Eigenstate thermalization scaling in approaching the classical limit

Goran Nakerst¹ and Masudul Haque^{1,2}

According to the eigenstate thermalization hypothesis (ETH), the eigenstate-to-eigenstate fluctuations of expectation values of local observables should decrease with increasing system size. In approaching the thermodynamic limit — the number of sites and the particle number increasing at the same rate — the fluctuations should scale as $\sim D^{-1/2}$ with the Hilbert space dimension D. Here, we study a different limit — the classical or semiclassical limit — by increasing the particle number in fixed lattice topologies. We focus on the paradigmatic Bose-Hubbard system, which is quantum-chaotic for large lattices and shows mixed behavior for small lattices. We derive expressions for the expected scaling, assuming ideal eigenstates having Gaussian-distributed random components. We show numerically that, for larger lattices, ETH scaling of physical mid-spectrum eigenstates follows the ideal (Gaussian) expectation, but for smaller lattices, the scaling occurs via a different exponent. We examine several plausible mechanisms for this anomalous scaling.

I. INTRODUCTION

During the last decade and half, a considerable amount of research has focused on understanding how isolated quantum systems can relax and thermalize. A cornerstone of this understanding is the Eigenstate Thermalization Hypothesis (ETH) [1–9]. According to the ETH, the diagonal matrix elements of observables in the eigenstate basis of the Hamiltonian, the eigenstate expectation values (EEV), coincide locally with the micro-canonical ensemble. This means that the EEVs vary smoothly as a function of energy eigenvalues. The smoothness is quantified as the fluctuations of the EEVs being exponentially small as a function of the system size.

A standard quantitative statement of ETH states that the matrix elements of an operator A representing a typical physical observable should have the form [3, 4]

$$\langle E_{\alpha}|A|E_{\beta}\rangle = \delta_{\alpha\beta}f_A^{(1)}(\bar{E}) + e^{-S(\bar{E})/2}f_A^{(2)}(\bar{E},\omega)R_{\alpha\beta} \quad (1)$$

where S is the entropy, $|E_{\alpha}\rangle$ is an energy eigenstate with eigenenergy E_{α} , $\bar{E} = (E_{\alpha} + E_{\beta})/2$, and $\omega = E_{\beta} - E_{\alpha}$. The $f_A^{(1/2)}$ are smooth functions, and $R_{\alpha\beta}$ is a (pseudo) random variable with zero mean and unit variance. For systems with finite Hilbert space dimension D, the entropy scales as $S \sim \log D$. Thus a crucial aspect of ETH is the scaling of the width of the distribution of either diagonal or off-diagonal matrix elements: when approaching the thermodynamic limit, this width falls off as $e^{-S/2} \sim D^{-1/2}$, i.e., exponentially with system size. This scaling can be understood using the similarity between typical many-body eigenstates and random states [10–12]. This behavior contrasts sharply with integrable systems, which do not obey ETH scaling — the width of diagonal matrix element distributions generally have power law decay with system size [11, 13–19], and the off-diagonal matrix element generally has a non-gaussian distribution [12, 18, 20].

Evidence from a large number of numerical studies strongly suggests that ETH is satisfied for typical states of generic nonintegrable systems and for physical observables [5, 9–12, 18, 19, 21–45]. In particular, several studies have examined the decrease of the width of matrix element distributions as the thermodynamic limit is approached, both for chaotic and integrable systems [11–19, 28, 31, 36, 40, 43–48]. For lattice systems, the thermodynamic limit involves increasing both the lattice size and the particle number simultaneously, keeping the average density fixed. In the present work, we will instead explore approaching the *classical limit* — we consider increasing particle numbers in fixed lattice topologies. We investigate the scaling behavior of EEV fluctuations as a function of Hilbert space dimension, as the classical limit is approached.

We focus on bosonic systems and study the Bose-Hubbard Hamiltonian on fixed numbers of sites. The dynamics of these systems in the classical limit have been extensively studied, where they are described by a discrete nonlinear Schroedinger equation. Because of the existence of a classical limit, Bose-Hubbard systems have also been extensively used as a testbed for semi-classical methods [49–68]. Quantum dynamics of these systems have also been compared with the dynamics of the corresponding classical limit [69–75].

We consider N particles in k sites, arranged linearly. We approach the large limit of large Hilbert space dimension $(D \to \infty)$ by keeping k fixed and increasing N, as opposed to the usual thermodynamic limit for which the ratio N/k would be kept fixed. The 2-site system (Bose-Hubbard dimer, k=2) is integrable, and hence we omit this case in the present work on ETH scaling. We focus on lattices with sizes from k=3 up to k=10. The case of k=3 (and to a lesser extent k=4) is particularly interesting: although not integrable, the classical phase space in this case is known to be highly 'mixed' [52, 71, 75–80], and the behavior of the finite-N quantum system shows deviations from fully chaotic behavior. For larger site numbers, $k \gtrsim 5$, the Bose-Hubbard systems tend to behave as quantum-chaotic systems [73, 81–84].

Assuming the eigenstates to be perfectly ergodic "infinite-temperature" states in the sense of having Gaussian-distributed random coefficients, an exact for-

 $^{^1}Department$ of Theoretical Physics, Maynooth University, Co. Kildare, Ireland $^2Max\text{-}Planck\text{-}Institut$ für Physik komplexer Systeme, D-01187 Dresden, Germany

mula for the EEV fluctuations is available. This relation [Eq. (7)] only depends on trace expressions of operators, and holds for all operators, as long as the "Gaussian eigenstates" approximation is valid. It mathematically connects the Gaussianity of eigenstates to finite size scaling of ETH, and provides an alternate derivation of ETH scaling.

The class of observables generally considered in ETH investigations (observables of physical interest) are fewbody operators or sums thereof, i.e., 'local' operators. However, having k fixed and small means that the notion of 'local' operators has to be re-examined. When considering $k \to \infty$, two-site operators are local in the sense that $2 \ll k$. This is no longer true for our constant k; in particular it is strongly violated for k=3. Therefore we re-examine the expected scaling of EEV fluctuations, first for Gaussian eigenstates. We show that the expected behavior for $N \gg k$ is $\sim D^{-e_0}$ with $e_0 = \frac{1}{2} - \frac{1}{k-1}$. The exponent ranges from $e_0 = 0$ for the 3-site case (k=3) to $e_0 \to \frac{1}{2}$ for large k $(N \gg k \gg 1)$.

The scaling exponent e_0 being different from 1/2 for moderate k can be traced to the unboundedness of the operators in the $N \to \infty$ limit. The $D^{-1/2}$ scaling could be retrieved by rescaling the operators A and considering operators $\bar{A} = A/N$.

For the actual mid-spectrum eigenstates of the Bose-Hubbard chains, we show numerically that the EEV fluctuations match the behavior predicted by the Gaussian ansatz, once k is large enough. For small k, the dependence on D is seen to be a power law but with a different exponent $e \neq e_0$. This is a remarkable manifestation of the 'mixed' nature of the 3-site and 4-site Bose-Hubbard systems.

At present, we do not have a predictive explanation for the observed exponents. Small or subleading deviations from Gaussian behavior (from random-matrix behavior) has been noted previously in mid-spectrum eigenstates of chaotic many-body systems. Such deviations can be seen in the eigenstate coefficient distribution in the configuration basis [37, 85–87], in the entanglement entropy of eigenstates [20, 88–90], and in the fluctuations of offdiagonal matrix elements [45]. Since the present k=3system has strong non-chaotic features, it is not surprising that there is even stronger deviation from Gaussianstate behavior. Perhaps unexpectedly, EEV fluctuations show a clear power law dependence on the Hilbert space dimension, but the exponent is markedly different from the random-matrix prediction. This demonstrates that eigenstates of small-k Bose-Hubbard chains have nonrandom correlations. We examine the most straightforward ways in which physical eigenstates could be different from ideally ergodic eigenstates, e.g., non-Gaussian coefficient distributions and two-point correlations between the eigenstate coefficients. We demonstrate that none of these effects explain our observed scaling. The observed scaling exponent thus results from more subtle structures in the eigenstate coefficients, which seems challenging to characterize or identify.

In Section II we introduce the system, notations, and the numerical procedure for extracting EEV fluctuations Section III is the heart of the paper and reports the main results: analytically derived trace expressions and scaling laws for σ in the case of idealized (Gaussian) eigenstates (IIIA), and results of extensive numerics showing where these predictions succeed and where they fail (IIIB). In Section IV we examine possible reasons for the few site systems (particularly k=3) deviating from the Gaussian prediction and displaying anomalous exponents in the D-dependence of σ . We conclude that the anomalous scaling is due to subtle higher-order correlations among the eigenfunction coefficients which cannot be captured by the 2-point correlations among eigenfunction components. Section V provides context and concluding discussion.

The appendices present further details, data and clarification. We show the dependence on the interaction parameter and justify the interaction value used in the main text (Appendices A and B). We display the structure of covariance matrices of the physical eigenstates (C). Appendices D and E contain the derivations of the analytic results announced in the main text.

II. PRELIMINARIES — SYSTEMS AND OBSERVABLES

In this section, we introduce the system Hamiltonian (IIA), discuss how to define the fluctuations of the eigenstate expectation values (EEVs) (IIB), and introduce notation to be used in this paper.

A. Hamiltonians

We will investigate Bose-Hubbard systems restricted to open-boundary chains of length k, with nearest neighbor hoppings and on-site interactions. The Hamiltonian is

$$H = -1/2 \sum_{\langle i,j \rangle} J_{i,j} a_i^{\dagger} a_j + \frac{U}{2} \sum_{j=1}^k n_j (n_j - 1), \qquad (2)$$

where $\langle i,j \rangle$ denotes summation over adjacent sites $(j=i\pm 1), a_j^{\dagger}$ and a_j are the bosonic creation and annihilation operators for the j-th site and $n_j=a_j^{\dagger}a_j$ is the corresponding occupation number operator. $J_{i,j}=J_{j,i}$ is the symmetric tunneling coefficient and U is the two-particle on-site interaction strength. We choose $J_{1,2}=1.5$ and $J_{i,j}=1$ for $i,j\geq 2$ to avoid reflection symmetry. The particle number N is conserved by the Hamiltonian. We introduce the tuning parameter $\Lambda=UN$.

In the limits $\Lambda \to 0$ and $\Lambda \to \infty$ the system is integrable. If $\Lambda = 0$, then the free bosonic system is solvable in terms of single-particle eigenstates, while $\Lambda \to \infty$ effectively means J can be neglected, so that Eq. (2) is diagonal already. For intermediate Λ the systems behave chaotically, i.e., the spectrum approximately obeys

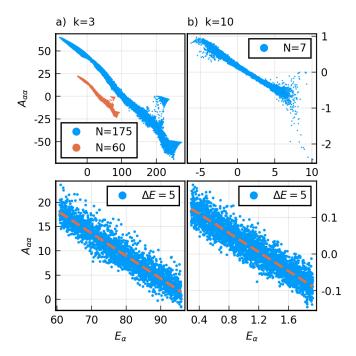


FIG. 1. Eigenstate expectation values $A_{\alpha\alpha}$ of the tunnel operator $A=a_2^{\dagger}a_1$, plotted against eigenenergies, for a Bose-Hubbard chain with a) k=3 sites and b) k=10 sites. The numbers of particles N are listed in legends. The interaction parameter is $\Lambda\approx 2.477$ as explained in the text. Top panels show full spectra. Bottom panels zoom into the 5th of ten equal energy intervals, as indicated by the shorthand label " $\Delta E=5$ ". Dotted lines are fitted linear functions.

