

# Quantum phase transition in a non-Hermitian XY spin chain with global complex transverse field

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In this work, we investigate the quantum phase transition in a non-Hermitian XY spin chain. The phase diagram shows that the critical points of Ising phase transition expand into a critical transition zone after introducing a non-Hermitian effect. By analyzing the non-Hermitian gap and long-range correlation function, one can distinguish different phases by means of different gap features and decay properties of correlation function, a tricky problem in traditional XY model. Furthermore, the results reveal the relationship among different regions of the phase diagram, non-Hermitian energy gap and long-range correlation function.

## I. INTRODUCTION

Quantum XY spin chain is a textbook model in exploring quantum magnetism and quantum phase transitions (QPTs) [1]. It is extended from one-dimensional transverse field Ising model by adding the spin-spin interaction along another direction. There are two types of QPTs in the XY model, i.e., Ising phase transition [2, 3] and anisotropic phase transition [3–5], which can be characterized by different critical behaviors. Since most XY models and their derivatives can be analytically solved by Jordan-Wigner transformation [6] or numerically solved by the renormalization group method [7–10], XY model has attracted wide attention and fruitful research results have been obtained during the past decades [5, 11–21]. On the other hand, great attention has been paid to the non-Hermitian systems as experimental techniques develop rapidly in recent years. Not only can non-Hermitian systems be readily realized through multiple existing table-top experimental platforms (such as cold atomic system [22, 23], optical system [24–26], nitrogen-vacancy center [27], etc.), but they can also trigger many novel physical phenomena (such as real eigenvalues with parity-time ( $\mathcal{PT}$ ) symmetry [28], non-Hermitian skin effect [29, 30], new topological properties corresponding to exceptional points (EPs) [31–37] and disorders [38–41]). What would happen if QPTs meet non-Hermitian effects?

In general, QPTs fall into two broad categories: traditional QPTs [1, 42–44] and topological QPTs [45–50]. The former can be depicted by local order parameters, while the latter are characterized by the global topo-

logical invariants. The non-Hermiticity gives rise to a brand-new phase transition, so called as non-Hermitian QPTs, characterized by energy spectrum. The new phase transition is closely related to particular symmetries, for instance,  $\mathcal{PT}$  symmetry and intrinsic rotation-time-reversal ( $\mathcal{RT}$ ) symmetry [42, 51–54]. The system features pure real energy spectrum in symmetry-preserving region, whereas it possesses complex energy spectrum in the region of broken symmetry [51, 52]. Recent years have witnessed extensive investigation of the non-Hermitian QPTs and there are also some works for the spin system in the complex field or with the non-Hermitian interactions [21, 55–57]. However, the research remains inadequate on the influence of non-Hermiticity on the traditional QPTs in spin system. Recently, there are some works that investigated non-Hermitian quantum criticality in real-spectrum region by biorthogonal fidelity susceptibility [44, 58]. But it is still an open question that how the non-Hermiticity affects the traditional QPT and quantum magnetism in the complex-spectrum region or the system without  $\mathcal{PT}$  or  $\mathcal{RT}$  symmetry.

This paper is devoted to the research on the traditional QPTs in the presence of non-Hermitian effects, which are derived from global complex transverse field. First, we define the non-Hermitian ground state and energy gap. Based on characteristics of the energy gap, phase diagram of the system can be divided into three regions, which are corresponding to pure real gap, pure imaginary gap and complex gap, respectively. Besides, all gapless points form an exceptional ring. By studying the non-analyticity of the ground state energy density, we find that the phase transition which occurs on the exceptional ring is actually a second-order phase transition. Moreover, through the exact solution and numerical fitting, we investigate the long-range correlation function (LRCF) in different regions of the phase diagram. We discover the

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corresponding relations among the QPT, non-Hermitian energy gap and LRCF, i.e., ferromagnetic phase (critical transition zone, paramagnetic phase) corresponds to pure real (pure imaginary, complex) gap, whose LRCF features no decay (polynomial decay, exponential decay).

The paper is organized as follows. In Sec. 2, the model, the exact solution and its non-Hermitian ground state are provided. In Sec. 3, we define the non-Hermitian gap, draw the phase diagram and point out that the QPT on the exceptional ring is the second-order phase transition. In Sec. 4, we study the LRCF and analyze its decay behavior. Sec. 5 is the conclusion.

