Quantum refrigerators in finite-time cycle duration

Guangqian Jiao¹, Shoubao Zhu¹, Jizhou He¹, Yongli Ma², and Jianhui Wang^{1,2*}

¹ Department of Physics, Nanchang University, Nanchang 330031, China

² State Key Laboratory of Surface Physics and Department of Physics,

Fudan University, Shanghai 200433, China

(Dated: February 9, 2022)

Abstract

We derive cooling rate and coefficient of performance as well as their variances for a quantum Otto engine proceeding in finite-time cycle period. This machine consists of two driven strokes, where the system isolated from the heat reservoir undergoes finite-time unitary transformation, and two isochoric steps, where the finite-time system-bath interaction durations take the system away from the equilibrium even at the ends of the two stages. We explicitly calculate the statistics of cooling rate and coefficient of performance for the machine operating with an analytically solvable two-level system. We clarify the role of finite-time durations of four processes on the machine performance. We show that there is the trade-off between the performance parameter and its corresponding variance, thereby indicating that the cooling rate or coefficient of performance can be enhanced, but at the cost of increasing the corresponding fluctuations.

PACS number(s): 05.70.Ln

^{*}Electronic address: wangjianhui@ncu.edu.cn

I. INTRODUCTION

A refrigerator as an inverse operation of a heat engine transfers energy from a cold thermal bath of temperature T_h to a a hot one with temperature T_c by consuming work. Quantum refrigerators well as quantum heat engines use quantum systems as their working substance and can be classified either cyclic [1–7] or steady-state [8–13] models. The quantum Otto cycle of operation, as a typical example of cyclic machines, is controlled by the segments of time that the working system is coupled to a hot and a cold bath, and by the time interval required to driving the control parameter of the system. It was most studied [1-5, 7, 14, 15] as it is easier to analyze and realize. To describe the performance characteristics of a refrigerator, one introduces the coefficient of performance (COP) that is defined as the ratio of heat absorbed from the cold reservoir and work input. An upper bound on the COP imposed by the second law of thermodynamic is given by the Carnot COP: $\varepsilon_C = T_c/(T_h - T_c)$, which, however, requires infinitesimally slow transitions between thermodynamic states and thus produces vanishing cooling rate. Hence, the refrigerators actually operate far from the infinite long time limit in order for positive cooling rate to be produced [16–19]. The finite cooling rate for a cyclic refrigerator consisting of a sequence of thermodynamic processes indicates that each process must proceed in finite time. For an adequate description of an actual machine, the effects induced by finite-time duration along any thermodynamic stroke on heat and work have to be considered.

While in a macroscopic system the work and heat are deterministic, for a microscopic quantum system (with a limited number of freedoms) these physical variables become random due to non-negligible thermal [20, 21] and quantum [22, 23] fluctuations. Theoretical and experimental investigation on the statistics of work [24–30] and heat [31–33] has attracted much interest in the literature. On the other hand, for heat engines the statistics of power [4, 34–36] and efficiency [37–43] has been analyzed, under the assumption that either system-bath interaction interval or unitary driving process is quasistatic. However, a unified thermodynamic description of quantum refrigerators with non-negligible quantum and thermal fluctuations, particularly when every thermodynamic stroke of these machines evolves in finite time, is available.

In the present paper, we study the thermodynamics of a quantum Otto refrigerator where all the four strokes proceed in finite time, within a framework of stochastic thermodynamics.



FIG. 1: (Color online) Schematic diagram of a quantum Otto refrigeration cycle operating with a two-level system in (ω, n) plane. The cycle consists of the two adiabatic strokes (connecting states A and B, and C and D), where the system isolated from the heat reservoir evolves unitary transformation, and two isochoric strokes (connecting states B and C, and D and A), where the system is kept in thermal contact with the cold and the hot reservoir, respectively. The average population n_3 (n_1) at the end of the cold (hot) isochoric stroke would approach its asymptotic value n_c^{eq} (n_h^{eq}) , when and only when the system-bath interaction duration τ_c (τ_h) tends to be infinitely long.

Having determined distribution functions for heat and work, we derive general formulae for the COP and cooling load as well as their variances. With these we then analyze a quantum Otto cycle working with a two-level system which is exactly solvable analytically. We discuss the effects of thermal and quantum fluctuations on finite-time performance and the statistics of the machine, and also demonstrate that there is trade-off of the physical variable (COP or cooling rate) and its fluctuations. We finally show that, the average COP $\langle \varepsilon \rangle$ can be always larger than the conventional thermodynamic COP ε_{th} for adiabatic driving, but it can be equal to or smaller than COP ε_{th} for nonadiabatic driving.

II. THE PROBABILITY OF STOCHASTIC COP FOR QUANTUM OTTO RE-FRIGERATORS

The quantum Otto cycle under consideration is sketched in Fig. 1. It consists of two isochoric branches, one with a cold and another with a hot heat reservoir where the Hamiltonian is kept constant, and two other strokes, where the system undergoes unitary transformation while isolating from the thermal reservoirs. In the adiabatic branch $A \to B$, the system is isolated from any heat reservoir and undergoes a unitary expansion from time t = 0 to $t = \tau_{hc}$. Initially, the system is assumed to be with local thermal equilibrium at inverse temperature $\beta_A (\geq \beta_h)$. The probability distribution of stochastic work done by the system, w_{hc} , can be given by [44]

$$p(w_{hc}) = \sum_{n,m} \delta[w_{hc} - (E_m^c - E_n^h)] p_{n \to m}^{\tau_{hc}} p_n^0(\beta_A).$$
(1)

where E_n^h and E_m^c are the respective energy eigenvalues at the initial and final instants along this expansion. Here $p_n^0(\beta_A) = \frac{e^{-\beta_A E_n^h}}{Z_A}$, with the partition function $Z_A = \sum_n e^{-\beta E_n}$, denotes the thermal occupation probability at instant A, and $p_{n\to m}^{\tau_{hc}} = |\langle n|U_{\exp}|m\rangle|^2$ is the transition probability from eigenstate $|n\rangle$ and $|m\rangle$, with the unitary time evolution operator U_{\exp} . and the population $\langle n\rangle$ remains unchanged in the adiabatic stage ($\xi = 0$). If the system Hamiltonian evolves slowly enough wherein the quantum adiabatic condition [45] is satisfied in time interval τ_{hc} , the the system remains in the same state and the transition probability therefore satisfies $p_{n\to m}^{\tau_{hc}} = \delta_{nm}$, with the Dirac's delta function δ .

