

Local number variances and hyperuniformity of the Heisenberg family of determinantal point processes

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Abstract

The bulk scaling limit of eigenvalue distribution on the complex plane \mathbb{C} of the complex Ginibre random matrices provides a determinantal point process (DPP). This point process is a typical example of disordered hyperuniform system characterized by an anomalous suppression of large-scale density fluctuations. As extensions of the Ginibre DPP, we consider a family of DPPs defined on the D -dimensional complex spaces \mathbb{C} , $D \in \mathbb{N}$, in which the Ginibre DPP is realized when $D = 1$. This one-parameter family ($D \in \mathbb{N}$) of DPPs is called the Heisenberg family, since the correlation kernels are identified with the Szegő kernels for the reduced Heisenberg group. For each D , using the modified Bessel functions, an exact and useful expression is shown for the local number variance of points included in a ball with radius R in $\mathbb{R}^{2D} \simeq \mathbb{C}^D$. We prove that any DPP in the Heisenberg family is in the hyperuniform state of Class I, in the sense that the number variance behaves as R^{2D-1} as $R \rightarrow \infty$. Our exact results provide asymptotic expansions of the number variances in large R .

Keywords Hyperuniformity; Local number variances; Determinantal point processes; Ginibre DPP; Heisenberg group; Heisenberg family of DPPs

1 Introduction and Main Results

We consider the d -dimensional Euclid space \mathbb{R}^d , $d \in \mathbb{N} := \{1, 2, \dots\}$, or the D -dimensional complex space \mathbb{C}^D , $D \in \mathbb{N}$ as a base space S . We assume that S is associated with a reference

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measure λ . We consider an *infinite point process* on S , which is expressed by an infinite sum of delta measures concentrated on a set of random points $X_i, i \in \mathbb{N}$,

$$\Xi = \sum_{i:i \in \mathbb{N}} \delta_{X_i}. \quad (1.1)$$

Here a delta measure $\delta_X(\{x\})$, $x \in S$, gives 1 if $x = X$, and 0 otherwise. Hence the number of points included in a domain $\Lambda \subset S$ is given by $\Xi(\Lambda) := \int_{\Lambda} \Xi(dx) = \sum_{i:X_i \in \Lambda} 1$. We assume that for any bounded domain $\Lambda \subset S$, $\Xi(\Lambda) < \infty$; that is, accumulation of points does not occur, and with respect to the reference measure $\lambda(dx)$ the point process has a finite density $\rho_1(x) < \infty$ at almost every $x \in S$. We consider a *homogeneous* point process in the sense that $\rho_1(x)\lambda(dx) = \text{const.} \times dx$, $x \in S$, where dx denotes the Lebesgue measure on S . The above assumption implies that for a bounded domain $\Lambda \subset S$ the expectation of $\Xi(\Lambda)$ is proportional to the volume $\text{vol}(\Lambda)$ of Λ ; $\mathbf{E}[\Xi(\Lambda)] \propto \text{vol}(\Lambda)$. Now we consider the number variance in the domain Λ ,

$$\text{var}[\Xi(\Lambda)] := \mathbf{E}[(\Xi(\Lambda) - \mathbf{E}[\Xi(\Lambda)])^2],$$

which represents local density fluctuation of point process Ξ . If the points are non-correlated and given by a Poisson process, then $\text{var}[\Xi(\Lambda)] \propto \text{vol}(\Lambda)$.

Recently in condensed matter physics and related material sciences, correlated particle systems are said to be in a *hyperuniform state* when density fluctuations are anomalously suppressed in large-scale limit. The bounded domain Λ is regarded as a observation *window* to measure density fluctuation of the system. For an infinite random point process Ξ , the *hyperuniformity* is defined by

$$\lim_{\Lambda \rightarrow S} \frac{\text{var}[\Xi(\Lambda)]}{\mathbf{E}[\Xi(\Lambda)]} = 0. \quad (1.2)$$

This means that the number variance of points grows more slowly than the window volume in the limit such that the window covers whole of the space $\Lambda \rightarrow S$. See [10, 30] and references therein. Moreover, Torquato [30] proposed the three hyperuniformity classes for point processes concerning asymptotics of number variances. In order to clearly state this classification, here we assume that $S = \mathbb{R}^d$ and $\Lambda = \mathbb{B}_R^{(d)}$, $d \in \mathbb{N}$, where $\mathbb{B}_R^{(d)}$ denotes a ball in \mathbb{R}^d with radius $R > 0$ centered at the origin; $\mathbb{B}_R^{(d)} := \{x \in \mathbb{R}^d : |x| < R\}$. The volume of the ball is given by

$$\text{vol}(\mathbb{B}_R^{(d)}) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} R^d, \quad (1.3)$$

where the gamma function is defined by $\Gamma(z) := \int_0^\infty e^{-u} u^{z-1} du$, $\text{Re } z > 0$ and it satisfies the functional equation $\Gamma(z + 1) = z\Gamma(z)$ with $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. We consider a series of balls with increasing R , $\{\mathbb{B}_R^{(d)}\}_{R>0}$, and the hyperuniform states are classified as follows;

$$\begin{aligned} \text{Class I :} & \quad \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-1}, \\ \text{Class II :} & \quad \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-1} \log R, \\ \text{Class III :} & \quad \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-\alpha}, \quad 0 < \alpha < 1, \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Here $f(R) \asymp g(R)$ means that there are finite positive constants c_1 and c_2 such that $c_1 g(R) < f(R) < c_2 g(R)$. The above characterization of these classes will be similarly described for any series of windows $\{\Lambda_R\}_{R>0}$ labeled by a linear scale R of window.

Determinantal point processes (DPPs) [26, 23, 24, 13, 14] studied in random matrix theory (RMT) [18, 9] provide a variety of examples of hyperuniform systems. In general a DPP is specified by a triplet $(\Xi, K, \lambda(dx))$ [17], where Ξ is a random measure (1.1) representing a point process, K is a continuous function $S \times S \rightarrow \mathbb{C}$ called the *correlation kernel*, and $\lambda(dx)$ is a reference measure defined on S . In Section 3.2 below a precise definition of DPP will be given.

The most studied DPP in RMT may be the *sinc (sine) determinantal point process* (DPP), $(\Xi_{\text{sinc}}, K_{\text{sinc}}, dx)$ on $S = \mathbb{R}$, where $K_{\text{sinc}}(x, y) = \sin(x - y)/\{\pi(x - y)\}$, $x, y \in \mathbb{R}$. This DPP is obtained as the bulk scaling limit of the eigenvalue distribution of Hermitian random matrices in the Gaussian unitary ensemble. It is known as a classical result in RMT that

$$\text{var}[\Xi_{\text{sinc}}(\mathbb{B}_R^{(1)})] \sim \frac{\log R}{\pi^2} \quad \text{as } R \rightarrow \infty.$$

See, for instance, [18, Section 16.1], [6], [27] [28] [23, Remark 5.8]. In the present paper $f(R) \sim g(R)$ as $R \rightarrow \infty$ means $\lim_{R \rightarrow \infty} f(R)/g(R) = 1$. That is, the sinc DPP is in Class II of hyperuniformity. Torquato et al.[31] and Torquato [30] studied one-parameter ($d \in \mathbb{N}$) family of DPPs called the *Fermi-sphere point processes*, which gives the sinc DPP when $d = 1$. They proved that the Fermi-sphere point processes are in Class II for general $d \in \mathbb{N}$.

An example of infinite DPP in Class I of hyperuniformity is also provided in RMT. It is the DPP on \mathbb{C} called the *Ginibre DPP*, $(\Xi_{\text{Ginibre}}, K_{\text{Ginibre}}, \lambda_{\text{N}(0,1;\mathbb{C})}(dx))$, which is obtained as the bulk scaling limit of eigenvalue distribution of non-Hermitian random matrices in the complex Ginibre ensemble [11]. Here the correlation kernel is given by $K_{\text{Ginibre}}(x, y) = e^{x\bar{y}}$, $x, y \in \mathbb{C}$, where \bar{y} denotes the complex conjugate of x , and $\lambda_{\text{N}(0,1;\mathbb{C})}(dx)$ is the *complex standard normal distribution*; $\lambda_{\text{N}(0,1;\mathbb{C})}(dx) = e^{-|x|^2} dx/\pi$. A disk on \mathbb{C} centered at the origin with radius R , $\mathbb{D}_R := \{x \in \mathbb{C} : |x| < R\}$, is identified with $\mathbb{B}_R^{(2)} \subset \mathbb{R}^2$. One of the present authors proved [21]

$$\text{var}[\Xi_{\text{Ginibre}}(\mathbb{B}_R^{(2)})] \sim \frac{R}{\sqrt{\pi}} \quad \text{as } R \rightarrow \infty.$$

See also [20, 22, 30]. In the present paper we will report extensions of this result in $S = \mathbb{C}$ to DPPs in the higher-dimensional complex spaces $S = \mathbb{C}^D$, $D = 2, 3, \dots$