## II. MODEL AND NON-HERMITIAN GROUND STATE

The Hamiltonian of a non-Hermitian quantum XY spin chain with a global complex transverse field reads

$$H = - \sum_{j=-N/2}^{N/2-1} \left( \frac{J+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{J-\gamma}{2} \sigma_j^y \sigma_{j+1}^y \right) - \sum_{j=-N/2}^{N/2} \left( \lambda \sigma_j^z + \frac{i\Gamma}{2} \sigma_j^u \right), \quad (1)$$

where  $N$  denotes the total sites number of the chain. Since  $N$  is large enough,  $N/2$  could be considered a “decent half” no matter  $N$  is even or odd. The Hilbert space on each site has a set of basis vectors of two spin states, i.e.,  $|\uparrow\rangle$  and  $|\downarrow\rangle$ .  $\sigma_j^x$ ,  $\sigma_j^y$  and  $\sigma_j^z$  are the Pauli matrices for spin  $j$  and  $\sigma^u$  denotes the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , which corresponds to the gain ( $\Gamma < 0$ ) or loss ( $\Gamma > 0$ ) of  $|\uparrow\rangle$  with the rate of  $\Gamma/2$ . The system can be reduced to Hermitian XY model when  $\Gamma = 0$ .  $\lambda$  parameterizes transverse magnetic field along  $z$  direction.  $J$  ( $\gamma$ ) represents the isotropic (anisotropic) nearest neighboring interaction strength between  $x$  and  $y$  directions. All the parameters are real numbers. The sign of  $J$  determines that the system is characterized by ferromagnetic or antiferromagnetic chains. Since we hereby just concentrate on the ferromagnetism case,  $J > 0$  and  $\gamma/J \in [-1, 1]$  are taken into consideration. Besides,  $J$  as the system character parameter is set as unit 1 later in this paper.

We can diagonalize the Hamiltonian in Eq. (1) by three steps. First, we map spin operators onto spinless Fermi operators by Jordan-Wigner transformation, i.e.,

$$\begin{aligned} \sigma_j &= e^{-i\pi \sum_{l<j} c_l^\dagger c_l} c_j, & \sigma_j^\dagger &= c_j^\dagger e^{i\pi \sum_{l<j} c_l^\dagger c_l}, \\ c_j &= e^{i\pi \sum_{l<j} \sigma_l^\dagger \sigma_l} \sigma_j, & c_j^\dagger &= \sigma_j^\dagger e^{-i\pi \sum_{l<j} \sigma_l^\dagger \sigma_l}, \end{aligned} \quad (2)$$

where  $\sigma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $\sigma^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , and  $c_j$  ( $c_j^\dagger$ ) denotes the annihilation (creation) operators of the fermion on the  $j$ th site. Then, we can obtain the Hamiltonian in the

spinless fermion representation as

$$H = - \sum_{j=-N/2}^{N/2-1} \left( J c_j^\dagger c_{j+1} + \gamma c_j^\dagger c_{j+1}^\dagger + h.c. \right) + \sum_{j=-N/2}^{N/2} \left[ \lambda \mathbf{1}_j - (2\lambda + i\frac{\Gamma}{2}) c_j^\dagger c_j \right]. \quad (3)$$

Second, we do Fourier transformation with  $c_k = \frac{e^{i\pi/4}}{\sqrt{N}} \sum_j \exp(-ikj) c_j$ . The corresponding Hamiltonian in momentum space reads,

$$H = \sum_{k=0}^{\pi} \left[ -d_0(k) + (c_k^\dagger \ c_{-k}) \mathcal{H}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \right]. \quad (4)$$

$\mathcal{H}(k)$  is the BdG Hamiltonian,

$$\mathcal{H}(k) = d_x(k) \tau_x + d_y(k) \tau_y + d_z(k) \tau_z, \quad (5)$$

where the  $\tau_i$  ( $i = x, y, z$ ) is the Pauli matrix of pseudo spin and

$$\begin{aligned} d_0(k) &= 2J \cos k + i\Gamma/2, \\ d_x(k) &= 2\gamma \sin k, \\ d_y(k) &= 0, \\ d_z(k) &= -(2\lambda + 2J \cos k + i\Gamma/2). \end{aligned} \quad (6)$$

Third, we use non-Hermitian Bogoliubov transformation to complete diagonalization of Hamiltonian in Eq. (4) following the method outlined in Ref. [42]. In Ref. [42], the authors investigate nonequilibrium steady state, which has minimum imaginary part of eigenvalue. Here, we focus on the state with minimum real part of energy, which is defined as the non-Hermitian ground state in this paper. Thus, our definitions of the  $u$ ,  $v$  factors of Bogoliubov transformation are different from that in Ref. [42].

