HOMOGENIZATION OF THE LANDAU-LIFSHITZ EQUATION

LENA LEITENMAIER * AND OLOF RUNBORG †

Abstract. In this paper, we consider homogenization of the Landau-Lifshitz equation with a highly oscillatory material coefficient with period ε modeling a ferromagnetic composite. We derive equations for the homogenized solution to the problem and the corresponding correctors and obtain estimates for the difference between the exact and homogenized solution as well as corrected approximations to the solution. Convergence rates in ε over times $O(\varepsilon^\sigma)$ with $0 \le \sigma \le 2$ are given in the Sobolev norm H^q, where q is limited by the regularity of the solution to the detailed Landau-Lifshitz equation and the homogenized equation. The rates depend on q, σ and the the number of correctors.

Keywords. Homogenization; Micromagnetics; Magnetization Dynamics; Multiscale

AMS subject classifications. 35B27; 65M15; 82D40

1. Introduction

The governing equation in micromagnetics is the Landau-Lifshitz equation [2, 9, 18, 23],

$$\partial_t \mathbf{m}^\varepsilon = -\mathbf{m}^\varepsilon \times \mathbf{H}(\mathbf{m}^\varepsilon) - \alpha \mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon \times \mathbf{H}(\mathbf{m}^\varepsilon), \tag{1.1}$$

where $\mathbf{m}^\varepsilon$ is the magnetization vector, $\mathbf{H}(\mathbf{m}^\varepsilon)$ the so-called effective field and α a positive damping constant. The first term on the right-hand side here is a precession term, while the second one is damping, with the damping parameter α determining the strength of the effect. The Landau-Lifshitz equation is important for describing magnetic materials and processes in applications like recording devices, discrete storage media, and magnetic sensors.

In this paper we consider a simplified version of the Landau-Lifshitz equation, where we assume that $\mathbf{H}(\mathbf{m}^\varepsilon)$ only consists of the exchange interaction contribution, which in many cases is the term dominating the effective field,

$$\mathbf{H}(\mathbf{m}^\varepsilon) = \boldsymbol{\nabla} \cdot (a^\varepsilon(x) \boldsymbol{\nabla} \mathbf{m}^\varepsilon).$$

We assume that $a^\varepsilon(x) = a(x/\varepsilon)$ is a smooth, periodic, highly oscillatory material coefficient. This could for instance be seen as a simple model for a magnetic multilayer [20], a ferromagnetic composite, consisting of thin layers of two different materials with different interaction behavior, with a^ε indicating the current material. The size of two of the layers then corresponds to ε. The most straight forward coefficient describing such a setup would have rather low regularity. However, to make the problem more suitable for mathematical treatment we suppose that $a \in C^\infty$.

Numerical simulations of the Landau-Lifschitz equation are of considerable interest in applications, [17, 25]. For the case when the material changes rapidly, as above with $\varepsilon \ll 1$, the computational cost of simulations becomes very high, since the ε-scales must be well resolved by the numerical approximation. For such problems, multiscale methods like the heterogeneous multiscale methods (HMM) [15] and equation free methods [21] become more efficient. These are inspired by homogenization theory [8, 13]. In the framework of HMM, one combines the approximation of a coarse macroscale model, similar to a homogenized equation, with simulations of the original detailed equation

* Department of Mathematics, KTH, Royal Institute of Technology, Lindstedsvägen 25, 100 44 Stockholm, Sweden (lenalei@kth.se).

† Department of Mathematics, KTH, Royal Institute of Technology (olofr@kth.se).

(1.1). The simulations of (1.1) are, however, restricted to small boxes in space and short time intervals, which reduces the computational cost. The motivation behind our choice of focus here is to do error analysis of HMM methods for magnetization dynamics. Such analysis relies on homogenization theory, and the behavior of solutions to (1.1) over short times, as $\varepsilon \to 0$. See [6, 7] for examples of HMM methods in the context of magnetization dynamics.

There are several articles dealing with the homogenization of (1.1) and related problems. In particular, a similar problem was considered in [19] and recently in [3], where the authors use two-scale convergence techniques to analyze (1.1) with a stochastic material coefficient a^ε, which can be seen as a model for so-called spring magnets, a special type of ferromagnetic composites. The corresponding stationary problem was studied in [4]. Furthermore, in [12], a high contrast composite medium is considered using two-scale convergence. In [26] homogenization for ferromagnetic multilayers in the presence of surface energies is studied, using a material coefficient to describe the magnetic field associated with the exchange energy. In all of these papers, the authors show convergence for weak solutions and do not focus on convergence rates in ε, which is of prime importance for HMM error analysis. In contrast, our goal is to study how classical solutions to (1.1) can be approximated by the homogenized solution and associated correction terms. We note that while existence of weak solutions to (1.1) is shown in [5], existence of classical solutions is only known for short times and/or for small initial data gradients, see for example [10, 11, 14, 16, 24]. In particular, in [10, 11], the authors prove local existence and global existence given that the gradient of the initial data is sufficiently small. In [16], existence of arbitrarily regular solutions with respect to space and time up to an arbitrary final time is shown on bounded 3D domains, assuming that the initial data is small enough and has high enough regularity. Although these works do not consider exactly the same Landau-Lifshitz problem as us — they do not include a varying material coefficient $a^\varepsilon(x)$ and use slightly different norms — we will in this paper assume existence of regular solutions to (1.1) and the corresponding homogenized equation and focus on convergence rates. Moreover, in Appendix B.1 and Appendix B.2 we generalize the energy estimates in [24] to the problem considered here.

In the main result of this paper we analyze the difference between the solution $\mathbf{m}^\varepsilon$ of (1.1) and the homogenized solution with arbitrary many correction terms. We provide rates for the convergence in terms of ε in Sobolev norms for dimensions $n = 1,2,3$. The rates that we obtain depend on the length of the time interval considered, and are centered on short times of length $O(\varepsilon^\sigma)$ with $0 \le \sigma < 2$. These short times are of main relevance for HMM analysis. Note that the temporal oscillation period in $\mathbf{m}^\varepsilon$ is of order ε^2 meaning that the times considered are still relatively long, and include an infinite number of oscillations in time as $\varepsilon \to 0$. The approach we use to achieve this is based on asymptotic multiscale expansions, together with careful estimates of the corrector terms, inspired by [1], which used a similar strategy to derive estimates for the wave equation over long time. Unlike that paper and the ones mentioned earlier, we include a fast time variable $\tau = t/\varepsilon^2$ in the multiscale expansions to capture the precise behavior of the initial transient of the solution. Our main assumption, besides existence and regularity of all solutions, is an L^∞ bound on $\boldsymbol{\nabla}\mathbf{m}^\varepsilon$, uniformly in ε. We note that such a uniform bound is easy to check in L^2, and that it also true in L^∞ for the homogenized solution with correction terms.

This paper is organized as follows: in the next section we introduce the notation used in this paper as well as some useful identities. Section 3 contains the main result of the paper and outlines the steps that are required to obtain it. In Section 4, we motivate our choice of homogenized equation corresponding to (1.1) as well as the form of the related correctors. We obtain linear partial differential equations describing the evolution of these correctors. In Section 5, we then show several properties of Bochner-Sobolev norms that simplify dealing with the multiscale character of the problem. Section 6 is devoted to a stability estimate for the error introduced when approximating the solution $\mathbf{m}^\varepsilon$ to (1.1) by the solution to a perturbed version of the original problem. We then derive specific bounds for the correctors and the corresponding approximation to $\mathbf{m}^\varepsilon$ in Section 7.