In the next step $B \to C$, the quantum system with constant frequency $\omega = \omega_c$ is kept in contact with a cold thermal bath of inverse temperature β_c during a period τ_c . The probability density of the stochastic heat q_c can be determined by conditional distribution to arrive at

$$p(q_c|w_{hc}) = \sum_{k,l} \delta[q_c - (E_l^c - E_k^c)] p_{k \to l}^{\tau_c} p_k^{\tau_{hc}}, \qquad (2)$$

where $p_k^{\tau_{hc}}$ is the occupation probability at time $t = \tau_{hc}$ and it satisfies the constraint $p_k^{\tau_{hc}} = \delta_{km}$. We assume that at the end of the system-bath interaction interval, the system is at the local thermal equilibrium state with inverse temperature $\beta_C (\geq \beta_c^r)$, thereby indicating that $p_{k\to l}^{\tau_c} = e^{-\beta_C E_l^c}/Z_C$ with partition function $Z_C = \sum_l e^{-\beta_C E_l^c}$. Without loss of generality, the internal energy of system along the isochoric process can be expressed as $\langle n(t) \rangle = [Z(t)]^{-1} \sum_n n e^{-\beta(t) E_n^c}$, where $\langle n \rangle$ is the population (which is also the the expectation value of the particle number operator in Appendix A). When the time duration τ_c tends to be infinity, the mean population n_C at end of the isochoric process approaches the equilibrium value,

$$\langle n_c \rangle^{eq} \equiv n(\tau_c \to \infty) = Z^{-1} \sum_n n e^{-\beta_c E_n},$$
(3)

with $Z_c = \sum_n n e^{-\beta_c E_n}$. It is shown in Appendix A that the dynamics of the system along the thermalization can be described by the motion equation of population, which gives

$$\langle n_c \rangle^{eq} = \langle n_c \rangle^{eq} + [\langle n_B \rangle - \langle n_c \rangle^{eq}] e^{-\gamma_c \tau_c}, \qquad (4)$$

where $\langle n_C \rangle = \langle n(\tau_{hc} + \tau_c) \rangle$ and $\langle n_B \rangle = \langle n(\tau_{hc}) \rangle$, and γ_c is the thermal conductivity when $\omega = \omega_c$.

On the adiabatic compression $C \to D$, the system is isolated in time duration τ_{ch} while the energy gap ω varies from ω_c to ω_h . For given work output w_{hc} and the heat q_c released from the system, the probability density of stochastic work input is given by

$$p(w_{ch}|q_c, w_{hc}) = \sum \delta[w_{ch} - (E_j^h - E_i^c)] p_{i \to j}^{\tau_{ch}} p_i^{\tau_{hc} + \tau_c},$$
(5)

where the occupation probability $p_i^{\tau_{hc}+\tau_c} = \delta_{il}$, and $p_{i\to j}^{\tau_{ch}} = |\langle i|U_{\rm com}|j\rangle|^2$ is transition probability from eigenstate $|i\rangle$ and $|j\rangle$, with the time evolution operator $U_{\rm com}$ along the compression.

In the fourth step $D \to A$, the system is coupled to a hot reservoir of inverse temperature β_h in time duration τ_h while keeping its frequency in a constant with $\omega = \omega_h$. Since the system returns to its initial state A after the cycle period $\tau_{cycle} = \tau_{hc} + \tau_c + \tau_{ch} + \tau_h$, we will do not derive the expression of the stochastic heat exchanged q_h along this process. As shown in Appendix A, the populations at the beginning and end of the heating process ($\langle n_D \rangle$ and $\langle n_A \rangle$) satisfies the constraint:

$$\langle n_A \rangle = \langle n_h \rangle^{eq} + [\langle n_D \rangle - \langle n_h \rangle^{eq}] e^{-\gamma_h \tau_h}, \tag{6}$$

where $\langle n_A \rangle = \langle (\tau_{cycle}), \langle n_D \langle = \langle n(\tau_{cycle} - \tau_h) \rangle$,

$$\langle n_h \rangle^{eq} \equiv \langle n(\tau_h \to \infty) \rangle^{eq} = Z_h^{-1} \sum_n n e^{-\beta_h E_n}$$
(7)

with $Z_h = \sum_n e^{-\beta_h E_n}$, and γ_h is the thermal conductivity between the system and the hot reservoir.

The probability $p(w_{ch}, q_c, w_{hc})$ for the machine which has certain values of w_{ch}, q_c, w_{hc} can be calculated from the the chain rule for condition probabilities $p(w_{ch}, q_c, w_{hc}) = p(w_{ch}|q_c, w_{hc})p(q_c|w_{hc})p(w_{hc})$:

$$p(w_{ch}, q_c, w_{hc}) = \sum \delta \left[q_c - (E_k^c - E_m^c) \right] \delta \left[w_{hc} - \left(E_m^c - E_n^h \right) \right] \delta \left[w_{ch} - \left(E_l^h - E_k^c \right) \right] \\ \times |\langle n | U_{\exp} | m \rangle|^2 |\langle i | U_{com} | j \rangle|^2 \frac{e^{-\beta_A E_n^h} e^{-\beta_C E_k^c}}{Z_A Z_C}.$$
(8)

In deriving this, we have used Eqs. (1), (2) and (5). For the quantum Otto refrigerator, the stochastic coefficient of performance reads $\varepsilon = q_c/(w_{hc} + w_{ch})$. It follows, integrating over all values of w_{ch}, q_c, w_{hc} , that the probability distribution $p(\varepsilon)$ becomes

$$p(\varepsilon) = \sum_{m,n,i,j} \delta\left(\epsilon - \frac{E_k^c - E_m^c}{E_m^c - E_n^h + E_l^h - E_k^c}\right) \times \frac{e^{-\beta_A E_n^h} e^{-\beta_C E_k^c}}{Z_A Z_C} |\langle n|U_{\exp}|m\rangle|^2 |\langle i|U_{\cos}|j\rangle|^2.$$
(9)

While for adiabatic driving, the system remains in the same state (n = m, i = j), in the nonadiabatic driving the transition probability $|\langle n|U_{\exp}|m\rangle|^2$ (or $|\langle i|U_{\cos}|j\rangle|^2$) is positive due to transitions between sates n and m (or i and j). For the quantum Otto refrigerator, its efficiency statistics is fully determined by the unitary time evolution for the adiabatic expansion and compression (U_{\exp} and U_{com}), and by the finite-time system dynamics along the thermalization processes, when the two temperatures of the two heat reservoirs (β_h and β_c) are given.

