.e., $|\mathbf{p}_k| = p = d$. The partition function is then given by

$$Z = \sum_k \int \prod_{\mathbf{r}_n} dA_{\mu,k}(\mathbf{r}_n) \sum_{j_{\mu,k}^b} \delta\left(\Delta_\mu j_{\mu,k}^b\right) \cdot e^{-S_{\text{M}}^{\text{cr}}}$$

$$\cdot e^{-\int dt \sum_{\mathbf{r}_n} \sum_k \left[\frac{m_p}{2}(\partial_t\varphi_k + pA_{0,k})^2 - d^2 E_{j_{\mu,k}^b}|j_{\mu,k}^b(\mathbf{r}_n)|^2 + dA_{\mu,k}j_{\mu,k}^b\right]}$$

$$\equiv \sum_k \int \prod_{\mathbf{r}_n} dA_{\mu,k}(\mathbf{r}_n)d\varphi_k(\mathbf{r}_n) \sum_{j_{\mu,k}^b} e^{-\tilde{S}_{\text{cr}}[A_{\mu,k}, j_{\mu,k}^b]}$$

$$(57)$$

with the action

$$\tilde{S}_{\text{cr}} = \int dt \sum_{\mathbf{r}_n} \sum_k \left[\frac{m_p}{2}(\partial_t\varphi_k + pA_{0,k})^2 - \tilde{E}_{j_{\mu,k}^b}|j_{\mu,k}^b(\mathbf{r}_n)|^2 \right.$$

$$\left. + d\left(A_{\mu,k} + \frac{1}{d}\Delta_\mu\varphi_k\right)j_{\mu,k}^b(\mathbf{r}_n)\right] + S_{\text{M}}^{\text{cr}},$$

$$(58)$$

where, m_p is the effective inertia mass of the dipoles, $\tilde{E}_{j_{\mu,k}^b} = d^2 E_{j_{\mu,k}^b}$ in the discrete lattice, and, $\Delta_\mu j_{\mu,k}^b = j_{\mu,k}^b(\mathbf{r} + \mu) - j_{\mu,k}^b(\mathbf{r})$ and $\Delta_\mu\varphi_k = \varphi_k(\mathbf{r} + \mu) - \varphi_k(\mathbf{r})$, $\mu = x, y$, are the discrete lattice derivatives. Note the continuity equation $\Delta_\mu j_{\mu,k}^b = 0$ is automatically satisfied when we integrate out φ_k, which is the phase of the $k-$flavor dipole field $\psi_k = |\psi_k|e^{i\varphi_k}$. After tracing over the 3-currents $j_{\mu,k}^b = (n_k^b, \mathbf{j}_k^b)$, we obtain

$$\tilde{S}_{\text{cr}} = \int dt \sum_{\mathbf{r}_n} \sum_k \left[\frac{m_p}{2}(\partial_t\varphi_k + pA_{0,k})^2 \right.$$

$$\left. - g_k^b \cos(\Delta_\mu\varphi_k + pA_{\mu,k})\right] + S_{\text{M}}^{\text{cr}}, \qquad (59)$$

where $g_k^b = 2e^{-\tilde{E}_{j_k^b}}$, and we have approximated the resulting Villain potential by its lowest harmonic.

In the continuum limit, we have

$$\tilde{S}_{\text{cr}} = \int dt \int d^2r \tilde{\mathcal{L}}_{\text{cr}}, \qquad (60)$$

with the Lagrangian density $\tilde{\mathcal{L}}_{\text{cr}}$ given by

$$\tilde{\mathcal{L}}_{\text{cr}} = \sum_k \left[\frac{\rho_k}{2}(\partial_t\varphi_k + pA_{0,k})^2 - \tilde{g}_k^b \cos(\boldsymbol{\nabla}\varphi_k + p\mathbf{A}_k)\right]$$

$$+ \mathcal{L}_{\text{M}}^{\text{cr}},$$

$$(61)$$

where, $\tilde{g}_k^b = g_k^b/d^2 = \frac{2}{d^2}e^{-\tilde{E}_{j_k^b}}$.

We then turn to an equivalent "soft-spin" description by noting that this ordered phase action emerges from a corresponding quantum Ginzburg-Landau theory for the complex order parameter, $\psi_k = |\psi_k|e^{i\varphi_k}$, and write $\tilde{\mathcal{L}}_{\text{cr}}$ as,

$$\tilde{\mathcal{L}}_{\text{cr}} = \sum_k \frac{1}{2}|(\partial_\mu + ipA_{\mu,k})\psi_k|^2 - V(\{|\psi_k|\}) + \mathcal{L}_{\text{M}}^{\text{cr}}, \quad (62)$$

where, ψ_k correspond to $\hat{\mathbf{x}}$- and $\hat{\mathbf{y}}$-oriented dipole fields ($\hat{\mathbf{y}}-$ and $\hat{\mathbf{x}}-$dislocations) in a square lattice, and $V(\{|\psi_k|\})$, is the Landau U(1)-invariant potential for the quantum crystal, the form of which controls the type and subsequence of phase transitions (See Section III B). It is straightforward to verify that Eq.(62) reduces to Eq.(61) when the gapped Higgs-like magnitude degrees of freedom, $|\psi_k|$, whose fluctuations are controlled by $V(\{|\psi_k|\})$, are integrated out, and therefore these two Lagrangians are equivalent.

2. Quantum smectic

Similar analysis applies also for the quantum smectic. The condensation of vacancies or interstitials in the super-smectic ("incommensurate" smectic) phase also endows the full mobility of dipoles (dislocations) in the smectic.

Then, starting with $\tilde{\mathcal{L}}_{\text{sm}}$ given by Eq.(39) and elevating dislocation and disclination defects into dynamical charges, we add corresponding kinetic energies, $\frac{\rho}{2}\left(\partial_t\varphi_x + pA_0\right)^2$, and follow the same procedure as what have been done for the quantum crystal above, which leads to the effective Lagrangian density of the super-smectic in the continuum given by,

$$
\begin{aligned}
\tilde{\mathcal{L}}_{\text{sm}} =& \frac{\rho}{2}\left(\partial_t\varphi_x + pA_0\right)^2 - \tilde{g}_b\cos\left(\boldsymbol{\nabla}\varphi_x + p\mathbf{A}\right) + \mathcal{L}_{\text{M}}^{\text{sm}} \\
=& \frac{1}{2}|\left(\partial_\mu + ipA_\mu\right)\psi_x|^2 - V\left(|\psi_x|\right) + \mathcal{L}_{\text{M}}^{\text{sm}},
\end{aligned}
$$
(63)

where for concreteness we have taken the layers to be along the $\hat{\mathbf{x}}$ axis, and replaced $\tilde{g}_x^b$ simply by $\tilde{g}_b$ for the $\hat{\mathbf{x}}$-dipole ($\hat{\mathbf{y}}$-dislocation) fugacity, and $A_{\mu,x}$ simply by A_μ. In the second form, we have written it as an equivalent "soft-spin" description in terms of $\psi_x = |\psi_x|e^{i\varphi_x}$, with the Landau U(1)-invariant potential, $V\left(|\psi_x|\right)$, which is equivalent to the first form when the gapped Higgs magnitude degree of freedom, $|\psi_x|$, whose fluctuations are controlled by $V\left(|\psi_x|\right)$, is integrated out.

B. Crystal-to-smectic and smectic-to-nematic transitions

The quantum crystal phase can go through a fully isotropic melting transition, mediated by dislocations, into a hexatic (or nematic) phase, or through a multistage anisotropic transition, first partially melting into a smectic phase, depending on the form of the Landau potential $V(\{|\psi_k|\})$. A simple discussion based on Ginzburg-Landau theory of continuous phase transitions is given in the following.

Considering a 2D square lattice, the U(1)-invariant Landau potential $V(\{|\psi_k|\})$, satisfying the symmetries of the system, expanded to fourth-order of the dipole fields ψ_k, is given by

$$
\begin{aligned}
V =& \frac{\alpha}{2}\sum_k|\psi_k|^2 + \frac{\beta}{4}\sum_k|\psi_k|^4 + \frac{\beta'}{2}|\psi_x|^2|\psi_y|^2 \\
\equiv& \frac{\alpha}{2}|\Psi|^2 + \frac{\beta}{4}|\Psi|^4 + \frac{1}{2}\left(\beta' - \beta\right)|\psi_x|^2|\psi_y|^2,
\end{aligned}
$$
(64)

where, $\beta, \beta' > 0$, and we have defined the vector complex order parameter $\Psi = (\psi_x, \psi_y)$. Note that by these two forms, we can regard this potential as two identical complex Ising models coupled together, or a complex XY model with the originally rotational symmetry broken down to a rectangular one. As usual, at mean-field level the phase transition takes place at $\alpha = 0$, as α changes its sign.

1. Crystal-to-nematic transition

We first consider the case of $\beta > \beta'$, when it is energetically favorable for both flavors of dipole fields to condense with the same expectation value $|\psi_x| = |\psi_y| = \sqrt{\frac{|\alpha|}{\beta+\beta'}}$, corresponding to the nematic phase. This crystal-to-nematic transition, analogue of the crystal-to-hexatic melting in a 2D classical crystal, has been discussed in Ref. 46, formulated as a Ginzburg-Landau theory of tensor superconductors, i.e., dipole fields coupled to the symmetric tensor gauge field. However, as pointed out in the Introduction, that tensor-only gauge theory formulation of the Mott-insulating "commensurate" crystal, fails to capture the full dipole mobility endowed by the condensation of vacancies and interstitials. Here, we take a complementary coupled vector gauge theory description[48] and condense both $\hat{\mathbf{x}}$- and $\hat{\mathbf{y}}$-dipoles, $\psi_x \neq 0, \psi_y \neq 0$. Via Anderson-Higgs mechanism this gaps out all gauge field flavors, $A_{\mu,k}, k = x, y$. These can then be safely integrated out at low energies at wavelengths longer than $\sqrt{\frac{2m_p}{\rho_k p^2}}$ (i.e., the penetration length of $A_{\mu,k}$), thereby reducing the Maxwell Lagrangian of the crystal to that of a form described by the rotational gauge field a_μ only. With details relegated to Appendix B, the result is given by,

$$
\begin{aligned}
\mathcal{L}_{\text{M}}^{\text{cr}}\left(A_{\mu,k} = 0, a_\mu\right) \approx& \mathcal{L}_{\text{M}}^{\text{nm}}\left(a_\mu\right) \\
=& \frac{1}{2}K^{-1}\left(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{a}\right)^2,
\end{aligned}
$$
(65)

which corresponds to setting, $A_{\mu,k} \approx 0$, to the lowest order, and neglecting anisotropies of the resulting nematic state.

