

Cavitation of a spherical body under mechanical and self gravitational forces

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Abstract

In this paper we look for minimizers of the energy functional for isotropic compressible elasticity taking into consideration the effect of a gravitational field induced by the body itself. We consider the displacement problem in which the outer boundary of the body is subjected to a Dirichlet type boundary condition. For a spherically symmetric body occupying the unit ball $\mathcal{B} \in \mathbb{R}^3$, the minimization is done within the class of radially symmetric deformations. We give conditions for the existence of such minimizers, for satisfaction of the Euler–Lagrange equations, and show that for large displacements the minimizer must develop a cavity at the centre. A numerical scheme for approximating these minimizers is given together with some simulations that show the dependence of the cavity radius and minimum energy on the displacement and mass density of the body.

Key words: nonlinear elasticity, cavitation, self-gravity

1 Introduction

The study of the shape of self gravitating bodies is extensive and dates back to the time of Newton itself. It is well known that depending on the density of a dying star, there are several possibilities for the resulting object: white dwarf, neutron star, black hole, etc. The case of a black hole forming is also referred to as *gravitational collapse*. The literature on these phenomena is extensive and we refer to [6] and [7] for a historical account.

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In this paper we consider the problem of a self gravitating spherical body. Apart from its apparent “simplicity”, this problem plays an important role on the study of the more complex phenomena described above. The proposed variational model combines both mechanical and gravitational responses, the mechanical part based on a model from nonlinear elasticity which allows for the characterization of large deformations. Under certain mathematically physical conditions, the extrema of the corresponding energy functional, can be characterized via the Euler–Lagrange equations. This combined model has been used by [5], [6] and [7] among others. In [5] the existence of solutions to the Euler–Lagrange equations, with a zero dead load boundary condition on the outer boundary of the body, is established via the implicit function theorem and is valid for “small bodies” of arbitrary shape.

The work in [7] is for spherically symmetric deformations with a zero dead load boundary on the outer boundary condition as well, and combines asymptotic analysis with numerics to get results for varying densities and reference configuration body radius. They used a stored energy function of the form

$$W(\mathbf{F}) = \frac{\mu}{2} (\|\mathbf{F}\|^2 - 3 - 2f_\alpha(\det \mathbf{F})) + \frac{\beta}{2} (\det \mathbf{F} - 1)^2, \quad (1)$$

where $\alpha \geq 0$, β and μ are positive constants, and $f_\alpha(d) = \ln(d) - \alpha d^{-\alpha}(d - 1)^4$. This material corresponds to a “soft” compressible material for $\alpha = 0$ or small, and to a “strong” compressible material otherwise. For constant reference configuration density ρ_0 , the authors in [7] show numerically that for α small there exists a critical density ρ_0^* such that if $\rho_0 \leq \rho_0^*$, then the Euler–Lagrange equations (cf. (22)) can have multiple solutions, most of them unstable, while if $\rho_0 > \rho_0^*$, then there are no solutions which could be interpreted as gravitational collapse. Moreover for α large, there are solutions for all densities ρ_0 , which appear to be unique.

By adapting the techniques in [1] for polyconvex stored energy functions, the authors in [6] show the existence of minimizers for the resulting energy functional, now for large deformations and arbitrary bodies, and for both, zero dead load and displacement boundary conditions. The stored energy functions used in [6] could be classified as corresponding to “strong” compressible materials (cf. eqns. (13) and (14) in [6]).

In this paper we look for minimizers of the energy functional for isotropic compressible elasticity and taking into consideration the effect of a gravitational field induced by the body itself. We consider the displacement problem in which the outer boundary of the body is subjected to a Dirichlet type boundary condition. For a spherically symmetric body occupying the unit ball $\mathcal{B} \in \mathbb{R}^3$ and with radially symmetric mass density, the minimization is done within the class of radially symmetric deformations. Contrary to previous works, the deformations we consider belong to $W^{1,p}(\mathcal{B})$ with $p < 3$, and thus may develop singularities. For the particular case of radially symmetric deformations, we study the occurrence or initiation of a cavitation at the centre of the ball and its dependence on the boundary displacement and gravitational related constants.

In Section 2 we introduce the basic model, with energy functional and admissible function space, for radial deformations (cf. (7)) of a spherically symmetric body. These

deformations are characterized by a function $r : [0, 1] \rightarrow [0, \infty)$, the Dirichlet boundary condition taking the form $r(1) = \lambda$. After this we show in Section 3 that under certain growth conditions on the stored energy function (cf. (16) with H1–H3) and for any reference configuration density function ρ_0 that is bounded, nonnegative and bounded away from zero, a minimizer of the energy functional (9)–(11) exists over the admissible set (15). Under the additional constitutive assumption (21), these minimizers satisfy the Euler–Lagrange equations (22) where either $r(0) = 0$ or $r(0) > 0$ (*cavitation*) with zero Cauchy stress at the origin (cf. (24)). In Section 4 we show that for λ sufficiently large, these minimizers must satisfy $r(0) > 0$. This result is an adaptation to the problem with self gravity of a similar result in [10] for compressible inhomogeneous materials.

In Section 5 we collect several results for λ small where the minimizers must have the centre intact. In addition we show in Theorem 5.3 that any minimizer which leaves the centre intact must have strains at the origin less than the critical boundary displacement corresponding to an isotropic material made of the material at the centre of the original body. Once again this result is an adaptation to the problem with self gravity of a similar result in [10] for compressible inhomogeneous materials.

Finally in Section 6 we present a numerical scheme for the computation of the minimizers of our energy functional. This method is based on a combination of a gradient flow iteration which works as a predictor, together with a shooting method to solve the EL-equations, that works as a corrector. For constant reference configuration densities we present several simulations that show the dependence of the cavity radius and minimum energy on the displacement λ and density ρ_0 .

2 Problem formulation

Consider a body which in its reference configuration occupies the region

$$\mathcal{B} = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| < 1\}, \quad (2)$$

where $\|\cdot\|$ denotes the Euclidean norm. Let $\mathbf{u} : \mathcal{B} \rightarrow \mathbb{R}^3$ denote a deformation of the body and let its *deformation gradient* be

$$\nabla \mathbf{u}(\mathbf{x}) = \frac{d\mathbf{u}}{d\mathbf{x}}(\mathbf{x}). \quad (3)$$

For smooth deformations, the requirement that $\mathbf{u}(\mathbf{x})$ is locally *invertible and preserves orientation* takes the form

$$\det \nabla \mathbf{u}(\mathbf{x}) > 0, \quad \mathbf{x} \in \mathcal{B}. \quad (4)$$

Let $W : M_+^{3 \times 3} \rightarrow \mathbb{R}$ be the *stored energy function* of the material of the body where $M_+^{3 \times 3} = \{\mathbf{F} \in M^{3 \times 3} \mid \det \mathbf{F} > 0\}$ and $M^{3 \times 3}$ denotes the space of real 3 by 3 matrices. Since we are interested in modelling large deformations, we assume that the stored energy function W satisfies that $W \rightarrow \infty$ as either $\det \mathbf{F} \rightarrow 0^+$ or $\|\mathbf{F}\| \rightarrow \infty$.

We consider the problem of determining the equilibrium configuration of the body that satisfies (4) a.e., and satisfying the boundary condition:

$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x}, \quad \mathbf{x} \in \partial \mathcal{B}, \quad (5)$$

where $\lambda > 0$ is given.

We assume that the stored energy function, in units of energy per unit volume, describing the mechanical response of the body is given by

$$W(\mathbf{x}, \mathbf{F}) = \Phi(\mathbf{x}, v_1, v_2, v_3), \quad \mathbf{F} \in M_+^{3 \times 3}, \quad \mathbf{x} \in \mathcal{B}, \quad (6)$$

for some function $\Phi : \mathcal{B} \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ symmetric in its last three arguments, and where v_1, v_2, v_3 are the eigenvalues of $(\mathbf{F}^t \mathbf{F})^{1/2}$ known as the *principal stretches*. Note that for any fixed \mathbf{x} , the material response $W(\mathbf{x}, \cdot)$ corresponds to an isotropic and frame indifferent material.