III. A QUANTUM OTTO REFRIGERATOR USING A TWO-LEVEL SYSTEM

We now consider a quantum Otto refrigerator operating with a two-level system of the eigenenergies $E_+ = -\hbar\omega/2$ and $E_- = -\hbar\omega/2$. If the unitary expansion and compression during the Otto cycle (from ω_h to ω_c to ω_h and vice versa) is such that there is a probability of level transitions due to quantum fluctuations, then there is a probability that population $\langle n \rangle$ may change with varying time. After a simple calculation (see Appendix B), we find that

$$\langle n_B \rangle = (1 - 2\xi) \langle n_A \rangle, \ \langle n_D \rangle = (1 - 2\xi) \langle n_C \rangle$$
 (10)

where $\xi = |\langle \pm |U_{exp} | \mp \rangle|^2 = |\langle \pm |U_{com} | \mp \rangle|^2$ is called the adiabacity parameter indicating the probability of transition between state $|+\rangle$ and $|-\rangle$ during the compression or expansion. As shown in Fig. 1, the populations at any instant along the cycle is negative, which means that ξ must be situated between $0 \leq \xi < 1/2$. The probability of no state transition along either driving phase is accordingly $|\langle \pm |U_{exp} | \pm \rangle|^2 = |\langle \pm |U_{com} | \pm \rangle|^2 = 1 - \xi$. The adiabaticity parameter ξ depends on the the speed at which the driving process is performed [5, 7, 37]. When the time scale of the state change is much larger than that of the dynamical one, the quantum adiabatic condition is satisfied and the population $\langle n \rangle$ remains unchanged in the adiabatic stage ($\xi = 0$). Rapid change in the control field ω , however, leads to nonadiabatic behavior ($\xi > 0$) which can understood as inner friction [1, 2, 14, 46–48] causing state

transitions. Equation (10) shows that $\langle n_C \rangle > \langle n_B \rangle$ and $\langle n_D \rangle > \langle n_C \rangle$ for $\xi > 0$ (see also Fig. 1), as lies in the fact that the finite time duration of the expansion and compression accounts for the nonadiabatic inner friction related to the irreversible entropy production.

Using Eqs. (4), (6), and (10), it follows that the populations $\langle n_A \rangle$ and $\langle n_C \rangle$ can be expressed in terms of the equilibrium populations $\langle n_h \rangle^{eq}$ and $\langle n_c \rangle^{eq}$,

$$\langle n_A \rangle = \langle n_h \rangle^{eq} + \Delta_h, \langle n_C \rangle = \langle n_c \rangle^{eq} + \Delta_c$$
 (11)

where

$$\Delta_h = \frac{\left(2\xi - 1\right)\left[\left(2\xi - 1\right)\langle n_h \rangle + \langle n_c \rangle^{eq}\right] - y\left[\langle n_h \rangle + \left(2\xi - 1\right)\langle n_c \rangle^{eq}\right]}{xy - \left(2\xi - 1\right)^2},\tag{12}$$

$$\Delta_c = \frac{\left(2\xi - 1\right)\left[\left(2\xi - 1\right)\langle n_c \rangle + \langle n_h \rangle^{eq}\right] - x\left[\langle n_c \rangle + \left(2\xi - 1\right)\langle n_h \rangle^{eq}\right]}{xy - \left(2\xi - 1\right)^2},\tag{13}$$

with $x = e^{\gamma_h \tau_h}$ and $y = e^{\gamma_c \tau_c}$. Hereafter we will refer x and y rather than τ_h and τ_c as the time durations along the hot and cold isochoric branches for simplicity. Here $\langle n_c \rangle^{eq}$ defined in Eq. (3) and $\langle n_h \rangle^{eq}$ in Eq. (7) can be obtained using the same approach as that described in Appendix A for the derivation of Eq. (B.2) to arrive at ($\hbar \equiv 2$)

$$\langle n_c \rangle^{eq} = -\frac{1}{2} \tanh(\beta_c \omega_c), \langle n_h \rangle^{eq} = -\frac{1}{2} \tanh(\beta_h \omega_h),$$
 (14)

which is achieved in quasi-static limit $(x, y \to \infty)$ when $\Delta_{c,h} \to 0$. While for the finitetime system-bath interaction interval the system is away from the thermal equilibrium, the populations $\langle n_C \rangle$ and $\langle n_A \rangle$ approach the thermal values $\langle n_c \rangle^{eq}$ and $\langle n_h \rangle^{eq}$, respectively, when these intervals go to the infinite long time limit. Therefore, Δ_c and Δ_h indicate how far the two isochoric processes deviates from the quasistatic limit.

From Eqs. (1) and (2), the average heat injection, $\langle q_c \rangle = \int \int q_c p(q_c | w_{hc}) p(w_{hc}) dw_{hc} dq_c$ can be obtained as $(\hbar \equiv 2)$

$$\langle q_c \rangle = 2\omega_c \left[\langle n_c \rangle + \Delta_c + (2\xi - 1) \left(\langle n_h \rangle + \Delta_h \right) \right],$$
 (15)

Integrals over the distribution function $p(q_c|w_{hc}) p(w_{hc})$ yield the second moments of absorbed heat q_c , $\langle q_c^2 \rangle = \int \int q_c^2(w_{hc}) p(q_c|w_{hc}) dw_{hc} dq_c$, which reads $\langle q_c^2 \rangle = 8\omega_c^2 [1/4 + (2\xi - 1)(\langle n_h \rangle + \Delta_h)(\langle n_c \rangle + \Delta_c)]$. The variance of absorbed heat q_c , $\delta q_c^2 = \langle q_c^2 \rangle - \langle q_c \rangle^2$, then becomes

$$\delta q_c^2 = 4\omega_c^2 \left[\frac{1}{2} - (2\xi - 1)^2 \left(\langle n_h \rangle + \Delta_h \right)^2 - \left(\langle n_c \rangle + \Delta_c \right)^2 \right].$$
(16)



FIG. 2: Contour plots of relative heat variance, $f_{\dot{q}_c}$ in the effective time duration (x, y) plane for an adiabatic (a) and nonadiabatic (b) driving, with $\xi = 0$ and $\xi = 0.02$, respectively. The values of the parameters are $\beta_h = 0.8$, $\beta_c = 1$, $\omega_c = 0.6$, and $\omega_h = 1$.