As demonstrated by foundational papers on duality[40,41], such Abelian gauge theory is dual to a quantum XY model of the nematic given by,

$$
\mathcal{L}_{\text{nm}} = \frac{1}{2}\left(\partial_t\theta\right)^2 - \frac{1}{2}K\left(\boldsymbol{\nabla}\theta\right)^2,
$$
(66)

where θ is the layer orientation, confirming a consistency with our physical expectations.

2. Crystal-to-smectic and smectic-to-nematic transitions

We next consider the case of $\beta < \beta'$, when it is favorable to have only one flavor of dipole fields condensed, with $\psi_x = 0$ and $|\psi_y| = \sqrt{\frac{|\alpha|}{\beta}}$, or, $\psi_y = 0$ and $|\psi_x| = \sqrt{\frac{|\alpha|}{\beta}}$, corresponding to the crystal-to-smectic phase transition, which restores translational symmetry in only one direction. With our interest in the quantum smectic, in the following we will thus focus on the $\beta < \beta'$ case, and dualize the quantum smectic alternatively, via Anderson-Higgs mechanism, gapping out one of the flavors of the gauge fields in this smectic phase, formulated in a "soft-spin" description of a 2D quantum crystal.

The crystal-smectic partial melting transition corresponds to condensation of one flavor of the dipole fields, that according to Fig. 2, we take to be $\hat{\mathbf{y}}$-dipoles, i.e. $\psi_y \neq 0$. Within this Higgs phase, the corresponding y flavor gauge field $A_{\mu,y}$ is gapped out, and can be safely integrated out in the low-energy regime with wavelengths of excitations much greater than $\sqrt{\frac{2m_p}{\rho_y p^2}}$ (i.e., the penetration length of $A_{\mu,y}$). To lowest order, it corresponds to $A_{\mu,y} \approx 0$, reducing the crystal's Maxwell Lagrangian to that of a smectic,

$$\mathcal{L}_M^{cr}\left(A_{\mu,x}, A_{\mu,y}=0, a_\mu\right) \approx \mathcal{L}_M^{sm}\left(A_{\mu,x}, a_\mu\right). \tag{67}$$

The detailed derivations are given in Appendix B. Subsequent melting with a condensation of $\hat{\mathbf{x}}$-dipoles ($\hat{\mathbf{y}}$-dislocations) leads to a single vector gauge theory for a_μ, with both $A_{\mu,k}$ gapped out, in the low-energy regime with wavelengths of excitations much greater than $\sqrt{\frac{2m_p}{\rho_x p^2}}$, i.e.,

$$\mathcal{L}_M^{sm}\left(A_{\mu,x}=0, a_\mu\right) \approx \mathcal{L}_M^{nm}\left(a_\mu\right)$$
$$= \frac{1}{2}K^{-1}\left(\partial_t \mathbf{a} - \boldsymbol{\nabla} a_0\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla} \times \mathbf{a}\right)^2, \tag{68}$$

to the lowest order. As expected, this corresponds to the dual of the quantum XY model of the nematic state.

C. Vector to tensor gauge theory redux in low-energy limit

The Maxwell part of a crystal, given by Eq.(54), is gauge invariant under the following transformation,

$$\left(A_{0k}, \mathbf{A}_k\right) \to \left(A_{0k} + \partial_t \chi_k, \mathbf{A}_k + \boldsymbol{\nabla}\chi_k\right),$$
$$\left(a_0, a_k\right) \to \left(a_0 + \partial_t \phi, a_k + \partial_k \phi + \chi_k\right). \tag{69}$$

As demonstrated explicitly in Ref. 48, the enlarged gauge redundancy allows us to completely eliminate $\mathbf{a}$ by choosing $\chi_k = a_k$, as a result of which, the term $\frac{1}{2}\left(\boldsymbol{\nabla} \times \mathbf{a} - \hat{\mathbf{z}} \times \mathbf{A}_k\right)^2$ reduces to $\frac{1}{2}\left(\epsilon_{ik}A_{ik}\right)^2$, thereby gapping out the antisymmetric component $\epsilon_{ik}A_{ik}$ at energies well below this gap, i.e., with length scales greater than $\sqrt{\chi}$. Furthermore, the electric field term $\frac{1}{2}K^{-1}(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0 - A_{0k}\hat{\mathbf{e}}_k)^2$ reduces to $\frac{1}{2}K^{-1}(\boldsymbol{\nabla}a_0 + A_{0k}\hat{\mathbf{e}}_k)^2$ under this transformation, enforcing $A_{0k} = -\partial_k a_0$ at low enough energies with length scales greater than $\sqrt{\frac{K}{\chi}}$. Therefore, the dual coupled U(1) vector gauge theory for a quantum crystal, reduces to the dual tensor gauge theory in the low-energy limit, with, $\mathcal{L}_M^{cr}$, reduces to that in the tensor gauge theory, described by,

$$\mathcal{L}_M^{cr} = \frac{1}{2}\chi^{-1}\left(\partial_t A_{ik} + \partial_i\partial_k a_0\right)^2 - \frac{1}{2}\left(\epsilon_{ji}\partial_j A_{ik}\right)^2$$
$$= E_{ik}\left(\partial_t A_{ik} + \partial_i\partial_k a_0\right) - \frac{1}{2}\chi E_{ik}^2 - \frac{1}{2}B_k B^k, \tag{70}$$

where A_{ik}, is a rank-2 symmetric tensor field, which corresponds to the ith component of k-flavor vector gauge

field $\mathbf{A}_k$ in the coupled vector gauge theory, a_0 is a scalar field with $\partial_k a_0$ corresponding to A_{0k} in the vector gauge theory, $B_k = \epsilon_{ji}\partial_j A_{ik}$, and E_{ik} is the electric tensor field canonically conjugate to A_{ik}.

In contrast, within the smectic Higgs phase, corresponding to the condensation of $\hat{\mathbf{y}}$-dipoles ($\hat{\mathbf{x}}$-dislocations), we cannot eliminate $\mathbf{a}$ completely, since we have already made a gauge choice with $\chi_y = \frac{1}{p}\varphi_y$,

$$A_{iy} \to A_{iy} + \frac{1}{p}\partial_i\varphi_y, \ A_{0y} \to A_{0y} + \frac{1}{p}\partial_t\varphi_y, \ a_y \to a_y + \frac{1}{p}\varphi_y, \tag{71}$$

to absorb the phase φ_y of the condensed dipole field $\psi_y = \sqrt{\rho_y}e^{i\varphi_y}$ into the gauge fields $A_{\mu,y}$. Integrating out the gapped gauge field components, A_{iy} and $\partial_y a_0$, reduces $\mathcal{L}_M^{cr}$, in a condensate of $\hat{\mathbf{y}}$-dipoles, to that of a smectic,

$$\mathcal{L}_M^{cr} = \frac{1}{2}\chi^{-1}\left(\partial_t A_{ix} + \partial_i\partial_x a_0\right)^2 - \frac{1}{2}\left(\epsilon_{ji}\partial_j A_{ix}\right)^2$$
$$+ \frac{1}{2}K^{-1}\left(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0 - A_{0x}\hat{\mathbf{x}}\right)^2 + \frac{1}{2}\left(\epsilon_{ij}\partial_i a_j + A_{yx}\right)^2$$
$$= -\mathbf{E}_x \cdot \left(\partial_t\mathbf{A}_x - \boldsymbol{\nabla}A_{0x}\right) - \frac{1}{2}\chi\mathbf{E}_x^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{A}_x\right)^2$$
$$- \mathbf{e}\cdot\left(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0 - A_{0x}\hat{\mathbf{x}}\right) - \frac{1}{2}K\mathbf{e}^2$$
$$- \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{a} + \hat{\mathbf{x}}\times\mathbf{A}_x\right)^2, \tag{72}$$

which matches exactly with Eq.(40) after setting $A_{0x} = -\partial_x a_0$ and dropping the 'x' index.

We may also explore a 2D quantum smectic at low energies by similar analysis, as what has been done for a crystal in Eq.(70). In the smectic case, choosing $\chi = a_x$ in the gauge transformation (46) allows us to eliminate a_x completely, as a result of which, the term $\frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{a} + \hat{\mathbf{x}}\times\mathbf{A}\right)^2$ of $\mathcal{L}_M^{sm}$ given by (40), reduces to, $\frac{1}{2}\left(\partial_x a_y + A_y\right)^2$, thereby enforcing $A_y = -\partial_x a_y$ at sufficiently low energies, with length scales greater than $\sqrt{\chi}$. Furthermore, the term $\frac{1}{2}K^{-1}\left(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0 - A_0\hat{\mathbf{x}}\right)^2$, reduces to $\frac{1}{2}K^{-1}\left[\left(\partial_t a_y - \partial_y a_0\right)^2 + \left(\partial_x a_0 + A_0\right)^2\right]$ under this transformation, enforcing $A_0 = -\partial_x a_0$ in low energy regime with length scales greater than $\sqrt{\frac{K}{\chi}}$. Therefore, in the low-energy limit, $\mathcal{L}_M^{sm}$ reduces to,

$$\mathcal{L}_M^{sm} = \frac{1}{2}\chi^{-1}\left(\partial_t A_i + \partial_i\partial_x a_0\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{A}\right)^2$$
$$+ \frac{1}{2}K^{-1}\left(\partial_t a_y - \partial_y a_0\right)^2$$
$$= -E_{ix}\left(\partial_t A_i + \partial_x\partial_x a_0\right) - \frac{1}{2}\chi E_{ix}^2 - \frac{1}{2}B_x^2$$
$$+ \frac{1}{2}K^{-1}\left(\partial_t a_y - \partial_y a_0\right)^2, \tag{73}$$

where E_{ix} is a tensor gauge field, corresponding to the ith component of the vector field $\mathbf{E}_x$. This vector- to tensor-gauge theory reduction at low energies is illustrated in Fig.9.
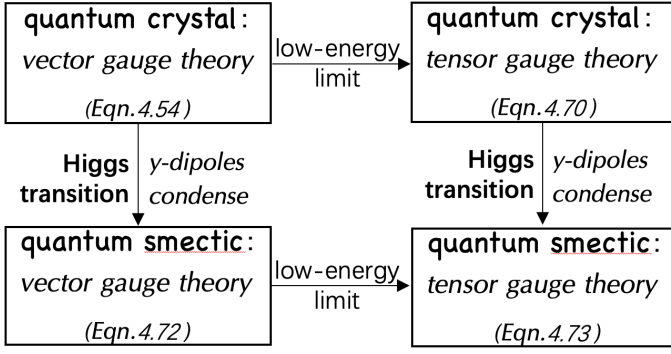
FIG. 9: Vector gauge theory of a 2D quantum crystal and smectic, in the low-energy limit reduce to their corresponding tensor gauge theory forms.