We now restrict attention to the special case in which the deformation $\mathbf{u}(\cdot)$ is *radially symmetric*, so that

$$\mathbf{u}(\mathbf{x}) = r(R) \frac{\mathbf{x}}{R}, \quad \mathbf{x} \in \mathcal{B}, \quad (7)$$

for some scalar function $r(\cdot)$, where $R = \|\mathbf{x}\|$. In this case one can easily check that

$$v_1 = r'(R), \quad v_2 = v_3 = \frac{r(R)}{R}. \quad (8)$$

Assuming that the dependence of Φ on \mathbf{x} in (6) is only on $R = \|\mathbf{x}\|$, the total stored energy functional, due to internal mechanical and gravitational forces (see [6]), is given by (up to a multiplicative constant of 4π):

$$I(r) = I_{\text{mec}}(r) - I_{\text{pot}}(r), \quad (9)$$

where

$$I_{\text{mec}}(r) = \int_0^1 \Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) R^2 \, dR, \quad (10)$$

$$I_{\text{pot}}(r) = \int_0^1 \rho_0(R) \frac{M_R}{r(R)} R^2 \, dR, \quad (11)$$

are the mechanical and potential energy functionals respectively. Here ρ_0 is the mass density of the body (mass per unit volume) in the reference configuration, and

$$M_R = 4\pi \int_0^R \rho_0(u) u^2 \, du,$$

is the mass of the ball in the reference configuration of radius R and centered at the origin. We assume that

$$k_0 \leq \rho_0(R) \leq k_1, \quad 0 \leq R \leq 1, \quad (12)$$

for some positive constants k_0 and k_1 .

In accord with (4) we have the inequalities

$$r'(R), \frac{r(R)}{R} > 0, \quad 0 < R < 1, \quad (13)$$

and (5) reduces to:

$$r(1) = \lambda. \quad (14)$$

Our problem now is to minimize the functional $I(\cdot)$ over the set

$$\mathcal{A}_\lambda = \left\{ r \in W^{1,1}(0,1) \mid r(1) = \lambda, r'(R) > 0 \text{ a.e. for } R \in (0,1), \right. \\ \left. r(0) \geq 0, I_{\text{mec}}(r) < \infty \right\}. \quad (15)$$

Note that $\mathcal{A}_\lambda \neq \emptyset$ as $r_\lambda \in \mathcal{A}_\lambda$ where $r_\lambda(R) = \lambda R$.

3 Existence of minimizers

In this section we show that the functional $I(\cdot)$ in (9) has a minimizer over the set \mathcal{A}_λ in (15). The proofs of the results in this section are adaptations of the corresponding ones in [2] due to the presence of the potential energy functional (11). We do emphasize that they are not direct consequence of those in [6] as these are for maps in Sobolev spaces $W^{1,p}$ with $p > 3$ and thus they represent continuous deformations.

Throughout this section and the rest of the paper we assume that the stored energy function Φ in (10) satisfies that

$$\Phi(R, v_1, v_2, v_3) \geq \phi(v_1) + \phi(v_2) + \phi(v_3) + h(v_1 v_2 v_3), \quad R \in [0, 1], \quad (16)$$

where $\phi, h : (0, \infty) \rightarrow (0, \infty)$ are strictly convex and such that

H1: $\phi(v) \geq C v^\gamma$ for some positive constant C and $1 < \gamma < 3$;

H2: $\frac{h(d)}{d} \rightarrow \infty$, as $d \rightarrow \infty$;

H3: $h(d) \geq K d^{-s}$, $d > 0$, for some positive constant K and $s \geq \gamma^* = \frac{\gamma}{\gamma-1}$.

If we let

$$\delta_r(R) = r'(R) \left(\frac{r(R)}{R} \right)^2,$$

then the specialization of [6, Eqn. (31)] to the radial map (7) together with (12) gives that

$$\left| \int_0^1 \rho_0(R) \frac{M_R}{r(R)} R^2 dR \right| \leq C \left(\int_0^1 \delta_r(R)^{-s} R^2 dR \right)^{\frac{1}{3s}}, \quad (17)$$

for some positive constant C independent of $r \in \mathcal{A}_\lambda$. Using this we now have the following:

Lemma 3.1. *Under the growth assumption (16) with H3, the functional $I(\cdot)$ is bounded below on \mathcal{A}_λ .*

Proof: Combining (12), (16) with H3, and (17) we get for some positive constants K_1, K_2 that

$$I(r) \geq K_1 \int_0^1 \delta_r(R)^{-s} R^2 \, dR - K_2 \left(\int_0^1 \delta_r(R)^{-s} R^2 \, dR \right)^{\frac{1}{3s}},$$

for all $r \in \mathcal{A}_\lambda$. Since the function $g(x) = K_1 x - K_2 x^{\frac{1}{3s}}$ is bounded below for $x \geq 0$, the result follows. \square

Using this we can now establish the existence of minimizers for I over \mathcal{A}_λ .

Theorem 3.2. *Let the stored energy function Φ in (10) satisfy (16) with H1–H3. Then there exists $r_\lambda \in \mathcal{A}_\lambda$ such that*

$$I(r_\lambda) = \inf_{r \in \mathcal{A}_\lambda} I(r).$$

Proof: Since $\mathcal{A}_\lambda \neq \emptyset$, it follows from Lemma 3.1 that $\inf_{r \in \mathcal{A}_\lambda} I(r) \in \mathbb{R}$. Let (r_j) with $r_j \in \mathcal{A}_\lambda$ for all j , be an infimizing sequence, i.e.,

$$\inf_{r \in \mathcal{A}_\lambda} I(r) = \lim_{j \rightarrow \infty} I(r_j).$$

Since $(I(r_j))$ is bounded, it follows from the proof of Lemma 3.1 that the sequence

$$\left(K_1 \int_0^1 \delta_{r_j}(R)^{-s} R^2 \, dR - K_2 \left(\int_0^1 \delta_{r_j}(R)^{-s} R^2 \, dR \right)^{\frac{1}{3s}} \right), \quad (18)$$

is bounded. Hence the sequence

$$\left(\int_0^1 \delta_{r_j}(R)^{-s} R^2 \, dR \right),$$

must be bounded as well, and thus from (17) that $(I_{\text{pot}}(r_j))$ is bounded. From this and the boundedness of $(I(r_j))$, we get that $(I_{\text{mec}}(r_j))$ is bounded.

From the boundedness of $(I_{\text{mec}}(r_j))$ and (16), we get that

$$\left(\int_0^1 h(\delta_{r_j}(R)) R^2 \, dR \right),$$

is bounded. Let $\rho = R^3$ and $u_j(\rho) = r_j^3(\rho^{1/3})$. It follows now that

$$\dot{u}_j(\rho) = \frac{du_j}{d\rho}(\rho) = \delta_{r_j}(\rho^{1/3}),$$

and that the sequence

$$\left(\int_0^1 h(\dot{u}_j(\rho)) \, d\rho \right),$$

is bounded. It follows now from H1 and De La Vallée–Poussin Criterion that for some subsequence (\dot{u}_k) of (\dot{u}_j) , we have $\dot{u}_k \rightharpoonup w$ in $L^1(0, 1)$ for some $w \in L^1(0, 1)$, and that (\dot{u}_j) is equi-integrable. Using H3 is easy to show that $w > 0$ a.e. Letting

$$u(\rho) = \lambda^3 - \int_{\rho}^1 w(s) \, ds,$$

we get from the equi-integrability of (\dot{u}_j) that $u_k \rightarrow u$ in $C[0, 1]$. Thus $r_k \rightarrow r_{\lambda}$ in $C[0, 1]$ where $r_{\lambda}(R) = u(R^3)^{1/3}$. From these we can conclude $r_k \rightharpoonup r_{\lambda}$ in $W^{1,1}(\varepsilon, 1)$ and that $\delta_{r_j} \rightharpoonup \delta_{r_{\lambda}}$ in $L^1(\varepsilon, 1)$ for any $\varepsilon \in (0, 1)$. By the weak lower semi-continuity properties of $I_{\text{mec}}(\cdot)$ (cf. [3]), we get that

$$\begin{aligned} \int_{\varepsilon}^1 \Phi \left[R, r'_{\lambda}(R), \frac{r_{\lambda}(R)}{R}, \frac{r_{\lambda}(R)}{R} \right] R^2 \, dR &\leq \liminf_k \int_{\varepsilon}^1 \Phi \left[R, r'_k(R), \frac{r_k(R)}{R}, \frac{r_k(R)}{R} \right] R^2 \, dR, \\ &\leq \liminf_k I_{\text{mec}}(r_k) < \infty. \end{aligned}$$

We get now from the Monotone Convergence Theorem and the arbitrariness of ε that

$$I_{\text{mec}}(r_{\lambda}) \leq \liminf_k I_{\text{mec}}(r_k). \quad (19)$$

This together with the facts that $r_{\lambda}(0) \geq 0$, $r'_{\lambda}(R) \geq 0$ a.e., and $r_{\lambda}(1) = \lambda$, show that $r_{\lambda} \in \mathcal{A}_{\lambda}$.