2. Preliminaries

Throughout this paper, the problems are set on a domain $\Omega \subset \mathbb{R}^n$, with $n=1,2,3$ and periodic boundary conditions. Moreover, for the fast variations we define also Y as the n-dimensional unit cell, $Y=[0,1]^n$.

In this section, we introduce notation for working with vector functions $\mathbf{v}(x,t)$: $\Omega \times \mathbb{R} \mapsto \mathbb{R}^3$ and their gradients. We moreover introduce suitable norms and useful identities for working with multiscale problems and matrix valued functions.

2.1. Basic notation and differential operators. Let $\mathbf{m}: \Omega \times \mathbb{R} \mapsto \mathbb{S}^2 \subset \mathbb{R}^3$ denote the magnetization vector, which is a function of time t and space $x \in \mathbb{R}^n$. The components of $\mathbf{m}$ will be called $m^{(j)}$, hence $\mathbf{m}=[m^{(1)},m^{(2)},m^{(3)}]^T$. In accordance with standard notation in the area we denote its Jacobian matrix by $\boldsymbol{\nabla}\mathbf{m}$. We consider this as an element in $\mathbb{R}^{3\times n}$, such that

$$\boldsymbol{\nabla}\mathbf{m} := \begin{bmatrix} \left(\nabla m^{(1)}\right)^T \\ \left(\nabla m^{(2)}\right)^T \\ \left(\nabla m^{(3)}\right)^T \end{bmatrix}.$$

Suppose that $\mathbf{A}: \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ gives a symmetric positive definite matrix, uniformly in x. Then we define L for a function $u: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and the corresponding vector-operator $\mathcal{L}$ according to

$$Lu := \nabla\cdot(\mathbf{A}(x)\nabla u), \qquad \mathcal{L}\mathbf{m} := \left[Lm^{(1)}\ Lm^{(2)}\ Lm^{(3)}\right]^T.$$

In general, all linear operators returning scalars are to be applied element-wise to vector-valued functions if not explicitly stated otherwise. As a convention, the cross product and scalar product between a vector-valued function $\mathbf{v}\in\mathbb{R}^3$ and $\mathbf{B}$ are done column-wise, and the divergence-operator is applied row-wise.

Moreover, consider $\mathbf{B},\mathbf{C}\in\mathbb{R}^{3\times n}$ with elements b_{ij} and c_{ij} where i and j denote row and column, then we define

$$\mathbf{B}:\mathbf{C} := \sum_{i=1}^{3}\sum_{j=1}^{n} b_{ij}c_{ij}.$$

Finally, note that the operator $\mathcal{L}\mathbf{m}$ could also be defined as $\mathcal{L}\mathbf{m} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{m}\mathbf{A})$ using the notation introduced above. This is equivalent to the component-wise definition.

2.2. Function spaces and norms. In the following, we denote by $C(I)$ the space of continuous functions on an interval I and by $C^\infty(\Omega)$ the space of smooth functions on Ω. By $H^q(\Omega)$ we denote the standard periodic Sobolev spaces on Ω, with norm $\|\cdot\|_{H^q}$,

$$\|v\|_{H^q}^2 = \sum_{|\beta|\le q} \int_\Omega |\partial_x^\beta v(x)|^2 dx.$$

Moreover, by $H^{q,p}(\Omega;Y)$ we denote the periodic Bochner–Sobolev spaces on $\Omega\times Y$, with norm $\|\cdot\|_{H^{q,p}}$, defined as

$$\|v\|_{H^{q,p}}^2 = \sum_{|\beta|\le q} \int_\Omega ||\partial_x^\beta v(x,\cdot)||_{H^p(Y)}^2 dx = \sum_{|\beta|\le q, |\gamma|\le p} \int_\Omega \int_Y |\partial_x^\beta \partial_y^\gamma v(x,y)|^2 dy dx$$

and we define the multiscale-norm

$$\|v\|_{H_\varepsilon^q} := \sum_{j=0}^q \varepsilon^j \|v\|_{H^j},$$

where we assume $0<\varepsilon\le 1$. All previous norm definitions are analogous for vector valued functions. Furthermore, let $|\cdot|$ denote the norm on $\mathbb{R}^{3\times n}$, for a matrix-valued function $\mathbf{B}\in\mathbb{R}^{3\times n}$,

$$|\mathbf{B}|^2 := \mathbf{B}:\mathbf{B},$$

and the corresponding L^2-norm on $\Omega\mapsto\mathbb{R}^{3\times n}$ is given by

$$\|\mathbf{B}\|_{L^2}^2 = \int_\Omega |\mathbf{B}|^2 dx = \int_\Omega \mathbf{B}:\mathbf{B} dx.$$

2.3. Useful identities. In the following, we will make frequent use of the standard triple product identities, stating that it holds for three vector-valued functions $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ that

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}), \tag{2.1}$$

$$\mathbf{u}\times\mathbf{v}\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}. \tag{2.2}$$

From (2.1) it follows directly that

$$\boldsymbol{\nabla}\mathbf{u}:(\mathbf{v}\times\boldsymbol{\nabla}\mathbf{u}) = \sum_{j=1}^n \partial_{x_j}\mathbf{u}\cdot(\mathbf{v}\times\partial_{x_j}\mathbf{u}) = \sum_{j=1}^n \mathbf{v}\cdot(\partial_{x_j}\mathbf{u}\times\partial_{x_j}\mathbf{u}) = \mathbf{0}. \tag{2.3}$$

In accordance to integration by parts for scalar functions, it holds given a periodic matrix-valued function $\mathbf{B}\in H^1(\Omega;\mathbb{R}^{3\times n})$, a periodic vector-valued $\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)$ and a periodic real valued function $f\in H^1(\Omega;\mathbb{R})$ that

$$\int_\Omega \mathbf{v}\cdot(\boldsymbol{\nabla}\cdot\mathbf{B})dx = -\int_\Omega \mathbf{B}:\boldsymbol{\nabla}\mathbf{v}dx, \qquad \text{and} \qquad \int_\Omega f\boldsymbol{\nabla}\cdot\mathbf{B}dx = -\int_\Omega \mathbf{B}\boldsymbol{\nabla}f dx.$$

Moreover, integration by parts implies that

$$\int_\Omega \mathbf{v}\times\mathcal{L}\mathbf{v}dx = 0 \qquad \text{and} \qquad \int_\Omega \mathbf{u}\cdot\mathcal{L}\mathbf{v}dx = -\int_\Omega a\boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{v}dx. \tag{2.4}$$