IV. CLASSICAL LIMIT OF SMECTIC-GAUGE THEORY DUALITY

A. 2D classical smectic duality

As a consistency check on our quantum dual theory for a 2D quantum smectic, we anticipate that the classical smectic theory must emerge as the classical limit of the above duality, as we demonstrate explicitly below.

The elasticity of a 2D classical smectic is given by the Hamiltonian density,

$$
\begin{aligned}
\mathcal{H}_{\rm sm} &= \frac{1}{2}\chi\left(\boldsymbol{\nabla}u - \theta\hat{\mathbf{x}}\right)^2 + \frac{1}{2}K\left(\boldsymbol{\nabla}\theta\right)^2 \\
&= \frac{1}{2}\chi^{-1}\sigma^2 - i\sigma\cdot\left(\boldsymbol{\nabla}u - \theta\hat{\mathbf{x}}\right) + \frac{1}{2}K^{-1}\mathbf{j}^2 - i\mathbf{j}\cdot\boldsymbol{\nabla}\theta.
\end{aligned}
\tag{74}
$$

where we have introduced two Hubbard-Stratonovich fields σ and $\mathbf{j}$ to decouple the two elastic terms.

We derive its classical dual by integrating out the smooth part of u and θ, which leads to the constraints:

$$
\begin{aligned}
\partial_i\sigma_i &= 0, \\
\partial_i j_i + \sigma_x &= 0.
\end{aligned}
\tag{75}
$$

The first equation can be solved in terms of a scalar potential ϕ, $\sigma_i = \epsilon_{ij}\partial_j\phi$, which inside the second constraint gives,

$$
\partial_i(j_i + \epsilon_{xi}\phi) = 0,
\tag{76}
$$

that is then solved by introducing another potential α, $j_i = \epsilon_{ij}\partial_j\alpha - \epsilon_{xi}\phi$.

Substituting σ_i and j_i back into the original Hamilto-

nian, and integrating by parts, lead to

$$
\begin{aligned}
\tilde{\mathcal{H}}_{\rm sm} &= \frac{1}{2}\chi^{-1}\left(\boldsymbol{\nabla}\phi\right)^2 - i\epsilon_{ij}\partial_j\phi\left(\partial_i u - \delta_{ix}\theta\right) \\
&\quad + \frac{1}{2}K^{-1}\left(\epsilon_{ij}\partial_j\alpha - \epsilon_{xi}\phi\right)^2 - i\left(\epsilon_{ij}\partial_j\alpha - \epsilon_{xi}\phi\right)\partial_i\theta \\
&= \frac{1}{2}\chi^{-1}\left(\boldsymbol{\nabla}\phi\right)^2 + i\phi\epsilon_{ij}\partial_j\left(\partial_i u - \delta_{ix}\theta\right) \\
&\quad + \frac{1}{2}K^{-1}\left(\boldsymbol{\nabla}\alpha + \phi\hat{\mathbf{x}}\right)^2 + i\theta\epsilon_{ij}\partial_i\partial_j\alpha + i\phi\epsilon_{xi}\partial_i\theta \\
&= \frac{1}{2}\chi^{-1}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}K^{-1}\left(\boldsymbol{\nabla}\alpha + \phi\hat{\mathbf{x}}\right)^2 - i\phi b - i\alpha s,
\end{aligned}
\tag{77}
$$

where we have defined $b = \epsilon_{ij}\partial_i\partial_j u$ and $s = \epsilon_{ij}\partial_i\partial_j\theta$, as the dislocation and disclination densities respectively.

Low-energy regime, $\frac{1}{2}K^{-1}\left(\boldsymbol{\nabla}\alpha + \phi\hat{\mathbf{x}}\right)^2 \approx k_B T$, i.e., at length scales greater than $\sqrt{Kk_B T}$, integrating over ϕ, to lowest-order, sets $\phi = \partial_x\alpha$, and therefore, gives

$$
\tilde{\mathcal{H}}_{\rm sm} = \frac{1}{2}\chi^{-1}\left(\partial_x^2\alpha\right)^2 + \frac{1}{2}K^{-1}\left(\partial_y\alpha\right)^2 - i\partial_x\alpha b - i\alpha s, \tag{78}
$$

which, as expected, turns out to be the electrostatic limit of the quantum smectic duality, i.e., Eq.(39) and Eq. (40), with $\phi = A_0$ and $\alpha = a_0$.

Focusing on dislocations and neglecting the high energy disclination defects, we can straightforwardly integrate out $\alpha(\mathbf{r})$ in the partition function, obtaining a dislocation Coulomb gas Hamiltonian

$$
H_{\rm b} = \frac{1}{2}\int\frac{d^2 q}{(2\pi)^2}b(\mathbf{q})\tilde{K}(\mathbf{q})b(-\mathbf{q}),
\tag{79}
$$

with,

$$
\tilde{K}(\mathbf{q}) = \frac{d^2}{a^2}\frac{Kq_x^2}{q_y^2 + \lambda^2 q_x^4} + 2E_b,
\tag{80}
$$

where, $E_{\rm b}$ is the defect core energy, d is the layer spacing, and a is the lattice spacing between atoms within the layers, and "penetration" length λ is defined as, $\lambda^2 \equiv K/\chi$. Thus, the dislocations Coulomb gas Hamiltonian in real space reduces to,

$$
H_{\rm b} = \frac{1}{2}\int_{\mathbf{r}_1,\mathbf{r}_2}U(\mathbf{r}_1 - \mathbf{r}_2)b(\mathbf{r}_1)b(\mathbf{r}_2) + E_{\rm b}\int_{\mathbf{r}}b^2(\mathbf{r}),
\tag{81}
$$

where,

$$
U(\mathbf{r}) = \frac{1}{4}\frac{d^2}{a^2}\chi\left(\frac{\lambda}{\pi|y|}\right)^{1/2}e^{-x^2/4\lambda|y|},
\tag{82}
$$

as first found by Toner and Nelson in Ref. 21. The complete Hamiltonian contains also a smooth phonon part, H_0, depending only the smooth, single-valued part, $\tilde{u}$, of the displacement, $u = \tilde{u} + u^s$, i.e.,

$$
H = H_0 + H_b.
\tag{83}
$$

with H_0 given by

$$H_0 = \frac{1}{2} \int d^2r \left[\chi \left(\partial_y \tilde{u} \right)^2 + K \left(\partial_x^2 \tilde{u} \right)^2 \right] \qquad (84)$$

With this Hamiltonian, we can study the effects of phonons and dislocations at finite temperatures on translational and orientational orders.

Effect of phonon fluctuations on the translation order is expressed in terms of correlations in the order parameter, $\psi(\mathbf{r}) = |\psi| e^{-i q_0 u(\mathbf{r})}$ as,

$$\langle \psi(\mathbf{r}) \psi^*(\mathbf{0}) \rangle \sim \langle e^{-i q_0 [u(\mathbf{r}) - u(\mathbf{0})]} \rangle$$
$$= e^{-\frac{1}{2} q_0^2 \langle [u(\mathbf{r}) - u(\mathbf{0})]^2 \rangle}$$
$$\sim \begin{cases} \exp\left(-\frac{q_0^2 k_B T}{\chi} \sqrt{\frac{|y|}{4\pi\lambda}} \right), & \text{for } x^2 \ll \lambda y, \\ \exp\left(-\frac{q_0^2 k_B T}{4\chi\lambda} |x| \right), & \text{for } x^2 \gg \lambda y, \end{cases}$$
$$(85)$$

where we have used the fact that

$$\langle [u(\mathbf{r}) - u(\mathbf{0})]^2 \rangle = \int_{\mathbf{q}} \int_{\mathbf{q}'} \left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1 \right) \left(e^{i\mathbf{q}'\cdot\mathbf{r}} - 1 \right) \langle u(\mathbf{q}) u(\mathbf{q}') \rangle$$
$$= \int_{\mathbf{q}} \left(2 - 2\cos(\mathbf{q}\cdot\mathbf{r}) \right) \frac{k_B T}{\chi \left(q_y^2 + \lambda^2 q_x^4 \right)}$$
$$\sim \begin{cases} \frac{k_B T}{\chi} \sqrt{\frac{|y|}{\pi\lambda}}, & \text{for } x^2 \ll \lambda y, \\ \frac{k_B T}{2\chi\lambda} |x|, & \text{for } x^2 \gg \lambda y. \end{cases}$$
$$(86)$$

