The solutions of the Yang-Baxter equation for the $(n+1)(2n+1)$ -vertex models through a differential approach.

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Abstract

The formal derivatives of the Yang-Baxter equation with respect to its spectral parameters, evaluated at some fixed point of these parameters, provide us with two systems of dierential equations. The derivatives of the R matrix elements, however, can be regarded as independent variables and eliminated from the systems, after which two systems of polynomial equations are obtained in place. In general, these polynomial systems have a non-zero Hilbert dimension, which means that not all elements of the R matrix can be fixed through them. Nevertheless, the remaining unknowns can be found by solving a few number of simple differential equations that arise as consistency conditions of the method. The branches of the solutions can also be easily analyzed by this method, which ensures the uniqueness and generality of the solutions. In this work we considered the Yang-Baxter equation for the $(n+1)(2n+1)$-vertex models with a generalization based on the A_{n-1} symmetry. This differential approach allowed us to solve the Yang-Baxter equation in a systematic way. .

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1 Introduction

A short life story:

- "I was privileged to work with Ricardo Soares Vieira or Ricardinho, as he was called by colleagues and friends.

Ricardinho died prematurely on October 21, 2020, due to a post-operative complication to remove stomach cancer.

I followed his graduate studies in physics at the Federal University of São Carlos and was his advisor for the master's [38]and doctorate [37].

During this period it was possible to observe his interests in the various areas of knowledge.

I was often impressed by his ability to solve complex problems quickly".

I'm very sorry for his death!

A few months before his death, Vieira came to me to propose me to work with the Yang-Baxter equation.The problem was to consider the possible vertex models, taking into account the structure of the R matrices associated with the symmetries of the non-exceptional affine Lie algebras. The new R matrix solution is recalculated by the Yang Baxter equation using a differential approach. He had just solved this problem for two-state models[39] . Then we started looking at the $A_{n-1}^{(1)}$models , in the certainty that they are the simplest and due to their current interest [40, 41]. We have organized this paper as follows. In sections 2 and 3 we make a usual presentation of Yang-Baxter's equation and its corresponding differential equations , respectively [39]. In section 4, we presente the calculos for the 15-vertex model. In section 5 we conside the case $n = 3$, or 28-vertex model and in section 6 we presente the general case. We conclude in the section 7.

2 The Yang-Baxter equation

The Yang-Baxter equation (YBE) is one of the most important equations of contemporary mathematical-physics. It originally emerged in two different contexts of theoretical physics: in quantum field theory, the YBE appeared as a sufficient condition for the many-body scattering amplitudes to factor into the product of pairwise scattering amplitudes [1,2,3]; in statistical mechanics it represented a sufficient condition for the transfer matrix of a given statistical model to commute for different values of the spectral parameters [4, 5]. Since the pioneer works in quantum integrable systems – see [6, 7, 8] for a historical background –, the YBE has become a cornerstone in several fields of physics and mathematics: it is most known for its fundamental role in the quantum inverse scattering method and in the algebraic Bethe Ansatz [9, 10, 11], although it also revealed to be important in the formulation of Hopf algebras and quantum groups [12, 13, 14, 15, 16], in knot theory [17], in quantum computation [18], in AdS-CFT correspondence [19, 20] and, more recently, in gauge theory [21, 22, 23]. The YBE can be seen as a matrix relation defined in $End(V \otimes V \otimes V)$, where V is an n- dimensional complex vector space. In the most general case, it reads:

$$YB = R_{12}(u)R_{13}(u+v)R_{23}(v) - R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{1}$$

where the arguments u and v, called spectral parameters, have values in C. The solution of the YBE is an R matrix defined in End (V $\otimes$ V). The indexed matrices Rij appearing in (1) are defined in End (V $\otimes$ V $\otimes$ V) through the formulas

$$R_{12} = R \otimes I, \qquad R_{23} = I \otimes R, \qquad R_{13} = P_{23}R_{12}P_{23} \tag{2}$$

where $I \in End(V)$ is the identity matrix, $P \in End(V \otimes V)$ is the permutator matrix (defined by the relation $P(A \otimes B)P = B \otimes A$ for $\forall A, B \in End(V)$) and $P_{12} = P \otimes I, P_{23} = I \otimes P$.