Wigner-Dyson statistics. For the smallest chain length k=3, there are stronger deviations, but nevertheless the spectral statistics near the middle of the spectrum is close to Wigner-Dyson form (Appendix A).

The Hilbert space size is $D = \binom{N+k-1}{k-1} = \binom{N+k-1}{N}$, which scales for fixed k in the large N limit as $D \sim N^{k-1}$. Therefore, for fixed k in the large D limit, N scales as $N \sim D^{1/(k-1)}$.

The operators we focus on are the tunnel operator $a_2^{\dagger}a_1$ from site 1 to site 2 and the number operator n_1 on site 1. We have checked that the overall scaling behaviors are the same for $a_i^{\dagger}a_j$ with other i, j. Note that $a_2^{\dagger}a_1$ is non-hermitian. If hermiticity is required, we will instead investigate $(a_2^{\dagger}a_1 + a_1^{\dagger}a_2)$. This change results in an additional factor of 2 in the corresponding EEV fluctuation.

B. Defining EEV fluctuations

We denote by $|E_{\alpha}\rangle$ the eigenstate of the Hamiltonian H belonging to the eigenvalue E_{α} . If the ETH holds, $A_{\alpha\alpha} = \langle E_{\alpha} | A | E_{\alpha} \rangle$ is constant on small energy windows $[E_{\alpha} - \delta, E_{\alpha} + \delta]$ around E_{α} for suitable (small) δ , up to finite size fluctuations. Further, $A_{\alpha\alpha}$ coincides with the

microcanonical average

$$\langle A \rangle_{(E_{\alpha},\delta)} = \frac{1}{N_{E_{\alpha},\delta E_{\alpha}}} \sum_{\beta: E_{\beta} \in [E_{\alpha} - \delta, E_{\alpha} + \delta]} A_{\beta\beta}.$$
 (3)

A quantitative measure of how well the EEVs coincide with the microcanonical average is the magnitude of the finite size fluctuations, i.e., the statistical standard deviation of $A_{\alpha\alpha} - \langle A \rangle_{(E_{\alpha}, \delta)}$,

$$\left[\frac{1}{N_{\Delta E}} \sum_{E_{\alpha} \text{ in } \Delta E} \left(A_{\alpha \alpha} - \langle A \rangle_{(E_{\alpha}, \delta)} \right)^{2} \right]^{1/2} \tag{4}$$

over some energy window ΔE , where $N_{\Delta E}$ is the number of eigenvalues E_{α} in ΔE . Graphically this measures the width of the distribution of $A_{\alpha\alpha}$ values.

The above definition involves a choice of the window width δ ; this choice may depend on various parameters of the physical model. To avoid larger scale changes of $A_{\alpha\alpha}$ (e.g, as visible in Figure 1), the window should not be too wide. The window should also not be too small, so as to ensure that sufficiently many EEVs are within the window around E_{α} to get a statistically significant estimate of the fluctuations. Choosing δ can thus be a tricky balancing act.

We can avoid these technicalities by noting that ETH implies that the quantity $A_{\alpha\alpha}$ will vary smoothly as a function of E_{α} , up to finite size fluctuations. If the ETH holds in this sense, then, by definition of smoothness, $A_{\alpha\alpha}$ should be locally linear. We divide the energy spectrum into 10 equal-length, disjoint intervals ΔE . In these intervals the large scale change of $A_{\alpha\alpha}$ in our case is indeed linear to an excellent approximation, as seen in the lower panels of Figure 1. We fit linear functions $E_{\alpha} \to b + m E_{\alpha}$, on each interval to $A_{\alpha\alpha}$. We then investigate the fluctuations around these functions, i.e.,

$$\sigma^{2}(A, \Delta E) = \frac{1}{N_{\Delta E}} \sum_{E_{\alpha} \text{ in } \Delta E} |A_{\alpha \alpha} - b - mE_{\alpha}|^{2}.$$
 (5)

Since we are primarily interested in mid-spectrum eigenstates, we will show data from the 5th, 6th and 7th energy intervals. In shorthand these will be labeled as $\Delta E = 5$, $\Delta E = 6$, $\Delta E = 7$, with ΔE referring to the label and not the interval width. In Figure 1, we display the EEV's for the full spectrum and for the $\Delta E = 5$ interval, for two different values of k. Unless indicated otherwise, we present data for an intermediate value of the interaction parameter $\Lambda = UN$ around which the systems are found to be significantly chaotic, namely, $\Lambda = 10^{13/33} \approx 2.477$. Appendix A provides further details on this choice of Λ .

III. EEV FLUCTUATION SCALING

In this section we first (IIIA) consider Gaussian-random states, e.g. eigenstates of matrices drawn from

the GOE ensemble. We present expressions for the EEV fluctuations for such idealized eigenstates. For operators of the type $A=a_i^{\dagger}a_j$, we show that the scaling of EEV fluctuations, $\sigma\sim D^{-e_0}$, occurs with exponent $e_0=\frac{1}{2}-\frac{1}{k-1}$ in the classical limit $(N\to\infty,$ fixed k). For scaled operators $\bar{A}=A/N$, the exponent is $e_0=\frac{1}{2}$. In comparison, in the usual thermodynamic limit $(N\to\infty,$ fixed k/N), the exponent is $e_0=\frac{1}{2}$ for local operators.

Next (IIIB), we present numerically calculated scaling results for Bose-Hubbard chains. We show that the EEV fluctuations have power-law dependence on the Hilbert-space dimension, $\sim D^{-e}$. The exponent e matches the Gaussian-state prediction e_0 for larger chains, but differs substantially for small k.

A. Derivation for random Gaussian states

The EEV's calculated for random states do not have large-scale smooth variation as a function of energy, in contrast to Figure 1. Thus, the statistical standard deviation of EEV's, σ , can be directly compared with our measurement of EEV fluctuations in the physical eigenstates. The assumption of eigenstates being effectively random has been previously used to derive scaling properties of EEV's in the thermodynamic limit [10, 11, 45]. Here, we provide explicit expressions for σ in terms of trace properties of the operator matrix, and then specialize to both thermodynamic and classical limits. We present the important expressions here, and relegate derivation details to Appendices D and E.

1. Trace expressions

Let A be a $D \times D$ square matrix representing the operator of interest, and $|Z\rangle$ be a D-dimensional, multivariate random state with identically and independently distributed (i.i.d.) components Z_i , each with mean 0. Then the statistical variance of $\langle A \rangle = \langle Z|A|Z\rangle$ can be expressed as

$$\sigma^{2}(\langle A \rangle) = \left(E[Z_{1}^{4}] - 3E[Z_{1}^{2}]^{2} \right) \sum_{i} A_{ii}^{2}$$

$$+ E[Z_{1}^{2}]^{2} \left[\operatorname{tr}(A^{2}) + \operatorname{tr}(AA^{\dagger}) \right]. \tag{6}$$

Here $E[\cdot]$ represents the expectation value. The manipulations leading to this expression can be found in Appendix D.

Eq. (6) is valid irrespective of the distribution of the i.i.d. variables Z_i . Simplifications are achieved by specializing to the case of GOE eigenstates, widely considered as reasonable models for the behavior of mid-spectrum eigenstates of chaotic Hamiltonians. Eigenstates of D-dimensional Gaussian matrices are uniformly distributed on the D-dimensional unit sphere. In the large D limit their components can be regarded as independently normally distributed with mean 0 and variance 1/D. If we

take Z_i to be normally distributed with variance 1/D, in addition to the previous constraints, then the first term in Eq. (6) vanishes. Thus

$$\sigma^{2}(\langle A \rangle) = \frac{1}{D^{2}} \left[\operatorname{tr}(A^{2}) + \operatorname{tr}(AA^{\dagger}) \right] \tag{7}$$

for random states with Gaussian-distributed coefficients. The second of the two traces, $\operatorname{tr}(AA^{\dagger})$, is the squared Hilbert-Schmidt norm or the Frobenius norm of the operator. For Hermitian A the two trace terms are equal: $\operatorname{tr}(A^2) = \operatorname{tr}(AA^{\dagger})$. As $\operatorname{tr}(A^2)$ and $\operatorname{tr}(AA^{\dagger})$ are invariant under a basis change, so is the variance Eq. (7). In contrast, Eq. (6) is not basis-invariant, due to the first term. Expressions equivalent or analogous to Eq. (7) have appeared previously in the literature, e.g., in Refs. [91–94].

For observables of the form $A = a_j^{\dagger} a_i$, the trace expression is shown in Appendix E to be given by

$$\operatorname{tr}(A^{2}) + \operatorname{tr}(AA^{\dagger}) = \frac{N(N+k)}{k(k+1)}D \qquad i \neq j, \quad (8a)$$

$$\operatorname{tr}(A^2) + \operatorname{tr}(AA^{\dagger}) = \frac{2N(2N+k-1)}{k(k+1)}D$$
 $i = j$. (8b)

In the usual thermodynamic limit, the fraction multiplying D has $O(D^0)$ scaling in either case. Thus one obtains $\sigma^2 \sim \frac{D}{D^2}$, i.e., the usual $\sigma \sim D^{-1/2}$ scaling for EEV fluctuations in the thermodynamic limit.

2. Scaling in the classical limit

We now consider the classical limit, $k \ll N$. If A is a linear combination of terms like $a_j^{\dagger}a_i$, the trace expression scales as

$$\operatorname{tr}(A^2) + \operatorname{tr}(AA^{\dagger}) \sim N^2 D. \tag{9}$$

Since $N \sim D^{1/(k-1)}$ in the classical limit, the variance scales as

$$\sigma^2(\langle A \rangle) \sim \frac{D \cdot D^{2/(k-1)}}{D^2} = D^{-2e_0},$$
 (10)

where

$$e_0 = e_0(k) = \frac{1}{2} - \frac{1}{(k-1)}$$
 (11)

is the scaling exponent announced previously. For $k\gg 1$, but still $k\ll N$, the second term becomes negligible and we obtain $\sigma\sim D^{-1/2}$ scaling similar to the thermodynamic limit

For numbers of sites that are not too large, the EEV scaling of two-point operators (of the type $a_j^{\dagger}a_i$ or their linear combinations) is different for the classical limit compared to the thermodynamic limit. Mathematically, this difference can be attributed to the operators A scaling with N. If we rescale A to $\bar{A} = A/N$, then the traces

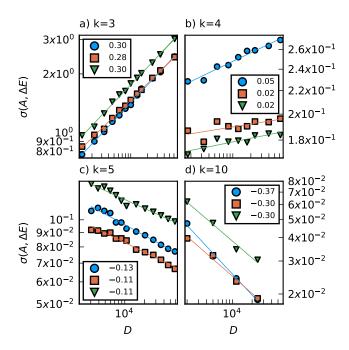


FIG. 2. EEV fluctuations σ of the operator $A=a_2^\dagger a_1$ against Hilbert space size D for various chain lengths k. Data shown for eigenstates in energy windows $\Delta E=5$ (blue circles), 6 (red squares), 7 (green inverted triangles). The σ vs D data sequences are arranged reasonably linearly in all cases in the log-log plots, suggesting $\sigma \sim D^{-e}$ behavior. The slopes of fitted lines (i.e., numerical estimates of -e) are given in the legends. The Gaussian predictions e_0 for the exponents are 0, $\frac{1}{6} \approx 0.1667$, 0.25, and $\frac{7}{18} \approx 0.3889$ respectively for k=3, 4, 5, and 10.