Smectic layers orientational order is expressed in terms of correlations in the nematic-like order parameter, $\mathbf{N}(\mathbf{r}) = (\cos\theta(\mathbf{r}), \sin\theta(\mathbf{r}))$, as,

$$\lim_{r\to\infty} \langle \mathbf{N}(\mathbf{r}) \cdot \mathbf{N}(\mathbf{0}) \rangle = \lim_{r\to\infty} \langle \cos[\theta(\mathbf{r}) - \theta(\mathbf{0})] \rangle$$
$$= \lim_{r\to\infty} e^{-\frac{1}{2}\langle [\theta(\mathbf{r}) - \theta(\mathbf{0})]^2 \rangle} \qquad (87)$$
$$= e^{-\frac{K_B T \Lambda}{\chi\lambda a}} = \text{constant},$$

where we have used

$$\langle [\theta(\mathbf{r}) - \theta(\mathbf{0})]^2 \rangle = \langle [\partial_x u(\mathbf{r}) - \partial_x u(\mathbf{0})]^2 \rangle$$
$$= -\int_{\mathbf{q}} \int_{\mathbf{q}'} \left(e^{i\mathbf{q}\cdot\mathbf{r}} - 1 \right) \left(e^{i\mathbf{q}'\cdot\mathbf{r}} - 1 \right) q_x q_x' \langle u(\mathbf{q}) u(\mathbf{q}') \rangle$$
$$= \int_{\mathbf{q}} \left(2 - 2\cos(\mathbf{q}\cdot\mathbf{r}) \right) \frac{q_x^2 k_B T}{\chi q_y^2 + K q_x^4}$$
$$\sim \frac{2 k_B T}{\chi\lambda a}, \quad \text{for } r \to \infty.$$
$$(88)$$

with $\Lambda = \frac{2\pi}{a}$, a convenient cutoff. Therefore, in a 2D classical smectic at nonzero temperature, the translational order is destroyed by thermal phonon fluctuations[19–21], while the orientational order persists even in the presence of thermally excited phonons, destroyed only at higher temperatures by proliferation of dislocations.

In presence of unbound dislocations, appearing at density, $n_d \sim e^{-E_c/(k_B T)}$, in thermal equilibrium, the effective elasticity in Debye-Huckel approximation, reduces to that of a nematic at scales greater than, $\xi_D \sim e^{-E_c/(2k_B T)}$,

$$F_N = \frac{1}{2} K(T) \int d^2r \left(\boldsymbol{\nabla}\theta \right)^2, \qquad (89)$$

and the correlations in orientational order become decay algebraically,

$$\lim_{r\to\infty} \langle \mathbf{N}(\mathbf{r}) \cdot \mathbf{N}(\mathbf{0}) \rangle \sim r^{-\eta(T)}, \qquad (90)$$

with, $\eta(T) = 2k_B T/[\pi K(T)]$. Therefore, a 2D smectic is unstable to thermal fluctuations, driven into a nematic fluid at any nonzero temperatures. At high temperature, the nematic to isotropic liquid transition, driven by unbinding of disclinations, is described by Kosterlitz and Thouless, with[18,49],

$$\eta\left(T_c^- \right) = \frac{1}{4}, \qquad (91)$$

at the critical temperature T_c.

Motivated by our formulation of two-dimensional melting of a classical crystal[17,18,58–60], via a dual theory in terms of a higher derivative vector sine-Gordon model[61], we expect to find the analogous description for the 2D smectic. To this end, we express the dislocation and disclination densities in terms of a sum of their discrete charges as,

$$b(\mathbf{r}) = \sum_{\mathbf{r}_n} b_{\mathbf{r}_n} \delta^2(\mathbf{r} - \mathbf{r}_n),$$
$$s(\mathbf{r}) = \sum_{\mathbf{r}_n} s_{\mathbf{r}_n} \delta^2(\mathbf{r} - \mathbf{r}_n). \qquad (92)$$

In terms of these discrete topological defect charges, the Hamiltonian is given by

$$H_{sm} = \frac{1}{2} \int_{\mathbf{r}} \left[\chi^{-1} (\partial_x^2 \alpha)^2 + K^{-1} (\partial_y \alpha)^2 \right]$$
$$+ \sum_{\mathbf{r}_n} \left[E_b b_{\mathbf{r}_n}^2 + E_s s_{\mathbf{r}_n} \right] - \sum_{\mathbf{r}_n} \left[i\partial_x \alpha b_{\mathbf{r}_n} + i2\pi\alpha s_{\mathbf{r}_n} \right].$$
$$(93)$$

Following a standard analysis, summing over the charges, we obtain the dual sine-Gordon Hamiltonian,

$$\tilde{H}_{sm} = \int_{\mathbf{r}} \left[\frac{1}{2} \chi^{-1} (\partial_x^2 \alpha)^2 + \frac{1}{2} K^{-1} (\partial_y \alpha)^2 - g_b \cos(b\partial_x \alpha) \right.$$
$$\left. - g_s \cos(2\pi\alpha) \right].$$
$$(94)$$

where $g_b = \frac{2}{a^2} e^{-a^2 E_b}$ and $g_s = \frac{2}{a^2} e^{-E_s}$, which provides a transparent description of the continuous two-stage melting in terms of the renormalization-group relevance of two

cosine operators that control the sequential unbinding of dislocations and disclinations, respectively corresponding to the smectic-to-nematic and nematic-to-isotropic fluid transitions. The resulting phase diagram is illustrated in Fig.10.
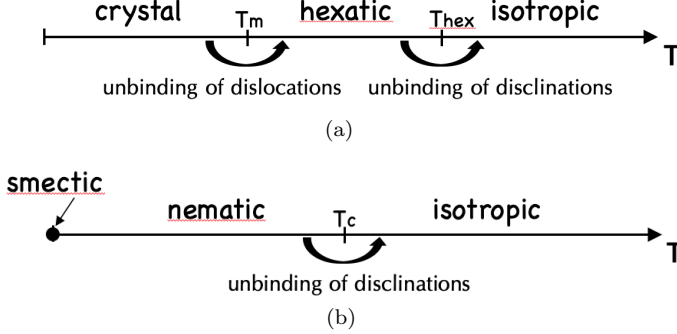

(a)

(b)

FIG. 10: Different phases involved in the melting transitions of a triangular crystal and a smectic in two dimensions. (a) A triangular crystal at low temperatures, which first melts at T_m via an unbinding of dislocations into a hexatic, and the sixfold orientational order in the hexatic phase is destroyed at a higher temperature T_{hex} by unbinding of disclinations. (b) A zero-temperature smectic, that is unstable and driven into a nematic at any nonzero temperatures. The nematic phase transforms into an isotropic fluid via a further disclination unbinding transition at T_c. [Figures adapted from Reference 21.]

Because of the second-order Laplacian elasticity, standard analysis around the Gaussian fixed line $g_b = g_s = 0$ shows that, the mean-squared fluctuations of $\alpha(\mathbf{r})$ is given by

$$\langle [\alpha(\mathbf{r}) - \alpha(\mathbf{0})]^2 \rangle \sim \begin{cases} \frac{k_B T}{\chi^{-1}} \sqrt{\frac{\lambda|y|}{\pi}}, & \text{for } x^2 \ll \lambda^{-1}y, \\ \frac{k_B T}{2\chi^{-1}} \lambda|x|, & \text{for } x^2 \gg \lambda^{-1}y, \end{cases} \quad (95)$$

which leads to an exponentially (as opposed to power-law in a conventional sine-Gordon model) vanishing Debye-Waller factor, $\langle e^{i2\pi\alpha(\mathbf{r})} e^{-i2\pi\alpha(\mathbf{0})} \rangle = e^{-2\pi^2 \langle [\alpha(\mathbf{r}) - \alpha(\mathbf{0})]^2 \rangle}$, and in turn to a strongly irrelevant disclination cosine, g_s, that can therefore be neglected. In contrast, mean-squared fluctuations of $\partial_x \alpha(\mathbf{r})$ is,

$$\langle [\partial_x \alpha(\mathbf{r}) - \partial_x \alpha(\mathbf{0})]^2 \rangle \sim \text{constant}, \quad (96)$$

for large r, and orientational correlation therefore given by,

$$\langle e^{ib \cdot \partial_x \alpha(\mathbf{r})} e^{-ib \cdot \partial_x \alpha(\mathbf{0})} \rangle = e^{-\frac{1}{2}b^2 \langle [\partial_x \alpha(\mathbf{r}) - \partial_x \alpha(\mathbf{0})]^2 \rangle}$$
$$\sim \text{constant}. \quad (97)$$

This therefore leads to the conclusion that the dislocation cosine, g_b, is always relevant. At sufficiently long scales, dislocation cosine in Eq.(94) reduces to a harmonic potential for $\partial_x \alpha$, $-g_b \cos(b \partial_x \alpha) \simeq \frac{1}{2} g_b b^2 (\partial_x \alpha)^2$.

The effective Hamiltonian is then given by

$$\tilde{H}_{nm} \simeq \int_{\mathbf{r}} \left[\frac{1}{2} K^{-1} (\partial_y \alpha)^2 + \frac{1}{2} g_b b^2 (\partial_x \alpha)^2 - g_s \cos(2\pi\alpha) \right], \quad (98)$$

where we have neglected the χ^{-1} "curvature" elasticity relative to the gradient one encoded in g_b, and restored the disclination cosine operator $g_s \cos(2\pi\alpha)$. The resulting conventional sine-Gordon model in α can then exhibit the second KT-like "roughening" transition, capturing the nematic-to-isotropic fluid transition, associated with the unbinding of disclinations, with well-known standard KT phenomenology.

B. 3D classical smectic duality

Motivated by the correspondence of a (2+1)D quantum smectic and a 3D classical smectic, and the extensively studied 3D nematic to smectic-A transition[50–56], we formulate a dual gauge theory of a 3D classical smectic, akin to a mapping of a 3D classical XY model onto a classical charged superconductor.[40,41]

The elasticity of a 3D classical smectic with its layers along xy plane, is captured by the Hamiltonian density,

$$\mathcal{H}_{sm}^{3d} = \frac{1}{2} \chi (\nabla u + \delta\mathbf{n})^2 + \frac{1}{2} K (\nabla \delta\mathbf{n})^2$$
$$= \frac{1}{2\chi} \sigma^2 - i\sigma \cdot (\nabla u + \delta\mathbf{n}) - \frac{1}{2K} \mathbf{j} \frac{1}{\nabla^2} \mathbf{j} - i\mathbf{j} \cdot \delta\mathbf{n}, \quad (99)$$

where $\delta\mathbf{n} = \mathbf{n} - \hat{\mathbf{z}} = (\delta n_x, \delta n_y, 0)$ represents fluctuations in layer orientation, and we introduced two Hubbard-Stratonovich fields σ and $\mathbf{j}$ to decouple the two elastic terms. The Hamiltonian density in Eq. (99) is equivalent to the standard smectic form,

$$\mathcal{H}_{sm}^{3d} = \frac{1}{2} \chi (\nabla_z u)^2 + \frac{1}{2} K (\nabla_\perp^2 u)^2, \quad (100)$$

in the low-energy limit, where the orientational degree of freedom, $\delta\mathbf{n}$, locks to the layer normals with $\delta\mathbf{n} = -\nabla_\perp u$.