For each solution of the YBE, a given integrable system can be associated. In fact, in statistical mechanics, the R matrix represents the Boltzmann weights of a given statistical model while, in quantum field theory, the R matrix is associated with factorizable scattering amplitudes between relativistic particles. From the YBE we can prove that systems described by an R matrix possess infinitely many conserved quantities in involution – the Hamiltonian being one of them –, the reason why they are called integrable [24] We say that a given solution $R(u)$ of the YBE (1) is regular if $R(0) = P$. Regular solutions of the YBE have several important properties [8].

3 The differential Yang-Baxter equation

The YBE corresponds to a system of non-linear functional equations. Several particular solutions of the YBE are known [6,7,8]. The first solutions were found by a direct inspection of the functional equations, which are in fact very simple because the R matrix is assumed to have many symmetries. Nevertheless, there are other more advanced methods for solving the YBE: we can cite, for instance, the Baxterization of braid relations [25], the use of Lie algebras and superalgebras [26, 27, 28], the construction of Hopf algebras and quantum groups [13, 14, 15], and also techniques relying on algebraic geometry [29], see also [30]. The methods mentioned above usually require that the R matrix presents one or more symmetries from the very start. From a mathematical point of view, would be desirable to develop a method that requires in principle as few as possible symmetries and, at the same time, that is powerful enough in order to find and classify the solutions of the YBE. This paper is concerned with the development and extensive use of such a method, which is based on a differential approach.

From the quantum grup invariant representation for non-excepitional affine Lie algebra $A^{(1)}_{n-1}$ [26],we consider the following regeralization for a R-matrix solution of the Yang-Baxter equation (YBE)

$$R(u)=\sum_{i=1}^{n+1} a_{ii}(u)\mathrm{e}_{ii}\otimes\mathrm{e}_{ii}+\sum_{i\neq j}^{n+1} b_{ij}(u)\mathrm{e}_{ii}\otimes\mathrm{e}_{jj}+\sum_{i\neq j}^{n+1} c_{ij}(u)\mathrm{e}_{ij}\otimes\mathrm{e}_{ji} \tag{3}$$

where $e_{ij}^{(n)}$ are the Weyl matrices acting for a $n+1$ dimensional vector space V at the site n . The R-matrix elementes $a_{ii}(u)$, $b_{ij}(u)$ and $c_{ij}(u)$ are fixed.

To be more precise, this method consists mainly of the following: if we take the formal derivatives of the (1) with respect to the spectral parameters u and v and then evaluate the derivatives at some fixed point of those variables (say at zero), then we shall get two systems of ordinary non-linear differential equations for the elements of the R matrix. The derivatives of the R matrix elements, however, can be regarded as independent variables, so that, after they are eliminated, two systems of polynomial equations are obtained in place. Thus, these polynomial systems can be analyzed – for instance, through techniques of the computational algebraic geometry [32] – and eventually completely solved. It happens, however, that these polynomial systems usually have a positive Hilbert dimension, which means that the systems are satisfied even when some of the R matrix elements are still arbitrary. The remaining unknowns, nonetheless, can be found by solving a few number of differential equations that arise from the expressions for the derivatives we had eliminated before. These auxiliary differential equations, therefore, can be thought as consistency conditions of the method. For example, if we take the formal derivative of (1) with respect to v and then evaluate the result at the point $v=0$, then we shall get the equation,

$$YB_v=R_{12}(u)_{13}(u)P_{23}+R_{12}(u)R_{13}(u)H_{23}-H_{23}R_{13}(u)R_{12}(u)-P_{23}R_{13}(u)R_{12}(u) \tag{4}$$

and from its derivative with respect u at the point $u=0$, we get

$$YB_u=H_{12}R_{13}(v)R_{23}(v)+P_{12}D_{13}(v)R_{23}(v)-R_{23}(v)D_{13}(v)P_{12}-R_{23}(v)R_{13}(v)H_{12} \tag{5}$$

where$P$ is the permutator matrix. $P=R(0)$ and $H=D(0)$.