3. Main results

Assume that $\mathbf{m}^\varepsilon$ is a classical solution to the Landau-Lifshitz equation on a domain $\Omega \subset \mathbb{R}^n$, $n=1,2,3$ with periodic boundary conditions,

$$\partial_t \mathbf{m}^\varepsilon(x,t) = -\mathbf{m}^\varepsilon(x,t) \times \mathcal{L}\mathbf{m}^\varepsilon(x,t) - \alpha \mathbf{m}^\varepsilon(x,t) \times \mathbf{m}^\varepsilon(x,t) \times \mathcal{L}\mathbf{m}^\varepsilon(x,t), \tag{3.1a}$$

$$\mathbf{m}^\varepsilon(x,0) = \mathbf{m}_{\text{init}}(x), \tag{3.1b}$$

where $\mathcal{L}\mathbf{m}^\varepsilon := \boldsymbol{\nabla} \cdot (a^\varepsilon \boldsymbol{\nabla} \mathbf{m})$ and $a^\varepsilon(x) := a(x/\varepsilon)$ is a highly oscillatory, scalar material coefficient. Moreover, let $\mathbf{m}_0$ satisfy the homogenized equation corresponding to (3.1) on Ω, which is derived in Section 4,

$$\partial_t \mathbf{m}_0(x,t) = -\mathbf{m}_0(x,t) \times \bar{\mathcal{L}}\mathbf{m}_0(x,t) - \alpha \mathbf{m}_0(x,t) \times \mathbf{m}_0(x,t) \times \bar{\mathcal{L}}\mathbf{m}_0(x,t), \tag{3.2a}$$

$$\mathbf{m}_0(x,0) = \mathbf{m}_{\text{init}}(x), \tag{3.2b}$$

where $\bar{\mathcal{L}}\mathbf{m}_0 := \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \mathbf{m}_0 \mathbf{A}^H)$ and $\mathbf{A}^H \in \mathbb{R}^{n\times n}$ is the constant homogenized coefficient matrix. Let furthermore $\tilde{\mathbf{m}}_J^\varepsilon$ be a corrected approximation to $\mathbf{m}^\varepsilon$, defined as

$$\tilde{\mathbf{m}}_J^\varepsilon(x,t) = \mathbf{m}_0(x,t) + \sum_{j=1}^{J} \varepsilon^j \mathbf{m}_j(x, x/\varepsilon, t, t/\varepsilon^2), \tag{3.3}$$

where $\mathbf{m}_j$ are higher order correctors obtained by solving linear equations as given in (4.7). Our main goal in this paper then is to investigate the difference in terms of ε between $\mathbf{m}^\varepsilon$ and $\mathbf{m}_0$ as well as between $\mathbf{m}^\varepsilon$ and $\tilde{\mathbf{m}}_J^\varepsilon$. We assume that the homogenized solution $\mathbf{m}_0$ exists up to time T. For $\mathbf{m}^\varepsilon$ and the error estimates we mainly consider shorter time intervals $t \in [0, T^\varepsilon]$, where

$$T^\varepsilon := \varepsilon^\sigma T, \qquad 0 \le \sigma \le 2. \tag{3.4}$$

We make the following precise assumptions.

(A1) The material coefficient function $a(x)$ is in $C^\infty(\Omega)$ and such that $a_{\min} \le a(x) \le a_{\max}$ for constants $a_{\min}, a_{\max} > 0$.

(A2) The initial data $\mathbf{m}_{\text{init}}(x)$ satisfies $|\mathbf{m}_{\text{init}}(x)| \equiv 1$, constant in space. Note that the Landau-Lifshiz equation is norm preserving,

$$\tfrac{1}{2}\partial_t |\mathbf{m}|^2 = \mathbf{m} \cdot \partial_t \mathbf{m} = \mathbf{m} \cdot (\mathbf{m} \times \mathcal{L}\mathbf{m} - \alpha \mathbf{m} \times \mathbf{m} \times \mathcal{L}\mathbf{m}) = 0. \tag{3.5}$$

Hence, this assumption implies that $|\mathbf{m}^\varepsilon(x,t)| \equiv 1$ and $|\mathbf{m}_0(x,t)| \equiv 1$ for all time.

(A3) The damping coefficient α and the oscillation period ε are small, $0 < \alpha \le 1$ and $0 < \varepsilon < 1$.

(A4) The solution $\mathbf{m}^\varepsilon$ is such that

$$\mathbf{m}^\varepsilon \in C^1([0,T^\varepsilon]; H^{s+1}(\Omega)), \qquad \text{for some } s \ge 1,$$

and there is a constant M independent of ε such that

$$\|\boldsymbol{\nabla} \mathbf{m}^\varepsilon(\cdot,t)\|_{L^\infty} \le M, \qquad 0 \le t \le T^\varepsilon.$$

(A5) The homogenized solution $\mathbf{m}_0$ is such that, for some $r \ge 5$,

$$\partial_t^k \mathbf{m}_0 \in C([0,T]; H^{r-2k}(\Omega)), \qquad 0 \le 2k \le r, \tag{3.6}$$

which implies that

$$\|\boldsymbol{\nabla} \mathbf{m}_0(\cdot,t)\|_{L^\infty} \le C, \qquad 0 \le t \le T.$$

We then obtain the following result.

THEOREM 3.1. *Assume that* $\mathbf{m}^\varepsilon$ *is a classical solution to* (3.1), $\mathbf{m}_0$ *is a classical solution to* (3.2) *and that the assumptions (A1)-(A5) are satisfied. Let* $\tilde{\mathbf{m}}_J^\varepsilon$ *be the corrected approximation to* $\mathbf{m}^\varepsilon$ *as given by* (3.3) *and consider the final time* T^ε *in* (3.4) *with* σ *satsifying*

$$\begin{cases} 0 \le \sigma \le 2, & J \le 2, \\ 1 - \frac{1}{J-2} \le \sigma \le 2, & J \ge 3. \end{cases} \tag{3.7}$$

Moreover, let $q_J = \min(s, r-3-\max(2,J))$. *Then we have results for three different cases:*

- *Fixed time,* $\sigma = 0$*:*

$$\|\mathbf{m}^\varepsilon(\cdot,t) - \tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{L^2} \le C\varepsilon, \qquad \|\mathbf{m}^\varepsilon(\cdot,t) - \tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^1} \le C, \tag{3.8}$$

for $0 \le t \le T$ *and* $0 \le J \le 2$, *provided* $r \ge 6$ *for the* H^1 *case.*

- *Short time,* $0 < \sigma \le 1$*:*

$$\|\mathbf{m}^\varepsilon(\cdot,t) - \tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^q} \le C \begin{cases} \varepsilon^{1+\sigma/2-q}, & J = 1, \\ \varepsilon^{2-(1-\sigma)(J-1)-\sigma/2-q}, & J \ge 2, \end{cases} \tag{3.9}$$

for $0 \le t \le T^\varepsilon$, *provided* $q \le q_J$.

- *Very short time,* $1 < \sigma \le 2$*:*

$$\|\mathbf{m}^\varepsilon(\cdot,t) - \tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^q} \le C\varepsilon^{2+(\sigma-1)(J-1)-\rho(\sigma,q,r,J)-q}, \tag{3.10}$$

for $0 \le t \le T^\varepsilon$, *provided* $q \le q_J$ *and* $J \ge 1$. *Here* $\rho(\sigma,q,r,J) := \max(0, \frac{1}{2}\sigma - (\sigma - 1)(r-3-J-q))$.

In all cases, the constant C *is independent of* ε *and* t *but depends on* M *and* T.

For fixed final times of order $\mathcal{O}(1)$ this theorem shows the expected strong L^2 convergence rate of ε for $\mathbf{m}_0$ and also for the higher order approximations $\tilde{\mathbf{m}}_1^\varepsilon = \mathbf{m}_0 + \varepsilon\mathbf{m}_1$ and $\tilde{\mathbf{m}}_2^\varepsilon = \mathbf{m}_0 + \varepsilon\mathbf{m}_1 + \varepsilon^2\mathbf{m}_2$. Moreover, the errors with $\mathbf{m}_0$, $\tilde{\mathbf{m}}_1^\varepsilon$ and $\tilde{\mathbf{m}}_2^\varepsilon$ have bounded H^1-norms, suggesting weak H^1 convergence for these three approximations.

For the short and very short time cases where $\sigma > 0$ we note that since the temporal oscillation period in the problem is of order ε^2, as is shown in Section 4, final times with $0 < \sigma < 2$ are still relatively long, and include an infinite number of oscillations in time as $\varepsilon \to 0$.