Integrating out the smooth part of u and $\delta\mathbf{n}$, leads to the constraints:

$$\nabla \cdot \sigma = 0,$$
$$\mathbf{j} + \sigma^\perp = 0. \quad (101)$$

The first equation can be solved in terms of a vector potential $\mathbf{A}$, $\sigma_i = \epsilon_{ijk} \partial_j A_k$, $(\sigma = \nabla \times \mathbf{A})$, which inside the second constraint gives,

$$j_i + \epsilon_{ijk}^\perp \partial_j A_k = 0, \quad (102)$$

such that j_i is solved as $j_i = -\epsilon_{ijk}^\perp \partial_j A_k$.

Substituting σ_i and j_{ij} back into the original Hamiltonian, and integrating by part, lead to

$$
\begin{aligned}
\mathcal{H}_{\mathrm{sm}}^{\mathrm{3d}} =& \frac{1}{2\chi} \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^2 - i\epsilon_{ijk}\partial_j A_k \left(\partial_i u + \delta n_i\right) \\
& -\frac{1}{2K}\epsilon_{ijk}^{\perp}\partial_j A_k \frac{1}{\boldsymbol{\nabla}^2}\epsilon_{imn}^{\perp}\partial_m A_n + i\epsilon_{ijk}^{\perp}\partial_j A_k \delta n_i \\
=& \frac{1}{2\chi} \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^2 + iA_k\epsilon_{ijk}\partial_j \left(\partial_i u + \delta n_i\right) \\
& -\frac{1}{2K}\epsilon_{ijk}^{\perp}\partial_j A_k \frac{1}{\boldsymbol{\nabla}^2}\epsilon_{imn}^{\perp}\partial_m A_n - iA_k\epsilon_{ijk}^{\perp}\partial_j \delta n_i \\
=& \frac{1}{2\chi} \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^2 + iA_k\epsilon_{ijk}\partial_j \partial_i u \\
& -\frac{1}{2K}\epsilon_{ijk}^{\perp}\partial_j A_k \frac{1}{\boldsymbol{\nabla}^2}\epsilon_{imn}^{\perp}\partial_m A_n + iA_k\left(\epsilon_{ijk} - \epsilon_{ijk}^{\perp}\right)\partial_j \delta n_i \\
=& \frac{1}{2\chi} \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^2 - \frac{1}{2K}\epsilon_{ijk}^{\perp}\partial_j A_k \frac{1}{\boldsymbol{\nabla}^2}\epsilon_{mnk}^{\perp}\partial_m A_n - iA_k b_k,
\end{aligned}
\tag{103}
$$

where dislocation density (see Fig.11) is given by $b_k = \epsilon_{ijk}\partial_i\partial_j u$.
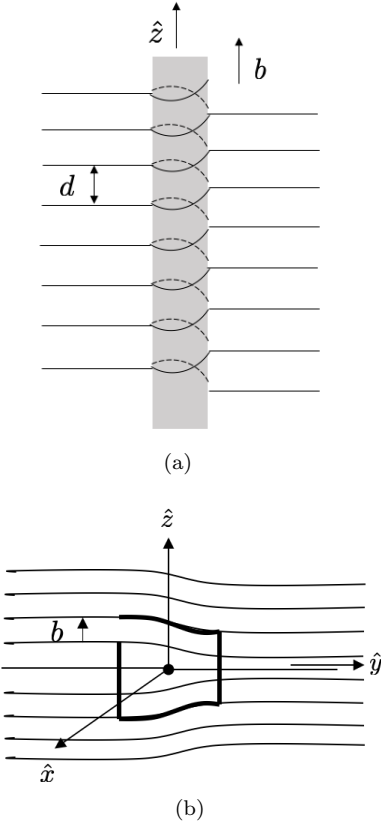

(a)

(b)

FIG. 11: Different types of dislocations in a 3D smectic. (a) A screw dislocation with its line core tangent along $\hat{\mathbf{z}}$. (b) An edge dislocation with its line core tangent along $\hat{\mathbf{x}}$ axis, i.e., perpendicular to $\hat{\mathbf{z}}$.

In momentum space, we have

$$
\begin{aligned}
\mathcal{H}_{\mathrm{sm}}^{\mathrm{3d}} =& \frac{1}{2\chi} |\mathbf{q} \times \mathbf{A}(\mathbf{q})|^2 + \frac{1}{2K}\frac{|\left(\mathbf{q} \times \mathbf{A}(\mathbf{q})\right)_{\perp}|^2}{q^2} \\
& - iA_k(\mathbf{q})b_k(-\mathbf{q}) \\
=& \frac{1}{2\chi} |\mathbf{q}_{\perp} \times \mathbf{A}_{\perp}(\mathbf{q})|^2 - i\mathbf{A}_{\perp}(\mathbf{q})\mathbf{b}_{\perp}(-\mathbf{q}) \\
& +\frac{1}{2}\left(\frac{1}{\chi} + \frac{1}{Kq^2}\right)|\left(q_z\mathbf{A}_{\perp} - A_z\mathbf{q}_{\perp}\right)|^2 - iA_z(\mathbf{q})b_z(-\mathbf{q}),
\end{aligned}
\tag{104}
$$

Functionally integrating out A_z, leads to,

$$
\begin{aligned}
\mathcal{H}_{\mathrm{sm}}^{\mathrm{3d}} =& \frac{1}{2\chi}\left[|\mathbf{q}_{\perp} \times \mathbf{A}_{\perp}(\mathbf{q})|^2 + \left(1 + \frac{1}{\lambda^2 q^2}\right)q_z^2|\mathbf{A}_{\perp}|^2\right] \\
& - i\mathbf{A}_{\perp}(\mathbf{q})\mathbf{b}_{\perp}(-\mathbf{q}) + \frac{1}{2}\frac{\chi}{q_{\perp}^2\left[1 + 1/(\lambda^2 q^2)\right]}|b_z(\mathbf{q})|^2 \\
=& \frac{1}{2\chi}A_i^{\perp}\left[q_{\perp}^2 P_{ij}^{\mathrm{T}} + \left(q_z^2 + \frac{q_z^2}{\lambda^2 q^2}\right)\left(P_{ij}^{\mathrm{L}} + P_{ij}^{\mathrm{T}}\right)\right]A_j^{\perp} \\
& - i\mathbf{A}_{\perp}(\mathbf{q})\mathbf{b}_{\perp}(-\mathbf{q}) + \frac{1}{2}\Gamma(\mathbf{q})|b_z(\mathbf{q})|^2 \\
=& \frac{1}{2\chi}A_i^{\perp}(\mathbf{q})\left(q^2 + \frac{q_z^2}{\lambda^2 q^2}\right)P_{ij}^{\mathrm{T}}A_j^{\perp}(-\mathbf{q}) \\
& - i\mathbf{A}_{\perp}(\mathbf{q}) \cdot \mathbf{b}_{\perp}(-\mathbf{q}) + \frac{1}{2}\Gamma(\mathbf{q})|b_z(\mathbf{q})|^2,
\end{aligned}
\tag{105}
$$

where we have chosen the Coulomb (transverse) gauge $\boldsymbol{\nabla}_{\perp} \cdot \mathbf{A}_{\perp} = 0$, i.e., $\mathbf{q}_{\perp} \cdot \mathbf{A}_{\perp} = 0$, $P_{ij}^{\mathrm{T}} = \delta_{ij} - q_i q_j/q^2$, and, $P_{ij}^{\mathrm{L}} = q_i q_j/q^2$, are transverse and longitudinal projection operators respectively, $\Gamma(\mathbf{q}) \equiv \chi q_{\perp}^{-2}\left[1 + 1/(\lambda^2 q^2)\right]^{-1}$ is the interaction potential for screw dislocation b_z's, and we have used the fact that, $P_{ij}^{\mathrm{L}}A_j^{\perp} = 0$, in the Coulomb gauge. If we integrate out $\mathbf{A}_{\perp}$ further, we get the dislocation Coulomb gas model, given by,

$$
\begin{aligned}
H_b^{\mathrm{3d}} =& \frac{1}{2}\int_{\mathbf{q}}\left(\frac{Kq^2}{q_z^2 + \lambda^2 q^4}P_{ij}^{\perp} + 2E_b^i\delta_{ij}\right)b_i(\mathbf{q})b_j(-\mathbf{q}) \\
& + \frac{1}{2}\int_{\mathbf{q}}\Gamma(\mathbf{q})|b_z(\mathbf{q})|^2 \\
=& \frac{1}{2}\int_{\mathbf{q}}\left(\frac{K\left[\hat{\mathbf{z}} \cdot \mathbf{q} \times \mathbf{b}(\mathbf{q})\right]^2}{q_z^2 + \lambda^2 q_{\perp}^4} + 2E_b^i\mathbf{b}(\mathbf{q})\mathbf{b}(-\mathbf{q})\right),
\end{aligned}
\tag{106}
$$

where, $P_{ij}^{\perp} = \left(\delta_{ij} - q_i^{\perp}q_j^{\perp}/q_{\perp}^2\right)\left(1 - \delta_{iz}\delta_{jz}\right)$, is the transverse projection operator for edge dislocations, i.e. screw dislocations b_z−components projected away, and E_b^i's are core energies of dislocations[56]. We note that $\Gamma(\mathbf{q})$ is approximately a constant at small q, where q is smaller than $1/\lambda$, and therefore contributes to the core energy of a screw dislocation, i.e., $E_b^z \to E_b^z + \Gamma(\mathbf{q})$. Note that for simplicity, we have assumed that the lattice spacing between atoms within the layers is equal to the layer

spacing, i.e., $a = d$, such that we have no factor like '$\frac{d^2}{a^2}$' as in Eq. (80) for the 2D case.