We highlight that $\mathcal{H}=PH$, where $\mathcal{H}$ is nothing but the local Hamiltonian associated with the model – see, for instance, [6,8].

Therefore

$$D(u)=\frac{\partial R(u+v)}{\partial v}\mid_{v=0}, \qquad D(v)=\frac{\partial R(u+v)}{\partial u}\mid_{u=0}, \qquad P=R(0), \qquad H=D(0) \tag{6}$$

Using the notation

$$\frac{da_{ii}(u)}{du}\mid_{u=0}=\alpha_{ii}, \qquad \frac{db_{ij}(u)}{du}\mid_{u=0}=\beta_{ii} \qquad \text{and} \qquad \frac{dc_{ij}(u)}{du}\mid_{u=0}=\mu_{ij} \tag{7}$$

we can write

$$D(u)=\sum_{i=1}^{n+1} da_{ii}(u)\mathrm{e}_{ii}\otimes\mathrm{e}_{ii}+\sum_{i\neq j}^{n+1} db_{ij}(u)\mathrm{e}_{ii}\otimes\mathrm{e}_{jj}+\sum_{i\neq j}^{n+1} dc_{ij}(u)\mathrm{e}_{ij}\otimes\mathrm{e}_{ji} \tag{8}$$

and

$$H=\sum_{i=1}^{n+1}\alpha_{ii}\mathrm{e}_{ii}\otimes\mathrm{e}_{ii}+\sum_{i\neq j}^{n+1}\beta_{ij}\mathrm{e}_{ii}\otimes\mathrm{e}_{jj}+\sum_{i\neq j}^{n+1}\mu_{ij}\mathrm{e}_{ij}\otimes\mathrm{e}_{ji} \tag{9}$$

where da, db, dc are the derivatives of a, b, c in respect to u or v.

The idea of transforming a functional equation into a differential one goes back to the works of the Niels Henrik Abel, who solved several functional equations in this way . Abel's method presents many advantages when compared with other methods of solving functional equations. For instance, it consists in a general method that can be applied to a huge class of functional equations; it establishes the generality and uniqueness of the solutions (which would be difficult, if not impossible, to establish in other ways) by reducing the problem to the theory of differential equations and so on – see [32] for more. Notice moreover that although Abel's method requires the solutions to be differentiable (there can be non-differentiable solutions of some functional equations), this restriction is not a problem when dealing with the YBE, as its solutions are always assumed to be differentiable because of the connection between the R matrix and the corresponding local Hamiltonian. Concerning the theory of integrable systems, the differential method is perhaps most known in connection with boundary YBE [34, 35, 36].

Now we can look for the matrices (3) what are solutions of (1). First, let's explain the calculations for the deformed $A_1^{(1)}$, or 15 -vertex model

4 The 15-vertex models

The correspondig matrices are

$$R(u)=\begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b_{12} & 0 & c_{12} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b_{13} & 0 & 0 & 0 & c_{13} & 0 & 0\\
0 & c_{21} & 0 & b_{21} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b_{23} & 0 & c_{23} & 0\\
0 & 0 & c_{31} & 0 & 0 & 0 & b_{31} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & c_{32} & 0 & b_{32} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{33}
\end{pmatrix} \tag{10}$$

$$D(u) = \begin{pmatrix} da_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & db_{12} & 0 & dc_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & db_{13} & 0 & 0 & 0 & dc_{13} & 0 & 0 \\ 0 & dc_{21} & 0 & db_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & da_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & db_{23} & 0 & dc_{23} & 0 \\ 0 & 0 & dc_{31} & 0 & 0 & 0 & db_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & dc_{32} & 0 & db_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & da_{33} \end{pmatrix} \tag{11}$$

$$H = \begin{pmatrix} \alpha 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{12} & 0 & \mu_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{13} & 0 & 0 & 0 & \mu_{13} & 0 & 0 \\ 0 & \mu_{21} & 0 & \beta_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_{23} & 0 & \mu_{23} & 0 \\ 0 & 0 & \mu_{31} & 0 & 0 & 0 & \beta_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{32} & 0 & \beta_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{33} \end{pmatrix} \tag{12}$$