These variances are upper limited by the value of $2\omega_c^2$ and they become $\delta q_c^2 = 4\omega_c^2 \left[\frac{1}{2} - \langle (n_h \rangle)^2 - (\langle n_c \rangle^{eq})^2 \right]$ for the quasistatic cycle.

As $\langle \dot{q}_c \rangle = \langle q_c \rangle / \tau_{cycle}$ and $\delta \dot{q}_c = \langle q_c \rangle / \tau_{cycle}$, the relative variance of cooling rare $f_{\dot{q}_c}$ can be obtained by using Eqs. (15) and (16) to arrive at

$$f_{\dot{q}_{c}} = \frac{\delta q_{c}}{\langle q_{c} \rangle} = \frac{\left[\frac{1}{2} - (2\xi - 1)^{2} \left(\langle n_{h} \rangle^{eq} + \Delta_{h} \right)^{2} - \left(\langle n_{c} \rangle + \Delta_{c} \right)^{2}\right]^{\frac{1}{2}}}{(2\xi - 1) \left(\langle n_{h} \rangle + \Delta_{h} \right) + \langle n_{c} \rangle^{eq} + \Delta_{c}}.$$
 (17)

This is a monotonically decreasing function of time durations (x and y) of system-bath interaction intervals, both for adiabatic and nonadiabatic driving [see Figs. 2(a) and 2(b)]. If the isochoric hot (or cold) branch is completed in an finite time τ_c (or τ_h), with finite value of x_c (or x_h), this isochoric step is out of equilibrium and the fluctuations are inevitably avoidable. In order to decrease the relative fluctuations, we thus need to slow down the machine, which, however, must make the average cooling rate $\langle \dot{q}_c \rangle$ down. Comparison between 2(a) and 2(b) also shows that the relative fluctuations are larger in nonadiabatic driving with finite time (τ_{hc} or τ_{ch}) than in adiabatic, quasistatic evolution.

Since no work is produced in the two isochoric processes, the average total work per cycle is $\langle w_{hc} \rangle + \langle w_{ch} \rangle$, where $\langle w_{hc} \rangle = \int w_{hc} p(w_{hc}) dw_{hc}$ and $\langle w_{ch} \rangle = \int w_{ch} p(w_{ch}, q_c, w_{hc}) dw_{hc} dq_c dw_{ch}$. It follows, using Eqs. (1) and (5), that the total work

can be obtained,

$$\langle w \rangle = 2 \left(\omega_h - \omega_c \right) \left[\left(\langle n_c \rangle^{eq} + \Delta_c \right) - \left(\langle n_h \rangle + \Delta_h \right) \right] - 4\xi \left[\omega_c \left(\langle n_h \rangle^{eq} + \Delta_h \right) + \omega_h \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right].$$
(18)

The thermodynamic coefficient of performance, defined by $\varepsilon_{th} = \langle q_c \rangle / \langle w \rangle$, is the given by

$$\varepsilon_{th} = \frac{\omega_c}{\omega_h \frac{1-\xi\mathcal{F}}{1+\xi\mathcal{G}} - \omega_c},\tag{19}$$

where $\mathcal{F} = 2(\langle n_c \rangle^{eq} + \Delta_c)/[(\langle n_c \rangle + \Delta_c) - (\langle n_h \rangle^{eq} + \Delta_h)]$ and $\mathcal{G} = 2(\langle n_h \rangle^{eq} + \Delta_h)/[(\langle n_c \rangle + \Delta_c) - (\langle n_h \rangle^{eq} + \Delta_h)]$. As $\mathcal{F}, \mathcal{G} < 0$ and $\xi \geq 0$, the thermodynamic coefficient of performance ε_{th} increases as the adiabacity parameter ξ decreases, and it reaches its upper bound $\varepsilon_{th}^{ad} = \omega_c/(\omega_h - \omega_c)$ in the ideal adiabatic case when $\xi = 0$. The fact that the additional heat is dissipated into the hot reservoir due to finite time realization of the compression or expansion, so that the additional work is input to overcome such heat loss, suggests that cycles consisting of nonadiabatic transformation along the expression and compression runs less efficiently than those with ideal adiabatic strokes. While the stochastic COP is defined by $\varepsilon = q_c/(w_{hc} + w_{ch})$, its probability distribution $p(\varepsilon)$ can be determined by $p(\varepsilon) = \int \int \int dw_{hc} dq_c dw_{hc} p(w_{ch}, q_c, w_{hc}) \delta\left(\varepsilon - \frac{q_c}{w_{hc} + w_{ch}}\right)$ to arrive at

$$p(\varepsilon) = 2\left\{ \left[\frac{1}{4} + \left(\langle n_h \rangle + \Delta_h \right) \left(\langle n_c \rangle + \Delta_c \right) \right] (1 - \xi)^2 + \left[\frac{1}{4} - \left(\langle n_h \rangle^{eq} + \Delta_h \right) \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right] \xi^2 \right\} \delta(\varepsilon) + 2 \left[\frac{1}{4} + \left(\langle n_h \rangle^{eq} + \Delta_h \right) \left(\langle n_c \rangle + \Delta_c \right) \right] \xi^2 \delta\left(\varepsilon + \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right) + 2 \left[\frac{1}{4} - \left(\langle n_h \rangle^{eq} + \Delta_h \right) \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right] (1 - \xi)^2 \delta\left(\varepsilon - \varepsilon_{th}^{ad} \right) + (1 - \xi) \xi \left[\delta\left(\varepsilon + 1 \right) + \delta\left(\varepsilon \right) \right]$$

$$(20)$$

We examine the statistics of stochastic COP in Fig. 3 at different ξ for given time durations allocated to the two isochoric strokes (x and y). The statistics of COP depends on the adiabacity parameter ξ determined merely by the driving time (τ_{hc} or τ_{ch}). For adiabatic driving with $\xi = 0$ (blue squares), the stochastic COP may be zero or equal to the adiabatic value ε_{th}^{ad} , with the largest peak at zero and the second largest at ε_{th}^{ad} . By contrast, for nonadiabatic driving with $\xi > 0$ (red dots), the negative values $[-\varepsilon_{th}^{ad}/(2\varepsilon_{th}^{ad}+1)$ and -1]of $p(\varepsilon)$ are visible due to quantum determinacy, in addition to nonnegative ones (zero and



FIG. 3: The probability distribution $p(\varepsilon)$ of the quantum stochastic COP for both adiabatic (blue dots) and nonadiabatic (red squares) driving. We observe the appearance of peaks at negative COP in the nonadiabatic case.