Interested in the nature of the nematic to smectic-A transition, Toner[56] mapped a model of a smectic onto a Coulomb gas of dislocation loops, which he then transformed into an anisotropic superconductor in a vector gauge field $\mathbf{A}$, and analyzed it with a momentum-shell renormalization group. In the long-wavelength limit of $q \ll 1/\lambda$, indeed our model reduces to Toner's, with a generalization that screw dislocations in our model have a finite interaction.

In analogy to what we have done for a 2D smectic, we transform the Coulomb gas Hamiltonian, Eq.(106) into a classical gauge theory. The partition function for the dislocation-loop Coulomb gas on a lattice is given by,

$$
Z = \int \prod_{\mathbf{r}_n} d\mathbf{A}(\mathbf{r}_n) \sum_{\{b_{\mathbf{r}_n}\}} e^{-H_A[\{\mathbf{A}(\mathbf{r}_n)\}] + \sum_{\mathbf{r}_n} ib_{\mathbf{r}_n} d\cdot\mathbf{A}(\mathbf{r}_n)}
$$
$$
\cdot e^{-\sum_{\mathbf{r}_n} E_b^i d^2 |b_{\mathbf{r}_n}^i|^2} \delta(\Delta\cdot\mathbf{b}_{\mathbf{r}_n})\delta(A_z)\delta(\Delta\cdot\mathbf{A}_{\mathbf{r}_n})
$$
$$
= \int \prod_{\mathbf{r}_n} d\mathbf{A}(\mathbf{r}_n) d\varphi(\mathbf{r}_n)\delta(A_z)\delta(\Delta\cdot\mathbf{A}_{\mathbf{r}_n})
$$
$$
\cdot e^{-H_A[\{\mathbf{A}(\mathbf{r}_n)\}] + \sum_{\mathbf{r}_n} ib_{\mathbf{r}_n}\cdot[d\mathbf{A}(\mathbf{r}_n) - \Delta\varphi(\mathbf{r}_n)] - E_b^i d^2 |b_{\mathbf{r}_n}^i|^2},
$$
(107)

with,

$$
H_A[\{\mathbf{A}(\mathbf{r}_n)\}] = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\left(\frac{q_z^2}{Kq_\perp^2} + \frac{q_\perp^2}{\chi}\right)|\mathbf{A}(\mathbf{q})|^2,
$$
(108)

and we have introduced an auxiliary scalar field $\varphi(\mathbf{r})$, such that integrating out φ recovers the constraint, $\Delta\cdot\mathbf{b}_{\mathbf{r}_n} = 0$.

After tracing over the dislocation charges $\mathbf{b}_{\mathbf{r}_n}$, we obtain

$$
\tilde{H}_{\text{sm}}^{3d} = H_A - g_{b_i}^{3d}\sum_{\mathbf{r}_n}\cos\left[\Delta_i\varphi(\mathbf{r}_n) - d\cdot A_i(\mathbf{r}_n)\right]
$$
(109)

where $g_{b_i}^{3d} = 2e^{-E_b^i d^2}$, and we have approximated the resulting Villain potential by its lowest harmonic. In the continuum limit, it becomes

$$
\tilde{H}_{\text{sm}}^{3d} = H_A - \tilde{g}_{b_i}^{3d}\int d^3r \cos\left(\partial_i\varphi - d\cdot A_i\right)
$$
$$
= H_A - \int d^3r\left[\frac{1}{2}|(\boldsymbol{\nabla} - id\cdot\mathbf{A})\psi|^2 + V(|\psi|)\right],
$$
(110)

with, $\tilde{g}_{b_i}^{3d} = \frac{g_{b_i}^{3d}}{a^2 d} = \frac{2}{a^2 d}e^{-E_b^i d^2}$, and in the second form we have written it as an equivalent "soft-spin" description in terms of $\psi = |\psi|e^{i\varphi}$, with the Landau U(1)-invariant potential, $V(|\psi|)$, which is equivalent to the first form below the energy scale of the gapped Higgs-like magnitude degree of freedom, $|\psi|$. Thus, we reproduce Toner's anisotropic superconductor model[56] expected to have the same critical properties as the above dislocation-loop Coulomb gas model.

V. SUMMARY AND CONCLUSION

In this paper, after a brief review of smectic elasticity, we developed a coupled U(1) vector gauge theory for a two-dimensional quantum smectic, where the phonons and orientational Goldstone modes map onto coupled gauge fields, and topological defects correspond gauge charges and dipoles. We discovered that charges (disclinations) exhibit subdimensional lineon dynamics, restricted to move transverse to the layer. Motivated by the partial quantum melting of a crystal into a smectic, and the subsequent smectic-to-nematic transition, we reproduced the dual description of a quantum smectic by condensing the one flavor species of dipoles within the generalized Abelian-Higgs model of a 2D quantum crystal.

We also applied this duality to treat a classical smectic liquid crystal. To this end, we formulated the smectic-to-nematic and nematic-to-isotropic fluid transitions as a higher-derivative sine-Gordon model of a 2D classical smectic. Motivated by the correspondence between a (2+1)D quantum system and a 3D classical system, we also derived a dual theory for a 3D classical smectic, and reproduced smectic's dislocation-loop Coulomb gas description for the nematic-smectic transition, which we then mapped onto an anisotropic Abelian-Higgs model.

We expect this fractonic gauge theory reformulation of smectics will be useful for further detailed explorations, e.g., subjected to an external stress and in presence of a substrate. We leave a study of the true critical behavior (beyond mean-field) of the crystal-smectic and smectic-nematic transitions using the dual gauge theory for future studies. The duality analysis in the presence of elastic nonlinearities also remains a challenging open problem.

Acknowledgments

Appendix A: Crystal-to-supersolid transition

In the Mott-insulating "commensurate" crystal phase associated with the particle-number conservation symmetry, the Lagrangian density of a square-lattice quantum-crystal with lattice spacing d, for the two dipole fields $\psi_{\mathbf{p}} = \psi_x, \psi_y$, corresponding to the two minimal dipole species $\mathbf{p}_x = p\hat{\mathbf{x}}, \mathbf{p}_y = p\hat{\mathbf{y}}$, plus the Maxwell gauge field

part, takes the form

$$\mathcal{L}_{\text{dis}} = \sum_{\mathbf{p}} i\psi_{\mathbf{p}}^{\dagger} D_0 \psi_{\mathbf{p}} - \frac{1}{2m_p} \sum_{\mathbf{p}} |\Pi_{ij}^{\perp \mathbf{p}} D_j \psi_{\mathbf{p}}|^2 - V(\{\psi_k\})$$
$$+ \mathcal{L}_{\text{M}}^{\text{cr}},$$
(A1)

where, m_p is the effective mass of the dipole, $D_0 = \partial_t + ip_k A_{0k}$ and $D_j = \partial_j + ip_k A_{jk}$ (i.e., $\mathbf{D} = \boldsymbol{\nabla} + ip_k \mathbf{A}_k$) are the covariant derivatives[47], $V(\{\psi_k\})$ is the U(1)-invariant Ginzburg-Landau potential, $\mathcal{L}_{\text{M}}^{\text{cr}}$ the Lagrangian density of the Maxwell part, and $\Pi_{ij}^{\perp \mathbf{p}} = \delta_{ij} - \frac{p_i p_j}{p^2}$ is the projection operator since in this Mott-insulating crystal phase, the dipole can only move in the direction perpendicular to $\mathbf{p}$ while the along-dipole climbs are forbidden due to the U(1) particle-number conservation symmetry (" glide-constraint").

However, as discussed in the main text, the crystal also exhibits scalar non-topological point defects, corresponding to deficiency and excess in atom density, which permits the climb process of the dislocations (See Fig. 7). Combined with the bosonic statistics of the underlying particles, the quantum crystal can first develop into a super-solid phase (incommensurate crystal), featuring both the crystalline order and the superfluid order. The condensation of vacancies or interstitials in the super-solid phase, therefore, frees these symmetry-forbidden climb events. Therefore, for a complete description, we also need to add the superfluid part of the underlying bosonic particles, $\mathcal{L}_{\text{sf}}$, and the minimal gauge-invariant coupling between the dislocation climb operators and superfluid order parameter, $\mathcal{L}_{\text{sf-dis}}$, into the full Lagrangian[47],

$$\mathcal{L}_{\text{sf}} = i\Psi_{\text{sf}}^{\dagger} \partial_t \Psi_{\text{sf}} - \frac{1}{2m_s} |\boldsymbol{\nabla}\Psi_{\text{sf}}|^2 - \mu |\Psi_{\text{sf}}|^2 - \frac{U}{2}|\Psi_{\text{sf}}|^4, \text{ (A2)}$$

$$\mathcal{L}_{\text{sf-dis}} = \gamma \Psi_{\text{sf}} \sum_{\mathbf{p}} \mathcal{O}_{\text{climb},\mathbf{p}} + h.c., \quad (A3)$$

where, Ψ_{sf} is the superfluid order parameter, $\mathcal{O}_{\text{climb},\mathbf{p}} = \psi_{\mathbf{p}}^{\dagger}(\mathbf{r} + \mathbf{p}) e^{ip_i A_{ij} p_j} \psi_{\mathbf{p}}(\mathbf{r})$ is the $\mathbf{p}$-dipole climb operator, and γ is the coupling constant[47]. Fig. 8 shows an example of terms in $\mathcal{L}_{\text{sf-dis}}$.