When $S = \mathbb{C}^D$, $D \in \mathbb{N}$, each coordinate $x \in \mathbb{C}^D$ has D complex components; $x = (x^{(1)}, \dots, x^{(D)})$ with $x^{(\ell)} = \text{Re } x^{(\ell)} + \sqrt{-1}\text{Im } x^{(\ell)}$, $\ell = 1, \dots, D$. In order to clearly describe such a complex structure, we set $x_{\text{R}} := (\text{Re } x^{(1)}, \dots, \text{Re } x^{(D)})$, $x_{\text{I}} := (\text{Im } x^{(1)}, \dots, \text{Im } x^{(D)}) \in \mathbb{R}^D$, and write $x = x_{\text{R}} + \sqrt{-1}x_{\text{I}}$ in this paper. The Lebesgue measure on \mathbb{C}^D is given by $dx = dx_{\text{R}} dx_{\text{I}} := \prod_{\ell=1}^D d\text{Re } x^{(\ell)} d\text{Im } x^{(\ell)}$. For $x = x_{\text{R}} + \sqrt{-1}x_{\text{I}}$, $y = y_{\text{R}} + \sqrt{-1}y_{\text{I}} \in \mathbb{C}^D$, we use the *standard Hermitian inner product*;

$$x \cdot \bar{y} := (x_{\text{R}} + \sqrt{-1}x_{\text{I}}) \cdot (y_{\text{R}} - \sqrt{-1}y_{\text{I}}) = (x_{\text{R}} \cdot y_{\text{R}} + x_{\text{I}} \cdot y_{\text{I}}) - \sqrt{-1}(x_{\text{R}} \cdot y_{\text{I}} - x_{\text{I}} \cdot y_{\text{R}}).$$

Notice that if $x = x_{\text{R}}, y = y_{\text{R}} \in \mathbb{R}^D$, then $x \cdot \bar{y} = x_{\text{R}} \cdot y_{\text{R}} := \sum_{\ell=1}^D \text{Re } x^{(\ell)} \text{Re } y^{(\ell)}$. We define the norm by $|x| := \sqrt{x \cdot \bar{x}} = \sqrt{|x_{\text{R}}|^2 + |x_{\text{I}}|^2}$, $x \in \mathbb{C}^D$. Hence the D -dimensional disk

$\{x \in \mathbb{C}^D : |x| < R\}$ centered at the origin with radius R on \mathbb{C}^D will be identified with $\mathbb{B}_R^{(d)}$ in \mathbb{R}^d with $d = 2D, D \in \mathbb{N}$. On \mathbb{C}^D the reference measure is given by the D -dimensional extension of $\lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}(dx)$,

$$\begin{aligned} \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}(dx) &:= \prod_{i=1}^D \lambda_{\mathbb{N}(0,1;\mathbb{C})}(dx^{(i)}) \\ &= \frac{e^{-|x|^2}}{\pi^D} dx = \frac{e^{-(|x_{\mathbb{R}}|^2 + |x_{\mathbb{I}}|^2)}}{\pi^D} dx_{\mathbb{R}} dx_{\mathbb{I}}. \end{aligned} \quad (1.4)$$

The one-parameter family ($D \in \mathbb{N}$) of DPPs studied in this paper is the *Heisenberg family of DPPs* defined on \mathbb{C}^D as follows.

Definition 1.1 *The Heisenberg family of DPPs is defined by $(\Xi_{\mathbb{H}_D}, K_{\mathbb{H}_D}, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})$ on \mathbb{C}^D , $D \in \mathbb{N}$ with the correlation kernel*

$$K_{\mathbb{H}_D}(x, y) = e^{x \cdot \bar{y}}, \quad x, y \in \mathbb{C}^D. \quad (1.5)$$

Note that $K_{\mathbb{H}_D}$ is hermitian; $\overline{K_{\mathbb{H}_D}(x, y)} = K_{\mathbb{H}_D}(y, x)$, $x, y \in \mathbb{C}^D$. The kernels in this form on \mathbb{C}^D , $D \in \mathbb{N}$ have been studied by Zelditch and his coworkers (see [35, 4] and references therein), who identified them with the Szegő kernels for the *reduced Heisenberg group* $\mathbb{H}_D^{\text{red}}$. This is the reason why we call the DPPs associated with (1.5) the *Heisenberg family of DPPs* on \mathbb{C}^D , $D \in \mathbb{N}$ [17]. This family includes the complex Ginibre DPP [11, 13, 20, 14, 22, 30, 15] as the lowest dimensional case with $D = 1$. A brief review of the representation theory of the Heisenberg group \mathbb{H}_D is given in Appendix A. There the Bargmann–Fock representation of \mathbb{H}_D is explained and the correlation kernel (1.5) is realized as the *reproducing kernel* of the Bargmann–Fock space \mathcal{F}_D . It should be noted that [17] if we follow a similar reasoning, the Fermi-sphere point processes studied by Torquato et al. [31, 30] can be called the *Euclidean family of DPPs*, since the correlation kernels in these DPPs can be regarded as the Szegő kernels for the reduced Euclidean motion group [35, 25, 36, 5].

Define the modified Bessel function of the first kind [34, 19] by

$$I_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (1.6)$$

We prove the following.

Proposition 1.2 *For the Heisenberg family of DPPs, $(\Xi_{\mathbb{H}_D}, K_{\mathbb{H}_D}, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})$ on \mathbb{C}^D , $D \in \mathbb{N}$,*

$$\begin{aligned} \text{var}[\Xi_{\mathbb{H}_D}(\mathbb{B}_R^{(2D)})] &= \frac{R^{2D} e^{-2R^2}}{D!} \sum_{n=0}^{D-1} [I_n(2R^2) + I_{n+1}(2R^2)] \\ &= \frac{R^{2D} e^{-2R^2}}{D!} \left[I_0(2R^2) + 2 \sum_{n=1}^{D-1} I_n(2R^2) + I_D(2R^2) \right], \quad R > 0. \end{aligned} \quad (1.7)$$

Remark 1 When $D = 1$, (1.7) gives

$$\text{var}[\Xi_{\text{H}_1}(\mathbb{B}_R^{(2)})] = \text{var}[\Xi_{\text{Ginibre}}(\mathbb{B}_R^{(2)})] = R^2 e^{-2R^2} [I_0(2R^2) + I_1(2R^2)],$$

which is identified with Eq.(249) in [30] calculated for the complex Ginibre DPP.

Remark 2 If we use the following type of hypergeometric function

$${}_2F_2(a_1, a_2; b_1, b_2; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n x^n}{(b_1)_n (b_2)_n n!} \quad (1.8)$$

with the Pochhammer symbols, $(a)_0 := 1$, $(a)_n := a(a+1)\cdots(a+n-1)$, $n \in \mathbb{N}$, the number variances (1.7) are expressed as

$$\begin{aligned} & \text{var}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})] \\ &= \frac{R^{2D}}{\Gamma(D+1)} \left[1 - \frac{R^{2D}}{\Gamma(D+1)} {}_2F_2(D, D+1/2; D+1, 2D+1; -4R^2) \right], \end{aligned} \quad (1.9)$$

$D \in \mathbb{N}$, $R > 0$. We found that the expressions (1.7) using the modified Bessel functions with argument $2R^2$ are more useful than (1.9) to derive the following results.

Theorem 1.3 Any DPP in the Heisenberg family, $(\Xi_{\text{H}_D}, K_{\text{H}_D}, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})$ on \mathbb{C}^D , $D \in \mathbb{N}$, is in the hyperuniform state of Class I such that

$$\lim_{R \rightarrow \infty} R \frac{\text{var}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})]}{\mathbf{E}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})]} = \frac{D}{\sqrt{\pi}}. \quad (1.10)$$

Moreover, for each $D \in \mathbb{N}$, the following asymptotic expansion holds,

$$\frac{\text{var}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})]}{\mathbf{E}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})]} \sim \frac{D}{\sqrt{\pi}} R^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k(D)}{(2k+1)k!2^{4k}} R^{-2k} \quad \text{as } R \rightarrow \infty, \quad (1.11)$$

where

$$\alpha_k(D) = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{\ell=1}^k \{4D^2 - (2\ell - 1)^2\} = \prod_{\ell=-k+1}^k (2D + 2\ell - 1), & \text{if } k \in \mathbb{N}. \end{cases} \quad (1.12)$$

Remark 3 The Heisenberg family of DPPs belongs to a wider class of DPPs called the infinite Weyl–Heisenberg ensemble studied by Abreu et al. [1, 3, 2]. In the present setting and notations, a DPP in the Weyl–Heisenberg ensemble is expressed by $(\Xi_{\text{WH}}, K_{\text{WH}}^g, dx)$ on $S = \mathbb{C}^D \simeq \mathbb{R}^{2D}$, $D \in \mathbb{N}$, with the correlation kernel in the form,

$$K_{\text{WH}}^g(x, y) = \int_{\mathbb{R}^D} g(u - x_{\mathbb{R}}) \overline{g(u - y_{\mathbb{R}})} e^{2\sqrt{-1}(x_{\mathbb{I}} - y_{\mathbb{I}}) \cdot u} du, \quad x, y \in \mathbb{C}^D,$$

where a function g on \mathbb{R}^D satisfies some conditions [1, 3, 2]. See also Section 2.6 of [17]. We can verify that if g is chosen as

$$G(\zeta) = \left(\frac{2}{\pi}\right)^{D/4} \frac{e^{-\zeta^2}}{\pi^{D/2}}, \quad \zeta \in \mathbb{R}^D, \quad (1.13)$$

then the conditions are satisfied and the following equality is obtained,

$$K_{\text{WH}}^G(x, y) = \frac{e^{\sqrt{-1}x_{\text{R}} \cdot x_{\text{I}}}}{e^{\sqrt{-1}y_{\text{R}} \cdot y_{\text{I}}}} \sqrt{\frac{e^{-|x|^2}}{\pi^D}} K_{\text{H}_D}(x, y) \sqrt{\frac{e^{-|y|^2}}{\pi^D}}, \quad x, y \in \mathbb{C}^D.$$

The factor $e^{\sqrt{-1}x_{\text{R}} \cdot x_{\text{I}}}/e^{\sqrt{-1}y_{\text{R}} \cdot y_{\text{I}}}$ is irrelevant for DPP and this equality proves the equivalence between $(\Xi_{\text{H}_D}, K_{\text{H}_D}, \lambda_{\text{N}(0,1;\mathbb{C}^D)})$ and $(\Xi_{\text{WH}}, K_{\text{WH}}^G, dx)$ on \mathbb{C}^D , $D \in \mathbb{N}$. It was proved in [3, Theorem 5.8] that any DPP in the Weyl-Heisenberg ensemble is in the hyperuniform state of Class I. Notice that we have determined the coefficient $D/\sqrt{\pi}$ of the dominant term in $R \rightarrow \infty$ as (1.10) and completed the asymptotic expansion (1.11) with (1.12) for the Heisenberg family of DPPs in the above theorem.