(9) scale as D rather than as N^2D , so that the variance scales for all k as

$$\sigma^2(\langle \bar{A} \rangle) \sim \frac{D}{D^2} = D^{-1}. \tag{12}$$

Summarizing: in the classical limit, the EEV fluctuation scaling is $\sim D^{-1/2}$ for scaled operators $\bar{A}=A/N$ for all k, and also for un-scaled operators A in the $k\gg 1$ limit. This is the same exponent $e_0=\frac{1}{2}$ familiar from the thermodynamic limit [11, 45]. However, for moderate k and for the operator A, the scaling is according to the exponent $e_0=\frac{1}{2}-\frac{1}{(k-1)}$ of Eq. (11)

B. Numerical results: Bose-Hubbard eigenstates

In Figure 2 we show the fluctuations σ of the EEVs for different energy windows near the middle of the spectrum, plotted against D. Each panel shows a different (fixed) number of sites k; in each case the classical limit is approached by increasing N. Generally, the sequences follow clear power-law dependencies, $\sigma \sim D^{-e}$. The power-law behavior sets in at relatively small values of D already.

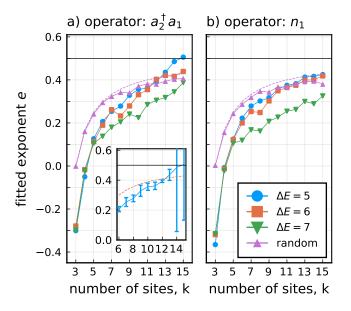


FIG. 3. The exponent e versus the chain length k for Bose-Hubbard eigenstates in different energy windows, and for Gaussian-random states. The pink dashed curve is the predicted formula for exponents, e_0 , which tends to 1/2 (solid horizontal line) for large k. Inset to left panel shows the error bars for the estimation of e from $\Delta E = 5$ data. The error bars are omitted elsewhere and will be omitted in later figures.

It is clear from the k=3 data, panel (a), that the exponent e does not match the value predicted for Gaussian-random states, Eq. (11), which is $e_0=0$ for k=3. The EEV fluctuation for the system eigenstates increases with a positive exponent (e<0) instead of being flat as a function of D. Similarly, for the 4-site chain the exponent e is seen to be slightly negative — σ increases slowly with system size — whereas the predicted value is $e_0=+1/6$.

The calculations rely on full numerical diagonalization, and hence are limited by the Hilbert space size D. Our limit was $D \lesssim 10^5$. For each k, we increased the particle number N as far as possible such that D did not exceed 100,000. For small k, this provides a satisfactory number of available N values, and extracting the exponent e from a fit to $\sigma \sim D^{-e}$ is quite reliable. For large k, only a few N values are available. For the largest lattice (k=15), only three data points (N=4, N=5, and N=6) were used. This means a large uncertainty in the estimation of e (Figure 3 inset). It also means that the regime $N\gg k$ is not reached.

In Figure 3 we present the exponents e extracted from the numerical data. In addition to the exponents for the operator $a_2^{\dagger}a_1$ (corresponding to Figure 2), we also show the exponents for the operator n_1 . For small k, the numerically observed exponents e fall significantly below the Gaussian-random case, for both operators. For larger k values, the Bose-Hubbard systems show EEV fluctuations closer to the Gaussian-random case, at least

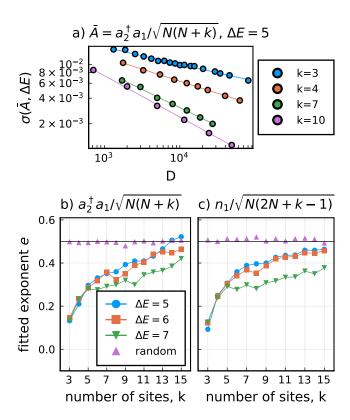


FIG. 4. Similar data as in Figures 2 and 3 but with scaled operators \bar{A} . The operators are scaled by factors $\sim N$; the precise factors are explained in the text.

for $\Delta E=5,6$. (The $\Delta E=7$ window shows larger deviation, presumably because it is closer to the edges of the spectrum.) We interpret this as a signature of the large-k Bose-Hubbard systems being more chaotic, so that mid-spectrum eigenstates are better approximated by Gaussian-random states. The deviation for small k represents the 'mixed' (chaotic+regular) nature of the few-site Bose-Hubbard Hamiltonians.

Figure 3 also shows numerically calculated exponents for EEV fluctuations in Gaussian-random states (pink triangles), and compares with the $k \ll N$ prediction, Eq. (11) (pink dashed curve). The agreement is good for all k and excellent for small k. At larger k, computer memory limitations prevent our calculations from reaching particle numbers $N \gg k$. This explains the (minor) deviation of the numerical exponents from the $N \gg k$ prediction.

One can view the same effects through the fluctuations of the scaled (\bar{A}) operators $a_2^{\dagger}a_1/N$ and n_1/N . For these operators, the predicted exponent is 1/2 for all k. We present the numerical exponents for such operators in Figure 4; however we scale by factors slightly different from N. The prediction $e_0 = 1/2$ was obtained in the previous subsection by assuming $N \gg k$. In particular in the trace expressions of Eq. (8), this led to $N(N+k) \approx N^2$ and $2N(2N+k-1) \approx 4N^2$. In our

numerical calculations, for larger k we do not have access to N values in this regime. Therefore we scale the operators by the factor $\sqrt{N(N+k)}$ for $A=a_i^\dagger a_j$ with $i\neq j$ and by the factor $\sqrt{N(2N+k-1)}$ for i=j. With this modification, the numerically calculated exponents using Gaussian-random states (pink triangles) do not deviate systematically from 1/2 at large k, even though the $N\gg k$ regime is not reached. The observed physical exponents (for Bose-Hubbard eigenstates) are significantly different from the predicted $e_0=1/2$ for small k, but approach this value as k is increased.

Summarizing our numerical findings: power-law dependence of EEV fluctuations on D is seen for all k. The exponent at larger k (more fully chaotic systems) matches well the Gaussian prediction. For smaller k (mixed systems), there are significant deviations from the Gaussian prediction. Perhaps the most remarkable aspect of these numeric results is that, for small k, the departure of the eigenstates from Gaussian-random or ergodic behavior does not destroy the power-law dependence of EEV fluctuations with D but changes the exponent substantially. In the following section we will examine possible explanations for this phenomenon.

IV. NON-REASONS FOR ANOMALOUS SCALING

The mismatch with predicted scaling for small k must be due to the eigenstates of few-site Bose-Hubbard systems deviating from idealized Gaussian-random states, which we refer to as 'ergodic' states. In this section, we examine various types of deviation from the idealized case, and rule out several plausible explanations for the anomalous scaling.

One could imagine that the eigenstates effectively occupy a smaller fraction of the Hilbert space than a random state, and that this fraction has sublinear scaling with D. This can be quantified through analysis of the participation ratio or multifractal dimensions of the eigenstates. Therefore, in Subsection IV A, we discuss the participation ratio P_{α} . We show how $P_{\alpha} \sim D^1$ scaling implies the expected EEV fluctuation scalings that we have derived previously in III A. We also show that Hilbert space occupancy is not the explanation for our anomalous EEV scaling exponents, because eigenstates of the k=3 system have $P_{\alpha} \sim D^1$ scaling.

One could also imagine that the anomaly of scaling exponents is due to the eigenstate components not being identically distributed. In IV B we present data showing that this is not the reason for the anomalous scaling of EEV fluctuations.

These results show that the non-ergodic scaling must be due to correlations present in the eigenstates. In IV C we examine the simplest (and most prominent) type of correlations between eigenstate coefficients, namely, those captured in the covariance matrix. Perhaps surprisingly, we find that these correlations do not explain the anomalous scaling either. The anomalous scaling exponent for k=3 EEV fluctuations is thus caused by more subtle (higher-order) correlations.

A. Participation Ratios

For this Subsection it will be convenient to consider our operator A to be Hermitian. We expand the eigenstates $|E_{\alpha}\rangle$ of the Hamiltonian in the eigenstate basis $|\phi_{\gamma}\rangle$ of the operator A

$$|E_{\alpha}\rangle = \sum_{\gamma} c_{\gamma}^{(\alpha)} |\phi_{\gamma}\rangle,$$
 (13)

where $c_{\gamma}^{(\alpha)} = \langle \phi_{\gamma} | E_{\alpha} \rangle$. If we denote the eigenvalues of A as a_{γ} , then the EEVs can be written as

$$A_{\alpha\alpha} = \sum_{\gamma=1}^{D} |c_{\gamma}^{(\alpha)}|^2 a_{\gamma}. \tag{14}$$

We regard the coefficients $c_{\gamma}^{(\alpha)}$ to be random variables, with each eigenstate index (α) denoting a different sample from the same underlying distribution. The distribution is assumed to be the same for every γ . As usual, this framework will not capture large-scale dependences of the EEVs $A_{\alpha\alpha}$ on the energies E_{α} . This is acceptable because we are interested in the fluctuations only.

The variance of the EEVs is

$$\operatorname{var}\left(\sum_{\gamma=1}^{D}|c_{\gamma}|^{2}a_{\gamma}\right) = \sum_{\gamma=1}^{D}\operatorname{var}\left(|c_{\gamma}|^{2}\right)a_{\gamma}^{2}$$
$$= \operatorname{var}\left(|c_{\gamma}|^{2}\right)\operatorname{tr}(A^{2}). \quad (15)$$

The variance of $|c_{\gamma}|^2$ can be written as

$$\operatorname{var}(|c_{\gamma}|^{2}) = E[|c_{\gamma}|^{4}] - (E[|c_{\gamma}|^{2}])^{2} = \frac{1}{DP} - \frac{1}{D^{2}}$$
 (16)

where we have used the definition of the participation ratio to be

$$P_{\alpha} = \left(\sum_{\gamma=1}^{D} |c_{\gamma}^{(\alpha)}|^4\right)^{-1} = \left(\sum_{\gamma=1}^{D} D \times E\left[|c_{\gamma}^{(\alpha)}|^4\right]\right)^{-1}.$$
(17)

We thus have the prediction for the ${\rm EEV}$ variance

$$\sigma^2 = \left(\frac{1}{DP} - \frac{1}{D^2}\right) \operatorname{tr}(A^2) = \left(\frac{1}{DP} - \frac{1}{D^2}\right) \operatorname{tr}(AA^{\dagger})$$
(18)

for Hermitian operators. For Gaussian states, P = D/3, so that this expression for σ^2 reduces to Eq. (7). More generally, as long as the participation ratio scales linearly with D, the factor in brackets $\sim 1/D^2$, so that we obtain the same scaling as for Gaussian states. For non-ergodic states $P \sim D^K$ with K < 1, the first term in brackets would dominate and one would obtain different scaling.

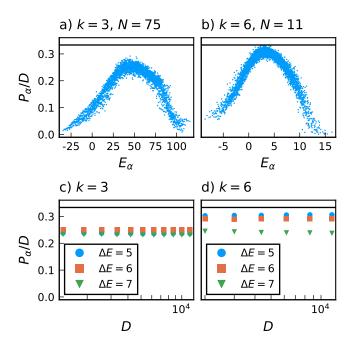


FIG. 5. Participation ratio in the basis of eigenstates of the operator $a_2^{\dagger}a_1 + a_1^{\dagger}a_2$. Horizontal lines indicate the Gaussian expectation (= $\frac{1}{3}$). Top panels: Scaled participation ratio P_{α}/D of energy eigenstates, versus corresponding energy eigenvalues. Lower panels: average scaled participation ratio for different energy intervals plotted against Hilbert space dimension D, for fixed chain length k and increasing particle number N.