The Mott insulator-to-superfluid transition, described by $\mathcal{L}_{\text{sf}}$, occurs at the critical point $\mu = 0$, with $|\Psi_{\text{sf}}| = \sqrt{\frac{|\mu|}{U}} \equiv \sqrt{\rho_s^0}$. Writing $\Psi_{\text{sf}} = \sqrt{\rho_s} e^{i\varphi_s} = \sqrt{\rho_s^0 + \delta\rho_s} e^{i\varphi_s}$, the superfluid part becomes.

$$\mathcal{L}_{\text{sf}} = -\rho_s \partial_t \varphi_s - \frac{1}{2m_s}\left[\frac{(\boldsymbol{\nabla}\rho_s)^2}{4\rho_s} + \rho_s(\boldsymbol{\nabla}\varphi_s)^2\right] - \mu\rho_s - \frac{U}{2}\rho_s^2.$$
(A4)

Integrating out the massive magnitude fluctuations $\delta\rho_s$,

leads to,

$$\mathcal{L}_{\text{sf}} = \frac{1}{2}\partial_t \varphi_s \frac{1}{U - 1/(4m_s\rho_s^0) \cdot \boldsymbol{\nabla}^2} \partial_t \varphi_s - \frac{\rho_s^0}{2m_s}(\boldsymbol{\nabla}\varphi_s)^2$$
$$\approx \frac{1}{2U}(\partial_t \varphi_s)^2 - \frac{\rho_s^0}{2m_s}(\boldsymbol{\nabla}\varphi_s)^2$$
$$\equiv \frac{\rho_s^0}{2m_s}(\partial_\mu \varphi_s)^2,$$
(A5)

where, in the second line, we have assumed that $\varphi_s(\mathbf{r})$ varies slowly in space and dropped the term, $\frac{1}{4m_s\rho_s^0}\boldsymbol{\nabla}^2$, and in the last line, we have written it as that of a sound mode, in a Lorentz-invariant form for simplicity, with $\partial_\mu \equiv \left(\frac{1}{c_{\text{ph}}}\partial_t, \boldsymbol{\nabla}\right)$ and $c_{\text{ph}} = \sqrt{\frac{\rho_s^0 U}{m_s}}$.

Combined, $\mathcal{L}_{\text{dis}}$, with, $\mathcal{L}_{\text{sf}}$ and $\mathcal{L}_{\text{sf-dis}}$, the resulting phase is a supersolid with the spontaneous breaking of both partice number conservation U(1) symmetry and translational symmetry. Writing $\psi_k = \sqrt{\rho_k}e^{i\varphi_k}$, and integrating out the massive magnitude fluctuations, the effective Lagrangian density for the super-solid phase is given by

$$\tilde{\mathcal{L}}_{\text{cr}} = \frac{\rho_k}{2}\left(\partial_t \varphi_k + pA_{0k}\right)^2 - \frac{\rho_k}{2m_p}\cos\left[\Pi_{ij}^{\perp k}\left(\partial_j \varphi_k + pA_{jk}\right)\right]$$
$$- \rho_p^{\parallel}\cos\left[\Pi_{ij}^{\parallel k}\left(\partial_j \varphi_k + pA_{jk}\right) + \varphi_s\right] + \mathcal{L}_{\text{M}}^{\text{cr}} + \mathcal{L}_{\text{sf}}.$$
(A6)

Freezing the superfluid phase by fixing $\varphi_s = 0$, and rescaling the longitudinal and transverse gradients, we get (correct to quadratic order in the argument of cosine terms),

$$\tilde{\mathcal{L}}_{\text{cr}} = \frac{\rho_k}{2}\left(\partial_t \varphi_k + pA_{0k}\right)^2 - \frac{\rho_k}{2m_p}\cos\left(\boldsymbol{\nabla}\varphi_k + p\mathbf{A}_k\right) + \mathcal{L}_{\text{M}}^{\text{cr}}$$
$$= \frac{1}{2}|\left(\partial_\mu + ipA_{\mu,k}\right)\psi_k|^2 - V\left(\{\psi_k\}\right) + \mathcal{L}_{\text{M}}^{\text{cr}},$$
(A7)

where in the second form, we have written it as an equivalent "soft-spin" description in terms of ψ_k, with the landau U(1)-invariant potential, $V\left(\{\psi_k\}\right)$. Note that the fugacity $e^{-\tilde{E}_k^b}$ and density ρ_k are related by the relation: $\frac{m_p}{d^2}e^{-\tilde{E}_k^b} = \rho_k$ to the lowest order, such that $\frac{\rho_k}{m_p} = 2\tilde{g}_k^b$. Therefore, Eq.(A7) is in the same form as Eq.(61).

Similar analysis applies for the quantum smectic phase with just one species of dipoles, i.e., $\hat{\mathbf{x}}$−dipoles ($\hat{\mathbf{y}}$−dislocations), which leads to the full mobility of dipoles (dislocations) in the smectic and the effective Lagrangian density of the super-smectic phase given by,

$$\tilde{\mathcal{L}}_{\text{sm}} = \frac{\rho}{2}\left(\partial_t \varphi_x + pA_0\right)^2 - \frac{\rho}{2m_p}\cos\left(\boldsymbol{\nabla}\varphi_x + p\mathbf{A}\right) + \mathcal{L}_{\text{M}}^{\text{sm}}$$
$$= \frac{1}{2}|\left(\partial_\mu + ipA_\mu\right)\psi_x|^2 - V\left(|\psi_x|\right) + \mathcal{L}_{\text{M}}^{\text{sm}},$$
(A8)

which is in the same form as Eq.(63).

Appendix B: From Crystal dual to Smectic dual

The crystal-smectic partial melting transition corresponds to condensation of one flavor of the dipole fields, that according to Fig. 2, we take to be $\hat{\mathbf{y}}$-dipoles, i.e. $\psi_y \neq 0$. In this $\hat{\mathbf{y}}$-dipole condensate, the vortices in the phase field φ_y are suppressed, i.e., $\boldsymbol{\nabla}\varphi_y$ is small, and thus, we can expand the corresponding cosine term in (A7) to the quadratic order in its argument, which leads to

$$
\tilde{\mathcal{L}}_{\text{cr}} = \frac{\rho_x}{2} \left(\partial_t\varphi_x + pA_{0x}\right)^2 - \frac{\rho_x}{2m_p} \cos\left(\boldsymbol{\nabla}\varphi_x + p\mathbf{A}_x\right) + \mathcal{L}_{\text{M}}^{\text{cr}}
$$
$$
+ \frac{\rho_y}{2} \left(\partial_t\varphi_y + pA_{0y}\right)^2 + \frac{1}{2}\cdot\frac{\rho_y}{2m_p}\left(\boldsymbol{\nabla}\varphi_y + p\mathbf{A}_y\right)^2.
$$
(B1)

With the gauge transformation by choosing $\chi_y \to \frac{1}{p}\varphi_y$ in Eq.(46),

$$
A_{iy} \to A_{iy} + \frac{1}{p}\partial_i\varphi_y, \ A_{0y} \to A_{0y} + \frac{1}{p}\partial_t\varphi_y, \ a_y \to a_y + \frac{1}{p}\varphi_y,
$$
(B2)

we absorb the gradients of the phase φ_y into the gauge fields $A_{\mu,y}$, and the last two terms in Eq.(B1) become quadratic terms of $\hat{\mathbf{y}}$-flavor gauge fields, $A_{\mu,y}^2$, which make the original massless modes $A_{\mu,y}$ become massive. And, $\tilde{\mathcal{L}}_{\text{cr}}$ can be written as

$$
\tilde{\mathcal{L}}_{\text{cr}} = \frac{\rho_x}{2}\left(\partial_t\varphi_x + pA_{0x}\right)^2 - \frac{\rho_x}{2m_p}\cos\left(\boldsymbol{\nabla}\varphi_x + p\mathbf{A}_x\right) + \mathcal{L}_{\text{M}}^{\text{eff}},
$$
(B3)

where the effective Maxwell part $\mathcal{L}_{\text{M}}^{\text{eff}}$ in terms of newly defined $A_{\mu,y}$ is given by

$$
\mathcal{L}_{\text{M}}^{\text{eff}} = \mathcal{L}_{\text{M}}^{\text{cr}} + \frac{1}{2}\rho_y p^2 A_{0y}^2 + \frac{1}{2}\cdot\frac{\rho_y p^2}{2m_p}\mathbf{A}_y^2
$$
(B4)

Therefore, within this condensate phase, the gauge field components $A_{y,\mu}$ become gapped via the Anderson-Higgs mechanism by coupling to the $\hat{\mathbf{y}}$-dipole ($\hat{\mathbf{x}}$-dislocation) condensate , and can be safely integrated out in the low-energy regime with wavelengths of excitations much greater than $\sqrt{\frac{2m_p}{\rho_y p^2}}$ (i.e., penetration length of $A_{\mu,y}$) , which leads to

$$
\mathcal{L}_{\text{M}}^{\text{eff}} = \frac{1}{2}\chi\mathbf{E}_x^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{A}_x\right)^2 + \frac{1}{2}K_x e_x^2 + \frac{1}{2}K_y e_y^2
$$
$$
- \frac{1}{2}\left(\frac{\rho_y p^2}{\rho_y p^2 - m_p}\right)\left(\epsilon_{ij}\partial_i a_j + \epsilon_{ix}A_{ix}\right)^2,
$$
(B5)

where, the modified bend modulus $\mathbf{K} = (K_x, K_y)$, with $K_x^{-1} = K^{-1}$ and $K_y^{-1} = K^{-1} - \frac{K^{-1}}{K\rho_y p^2 + 1}$, becomes anisotropic in this smectic case. In the lowest order approximation, making $\rho_y \to \infty$ (i.e., the condensate is very dense), $K_x = K_y = K$, and $\mathcal{L}_{\text{M}}^{\text{eff}}$ reduces to the Maxwell

Lagrangian of the smectic case,

$$
\mathcal{L}_{\text{M}}^{\text{eff}} \approx \frac{1}{2}\chi\mathbf{E}_x^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{A}_x\right)^2 + \frac{1}{2}K\mathbf{e}^2 - \frac{1}{2}\left(\epsilon_{ij}\partial_i a_j + \epsilon_{ix}A_{ix}\right)^2
$$
$$
= \mathcal{L}_{\text{M}}^{\text{sm}}.
$$
(B6)