To get that r_{λ} is a minimizer of I over \mathcal{A}_{λ} , we must still have to deal with the potentials $(I_{\text{pot}}(r_k))$. First note that

$$\begin{aligned} \left| I_{\text{pot}}(r_k) - I_{\text{pot}}(r_{\lambda}) \right| &= \left| \int_0^1 \frac{\rho_0(R) M_R R^2}{r_k(R) r_{\lambda}(R)} (r_{\lambda}(R) - r_k(R)) \, dR \right|, \\ &\leq \|r_{\lambda} - r_k\|_{C[0,1]} \left[\int_0^1 \frac{\rho_0(R) M_R R^2}{r_k^2(R)} \, dR \right]^{\frac{1}{2}} \left[\int_0^1 \frac{\rho_0(R) M_R R^2}{r_{\lambda}^2(R)} \, dR \right]^{\frac{1}{2}}, \end{aligned} \quad (20)$$

where in the last step we used a weighted Holder's inequality with weight $\rho_0(R) M_R R^2$. We now show that each of the two integrals on the right hand side of this inequality are bounded. Upon recalling (12), we can take $M_R \leq C R^2$ for some constant C . Hence

$$\begin{aligned} \int_0^1 \frac{\rho_0(R) M_R R^2}{r_k^2(R)} \, dR &\leq (\text{const}) \int_0^1 \frac{R^4}{r_k^2(R)} \, dR = (\text{const}) \int_0^1 \frac{R^2}{\delta_{r_k}(R)} r'_k(R) \, dR, \\ &\leq (\text{const}) \left[\int_0^1 \frac{R^2}{\delta_{r_k}(R)^{\gamma^*}} \, dR \right]^{\frac{1}{\gamma^*}} \left[\int_0^1 R^2 (r'_k(R))^{\gamma} \, dR \right]^{\frac{1}{\gamma}}. \end{aligned}$$

However by the weighted Holder's inequality (with weight R^2),

$$\left[\int_0^1 \frac{R^2}{\delta_{r_k}(R)^{\gamma^*}} \, dR \right]^{\frac{1}{\gamma^*}} \leq (\text{const}) \left[\int_0^1 \frac{R^2}{\delta_{r_k}(R)^s} \, dR \right]^{\frac{1}{s}},$$

with the sequence on the right hand side bounded. From (16) and H1 it follows that

$$\int_0^1 R^2 (r'_k(R))^\gamma dR \leq \int_0^1 R^2 \phi(r'_k(R)) dR \leq I_{\text{mec}}(r_k),$$

with the sequence on the right hand side bounded. Combining these results we can conclude that the sequence

$$\left(\int_0^1 \frac{\rho_0(R) M_R R^2}{r_k^2(R)} dR \right),$$

is bounded. Combining this with (20) we get that

$$\left| I_{\text{pot}}(r_k) - I_{\text{pot}(r_\lambda)} \right| \leq (\text{const}) \|r_\lambda - r_k\|_{C[0,1]} \rightarrow 0,$$

as $k \rightarrow \infty$, which together with (19) imply that

$$I(r_\lambda) \leq \liminf_k I(r_k) = \inf_{r \in \mathcal{A}_\lambda} I(r),$$

i.e., that r_λ is a minimizer. □

For our next result we shall need the following assumption: there exist constants $M, \varepsilon_0 \in (0, \infty)$ such that (cf. [4])

$$\left| \frac{\partial \Phi}{\partial v_k}(R, \alpha_1 v_1, \alpha_2 v_2, \alpha_3 v_3) v_k \right| \leq M [\Phi(R, v_1, v_2, v_3) + 1], \quad (21)$$

for all $R \in [0, 1]$, $k = 1, 2, 3$, and $|\alpha_i - 1| < \varepsilon_0$ for $i = 1, 2, 3$. The techniques in [2] can now be adapted to show the following result.

Theorem 3.3. *Let r be any minimizer of I over \mathcal{A}_λ . Assume that the function Φ satisfies (21). Then $r \in C^1(0, 1]$, $r'(R) > 0$ for all $R \in (0, 1]$, $R^{n-1} \Phi_1(R, r(R))$ is $C^1(0, 1]$, and*

$$\frac{d}{dR} [R^2 \Phi_{,1}(R, r(R))] = 2R \Phi_{,2}(R, r(R)) + R^2 \frac{\rho_0(R) M_R}{r^2(R)}, \quad 0 < R < 1, \quad (22)$$

subject to (14) and $r(0) \geq 0$, where:

$$\Phi_{,i}(R, r(R)) = \frac{\partial \Phi}{\partial v_i} \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right), \quad i = 1, 2. \quad (23)$$

Moreover, if $r(0) > 0$, then

$$\lim_{R \rightarrow 0^+} R^2 \Phi_{,1}(R, r(R)) = 0. \quad (24)$$

The radial component of the Cauchy stress is given by

$$T(R, r(R)) = \frac{R^2}{r^2(R)} \Phi_{,1}(R, r(R)). \quad (25)$$

Using (22) we now get that

$$\begin{aligned} \frac{dT}{dR}(R, r(R)) &= 2 \frac{R^2}{r^3(R)} \left[\frac{r(R)}{R} \Phi_{,2}(R, r(R)) - r'(R) \Phi_{,1}(R, r(R)) \right] \\ &\quad + R^2 \frac{\rho_0(R) M_R}{r^4(R)}. \end{aligned} \quad (26)$$

By the Baker–Ericksen inequality, the right hand side of this equation is positive whenever $r'(R) < \frac{r(R)}{R}$.

The material of the body \mathcal{B} is *homogeneous* if $\rho_0(R)$ is constant, still denoted by ρ_0 , and

$$\Phi(R, v_1, v_2, v_3) = \tilde{\Phi}(v_1, v_2, v_3). \quad (27)$$

In this case (22) reduces to

$$\frac{d}{dR} \left[R^2 \tilde{\Phi}_{,1}(r(R)) \right] = 2R \tilde{\Phi}_{,2}(r(R)) + \frac{4\pi}{3} \rho_0^2 \frac{R^5}{r^2(R)}, \quad 0 < R < 1, \quad (28)$$

where now

$$\tilde{\Phi}_{,i}(r(R)) = \frac{\partial \tilde{\Phi}}{\partial v_i} \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right), \quad i = 1, 2. \quad (29)$$

The radial component of the Cauchy stress is now given by

$$\tilde{T}(r(R)) = \frac{R^2}{r^2(R)} \tilde{\Phi}_{,1}(r(R)). \quad (30)$$

and (26) reduces to

$$\frac{d\tilde{T}}{dR}(r(R)) = 2 \frac{R^2}{r^3(R)} \left[\frac{r(R)}{R} \tilde{\Phi}_{,2}(r(R)) - r'(R) \tilde{\Phi}_{,1}(r(R)) \right] + \frac{4\pi}{3} \rho_0^2 \frac{R^5}{r^4(R)}. \quad (31)$$

4 Cavitation

In this section we show that when λ is sufficiently large, the minimizer r of $I(\cdot)$ over \mathcal{A}_λ has to have $r(0) > 0$. The proof given here of this fact is an adaptation (to the problem with self gravity) of the technique used in [10] to establish a similar fact for compressible inhomogeneous materials.