In Figure 5 we show the behavior of the participation ratio in the basis of eigenstates of $(a_2^{\dagger}a_1 + a_1^{\dagger}a_2)$. The mid-spectrum P_{α} is close to the Gaussian expectation for highly chaotic (larger k) systems. For k=3 the deviation from Gaussianity (P=D/3) is strong. However, in both cases the scaling of P_{α} with Hilbert space dimension D is very linear (lower panels).

Thus, Hilbert space occupancy does not explain the observed anomalous exponent of EEV fluctuations.

B. Non-identical distribution of eigenstate coefficients

The analysis in Subsections IV A and III A are based on eigenstate coefficients being *identically distributed* and *independent*. Could it be that the failure to capture the EEV fluctuation scaling at small k results from the eigenstate components having non-identical distributions?

To check, we drop the condition that the eigenstates have identically distributed components and only assume that the components are independent. In Figure 6 we show the EEV fluctuations obtained from an estimation of the underlying distributions of the eigenstate components of the Bose-Hubbard systems. We sampled uniformly from components of all eigenstates of the Bose-

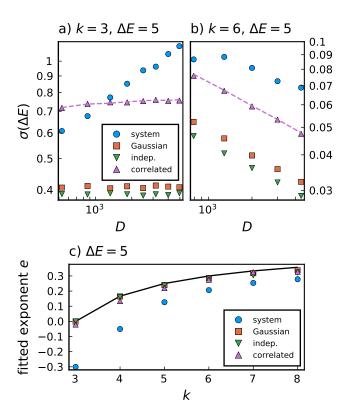


FIG. 6. The operator in all plots is $a_2^{\dagger}a_1$. Blue dots: eigenstates of physical Bose-Hubbard system. Red squares: Gaussian states with i.i.d. components. Green down triangles: vectors with independent but non-identically distributed components, each component sampled from system eigenstates. Purple up triangles: multivariate Gaussian states with covariance matrix estimated from system eigenstates. a)-b) EEV fluctuations. The dashed purple line is the prediction by Eq. (20). c) The exponent e, such that $\sigma \sim D^{-e}$, versus the number of sites k for the same distributions as in a)-b).

Hubbard model in a specific energy interval. Effectively, for each component of the state, we randomly picked an eigenstate in the energy interval and used the entry of the corresponding component. The EEV fluctuations are then calculated from such sampled states. The resulting data are marked 'independent' in Figure 6. The results match well with the EEV fluctuations obtained from Gaussian states and do not match the EEV fluctuations of the actual physical systems for small k.

We conclude that that any model for the wavefunctions with independent components does not explain the observed anomalous scaling at small k. In other words, the root cause of the phenomenon is not the eigenstate coefficients having non-identical or non-Gaussian distributions, or insufficient occupancy of the Hilbert space. Rather, we have traced the cause to the fact that the eigenstate coefficients are not independent.

C. Eigenstate Correlations

Continuing our effort to identify what feature of small-k eigenstates is responsible for the anomalous ETH scaling, we relax the constraints on the model states even further. Now, we assume that eigenstates are drawn independently from a multivariate distribution Z, but the D components of Z are allowed to be correlated. Given a multivariate distribution Z, the statistical correlations between components are quantified by the covariance matrix Σ . The covariance matrix in the case of eigenstates can be estimated by regarding the eigenstates within a specific energy interval as different samples of Z.

It is reasonable to assume that the mean of all components of Z is zero, as components of mid-spectrum eigenstates generally have zero mean. Then an estimate of Σ is given by

$$\Sigma = N_{\Delta E}^{-1} \sum_{E_{\alpha} \text{ in } \Delta E} |E_{\alpha}\rangle \langle E_{\alpha}|, \qquad (19)$$

where E_{α} and $|E_{\alpha}\rangle$ denote eigenvalues and eigenstates, respectively, and $N_{\Delta E}$ is the number of eigenstates in the energy window ΔE . Eq. (19) follows directly from the definition of the sample covariance matrix of Z, where the samples are the eigenstates $|E_{\alpha}\rangle$.

In Appendix C, Figure 9, we have provided visualizations of estimated covariance matrices for k=3 and k=6. The k=3 case is seen to have significant off-diagonal elements (arranged in intriguing patterns), indicating non-negligible correlations between the eigenstate components.

Once we have constructed the correlation matrix Σ fom the actual eigenstates, we can sample D-component vectors whose components are distributed and correlated according to Σ . In Figure 6 the EEV fluctuations obtained from such sampled states are marked as 'correlated'. The values of the fluctuations thus obtained are larger than those obtained from the independent-component random states, and more comparable to the fluctuations obtained from the physical eigenstates. In the chaotic cases (large k), all of these cases have the same scaling. However, in the k=3 case, the scaling exponent is close to the ergodic (Gaussian) case and does not reproduce the anomalous scaling at all. This is seen in panel a) of Figure 6, and also in panel c) where the fitted exponents are plotted. The fitted exponent is slightly off the Gaussian value for small k, but far from the anomalous values of the physical

These results show that the deviation of the physical system from expected 'ergodic' behavior is only partially captured by the two-point correlations between the eigenstate components. This suggests that the small-k eigenstates deviate from randomness in some more drastic manner, which does not seem easy to quantify.

In addition to the direct numerical verification discussed above, one can use analytic results to argue (nonrigorously) that inclusion of reasonable two-point correlations in the model of random states should not change the EEV fluctuation scaling exponent. Given a multivariate Gaussian distribution with mean zero and covariance matrix Σ , we can show (details in Appendix D) that the variance of EEV fluctuations is given by

$$\operatorname{var}(\langle A \rangle) = \frac{1}{D^2} \left[\operatorname{tr}((D\Sigma A)^2) + \operatorname{tr}(D\Sigma A D\Sigma A^t) \right]. \quad (20)$$

This is Eq. (7) with the change $A \to D\Sigma A$. The variance of components of the wavefunction is not fixed by normalization of the eigenstates any more. But it is reasonable to assume that it still scales as $\sim 1/D$, and we have checked numerically that this scaling holds for the mid-spectrum eigenstates of all our physical systems, including k=3. Since the variances of the wavefunction components are the diagonal entries of Σ , the diagonals of $D\Sigma$ scale (at most) as constant in D. By the Cauchy-Schwartz theorem the off-diagonal terms of Σ are bounded by the diagonal, so every component of $D\Sigma$ is (at most) constant in D.

Without making assumptions about the detailed structure of Σ , we cannot derive rigorously the scaling of the traces in Eq. (20), which was possible for Eq. (8) or (9). However, since $D\Sigma$ is elementwise at most $\sim D^0$, and assuming Σ is not too exotic, one can argue that the derivation of Eq. (E) should hold for this case as well. In other words, for 'reasonable' Σ , one expects the same scaling as in the case of independent Gaussian eigenstates. This is consistent with Figure 6, where the Σ is estimated numerically from the physical eigenstates.

V. DISCUSSION AND CONTEXT

Motivated by questions relating quantum dynamics to statistical mechanics, we have undertaken a study of the eigenstate thermalization hypothesis in the scaling sense, but considering increasing Hilbert space dimensions along the (semi)classical limit rather than the usual thermodynamic limit. This has led to a characterization of the distinctive properties of few-site Bose-Hubbard systems in terms of anomalous scaling exponents.

Summary of analytic results. For GOE eigenstates, i.e., states whose components can be approximated by Gaussian-distributed random variables, we have used trace expressions for the EEV fluctuation σ , Eq. (7). For operators of the type $A = a_j^{\dagger} a_i$, the trace operators can be expressed as Eq. (8). Based on these main expressions, we are able to predict ideal scaling behaviors of EEV fluctuations in the classical limit, for both un-scaled operators of the type A and scaled operators $\bar{A} = A/N$. Of course, the usual ETH scaling of the thermodynamic limit also follows from these expressions.

In the classical limit $N\gg k$, the EEV fluctuations are found for such idealized eigenstates to behave as $\sigma\sim D^{-e_0}$, with $e_0=\frac{1}{2}-\frac{1}{(k-1)}$ for un-scaled A operators and $e_0=1/2$ for scaled \bar{A} operators.

In addition, we have presented expressions for σ for a number of related cases, e.g., for i.i.d. distributed eigen-

state components with the distribution not assumed to be Gaussian, Eq. (6), in terms of the participation ratio, Eq. (18), and for the more general case where the eigenstate components are allowed to be correlated according to a covariance matrix, Eq. (20).

Summary of numerical results. We have explored the scaling exponent for various lattice lengths k, increasing the boson number N with fixed k to approach the classical limit. At larger k, the exponent matches the random-eigenstate prediction. At small k, the fluctuation appears to have power-law dependence $\sigma \sim D^{-e}$ on the Hilbert space dimension, i.e., e is well-defined, but the value of e differs markedly from the ergodic prediction. Through a series of additional numerical tests, we have shown that this anomalous scaling is not explained by 2-point correlations between eigenstate components.

The small-size Bose-Hubbard systems thus have midspectrum eigenstates which violate the usual randomness approximation in some subtle higher-order manner.

Deviation from Ergodicity. The deviation of quantum many-body systems from ergodicity has been the subject of interest from multiple viewpoints in recent years. Other than the strong violations of ETH/ergodicity due to integrability or many-body localization, more subtle departures have also been addressed or observed. In many-body systems that are nominally chaotic, mid-spectrum states are largely well-modeled by random states, but small or subleading deviations have been observed in various properties [20, 37, 45, 85–90]. However, scaling properties in these systems generally follow random-state predictions. In the small-k Bose-Hubbard systems, we have shown a striking exception: a system which is not integrable or many-body localized, but nevertheless violates the usual scaling behavior expected in chaotic systems.

The three-site and four-site Bose-Hubbard systems are widely known to be imperfectly chaotic, in particular because their classical phase space has been explored and is known to have both chaotic and regular regions [52, 71, 75–80]. With the current interest in chaos in many-body eigenstates, it is important to characterize such deviations from chaoticity. In this work, ETH scaling (i.e., the scaling of EEV fluctuations) has turned out to be fruitful approach to characterize these special systems.

Open questions and ideas arising from this work. The present work opens up a number of new questions deserving investigation:

(1) We have found that the small-k Bose-Hubbard systems display EEV fluctuations scaling with exponents that appear numerically well-defined but very clearly different from the random-state prediction. An analytic explanation for these observed new exponents is currently not available, and remains an open question. The anomalous scaling is related to the insufficient chaoticity of few-site systems, which is closely connected to the mixed phase space of the corresponding classical system. Hence a tempting conjecture is that some property measuring

the degree of chaos in the classical limit might explain the exponents.