Note the entries of $R(u)$ and $D(u)$ are functions of u and the conditions $a_{ii}(0) = 1, b_{ij}(0) = 0$ and $c_{ij}(0) = 1$, define the matrix P, $P = R(0)$.

For this model we have three 27 by27 matrices equations $YB = 0, YB_u = 0$ and $YB_v = 0$. Looking at their diagonals $YB[i.i] = 0$, we can find several equations containing only the $c_{ij}(u)$ amplitudes. For their derivatives, $YB_u[i,i] = 0$, we find

$$c_{ij}(u)c_{ji}(u)\left(\mu_{ij} - \mu_{ji}\right) + c_{ij}(u)dc_{ji}(u) - dc_{ij}(u)c_{ji}(u) = 0, \quad i \neq j = \{1,2,3\} \tag{13}$$

With the regular condions, the solutions are

$$c_{ij}(u) = \exp(\mu_{ij}u) \tag{14}$$

After replace the $c_{ij}(u)$in all remained equations , the conjugated equations $YB[i, 28-i] = 0, YB_u[i, 28-i] = 0$ and $YB_v[i, 28-i] = 0$ are solved by the following relations

$$b_{23}(u) = \frac{\beta_{23}}{\beta_{21}} b_{21}(u), \qquad b_{32}(u) = \frac{\beta_{32}}{\beta_{12}} b_{12}(u) \tag{15}$$

with the constraint $\beta_{23}\beta_{32} = \beta_{12}\beta_{21}$. From other equations find $b_{13}(u), b_{31}(u)$and its derivatives

$$b_{13}(u) = \frac{\beta_{13}}{\beta_{12}} b_{12}(u), \qquad b_{31}(u) = \frac{\beta_{31}}{\beta_{21}} b_{21}(u), \tag{16}$$

with a second constraind $\beta_{13}\beta_{31} = \beta_{12}\beta_{21}$.

We notice that these relations can be write in a more compact form

$$b_{ij}(u) = \beta_{ij} K(u), \qquad \beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}}, \quad i \neq j = \{1,2,3\} \tag{17}$$

where $K(u)$ is an srbitrary funcion, to be fixed.

Using (17), all remaind equations contain only the $a_{ii}(u)$ and $K(u)$, as well as its derivatives. In particular, the differential equation $YB_u[6,20] = 0$ allow us to find the function $K(u)$:

$$-\frac{d}{du}K(u) + (\mu_{13} + \mu_{21} - \mu_{23})\, K(u) + \exp((\mu_{12} + \mu_{23} - \mu_{13})u) = 0 \tag{18}$$

For regular solutions we have

$$K(u) = \frac{\exp((\mu_{12} - \mu_{13} + \mu_{23})u) - \exp((\mu_{21} + \mu_{13} - \mu_{23})u)}{(\mu_{12} - 2\mu_{13} - \mu_{21} + 2\mu_{23})} \tag{19}$$

Substituing in the Yang-Baxter equations we have to fix two μ_{ij}

$$\mu_{31} = \mu_{21} + \mu_{12} - \mu_{13}. \qquad \mu_{32} = \mu_{21} + \mu_{12} - \mu_{23} \tag{20}$$

Before the computation of the $a_{ii}(u), i = 1,2,3$,we can still simplify the notation by defining two parameters

$$\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}, \qquad \kappa_2 = \mu_{12} - \mu_{13} + \mu_{23} \tag{21}$$