The second bullet in the theorem shows that for times from $\mathcal{O}(\varepsilon)$ and up to $\mathcal{O}(1)$,$0 < \sigma \le 1$, one gets strong convergence of the L^2 and H^1-norms when considering the corrected approximation $\tilde{\mathbf{m}}_1^\varepsilon$,

$$\|\mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_1^\varepsilon\|_{L^2} \le C\varepsilon^{1+\sigma/2}, \qquad \|\mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_1^\varepsilon\|_{H^1} \le \varepsilon^{\sigma/2}.$$

However, one does not get better approximations by including more correctors.

For final times shorter than $\mathcal{O}(\varepsilon)$, on the other hand, one gets better approximations by including more correctors, as (3.10) shows. This is especially relevant since these are the times that are most interesting in the context of HMM. For these short times, the regularity of $\mathbf{m}_0$ determines which converge rate one obtains. In particular, if

$$r \ge J + 3 + q + \left\lceil \frac{\sigma}{2(\sigma-1)} \right\rceil,$$

the penalty term ρ in (3.10) becomes zero and one obtains the optimal estimate for short times. The longer the time considered, which means the closer σ is to one, the higher is the required regularity. In particular, if $\partial_t^k \mathbf{m}_0 \in C([0,T];H^\infty(\Omega))$, $k \geq 0$, we get

$$\|\mathbf{m}^\varepsilon(\cdot,t) - \tilde{\mathbf{m}}_J^\varepsilon\|_{H^q} \leq C\varepsilon^{2+(\sigma-1)(J-1)-q}, \qquad \sigma > 1,\ J > 0.$$

This entails for example the following bounds for $\sigma = 3/2$ and $\sigma = 2$,

$$\sigma = 3/2:\ \|\mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_J^\varepsilon\|_{H^q} \leq C\varepsilon^{0.5J+1.5-q}, \qquad \sigma = 2:\ \|\mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_J^\varepsilon\|_{H^q} \leq C\varepsilon^{J+1-q}.$$

Choosing J high enough, we can obtain any convergence rate we want for these errors.

Note that the first corrected approximation is of the form

$$\tilde{\mathbf{m}}_1(x,t) = \mathbf{m}_0(x,t) + \varepsilon \nabla \mathbf{m}_0(x,t)\boldsymbol{\chi}(x/\varepsilon) + \varepsilon \mathbf{v}(x, x/\varepsilon, t, t/\varepsilon^2).$$

The part $\nabla \mathbf{m}_0 \boldsymbol{\chi}$ is familiar from homogenization of elliptic operators with $\boldsymbol{\chi}$ being the solution of the cell problem (4.13). The second part $\mathbf{v}$ is special for (3.1). It satisfies the linear PDE (4.15) and oscillates both in time and space, with the time variations decaying exponentially. See Section 4.

3.1. Proof of Theorem 3.1

We begin with a preliminary estimate, based on Theorem 6.1, which we subsequently improve to obtain the results in Theorem 3.1. In Theorem 7.4 we show that the approximation $\tilde{\mathbf{m}}_J^\varepsilon$, (3.3), satisfies a perturbed version of (3.1),

$$\partial_t \tilde{\mathbf{m}}_J^\varepsilon(x,t) = -\tilde{\mathbf{m}}_J^\varepsilon(x,t) \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon(x,t) - \alpha \tilde{\mathbf{m}}_J^\varepsilon(x,t) \times \tilde{\mathbf{m}}_J^\varepsilon(x,t) \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon(x,t) + \boldsymbol{\eta}_J^\varepsilon, \tag{3.11a}$$

$$\tilde{\mathbf{m}}_J^\varepsilon(x,0) = \mathbf{m}_{\text{init}}(x), \tag{3.11b}$$

and that the norm of the residual $\boldsymbol{\eta}_J^\varepsilon$ can be bounded as

$$\|\boldsymbol{\eta}_J^\varepsilon(\cdot,t)\|_{H_\varepsilon^q} \leq C\varepsilon^{1+(\sigma-1)(J-2)}, \qquad 0 \leq t \leq T^\varepsilon, \tag{3.12}$$

if we include at least two correctors in the expansion, $J \geq 2$, and if $0 \leq q \leq r-2-J$. Furthermore, using (7.28) after Lemma 7.3, we show that

$$\|\boldsymbol{\nabla}|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)|^2\|_{H_\varepsilon^q} \leq C\varepsilon^{2+(\sigma-1)(J-2)}, \qquad 0 \leq t \leq T^\varepsilon, \tag{3.13}$$

under the same conditions. This last estimate can be seen as a measure for how rapidly the length of $\tilde{\mathbf{m}}_J^\varepsilon$ changes. Theorem 6.1 now says that the error $\mathbf{e}_J := \mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_J^\varepsilon$ satisfies

$$\|\mathbf{e}_J(\cdot,t)\|_{H^q}^2 \leq C\frac{t}{\varepsilon^{2q}} \sup_{0\leq s\leq t} \left(\|\nabla|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,s)|^2\|_{H_\varepsilon^q}^2 + \|\boldsymbol{\eta}_J^\varepsilon(\cdot,s)\|_{H_\varepsilon^q}^2 \right), \quad 0 \leq t \leq T^\varepsilon, \tag{3.14}$$

when $q \leq s$ and

$$\|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{W^{k,\infty}} \leq C\varepsilon^{\min(0,1-k)}, \qquad 0 \leq k \leq q+1,$$

uniformly for $t \in [0,T^\varepsilon]$. The latter estimates are true by Theorem 7.3 when $0 \leq q \leq r-3-J$. Therefore, combining (3.12), (3.13), (3.14), and (3.7) we get

$$\|\mathbf{e}_J(\cdot,t)\|_{H^q} \leq C\varepsilon^{2+(\sigma-1)(J-1)-\sigma/2-q}, \quad 0 \leq t \leq T^\varepsilon, \tag{3.15}$$

as long as $0 \le q \le \min(s, r-3-J)$ and $J \ge 2$. This completes the preliminary estimate.

To improve the estimate, we consider the difference between $\tilde{\mathbf{m}}_J^\varepsilon$ and higher order corrections $\tilde{\mathbf{m}}_{J'}^\varepsilon$ with $J' > J$ and $J' \ge 2$. We write, using Lemma 5.1,

$$\begin{aligned}\|\mathbf{e}_J\|_{H^q} &\le \|\mathbf{m}^\varepsilon - \tilde{\mathbf{m}}_{J'}^\varepsilon\| + \|\tilde{\mathbf{m}}_{J'}^\varepsilon - \tilde{\mathbf{m}}_J^\varepsilon\|_{H^q} \le \|\mathbf{e}_{J'}\|_{H^q} + \sum_{j=J+1}^{J'} \varepsilon^j \left\|\mathbf{m}_j(\cdot,\cdot/\varepsilon,t,t/\varepsilon^2)\right\|_{H^q} \\ &\le \|\mathbf{e}_{J'}\|_{H^q} + C\sum_{j=J+1}^{J'} \varepsilon^{j-q}\|\mathbf{m}_j(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{q,q+2}}. \end{aligned} \tag{3.16}$$

We then need to use Theorem 7.2, where it is shown that the norms of the first two correctors, $\mathbf{m}_1$ and $\mathbf{m}_2$, are uniformly bounded in τ, while higher order correctors grow algebraically. In particular, it holds for all $p \ge 0$ and $j \le r$ that

$$\|\mathbf{m}_j(\cdot,\cdot,t,\tau)\|_{H^{r-j,p}} \le C(1+\tau^{\max(0,j-2)}) \le C\varepsilon^{(\sigma-2)\max(0,j-2)}, \tag{3.17}$$

for $0 \le t \le T^\varepsilon$ and $0 \le \tau \le \varepsilon^{-2}T^\varepsilon$. Entering (3.15) and (3.17) in (3.16) then shows that

$$\|\mathbf{e}_J\|_{H^q} \le C\varepsilon^{2+(\sigma-1)(J'-1)-\sigma/2-q} + C\sum_{j=J+1}^{J'} \varepsilon^{j+(\sigma-2)\max(0,j-2)-q}, \tag{3.18}$$

when $q \le \min(s, r-3-J')$.