- (2) Our analytic results have focused on essentially infinite-temperature states. It would be interesting to develop trace expressions for finite temperatures. This is likely not possible to do in complete generality without making assumptions on the system Hamiltonian, but perhaps some results can be derived with minimal assumptions, such as locality of the Hamiltonian.
- (3) For the thermodynamic limit, the EEV fluctuations of *integrable* systems generally show power-law scaling in N or k [11, 13–19], which translates into $\sim \ln D$ scaling. For the classical limit, however, power-law scaling in N would mean power-law scaling in D as well. Our numerics (Appendix B) shows that fitting $\sigma \sim D^{-e}$ in the integrable regime yields e = -1/(k-1) for A-type operators and e = 0 for \bar{A} -type operators. A detailed understanding and explanation of these exponents remains a task for future research.
- (4) Bose-Hubbard systems are, of course, not the only quantum systems with a classical limit. It remains to be discovered how generic our findings are. Comparing chaos-related properties between quantum systems and the corresponding classical systems has also been of interest in few-spin systems [95–106] and in spin-boson systems [103, 104, 107–116]. (The classical system is obtained in the limit of large spin quantum number.) Studying the behavior of EEV fluctuations in such systems when approaching the classical limit would provide interesting characterizations of ergodicity, e.g., of how well randomness approximations work.
- (5) As part of our effort to address the anomalous scaling at small k, we have briefly examined the covariance matrix of eigenstates, treating each eigenstate as a sample drawn from the distribution of eigenstates, according to Eq. (19). Studying the thus-defined covariance matrix might be fruitful for various quantum systems, as the departure of this matrix from the identity matrix tells us how different the eigenstates are from infinite-temperature states. A further significance of this covariance matrix is that the same object is the microcanonical density matrix, and hence its structure should provide insights into the connection betwen quantum eigenstates and thermodynamics.

ACKNOWLEDGMENTS

MH thanks P. Ribeiro and R. Mondaini for enlightening discussions. We thank J. D. Urbina and K. Richter for useful correspondence. Some computations were performed on the Irish Center for High-End Computing. GN is supported by the Irish Research Council Government of Ireland Postgraduate Scholarship Scheme (GOIPG/2019/58).

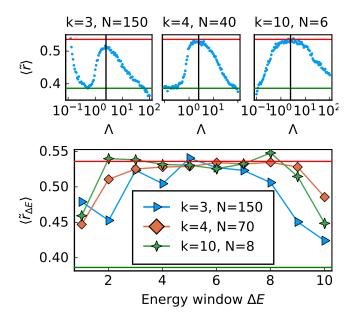


FIG. 7. Top: Level ratios \tilde{r} averaged over the whole spectrum, as a function of interaction strength $\Lambda = UN$, shown in the range $\Lambda \in (0.1,100)$. The horizontal lines are $\langle \tilde{r} \rangle_{\rm integr}$ and $\langle \tilde{r} \rangle_{\rm GOE}$. The vertical line indicates $\Lambda = 10^{13/33} \approx 2.477$. Bottom: Mean level ratio against ten evenly spaced energy windows ΔE , labeled 1 through 10 and plotted against these labels, for $\Lambda \approx 2.477$, various system sizes. The match to the GOE value is better away from spectral edges.

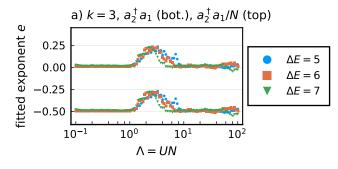
Appendix A: Level spacing statistics

The distribution of energy eigenvalue spacings, $s_{\alpha} = E_{\alpha+1} - E_{\alpha}$ for ordered $E_{\alpha} < E_{\alpha+1}$, is often used as an indicator for quantum chaos. Measuring this distribution involves unfolding the spectrum. This is bypassed by investigating the distribution of spacing ratios [117, 118]

$$r_{lpha} = rac{s_{lpha+1}}{s_{lpha}} \quad ext{and} \quad ilde{r}_{lpha} = \min \left(r_{lpha}, rac{1}{r_{lpha}}
ight).$$

The latter has the advantage that it is bounded. Quantum Hamiltonians are considered chaotic when the distributions follow that of the relevant Wigner-Dyson ensemble, which in our case is the GOE ensemble. The mean \tilde{r} for GOE is $\langle \tilde{r} \rangle_{\text{GOE}} \approx 4-2\sqrt{3} \approx 0.53590$. For an integrable Hamiltonian, the level spacing distribution is Poissonian (exponential); this leads to $\langle \tilde{r} \rangle_{\text{integr}} = 2 \log 2 - 1 \approx 0.38629$.

Figure 7 (top) shows that, with increasing interaction strength $\Lambda = UN$, the Bose-Hubbard systems turn from integrable to chaotic and back to integrable. Some anomalies are visible for short chains (small k). First, the distributions of the level ratios for very small Λ or very large Λ match neither GOE nor Poisson statistics. Also, for small k, in the most chaotic regime, the peak of $\langle \tilde{r} \rangle$ is still noticeably lower than the GOE value ≈ 0.534 . As k increases, the peak gets closer and closer to the GOE



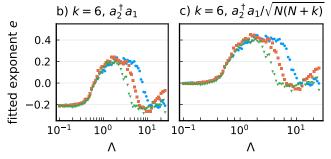


FIG. 8. Exponent e obtained from $\sigma \sim D^{-e}$ fits, as a function of the interaction parameter $\Lambda = UN$. Results are shown for both un-scaled operator $A = a_2^{\dagger}a_1$ and its scaled version \bar{A} , with $\bar{A} = A/N$ for the k=3 system (top) and $\bar{A} = A/\sqrt{N(N+k)}$ for the k=6 system (bottom).

value, consistent with the intuition that larger-k chains are more chaotic.

Our grid of Λ values contains 100 values spaced logarithmically from 0.1 to 100. Among these values, $\Lambda = 10^{13/33} \approx 2.477$ is found to be close to the location of the chaotic peak, for all k. Therefore, in this work we have shown data for this value of the interaction parameter.

Of course, we do not expect that the full spectrum obeys GOE statistics — the spectral edges are expected to deviate. This can be observed in Eq. (7) (bottom panel), where we plot $\langle \tilde{r} \rangle$ against different energy windows.

Appendix B: Dependence of EEV Scaling on interaction strength $\boldsymbol{\Lambda}$

In the main text we focused on the value of the interaction parameter $\Lambda=UN$ at which our systems are most chaotic. Here, we show how changing Λ affects the scaling of EEV fluctuations. Fitting the fluctuation data to $\sigma \sim D^{-e}$, we extract exponents e which we show in Figure 8 as a function of Λ . We compare the smallest Bose-Hubbard chain with k=3 sites with a larger, thus more chaotic chain with k=6.

We show data both for the operator $A = a_2^{\dagger} a_1$ and the scaled operator $\bar{A} \approx a_2^{\dagger} a_1/N$. For k = 3, we show results for $\bar{A} = a_2^{\dagger} a_1/N$. (As the data for small k ex-

tends into the $N\gg k$ regime, it is reasonable to approximate $\sqrt{N(N+k)}\approx N$.) For larger k values, the regime $N\gg k$ is difficult to reach numerically. Therefore it is appropriate to scale with the factor $\sqrt{N(N+k)}$ instead of the factor N, as the approximation $\sqrt{N(N+k)}\approx N$ made for Eq. (8) or Eq. (E9) may not be accurate.

Since $N \sim D^{1/(k-1)}$, the exponent e for $\sigma(\bar{A})$ is shifted upwards compared to that for $\sigma(A)$ by 1/(k-1). This is seen in Figure 8 for both k=3 (shift of $\frac{1}{3-1}=0.5$) and for k=6 (shift of $\frac{1}{6-1}=0.2$).

At intermediate values of Λ , when the system is (partially) chaotic, the value of e is near that discussed in the main text. If we had fully chaotic (Gaussian) eigenstates, we would expect e=0.5 for the scaled operators \bar{A} and e=0.5-0.5=0 (k=3) or e=0.5-0.2=0.3 (k=6) for the un-scaled operators A. Compared to the Gaussian prediction, the observed exponents are significantly smaller for the k=3 system, and somewhat smaller for the k=6 system, as discussed in detail in the main text.

For small or large Λ , the system is (near-)integrable. In these limits, the fluctuations for scaled operators \bar{A} do not have power-law dependence on D, and the fit to D^{-e} results in e=0. Thus eigenstate fluctuation scaling of scaled operators $\bar{A}=A/N$ in the classical limit is analogous to that of local operators in the thermodynamic limit: $e\to 0$ in the integrable limits and $e\approx 0.5$ in the chaotic regime [11]. For local operators in the thermodynamic limit, the fluctuations have power-law dependence on system size [11, 13–19], i.e., logarithmic dependence on D. A detailed study of the (near-)integrable models in the classical limit remains an interesting task for future studies.

Because e for \bar{A} settles to zero for $\Lambda \to 0$ and $\Lambda \to \infty$, the exponent for the unscaled operator $A = a_2^{\dagger} a_1$ settles to -1/(k-1) in these limits, i.e., to -0.5 for the trimer and to -0.2 for k=6. This is clearly seen in Figure 8.

Appendix C: Estimated covariance matrices from physical eigenstates

In Subection IV C we discussed results incorporating the 2-point correlations between eigenstate components, as embodied in the covariance matrix. The covariance matrix for physical mid-spectrum eigenstates is calculated according to Eq. (19), i.e., as an equal-weight mixture of the density matrices $|E_{\alpha}\rangle\langle E_{\alpha}|$ corresponding to each eigenstate in the energy window. Here we show the structure of the covariance matrices obtained in this way, i.e., we visualize the 2-point correlations present in the physical eigenstates.

To visualize the structure of covariance matrices one has to fix a basis of the Hilbert space. In Eq. (9) we show covariance matrices with respect to the defining basis \mathcal{B} . The elements of \mathcal{B} are the mutual eigenstates of all number operators $n_j = a_j^{\dagger} a_j$, denoted by $|n_1, \ldots, n_k\rangle$ where $n_1 + \cdots + n_k = N$. For two states $|\psi\rangle = |n_1, \ldots, n_k\rangle$

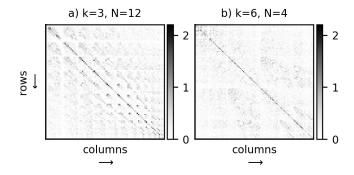


FIG. 9. Scaled estimated covariance matrix $D\Sigma$ of midspectrum eigenstates (energy window $\Delta E = 5$) for Bose-Hubbard chains with k sites and N particles. The estimate Σ is defined in Eq. (19), The absolute values of matrix entries, $|D \cdot \Sigma_{ij}|$, corresponding to basis $\mathcal B$ described in text, are shown. The arrows indicate the ordering of the basis, e.g. in the top left is the entry $|D \cdot \Sigma_{11}|$. N had to be chosen significantly smaller than in the rest of the paper to visualize the patterns without zooming. The patterns are stable for increasing N.

and $|\phi\rangle = |m_1, \dots, m_k\rangle$ we say that $|\psi\rangle < |\phi\rangle$ if and only if the states interpreted as (N+1)-adic numbers satisfy $(\psi)_{N+1} < (\phi)_{N+1}$, i.e.

$$|\psi\rangle < |\phi\rangle \iff \sum_{j=1}^{k} n_j (N+1)^j < \sum_{j=1}^{k} m_j (N+1)^j$$
. (C1)

The basis states in \mathcal{B} are in descendent order with respect to the ordering given by Eq. (C1). For example, for k=3 and N=2, \mathcal{B} is ordered as

$$\mathcal{B} = (|0,0,2\rangle\,, |0,1,1\rangle\,, |0,2,0\rangle\,, |1,0,1\rangle\,, |1,1,0\rangle\,, |2,0,0\rangle)\,. \tag{C2}$$

If the components of eigenstates of Bose-Hubbard chains would be independent, their estimated covariance matrix would be close to the identity matrix. Figure 9 shows that, even for k=6, there are significant off-diagonal terms, indicating that the eigenstate components are not independent. The deviation from identity matrix suggests that the mid-spectrum states deviate from infinite-temperature states, even for significantly chaotic states, consistent with Refs. [20, 37, 45, 85–90]. For small k, nonzero off-diagonal elements are even more pronounced, and these appear in intricate patterns (Figure 9(a)).