We assume for some positive constants c_0 and c_1 ,

$$c_0 \tilde{\Phi}(v_1, v_2, v_3) \leq \Phi(R, v_1, v_2, v_3) \leq c_1 \tilde{\Phi}(v_1, v_2, v_3), \quad (32)$$

for all $R \in [0, 1]$ and where $\tilde{\Phi}$ corresponds to an isotropic and frame indifferent material that satisfies (16) with H1–H3. We further assume that:

H4: For any $\eta > 1$,

$$\frac{v^2}{(v^3 - 1)^2} \tilde{\Phi} \left(\frac{1}{v^2}, v, v \right) \in L^1(\eta, \infty).$$

Theorem 4.1. *Let r be a minimizer of the functional (9) over (15) and assume that (32) holds where $\tilde{\Phi}$ satisfies (16) with H1–H4. Then for λ sufficiently large we must have that $r(0) > 0$.*

Proof: We consider an incompressible deformation given by

$$r_{inc}(R) = \sqrt[3]{R^3 + \lambda^3 - 1}.$$

It follows that $r_{inc} \in \mathcal{A}_\lambda$. If r_λ is any minimizer of $I(\cdot)$ over \mathcal{A}_λ , then

$$\begin{aligned} \Delta I &= I(r_{inc}) - I(r_\lambda) \\ &\leq c_1 k_1 \int_0^1 \tilde{\Phi}(r_{inc}(R)) R^2 \, dR - c_0 k_0 \int_0^1 \tilde{\Phi}(r_\lambda(R)) R^2 \, dR \\ &\quad - \int_0^1 \rho_0(R) \frac{M_R}{r_{inc}(R)} R^2 \, dR + \int_0^1 \rho_0(R) \frac{M_R}{r_\lambda(R)} R^2 \, dR. \end{aligned}$$

By [9, Pro. 4.10] we have that for $\lambda_1 \leq \lambda_2$

$$r_{\lambda_1}(R) \leq r_{\lambda_2}(R), \quad 0 \leq R \leq 1.$$

Hence for λ_0 fixed, we get that for $\lambda \geq \lambda_0$,

$$\int_0^1 \rho_0(R) \frac{M_R}{r_\lambda(R)} R^2 \, dR \leq \int_0^1 \rho_0(R) \frac{M_R}{r_{\lambda_0}(R)} R^2 \, dR.$$

It follows now that

$$\begin{aligned} \Delta I &\leq c_1 k_1 \int_0^1 \tilde{\Phi}(r_{inc}(R)) R^2 \, dR - c_0 k_0 \int_0^1 \tilde{\Phi}(r_\lambda(R)) R^2 \, dR \\ &\quad + \int_0^1 \rho_0(R) \frac{M_R}{r_{\lambda_0}(R)} R^2 \, dR. \end{aligned}$$

If $r_\lambda(0) = 0$, it follows (cf. [2]) that

$$\int_0^1 \tilde{\Phi}(r_\lambda(R)) R^2 \, dR \geq \int_0^1 \tilde{\Phi}(r_{hom}(R)) R^2 \, dR,$$

where $r_{hom}(R) = \lambda R$. Hence

$$\begin{aligned} \Delta I &\leq c_1 k_1 \int_0^1 \tilde{\Phi}(r_{inc}(R)) R^2 \, dR - c_0 k_0 \int_0^1 \tilde{\Phi}(r_{hom}(R)) R^2 \, dR \\ &\quad + \int_0^1 \rho_0(R) \frac{M_R}{r_{\lambda_0}(R)} R^2 \, dR. \end{aligned}$$

Since the third integral on the right hand side of this inequality is fixed, the result now follows as in [10] using H2 and H4. \square

5 No cavitation results

In this section we give conditions under which the minimizer of $I(\cdot)$ over \mathcal{A}_λ for λ sufficiently small, must satisfy that $r(0) = 0$.

We first consider the case of a homogeneous material for which (27) holds. The functional I is now given by

$$I(r) = \int_0^1 \left[\tilde{\Phi} \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \rho_0(R) \frac{M_R}{r(R)} \right] R^2 dR \quad (33)$$

We denote by λ_c^h the critical boundary displacement for the cavitation problem considered in [2] with stored energy function $\tilde{\Phi}$.

Theorem 5.1. *Let r be a minimizer of (33) over \mathcal{A}_λ . If $\lambda < \lambda_c^h$, then $r(0) = 0$.*

Proof: To argue by contradiction, assume that $r(0) > 0$. For $\hat{\lambda} = (\lambda + \lambda_c^h)/2$, we let

$$R_0 = \inf \left\{ R \mid r(R) = \hat{\lambda}R \right\}.$$

Since $\lambda < \lambda_c^h$ and $\frac{r(R)}{R} \rightarrow \infty$ as $R \rightarrow 0^+$, we get that R_0 is well defined and $R_0 > 0$. We define

$$\hat{r}(R) = \begin{cases} \hat{\lambda}R & , \quad R \leq R_0, \\ r(R) & , \quad R > R_0. \end{cases}$$

It follows that $\hat{r} \in \mathcal{A}_\lambda$. Now

$$\begin{aligned} \Delta I &= I(r) - I(\hat{r}) = \int_0^{R_0} \left[\tilde{\Phi} \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \tilde{\Phi} \left(\hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 dR \\ &\quad + \int_0^{R_0} \rho_0(R) M_R \left[\frac{1}{\hat{r}(R)} - \frac{1}{r(R)} \right] R^2 dR, \\ &\geq \int_0^{R_0} \left[\tilde{\Phi} \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \tilde{\Phi} \left(\hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 dR, \end{aligned}$$

as $\hat{r}(R) \leq r(R)$ for $R \leq R_0$. Since $\hat{\lambda} < \lambda_c^h$ and $r(0) > 0$, the results in [2] imply that

$$\int_0^{R_0} \tilde{\Phi} \left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) R^2 dR > \int_0^{R_0} \tilde{\Phi} \left(\hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) R^2 dR,$$

and thus that $\Delta I > 0$ which contradicts the minimality of r . \square

For the next result we take $\beta \equiv 0$ and $\gamma \equiv 1$ in (34). We let d_0 be the value of the argument at which h assumes its global minimum value. Note that $h'(d) < 0$ for $d < d_0$.

Theorem 5.2. *Let r be a minimizer of (9) over \mathcal{A}_λ corresponding to the stored energy function (34). Assume that $\alpha'(R) \leq 0$ for all R and that $\phi'(t) \geq 0$ for all t . Then if $\lambda^3 < d_0$ we must have that $r(0) = 0$.*

Proof: As in the proof of Theorem 5.1, we argue by contradiction. Thus we assume that $r(0) > 0$ and let $\hat{\lambda} = (\lambda + d_0^{\frac{1}{3}})/2$. Define R_0 and \hat{r} as in the proof of Theorem 5.1. Then

$$\begin{aligned}\Delta I &= I(r) - I(\hat{r}) = \int_0^{R_0} \left[\Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \Phi \left(R, \hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 dR \\ &\quad + \int_0^{R_0} \rho_0(R) M_R \left[\frac{1}{\hat{r}(R)} - \frac{1}{r(R)} \right] R^2 dR, \\ &\geq \int_0^{R_0} \left[\Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \Phi \left(R, \hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 dR,\end{aligned}$$

as $\hat{r}(R) \leq r(R)$ for $R \leq R_0$. Using the convexity of ϕ , ψ , and h in (34), we have now that (see [2, Page 589]):

$$\begin{aligned}\left[\Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \Phi \left(R, \hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 &\geq \\ \alpha(R) \phi'(\hat{\lambda}) \left(R^2 r'(R) + 2Rr(R) - 3\hat{\lambda}R^2 \right) + h'(\hat{\lambda}^3) \left(r^2(R)r'(R) - \hat{\lambda}^3 R^2 \right), \\ &= \alpha(R) \phi'(\hat{\lambda}) (r(R)R^2 - \hat{\lambda}R^3)' + \frac{1}{3} h'(\hat{\lambda}^3) (r^3(R) - \hat{\lambda}^3 R^3)'.\end{aligned}$$

It follows now, after integrating by parts, that

$$\begin{aligned}\int_0^{R_0} \left[\Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) - \Phi \left(R, \hat{\lambda}, \hat{\lambda}, \hat{\lambda} \right) \right] R^2 dR &\geq \\ -\phi'(\hat{\lambda}) \int_0^{R_0} \alpha'(R) (r(R)R^2 - \hat{\lambda}R^3) dR - \frac{1}{3} h'(\hat{\lambda}^3) r^3(0) &\geq -\frac{1}{3} h'(\hat{\lambda}^3) r^3(0) > 0.\end{aligned}$$

Thus $\Delta I > 0$ which contradicts the minimality of r . \square

Our next results is for inhomogeneous materials of the form

$$\Phi(R, v_1, v_2, v_3) = \alpha(R) \sum_i \phi(v_i) + \beta(R) \sum_{i<j} \psi(v_i v_j) + \gamma(R) h(v_1 v_2 v_3), \quad (34)$$

where α , β , and γ are smooth positive functions over $[0, 1]$, ϕ and ψ are nonnegative convex functions and with h strictly convex. In this case it easy to see that for some constant $C > 0$,

$$\left| \frac{\partial \Phi}{\partial R}(R, v_1, v_2, v_3) \right| \leq C \tilde{\Phi}(v_1, v_2, v_3), \quad (35)$$

where

$$\tilde{\Phi}(v_1, v_2, v_3) = \sum_i \phi(v_i) + \sum_{i<j} \psi(v_i v_j) + h(v_1 v_2 v_3). \quad (36)$$

We now denote by λ_c^h the critical boundary displacement corresponding to the stored energy function $\Phi(0, v_1, v_2, v_3)$. Note that any deformation with finite $\Phi(0, v_1, v_2, v_3)$

energy, has finite $\tilde{\Phi}$ energy as well. The following result is reminiscent to [10, Proposition 12]. It shows that any minimizer which leaves the centre intact must have strains at the origin less than λ_c^h .