Remark 4 Since the present Heisenberg family of point processes is determinantal, higher cumulants of $\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})$ can be directly calculated for finite values of R and their asymptotics in $R \rightarrow \infty$ will be evaluated. See [6, 8, 26, 28, 23, 21, 20] for general formulas of cumulants of linear statistics and their generating functions as well as applications to the sinc DPP and the Ginibre DPP (*i.e.*, the Heisenberg DPP with $D = 1$). In the present paper we concentrated on expectations and variances, since hyperuniformity (1.2) defined by these first two cumulants is focused. By Proposition 2.4 in [21] (see also Theorem 24 in [13]), the divergence $\text{var}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})] \rightarrow \infty$ as $R \rightarrow \infty$ implies that $\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})/\mathbf{E}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})] \rightarrow 1$ almost surely and the *central limit theorem* holds; as $R \rightarrow \infty$, $(\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)}) - \mathbf{E}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})])/\text{var}[\Xi_{\text{H}_D}(\mathbb{B}_R^{(2D)})]$ converges in distribution to the standard normal distribution $\text{N}(0, 1)$.

Remark 5 Applying the *duality relation between DPPs* (see Theorem 2.6 in [17]), we can evaluate $\text{var}[\Xi_{\text{H}_D}(\Lambda)]/\mathbf{E}[\Xi_{\text{H}_D}(\Lambda)]$ for windows which are different from balls. A *polydisk* $\Delta_R^{(D)}$ of radius $R > 0$ in \mathbb{C}^D , $D \in \mathbb{N}$ is defined by $\Delta_R^{(D)} := \{x = (x_1, \dots, x_D) \in \mathbb{C}^D : |x_i| < R, i = 1, \dots, D\}$. We can show that

$$\begin{aligned} \frac{\text{var}[\Xi_{\text{H}_D}(\Delta_R^{(D)})]}{\mathbf{E}[\Xi_{\text{H}_D}(\Delta_R^{(D)})]} &= 1 - \left(1 - \frac{\text{var}[\Xi_{\text{H}_1}(\mathbb{B}_R^{(2)})]}{\mathbf{E}[\Xi_{\text{H}_1}(\mathbb{B}_R^{(2)})]}\right)^D \\ &\sim \frac{D}{\sqrt{\pi}} R^{-1} \left[1 - \frac{D-1}{2\sqrt{\pi}} R^{-1} + \frac{1}{2} \left\{ \frac{(D-1)(D-2)}{3\pi} - \frac{1}{8} \right\} R^{-2} + \text{O}(R^{-3})\right] \end{aligned} \quad (1.14)$$

as $R \rightarrow \infty$. The leading term in $R \rightarrow \infty$ is exactly the same as (1.10) for balls $\mathbb{B}_R^{(2D)}$, but the correction terms with R^{-k} , $k \geq 1$ are different from (1.11).

The paper is organized as follows. In Section 2 we will give preliminaries for linear statistics of infinite point processes which are translationally invariant in distribution [30].

There the formulas of Bessel functions which we use in this paper are also summarized. In Section 3 we show useful formulas for local number variances and the definition of DPP with additional assumptions is given. Then the Heisenberg family of DPPs on $\mathbb{C}^D \simeq \mathbb{R}^{2D}$, $D \in \mathbb{N}$ is studied. The proofs of Proposition 1.2, the formula (1.9) in Remark 2, Theorem 1.3, and the formula (1.14) in Remark 5 are given in Section 4. In Appendix A a brief review of the representation theory of the Heisenberg group \mathbf{H}_D [7, 29, 12] is given.

2 Preliminaries

2.1 Correlation functions and variances

The configuration space of point process $\Xi = \Xi(\cdot)$ is given by

$$\text{Conf}(S) = \left\{ \xi = \sum_i \delta_{x_i} : x_i \in S, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

If $\Xi(\{x\}) \in \{0, 1\}$ for any point $x \in S$, then the point process is said to be *simple*. Let $\mathcal{B}_c(S)$ be the set of all bounded measurable complex functions on S of compact support and for $\xi \in \text{Conf}(S)$ and $\phi \in \mathcal{B}_c(S)$ we set

$$\langle \xi, \phi \rangle := \int_S \phi(x) \xi(dx) = \sum_i \phi(x_i).$$

Random variables written in this form are generally called *linear statistics* of a point process Ξ [18, Section 16.1] [9, Definition 14.3.1]. For a point process Ξ , if there exists a non-negative measurable function ρ_1 such that

$$\mathbf{E}[\langle \Xi, \phi \rangle] = \int_S \phi(x) \rho_1(x) \lambda(dx) \quad \forall \phi \in \mathcal{B}_c(S), \quad (2.1)$$

ρ_1 is called the *first correlation function* of Ξ with respect to the reference measure λ . By definition, $\rho_1(x)$ gives the density of point at $x \in S$ with respect to $\lambda(dx)$. For $n \in \mathbb{N}$, from $\xi \in \text{Conf}(S)$ we define $\xi_n := \sum_{i_1, \dots, i_n: i_j \neq i_k, j \neq k} \delta_{x_{i_1}} \cdots \delta_{x_{i_n}}$ and denote the n -product measure of λ by $\lambda^{\otimes n}$; $\lambda^{\otimes n}(dx_1 \cdots dx_n) := \prod_{i=1}^n \lambda(dx_i)$. For a point process Ξ , if there exists a symmetric, non-negative measurable function ρ_n on S^n such that

$$\mathbf{E}[\langle \Xi_n, \phi \rangle] = \int_{S^n} \phi(x_1, \dots, x_n) \rho_n(x_1, \dots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n) \quad \forall \phi \in \mathcal{B}_c(S^n),$$

we say that ρ_n is the *n -th correlation function* of Ξ with respect to λ .

We put the assumptions.

(A1) The point process Ξ on $(S, \mathcal{B}_c(S), \lambda)$ has the first and the second correlation functions.

The following is readily proved by the definition of correlation functions given above (see, for instance, [32, 33, 20, 10, 30]).

Lemma 2.1 Assume **(A1)**. For $\phi \in \mathcal{B}_c(S)$, the variance

$$\text{var}[\langle \Xi, \phi \rangle] := \mathbf{E}[|\langle \Xi, \phi \rangle - \mathbf{E}[\langle \Xi, \phi \rangle]|^2]$$

is expressed as

$$\text{var}[\langle \Xi, \phi \rangle] = \int_S |\phi(x)|^2 \rho_1(x) \lambda(dx) + \int_{S \times S} \phi(x) \overline{\phi(y)} (\rho_2(x, y) - \rho_1(x) \rho_1(y)) \lambda^{\otimes 2}(dxdy). \quad (2.2)$$

From now on we consider the case in which $S = \mathbb{R}^d$ with $d \in \mathbb{N}$. We put a further assumption.

(A2) The system is translationally invariant with respect to the Lebesgue measure dx on \mathbb{R}^d in the following sense.

- (i) The reference measure has a density $\ell(x)$ with respect to the Lebesgue measure dx on \mathbb{R}^d ; $\lambda(dx) = \ell(x)dx$, $x \in \mathbb{R}^d$, and

$$\rho_1(x)\ell(x) = \text{constant} =: \tilde{\rho} \quad \forall x \in \mathbb{R}^d.$$

- (ii) There is a measurable even function $g_2(x) = g_2(-x)$, $x \in \mathbb{R}^d$ so that the second correlation function is written in the form

$$\rho_2(x, y)\ell(x)\ell(y) = \tilde{\rho}^2 g_2(x - y), \quad x, y \in \mathbb{R}^d.$$

The two-point function $g(x)$ is called the *unfolded 2-correlation function* [9]. We define the following function which is called the *total correlation function* [30],

$$C(x) = g_2(x) - 1, \quad x \in \mathbb{R}^d. \quad (2.3)$$

Under the assumptions **(A1)** and **(A2)**, (2.2) is written as

$$\begin{aligned} \text{var}[\langle \Xi, \phi \rangle] &= \tilde{\rho} \left[\int_{\mathbb{R}^d} |\phi(x)|^2 dx + \tilde{\rho} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x) \overline{\phi(y)} C(x - y) dxdy \right] \\ &= \tilde{\rho} \left[\int_{\mathbb{R}^d} dx |\phi(x)|^2 + \tilde{\rho} \int_{\mathbb{R}^d} dz C(z) \int_{\mathbb{R}^d} dx \phi(x) \overline{\phi(x - z)} \right], \end{aligned}$$

where the integral variables are changed as $(x, y) \rightarrow (x, z)$ with $z = x - y$. Define

$$\mathcal{I}_\phi(z) := \int_{\mathbb{R}^d} \phi(x) \overline{\phi(x - z)} dx, \quad \phi \in \mathcal{B}_c(\mathbb{R}^d), \quad z \in \mathbb{R}^d. \quad (2.4)$$

This may be called the *intersection integral* of ϕ with displacement z . Remark that if $\phi \in \mathcal{B}_c(\mathbb{R}^d)$, then $\mathcal{I}_\phi \in \mathcal{B}_c(\mathbb{R}^d)$. Then Lemma 2.1 gives the following.