We define the annihilation ( $\alpha$ ) and creation ( $\bar{\alpha}$ ) operators of non-Hermitian Bogoliubov quasiparticles as

$$\begin{aligned} \alpha_k &= u_k c_k + v_k c_{-k}^\dagger, & \alpha_{-k} &= -v_k c_k^\dagger + u_k c_{-k}, \\ \bar{\alpha}_k &= u_k c_k^\dagger + v_k c_{-k}, & \bar{\alpha}_{-k} &= -v_k c_k + u_k c_{-k}^\dagger. \end{aligned} \quad (7)$$

The  $u$ ,  $v$  factors are defined as  $u_k = \cos(\theta_k/2)$  and  $v_k = \sin(\theta_k/2)$  with the relations  $\sin \theta_k = d_x(k)/E_k$ ,  $\cos \theta_k = d_z(k)/E_k$ , where

$$E_k = \sqrt{d_x(k)^2 + d_z(k)^2}. \quad (8)$$

Then, the Hamiltonian is diagonalized as

$$H = - \sum_{k=0}^{\pi} (d_0(k) + E_k) + \sum_{k=0}^{\pi} E_k (\bar{\alpha}_k \alpha_k + \bar{\alpha}_{-k} \alpha_{-k}). \quad (9)$$

Note that, since  $u$ ,  $v$  are complex numbers,  $\alpha_k^\dagger \neq \bar{\alpha}_k$ . The fermionic commutation relation is still held, i.e.,  $\{\bar{\alpha}_k, \alpha_{k'}\} = \delta_{kk'}$  and  $\{\alpha_k, \alpha_{k'}\} = \{\bar{\alpha}_k, \bar{\alpha}_{k'}\} = 0$ . As for

the excited state energy  $E_k$ , we choose the branch cut of square root along the negative x-axis so that the real part of  $E_k$  is always nonnegative. Therefore, we can define the non-Hermitian ground state  $|\mathcal{G}\rangle$  as the state with the minimum real part of energy.  $|\mathcal{G}\rangle$  can be obtained by the equation  $\alpha_k|\mathcal{G}\rangle = \alpha_{-k}|\mathcal{G}\rangle = 0$ , i.e.,

$$|\mathcal{G}\rangle = \prod_{k=0}^{\pi} \frac{1}{\sqrt{\mathcal{N}}} (u_k - v_k c_k^\dagger c_{-k}^\dagger) |0\rangle, \quad (10)$$

where  $|0\rangle$  is the vacuum state,  $\mathcal{N} = \prod_{k=0}^{\pi} (|u_k|^2 + |v_k|^2)$  and the ground state energy  $E_G$  is given as

$$E_G = - \sum_{k=0}^{\pi} (d_0(k) + E_k). \quad (11)$$

### III. NON-HERMITIAN GAP AND PHASE DIAGRAM

Due to the nature of non-Hermiticity, the excited state  $\bar{\alpha}_{\pm k}|0\rangle$  has a complex energy  $E_k$  described in Eq. (8). Notably, the  $E_k$  have nonnegative real part by the negative-x-axis branch cut of square root, so we can find a momentum  $k_m$  to corresponds to the least real part of excitation energy, i.e., minimum value of  $\text{Re}[E(k_m)]$ . We define the minimum value as the non-Hermitian gap, which is denote as  $\Delta$ . In Hermitian systems, gapless points are usually the boundary of different phases. A natural question is whether the  $\Delta$  will also become a landmark in non-Hermitian systems. Therefore, we divide the parameter space  $(\lambda, \gamma, \Gamma)$  of the Hamiltonian (1) by  $\Delta$ .

First, by solving the equation  $E_k = 0$ , one can get

$$\frac{\lambda^2}{J^2} + \frac{\Gamma^2}{(4\gamma)^2} = 1, \quad (12a)$$

$$\cos k_m = -\frac{\lambda}{J}. \quad (12b)$$

The Eq. (12a) depicts an elliptical exceptional ring, and the Eq. (12b) implies a limitation  $|\lambda| \leq J$ . Therefore, the system can be separated into three regions as shown in Fig. 1.

Region I:  $|\lambda| > J$ . This region is characterized by a complex gap with  $\text{Re}[\Delta] > 0$  and  $|\text{Im}[\Delta]| = |\Gamma|/2$ , where the  $\text{Re}[\Delta]$  and  $\text{Im}[\Delta]$  represent the real and imaginary part of complex gap  $\Delta$ , respectively. We take  $\lambda > J$  as an example. Under this condition,  $k_m = \pi$  and  $\Delta = 2\lambda - 2J + i\Gamma/2$ .

Region II: inside the exceptional ring. This region has pure real gap, i.e.,  $\text{Re}[\Delta] > 0$  and  $\text{Im}[\Delta] = 0$ . In this region,  $k_m = \arccos(-\frac{\lambda}{J})$  and  $\Delta = \sqrt{4\gamma^2(1 - \lambda^2/J^2) - \Gamma^2/4}$ .