We are now ready to show the final estimates as given in Theorem 3.1. For the first case, where $\sigma = 0$, we take $0 \le J < J' = 2$ and $0 \le q \le 1$. Then (3.18) gives us

$$\|\mathbf{e}_J^\varepsilon\|_{H^q} \le C\varepsilon^{1-q} + C\sum_{j=J+1}^{2} \varepsilon^{j-q} \le C\varepsilon^{1-q},$$

when $q \le \min(s, r-5)$, which is automatically satisfied for $q = 0$ by (A4) and (A5) but requires $r \ge 6$ for $q = 1$. The result for $J = 2$ follows directly from (3.15).

For the second case, where $0 < \sigma \le 1$, we cannot improve the preliminary estimate (3.15) using (3.18) when $J \ge 2$. However, for $J = 1$ and $J' = 2$, (3.18) gives

$$\|\mathbf{e}_1\|_{H^q} \le C\varepsilon^{1+\sigma/2-q} + C\varepsilon^{2-q} \le \varepsilon^{1+\sigma/2-q}.$$

This is valid as long as $q \le \min(s, r-3-\max(2,J)) = q_J$.

Finally, for the third case in Theorem 3.1, where $1 < \sigma \le 2$, we only consider (3.18) with $J \ge 1$. Then $j + (\sigma-2)(j-2) = 2 + (\sigma-1)(j-2)$ and we get

$$\begin{aligned}\|\mathbf{e}_J\|_{H^q} &\le C\varepsilon^{2+(\sigma-1)(J'-1)-\sigma/2-q} + C\sum_{j=J+1}^{J'} \varepsilon^{j+(\sigma-2)(j-2)-q} \\ &\le C\varepsilon^{2+(\sigma-1)(J'-1)-\sigma/2-q} + C\varepsilon^{2+(\sigma-1)(J-1)-q} \\ &\le C\varepsilon^{2+\min\left[(\sigma-1)(J'-1)-\sigma/2,\ (\sigma-1)(J-1)\right]-q} \\ &= C\varepsilon^{2+(\sigma-1)(J-1)-\max\left[\sigma/2-(\sigma-1)(J'-J),\ 0\right]-q},\end{aligned}$$

where the possible choices of J' are limited by the restrictions $q \le \min(s, r-3-J')$, $J' \ge 2$ and $J' > J$. When $q = r-3-\max(2,J)$ we can therefore not choose J' such that we get an improvement. Hence (3.10) is the same as the preliminary estimate (3.15) in that case. It thus only remains to prove the case $q < r-3-\max(2,J)$. We are then allowed to take $J' = r-3-q > \max(2,J)$ and get

$$\|\mathbf{e}_J\|_{H^q} \le C\varepsilon^{2+(\sigma-1)(J-1)-\max(\sigma/2-(\sigma-1)(r-3-q-J),0)-q}.$$

The theorem is proved.

4. Homogenization

In this section we derive differential equations for the homogenized solution $\mathbf{m}_0$ to (3.1) and the corresponding correction terms. We aim to motivate our choice of equations but do not include any proofs in this section. Precise energy estimates will be done in Section 7.

4.1. Multiscale expansion. We consider the Landau-Lifshitz (3.1) and assume that we are looking for an asymptotic solution to (3.1) of the form

$$\mathbf{m}^\varepsilon(x,t) = \mathbf{m}\left(x, x/\varepsilon, t, t/\varepsilon^2; \varepsilon\right)$$

for a suitable function $\mathbf{m}(x,y,t,\tau)$. Numerical experiments suggest that this is the form that is required for our problem. One example for this is shown in fig. 4.1 where one can clearly observe oscillations in space on an ε-scale and oscillations in time on an ε^2 scale when taking the difference between $\mathbf{m}^\varepsilon$ satisfying (3.1) and the suggested $\mathbf{m}_0$.

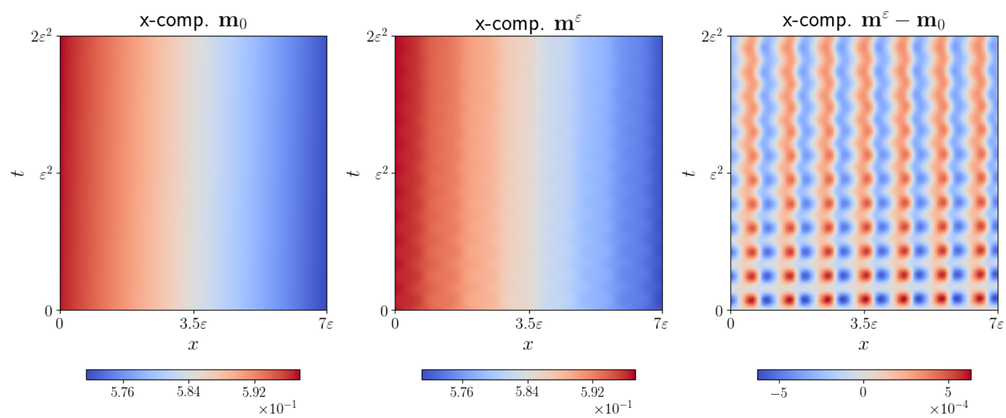

Fig. 4.1. *Numerical example: x-component of the solution $\mathbf{m}^\varepsilon$ to (3.1) in 1D and the corresponding $\mathbf{m}_0$ according to (3.2) when choosing $a^\varepsilon(x) = 1+0.5\sin(2\pi x/\varepsilon)$, $\varepsilon = 1/70$, $\alpha = 0.02$ and initial data $\mathbf{m}_{\text{init}}(x) = \mathbf{m}_{nn}(x)/|\mathbf{m}_{nn}(x)|$ where $\mathbf{m}_{nn}(x) = 0.5 + [\exp(-0.1\cos(2\pi(x-0.2))), \exp(-0.2\cos(2\pi x)), \exp(-0.1\cos(2\pi(x-0.8)))]^T$ on a subset $[0,7\varepsilon]$ of the domain $\Omega = [0,1]$ and a short time interval $0 \le t \le 2\varepsilon^2$.*

Taking derivatives of $\mathbf{m}^\varepsilon(x,t)$, one obtains

$$\nabla \mathbf{m}^\varepsilon(x,t) = \nabla_x \mathbf{m}(x,y,t,\tau;\varepsilon) + \frac{1}{\varepsilon}\nabla_y \mathbf{m}(x,y,t,\tau;\varepsilon),$$
$$\partial_t \mathbf{m}^\varepsilon(x,t) = \partial_t \mathbf{m}(x,y,t,\tau;\varepsilon) + \frac{1}{\varepsilon^2}\partial_\tau \mathbf{m}(x,y,t,\tau;\varepsilon),$$

where $y := \frac{x}{\varepsilon}$ is the fast variable in space and $\tau := \frac{t}{\varepsilon^2}$ the fast variable in time. The differential operator $\mathcal{L}$ can accordingly be rewritten in the form

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{\varepsilon}\mathcal{L}_1 + \frac{1}{\varepsilon^2}\mathcal{L}_2,$$

where $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$ are the vector-operators corresponding to the scalar operators

$$L_0 := \nabla_x \cdot (a(y)\nabla_x), \qquad L_1 := \nabla_x \cdot (a(y)\nabla_y) + \nabla_y \cdot (a(y)\nabla_x), \qquad L_2 := \nabla_y \cdot (a(y)\nabla_y).$$

We are looking for an asymptotic expansion for $\mathbf{m}$,

$$\mathbf{m}(x,y,t,\tau;\varepsilon) = \mathbf{m}_0(x,t) + \sum_{j=1}^{\infty} \varepsilon^j \mathbf{m}_j(x,y,t,\tau), \tag{4.1}$$

where we assume that $\mathbf{m}_0 = \mathbf{m}_0(x,t)$ only depends on the slow variables, x and t, and that the correctors $\mathbf{m}_j$, $j = 1,2,\dots$ are 1-periodic in y.