Proposition 2.2 Assume **(A1)** and **(A2)**. For $\phi \in \mathcal{B}_c(\mathbb{R}^d)$, the variance is given by

$$\text{var}[\langle \Xi, \phi \rangle] = \tilde{\rho} \left[\int_{\mathbb{R}^d} |\phi(x)|^2 dx + \tilde{\rho} \int_{\mathbb{R}^d} \mathcal{I}_\phi(x) C(x) dx \right].$$

For $k = (k^{(1)}, \dots, k^{(d)})$, $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$, $k \cdot x := \sum_{\ell=1}^d k^{(\ell)}x^{(\ell)}$, and with an integrable function φ the Fourier transform is defined by

$$\widehat{\varphi}(k) = \mathbf{F}[\varphi](k) := \int_{\mathbb{R}^d} e^{\sqrt{-1}k \cdot x} \varphi(x) dx, \quad (2.5)$$

and the inverse Fourier transform is given by

$$\varphi(x) = \mathbf{F}^{-1}[\widehat{\varphi}](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\sqrt{-1}k \cdot x} \widehat{\varphi}(k) dk. \quad (2.6)$$

Note that $\varphi(-x) = \varphi(x) \iff \widehat{\varphi}(-k) = \widehat{\varphi}(k)$. If $\varphi(x)$ and $\psi(x)$ are square integrable, then the *Parseval formula* holds,

$$\int_{\mathbb{R}^d} \varphi(x) \overline{\psi(x)} dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\varphi}(k) \overline{\widehat{\psi}(k)} dk. \quad (2.7)$$

By the definition (2.5), we have $\mathbf{F}[\phi(\cdot - z)](k) = \widehat{\phi}(k) e^{\sqrt{-1}k \cdot z}$. Hence, using the Parseval formula (2.7) for $\phi \in \mathcal{B}_c(\mathbb{R}^d)$, (2.4) is written as

$$\mathcal{I}_\phi(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\phi}(k) \overline{\widehat{\phi}(k)} e^{\sqrt{-1}k \cdot z} dk = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\sqrt{-1}k \cdot z} |\widehat{\phi}(k)|^2 dk, \quad z \in \mathbb{R}^d.$$

Comparing this with (2.6), we see that

$$\widehat{\mathcal{I}_\phi}(k) = |\widehat{\phi}(k)|^2, \quad \phi \in \mathcal{B}_c(\mathbb{R}^d), \quad k \in \mathbb{R}^d. \quad (2.8)$$

We put the third assumption.

(A3) Provided $S = \mathbb{R}^d$ with $d \in \mathbb{N}$, the total correlation function $C(x)$, $x \in \mathbb{R}^d$ is square integrable, and thus so is its Fourier transform $\widehat{C}(k)$, $k \in \mathbb{R}^d$.

Define

$$\widehat{S}(k) = 1 + \widetilde{\rho} \widehat{C}(k), \quad k \in \mathbb{R}^d, \quad (2.9)$$

which is called the *structure factor*. By definition, this is even; $\widehat{S}(-k) = \widehat{S}(k)$, $k \in \mathbb{R}^d$. By the Parseval formula (2.7), Proposition 2.2 gives the following.

Proposition 2.3 *Assume (A1)–(A3). For $\phi \in \mathcal{B}_c(\mathbb{R}^d)$, the variance is given by*

$$\text{var}[\langle \Xi, \phi \rangle] = \frac{\widetilde{\rho}}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\mathcal{I}_\phi}(k) \widehat{S}(k) dk.$$

2.2 Bessel functions

The *Bessel function of the first kind* is defined as [34, 19]

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (2.10)$$

If $\varphi(x)$, $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$ depends only on the modulus $|x| = \sqrt{\sum_{\ell=1}^d (x^{(\ell)})^2}$, it is said to be *radial*. The following lemma is well known (see, for instance, [30, Section 2.1]).

Lemma 2.4 *If the integrable function $\varphi(x)$, $x \in \mathbb{R}^d$ is radial and expressed as $\varphi(x) = f(r)$ with $r = |x|$, then its Fourier transform (2.5) is also radial and given by a function of $\kappa := |k|$ as*

$$\begin{aligned} \widehat{\varphi}(k) &= \widehat{f}(\kappa) = (2\pi)^{d/2} \int_0^\infty r^{d-1} \frac{J_{(d-2)/2}(\kappa r)}{(\kappa r)^{(d-2)/2}} f(r) dr \\ &= \frac{(2\pi)^{d/2}}{\kappa^{(d-2)/2}} \int_0^\infty r^{d/2} J_{(d-2)/2}(\kappa r) f(r) dr. \end{aligned}$$

The inverse transform of $\widehat{\varphi}(k) = \widehat{f}(\kappa)$ is given by

$$\begin{aligned} \varphi(x) = f(r) &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \kappa^{d-1} \frac{J_{(d-2)/2}(\kappa r)}{(\kappa r)^{(d-2)/2}} \widehat{f}(\kappa) d\kappa \\ &= \frac{1}{(2\pi)^{d/2} r^{(d-2)/2}} \int_0^\infty \kappa^{d/2} J_{(d-2)/2}(\kappa r) \widehat{f}(\kappa) d\kappa. \end{aligned} \quad (2.11)$$

We will use the following formulas [19, 34] for an indefinite integral,

$$\int \frac{J_\nu(ax)^2}{x^{2\nu-1}} dx = -\frac{1}{2(2\nu-1)} \frac{J_{\nu-1}(ax)^2 + J_\nu(ax)^2}{x^{2(\nu-1)}}, \quad \nu \neq 1/2, \quad (2.12)$$

and for definite integrals,

$$\int_0^\infty x^{-1} J_\nu(ax)^2 dx = \frac{1}{2\nu}, \quad (2.13)$$

$$\int_0^\infty x^{\nu+1} e^{-p^2 x^2} J_\nu(ax) dx = \frac{a^\nu}{(2p^2)^{\nu+1}} e^{-a^2/(4p^2)}, \quad (2.14)$$

$$\int_0^\infty x e^{-p^2 x^2} J_\nu(ax)^2 dx = \frac{1}{2p^2} e^{-a^2/(2p^2)} I_\nu(a^2/(2p^2)), \quad (2.15)$$

$$\begin{aligned} \int_0^\infty x^{-1} e^{-p^2 x^2} J_\nu(ax)^2 dx \\ = \frac{(a/p)^{2\nu}}{2^{2\nu+1} \nu^2 \Gamma(\nu)} {}_2F_2(\nu, \nu + 1/2; \nu + 1, 2\nu + 1; -(a/p)^2), \end{aligned} \quad (2.16)$$

$\operatorname{Re} \nu > -1, \operatorname{Re} p^2 > 0$, where I_ν and ${}_2F_2$ are defined by (1.6) and (1.8), respectively. Note that (2.16) is a special case of the integral formula given in Section 13.32 of [34] which is expressed using ${}_3F_3$. The following asymptotics will be also used [34, 19],

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}} \left\{ \cos \omega_\nu(x) \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_{2k}(\nu)}{(2k)! 2^{6k}} x^{-2k} \right. \\ &\quad \left. - \sin \omega_\nu(x) \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_{2k+1}(\nu)}{(2k+1)! 2^{3(2k+1)}} x^{-2k-1} \right\} \\ &\sim \sqrt{\frac{2}{\pi x}} \cos \omega_\nu(x), \quad \text{as } x \rightarrow \infty, \end{aligned} \tag{2.17}$$

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k(\nu)}{k! 2^{3k}} x^{-k}, \quad \text{as } x \rightarrow \infty, \tag{2.18}$$

where $\omega_\nu(x) = x - (2\nu + 1)\pi/4$ and $\alpha_k, k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ are defined by (1.12).

3 Local Number Variances

3.1 General formulas

An indicator function of a domain $\Lambda \subset S$ is defined by

$$1_\Lambda(x) := \begin{cases} 1, & \text{if } x \in \Lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Here we consider the case that $S = \mathbb{R}^d, d \in \mathbb{N}$ and $\Lambda = \mathbb{B}_R^{(d)}$ with $R > 0$. By definition $1_{\mathbb{B}_R^{(d)}}(x)$ is radial and we write $1_{\mathbb{B}_R^{(d)}}(x) = \chi_{\mathbb{B}_R^{(d)}}(|x|)$. For $\phi = 1_{\mathbb{B}_R^{(d)}}$, the intersection integral (2.4) becomes

$$\mathcal{I}_{1_{\mathbb{B}_R^{(d)}}}(x) = \int_{\mathbb{R}^d} 1_{\mathbb{B}_R^{(d)}}(y) 1_{\mathbb{B}_R^{(d)}}(y-x) dy, \quad x \in \mathbb{R}^d. \tag{3.1}$$

This is called the *intersection volume* of two balls with radius R whose centers are separated by x [32, 33, 30]. By definition $\mathcal{I}_{1_{\mathbb{B}_R^{(d)}}}(x) = 0$ for $x \in \mathbb{R}^d \setminus \mathbb{B}_R^{(d)}$.