Region III: outside the exceptional ring and  $|\lambda| < J$ . The gap is pure imaginary number ( $\text{Re}[\Delta] = 0$  and  $\text{Im}[\Delta] \neq 0$ ). In this region, we still have  $k_m = \arccos(-\frac{\lambda}{J})$ . However,  $\Delta$  becomes a pure imaginary number because

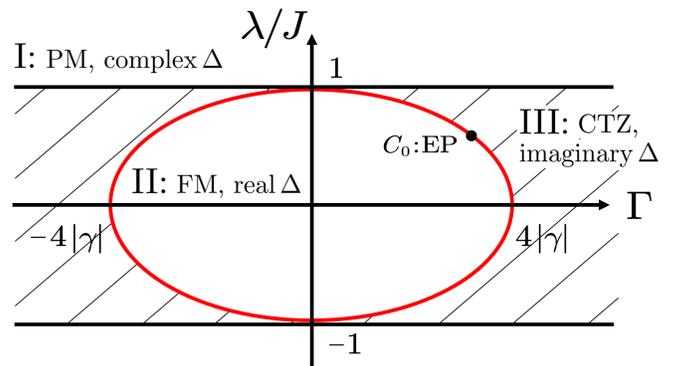


FIG. 1. Phase diagram of the non-Hermitian XY model. The red line denotes the exceptional ring, which corresponds elliptic equation (Eq. (12a)). In region I ( $|\lambda/J| > 1$ ), it is in the paramagnetic (PM) phase with a complex non-Hermitian gap. In region II, it is in the ferromagnetic (FM) phase with a pure real gap. In region III, the shaded area, the non-Hermitian gap is a pure imaginary number. We call region III as critical transition zone (CTZ).

the non-Hermitian strength  $\Gamma$  is large enough to be dominant, then  $\Delta = \pm i\sqrt{\Gamma^2/4 - 4\gamma^2(1 - \lambda^2/J^2)}$ .

Second, we investigate non-analyticity of ground state energy density  $U_G$ , which implies a QPT at zero temperature. Under the thermodynamic limit, we have

$$U_G = \lim_{N \rightarrow \infty} E_G/N = -\frac{1}{2\pi} \int_0^\pi (d_0(k) + E_k) dk. \quad (13)$$

Moreover, the corresponding first-order derivatives reads

$$\frac{\partial U_G}{\partial \lambda} = \frac{-1}{\pi} \int_0^\pi \cos \theta_k dk, \quad (14a)$$

$$\frac{\partial U_G}{\partial \gamma} = \frac{-1}{\pi} \int_0^\pi \sin k \sin \theta_k dk, \quad (14b)$$

$$\frac{\partial U_G}{\partial \Gamma} = \frac{-i}{4} - \frac{1}{4\pi} \int_0^\pi \cos \theta_k dk, \quad (14c)$$

and the second-order derivatives can be obtained as

$$\frac{\partial^2 U_G}{\partial \lambda^2} = \frac{-2}{\pi} \int_0^\pi \frac{\sin^2 \theta_k}{E_k} dk, \quad (15a)$$

$$\frac{\partial^2 U_G}{\partial \gamma^2} = \frac{-2}{\pi} \int_0^\pi \frac{\sin^2 k \cos^2 \theta_k}{E_k} dk, \quad (15b)$$

$$\frac{\partial^2 U_G}{\partial \Gamma^2} = \frac{1}{8\pi} \int_0^\pi \frac{\sin^2 \theta_k}{E_k} dk. \quad (15c)$$

By numerical calculations, we find that  $U_G$  and its first derivatives are continuous while there are divergent points in the second derivatives. The results of second derivatives are plotted in Fig. 2. It is obvious that the second derivatives diverge at the exceptional point  $C_0$ , which implies that a second-order QPT occurs on the exceptional ring.

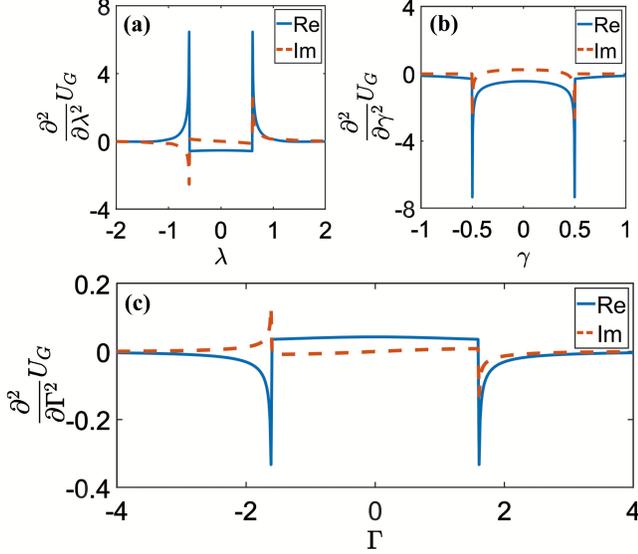


FIG. 2. Second derivatives of ground state energy density versus (a)  $\lambda$  with  $(\gamma, \Gamma) = (0.5, 1.6)$ , (b)  $\gamma$  with  $(\lambda, \Gamma) = (0.6, 1.6)$  and (c)  $\Gamma$  with  $(\lambda, \gamma) = (0.6, 0.5)$ . The blue solid (red dashed) line represent the real (imaginary) part of the derivatives. Throughout, we set  $J = 1$ .