Theorem 5.3. *Let $r \in \mathcal{A}_\lambda$ satisfy $r(0) = 0$ and that $\ell \in (0, \infty]$ where*

$$\lim_{R \rightarrow 0^+} r'(R) = \ell.$$

If $\ell > \lambda_c^h$, then r can not be a minimizer of $I(\cdot)$ over \mathcal{A}_λ .

Proof: The following proof is similar to the one of [10, Proposition 12] except for the treatment of the gravitational potential and the specific dependence on R of the stored energy function.

For $\varepsilon \in (0, 1)$ we let

$$\lambda(\varepsilon) = \frac{r(\varepsilon)}{\varepsilon}.$$

From the given hypotheses, it follows that

$$\lambda(\varepsilon) \rightarrow \ell, \quad \text{as } \varepsilon \rightarrow 0^+.$$

Assume for the moment that ℓ is finite, and let r_c be a cavitating extrema corresponding to $\Phi(0, v_1, v_2, v_3)$. We define

$$r_\varepsilon(R) = \begin{cases} \alpha_\varepsilon r_c\left(\frac{R}{\alpha_\varepsilon}\right) & , \quad R \in [0, \varepsilon], \\ r(R) & , \quad R \in (\varepsilon, 1], \end{cases} \quad (37)$$

where α_ε is such that $\alpha_\varepsilon r_c(\varepsilon/\alpha_\varepsilon) = \lambda(\varepsilon)\varepsilon$. That α_ε exists follows from the fact that $\lambda(\varepsilon) > \lambda_c^h$ for ε sufficiently small. Now

$$\begin{aligned} \Delta I &= I(r_\varepsilon) - I(r) \\ &= \int_0^\varepsilon R^2 \left[\Phi \left(R, r'_c\left(\frac{R}{\alpha_\varepsilon}\right), \frac{\alpha_\varepsilon r_c\left(\frac{R}{\alpha_\varepsilon}\right)}{R}, \frac{\alpha_\varepsilon r_c\left(\frac{R}{\alpha_\varepsilon}\right)}{R} \right) \right. \\ &\quad \left. - \Phi \left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R} \right) \right] dR \\ &\quad - \int_0^\varepsilon \rho_0(R) \frac{M_R}{\alpha_\varepsilon r_c\left(\frac{R}{\alpha_\varepsilon}\right)} R^2 dR + \int_0^\varepsilon \rho_0(R) \frac{M_R}{r(R)} R^2 dR \end{aligned}$$

With the change of variables $R = \varepsilon U$ with $U \in [0, 1]$, we can write the above as

$$\begin{aligned} \Delta I &= \varepsilon^3 \int_0^1 U^2 \left[\Phi \left(\varepsilon U, r'_c\left(\frac{\varepsilon U}{\alpha_\varepsilon}\right), \frac{\alpha_\varepsilon r_c\left(\frac{\varepsilon U}{\alpha_\varepsilon}\right)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c\left(\frac{\varepsilon U}{\alpha_\varepsilon}\right)}{\varepsilon U} \right) \right. \\ &\quad \left. - \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \right] dU \end{aligned}$$

$$- \int_0^\varepsilon \rho_0(R) \frac{M_R}{\alpha_\varepsilon r_c(R/\alpha_\varepsilon)} R^2 dR + \int_0^\varepsilon \rho_0(R) \frac{M_R}{r(R)} R^2 dR$$

We first examine the gravitational integrals. For this we use that $\rho_0(\cdot)$ is nonnegative and bounded above, and that M_R is bounded by a constant times R^3 . Since

$$\frac{r_c(\varepsilon/\alpha_\varepsilon)}{\varepsilon/\alpha_\varepsilon} = \lambda(\varepsilon) \rightarrow \ell \quad \text{as } \varepsilon \rightarrow 0^+,$$

we get that

$$\frac{\varepsilon}{\alpha_\varepsilon} \rightarrow \mu,$$

where $\mu > 0$ and $r_c(\mu)/\mu = \ell$. Upon recalling that $\frac{r_c(S)}{S}$ is a decreasing function of S , we have that:

$$\begin{aligned} \int_0^\varepsilon \rho_0(R) \frac{M_R}{\alpha_\varepsilon r_c(R/\alpha_\varepsilon)} R^2 dR &= \varepsilon^3 \int_0^1 \rho_0(\varepsilon U) \frac{M_{\varepsilon U}}{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)} U^2 dU \\ &\leq K_1 \varepsilon^5 \int_0^1 \frac{U^4}{\alpha_\varepsilon r_c(\varepsilon/\alpha_\varepsilon)/\varepsilon} dU \leq \frac{K \varepsilon^5}{5\ell}. \end{aligned}$$

As $r(0) = 0$ and $\ell > 0$, the function $r(R)/R$ is positive and continuous in $[0, 1]$. Thus if v_0 is its minimum value, we have that for some positive constant L :

$$\int_0^\varepsilon \rho_0(R) \frac{M_R}{r(R)} R^2 dR = \varepsilon^3 \int_0^1 \rho_0(\varepsilon U) \frac{M_{\varepsilon U}}{r(\varepsilon U)} U^2 dU \leq \frac{M \varepsilon^5}{5v_0}.$$

Thus both gravitational potential terms go to zero faster than ε^3 .

We now examine the mechanical potential terms in ΔI . For this we note that

$$\begin{aligned} &\int_0^1 U^2 \left[\Phi \left(\varepsilon U, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right. \\ &\quad \left. - \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \right] dU = \\ &\int_0^1 U^2 \left\{ \left[\Phi \left(\varepsilon U, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right. \right. \\ &\quad \left. \left. - \Phi \left(0, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right] \right. \\ &\quad \left. + \left[\Phi \left(0, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right. \right. \\ &\quad \left. \left. - \Phi(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon)) \right] \right\} \\ &\quad + \left[\Phi(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon)) - \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \right] \} dU \end{aligned} \quad (38)$$

From (35) and Taylor's Theorem, we have that

$$\begin{aligned} & \int_0^1 U^2 \left[\Phi \left(\varepsilon U, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right. \\ & \quad \left. - \Phi \left(0, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right] dU \\ & \leq C_1 \varepsilon \int_0^1 U^3 \tilde{\Phi} \left(r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) dU \leq C_2 \varepsilon, \end{aligned}$$

where we used that r_c has finite $\tilde{\Phi}$ energy. Thus the first bracketed term in (38) goes to zero with ε . For the third term, we note that the functions $r'(S)$ and $\frac{r(S)}{S}$ are $C[0, 1]$ and positive. Hence for some $M > 0$,

$$\left| \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \right| \leq M, \quad \forall U,$$

and since

$$\Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \rightarrow \Phi(0, \ell, \ell, \ell),$$

pointwise, we get by the Lebesgue dominated convergence theorem that

$$\int_0^1 U^2 \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) dU \rightarrow \int_0^1 U^2 \Phi(0, \ell, \ell, \ell) dU,$$

as $\varepsilon \rightarrow 0^+$. This together with $\lambda(\varepsilon) \rightarrow \ell$ yields that

$$\int_0^1 U^2 \left[\Phi(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon)) - \Phi \left(\varepsilon U, r'(\varepsilon U), \frac{r(\varepsilon U)}{\varepsilon U}, \frac{r(\varepsilon U)}{\varepsilon U} \right) \right] dU \rightarrow 0,$$

as $\varepsilon \rightarrow 0^+$. Thus the third term in (38) goes to zero with ε as well.