 ε_{th}^{ad}). Unlike in a quantum heat engine [37] where the stochastic efficiency can not be defined for $\xi > 0$, for the quantum refrigerator the average COP $\langle \varepsilon \rangle$ converges and can thus be well defined. Using the distribution function (20), we find that the first two central moments are

$$\langle \varepsilon \rangle = 2 \left[\frac{1}{4} - \left(\langle n_h \rangle + \Delta_h \right) \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right] (1 - \xi)^2 \varepsilon_{th}^{ad}$$

$$- 2 \left[\frac{1}{4} + \left(\langle n_h \rangle^{eq} + \Delta_h \right) \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right] \xi^2 \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} - (1 - \xi) \xi$$

$$(21)$$

and $\langle \varepsilon^2 \rangle = 2 \left[\frac{1}{4} - (\langle n_h \rangle + \Delta_h) \left(\langle n_c \rangle + \Delta_c \right) \right] (1 - \xi)^2 \left(\varepsilon_{th}^{ad} \right)^2 + 2 \left[\frac{1}{4} + (\langle n_h \rangle^{eq} + \Delta_h) \left(\langle n_c \rangle^{eq} + \Delta_c \right) \right] \xi^2 \left[\varepsilon_{th}^{ad} / (2\varepsilon_{th}^{ad} + 1) \right]^2 + (1 - \xi) \xi.$ This, combining with Eq. (21), gives rise to the variance of stochastic COP, $\delta \varepsilon^2 = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$, leading to

$$\delta \varepsilon^{2} = 2 \left[\frac{1}{4} - \left(\langle n_{h} \rangle + \Delta_{h} \right) \left(\langle n_{c} \rangle^{eq} + \Delta_{c} \right) \right] \left(1 - \xi \right)^{2} \left(\varepsilon_{th}^{ad} \right)^{2} + 2 \left[\frac{1}{4} + \left(\langle n_{h} \rangle^{eq} + \Delta_{h} \right) \left(\langle n_{c} \rangle^{eq} + \Delta_{c} \right) \right] \xi^{2} \left(\frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right)^{2} - \left\{ \xi \left(1 - \xi \right) + \left(1 - \xi \right)^{2} \left[2 \left(\langle n_{h} \rangle + \Delta_{h} \right) \left(\langle n_{c} \rangle^{eq} + \Delta_{c} \right) - \frac{1}{2} \right] \varepsilon_{th}^{ad} + \xi^{2} \frac{\left[2 \left(\langle n_{h} \rangle^{eq} + \Delta_{h} \right) \left(\langle n_{c} \rangle^{eq} + \Delta_{c} \right) + \frac{1}{2} \right] \varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right\}^{2} + \xi \left(1 - \xi \right).$$

$$(22)$$

For a cycle with either adiabatic or nonadiabatic driving branches, the average COP $\langle \varepsilon \rangle$ increases as time duration $y = e^{\gamma_c \tau_c}$ increases, but it decreases as time duration $x = e^{\gamma_h \tau_h}$



FIG. 4: Contour plots of the average COP $\langle \varepsilon \rangle$ and its variance $\langle \delta \varepsilon^2 \rangle$ in the effective time duration (x, y) plane for an adiabatic (a and c) and nonadiabatic (b and d) driving, with $\xi = 0$ and $\xi = 0.02$, respectively. The values of the parameters are $\beta_h = 0.8$, $\beta_c = 1$, $\omega_c = 0.6$, and $\omega_h = 1$.

increases, see Figs. 4(a) and 4(b). This follows from the fact that, for the machine with adiabatic or nonadiabatic processes, the heat absorbed by the system along the cold isochoric stroke increases as y increases, and the heat released to the hot reservoir increases as xincreases. Figures 4(c) and 4(d) show that, in contrast to $\langle \varepsilon \rangle$, the variance $\langle \delta \varepsilon^2 \rangle$ increases as x increases but decreases as y increases, thereby confirming that there is trade-off between average $\langle \varepsilon \rangle$ and the COP fluctuations $\langle \delta \varepsilon^2 \rangle$. We also observe that the internal dissipation along the adiabats results in performance deterioration for the machine by reducing the average COP $\langle \varepsilon \rangle$ but increasing fluctuations of COP $\langle \delta \varepsilon^2 \rangle$.

For given time durations (x and y) of two isochoric processes, both the average and the variance of the stochastic COP as a function of the inverse temperature β_c of cold reservoir is shown in Fig. 5. When decreasing inverse temperature, both $\langle \varepsilon \rangle$ and $\langle \delta \varepsilon^2 \rangle$ grow,



FIG. 5: (Color online) The variance of stochastic COP, $\langle \delta \varepsilon^2 \rangle$, and average COP $\langle \varepsilon \rangle$ (inset) as a function of the inverse temperature of cold reservoir $\beta_c = 0.8\beta_h$ for $\xi = 0$ (blue lines) and $\xi = 0.02$ (red lines). The red dashed line corresponds to the variance $\langle \delta \varepsilon^2 \rangle$ in the region of nonphysical, negative mean COP. The parameters are x = y = 20, $\omega_c = 0.6$, and $\omega_h = 1$.

and as expected, $\langle \delta \varepsilon^2 \rangle$ would vanish and $\langle \varepsilon \rangle$ would be even negative in the low temperature limit. The non-positive mean COP in the low-temperature regime can be understood that the stochastic COP may be negative due to quantum indeterminacy dominating at low temperatures (see also Fig. 3). While the low temperature domain is characterized by quantum fluctuations, the high temperature region is dominated by larger thermal fluctuations. Therefore, the variance $\langle \delta \varepsilon^2 \rangle$ gets increased while the temperature is increasing and vice versa.