#### IV. LONG-RANGE CORRELATION FUNCTION IN DIFFERENT PHASE

The LRCFs are the order parameters of magnetic system in Hermitian spin models. For example, in ferromagnetic phase, LRCF  $C_{xx}(r) = \langle G | \sigma_0^x \sigma_r^x | G \rangle$  or  $C_{yy}(r) = \langle G | \sigma_0^y \sigma_r^y | G \rangle$  is a non-zero constant, where the parameter  $r$  represents the distance between two spins. In paramagnetic phase, LRCF decays exponentially with the increase of  $r$ . At critical points, correlation functions have asymptotic behavior of polynomial decay, i.e.  $C_{xx}(r) \propto r^{2-d-\eta}$ , where  $d$  and  $\eta$  denote the spatial dimension and the critical exponent, respectively.  $\eta$  varies in different universality classes. As for XY spin chain,  $\eta$  is  $5/4$  for the Ising phase transition at  $\lambda/J = 1$  and  $3/2$  for the anisotropic phase transition at  $\gamma = 0$  [15, 16].

In our model,  $C_{xx}(r)$  can be calculated through the Pfaffian of a  $2r \times 2r$  skew symmetric matrix  $M$  ( $M^T = -M$ ) [42]

$$C_{xx}(r) = \text{pf}(M_{2r \times 2r}) = \text{pf} \begin{pmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{pmatrix}, \quad (16)$$

where the ‘‘pf’’ means the Pfaffian of a matrix. The numerical calculation codes for Pfaffian are from ‘‘Algorithm 923’’ [59]. The elements of matrix  $M$  are as following:

$$M_{11} = \begin{pmatrix} 0 & \langle B_0 B_1 \rangle & \langle B_0 B_2 \rangle & \cdots & \langle B_0 B_{r-1} \rangle \\ -\langle B_0 B_1 \rangle & 0 & \langle B_1 B_2 \rangle & \cdots & \langle B_1 B_{r-1} \rangle \\ -\langle B_0 B_2 \rangle & -\langle B_1 B_2 \rangle & 0 & \cdots & \langle B_2 B_{r-1} \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\langle B_0 B_{r-2} \rangle & -\langle B_1 B_{r-2} \rangle & \cdots & 0 & \langle B_{r-2} B_{r-1} \rangle \\ -\langle B_0 B_{r-1} \rangle & -\langle B_1 B_{r-1} \rangle & \cdots & -\langle B_{r-2} B_{r-1} \rangle & 0 \end{pmatrix}, \quad (17)$$

$$M_{22} = \begin{pmatrix} 0 & \langle A_1 A_2 \rangle & \langle A_1 A_3 \rangle & \cdots & \langle A_1 A_r \rangle \\ -\langle A_1 A_2 \rangle & 0 & \langle A_2 A_3 \rangle & \cdots & \langle A_2 A_r \rangle \\ -\langle A_1 A_3 \rangle & -\langle A_2 A_3 \rangle & 0 & \cdots & \langle A_3 A_r \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\langle A_1 A_{r-1} \rangle & -\langle A_2 A_{r-1} \rangle & \cdots & 0 & \langle A_{r-1} A_r \rangle \\ -\langle A_1 A_r \rangle & -\langle A_2 A_r \rangle & \cdots & -\langle A_{r-1} A_r \rangle & 0 \end{pmatrix}, \quad (18)$$

$$M_{12} = \begin{pmatrix} \langle B_0 A_1 \rangle & \langle B_0 A_2 \rangle & \cdots & \langle B_0 A_r \rangle \\ \langle B_1 A_1 \rangle & \langle B_1 A_2 \rangle & \cdots & \langle B_1 A_r \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle B_{r-2} A_1 \rangle & \langle B_{r-2} A_2 \rangle & \cdots & \langle B_{r-2} A_r \rangle \\ \langle B_{r-1} A_1 \rangle & \langle B_{r-1} A_2 \rangle & \cdots & \langle B_{r-1} A_r \rangle \end{pmatrix}. \quad (19)$$