For the second term in (38), note that with the change of variables $Z = (\varepsilon/\alpha_\varepsilon)U$, we get

$$\begin{aligned} & \int_0^1 U^2 \Phi \left(0, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) dU = \\ & \quad \left(\frac{\alpha_\varepsilon}{\varepsilon} \right)^3 \int_0^{\varepsilon/\alpha_\varepsilon} Z^2 \Phi \left(0, r'_c(Z), \frac{r_c(Z)}{Z}, \frac{r_c(Z)}{Z} \right) dZ \\ & \quad \rightarrow \frac{1}{\mu^3} \int_0^\mu Z^2 \Phi \left(0, r'_c(Z), \frac{r_c(Z)}{Z}, \frac{r_c(Z)}{Z} \right) dZ \\ & \quad = \int_0^1 U^2 \Phi \left(0, r'_c(\mu U), \frac{r_c(\mu U)}{\mu U}, \frac{r_c(\mu U)}{\mu U} \right) dU. \end{aligned}$$

It follows now that

$$\int_0^1 U^2 \left[\Phi \left(0, r'_c(\varepsilon U/\alpha_\varepsilon), \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U}, \frac{\alpha_\varepsilon r_c(\varepsilon U/\alpha_\varepsilon)}{\varepsilon U} \right) \right]$$

$$\begin{aligned}
& -\Phi(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon)) \Big] dR \rightarrow \\
& \int_0^1 U^2 \left[\Phi \left(0, r'_c(\mu U), \frac{r_c(\mu U)}{\mu U}, \frac{r_c(\mu U)}{\mu U} \right) - \Phi(0, \ell, \ell, \ell) \right] dR < 0,
\end{aligned}$$

where the last inequality follows since $\ell > \lambda_c^h$ and $\tilde{r}(U) = \mu^{-1}r_c(\mu U)$ is the minimizer for the functional with stored energy $\Phi(0, v_1, v_2, v_3)$ and boundary condition $\tilde{r}(1) = \ell$. Collecting all of the intermediate results so far, we get that $\varepsilon^{-3}\Delta I < 0$ for ε sufficiently small, which contradicts the minimality of r .

The case $\ell = \infty$, can be handled in a similar fashion using a suitable incompressible deformation on $[0, \varepsilon]$ in (37). See [10, Proposition 12] for details. \square

6 Numerical results

In this section we present some numerical results that confirm some of the results of previous sections. We employ two numerical schemes: a descent method for the minimization of (9) based on a gradient flow iteration; and a shooting method that solves directly the Euler-Lagrange boundary value problem (14), (22), and (24). The use of adaptive ode solvers in the shooting method allows for a more precise computation of the equilibrium states, especially near $R = 0$ where both strains in our problem tend to develop boundary layers. After the equilibrium is computed via the shooting method, it is compared to the results of the descent iteration in order to get some assurance of its minimizing character.

A *gradient flow* iteration (cf. [8]) assumes that r depends on a flow parameter t , and that $r(R, t)$ satisfies

$$\begin{aligned}
\frac{d^2}{dR^2}(r_t(R, t)) &= -\frac{d}{dR} [R^2\Phi_{,1}(R, r(R, t))] + 2R\Phi_{,2}(R, r(R, t)) \\
&+ R^2 \frac{\rho_0(R)M_R}{r^2(R, t)}, \quad 0 < R < 1, \quad t > 0,
\end{aligned} \tag{39}$$

$$r(1, t) = \lambda, \quad \lim_{R \rightarrow 0^+} \left[\frac{d}{dR}(r_t(R, t)) + R^2\Phi_{,1}(R, r(R, t)) \right] = 0, \quad t \geq 0. \tag{40}$$

(Here $r_t = \frac{\partial r}{\partial t}$.) The gradient flow equation leads to a descent method for the minimization of (9) over (15). After discretization of the partial derivative with respect to “ t ”, one can use a finite element method to solve the resulting flow equation. In particular, if we let $\Delta t > 0$ be given, and set $t_{i+1} = t_i + \Delta t$ where $t_0 = 0$, we can approximate $r_t(R, t_i)$ with:

$$z_i(R) = \frac{r_{i+1}(R) - r_i(R)}{\Delta t},$$

where $r_i(R) = r(R, t_i)$, etc. (We take $r_0(R)$ to be some initial deformation satisfying the boundary condition at $R = 1$, e.g., λR .) Inserting this approximation into the weak

form of (39), (40), and evaluating the right hand side of (39) at $r = r_i$, we arrive at the following iterative formula:

$$\int_0^1 z'_i(R)v'(R) dR + \int_0^1 \left[R^2\Phi_{,1}(R, r_i(R))v'(R) + \left(2R\Phi_{,2}(R, r_i(R)) + R^2 \frac{\rho_0(R)M_R}{r_i^2(R)} \right) v(R) \right] dR = 0, \quad (41)$$

for all functions v such that $v(1) = 0$ and sufficiently smooth so that the integrals above are well defined. Given r_i , one can solve the above equation for z_i via some finite element scheme, and then set $r_{i+1} = r_i + \Delta t z_i$. This process is repeated for $i = 0, 1, \dots$, until $r_{i+1} - r_i$ is “small” enough, or some maximum value of “ t ” is reached, declaring the last r_i as an approximate minimizer of (9) over (15).

In the shooting method technique, for given $\nu > 0$, we solve the initial value problem

$$\frac{d}{dR} [R^2\Phi_{,1}(R, r(R))] = 2R\Phi_{,2}(R, r(R)) + R^2 \frac{\rho_0(R)M_R}{r^2(R)}, \quad 0 < R < 1, \quad (42a)$$

$$r(1) = \lambda, \quad r'(1) = \nu, \quad (42b)$$

from $R = 1$ to $R = 0$. The value of ν is adjusted so that

$$\lim_{R \rightarrow 0^+} R^2\Phi_{,1} \left(R, \nu, \frac{r(R)}{R}, \frac{r(R)}{R} \right) = 0. \quad (43)$$

In actual calculations we solve (42) from $R = 1$ to $R = \varepsilon$, where $\varepsilon > 0$ is small, and replace (43) with

$$\Phi_{,1} \left(\varepsilon, \nu, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon} \right) = 0. \quad (44)$$

This equation is solved for ν via a secant type iteration which requires repeated solutions of the initial value problem (42) from $R = 1$ to $R = \varepsilon$. These intermediate initial value problems are solved with the routine `ode45` of the MATLAB™ ode suite.

Our first set of simulations are for the homogeneous case (33) and for which the stored energy function $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(v_1, v_2, v_3) = \frac{\kappa}{p} (v_1^p + v_2^p + v_3^p) + h(v_1 v_2 v_3),$$

for which

$$h(d) = C d^\gamma + D d^{-\delta},$$

where $p < 3$, $C \geq 0$, $D \geq 0$ and $\gamma, \delta > 0$. The reference configuration is mechanically stress free provided:

$$D = \frac{\kappa + C\gamma}{\delta}.$$

The mass density function ρ_0 is taken to be a constant. For the simulations we used the following values of the mechanical parameters in $\tilde{\Phi}$:

$$p = 2, \quad \kappa = 1, \quad C = 1, \quad \gamma = \delta = 2,$$

with a value of $\varepsilon = 0.001$ in (44) and in the shooting method. The gradient flow iteration was used as a predictor for the shooting method, with the integrals in (41) computed over $(\varepsilon, 1)$ as well.

In our first simulation we show the approximate cavity radius as a function of ρ_0 and λ , for values of $\rho_0 \in [0.5, 1.5]$ and $\lambda \in [0.9, 1.2]$. The resulting surface is shown in Figure 1. We note that for fixed values of ρ_0 , the cavity size is essentially zero up to some certain value of λ (the critical boundary displacement corresponding to ρ_0), after which the graph becomes concave. This critical boundary displacement appears to be an increasing function of ρ_0 . In Figure 2 we show the corresponding surface for the energies of the approximate minimizers. For fixed values of λ , the energy is an increasing function of ρ_0 , while for ρ_0 constant the energy is non monotone with respect to λ with a convex shape.