It follows

$$K(u) = \frac{e^{\kappa_1 u} - e^{\kappa_2 u}}{\kappa_1 - \kappa_2}, \qquad \mu_{3j} = \kappa_1 + \kappa_2 - \mu_{j3}, \qquad j = 1,2 \tag{22}$$

and from $YB_u[2,10] = 0$, we get the the relation between the parameters $\alpha_{11,}\beta_{ij}$ and μ_{ij}:

$$\beta_{21}\beta_{12} = (\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11}) \tag{23}$$

After we find the $a_{ii}(u)$ terms we get the following recurrence:

$$a_{ii}(u) = a_{11}(u) + (\alpha_{ii} - \alpha_{11})K(u), \qquad i = 2,3 \tag{24}$$

with

$$a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\mathrm{e}^{\kappa_2 \mathrm{u}} - (\kappa_2 - \alpha_{11})\mathrm{e}^{\kappa_1 \mathrm{u}}}{\kappa_1 - \kappa_2} \tag{25}$$

Finally, we have a system of two equations whose solutions will determine the R matrices of the model.

$$(\alpha_{11} - \alpha_{kk})(\kappa_1 + \kappa_2 - \alpha_{11} - \alpha_{kk}) = 0, \qquad k = \{2,3\} \tag{26}$$

n this case we get four solutions:

4.1 Solution 1: $\alpha_{22} = \alpha_{11}, \qquad \alpha_{33} = \alpha_{11}$

For this solution we have (10) with the following entries

$$c_{ij}(u) = \exp(\mu_{ij}(u)), \qquad b_{ij}(u) = \beta_{ij}K(u), \qquad i \neq j = \{1, 2, 3\} \tag{27}$$

and

$$a_{33}(u) = a_{22}(u) = a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{28}$$

where

$$\kappa_1 = \mu_{13}+\mu_{21}-\mu_{23}, \qquad \kappa_2 = -\mu_{13}+\mu_{12}+\mu_{23}, \qquad K(u) = \frac{\exp(\kappa_1 u) - \exp(\kappa_2 u)}{\kappa_1 - \kappa_2} \tag{29}$$

and the fixed parameters

$$\mu_{31} = \mu_{12} + \mu_{21} - \mu_{13}, \qquad \mu_{32} = \mu_{12} + \mu_{21} - \mu_{32} \tag{30}$$

$$\beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}}, \quad i < j = \{1, 2, 3\}, \qquad \beta_{12}\beta_{21} = (\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11}) \tag{31}$$

For a particular choice of parameters

$$\mu_{ij} = \eta, \qquad \mu_{ji} = 0, \quad (i < j), \qquad \beta_{ij} = \xi, \quad i \neq j = \{1, 2, 3\} \tag{32}$$

with

$$\eta = \frac{\xi^2 - \alpha_{11}^2}{\alpha_{11}} \tag{33}$$

we get the quantum group invariant solution of [26]

4.2 Solution 2: $\alpha_{22} = \alpha_{11}, \alpha_{33} = \kappa_1 + \kappa_2 - \alpha_{11}$

In this case we get

$$a_{22}(u) = a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{34}$$

and

$$a_{33}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_1 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_2 u)}{\kappa_1 - \kappa_2} \tag{35}$$

4.3 Solution 3 : $\alpha_{22} = \kappa_1 + \kappa_2 - \alpha_{11}, \qquad \alpha_{33} = \alpha_{11}$

Here we get

$$a_{33}(u) = a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{36}$$

and

$$a_{22}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_1 u) + (\kappa_2 - \alpha_{11})\exp(\kappa_2 u)}{\kappa_2 - \kappa_1} \tag{37}$$

4.4 Solution 4: $\alpha_{22} = \kappa_1 + \kappa_2 - \alpha_{11}, \qquad \alpha_{33} = \kappa_1 + \kappa_2 - \alpha_{11}$