The pair contractions for  $A_m$  and  $B_n$  are

$$\begin{aligned} \langle A_m A_n \rangle &= \delta_{mn} + \frac{1}{\pi} \int_0^\pi dk \sin k(n-m) \left( \frac{u_k v_k^* - u_k^* v_k}{|u_k|^2 + |v_k|^2} \right), \\ \langle B_m B_n \rangle &= -\delta_{mn} + \frac{1}{\pi} \int_0^\pi dk \sin k(n-m) \left( \frac{u_k v_k^* - u_k^* v_k}{|u_k|^2 + |v_k|^2} \right), \\ \langle B_m A_n \rangle &= -\langle A_n B_m \rangle \\ &= -\frac{1}{\pi} \int_0^\pi dk \cos k(n-m) \left( \frac{|u_k|^2 - |v_k|^2}{|u_k|^2 + |v_k|^2} \right) \\ &\quad + \frac{1}{\pi} \int_0^\pi dk \sin k(n-m) \left( \frac{u_k v_k^* + u_k^* v_k}{|u_k|^2 + |v_k|^2} \right). \end{aligned} \quad (20)$$

It needs to be mentioned that in some works of non-Hermitian systems, the average value of an observable quantity is defined on the biorthogonal bases [60, 61]. Here, we do not use the biorthogonal bases because we think it is not easy to measure the biorthogonal average value. The left and right eigenstate are not the same quantum state, thus before measuring the biorthogonal average value, one has to prepare two states, then finds a way to link the two states and the observable quantity together. These processes are inconvenient for experiments. Thus, in this work, the LRCF  $C_{xx}(r)$  is only defined on conventional bases, i.e. the ground state  $|G\rangle$  (Eq. 10) and its Hermitian conjugation  $\langle G|$ , and all the pair contractions in Eq. 20 are also based on conventional bases of the ground state.

From Eq. (16) to Eq. (20), we can obtain the exact value of  $C_{xx}(r)$ . Here, we show the values of  $|C_{xx}(r)|$  in different  $\lambda$  with  $(J, \gamma, \Gamma) = (1, 0.5, 1.6)$ . The data is plotted in Fig. 3. When  $\lambda = 1$  and  $\lambda = 1.2$ , the system is in the region I and the LRCFs exponentially decay with the increase of  $r$ , which implies the system is in PM phase. When  $\lambda = 0.4$  and  $\lambda = 0.6$ , the system is in the region II and the LRCFs are constants when  $r$  is large, which shows the characteristic of FM phase.

An intriguing result which is unique for non-Hermitian case is obtained for the region III. When  $\lambda = 0.8$  and  $\lambda = 0.9$ , the system is in the region III, the decay of  $C_{xx}(r)$  is like polynomial type. In the Hermitian system,

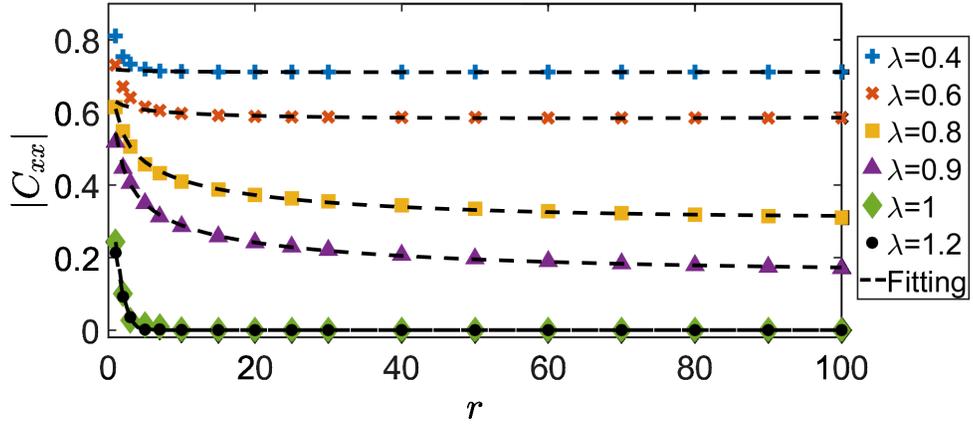


FIG. 3. Correlation function  $|C_{xx}(r)|$  in different  $\lambda$ . The other parameters are  $(J, \gamma, \Gamma) = (1, 0.5, 1.6)$ , corresponding to a gapless point  $C_0$  with  $\lambda_c = 0.6$ . Color points come from exact solution. Black dashed lines are obtained by numerical fitting with a function of the form  $|C_{xx}(r)| = Ar^{-B}e^{-Cr}$ .