As the covariance between the total stochastic work w and stochastic COP ε can be defined by [49]

$$\operatorname{Cov}\left(\frac{q_c}{w}, w\right) = (\varepsilon_{th} - \langle \varepsilon \rangle) \langle w \rangle, \tag{23}$$

the difference between the thermodynamic COP ε_{th} and average COP $\langle \varepsilon \rangle$ is determined according to $\varepsilon_{th} - \langle \varepsilon \rangle = \text{Cov}\left(\frac{q_c}{w}, w\right) / \langle w \rangle$. The ratio $\text{Cov}\left(\frac{q_c}{w}, w\right) / \langle w \rangle$ as a function of inverse temperature β_c is plotted in Figs. 6(a) and 6(b), where the time durations along two isochores are x = y = 20 and x = y = 2, respectively. We notice that, for $\xi = 0$ the ratio monotonically increases with increasing inverse temperature. It is moreover always positive, for either fast or slow isochoric branch, indicating that the thermodynamic COP ε_{th} must



FIG. 6: (Color online) The ratio $\operatorname{Cov}\left(\frac{q_c}{w}, w\right)/\langle w \rangle$ as a function of inverse temperature β_c as a function of the inverse temperature of cold reservoir $\beta_c = 0.8\beta_h$ for x = y = 20 (a) and x = y = 2 (b), where $\xi = 0$ and $\xi = 0.02$ are indicated by blue lines and red ones, respectively. The other parameters are $\omega_c = 0.6$ and $\omega_h = 1$.

be lager than the average COP $\langle \varepsilon \rangle$. By contrast, this ratio increases and then decreases as the temperature is lowered for $\xi > 0$. As this ratio can be either positive or negative, the thermodynamic COP ε_{th} can be larger or smaller than the corresponding average COP $\langle \varepsilon \rangle$ for nonadiabatic driving.

IV. CONCLUSION

In summary, we have developed a general scheme allowing to determine statistics of cooling rate and COP for a quantum Otto refrigerator by analyzing the time evolution of the two isochores and two adiabats. These performance parameters as well as their statistics are determined by the finite time durations required for completing the two nonadiabatic driving strokes and two isochoric branches with incomplete thermalization. When treating an analytically solvable two-level engine, we find that stochastic COP may be negative due to quantum indeterminacy, and but its average value converges and can thus be well defined. We show that there is trade-off between these variables and their corresponding fluctuations at zero and finite temperature, thereby indicating that the price for enhancing the machine performance is increasing fluctuations. We have additionally compared the average COP and the conventional thermodynamic COP, and we found that they are positive correlated for ideal adiabatic strokes, but their correlation may be negative for nonadibatic branches.

Appendix A TIME EVOLUTION OF POPULATION ALONG AN ISOCHORIC PROCESS

When a system under external control is weakly coupled to a heat reservoir, the quantum dynamics of the system generated by both thermal interaction and external fields can be described by a semigroup approach. The change in time of an operator \hat{X} for a system with Hamiltonian \hat{H} is determined according to the master equation [1, 36]:

$$\frac{d\hat{X}}{dt} = i[\hat{H}, \ \hat{X}] + \frac{\partial\hat{X}}{\partial t} + \mathcal{L}_D(\hat{X}), \tag{A.1}$$

where

$$\mathcal{L}_D(\hat{X}) = \sum_{\alpha} k_\alpha \left(\hat{V}^{\dagger}_{\alpha} \left[\hat{X}, \hat{V}^{\dagger}_{\alpha} \right] + \left[\hat{V}^{\dagger}_{\alpha}, \hat{X} \right] \hat{V}_{\alpha} \right)$$
(A.2)

is the Liouville dissipative generator due to the system-reservoir thermal interaction. \hat{V}_{α} are operators in the Hilbert space of the system and $\hat{V}_{\alpha}^{\dagger}$ are Hermitian conjugates, and k_{α} are phenomenological positive coefficients. In Eq. (A.2), $[\hat{X}, \hat{V}_{\alpha}^{\dagger}] = [\hat{X}, \hat{V}_{\alpha}^{\dagger}]_{-}$ denotes commutator for the Bose system, and $[\hat{X}, \hat{V}_{\alpha}^{\dagger}] = [\hat{a}, \hat{b}]_{+}$ is used for anticommutator for the Fermi system. Substituting $\hat{X} = \hat{H}$ into Eq. (A.1) leads to

$$\left\langle \frac{d\hat{H}}{dt} \right\rangle = \left\langle \frac{\partial\hat{H}}{\partial t} \right\rangle + \left\langle \mathcal{L}_D(\hat{H}) \right\rangle. \tag{A.3}$$

This reproduces the time derivative of quantum version of the first law of thermodynamics $d\langle \hat{H} \rangle/dt = d\langle w \rangle/dt + d\langle q \rangle/dt$, when the instantaneous average power and the average heat current are identified as, $d\langle w \rangle/dt = \langle \partial \hat{H}/\partial t \rangle$ and $d\langle q \rangle/dt = \langle \mathcal{L}_D(\hat{H}) \rangle$, respectively.

To proceed, we choose the operators \hat{V}^{\dagger} and \hat{V} as the bosonic (fermionic) creation operator \hat{a}^{\dagger} and annihilation operator \hat{a} for the Bose (Fermi) system. Inserting the system Hamiltonian $\hat{H} = \omega \hat{a}^{\dagger} \hat{a}$ into Eq. (A.1) and taking the expectation value, the motion of the population $\langle n \rangle = \langle a^{\dagger} a \rangle$ along an isochoric process with constant ω can be obtained as

$$\frac{d\langle n\rangle}{dt} = -\gamma(\langle n\rangle - \langle n\rangle^{eq}),\tag{A.4}$$

where $\gamma = k_{\downarrow} - k_{\uparrow}$ ($\gamma = k_{\downarrow} + k_{\uparrow}$) denotes the thermal conductivity for the Bose (Fermi) system. The detailed balance $k_{\uparrow} = k_{\downarrow}e^{-\beta\hbar\omega}$ is assumed to be satisfied, ensuring that the

system can achieve asymptotically the thermal state after an infinitely long system-bath interaction duration. Here $\langle n \rangle^{eq} = \frac{1}{2} \frac{k_{\downarrow} + k_{\uparrow}}{k_{\downarrow} - k_{\uparrow}} \left(\langle n \rangle^{eq} = \frac{1}{2} \frac{k_{\downarrow} - k_{\uparrow}}{k_{\downarrow} + k_{\uparrow}} \right)$ is the asymptotic value of $\langle n(t) \rangle$ with $t \to \infty$, and it corresponds to the equilibrium population: $\langle n \rangle = 1/(e^{\beta\omega} - 1)$ $[\langle n \rangle = -\frac{1}{2} \tanh(\beta\omega/2)]$. From Eq. (A.4), we find that instantaneous population $\langle n(t) \rangle$ along the thermalization process (staring at initial time t = 0) can be written in terms of the population $\langle n(0) \rangle$,

$$\langle n(t) \rangle = \langle n \rangle^{eq} + [\langle n(0) \rangle - \langle n \rangle^{eq}] e^{-\gamma t}.$$
(A.5)