In this case

$$a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{38}$$

and

$$a_{33}(u) = a_{22}(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{39}$$

Here we remember that $\kappa_1+\kappa_2 = \mu_{12}+\mu_{21}$ and from (26) that the $a_{ii}(u)$have only two values

$$a_{ii}(u) = A(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_2 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_1 u)}{\kappa_1 - \kappa_2} \tag{40}$$

when $\alpha_{ii} = \alpha_{11}$, and

$$a_{jj}(u) = B(u) = \frac{(\kappa_1 - \alpha_{11})\exp(\kappa_1 u) - (\kappa_2 - \alpha_{11})\exp(\kappa_2 u)}{\kappa_1 - \kappa_2} \tag{41}$$

when $\alpha_{jj} = \kappa_1 + \kappa_2 - \alpha_{11}$.

Therefore we have

	$a_{11}(u)$	$a_{22}(u)$	$a_{33}(u)$
solution 1	$A(u)$	$A(u)$	$A(u)$
solution 2	$A(u)$	$A(u)$	$B(u)$
solution 3	$A(u)$	$B(u)$	$A(u)$
solution 4	$A(u)$	$B(u)$	$B(u)$

The solutions 2 and 3 are equivalent. It means That there are 3 different solutions.

For these R matrix we start with 15 parameters, the derivates of matrix elentes at the point $u = 0$. We fix, $\{\alpha_{22}, \alpha_{33}, \beta_{21}, \beta_{31}, \beta_{32}, \mu_{31}, \mu_{32}\}$. Therefore these R matrices have 8 - free parameters.

5 The 28 vertex - model

In the $A_2^{(1)}$ model we have 28-vertex model and its Yang-Baxter solutions are obtained following the procedures used in the 15 -vertex model. The 16 by 16 matricesR, D, H are given (3), (8) and (9) , with $n = 3$, respectively.

The diagonal matrix entries equations

$$YB_u[i,i] = 0, \qquad and \qquad YB_v[i,i] = 0 \tag{42}$$

are verified by the amplitudes

$$c_{ij}(u) = \exp(\mu_{ij} u), \qquad i \neq j = \{1, 2.3, 4\} \tag{43}$$

where μ_{ij} are arbitary parameters to be fixed.

The $b_{ij}(u)$ vertices were computed as we did in the case of the 15 vertex model. Its form is

$$b_{ij}(u) = \beta_{ij} K(u), \qquad i \neq j = \{1,2,3,4\} \tag{44}$$

The parameters β_{ij}satisfy the relation

$$\beta_{ji} = \frac{\beta_{12}\beta_{21}}{\beta_{ij}} = \frac{(\kappa_1 - \alpha_{11})(\kappa_2 - \alpha_{11})}{\beta_{ij}}, \qquad i < j = \{1,2,3,4\} \tag{45}$$

Now we can get the relations between the parameters $\mu_{ij}$from the equations $\mathbf{YB_v[i,j] = 0}$ and $\mathbf{YB_u[i,j] = 0}$:

$$\mu_{3i} = \mu_{12} + \mu_{21} - \mu_{i3}, \quad i < 3 \quad and \quad \mu_{4i} = \mu_{12} + \mu_{21} - \mu_{i4} \quad i < 4 \tag{46}$$

Replacing the expressions in (3) and its derivatives (4) and (5), we find, for instance, from $YB_v[46, 55] = 0$ the same function $K(u)$ 22

$$K(u) = \frac{e^{\kappa_1 u} - e^{\kappa_2 u}}{\kappa_1 - \kappa_2}, \tag{47}$$

where $\kappa_1 = \mu_{13} + \mu_{21} - \mu_{23}, \qquad \kappa_2 = \mu_{12} - \mu_{13} + \mu_{23}$. The $a_{ii}(u)$ functions still satisfy the recurrence

$$a_{kk}(u) = a_{11}(u) + (\alpha_{kk} - \alpha_{11})K(u), \qquad k = 2,3,4 \tag{48}$$

with

$$a_{11}(u) = \frac{(\kappa_1 - \alpha_{11})\mathrm{e}^{\kappa_2 \mathrm{u}} - (\kappa_2 - \alpha_{11})\mathrm{e}^{\kappa_1 \mathrm{u}}}{\kappa_1 - \kappa_2} \tag{49}$$

As we can see, the results are the same of the 15 -vertex model, but including the index $n + 1 = 4$.