Before we consider an expanded version of the differential equation (3.1), we start by introducing suitable notation that will help us to keep track of terms of the same structure throughout the rest of this paper. First, we let $\mathbf{m}_{-1}(x,t) := 0$ and define

$$\mathbf{V}_j := \mathcal{L}_2 \mathbf{m}_j + \mathbf{Z}_{j-1}, \quad j \geq 1, \quad \text{and} \quad \mathbf{Z}_j := \begin{cases} \mathcal{L}_1 \mathbf{m}_0, & j = 0, \\ \mathcal{L}_0 \mathbf{m}_{j-1} + \mathcal{L}_1 \mathbf{m}_j, & j \geq 1. \end{cases} \tag{4.2}$$

Furthermore, let for $j \geq 1$,

$$\mathbf{T}_j := \sum_{k=1}^{j} \mathbf{m}_{j-k} \times \mathbf{V}_k = \mathbf{m}_0 \times \mathbf{V}_j + \mathbf{R}_{j-1}, \quad \mathbf{R}_j := \begin{cases} 0, & j = 0, \\ \sum_{k=1}^{j} \mathbf{m}_{j+1-k} \times \mathbf{V}_k, & j \geq 1, \end{cases} \tag{4.3}$$

and finally

$$\mathbf{S}_j := \begin{cases} 0, & j = 0, \\ \sum_{k=1}^{j} \mathbf{m}_{j+1-k} \times \mathbf{T}_k, & j \geq 1. \end{cases} \tag{4.4}$$

Note that in all of these quantities, j indicates the highest index of all $\mathbf{m}_j$ that are part of the quantity.

Consider now the expanded version of $\mathcal{L}\mathbf{m}^\varepsilon$, which becomes

$$\mathcal{L}\mathbf{m}^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L}_2 \mathbf{m}_0 + \frac{1}{\varepsilon}(\mathcal{L}_1 \mathbf{m}_0 + \mathcal{L}_2 \mathbf{m}_1) + \sum_{j=0}^{\infty} \varepsilon^j (\mathcal{L}_0 \mathbf{m}_j + \mathcal{L}_1 \mathbf{m}_{j+1} + \mathcal{L}_2 \mathbf{m}_{j+2}) =: \sum_{j=1}^{\infty} \varepsilon^{j-2} \mathbf{V}_j, \tag{4.5}$$

entailing that the precession term in (3.1) expands to

$$\mathbf{m}^\varepsilon(x,t) \times \mathcal{L}\mathbf{m}^\varepsilon(x,t) = \sum_{j=0}^{\infty} \varepsilon^j \mathbf{m}_j \times \sum_{k=1}^{\infty} \varepsilon^{k-2} \mathbf{V}_k = \sum_{j=1}^{\infty} \varepsilon^{j-2} \sum_{k=1}^{j} \mathbf{m}_{j-k} \times \mathbf{V}_k = \sum_{j=1}^{\infty} \varepsilon^{j-2} \mathbf{T}_j,$$

and the damping term takes the form

$$\mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon \times \mathcal{L}\mathbf{m}^\varepsilon = \sum_{\ell=0}^{\infty} \varepsilon^\ell \mathbf{m}_\ell \times \sum_{j=1}^{\infty} \varepsilon^{j-2} \mathbf{T}_j = \sum_{\ell=0}^{\infty} \varepsilon^{\ell-2} \sum_{j=1}^{\ell} \mathbf{m}_{\ell-j} \times \mathbf{T}_j.$$

For the time derivative of $\mathbf{m}^\varepsilon$, it moreover holds that

$$\partial_t \mathbf{m}^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j \partial_t \mathbf{m}_j + \varepsilon^{j-2} \partial_\tau \mathbf{m}_j.$$

We can then formally rewrite the differential equation (3.1) as

$$\sum_{j=1}^{\infty}\varepsilon^{j-2}(\partial_t\mathbf{m}_{j-2}+\partial_\tau\mathbf{m}_j)=-\sum_{j=1}^{\infty}\varepsilon^{j-2}\mathbf{T}_j-\alpha\sum_{j=0}^{\infty}\varepsilon^{j-2}\sum_{k=1}^{j}\mathbf{m}_{j-k}\times\mathbf{T}_k,$$

which implies that at scale ε^{j-2} and for $j\geq 1$, it holds that

$$\partial_t\mathbf{m}_{j-2}+\partial_\tau\mathbf{m}_j=-\mathbf{T}_j-\alpha\sum_{k=1}^{j}\mathbf{m}_{j-k}\times\mathbf{T}_k\,. \tag{4.6}$$

Note that as $\mathbf{m}_0(x,t)$ is independent of y and τ, both $\partial_\tau\mathbf{m}_0(x,t)=0$ and $\mathcal{L}_2\mathbf{m}_0(x,t)=0$. Based on (4.6), it is now possible to show that all the correctors $\mathbf{m}_j$, $j\geq 1$, satisfy linear differential equations of a similar structure as the one for $\mathbf{m}_0$. Since it holds that

$$\begin{aligned}\mathbf{T}_j&=\mathbf{m}_0\times\mathbf{V}_j+\mathbf{R}_{j-1}=\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j+\mathbf{m}_0\times\mathbf{Z}_{j-1}+\mathbf{R}_{j-1},\\ \sum_{k=1}^{j}\mathbf{m}_{j-k}\times\mathbf{T}_k&=\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j+\mathbf{m}_0\times\mathbf{m}_0\times\mathbf{Z}_{j-1}+\mathbf{m}_0\times\mathbf{R}_{j-1}+\mathbf{S}_{j-1},\end{aligned}$$

where $\mathbf{R}_{j-1}$, $\mathbf{S}_{j-1}$ and $\mathbf{Z}_{j-1}$ only contain lower order $\mathbf{m}_k$ with $k\leq j-1$, it follows that $\mathbf{m}_j$, with $j\geq 1$, satisfies the linear differential equation

$$\partial_\tau\mathbf{m}_j=-\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j-\alpha\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j+\mathbf{F}_{j-1}=\mathscr{L}\mathbf{m}_j+\mathbf{F}_j\,, \tag{4.7}$$

where the linear operator $\mathscr{L}$ is defined such that

$$\mathscr{L}\mathbf{m}_j:=-\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j-\alpha\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_j, \tag{4.8}$$

and all terms involving only $\mathbf{m}_k$ with $k<j$ are contained in $\mathbf{F}_j$, defined according to

$$\mathbf{F}_j:=-\mathbf{R}_{j-1}-\mathbf{m}_0\times\mathbf{Z}_{j-1}-\alpha(\mathbf{m}_0\times\mathbf{R}_{j-1}+\mathbf{m}_0\times\mathbf{m}_0\times\mathbf{Z}_{j-1}+\mathbf{S}_{j-1})-\partial_t\mathbf{m}_{j-2}, \tag{4.9}$$

for $j\geq 1$.