Therefore, to lowest order, the crystal's Maxwell Lagrangian reduces to that of a smectic,

$$
\mathcal{L}_{\text{M}}^{\text{cr}}\left(A_{\mu,x}, A_{\mu,y} = 0, a_\mu\right) \approx \mathcal{L}_{\text{M}}^{\text{sm}}\left(A_{\mu,x}, a_\mu\right),
$$
(B7)

which simply corresponds to setting $A_{\mu,y} \approx 0$. And, by replacing ρ_x simply with ρ in Eq.(B3), the dual Lagrangian of the crystal reduces to that of the smectic exactly, to the lowest order,

$$
\tilde{\mathcal{L}}_{\text{cr}}\left(A_{\mu,x}, A_{\mu,y} = 0, a_\mu, \psi_x, \psi_y = 0\right) \approx \tilde{\mathcal{L}}_{\text{sm}}\left(A_{\mu,x}, a_\mu, \psi_x\right).
$$
(B8)

Similarly analysis applies for the further melting with a condensation of the other, $\hat{\mathbf{x}}$-dipoles ($\hat{\mathbf{y}}$-dislocations). Within this $\psi_x \neq 0$ Higgs phase, corresponding to a condensation of unbound $\hat{\mathbf{x}}$-dipoles ($\hat{\mathbf{y}}$-dislocations) in the smectic, the gauge field components $A_{\mu,x}$ become gapped also, via coupling to the $\hat{\mathbf{x}}$-dipole ($\hat{\mathbf{y}}$-dislocation) condensate. This can be seen easily by making a further gauge transformation with $\chi_x \to \frac{1}{p}\varphi_x$ in Eq.(46),

$$
A_{ix} \to A_{ix} + \frac{1}{p}\partial_i\varphi_x, \ A_{0x} \to A_{0x} + \frac{1}{p}\partial_t\varphi_x, \ a_x \to a_x + \frac{1}{p}\varphi_x,
$$
(B9)

to absorb the gradients of the phase φ_x into the gauge fields $A_{\mu,x}$, and expanding the corresponding cosine term in (A6) to the quadratic order in its argument, which leads to,

$$
\tilde{\mathcal{L}}_{\text{sm}} = \mathcal{L}_{\text{M}}^{\text{sm}} + \frac{1}{2}\rho_x p^2 A_{0x}^2 + \frac{1}{2}\cdot\frac{\rho_x p^2}{2m_p}\mathbf{A}_x^2.
$$
(B10)

where the last two quadratic terms make the original massless modes $A_{\mu,x}$ become massive now. Integrating out $A_{\mu,x}$ in the low-energy regime with wavelengths of excitations much greater than $\sqrt{\frac{2m_p}{\rho_x p^2}}$ (i.e., penetration length of $A_{\mu,x}$), leads to,

$$
\tilde{\mathcal{L}}_{\text{sm}} = \frac{1}{2}\left(K^{-1} - \frac{1}{\rho_x p^2 + K^{-1}}\right)\left(\partial_t a_x - \partial_x a_0\right)^2
$$
$$
+ \frac{1}{2}K^{-1}\left(\partial_t a_y - \partial_y a_0\right)^2 - \frac{1}{2}\left(1 + \frac{2m_p}{\rho_x p^2}\right)\left(\boldsymbol{\nabla}\times\mathbf{a}\right)^2
$$
$$
\approx \frac{1}{2}K^{-1}\left(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{a}\right)^2
$$
$$
= \tilde{\mathcal{L}}_{\text{nm}},
$$
(B11)

in the lowest order approximation, making $\rho_x \to \infty$ (i.e., the condensate is very dense), which is just the dual Lagrangian density of the quantum xy model of a nematic. Therefore,

$$
\mathcal{L}_{\text{M}}^{\text{sm}}\left(A_{\mu,x} = 0, a_\mu\right) \approx \mathcal{L}_{\text{M}}^{\text{nm}}\left(a_\mu\right),
$$
(B12)

to the lowest order.

Appendix C: 3D smectic elasticity

In the main text, we have given the elasticity of a d-dimensional smectic in terms of the layer displacement u only, given by Eq.(11). For a 3D smectic, we just set $d = 3$. Here, we formulate the elasticity of a 3D smectic in terms of the displacement u and the Frank director $\mathbf{n}$ simultaneously. The elastic energy will not change, if all layers of molecules are rotated together rigidly. However, there will be an energy cost if the orientation directions of molecules, represented by Frank director $\mathbf{n}$, are rotated away from their equilibrium local orientation, normal to the layers. The elastic energy density of a 3D smectic, with its layers along xy plane, is given by

$$\mathcal{H}_{\text{sm}}^{\text{3d}} = \frac{1}{2}\chi (\partial_z u)^2 + \frac{1}{2}K (\boldsymbol{\nabla}_\perp u + \delta\mathbf{n})^2 + \frac{1}{2}K_1 (\boldsymbol{\nabla} \cdot \mathbf{n})^2 + \frac{1}{2}K_2 [\mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{n})]^2 + \frac{1}{2}K_3 [\mathbf{n} \times (\boldsymbol{\nabla} \times \mathbf{n})]^2, \quad (C1)$$

where u is the layer displacement, $\delta\mathbf{n} = \mathbf{n} - \hat{z} = (\delta n_x, \delta n_y, 0)$ represents the layer orientation degree of freedom, and the last three terms represents the slay, twist and bend distortions of the director respectively, with three independent, corresponding elastic constants K_1, K_2 and K_3. To linear order in $\delta\mathbf{n}$,

$$\mathcal{H}_{\text{sm}}^{\text{3d}} \simeq \frac{1}{2}\chi (\partial_z u)^2 + \frac{1}{2}K (\boldsymbol{\nabla}_\perp u + \delta\mathbf{n})^2 + \frac{1}{2}K_1 (\boldsymbol{\nabla} \cdot \mathbf{n})^2 + \frac{1}{2}K_2 [\mathbf{z} \cdot (\boldsymbol{\nabla} \times \mathbf{n})]^2 + \frac{1}{2}K_3 [\mathbf{z} \times (\boldsymbol{\nabla} \times \mathbf{n})]^2$$
$$= \frac{1}{2}\chi q_z^2 |u(\mathbf{q})|^2 + \frac{1}{2}K |\mathbf{q}_\perp u(\mathbf{q}) + \delta\mathbf{n}(\mathbf{q})|^2 + \frac{1}{2}K_1 \delta n_i P_{ij}^{\text{L}} \delta n_j + \frac{1}{2}K_2 \delta n_i P_{ij,z}^{\text{T}} \delta n_j + \frac{1}{2}K_3 \delta n_i P_{ij,\perp}^{\text{T}} \delta n_j, \quad (C2)$$

where, $P_{ij}^{\text{L}} = q_i q_j/q^2$ is the longitudinal projector, $P_{ij,z}^{\text{T}} = (\delta_{ij} - q_i^\perp q_j^\perp/q^2)(1 - \delta_{iz}\delta_{jz})$ and $P_{ij,\perp}^{\text{T}} = (\delta_{ij} - q_\perp^2/q^2)(\delta_{ix}\delta_{jx} + \delta_{iy}\delta_{jy})$ are the transverse-to-$\hat{\mathbf{z}}$ and transverse-to-layer projector, respectively, and in the second line, we have transformed into the momentum space. For long-wavelength limit, with the wave number $q \ll \sqrt{\chi/K_2}, \sqrt{\chi/K_3}$, we can integrate out the higher-energy terms, which sets $\delta\mathbf{n} = \boldsymbol{\nabla}_\perp u$ and reduces $\mathcal{H}_{\text{sm}}^{\text{3d}}$ into the form,

$$\mathcal{H}_{\text{sm}}^{\text{3d}} = \frac{1}{2}\chi (\partial_z u)^2 + \frac{1}{2}K_1 (\boldsymbol{\nabla}_\perp^2 u)^2, \quad (C3)$$

matching exactly with the standard form of the elastic energy of a 3D smectic. In Section IV, for a simple anal-

ysis with losing much qualitative physics, we have set $K = \chi$, and made the isotropic elasticity approximation with $K_1 = K_2 = K_3$, replacing them by K for simplicity.

Below, we give a briefly discussion of dislocations and their energies in a 3D smectic, based on $\mathcal{H}_{\text{sm}}^{\text{3d}}$ given by Eq.(C3). A more detailed discussion based on $\mathcal{H}_{\text{sm}}^{\text{3d}}$ given by Eq.(C1) can be found standard textbooks[2]. For a single positive screw dislocation with its line core, located in the origin of $xy-$plane, in the $\hat{z}$ direction, as shown in Fig. 11(a), from the Eq.(9), that determines the positions of the layer planes, we get the layer displacement given by,

$$u_{\text{scrw}}(\mathbf{r}) = \frac{d}{2\pi} \tan^{-1} \frac{y}{x}, \quad (C4)$$

taking place in the xy plane only, and then,

$$\partial_z u_{\text{screw}} = 0,$$
$$\boldsymbol{\nabla}_\perp u_{\text{screw}} = \frac{d}{2\pi} \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2 + y^2} = -\frac{d}{2\pi} \frac{\hat{\varphi}}{r_\perp}, \quad (C5)$$

shuch that,

$$\boldsymbol{\nabla}_\perp^2 u_{\text{screw}} = 0, \quad (C6)$$

and therefore, the energy of a single screw dislocation in a smectic is 0.

For a single positive edge dislocation with its line core perpendicular to the $\hat{z}$ direction, say in the $\hat{x}$ direction, as shown in Fig. 11(b), the layer displacement is then given by,

$$u_{\text{edge}}(\mathbf{r}) = \frac{d}{2\pi} \tan^{-1} \frac{z}{y}, \quad (C7)$$

and then,

$$\partial_z u_{\text{edge}} = \frac{y}{y^2 + z^2},$$
$$\boldsymbol{\nabla}_\perp u_{\text{edge}} = -\frac{z\hat{\mathbf{y}}}{y^2 + z^2}, \quad (C8)$$

such that,

$$\boldsymbol{\nabla}_\perp^2 u_{\text{edge}} = \frac{2yz}{(y^2 + z^2)^2}, \quad (C9)$$

and, the energy of a single edge dislocation in a smectic can be shown to be divergent as the length scale of the system, after integrating (C3) over space.

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