Appendix D: Variance and Covariance of EEVs of Random Eigenstates

In this section, we provide derivations of some of the equations that were announced and used in the main text without proof. First, we derive the results announced in Section III for random eigenstates with independent and identically distributed (i.i.d.) components (Subsection

D1). Next, we prove the result Eq. (20), announced in Subsection IVC, for random vectors whose components are not independent, with 2-point correlations described by a covariance matrix (Subsection D2).

1. Random vectors with i.i.d. components

Let A be a $D \times D$ square matrix and Z be a D-dimensional, multivariate random state with identically and independently distributed components Z_i , each with mean 0. The statistical variance of $\langle A \rangle = \langle Z|A|Z \rangle$ is given by

$$var(\langle A \rangle) = \sum_{i,j,i',j'=1}^{D} A_{i,j} A_{i',j'} cov(Z_i Z_j, Z_{i'} Z_{j'}), \quad (D1)$$

where

$$cov(Z_{i}Z_{j}, Z_{i'}Z_{j'}) = E[Z_{i}Z_{j}Z_{i'}Z_{j'}] - E[Z_{i}Z_{j}] \cdot E[Z_{i'}Z_{j'}]$$

$$= E[Z_{i}Z_{j}Z_{i'}Z_{j'}] - \delta_{ij}\delta_{i'j'}E[Z_{i}^{2}]^{2}.$$
(D2)

By the independence of Z_i, Z_j for $i \neq j$, $E[Z_i Z_j Z_{i'} Z_{j'}]$ is only non-zero if there is no index i, j, i', j' different to the other three. The only possibilities for this are

$$i = j = i' = j'$$

 $i = j$ and $i' = j'$
 $i = i'$ and $j = j'$
 $i = j'$ and $j = i'$, (D3)

SO

$$E[Z_{i}Z_{j}Z_{i'}Z_{j'}] = \delta_{ij}\delta_{ii'}\delta_{ij'}E[Z_{i}^{4}]$$

$$+ (1 - \delta_{ij}\delta_{ii'}\delta_{ij'})$$

$$\times \left[\delta_{ij}\delta_{i'j'}E[Z_{i}^{2}]^{2} + \delta_{ii'}\delta_{jj'}E[Z_{i}^{2}]^{2} + \delta_{ij'}\delta_{ji'}E[Z_{i}^{2}]^{2}\right].$$
(D4)

Then we get

$$\operatorname{var}(\langle A \rangle) = E[Z_1^4] \sum_{i} A_{ii}^2 + E[Z_1^2]^2 \times \left[\sum_{i \neq i'} A_{ii} A_{i'i'} + \sum_{i \neq j} A_{ij}^2 + \sum_{i \neq j} A_{ij} A_{ji} \right] - E[Z_1^2]^2 \sum_{i,i'} A_{ii} A_{i'i'}$$
(D5)

and

$$\operatorname{var}(\langle A \rangle) = \left(E[Z_1^4] - E[Z_1^2]^2 \right) \sum_{i} A_{ii}^2$$

$$+ E[Z_1^2]^2 \left[\sum_{i \neq j} A_{ij}^2 + \sum_{i \neq j} A_{ij} A_{ji} \right]$$

$$= \left(E[Z_1^4] - 3E[Z_1^2]^2 \right) \sum_{i} A_{ii}^2$$

$$+ E[Z_1^2]^2 \left[\operatorname{tr}(A^2) + \operatorname{tr}(AA^{\dagger}) \right].$$
 (D6)

This is Eq. (6) of the main text. It is a rather general result, not assuming a particular distribution of the components, only that they should be independent and identically distributed and have mean 0.

Now, we specialize to the case that the components Z_i are each normally distributed with mean 0 and variance 1/D. Then

$$E[Z_i^2] = \frac{1}{D}E[(\sqrt{D}Z_i)^2] = \frac{1}{D}$$
 (D7)

and

$$E[Z_i^4] = \frac{1}{D^2} E[(\sqrt{D}Z_i)^4] = \frac{3}{D^2},$$
 (D8)

because $\sqrt{DZ_i}$ is normally distributed with mean 0 and variance 1. Plugging this into the above formula, we get

$$\operatorname{var}(\langle A \rangle) = \left(\frac{3}{D^2} - \frac{3}{D^2}\right) \sum_{i} A_{ii}^2$$

$$+ \frac{1}{D^2} \left[\operatorname{tr}(A^2) + \operatorname{tr}(AA^{\dagger}) \right]$$

$$= \frac{1}{D^2} \left[\operatorname{tr}(A^2) + \operatorname{tr}(AA^{\dagger}) \right].$$
 (D10)

This concludes the proof of Eq. (7).

2. Including 2-point correlations

We now relax the assumption of independence and allow the wavefunction components to have 2-point correlations. The trace expression in this case is Eq. (20), which we will now prove.

Assume Z=LX, where X is a vector whose independent components are Gaussian-distributed with mean zero and variance 1, and L is the Cholesky root of a non-random $D \times D$ (covariance-) matrix Σ , i.e $\Sigma = LL^t$. Then Z has mean zero and the covariances between components of Z are given by Σ . We compute the statistical variance of the EEVs as

$$var(\langle A \rangle) = var((LX)^t A L X)$$

$$= var(X^t L^t A L X)$$

$$= var(1/\sqrt{D}X^t D L^t A L 1/\sqrt{D}X) \qquad (D11)$$

Because the components of $1/\sqrt{D}X$ are normally distributed with mean 0 and variance 1/D, we can use Eq. (7) with A replaced by L^tAL . This leads to

$$\operatorname{var}(\langle A \rangle) = \frac{1}{D^2} (\operatorname{tr}(D^2(L^t A L)^2) + \operatorname{tr}(D^2 L^t A L (L^t A L)^t))$$
$$= \frac{1}{D^2} (\operatorname{tr}((D \Sigma A)^2) + \operatorname{tr}(D \Sigma A D \Sigma A^t), \quad (D12)$$

which concludes the proof of Eq. (20).

Appendix E: Scaling of operators

In this section we derive results announced in the main text concerning the scaling of the traces. For $A = a_j^{\dagger} a_i$ we prove the trace expression of Eq. (8). For the classical limit, we prove the scaling relation, Eq. (9), for A being a (finite) linear combination of terms of the type $a_j^{\dagger} a_i$.

Considering A to be such a linear combination, A^2 and $A^{\dagger}A$ can be written as a sum of terms $a_j^{\dagger}a_ia_{j'}^{\dagger}a_{i'}$. Consider a defining basis state $|\psi\rangle = |n_1, \dots, n_k\rangle$, which has n_j particles on site j. Then

$$a_j^{\dagger} a_i |\psi\rangle = \delta_{ij} n_i |\psi\rangle$$

$$+ (1 - \delta_{ij}) \sqrt{n_j + 1} \sqrt{n_i} |\dots, n_i - 1, n_j + 1, \dots\rangle.$$
(E1)

Using Eq. (E1) twice we get

$$\langle \psi | a_j^{\dagger} a_i a_{j'}^{\dagger} a_{i'} | \psi \rangle = \delta_{i'j'} \delta_{ij} n_{i'} n_i + (1 - \delta_{i'j'}) (1 - \delta_{ij}) \delta_{ij'} \delta_{ji'} (n_{j'} + 1) n_{i'}. \quad (E2)$$

Now we will calculate $\operatorname{tr}(a_j^{\dagger}a_ia_{j'}^{\dagger}a_{i'})$. First let i=j=i'=j'. Using that there are $\binom{N-l+k-2}{k-2}$ states with l particles on site i, we get

$$\operatorname{tr}(n_i^2) = \sum_{l=0}^{N} l^2 \binom{N-l+k-2}{k-2}.$$
 (E3)

Writing l^2 in terms of binomial coefficients and exploiting an upper index Vandermonde identity, namely

$$\sum_{l=0}^{n} {l \choose c_1} {n-l \choose c_2} = {n+1 \choose c_1 + c_2 + 1},$$
(E4)

for constants n, c_1 and c_2 , we get

$$tr(n_i^2) = \frac{N(2N+k-1)}{k(k+1)}D.$$
 (E5)

Now let i=j and i'=j' but $i\neq i'$. There are $\binom{N-l-s+k-3}{k-3}$ many states with l particles on site i and s particles on site i', so

$$\operatorname{tr}(n_i n_{i'}) = \sum_{l=0}^{N} \sum_{s=0}^{N-l} ls \binom{N-l-s+k-3}{k-3}.$$
 (E6)

Invoking the Vandermonde identity Eq. (E4) twice gives us

$$\operatorname{tr}(n_i n_{i'}) = \frac{N(N-1)}{(k+1)k} D.$$
 (E7)

The case $i \neq j$ and $i' \neq j'$, but j = i' and i = j', works exactly the same as i = j and i' = j' but $i \neq i'$. Using Eq. (E4) twice on

$$\operatorname{tr}(a_j^{\dagger} a_i a_i^{\dagger} a_j) = \sum_{l=0}^{N} \sum_{s=0}^{N-l} (l+1) s \binom{N-l-s+k-3}{k-3}$$
(E8)

results in

$$\operatorname{tr}(a_j^{\dagger} a_i a_i^{\dagger} a_j) = \frac{N(N+k)}{k(k+1)} D.$$
 (E9)

In the classical limit, $k \ll N$, Eqs. (E5), (E7) and (E9) all scale as $\sim N^2 D$, so the corresponding trace expressions of all typical observables scale as

$$\operatorname{tr}(A^2) + \operatorname{tr}(A^{\dagger}A) \sim N^2 D. \tag{E10}$$

In the thermodynamic limit, $k = \rho N$ for a density ρ

not depending on N nor k and $k, N \to \infty$, the trace expressions scale as

$$\operatorname{tr}(A^2) + \operatorname{tr}(A^{\dagger}A) \sim D,$$
 (E11)

because the leading order of k and N is quadratic in both the numerators and the denumerators in Eqs. (E5), (E7) and (E9). By combining this with the results in Eq. (D) we rediscover the $D^{-1/2}$ dependence of the EEV fluctuations in the thermodynamic limit for Gaussian states and observables A, where $\operatorname{tr}(A^2)$ and $\operatorname{tr}(AA^{\dagger})$ are bounded in D.