4.2. Derivation homogenized equation. In order to derive a homogenized equation corresponding to (3.1), we now take a closer look at the differential equations for $\mathbf{m}_1$ and $\mathbf{m}_2$. As by definition $\mathbf{R}_0=\mathbf{S}_0=\mathbf{m}_{-1}:=0$,

$$\mathbf{F}_1=-\mathbf{m}_0\times\mathbf{Z}_0-\alpha\mathbf{m}_0\times\mathbf{m}_0\times\mathbf{Z}_0, \tag{4.10}$$

where $\mathbf{Z}_0=\mathcal{L}_1\mathbf{m}_0$, it holds according to (4.7) at scale ε^{-1} that

$$\partial_\tau\mathbf{m}_1=-\mathbf{m}_0\times\mathbf{V}_1-\alpha\mathbf{m}_0\times\mathbf{m}_0\times\mathbf{V}_1\,, \tag{4.11}$$

since $\mathbf{V}_1=\mathcal{L}_2\mathbf{m}_1+\mathcal{L}_1\mathbf{m}_0$. To find a solution for this equation, we assume that $\mathbf{m}_1$ takes the form

$$\mathbf{m}_1(x,y,t,\tau)=\nabla_x\mathbf{m}_0\boldsymbol{\chi}(y)+\mathbf{v}(x,y,t,\tau)\,, \tag{4.12}$$

where $\boldsymbol{\chi}(y)$ is the solution to the cell problem

$$\nabla_y\cdot(a(y)\nabla_y\boldsymbol{\chi}(y))=-\nabla_y a(y)\,. \tag{4.13}$$

Note that (4.13) only determines $\boldsymbol{\chi}$ up to a constant. In accordance with standard practice in the literature [8, 13], we assume in the following that this constant is chosen such that $\boldsymbol{\chi}(y)$ has zero average. As, by the definition of $\boldsymbol{\chi}(y)$ and the assumption (4.12),

$$\mathbf{V}_1 = \mathcal{L}_2\mathbf{m}_1 + \mathcal{L}_1\mathbf{m}_0 = \mathcal{L}_2\mathbf{v} + \mathcal{L}_2(\nabla_x\mathbf{m}_0\boldsymbol{\chi}) + \mathcal{L}_1\mathbf{m}_0 = \mathcal{L}_2\mathbf{v}, \tag{4.14}$$

it follows from (4.11) that

$$\partial_\tau\mathbf{v} = -\mathbf{m}_0\times\mathcal{L}_2\mathbf{v} - \alpha\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{v} = \mathscr{L}\mathbf{v}. \tag{4.15}$$

This is a linear differential equation with the same structure as (4.7), but with forcing $\mathbf{F}=0$. At the initial time, $\tau=0$, we set $\mathbf{m}_1(x,y,t,0)=0$ and hence have $\mathbf{v}(\tau=0,y) = -\boldsymbol{\nabla}_x\mathbf{m}_0\boldsymbol{\chi}(y)$. Note that $\mathbf{m}_1$ is biggest term in $\mathbf{m}^\varepsilon - \mathbf{m}_0$ and therefore determines the right figure in fig. 4.1: there we can observe oscillations around zero on a scale of approximately ε smaller then the variations in the homogenized solution. For short times, we clearly observe oscillations in both time and space while the oscillations in time reduce as t increases. This indicates that the $\mathbf{v}$-part of $\mathbf{m}_1$ gets damped away with time, while $\boldsymbol{\nabla}_x\mathbf{m}_0\boldsymbol{\chi}(y)$, which does not depend on t/ε^2 but oscillates in space, is preserved. This matches with the results for $\mathbf{v}$ and $\mathbf{m}_1$ in Section 7.2.

On the ε^0-scale, we have

$$\partial_\tau\mathbf{m}_2 = \mathscr{L}\mathbf{m}_2 + \mathbf{F}_2, \tag{4.16}$$

where the expression for $\mathbf{F}_2$ given by (4.9) becomes

$$\mathbf{F}_2 = -\mathbf{R}_1 - \mathbf{m}_0\times\mathbf{Z}_1 - \alpha[\mathbf{m}_0\times\mathbf{R}_1 + \mathbf{m}_0\times\mathbf{m}_0\times\mathbf{Z}_1 + \mathbf{S}_1] - \partial_t\mathbf{m}_0, \tag{4.17}$$

and the relation (4.14) gives the simplification

$$\mathbf{R}_1 = \mathbf{m}_1\times\mathcal{L}_2\mathbf{v}, \qquad \mathbf{S}_1 = \mathbf{m}_1\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{v}. \tag{4.18}$$

To obtain a homogenized equation, (4.16) is averaged over one period Y in y. Then all terms which are derivatives with respect to y of y-periodic terms cancel, and since $\mathbf{m}_0$ does not depend on y we get

$$\partial_\tau\int_Y\mathbf{m}_2dy = \int_Y\mathbf{F}_2dy = -\partial_t\mathbf{m}_0 - \mathbf{m}_0\times\int_Y\mathbf{Z}_1dy - \alpha\mathbf{m}_0\times\mathbf{m}_0\times\int_Y\mathbf{Z}_1dy - \mathbf{E}_1, \tag{4.19}$$

where

$$\mathbf{E}_1 := \int_Y\mathbf{R}_1 + \alpha[\mathbf{m}_0\times\mathbf{R}_1 + \mathbf{S}_1]dy. \tag{4.20}$$

Furthermore,

$$\begin{aligned}\int_Y\mathbf{Z}_1dy &= \int_Y\mathcal{L}_0\mathbf{m}_0 + \mathcal{L}_1\mathbf{m}_1dy = \int_Y\boldsymbol{\nabla}_x\cdot a(y)\boldsymbol{\nabla}_x\mathbf{m}_0 + \boldsymbol{\nabla}_x\cdot(a(y)\boldsymbol{\nabla}_y(\nabla_x\mathbf{m}_0\boldsymbol{\chi}+\mathbf{v}))\,dy\\ &= \int_Y\boldsymbol{\nabla}_x\cdot(a(y)\boldsymbol{\nabla}_x\mathbf{m}_0(\mathbf{I}+\boldsymbol{\nabla}_y\boldsymbol{\chi})\,dy + \int_Y\mathcal{L}_1\mathbf{v}dy.\end{aligned}$$

We therefore define the constant homogenized material coefficient matrix $\mathbf{A}^H\in\mathbb{R}^{n\times n}$ as

$$\mathbf{A}^H := \int_Y a(y)\,(\mathbf{I}+\boldsymbol{\nabla}_y\boldsymbol{\chi})\,dy$$

and let $\bar{L}u := \nabla_x \cdot (\mathbf{A}^H \nabla_x u)$ for any scalar function $u : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, with the corresponding vector-operator being denoted $\bar{\mathcal{L}}$. Plugging this into (4.19), we get

$$\partial_\tau \int \mathbf{m}_2 dy = -\partial_t \mathbf{m}_0 - \mathbf{m}_0 \times \bar{\mathcal{L}} \mathbf{m}_0 - \alpha \mathbf{m}_0 \times \mathbf{m}_0 \times \bar{\mathcal{L}} \mathbf{m}_0 - \mathbf{E}_1 - \mathbf{E}_2, \tag{4.21}$$

where

$$\mathbf{E}_2 := \mathbf{m}_0 \times \int_Y \mathcal{L}_1 \mathbf{v} dy + \alpha \mathbf{m}_0 \times \mathbf{m}_0 \times \int_Y \mathcal{L}_1 \mathbf{v} dy. \tag{4.22}$$

As we will see in Section 7.2, $\mathbf{v}$ oscillates and decays exponentially in τ, which means that so do $\mathbf{R}_1$ and $\mathbf{S}_1$ by (4.18). Therefore, if we average over a fixed interval in the fast time variable, the contributions of $\mathbf{E}_1$ and $\mathbf{E}_2$ will become negligible as the interval size increases, while $\mathbf{m}_0$ is unaffected. We therefore define $\mathbf{m}_0$ such that it satisfies

$$\partial_t \mathbf{m}_0 = -\mathbf{m}_0 \times \bar{\mathcal{L}} \mathbf{m}_0 - \alpha \mathbf{m}_0 \times \mathbf{m}_0 \times \bar{\mathcal{L}} \mathbf{m}_0 . \tag{4.23}$$

In contrast to the differential equations for $\mathbf{m}_j$, $j \geq 1$, this is a nonlinear differential equation with a matrix-valued coefficient in the operator $\bar{\mathcal{L}}$.