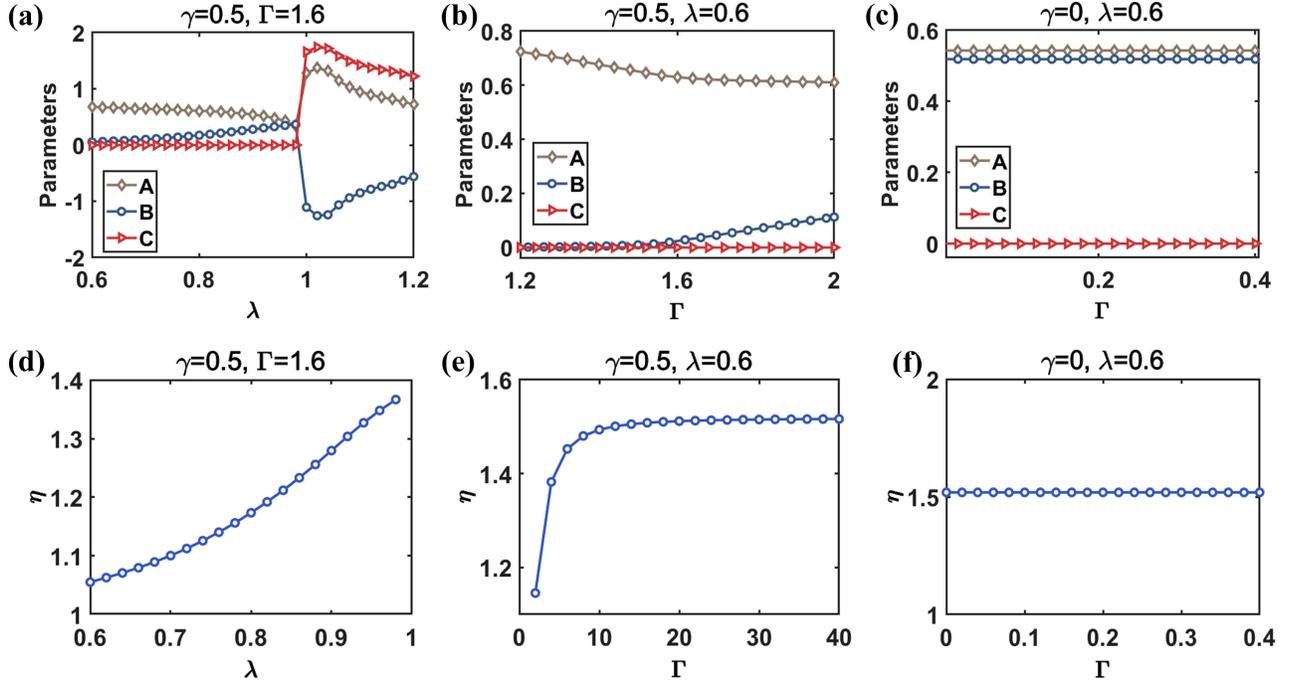


FIG. 4. Numerical fitting of LRCF. The fitting function is  $|C_{xx}(r)| = Ar^{-B}e^{-Cr}$ . (a)-(c) exhibit the fitting parameters  $A$  (grey diamond),  $B$  (blue circle) and  $C$  (red triangle). Panels (d)-(f) are the exponent  $\eta$ , which equals to  $1+B$  and reflects the critical behaviors in CTZ. In (a) the system changes from CTZ to PM phase. In (b) the system changes FM phase to CTZ. The transition point of both (a) and (b) is the EP  $C_0(\lambda = 0.6, \gamma = 0.5, \Gamma = 1.6)$ . In (c) and (f) the system is in CTZ. we set  $\gamma = 0$  and the system reduces to a non-Hermitian isotropic XY chain. Throughout,  $J$  is set as 1.

the polynomial decay only appears at critical point. In the non-Hermitian circumstances, the whole region III shows the similar behavior. Therefore, we call region III as CTZ. It is worth noting that, in Hermitian system, we can not distinguish ferromagnetic and paramagnetic phases only by energy gap. However, in this model, the different magnetic phases can be characterized by non-Hermitian gap.

To describe the decay more accurately, we fit data

points in Fig. 3 by the function

$$|C_{xx}(r)| = Ar^{-B}e^{-Cr}, \quad (21)$$

where  $A, B, C$  are the fitting parameters. Obviously,  $C > 0$  means an exponentially decay of correlation function, while the case  $C = 0$  and  $B > 0$  represents the polynomial decay. The fitting curves are shown by the black dashed lines in Fig. 3.