- J. M. Deutsch, "Quantum statistical mechanics in a closed system," Phys. Rev. A 43, 2046-2049 (1991).
- [2] Mark Srednicki, "Chaos and quantum thermalization," Phys. Rev. E 50, 888–901 (1994).
- [3] Mark Srednicki, "Thermal fluctuations in quantized chaotic systems," Journal of Physics A: Mathematical and General 29, L75 (1996).
- [4] Mark Srednicki, "The approach to thermal equilibrium in quantized chaotic systems," Journal of Physics A: Mathematical and General 32, 1163 (1999).
- [5] Marcos Rigol, Vanja Dunjko, and Maxim Olshanii, "Thermalization and its mechanism for generic isolated quantum systems," Nature 452, 854–858 (2008).
- [6] Anatoli Polkovnikov, Krishnendu Sengupta, Alessandro Silva, and Mukund Vengalattore, "Colloquium: Nonequilibrium dynamics of closed interacting quantum systems," Rev. Mod. Phys. 83, 863–883 (2011).
- [7] Marcos Rigol and Mark Srednicki, "Alternatives to eigenstate thermalization," Phys. Rev. Lett. 108, 110601 (2012).
- [8] Jens Eisert, Mathis Friesdorf, and Christian Gogolin, "Quantum many-body systems out of equilibrium," Nature Physics 11, 124–130 (2015).
- [9] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, "From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics," Advances in Physics 65, 239 (2016).
- [10] Clemens Neuenhahn and Florian Marquardt, "Thermalization of interacting fermions and delocalization in fock space," Phys. Rev. E 85, 060101 (2012).
- [11] W. Beugeling, R. Moessner, and Masudul Haque, "Finite-size scaling of eigenstate thermalization," Phys. Rev. E 89, 042112 (2014).
- [12] Wouter Beugeling, Roderich Moessner, and Masudul Haque, "Off-diagonal matrix elements of local operators in many-body quantum systems," Phys. Rev. E 91, 012144 (2015).
- [13] Simone Ziraldo and Giuseppe E. Santoro, "Relaxation and thermalization after a quantum quench: Why localization is important," Phys. Rev. B 87, 064201 (2013).
- [14] Tatsuhiko N. Ikeda, Yu Watanabe, and Masahito Ueda, "Finite-size scaling analysis of the eigenstate thermalization hypothesis in a one-dimensional interacting bose gas," Phys. Rev. E 87, 012125 (2013).
- [15] Vincenzo Alba, "Eigenstate thermalization hypothesis and integrability in quantum spin chains," Phys. Rev.

- B **91**, 155123 (2015).
- [16] Sourav Nandy, Arnab Sen, Arnab Das, and Abhishek Dhar, "Eigenstate gibbs ensemble in integrable quantum systems," Phys. Rev. B 94, 245131 (2016).
- [17] Javier M. Magán, "Random free fermions: An analytical example of eigenstate thermalization," Phys. Rev. Lett. 116, 030401 (2016).
- [18] M. Haque and P. A. McClarty, "Eigenstate thermalization scaling in majorana clusters: From chaotic to integrable sachdev-ye-kitaev models," Phys. Rev. B 100, 115122 (2019).
- [19] Marcin Mierzejewski and Lev Vidmar, "Quantitative impact of integrals of motion on the eigenstate thermalization hypothesis," Phys. Rev. Lett. 124, 040603 (2020).
- [20] Tyler LeBlond, Krishnanand Mallayya, Lev Vidmar, and Marcos Rigol, "Entanglement and matrix elements of observables in interacting integrable systems," Phys. Rev. E 100, 062134 (2019).
- [21] Marcos Rigol, "Breakdown of thermalization in finite one-dimensional systems," Phys. Rev. Lett. 103, 100403 (2009).
- [22] Giulio Biroli, Corinna Kollath, and Andreas M. Läuchli, "Effect of rare fluctuations on the thermalization of isolated quantum systems," Phys. Rev. Lett. 105, 250401 (2010).
- [23] Marcos Rigol and Lea F. Santos, "Quantum chaos and thermalization in gapped systems," Phys. Rev. A 82, 011604 (2010).
- [24] Lea F. Santos and Marcos Rigol, "Localization and the effects of symmetries in the thermalization properties of one-dimensional quantum systems," Phys. Rev. E 82, 031130 (2010).
- [25] Guillaume Roux, "Finite-size effects in global quantum quenches: Examples from free bosons in an harmonic trap and the one-dimensional Bose-Hubbard model," Phys. Rev. A 81, 053604 (2010).
- [26] Atsushi Motohashi, "Thermalization of atom-molecule bose gases in a double-well potential," Phys. Rev. A 84, 063631 (2011).
- [27] G. P. Brandino, A. De Luca, R. M. Konik, and G. Mussardo, "Quench dynamics in randomly generated extended quantum models," Phys. Rev. B 85, 214435 (2012).
- [28] R. Steinigeweg, J. Herbrych, and P. Prelovšek, "Eigenstate thermalization within isolated spin-chain sys-

- tems," Phys. Rev. E 87, 012118 (2013).
- [29] Hyungwon Kim, Tatsuhiko N. Ikeda, and David A. Huse, "Testing whether all eigenstates obey the eigenstate thermalization hypothesis," Phys. Rev. E 90, 052105 (2014).
- [30] S. Sorg, L. Vidmar, L. Pollet, and F. Heidrich-Meisner, "Relaxation and thermalization in the one-dimensional bose-hubbard model: A case study for the interaction quantum quench from the atomic limit," Phys. Rev. A 90, 033606 (2014).
- [31] R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, "Pushing the limits of the eigenstate thermalization hypothesis towards mesoscopic quantum systems," Phys. Rev. Lett. 112, 130403 (2014).
- [32] Keith R. Fratus and Mark Srednicki, "Eigenstate thermalization in systems with spontaneously broken symmetry," Phys. Rev. E 92, 040103 (2015).
- [33] Rahul Nandkishore and David A. Huse, "Many-Body Localization and Thermalization in Quantum Statistical Mechanics," Annual Review of Condensed Matter Physics 6, 15–38 (2015).
- [34] Abdellah Khodja, Robin Steinigeweg, and Jochen Gemmer, "Relevance of the eigenstate thermalization hypothesis for thermal relaxation," Phys. Rev. E 91, 012120 (2015).
- [35] Rubem Mondaini, Keith R. Fratus, Mark Srednicki, and Marcos Rigol, "Eigenstate thermalization in the two-dimensional transverse field ising model," Phys. Rev. E 93, 032104 (2016).
- [36] A. Chandran, Marc D. Schulz, and F. J. Burnell, "The eigenstate thermalization hypothesis in constrained hilbert spaces: A case study in non-abelian anyon chains," Phys. Rev. B **94**, 235122 (2016).
- [37] David J. Luitz and Yevgeny Bar Lev, "Anomalous thermalization in ergodic systems," Phys. Rev. Lett. 117, 170404 (2016).
- [38] Rubem Mondaini and Marcos Rigol, "Eigenstate thermalization in the two-dimensional transverse field ising model. ii. off-diagonal matrix elements of observables," Phys. Rev. E 96, 012157 (2017).
- [39] Julian Sonner and Manuel Vielma, "Eigenstate thermalization in the sachdev-ye-kitaev model," Journal of High Energy Physics 2017, 149 (2017).
- [40] Zhihao Lan and Stephen Powell, "Eigenstate thermalization hypothesis in quantum dimer models," Phys. Rev. B 96, 115140 (2017).
- [41] Toru Yoshizawa, Eiki Iyoda, and Takahiro Sagawa, "Numerical large deviation analysis of the eigenstate thermalization hypothesis," Phys. Rev. Lett. 120, 200604 (2018).
- [42] Anatoly Dymarsky, Nima Lashkari, and Hong Liu, "Subsystem eigenstate thermalization hypothesis," Phys. Rev. E 97, 012140 (2018).
- [43] Nicholas Hunter-Jones, Junyu Liu, and Yehao Zhou, "On thermalization in the syk and supersymmetric syk models," Journal of High Energy Physics 2018, 142 (2018).
- [44] David Jansen, Jan Stolpp, Lev Vidmar, and Fabian Heidrich-Meisner, "Eigenstate thermalization and quantum chaos in the holstein polaron model," Phys. Rev. B 99, 155130 (2019).
- [45] Ivan M. Khaymovich, Masudul Haque, and Paul A. McClarty, "Eigenstate thermalization, random matrix theory, and behemoths," Phys. Rev. Lett. 122, 070601

- (2019).
- [46] Ryusuke Hamazaki and Masahito Ueda, "Atypicality of most few-body observables," Phys. Rev. Lett. 120, 080603 (2018).
- [47] Vedika Khemani, Chris R. Laumann, and Anushya Chandran, "Signatures of integrability in the dynamics of rydberg-blockaded chains," Phys. Rev. B 99, 161101 (2019).
- [48] Marlon Brenes, Tyler LeBlond, John Goold, and Marcos Rigol, "Eigenstate thermalization in a locally perturbed integrable system," Phys. Rev. Lett. 125, 070605 (2020).
- [49] Anatoli Polkovnikov, "Quantum corrections to the dynamics of interacting bosons: Beyond the truncated wigner approximation," Phys. Rev. A 68, 053604 (2003).
- [50] Khan W. Mahmud, Heidi Perry, and William P. Reinhardt, "Quantum phase-space picture of bose-einstein condensates in a double well," Phys. Rev. A 71, 023615 (2005).
- [51] Moritz Hiller, Tsampikos Kottos, and T. Geisel, "Complexity in parametric bose-hubbard hamiltonians and structural analysis of eigenstates," Phys. Rev. A 73, 061604 (2006).
- [52] S. Mossmann and C. Jung, "Semiclassical approach to bose-einstein condensates in a triple well potential," Phys. Rev. A 74, 033601 (2006).
- [53] E. M. Graefe and H. J. Korsch, "Semiclassical quantization of an n-particle bose-hubbard model," Phys. Rev. A 76, 032116 (2007).
- [54] F. Trimborn, D. Witthaut, and H. J. Korsch, "Beyond mean-field dynamics of small bose-hubbard systems based on the number-conserving phase-space approach." Phys. Rev. A 79, 013608 (2009).
- [55] Anatoli Polkovnikov, "Phase space representation of quantum dynamics," Annals of Physics 325, 1790 (2010).
- [56] Krzysztof Pawłowski, Paweł Ziń, Kazimierz Rzażewski, and Marek Trippenbach, "Revivals in an attractive bose-einstein condensate in a double-well potential and their decoherence," Phys. Rev. A 83, 033606 (2011).
- [57] Lena Simon and Walter T. Strunz, "Analytical results for josephson dynamics of ultracold bosons," Phys. Rev. A 86, 053625 (2012).
- [58] Lena Simon and Walter T. Strunz, "Time-dependent semiclassics for ultracold bosons," Phys. Rev. A 89, 052112 (2014).
- [59] Hagar Veksler and Shmuel Fishman, "Semiclassical analysis of bose–hubbard dynamics," New Journal of Physics 17, 053030 (2015).
- [60] Thomas Engl, Juan Diego Urbina, and Klaus Richter, "Periodic mean-field solutions and the spectra of discrete bosonic fields: Trace formula for bose-hubbard models," Phys. Rev. E 92, 062907 (2015).
- [61] Thomas Engl, Juan Diego Urbina, and Klaus Richter, "The semiclassical propagator in fock space: dynamical echo and many-body interference," Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 374, 20150159 (2016).
- [62] Shouryya Ray, Paula Ostmann, Lena Simon, Frank Grossmann, and Walter T. Strunz, "Dynamics of interacting bosons using the herman-kluk semiclassical initial value representation," Journal of Physics A: Mathematical and Theoretical 49, 165303 (2016).