Appendix B RELATION BETWEEN POPULATIONS AT THE BEGIN AND THE END OF A UNITARY DRIVING PROCESS

We consider the unitary time evolution along the unitary adiabatic expansion $A \to B$ to identify the explicit relation of the populations at A and B. For a two level system, its eigenenergies are $E_+ = \frac{1}{2}\hbar\omega$, and $E_- = -\frac{1}{2}\hbar\omega$. The partition function at the initial instant A along this process can be given by

$$Z_A = e^{-\beta_A \hbar \omega_h/2} + e^{\beta_A \hbar \omega_h/2} = 2 \cosh\left(\frac{\beta_A \hbar \omega_h}{2}\right), \tag{B.1}$$

which, together with the occupation probabilities $p_+ = e^{-\beta \hbar \omega_h/2}/Z_A$ and $p_- = e^{\beta \hbar \omega_h/2}/Z_A$, gives the population at instant A,

$$\langle n_A \rangle = \frac{1}{2Z_A} \left(e^{-\beta \hbar \omega_h/2} - e^{\beta \hbar \omega_h/2} \right) = -\frac{1}{2} \tanh\left(\frac{\beta \hbar \omega_h}{2}\right).$$
 (B.2)

The average population at instant B can then be determined according to

$$\langle n_B \rangle = \sum_{n,m} m p_{n \to m}^{\tau_{hc}} p_n^0(A)$$

$$= \sum_{n,m} m |\langle n|U_{\exp}|m\rangle|^2 p_n^0(A)$$

$$= \frac{1}{Z_A} \Big[e^{-\beta_A \omega_h/2} \left(|\langle +|U_{\exp}|+\rangle|^2 - \langle +|U_{\exp}|-\rangle|^2 \right) + e^{\beta_A \omega_h/2} \left(|\langle -|U_{\exp}|+\rangle|^2 - \langle -|U_{\exp}|-\rangle|^2 \right) \Big]$$

$$= \frac{1}{Z_A} \Big[e^{-\beta_A \omega_h/2} \left(1 - 2\xi \right) + e^{\beta_A \omega_h/2} \left(2\xi - 1 \right) \Big]$$

$$= (1 - 2\xi) \langle n_A \rangle$$
(B.3)

where $\xi = |\langle \pm | U_{\text{exp}} | mp \rangle|^2$ and $\langle n_A \rangle$ was defined in Eq. (B.2). As for the two-level system $\langle n_B \rangle < 0$, ξ must be upper limited by 1/2. Similarly, for the unitary compression $C \to D$,

we have

$$\langle n_D \rangle = (1 - 2\xi) \langle n_C \rangle.$$
 (B.4)

where $\xi = |\langle \pm | U_{\text{com}} | \mp \rangle|^2$.

Acknowledgements This work is supported by National Science Foundation of China (Grant Nos. 11875034 and 11505091), and by the State Key Program of China under Grant NO. 2017YFA0304204. J. H. W. also acknowledges the financial support from the Major Program of Jiangxi Provincial Natural Science Foundation (Grant No. 20161ACB21006).

- T. Feldmann and R. Kosloff, Quantum four-stroke heat engine: Thermodynamic observables in a model with intrinsic friction, Phys. Rev. E 68, 016101 (2003); T. Feldmann and R. Kosloff, Performance of discrete heat engines and heat pumps in finite time, Phys. Rev. E 61, 4774 (2000).
- [2] Y. Rezek and R. Kosloff, Irreversible Performance of a Quantum Harmonic Heat Engine, New J. Phys. 8, 83 (2006).
- [3] S. H. Su, X. Q. Luo, J. C. Chen and C. P. Sun. Angle-dependent quantum Otto heat engine based on coherent dipole-dipole coupling. Europhy. Lett. 115, 30002 (2016).
- [4] Y. Hong, Y. Xiao, J. He, and J. H. Wang, Quantum Otto engine working with interacting spin systems: Finite power performance in stochastic thermodynamics, Phys. Rev. E 102, 022143 (2020).
- [5] O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Single ion heat engine with maximum efficiency at maximum power, Phys. Rev. Lett. 109, 203006 (2012).
- [6] H. Quan, H., Y. -X. Liu, C. P. Sun, F. Nori, Quantum thermodynamic cycles and quantum heat engines, Phys. Rev. E 76, 031105 (2007).
- [7] R. J. de Assis, T. M. de Mendonça, C. J. Villas-Boas, A. M. de Souza, R. S. Sarthour, I. S. Oliveira, and N. G. de Almeida, Efficiency of a Quantum Otto Heat Engine Operating under a Reservoir at Effective Negative Temperatures, Phys. Rev. Lett. **122**, 240602 (2019).
- [8] R. Kosloff and A. Levy, Quantum heat engines and refrigerators: Continuous devices, Annu. Rev. Phys. Chem. 65, 365 (2014).
- [9] B. Cleuren, B. Rutten, and C. Van den Broeck, Phys. Rev. Lett. 108, 120603 (2012).