Now, we study the decay of  $C_{xx}(r)$  by scanning the

parameters across an exceptional point  $C_0(\lambda = 0.6, \gamma = 0.5, \Gamma = 1.6)$  along  $\lambda$  and  $\Gamma$ . Fitting by Eq. (21), we get curves of parameters  $A, B, C$  in Fig. 4 (a) and (b). In (a),  $\lambda$  changes from 0.6 to 1.2. When  $\lambda < 1$ , the system is in CTZ, where the red and blue curves show that  $C = 0$  and  $B > 0$ . When  $\lambda \geq 1$ , the system is in region I, where the ground state is in PM phase characterized by  $C > 0$ . In (b), we scan  $\Gamma$  from 1.2 to 2, where  $C$  is always equal to zero. When  $\Gamma \leq 1.6$ , the system is in region II and  $B$  is near zero. Therefore, this region is corresponding to FM phase. In contrast,  $B$  is obviously larger than zero when the system is in CTZ with  $\Gamma > 1.6$ .

In CTZ, the parameter  $C$  is always equal to zero so that the correlation function  $|C_{xx}(r)|$  is proportional to  $r^{-B}$ . Compared with  $C_{xx}(r) \propto r^{1-\eta}$  at critical points in Hermitian XY chain, we have that  $\eta = 1 + B$ . Fig. 4(d) and (e) show  $\eta$  in CTZ with  $\lambda \in [0.6, 0.99]$  and  $\Gamma \in [2, 40]$ , which reflect that non-Hermiticity influences critical behavior. When we change  $\lambda$  in CTZ,  $\eta$  is no longer standard value (5/4) of the Ising transition. When we increase  $\Gamma$ ,  $\eta$  tends to be near 3/2, which corresponds to the anisotropic transition. It may imply that at large  $\Gamma$  limit, the critical behavior of the non-Hermitian system is similar to the critical point of anisotropic transition. The intuitive explanation is that when  $\Gamma$  is very large, the difference between the interactions in  $x$  and  $y$  directions, corresponding to  $(J + \gamma)/2$  and  $(J - \gamma)/2$ , becomes comparatively insignificant.

In addition, we point out that the non-Hermitian term  $i\frac{\Gamma}{2}\sigma_j^u$  does not influence anisotropic transition. This transition comes from the competition between  $\frac{1}{2}(J + \gamma)\sigma_j^x\sigma_{j+1}^x$  and  $\frac{1}{2}(J - \gamma)\sigma_j^y\sigma_{j+1}^y$ . However, the non-Hermitian term is along  $z$  direction ( $\sigma^u = \frac{1}{2}(\mathbf{1} + \sigma^z)$ ), which does not break the symmetry between  $x$  and  $y$  directions when  $\gamma = 0$ . We show the fitting with  $\gamma = 0$  in Fig. 4 (c) and (f). It is obvious that  $A, B, C$  and  $\eta$  remain unchanged when we increase  $\Gamma$ . Furthermore,  $\eta$  has always been 3/2, which is same as the value of anisotropic transition in Hermitian XY chain.

At last, we mention the phase diagram again. In Sec.3 we study the derivatives of ground state energy density and find that non-analyticity is only on the EP ring, where the complex gap is equal to 0. This result seems to indicate that the system has only two phases, i.e. inside and outside the EP ring. In fact, by analyzing LRCFs, we find that there are three phases. An extra QPT occurs at the line with  $|\lambda/J| = 1$ , where the gap is a pure imaginary number. This is a new property for non-Hermitian system that the QPT can occur without gap closing. The similar property has been found in the non-Hermitian Kitaev's toric-code model [62].

## V. DISCUSSION AND CONCLUSION

Experimentally, the non-Hermitian XY model can be realized by some artificial quantum systems, such as ultracold atoms in optical lattices, superconducting qubits and coupled cavity arrays. For example, one can use three-level atoms in optical lattices to realize the required Hamiltonian. Two metastable states of atoms represent spin up and spin down. The Hermitian part of the system can be realized by arraying atoms in optical lattices [63–65]. The non-Hermitian term can be generated by exciting one of the metastable states to an auxiliary state [22]. Besides the cold atom experimental scheme mentioned above, the results can also be realized in coupled cavity arrays [66], in which the non Hermiticity can be realized by active and passive cavities [25, 67, 68].

In summary, we have investigated the traditional QPT in a quantum XY spin chain with a global complex transverse field. This non-Hermitian transverse field changes the phase diagram of XY model. The results reveal that (i) the second-order QPT points change from Ising transition point to an exceptional ring. (ii) the critical points of Hermitian system are extended to a critical transition zone. In this zone, correlation function decays polynomially. Furthermore, the results reveal the correspondence among the different phases, non-Hermitian gap and LRCF.

Our results indicate the nontrivial influence of non-Hermiticity and our model offers a higher dimensional parameter space to study the critical behaviors and non-Hermitian quantum magnetism. Moreover, our findings in this paper can be readily realized with recent experimental techniques. We believe that our work will benefit the future research on traditional quantum phase transition of non-Hermitian systems.

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