Unlike the previous case, the parameters $\mu_{ji}, \qquad i < j$ are not sufficient to fix the solutions. looking at the equations $YB_v[i,j] = 0$ we see that many of them are type $(\alpha_{kk} - \alpha_{11})(\alpha_{11} + \kappa_1 + \kappa_2 - \alpha_{kk})G_{ij}(u) = 0$ and those that are not, will stay in that shape after we get some parameters μ_{ij} but now with $i < j$. These calculations are very annoying, but the results are simple. We found two possibilities:

$$\mu_{ij} = \kappa_1 - \mu_{1i} + \mu_{1j} \quad and \qquad \mu_{ij} = \kappa_2 - \mu_{1i} + \mu_{1j} \tag{50}$$

For this model we have to fix two parameters μ_{24} and μ_{34}. After this we have three equations

$$\alpha_{kk} - \alpha_{11})(\alpha_{11} + \kappa_1 + \kappa_2 - \alpha_{kk}) = 0 \tag{51}$$

and eight solutions for each set of fixed parameters

	$a_{11}(u)$	$a_{22}(u)$	$a_{33}(u)$	$a_{44}(u)$
sol 1	$A(u)$	$A(u)$	$A(u)$	$A(u)$
sol 2	$A(u)$	$A(u)$	$A(u)$	$B(u)$
sol 3	$A(u)$	$A(u)$	$B(u)$	$A(u)$
sol 4	$A(u)$	$A(u)$	$B(u)$	$B(u)$
sol 5	$A(u)$	$B(u)$	$A(u)$	$A(u)$
sol 6	$A(u)$	$B(u)$	$A(u)$	$B(u)$
sol 7	$A(u)$	$B(u)$	$B(u)$	$A(u)$
sol 8	$A(u)$	$B(u)$	$B(u)$	$B(u)$

Taking into account the equivalences for the solutions with the same number of A(u) and B(u), we have 4 different solutions.

Remember that

$$A(u)=\frac{(\kappa_1-\alpha_{11})\exp(\kappa_2 u)-(\kappa_2-\alpha_{11})\exp(\kappa_1 u)}{\kappa_1-\kappa_2} \tag{52}$$

for $\alpha_{kk}=\alpha_{11}$ and

$$B(u)=\frac{(\kappa_1-\alpha_{11})\exp(\kappa_1 u)-(\kappa_2-\alpha_{11})\exp(\kappa_2 u)}{\kappa_1-\kappa_2} \tag{53}$$

for $\alpha_{kk}=\alpha_{11}+\kappa_1+\kappa_2$.

Note that the second set of equations due to (50)has only two amplitudes $c_{ij}(u)$. different,c_{24} and c_{34}. Note that we have fixed 7 μ_{ij}, 6 β_{ij} and 3 α_{ii} . Therefore our R matrices solutions have 12 free-parameters.

Now we know how to generalize the results:

6 The $(n+1)(2n+1)$ -vertex models

For each value of $n{>}1$, the R matrix has $n+1$ diagonal entries $a_{ii}(u)$, that are determined by recurrence relative to $a_{11}(u)$

$$a_{kk}(u)=a_{11}(u)+(\alpha_{kk}-\alpha_{11})K(u), \qquad k=2,...,n+1 \tag{54}$$

where

$$\alpha_{jj}=\left(\frac{d}{du}a_{jj}(u)\right)_{u=0} \quad and \quad K(u)=\frac{e^{\kappa_1 u}-e^{\kappa_2 u}}{-\kappa_2+\kappa_1} \tag{55}$$

where $\kappa_1=\mu_{13}+\mu_{21}-\mu_{23}$ and $\kappa_2=-\mu_{13}+\mu_{12}+\mu_{23}$.