Under the assumptions **(A1)** and **(A2)**, (2.1) gives

$$\mathbf{E}[\Xi(\mathbb{B}_R^{(d)})] = \tilde{\rho} \int_{\mathbb{R}^d} 1_{\mathbb{B}_R^{(d)}}(x) dx = \operatorname{vol}(\mathbb{B}_R^{(d)}) \tilde{\rho}, \tag{3.2}$$

where $\operatorname{vol}(\mathbb{B}_R^{(d)})$ is given by (1.3).

As an application of Lemma 2.4, we have the following. See, for instance, [16, 17] for proof.

Lemma 3.1 *The Fourier transform of $1_{\mathbb{B}_R^{(d)}}(x)$*

$$\widehat{1_{\mathbb{B}_R^{(d)}}}(k) := \int_{\mathbb{R}^d} e^{\sqrt{-1}k \cdot x} 1_{\mathbb{B}_R^{(d)}}(x) dx = \int_{\mathbb{B}_R^{(d)}} e^{\sqrt{-1}k \cdot x} dx$$

is radial and given as a function of $\kappa := |k|$. If we write it as $\widehat{1_{\mathbb{B}_R^{(d)}}}(k) = \widehat{\chi_{\mathbb{B}_R^{(d)}}}(\kappa)$, then we have

$$\widehat{\chi_{\mathbb{B}_R^{(d)}}}(\kappa) = \frac{(2\pi)^{d/2}}{\kappa^{(d-2)/2}} \int_0^R r^{d/2} J_{(d-2)/2}(\kappa r) dr = (2\pi)^{d/2} \left(\frac{R}{\kappa}\right)^{d/2} J_{d/2}(\kappa R).$$

With the relation (2.8) the above lemma gives the Fourier transform of the intersection volume (3.1) as

$$\widehat{\mathcal{I}_{\mathbb{B}_R^{(d)}}}(k) = (2\pi)^d R^d \frac{J_{d/2}(\kappa R)^2}{\kappa^d} =: \widehat{\mathcal{I}_{\chi_{\mathbb{B}_R^{(d)}}}}(\kappa), \quad k \in \mathbb{R}^d, \quad \kappa = |k|. \quad (3.3)$$

The intersection volume (3.1) is then obtained as a function of the modulus $r = |x|$ by performing the inverse Fourier transform (2.11) of (3.3);

$$\begin{aligned} \mathcal{I}_{\mathbb{B}_R^{(d)}}(x) &= \mathbb{F}^{-1} \left[\widehat{\mathcal{I}_{\mathbb{B}_R^{(d)}}} \right] (x) \\ &= \frac{(2\pi)^{d/2}}{r^{(d-2)/2}} R^d \int_0^\infty \frac{J_{d/2}(\kappa R)^2 J_{(d-2)/2}(\kappa r)}{\kappa^{d/2}} d\kappa =: \mathcal{I}_{\chi_{\mathbb{B}_R^{(d)}}}(r), \quad r = |x| < 2R. \end{aligned}$$

By the definition (2.4), $\mathcal{I}_{\chi_{\mathbb{B}_R^{(d)}}}(r) = 0$ if $r \geq 2R$. Hence as a corollary of Propositions 2.2 and 2.3 we have the following.

Corollary 3.2 (i) *Assume (A1) and (A2). Then*

$$\text{var}[\Xi(\mathbb{B}_R^{(d)})] = \tilde{\rho} \left[\text{vol}(\mathbb{B}_R^{(d)}) + \tilde{\rho} \int_{\mathbb{R}^d} \mathcal{I}_{\chi_{\mathbb{B}_R^{(d)}}}(|x|) C(x) dx \right],$$

where $C(x)$ is the total correlation function (2.3). When $C(x)$ is radial and written as $C(x) = c(r)$ with $r = |x|$, then

$$\text{var}[\Xi(\mathbb{B}_R^{(d)})] = \tilde{\rho} \left[\text{vol}(\mathbb{B}_R^{(d)}) + \frac{2\pi^{d/2} \tilde{\rho}}{\Gamma(d/2)} \int_0^{2R} \mathcal{I}_{\chi_{\mathbb{B}_R^{(d)}}}(r) c(r) r^{d-1} dr \right]. \quad (3.4)$$

(ii) *Assume (A1)–(A3). Then*

$$\text{var}[\Xi(\mathbb{B}_R^{(d)})] = \frac{\tilde{\rho}}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\mathcal{I}_{\mathbb{B}_R^{(d)}}}(|k|) \widehat{S}(k) dk = \tilde{\rho} R^d \int_{\mathbb{R}^d} \frac{J_{d/2}(|k|R)^2}{|k|^d} \widehat{S}(k) dk, \quad (3.5)$$

where $\widehat{S}(k)$ is the structure factor (2.9). When $\widehat{S}(k)$ is radial and written as $\widehat{S}(k) = \widehat{s}(\kappa)$ with $\kappa = |k|$, then

$$\text{var}[\Xi(\mathbb{B}_R^{(d)})] = \frac{2\pi^{d/2} \tilde{\rho}}{\Gamma(d/2)} R^d \int_0^\infty \frac{J_{d/2}(\kappa R)^2}{\kappa} \widehat{s}(\kappa) d\kappa. \quad (3.6)$$

Proof The formulas (3.4) and (3.6) are obtained, if we use the polar coordinate expression of the Lebesgue measure for radial functions; $dx = r^{d-1} \sigma_{d-1} dr$ with $\sigma_{d-1} = 2\pi^{d/2} / \Gamma(d/2)$. ■

3.2 Determinantal point processes

Determinantal point process (DPP) is defined as follows [26, 23, 24, 13, 14, 17].

Definition 3.3 *A simple point process Ξ on $(S, \mathcal{B}_c(S), \lambda)$ is said to be a DPP with a measurable kernel $K : S \times S \rightarrow \mathbb{C}$, if it satisfies the assumption **(A1)** so that the correlation functions with respect to λ are given by*

$$\rho_n(x_1, \dots, x_n) = \det_{1 \leq i, j \leq n} [K(x_i, x_j)] \quad \text{for every } n \in \mathbb{N} \text{ and any } x_1, \dots, x_n \in S.$$

The integral kernel K is called the correlation kernel. The DPP is specified by the triplet (Ξ, K, λ) .

If the point process Ξ is a DPP, then (2.1) and (2.2) in Lemma 2.1 are given by

$$\begin{aligned} \mathbf{E}[\langle \Xi, \phi \rangle] &= \int_S \phi(x) K(x, x) \lambda(dx), \\ \text{var}[\langle \Xi, \phi \rangle] &= \frac{1}{2} \int_{S \times S} |\phi(x) - \phi(y)|^2 K(x, y) K(y, x) \lambda^{\otimes 2}(dxdy), \quad \phi \in \mathcal{B}_c(S). \end{aligned}$$

In particular, when $\phi = 1_\Lambda$ for a bounded domain $\Lambda \subset S$, the above give the following,

$$\begin{aligned} \mathbf{E}[\langle \Xi(\Lambda) \rangle] &= \int_\Lambda K(x, x) \lambda(dx), \\ \text{var}[\langle \Xi(\Lambda) \rangle] &= \int_\Lambda \int_{S \setminus \Lambda} K(x, y) K(y, x) \lambda(dx) \lambda(dy). \end{aligned} \quad (3.7)$$

Provided that $S = \mathbb{R}^d$, $d \in \mathbb{N}$, or $S = \mathbb{C}^D \simeq \mathbb{R}^d$ with $d = 2D$, $D \in \mathbb{N}$, we put additional assumptions.

(DPP) The point process Ξ is a DPP on S , (Ξ, K, λ) , and the following are satisfied.

- (i) The correlation kernel is hermitian,

$$\overline{K(x, y)} = K(y, x), \quad x, y \in S.$$

- (ii) The reference measure is given in the form $\lambda(dx) = \ell(x)dx$, $x \in S$, and

$$K(x, x)\ell(x) = \text{constant} =: \tilde{\rho} \quad \forall x \in S.$$

- (iii) There is a measurable even function $C(x) = C(-x)$, $x \in S$ such that

$$C(x - y) = -\frac{|K(x, y)|^2}{K(x, x)K(y, y)}, \quad x, y \in S.$$

Corollary 3.4 *Assume **(DPP)** and **(A3)**. Then the assertions of Corollary 3.2 (ii) hold.*

3.3 Heisenberg family of DPPs on \mathbb{C}^D

Lemma 3.5 *The Heisenberg family of DPPs, $(\Xi_{\mathbb{H}_D}, K_{\mathbb{H}_D}, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})$ on \mathbb{C}^D , $D \in \mathbb{N}$ satisfies (DPP) for $S = \mathbb{C}^D \simeq \mathbb{R}^d$ with $d = 2D$,*

$$\tilde{\rho} = \frac{1}{\pi^D}, \quad (3.8)$$

and

$$C(x) = c(|x|) = -e^{-|x|^2}, \quad (3.9)$$

where $|x|^2 = |x_{\mathbb{R}}|^2 + |x_{\mathbb{I}}|^2$, $x \in \mathbb{C}^D$.