- [63] Rémy Dubertrand and Sebastian Müller, "Spectral statistics of chaotic many-body systems," New Journal of Physics 18, 033009 (2016).
- [64] Steven Tomsovic, Peter Schlagheck, Denis Ullmo, Juan-Diego Urbina, and Klaus Richter, "Post-ehrenfest many-body quantum interferences in ultracold atoms far out of equilibrium," Phys. Rev. A 97, 061606 (2018).
- [65] Steven Tomsovic, "Complex saddle trajectories for multidimensional quantum wave packet and coherent state propagation: Application to a many-body system," Phys. Rev. E 98, 023301 (2018).
- [66] Josef Rammensee, Juan Diego Urbina, and Klaus Richter, "Many-Body Quantum Interference and the Saturation of Out-of-Time-Order Correlators," Physical Review Letters 121, 124101 (2018).
- [67] R. A. Kidd, M. K. Olsen, and J. F. Corney, "Quantum chaos in a bose-hubbard dimer with modulated tunneling," Phys. Rev. A 100, 013625 (2019).
- [68] Peter Schlagheck, Denis Ullmo, Juan Diego Urbina, Klaus Richter, and Steven Tomsovic, "Enhancement of many-body quantum interference in chaotic bosonic systems: The role of symmetry and dynamics," Phys. Rev. Lett. 123, 215302 (2019).
- [69] K. Nemoto, C. A. Holmes, G. J. Milburn, and W. J. Munro, "Quantum dynamics of three coupled atomic bose-einstein condensates," Phys. Rev. A 63, 013604 (2000).
- [70] Christoph Weiss and Niklas Teichmann, "Differences between mean-field dynamics and *n*-particle quantum dynamics as a signature of entanglement," Phys. Rev. Lett. **100**, 140408 (2008).
- [71] Thiago F Viscondi and K Furuya, "Dynamics of a bose-einstein condensate in a symmetric triple-well trap," Journal of Physics A: Mathematical and Theoretical 44, 175301 (2011).
- [72] Bettina Gertjerenken and Christoph Weiss, "Beyond-mean-field behavior of large bose-einstein condensates in double-well potentials," Phys. Rev. A 88, 033608 (2013).
- [73] Andrey R. Kolovsky, "Bose-hubbard hamiltonian: Quantum chaos approach," Int. J. Mod. Phys. B 30, 1630009 (2016).
- [74] Christoph Heinisch and Martin Holthaus, "Entropy production within a pulsed bose-einstein condensate," Zeitschrift für Naturforschung A 71, 875 (2016).
- [75] Michael Rautenberg and Martin Gärttner, "Classical and quantum chaos in a three-mode bosonic system," Phys. Rev. A 101, 053604 (2020).
- [76] Moritz Hiller, Tsampikos Kottos, and Theo Geisel, "Wave-packet dynamics in energy space of a chaotic trimeric bose-hubbard system," Phys. Rev. A 79, 023621 (2009).
- [77] Geva Arwas, Amichay Vardi, and Doron Cohen, "Triangular bose-hubbard trimer as a minimal model for a superfluid circuit," Phys. Rev. A 89, 013601 (2014).
- [78] Xizhi Han and Biao Wu, "Ehrenfest breakdown of the mean-field dynamics of bose gases," Phys. Rev. A 93, 023621 (2016).
- [79] R. Bürkle and J. R. Anglin, "Threshold coupling strength for equilibration between small systems," Phys. Rev. A 99, 063617 (2019).
- [80] Anna A. Bychek, Pavel S. Muraev, Dmitrii N. Maksimov, Evgeny N. Bulgakov, and Andrey R. Kolovsky, "Chaotic and regular dynamics in the three-site bose-

- hubbard model," AIP Conference Proceedings **2241**, 020007 (2020).
- [81] A. R Kolovsky and A Buchleitner, "Quantum chaos in the bose-hubbard model," Europhysics Letters (EPL) 68, 632 (2004).
- [82] Corinna Kollath, Guillaume Roux, Giulio Biroli, and Andreas M Läuchli, "Statistical properties of the spectrum of the extended bose-hubbard model," Journal of Statistical Mechanics: Theory and Experiment 2010, P08011 (2010).
- [83] Javier de la Cruz, Sergio Lerma-Hernández, and Jorge G. Hirsch, "Quantum chaos in a system with high degree of symmetries," Phys. Rev. E 102, 032208 (2020).
- [84] Lukas Pausch, Edoardo G Carnio, Alberto Rodríguez, and Andreas Buchleitner, "Chaos and ergodicity across the energy spectrum of interacting bosons," arXiv preprint arXiv:2009.05295 (2020).
- [85] Wouter Beugeling, Arnd Bäcker, Roderich Moessner, and Masudul Haque, "Statistical properties of eigenstate amplitudes in complex quantum systems," Phys. Rev. E 98, 022204 (2018).
- [86] Arnd Bäcker, Masudul Haque, and Ivan M Khaymovich, "Multifractal dimensions for random matrices, chaotic quantum maps, and many-body systems," Phys. Rev. E 100, 032117 (2019).
- [87] David J. Luitz, Ivan M. Khaymovich, and Yevgeny Bar Lev, "Multifractality and its role in anomalous transport in the disordered XXZ spin-chain," SciPost Phys. Core 2, 6 (2020).
- [88] W Beugeling, A Andreanov, and Masudul Haque, "Global characteristics of all eigenstates of local manybody hamiltonians: participation ratio and entanglement entropy," Journal of Statistical Mechanics: Theory and Experiment 2015, P02002 (2015).
- [89] Chunxiao Liu, Xiao Chen, and Leon Balents, "Quantum entanglement of the sachdev-ye-kitaev models," Phys. Rev. B 97, 245126 (2018).
- [90] Masudul Haque, Paul A McClarty, and Ivan M Khaymovich, "Entanglement of mid-spectrum eigenstates of chaotic many-body systems—deviation from random ensembles," arXiv preprint arXiv:2008.12782 (2020).
- [91] J. Gemmer, M. Michel, and G. Mahler, Quantum Thermodynamics: Emergence of Thermodynamic Behavior Within Composite Quantum Systems, Lecture Notes in Physics (Springer, 2009).
- [92] Peter Reimann, "Typicality for generalized microcanonical ensembles," Phys. Rev. Lett. 99, 160404 (2007).
- [93] Peter Reimann, "Typicality of pure states randomly sampled according to the gaussian adjusted projected measure," Journal of Statistical Physics 132, 921 (2008).
- [94] Seth Lloyd, "Pure state quantum statistical mechanics and black holes," arXiv preprint arXiv:1307.0378 (2013).
- [95] Mario Feingold and Asher Peres, "Regular and chaotic motion of coupled rotators," Physica D: Nonlinear Phenomena 9, 433 (1983).
- [96] Mario Feingold, Nimrod Moiseyev, and Asher Peres, "Ergodicity and mixing in quantum theory. ii," Phys. Rev. A 30, 509 (1984).
- [97] Daniel T. Robb and L. E. Reichl, "Chaos in a two-spin system with applied magnetic field," Phys. Rev. E 57, 2458–2459 (1998).

- [98] J. Emerson and L.E. Ballentine, "Characteristics of quantum-classical correspondence for two interacting spins," Phys. Rev. A 63, 052103 (2001).
- [99] L. E. Ballentine, "Quantum-to-classical limit in a hamiltonian system," Phys. Rev. A 70, 032111 (2004).
- [100] Yiyun Fan, Sven Gnutzmann, and Yuqi Liang, "Quantum chaos for nonstandard symmetry classes in the feingold-peres model of coupled tops," Phys. Rev. E 96, 062207 (2017).
- [101] Silvia Pappalardi, Angelo Russomanno, Bojan Žunkovič, Fernando Iemini, Alessandro Silva, and Rosario Fazio, "Scrambling and entanglement spreading in long-range spin chains," Phys. Rev. B 98, 134303 (2018).
- [102] Angelo Piga, Maciej Lewenstein, and James Q. Quach, "Quantum chaos and entanglement in ergodic and nonergodic systems," Phys. Rev. E 99, 032213 (2019).
- [103] Alessio Lerose and Silvia Pappalardi, "Bridging entanglement dynamics and chaos in semiclassical systems," Phys. Rev. A 102, 032404 (2020).
- [104] Tianrui Xu, Thomas Scaffidi, and Xiangyu Cao, "Does scrambling equal chaos?" Phys. Rev. Lett. 124, 140602 (2020).
- [105] Silvia Pappalardi, Anatoli Polkovnikov, and Alessandro Silva, "Quantum echo dynamics in the Sherrington-Kirkpatrick model," SciPost Phys. 9, 21 (2020).
- [106] Debabrata Mondal, Sudip Sinha, and S. Sinha, "Chaos and quantum scars in a coupled top model," Phys. Rev. E 102, 020101 (2020).
- [107] M. A. Bastarrachea-Magnani, S. Lerma-Hernández, and J. G. Hirsch, "Comparative quantum and semiclassical analysis of atom-field systems. ii. chaos and regularity," Phys. Rev. A 89, 032102 (2014).
- [108] Miguel Angel Bastarrachea-Magnani, Baldemar López del Carpio, Sergio Lerma-Hernández, and Jorge G Hirsch, "Chaos in the dicke model: quantum and semiclassical analysis," Physica Scripta 90, 068015 (2015).
- [109] S. Ray, A. Ghosh, and S. Sinha, "Quantum signature of chaos and thermalization in the kicked dicke model," Phys. Rev. E **94**, 032103 (2016).

- [110] M. A. Bastarrachea-Magnani, B. López-del Carpio, J. Chávez-Carlos, S. Lerma-Hernández, and J. G. Hirsch, "Delocalization and quantum chaos in atomfield systems," Phys. Rev. E 93, 022215 (2016).
- [111] RJ Lewis-Swan, Arghavan Safavi-Naini, John J Bollinger, and Ana M Rey, "Unifying scrambling, thermalization and entanglement through measurement of fidelity out-of-time-order correlators in the dicke model," Nature communications 10, 1–9 (2019).
- [112] Jorge Chávez-Carlos, B. López-del Carpio, Miguel A. Bastarrachea-Magnani, Pavel Stránský, Sergio Lerma-Hernández, Lea F. Santos, and Jorge G. Hirsch, "Quantum and classical lyapunov exponents in atom-field interaction systems," Phys. Rev. Lett. 122, 024101 (2019).
- [113] Qian Wang and Marko Robnik, "Statistical properties of the localization measure of chaotic eigenstates in the dicke model," Phys. Rev. E 102, 032212 (2020).
- [114] D Villaseñor, S Pilatowsky-Cameo, M A Bastarrachea-Magnani, S Lerma-Hernández, L F Santos, and J G Hirsch, "Quantum vs classical dynamics in a spin-boson system: manifestations of spectral correlations and scarring," New Journal of Physics 22, 063036 (2020).
- [115] Saúl Pilatowsky-Cameo, Jorge Chávez-Carlos, Miguel A. Bastarrachea-Magnani, Pavel Stránský, Sergio Lerma-Hernández, Lea F. Santos, and Jorge G. Hirsch, "Positive quantum lyapunov exponents in experimental systems with a regular classical limit," Phys. Rev. E 101, 010202 (2020).
- [116] Sudip Sinha and S. Sinha, "Chaos and quantum scars in bose-josephson junction coupled to a bosonic mode," Phys. Rev. Lett. 125, 134101 (2020).
- [117] Vadim Oganesyan and David A. Huse, "Localization of interacting fermions at high temperature," Phys. Rev. B 75, 155111 (2007).
- [118] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, "Distribution of the ratio of consecutive level spacings in random matrix ensembles," Phys. Rev. Lett. 110, 084101 (2013).