Our next simulations are for fixed values of λ and ρ_0 . In the first case $\lambda = 1$ and $\rho_0 = 1$. In Figure 3 we show the computed minimizer r compared to the affine deformation λR . The value of $r(0.001)$ is 7.7078×10^{-4} with an energy of 0.49074. Figure 4 shows plots of the strains $r'(R)$ and $\frac{r(R)}{R}$, and the determinant $r'(R)(r(R)/R)^2$ in this case. Also in Figure 5 we show the graph of the corresponding Cauchy stress (30). We note the boundary layer close to $R = \varepsilon$ in these plots. This boundary layer is a numerical artefact since the numerical scheme tries to make the Cauchy stress zero, while for this value of λ the value of $r(0)$ should be zero. Still in this case the numerical scheme converges to the solution with $r(0) = 0$ as $\varepsilon \rightarrow 0^+$ (cf. [9, Section 5]).

In our last simulation we take $\lambda = 1.15$ and $\rho_0 = 1$. In Figure 6 we show the computed minimizer r which corresponds to a cavitating solution. The value of $r(0.001)$ is 0.48346 with an energy of 0.91034. Figure 7 shows plots of the strains $r'(R)$ and $\frac{r(R)}{R}$, and the determinant $r'(R)(r(R)/R)^2$, and in Figure 8 we show the graph of the corresponding Cauchy stress. The boundary layer in $r(R)/R$ is now a “true” one associated with the computed cavitating solution.

7 Final comments

The usual self gravitating problem is that in which the centre of the body remains intact ($r(0) = 0$) and no condition is explicitly prescribed on the outer boundary. This is the problem considered in [7] and is a special case of one of the problems treated in [6]. It is straightforward to check that our results hold in this case as well where the admissible set is now given by

$$\mathcal{A} = \{r \in W^{1,1}(0, 1) \mid r(0) = 0, r'(R) > 0 \text{ a.e. for } R \in (0, 1), I_{\text{mec}}(r) < \infty\}.$$

In particular minimizers exist for all densities ρ_0 and satisfy the Euler–Lagrange equation (22) with natural boundary condition at $R = 1$ given by $T(1, r(1)) = 0$ (cf. (25)). In reference to (1), the growth condition (16) with H3 places our stored energy function under the “strong” compressibility category. Thus our result on the existence of minimizers with the centre intact for all densities ρ_0 , is consistent with the results in [7] which in turn suggest that there might be uniqueness of solutions of (22) in our problem as well.

A The gravitational potential

In this section we show how the potential term in the energy functional (9) is obtained or follows from the corresponding three dimensional potential energy functional.

Let \mathcal{B} be the unit ball with centre at the origin and ρ_0 be the mass density (per volume) in the reference configuration. We get an expression for the potential given in [6] by¹

$$V(\mathbf{u}) = \frac{1}{2} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{\rho_0(\mathbf{x})\rho_0(\mathbf{y})}{\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|} \, d\mathbf{y}d\mathbf{x},$$

when \mathbf{u} is radial (cf. (7)) and ρ_0 is radial as well. By symmetry, we can set the vertical or z axis in the inner integral to be along the direction of $\mathbf{u}(\mathbf{x})$. Then by considering a triangle with sides $\|\mathbf{u}(\mathbf{x})\| = r(R)$, $\|\mathbf{u}(\mathbf{y})\| = r(U)$, and $\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|$, we get that

$$\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\| = [r(R)^2 + r(U)^2 - 2r(R)r(U) \cos \phi]^{\frac{1}{2}},$$

where $R = \|\mathbf{x}\|$, $U = \|\mathbf{y}\|$, and ϕ is the angle between the vertical direction (along $\mathbf{u}(\mathbf{x})$) and $\mathbf{u}(\mathbf{y})$. Using these, and writing $\rho_0(\|\mathbf{x}\|)$ for $\rho_0(\mathbf{x})$, we get that $V(\mathbf{u}) = V(r)$ where

$$\begin{aligned} 2V(r) &= 4\pi \int_0^1 \rho_0(R)R^2 \left[2\pi \int_0^1 \int_0^\pi \frac{\rho_o(U) \sin \phi U^2}{[r(R)^2 + r(U)^2 - 2r(R)r(U) \cos \phi]^{\frac{1}{2}}} \, d\phi dU \right] dR, \\ &= 4\pi \int_0^1 \rho_0(R)R^2 \left[2\pi \int_0^1 \frac{\rho_o(U)U^2}{r(R)r(U)} (r(R) + r(U) - |r(R) - r(U)|) \, dU \right] dR, \\ &= 4\pi \int_0^1 \rho_0(R)R^2 \left[2\pi \int_0^R \frac{\rho_o(U)U^2}{r(R)r(U)} 2r(U) \, dU + 2\pi \int_R^1 \frac{\rho_o(U)U^2}{r(R)r(U)} 2r(R) \, dU \right] dR, \\ &= 4\pi \int_0^1 \rho_0(R)R^2 \left[\frac{M_R}{r(R)} + 4\pi \int_R^1 \frac{\rho_o(U)U^2}{r(U)} \, dU \right] dR. \end{aligned}$$

Integrating by parts we get that

$$\int_0^1 \rho_0(R)R^2 \left[4\pi \int_R^1 \frac{\rho_o(U)U^2}{r(U)} \, dU \right] dR = \int_0^1 \frac{\rho_0(R)M_R}{r(R)} R^2 \, dR.$$

Hence

$$V(r) = 4\pi \int_0^1 \frac{\rho_0(R)M_R}{r(R)} R^2 \, dR.$$

¹The gravitational constant G in this expression has been normalized to $\frac{1}{2}$.

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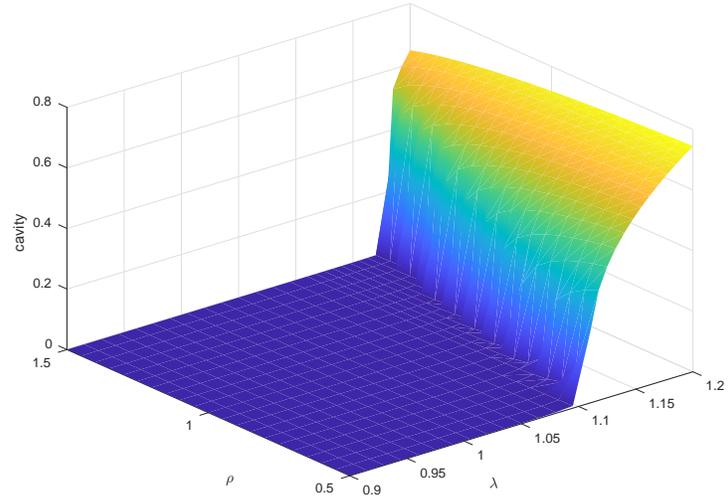


Figure 1: Cavity surface.

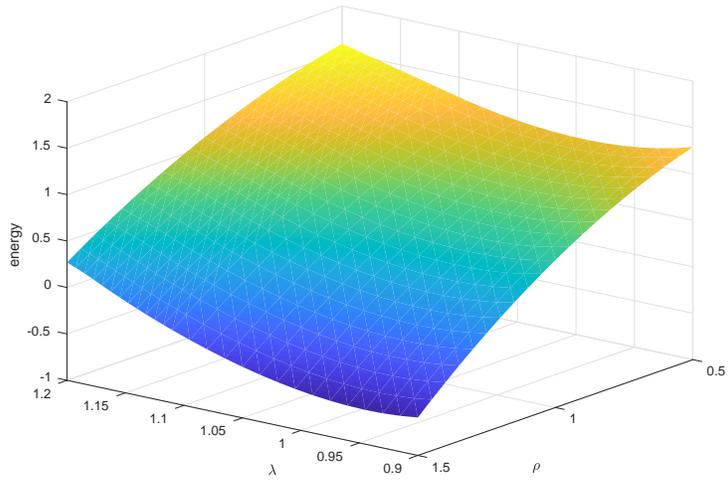


Figure 2: Energy surface.