Proof By Definition 1.1, $\rho_1(x) = K_{\mathbb{H}_D}(x, x) = e^{-|x|^2}$, $x \in \mathbb{C}^D$. Then $\tilde{\rho} = e^{-|x|^2} e^{-|x|^2} / \pi^D = 1/\pi^D$ proving (3.8). Since $K_{\mathbb{H}_D}(x, y) = e^{x \cdot \bar{y}}$, we see that

$$\begin{aligned} C(x-y) &= -\frac{|K_{\mathbb{H}_D}(x, y)|^2}{K_{\mathbb{H}_D}(x, x)K_{\mathbb{H}_D}(y, y)} = -e^{x \cdot \bar{y} + \bar{x} \cdot y - |x|^2 - |y|^2} \\ &= -e^{-|x-y|^2} =: c(|x-y|), \end{aligned}$$

which proves (3.9). ■

Since $\mathbb{C}^D \simeq \mathbb{R}^d$ with $d = 2D$, a disk centered at the origin with radius R on \mathbb{C}^D , $\{x \in \mathbb{C}^D : |x| < R\}$, is identified with $\mathbb{B}_R^{(2D)}$ in \mathbb{R}^{2D} .

Proposition 3.6 *For the Heisenberg family of DPPs, $(\Xi_{\mathbb{H}_D}, K_{\mathbb{H}_D}, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})$ on \mathbb{C}^D , $D \in \mathbb{N}$, the following hold,*

$$\mathbf{E}[\Xi_{\mathbb{H}_D}(\mathbb{B}_R^{(2D)})] = \frac{R^{2D}}{D!}, \quad (3.10)$$

$$\text{var}[\Xi_{\mathbb{H}_D}(\mathbb{B}_R^{(2D)})] = \frac{2R^{2D}}{(D-1)!} \int_0^\infty \frac{J_D(\kappa R)^2}{\kappa} (1 - e^{-\kappa^2/4}) d\kappa, \quad R > 0. \quad (3.11)$$

Proof Combining (3.2) with (1.3), and (3.8), (3.10) is proved. Since $C(x)$ is radial as given by (3.9) in Lemma 3.5, Lemma 2.4 determines its Fourier transform $\widehat{C}(k)$ as a radial function of $\kappa = |k|$ as

$$\widehat{c}(\kappa) = \frac{(2\pi)^D}{\kappa^{D-1}} \int_0^\infty r^D J_{D-1}(\kappa r) c(r) dr = -\frac{(2\pi)^D}{\kappa^{D-1}} \int_0^\infty r^D J_{D-1}(\kappa r) e^{-r^2} dr.$$

We use the integral formula (2.14) with $\nu = D-1$, $a = \kappa$, $p = 1$ and obtain $\widehat{c}(\kappa) = -\pi^D e^{-\kappa^2/4}$. In this sense, the assumption **(A3)** is satisfied for the present systems on $S = \mathbb{C}^D$, $D \in \mathbb{N}$. With (3.8) of Lemma 3.5, (2.9) gives

$$\widehat{s}(\kappa) = 1 + \frac{1}{\pi^D} (-\pi^D e^{-\kappa^2/4}) = 1 - e^{-\kappa^2/4}.$$

Then Corollary 3.4 proves (3.11). The proof is hence complete. ■

4 Proofs of Main Results

4.1 Proof of Proposition 1.2

For $n \in \mathbb{N}$, consider the integral

$$\begin{aligned} A_n(R) &:= \int_0^\infty \frac{J_n(\kappa R)^2}{\kappa} (1 - e^{-\kappa^2/4}) d\kappa \\ &= \int_0^\infty \frac{J_n(\kappa R)^2}{\kappa^{2n-1}} \kappa^{2(n-1)} (1 - e^{-\kappa^2/4}) d\kappa. \end{aligned} \quad (4.1)$$

By (2.12), we can perform the partial integration as

$$\begin{aligned} A_n(R) &= \left[-\frac{1}{2(2n-1)} [J_{n-1}(\kappa R)^2 + J_n(\kappa R)^2] (1 - e^{-\kappa^2/4}) \right]_0^\infty \\ &\quad + \frac{1}{2(2n-1)} \int_0^\infty \frac{J_{n-1}(\kappa R)^2 + J_n(\kappa R)^2}{\kappa^{2(n-1)}} \left[\frac{d}{d\kappa} \{ \kappa^{2(n-1)} (1 - e^{-\kappa^2/4}) \} \right] d\kappa. \end{aligned} \quad (4.2)$$

The asymptotic formula (2.17) implies $J_n(\kappa R)^2 + J_{n-1}(\kappa R)^2 \sim 2/(\pi\kappa R) \rightarrow 0$ as $\kappa R \rightarrow \infty$, and hence the first term in the RHS of (4.2) vanishes. Since

$$\frac{d}{d\kappa} \{ \kappa^{2(n-1)} (1 - e^{-\kappa^2/4}) \} = 2(n-1) \kappa^{2(n-1)-1} (1 - e^{-\kappa^2/4}) + \frac{1}{2} \kappa^{2(n-1)+1} e^{-\kappa^2/4},$$

(4.2) is written as

$$\begin{aligned} A_n(R) &= \frac{n-1}{2n-1} A_n(R) + \frac{n-1}{2n-1} A_{n-1}(R) \\ &\quad + \frac{1}{4(2n-1)} \int_0^\infty \kappa e^{-\kappa^2/4} [J_{n-1}(\kappa R)^2 + J_n(\kappa R)^2] d\kappa \\ &= \frac{n-1}{2n-1} A_n(R) + \frac{n-1}{2n-1} A_{n-1}(R) + \frac{1}{2(2n-1)} e^{-2R^2} [I_{n-1}(2R^2) + I_n(2R^2)], \end{aligned}$$

where (2.15) was used. This gives the following recurrence relation

$$nA_n(R) - (n-1)A_{n-1} = \frac{e^{-2R^2}}{2} [I_{n-1}(2R^2) + I_n(2R^2)], \quad n \in \mathbb{N}.$$

By taking the summation with respect to n from 1 to $D \in \mathbb{N}$, we have

$$\begin{aligned} DA_D(R) &= \frac{e^{-2R^2}}{2} \sum_{n=1}^D [I_{n-1}(2R^2) + I_n(2R^2)] \\ &= \frac{e^{-2R^2}}{2} \left[I_0(2R^2) + 2 \sum_{n=1}^{D-1} I_n(2R^2) + I_D(2R^2) \right]. \end{aligned}$$

With (3.11) this proves the assertion (1.7).

4.2 Proof of (1.9) in Remark 2

Consider (4.1) with $n = D \in \mathbb{N}$ and set

$$A_D(R) = A_D^{(1)}(R) - A_D^{(2)}(R)$$

with

$$A_D^{(1)}(R) = \int_0^\infty \frac{J_D(\kappa R)^2}{\kappa} d\kappa, \quad A_D^{(2)}(R) = \int_0^\infty \frac{J_D(\kappa R)^2}{\kappa} e^{-\kappa^2/4} d\kappa.$$

The integral formulas (2.13) and (2.16) give $A_D^{(1)}(R) = 1/(2D)$ and

$$A_D^{(2)}(R) = \frac{(2R)^{2D}}{2^{2D+1} D^2 \Gamma(D)} {}_2F_2(D, D + 1/2; D + 1, 2D + 1; -(2R)^2).$$

Putting the above results into (3.11), the formula (1.9) is obtained.

4.3 Proof of Theorem 1.3

Apply the asymptotic formula of the modified Bessel functions (2.18) with (1.12). Then (1.7) in Proposition 1.2 combined with (3.10) in Proposition 3.6 gives

$$\frac{\text{var}[\Xi_{\mathbb{H}_D}(\mathbb{B}_R^{(2D)})]}{\mathbf{E}[\Xi_{\mathbb{H}_D}(\mathbb{B}_R^{(2D)})]} \sim \frac{1}{2\sqrt{\pi}} R^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{\beta_k(D)}{k! 2^{4k}} R^{-2k},$$

where

$$\beta_k(D) := \alpha_k(0) + 2 \sum_{n=1}^{D-1} \alpha_k(n) + \alpha_k(D), \quad k \in \mathbb{N}_0.$$

Since $\alpha_0(n) \equiv 1$ by the definition (1.12), $\beta_0(D) = 1 + 2(D-1) + 1 = 2D$, and (1.10) is proved. For $k \in \mathbb{N}$, (1.12) gives

$$\begin{aligned} \alpha_k(n+1) &= \prod_{\ell=-k+1}^k \{2(n+1) + 2\ell - 1\} = \prod_{\ell'=-k+2}^{k+1} (2n + 2\ell' - 1) \\ &= \frac{2n + 2k + 1}{2n - 2k + 1} \alpha_k(n), \quad n \in \mathbb{N}_0. \end{aligned}$$

This equality is rewritten as

$$\alpha_k(n) + \alpha_k(n+1) = \frac{2}{2k+1} [(n+1)\alpha_k(n+1) - n\alpha_k(n)], \quad n \in \mathbb{N}_0.$$

If we take summation of the above from $n = 0$ to $n = D-1$, then we have

$$\sum_{n=0}^{D-1} \{\alpha_k(n) + \alpha_k(n+1)\} = \frac{2D}{2k+1} \alpha_k(D).$$

This implies $\beta_k(D) = \{2D\alpha_k(D)\}/(2k+1)$ and (1.11) is concluded. Then the proof is complete.