- [10] F. Tonner, G. Mahler, Autonomous quantum thermodynamic machines. Phys. Rev. E 72(6), 066118 (2005).
- [11] N. Linden, S. Popescu, and P. Skrzypczyk, How small can thermal machines be? the smallest possible refrigerator. Phys. Rev. Lett., 105(13), 130401 (2010).
- [12] A. Levy, and R. Kosloff, Quantum absorption refrigerator, Phys. Rev. Lett. 108, 070604 (2012).
- [13] G. Maslennikov, S. Ding, R.Hablützel, J. Gan, A. Roulet, S. Nimmrichter, J. Dai, V.Scarani, and D. Matsukevich, Quantum absorption refrigerator with trapped ions, Nature Commun. 10, 202 (2019).
- [14] A. Alecce, F. Galve, N. Lo Gullo, L. Dell' Anna, F. Plastina, and R. Zambrini, Quantum Otto cycle with inner friction: finite-time and disorder effects, New J. Phys. 17, 075007 (2015).
- [15] S. Lee, M. Ha, J. M. Park, and H. Jeong, Finite-time quantum Otto engine: Surpassing the quasistatic efficiency due to friction, Phys. Rev. E 101, 022127 (2020).
- [16] C. de Tomas, A. C. Hernandez, and J. M. M. Roco, Phys. Rev. E 85, 010104(R) (2012).
- [17] Y. Wang, M. Li, Z. C. Tu, A. C. Hernandez, and J. M. M. Roco, Phys. Rev. E 86, 011127 (2012).
- [18] Y. Hu, F. Wu, Y. Ma, J. Z. He, J. H. Wang, A. Calvo Hernández, and J. M. M. Roco, Coefficient of performance for a low-dissipation Carnot-like refrigerator with nonadiabatic dissipation, Phys. Rev. E 88, 062115 (2013).
- [19] Y. Izumida, K. Okuda, J. M. M. Roco, and A. Calvo Hernández, Heat devices in nonlinear irreversible thermodynamics Phys. Rev. E 91, 052140 (2015).
- [20] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, Rep. Prog. Phys. 75, 126001 (2012).
- [21] K. Sekimoto, Stochastic Energetics (Spinger, Berlin, 2010).
- [22] M. Esposito, U. Harbola and S. Mukamel, Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems, Rev. Mod. Phys. 81, 1665 (2009).
- [23] M. Campisi, P. Hänggi, and P. Talkner, Quantum fluctuation relations: Foundations and applications, Rev. Mod. Phys. 83, 771 (2011).
- [24] P. Solinas, D. V. Averin, and J. P. Pekola, Work and its uctuations in a driven quantum system, Phys. Rev. B 87, 060508(R) (2013).
- [25] F. W. J. Hekking and J. P. Pekola, Quantum jump approach for work and dissipation in a

two-level system, Phys. Rev. Lett. 111, 093602 (2013).

- [26] T. B. Batalhão, A. M. Souza, L. Mazzola, R. Auccaise, R. S. Sarthour, I. S. Oliveira, J. Goold, G. De Chiara, M. Paternostro, and R. M. Serra, Experimental reconstruction of work distribution and study of uctuation relations in a closed quantum system, Phys. Rev. Lett. 113, 140601 (2014).
- [27] G. Verley, C. Van den Broeck, and M. Esposito, Work statistics in stochastically driven systems, New J. Phys. 16, 095001 (2014).
- [28] Y. Qian, and F. Liu, Computing characteristic functions of quantum work in phase space Physical Review E 100, 062119 (2019).
- [29] Z. Fei, N. Freitas, V. Cavina, H. T. Quan, and M. Esposito, Work statistics across a quantum phase transition, Phys. Rev. Lett 124, 170603 (2020).
- [30] F. Cerisola, Y. Margalit, S. Machluf, A. J. Roncaglia, J. P. Paz, and R. Folman, Using a quantum work meter to test non-equilibrium uctuation theorems, Nat. Commun. 8, 1241 (2017).
- [31] S. Rahav, U. Harbola, and S. Mukamel, Heat fluctuations and coherences in a quantum heat engine, Phys. Rev. A 86, 043843 (2012).
- [32] T. Denzler and E. Lutz, Heat distribution of a quantum harmonic oscillator, Phys. Rev. E 98, 052106 (2018).
- [33] S. Gasparinetti, P. Solinas, A. Braggio, and M. Sassetti, Heat-exchange statistics in driven open quantum systems, New J. Phys. 16, 115001 (2014).
- [34] V. Holubec and A. Ryabov, Cycling Tames Power Fluctuations Near Optimum Efficiency, Phys. Rev. Lett. 121, 120601 (2018).
- [35] V. Holubec, An exactly solvable model of a stochastic heat engine: Optimization of power, power fluctuations and efficiency, J. Stat. Mech. P05022 (2014).
- [36] J. H. Wang, J. Z. He, and Y. L. Ma, Finite-time performance of a quantum heat engine with a squeezed thermal bath, Phys. Rev. E 100, 052126 (2019).
- [37] T. Denzler and E. Lutz, Efficiency fluctuations of a quantum heat engine, Phys. Rev. Research 2, 032062 (2020).
- [38] H. Vroylandt, A. Bonfils, and G. Verley, Efficiency fluctuations of small machines with unknown losses, Phys. Rev. E 93, 052123 (2016).
- [39] J.-M. Park, H.-M. Chun and J. D. Noh, Efficiency at maximum power and efficiency fluctua-

tions in a linear Brownian heat-engine model, Phys. Rev. E 94, 012127 (2016).

- [40] G. Verley and M. Esposito, Efficiency statistics at all times: Carnot limit at finite power M Polettini, Phys. Rev. Lett. 114, 050601 (2015).
- [41] G. Verley, T Willaert, C. Van den Broeck, and M. Esposito, Universal theory of efficiency fluctuations, Phys. Rev.E 90, 052145 (2014).
- [42] H. Vroylandt, M. Esposito, and G. Verley, Efficiency fluctuations of stochastic machines undergoing a phase transition, Phys. Rev. Lett. 124, 250603 (2020).
- [43] J. -H. Jiang, B. K. Agarwalla, and D. Segal, Efficiency Statistics and Bounds for Systems with Broken Time-Reversal Symmetry, Phys. Rev. Lett. 115, 040601 (2015).
- [44] P. Talkner, E. Lutz, and P. Hänggi, Fluctuation theorems: Work is not an observable, Phys. Rev. E 75, 050102 (2007).
- [45] M. Born and V. Fock, Z. Phys. **51**, 165 (1928).
- [46] Plastina, F. A. Alecce, T.J.G. Apollaro, G. Falcone, G. Francica, F. Galve, N. Lo Gullo, and R. Zambrini. Irreversible work and inner friction in quantum thermodynamic processes. Phys. Rev. Lett. 113, 260601 (2014).
- [47] L. A. Correa, J. P. Palao, and D. Alonso, Internal dissipation and heat leaks in quantum thermodynamic cycles, Phys. Rev. E 92, 032136 (2015).
- [48] P. A. Camati, J. F. G. Santos, and R. M. Serra, Coherence effects in the performance of the quantum Otto heat engine, Phys. Rev. A 99, 062103 (2019).
- [49] R. Heijmans, When does the expectation of a ratio equal the ratio of expectations?, Statistical Papers 40, 107 (1999).