The remaining $n(n+1)$diagonal entries

$$b_{ij}(u)=\beta_{ij}K(u), \qquad i\neq j=\{1,...,n+1\} \tag{56}$$

where

$$\beta_{ij}=\left(\frac{d}{du}b_{ij}(u)\right)_{u=0} \tag{57}$$

with the constraints

$$\beta_{ji}\beta_{ij}=\beta_{21}\beta_{12}=(\kappa_1-\alpha_{11})(\kappa_2-\alpha_{11}) \tag{58}$$

The number of fixed parameters β_{ij} is $n-1)(n+2)/2+1$.

The $n(n+1)$ off-diagonal matrix elements

$$c_{ij}(u)=e^{\mu_{ij}u} \tag{59}$$

All μ_{ij} of the $c_{ij}(u)$ below od main diagonal are fixed by the relation

$$\mu_{ji}=\kappa_1+\kappa_2-\mu_{ij}, \qquad j>i \tag{60}$$

The number is $(n-1)(n+2)/2$and some μ_{ij} above the main diagonal can be fixed by two relation

$$\mu_{ij}=\kappa_1-\mu_{1i}+\mu_{1j} \quad and \quad \mu_{ij}=\kappa_2-\mu_{1i}+\mu_{1j}, \quad j<i \tag{61}$$

The number is $(n+1)(n-2)/2$. It means that we have two sets of solutions. With these relations the Yang-Baxter equation and its derivatives are solved by two sets of 2^n solutions of the following n equations

$$(\alpha_{kk}-\alpha_{11})(\alpha_{kk}+\alpha_{11}-\kappa_1-\kappa_2)=0 \tag{62}$$

Therefore we have two differentes values for the $a_{kk}(u)$ and n parameters α_{ii} are fixed. It mean that our R matrix solutions have $n(n+3)/2+3$ free parameters

$$a_{kk}(u)=A(u)=\frac{(\kappa_1-\alpha_{11})\exp(\kappa_2 u)-(\kappa_2-\alpha_{11})\exp(\kappa_1 u)}{\kappa_1-\kappa_2} \tag{63}$$

when $\alpha_{ii}=\alpha_{11}$ and

$$a_{kk}(u)=B(u)=\frac{(\kappa_1-\alpha_{11})\exp(\kappa_1 u)-(\kappa_2-\alpha_{11})\exp(\kappa_2 u)}{\kappa_1-\kappa_2} \tag{64}$$

when $\alpha_{kk}=\kappa_1+\kappa_2-\alpha_{11}$.

Using the identity

$$2^n=\sum_{k=0}^{n+1}\left(\begin{array}{c} n \\ k \end{array}\right)=\sum_{k=0}^{n+1}\frac{n!}{k!(n-k)!} \tag{65}$$

we can identify $n+1$ different solutions

7 Conclusion

From our calculus for $n>2$, we have find two sets of $n+1$ R matrix with $\frac{n(n+3)}{2}+3$ free parameters as solution of the Yang-Baxter for the $(n+1)(2n+1)$vertex models. For a particular choice of the parameters, the solutions with $n+1$ $A(u)$ function, we recover the R of the affine Lie algebra $A_{n-1}^{(1)}$.

Several particular solutions of the Yang-Baxter equation associated with The 15 –vertex models are known. We can cite, for example, the fifteen-vertex R matrices of Cherednik [42], Babelon [43], Chudnovsky & Chudnovsky [44] and Perk & Schultz [45, 46] and [47] (these solutions hold for higher vertex models as well). These R matrices contain fewer parameters than the solutions we found, so that they can be thought of as reductions of a more general solution.

We believe that the results presented here are original.

Perhaps the calculation of the reflection matrices for these R matrices can also be interesting, as well as their Bethe's ansatz.

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