4.4 Proof of (1.14) in Remark 5

As shown in Appendix A, $\{\varphi_n\}_{n \in \mathbb{N}_0^D}$ with (A.8) gives a complete orthonormal system for the Bargmann–Fock space \mathcal{F}_D defined by (A.7). For $n = (n^{(1)}, \dots, n^{(D)})$, $m = (m^{(1)}, \dots, m^{(D)}) \in \mathbb{N}_0^D$ and $R > 0$, let

$$K_{\Delta_R^{(D)}}(n, m) := \int_{\Delta_R^{(D)}} \overline{\varphi_n(x)} \varphi_m(x) \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}(dx), \quad (4.3)$$

and consider the DPP $\Xi_{\Delta_R^{(D)}}$ on \mathbb{N}_0^D whose correlation kernel is given by (4.3). By the general theory of the *duality relations* between DPPs (Theorem 2.6 in [17]), the following equality holds,

$$\mathbf{P}(\Xi_{\mathbb{H}_D}(\Delta_R^{(D)}) = k) = \mathbf{P}(\Xi_{\Delta_R^{(D)}}(\mathbb{N}_0^D) = k) \quad \forall k \in \mathbb{N}_0. \quad (4.4)$$

By (A.8), we can show that [17, 22]

$$K_{\Delta_R^{(D)}}(n, m) = \delta_{nm} \prod_{\ell=1}^D p_{n^{(\ell)}}(R), \quad n, m \in \mathbb{N}_0^D,$$

where

$$p_k(R) := \int_0^{R^2} \frac{u^k e^{-u}}{k!} du = \sum_{j=k+1}^{\infty} \frac{R^{2j} e^{-R^2}}{j!}, \quad k \in \mathbb{N}_0.$$

Then the DPP $(\Xi_{\Delta_R^{(D)}}, K_{\Delta_R^{(D)}})$ on \mathbb{N}_0^D is the product measure $\bigotimes_{\ell=1}^D \bigotimes_{n^{(\ell)} \in \mathbb{N}_0} \mu_{p_{n^{(\ell)}}(R)}^{\text{Bernoulli}}$ under the natural identification between $\{0, 1\}^{\mathbb{N}_0^D}$ and the multivariate power set of \mathbb{N}_0^D , where $\mu_p^{\text{Bernoulli}}$ denotes the Bernoulli measure of probability $p \in [0, 1]$.

If we introduce a series of random variables $Y_{n^{(\ell)}}^{(R)} \in \{0, 1\}$, $n^{(\ell)} \in \mathbb{N}_0$, $\ell = 1, \dots, D$, which are mutually independent and $Y_{n^{(\ell)}}^{(R)} \sim \mu_{p_{n^{(\ell)}}(R)}^{\text{Bernoulli}}$. then the duality relation (4.4) implies the equivalences in distribution,

$$\Xi_{\mathbb{H}_D}(\Delta_R^{(D)}) \stackrel{d}{=} \Xi_{\Delta_R^{(D)}}(\mathbb{N}_0^D) \stackrel{d}{=} \sum_{n \in \mathbb{N}_0^D} \prod_{\ell=1}^D Y_{n^{(\ell)}}^{(R)}.$$

Then we have

$$\begin{aligned} \mathbf{E}[\Xi_{\mathbb{H}_D}(\Delta_R^{(D)})] &= \sum_{n \in \mathbb{N}_0^D} \prod_{\ell=1}^D p_{n^{(\ell)}}(R) = \left(\sum_{k=0}^{\infty} p_k(R) \right)^D, \\ \text{var}[\Xi_{\mathbb{H}_D}(\Delta_R^{(D)})] &= \text{var} \left[\sum_{n \in \mathbb{N}_0^D} \prod_{\ell=1}^D Y_{n^{(\ell)}}^{(R)} \right] = \sum_{n \in \mathbb{N}_0^D} \text{var} \left[\prod_{\ell=1}^D Y_{n^{(\ell)}}^{(R)} \right] \\ &= \sum_{n \in \mathbb{N}_0^D} \left[\prod_{\ell=1}^D p_{n^{(\ell)}}(R) - \left(\prod_{\ell=1}^D p_{n^{(\ell)}}(R) \right)^2 \right] = \left(\sum_{k=0}^{\infty} p_k(R) \right)^D - \left(\sum_{k=0}^{\infty} p_k(R)^2 \right)^D, \end{aligned}$$

and hence

$$\frac{\text{var}[\Xi_{\mathbf{H}_D}(\Delta_R^{(D)})]}{\mathbf{E}[\Xi_{\mathbf{H}_D}(\Delta_R^{(D)})]} = 1 - \left(\frac{\sum_{k=0}^{\infty} p_k(R)^2}{\sum_{k=0}^{\infty} p_k(R)} \right)^D.$$

When $D = 1$, the above gives

$$\begin{aligned} \frac{\text{var}[\Xi_{\mathbf{H}_1}(\Delta_R^{(1)})]}{\mathbf{E}[\Xi_{\mathbf{H}_1}(\Delta_R^{(1)})]} &= 1 - \frac{\sum_{k=0}^{\infty} p_k(R)^2}{\sum_{k=0}^{\infty} p_k(R)} \\ &= \frac{\text{var}[\Xi_{\mathbf{H}_1}(\mathbb{B}_R^{(2)})]}{\mathbf{E}[\Xi_{\mathbf{H}_1}(\mathbb{B}_R^{(2)})]}, \end{aligned}$$

where we have used the fact that $\Delta_R^{(1)} = \mathbb{D}_R \subset \mathbb{C}$ is identified with $\mathbb{B}_R^{(2)} \subset \mathbb{R}^2$. Hence the first equality of (1.14) is proved. The second equality of (1.14) is derived by the asymptotic expansion (1.11) for $D = 1$ in Theorem 1.3.

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A Representations of the Heisenberg Group

Following [7, 29, 12], we briefly review the representation theory of the *Heisenberg group* in order to explain the reason why we call the DPPs defined by Definition 1.1 the *Heisenberg family* of DPPs.

Consider the classical and quantum kinetics of a single particle moving in \mathbb{R}^D , $D \in \mathbb{N}$. We note that, if $D = 3k$, $k \in \mathbb{N}$, this represents a k -particle system in the three dimensional Euclidean space. The *phase space* is given by \mathbb{R}^{2D} with coordinates $(p, q) = (p_1, \dots, p_D, q_1, \dots, q_D)$. In order to describe the *Heisenberg Lie algebra* \mathfrak{h}_D , we consider \mathbb{R}^{2D+1} with coordinates $(p, q, \tau) = (p_1, \dots, p_D, q_1, \dots, q_D, \tau)$, in which a Lie bracket is given by

$$[(p, q, \tau), (p', q', \tau')] = (0, 0, p \cdot q' - q \cdot p') = (0, 0, [(p, q), (p', q')]).$$

The symplectic form of the Lie bracket $[(p, q), (p', q')] = p \cdot q' - q \cdot p'$ comes from the Poisson bracket in the classical mechanics and the commutator $[A, B] := AB - BA$ in quantum mechanics. The *Heisenberg group* \mathbf{H}_D is the Lie group on \mathbb{R}^{2D+1} satisfying the group law $ZZ' = Z + Z' + \frac{1}{2}[Z, Z']$, $Z, Z' \in \mathbb{R}^{2D+1}$; that is,

$$(p, q, \tau)(p', q', \tau') = \left(p + p', q + q', \tau + \tau' + \frac{1}{2}(p \cdot q' - q \cdot p') \right).$$

Let $L^2(\mathbb{R}^D)$ be the set of square integrable functions on \mathbb{R}^D , where the inner product is given by

$$\langle f, g \rangle_{L^2(\mathbb{R}^D)} := \int_{\mathbb{R}^D} f(\zeta) \overline{g(\zeta)} d\zeta, \quad f, g \in L^2(\mathbb{R}^D)$$

with the norm $\|f\|_{L^2(\mathbb{R}^D)} := \sqrt{\langle f, f \rangle_{L^2(\mathbb{R}^D)}}$, $f \in L^2(\mathbb{R}^D)$, where $\zeta = (\zeta^{(1)}, \dots, \zeta^{(D)}) \in \mathbb{R}^D$ and $d\zeta$ denotes the Lebesgue measure on \mathbb{R}^D . For a smooth function f , we introduce operators $X^{(\ell)}$ and $\mathcal{D}^{(\ell)}$ defined by

$$(X^{(\ell)} f)(\zeta) = \zeta^{(\ell)} f(\zeta), \quad (\mathcal{D}^{(\ell)} f)(\zeta) = \frac{1}{2\sqrt{-1}} \frac{\partial f}{\partial \zeta^{(\ell)}}(\zeta), \quad \ell = 1, \dots, D.$$

They satisfy the commutation relations

$$[X^{(\ell)}, \mathcal{D}^{(\ell')}] = \frac{\sqrt{-1}}{2} \delta_{\ell\ell'}, \quad \ell, \ell' = 1, \dots, D.$$

Note that the above will represent the *canonical commutation relations* in quantum mechanics, $[Q^{(\ell)}, P^{(\ell')}] = \sqrt{-1} \hbar \delta_{\ell\ell'}$. Here we should claim that the value of the Planck constant \hbar is specially chosen to be $1/2$. (This choice enables us to have the equality (A.5) below with the complex standard normal distribution $\lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}$ on \mathbb{C}^D defined by (1.4).) We consider a map from \mathbf{H}_D to the group of unitary operators acting on $L^2(\mathbb{R}^D)$ defined by

$$\rho(p, q, \tau) = e^{2\sqrt{-1}(p \cdot \mathcal{D} + qX + \tau I)},$$

where $\mathcal{D} := (\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(D)})$, $X := (X^{(1)}, \dots, X^{(D)})$ and I denotes the identity operator. We can show that

$$\rho(p, q, \tau) f(\zeta) = e^{2\sqrt{-1}(\tau + q \cdot \zeta + p \cdot q/2)} f(\zeta + p), \quad f \in L^2(\mathbb{R}^D). \quad (\text{A.1})$$

The map ρ is called the *Schrödinger representation* of \mathbf{H}_D . The kernel of ρ is $\{(0, 0, k\pi) : k \in \mathbb{Z}\}$, since $e^{2\pi k \sqrt{-1}} = 1, k \in \mathbb{Z}$. The *reduced Heisenberg group* $\mathbf{H}_D^{\text{red}}$ is defined by $\mathbf{H}_D^{\text{red}} := \mathbf{H}_D / \{(0, 0, k\pi) : k \in \mathbb{Z}\}$.