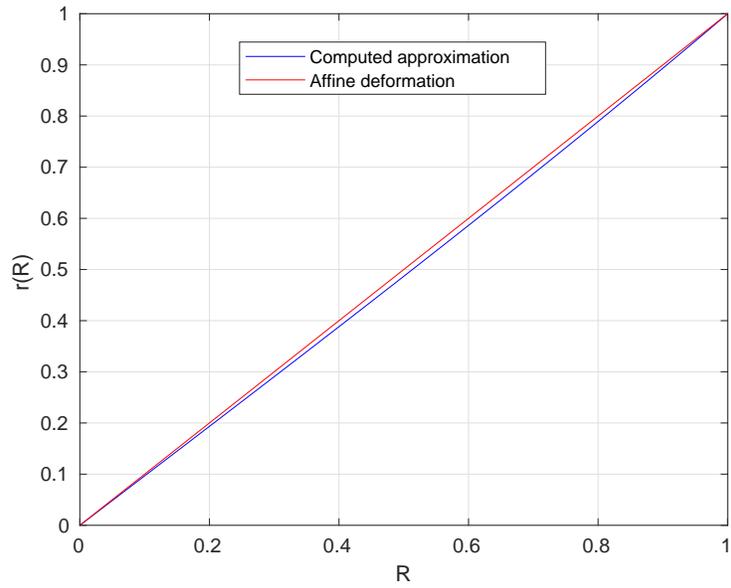


Figure 3: Radial displacement for $\lambda = 1$ and $\rho_0 = 1$.

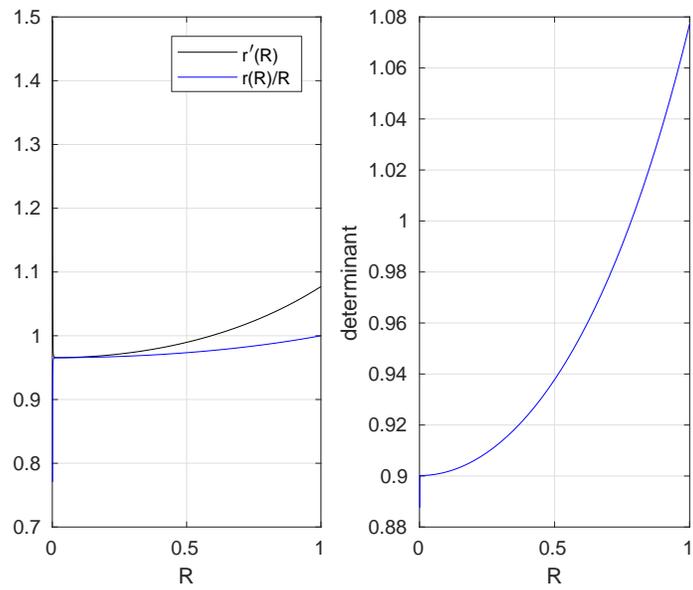


Figure 4: Strains and determinant for $\lambda = 1$ and $\rho_0 = 1$.

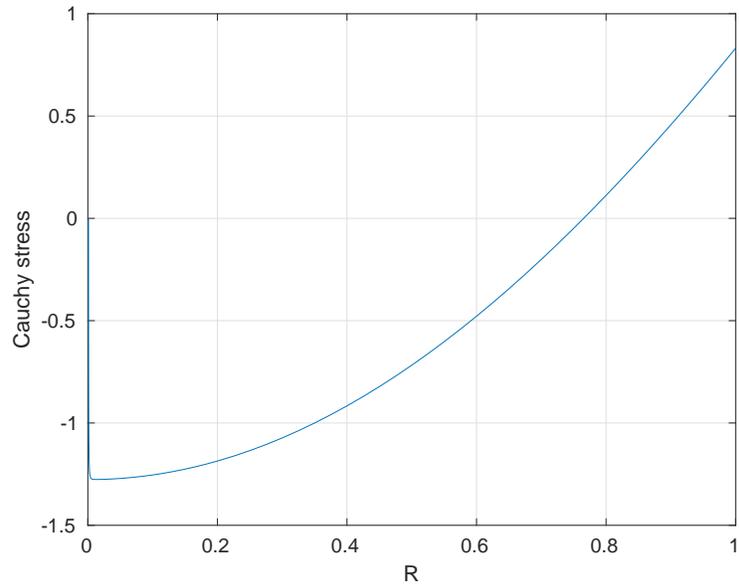


Figure 5: Cauchy stress for $\lambda = 1$ and $\rho_0 = 1$.

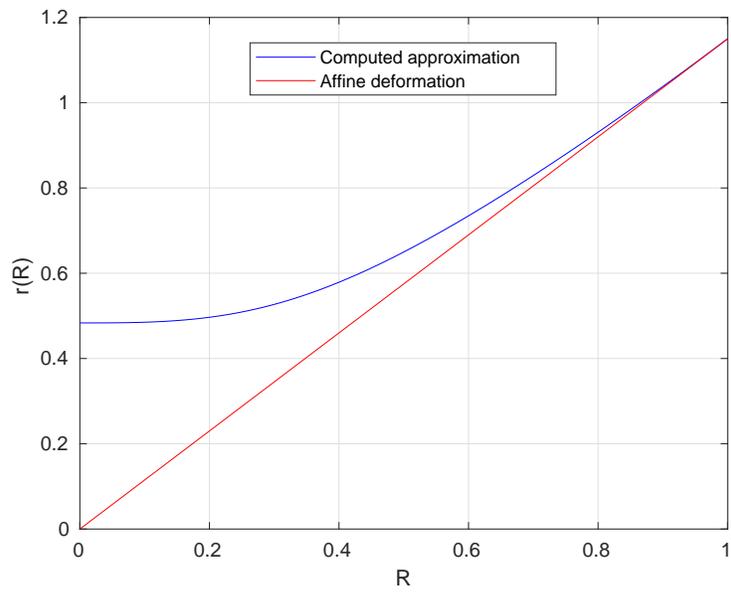


Figure 6: Radial displacement for $\lambda = 1.15$ and $\rho_0 = 1$.

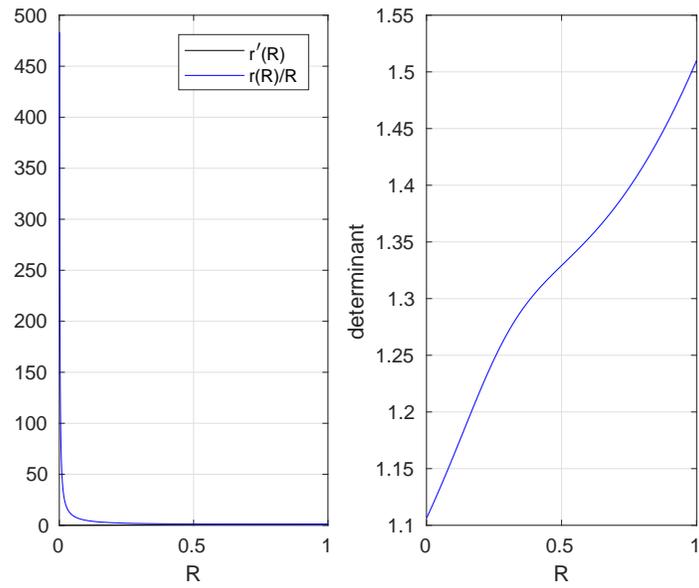


Figure 7: Strains and determinant for $\lambda = 1.15$ and $\rho_0 = 1$.

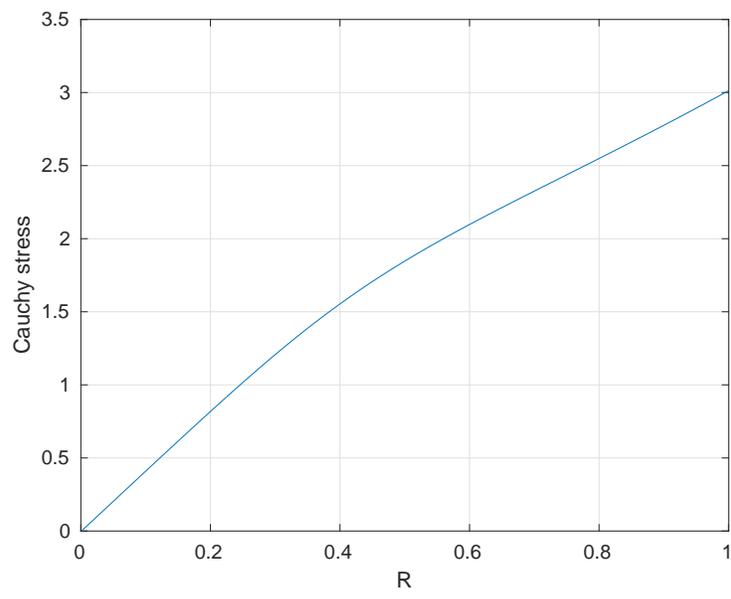


Figure 8: Cauchy stress for $\lambda = 1.15$ and $\rho_0 = 1$.