We calculate the matrix coefficients of $\rho(p, q, \tau)$ at $(f, g) \in (L^2(\mathbb{R}^D))^2$ and obtain the expression,

$$\begin{aligned} M_{f,g}(p, q, \tau) &:= \langle \rho(p, q, \tau) f, g \rangle_{L^2(\mathbb{R}^D)} \\ &= e^{2\sqrt{-1}\tau} \int_{\mathbb{R}^D} e^{2\sqrt{-1}q \cdot \zeta} f\left(\zeta + \frac{p}{2}\right) \overline{g\left(\zeta - \frac{p}{2}\right)} d\zeta, \quad f, g \in L^2(\mathbb{R}^D), \end{aligned} \quad (\text{A.2})$$

which is called the *Fourier–Wigner transform* [7]. For $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^D)$, the inner product of M_{f_1, g_1} and M_{f_2, g_2} in $L^2(\mathbb{R}^{2D})$ is calculated and the following equality is obtained,

$$\begin{aligned} \langle M_{f_1, g_1}, M_{f_2, g_2} \rangle_{L^2(\mathbb{R}^{2D})} &:= \int_{\mathbb{R}^D} dp \int_{\mathbb{R}^D} dq M_{f_1, g_1}(p, q, \tau) \overline{M_{f_2, g_2}(p, q, \tau)} \\ &= \pi^D \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^D)} \overline{\langle g_1, g_2 \rangle_{L^2(\mathbb{R}^D)}}. \end{aligned} \quad (\text{A.3})$$

If we put $g_1 = g_2 = G$ with (1.13) and define the complex variables

$$x = (x^{(1)}, \dots, x^{(D)}) := p + \sqrt{-1}q = (p^{(1)} + \sqrt{-1}q^{(1)}, \dots, p^{(D)} + \sqrt{-1}q^{(D)}) \in \mathbb{C}^D, \quad (\text{A.4})$$

then (A.2) and (A.3) become

$$M_{f,G}(p, q, \tau) = e^{2\sqrt{-1}\tau} \mathbf{B}[f](x) \frac{e^{-|x|^2/2}}{\pi^{D/2}},$$

and

$$\begin{aligned} \langle M_{f_1,G}, M_{f_2,G} \rangle_{L^2(\mathbb{R}^{2D})} &= \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^D)} \\ &= \langle \mathbf{B}[f_1], \mathbf{B}[f_2] \rangle_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})}, \quad f_1, f_2 \in L^2(\mathbb{R}^D), \end{aligned} \quad (\text{A.5})$$

with

$$\mathbf{B}[f](x) := \left(\frac{2}{\pi}\right)^{D/4} \int_{\mathbb{R}^D} f(\zeta) e^{2\zeta \cdot x - \zeta^2 - x^2/2} d\zeta, \quad f \in L^2(\mathbb{R}^D), \quad (\text{A.6})$$

which is called the *Bargmann transform*. Here the measure $\lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}$ on \mathbb{C}^D is defined by (1.4),

$$\langle F_1, F_2 \rangle_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})} := \int_{\mathbb{C}^D} F_1(x) \overline{F_2(x)} \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)}(dx),$$

and $\|F\|_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})} := \sqrt{\langle F, F \rangle_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})}}$. For $f \in L^2(\mathbb{R}^D)$, the integral of (A.6) converges uniformly for x in any compact subset of \mathbb{C}^D , and hence $\mathbf{B}[f]$ is an entire function on \mathbb{C}^D . The *Bargmann–Fock space* \mathcal{F}_D is defined by

$$\mathcal{F}_D := \left\{ F : F \text{ is entire on } \mathbb{C}^D \text{ and } \|F\|_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})} < \infty \right\}. \quad (\text{A.7})$$

Then (A.5) implies that the Bargmann transform is an isometry from $L^2(\mathbb{R}^D)$ into \mathcal{F}_D . The Schrödinger representation $\rho(p, q, \tau)$ of \mathbf{H}_D on $L^2(\mathbb{R}^D)$ can be transferred by the Bargmann transform to a representation β of \mathbf{H}_D on \mathcal{F}_D . The *Bargmann–Fock representation* β of \mathbf{H}_D is defined by

$$\beta(x, \tau) \mathbf{B} = \mathbf{B} \rho(p, q, \tau)$$

with (A.4). We can verify that (A.1) is mapped to

$$\beta(y, \tau) F(x) = e^{2\sqrt{-1}\tau - |y|^2/2 - x \cdot \bar{y}} F(x + y), \quad y \in \mathbb{C}^D, \quad F \in \mathcal{F}_D.$$

For $n = (n^{(1)}, \dots, n^{(D)}) \in \mathbb{N}_0^D$ and $x = (x^{(1)}, \dots, x^{(D)}) \in \mathbb{C}^D$, we use the notations, $n! := \prod_{\ell=1}^D n^{(\ell)}!$ and $x^n := \prod_{\ell=1}^D (x^{(\ell)})^{n^{(\ell)}}$. Then a complete orthonormal system (CONS) for \mathcal{F}_D is given by

$$\varphi_n(x) := \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0^D, \quad x \in \mathbb{C}^D, \quad (\text{A.8})$$

that is, $\langle \varphi_n, \varphi_m \rangle_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})} = \delta_{nm} := \prod_{\ell=1}^N \delta_{n^{(\ell)}m^{(\ell)}}$, $n, m \in \mathbb{N}_0^D$. Hence, if we define

$$\begin{aligned} k_y(x) &:= \sum_{n \in \mathbb{N}_0^D} \varphi_n(x) \overline{\varphi_n(y)} = \prod_{\ell=1}^D \sum_{n^{(\ell)} \in \mathbb{N}_0} \frac{(x^{(\ell)})^{n^{(\ell)}} (\overline{y^{(\ell)}})^{n^{(\ell)}}}{n^{(\ell)}!} \\ &= \prod_{\ell=1}^D e^{x^{(\ell)} \overline{y^{(\ell)}}} = e^{x \cdot \overline{y}}, \end{aligned} \quad (\text{A.9})$$

then it works as the *reproducing kernel* of \mathcal{F}_D ; $F(y) = \langle F, k_y \rangle_{L^2(\mathbb{C}^D, \lambda_{\mathbb{N}(0,1;\mathbb{C}^D)})} \forall F \in \mathcal{F}_D$, $\forall y \in \mathbb{C}^D$. This is identified with the *correlation kernel* $K_{\mathbb{H}_D}$ of the Heisenberg DPP (1.5) given in Definition 1.1.

A geometric picture of \mathbb{H}_D is given in Chapter XII in [29] as follows. Consider the unit ball in \mathbb{C}^{D+1} ,

$$\mathbb{B}_1^{(2(D+1))} := \left\{ w = (w^{(1)}, \dots, w^{(D+1)}) \in \mathbb{C}^{D+1} : \sum_{\ell=1}^{D+1} |w^{(\ell)}|^2 < 1 \right\}.$$

By the correspondence

$$z^{(\ell)} = \frac{2w^{(\ell)}}{1 + w^{(D+1)}}, \quad \ell = 1, \dots, D, \quad z^{(D+1)} = \sqrt{-1} \frac{1 - w^{(D+1)}}{1 + w^{(D+1)}},$$

$\mathbb{B}_1^{(2(D+1))}$ is mapped to an ‘upper half-space’ of \mathbb{C}^{D+1} ,

$$\mathcal{U}^D := \left\{ z = (z^{(1)}, \dots, z^{(D+1)}) \in \mathbb{C}^{D+1} : \text{Im } z^{(D+1)} > \frac{1}{4} \sum_{\ell=1}^D |z^{(\ell)}|^2 \right\}.$$

Note that the relations between $\mathbb{B}_1^{(2(D+1))}$ and \mathcal{U}^D , $D \in \mathbb{N}$ can be regarded as the higher dimensional extensions of the relation between the unit disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\} \subset \mathbb{C}$ and the upper half-plane $\mathbb{H} := \{z : \text{Im } z > 0\}$ via the Cayley transform, $z = \sqrt{-1}(1-w)/(1+w)$. Let $\mathcal{b}\mathcal{U}^D$ be the boundary of \mathcal{U}^D ; $\mathcal{b}\mathcal{U}^D := \{z \in \mathbb{C}^{D+1} : \text{Im } z^{(D+1)} = \frac{1}{4} \sum_{\ell=1}^D |z^{(\ell)}|^2\}$. We can identify \mathbb{H}_D with $\mathcal{b}\mathcal{U}^D$ by the correspondence,

$$\mathbb{H}_D \ni (x, \tau) \quad \longleftrightarrow \quad \left(x, \tau + \frac{\sqrt{-1}}{4} |x|^2 \right) \in \mathcal{b}\mathcal{U}^D$$

with (A.4). In this sense, the reproducing kernel of the Bargmann–Fock space \mathcal{F}_D will be interpreted as the Szegő kernel associated with an integral on the boundary $\mathcal{b}\mathcal{U}^D$ of \mathcal{U}^D [35, 4].

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