5. Sobolev norm estimates

The proofs in the following sections rely frequently on properties of the considered Bochner-Sobolev and multiscale norms as well as several bilinear Sobolev estimates. In this section, we therefore prove lemmas providing the required properties, making it possible to keep the subsequent sections mostly focused on specific estimates for the solution to the Landau-Lifshitz equation (3.1), corresponding homogenized solution and correctors.

If not stated otherwise, the estimates in this section apply to functions in arbitrary dimensions, not necessarily on Ω as considered previously. All the lemmas that are stated for scalar functions analogously apply to vector valued functions, with either scalar or cross products instead of products of scalar functions. Throughout this section, we suppose $0 < \varepsilon < 1$ in accordance with (A3).

In several of the subsequent estimates we use the Sobolev inequality which states that when $f \in H^2(D)$ and $D \subseteq \mathbb{R}^n$, for dimension $n \leq 3$, then

$$\sup_{x \in D} |f(x)| \leq C \|f\|_{H^2(D)} . \tag{5.1}$$

5.1. Multiscale norms.

In the present paper, a function $u(x,y)$ in the Bochner-Sobolev space $H^{q,p}(\Omega; Y)$ is often used to describe multiscale phenomena, where the x- and y-variables represent the slow and fast scales respectively. For such functions we have the following lemma.

Lemma 5.1. *Suppose $f^\varepsilon(x) := u(x, x/\varepsilon)$ and $n \leq 3$. Then*

$$\|f^\varepsilon\|_{H^q} \leq \frac{C}{\varepsilon^q} \|u\|_{H^{q,q+2}} , \qquad \|f^\varepsilon\|_{W^{q,\infty}} \leq \frac{C}{\varepsilon^q} \|u\|_{H^{q+2,q+2}} , \tag{5.2}$$

whenever the norms are bounded.

Proof. Using (5.1) and the definition of the norms, we find that

$$\|f^\varepsilon\|_{H^q}^2 \leq \sum_{\substack{|\alpha| \leq q \\ \gamma \leq \alpha}} \binom{\alpha}{\gamma} \int \varepsilon^{-2|\gamma|} |\partial_y^\gamma \partial_x^{\alpha - \gamma} u(x, x/\varepsilon)|^2 dx$$

$$\leq C\varepsilon^{-2q} \sum_{\substack{|\alpha|\leq q\\ \gamma\leq\alpha}} \|\partial_y^\gamma \partial_x^{\alpha-\gamma} u\|^2_{H^{0,2}} \leq C\varepsilon^{-2q}\|u\|^2_{H^{q,q+2}},$$

and accordingly,

$$\|f^\varepsilon\|^2_{W^{k,\infty}} \leq C\varepsilon^{-2q} \sum_{\substack{|\alpha|\leq q\\ \gamma\leq\alpha}} \sup_{x,y} |\partial_y^\gamma \partial_x^{\alpha-\gamma} u(x,y)|^2$$
$$\leq C\varepsilon^{-2q} \sum_{\substack{|\alpha|\leq q\\ \gamma\leq\alpha}} \|\partial_y^\gamma \partial_x^{\alpha-\gamma} u\|^2_{H^{2,2}} \leq C\varepsilon^{-2q}\|u\|^2_{H^{q+2,q+2}},$$

which shows the lemma. ☐

The weighted multiscale norm $\|\cdot\|_{H^q_\varepsilon}$ has the following properties that we will use:

LEMMA 5.2. *Consider $f \in H^q(\Omega)$ such that for $0\leq j\leq q$ and some constant $c\in\mathbb{R}$,*

$$\|f\|_{H^j} \leq C_j \varepsilon^{c-j},$$

then it follows that

$$\|f\|_{H^q_\varepsilon} \leq C\varepsilon^c, \tag{5.3}$$

where the constants C, C_j are independent of ε. Moreover, given a multi-index β, it holds for $0\leq q\leq r-|\beta|$ that

$$\|\partial^\beta f\|_{H^q_\varepsilon} \leq \varepsilon^{-|\beta|}\|f\|_{H^{q+|\beta|}_\varepsilon}. \tag{5.4}$$

Proof. The first claim, (5.3) holds, since by the definition of $\|\cdot\|_{H^q_\varepsilon}$ and the given assumption,

$$\|g\|_{H^q_\varepsilon} = \sum_{j=0}^q \varepsilon^j \|g\|_{H^j} \leq \varepsilon^c \sum_{j=0}^q C_j \leq C\varepsilon^c.$$

Similarly, we find that

$$\varepsilon^{|\beta|}\|\partial^\beta f\|_{H^q_\varepsilon} = \sum_{j=0}^q \varepsilon^{j+|\beta|}\|\partial^\beta f\|_{H^j} \leq \sum_{j=0}^q \varepsilon^{j+|\beta|}\|f\|_{H^{j+|\beta|}} = \sum_{j=|\beta|}^{q+|\beta|} \varepsilon^j\|f\|_{H^j} \leq \|f\|_{H^{q+|\beta|}_\varepsilon},$$

which implies (5.4). ☐

5.2. Bilinear estimates.

To obtain estimates for the product of two functions, the following bilinear Sobolev estimates are useful.

LEMMA 5.3. *Let $f, g \in C(\Omega)\cap H^q(\Omega)$. It then holds that*

$$\|(\partial^\beta f)(\partial^\gamma g)\|_{L^2} \leq C(\|f\|_{L^\infty}\|g\|_{H^q} + \|g\|_{L^\infty}\|f\|_{H^q}) \quad \text{for } |\beta|+|\gamma| = q, \tag{5.5}$$

and

$$\|fg\|_{H^q} \leq C(\|f\|_{L^\infty}\|g\|_{H^q} + \|f\|_{H^q}\|g\|_{L^\infty}). \tag{5.6}$$

Let $u\in H^{q_1,\infty}(\Omega;Y)$ *and* $v\in H^{q_2,\infty}(\Omega;Y)$ *where* $q_1,q_2\in\mathbb{Z}$. *Let* $q_0\le\min(q_1,q_2)$ *and* $n\le 3$. *Then, for all* $p\ge 0$,

$$\|uv\|_{H^{q_0,p}}\le C\|u\|_{H^{q_1,p+2}}\|v\|_{H^{q_2,p}}, \tag{5.7}$$

if either

$$q_1+q_2\ge\min(3+q_0,5) \qquad \textit{or} \qquad q_1\ge q_0+2. \tag{5.8}$$

The constants C *are independent of* f, g, u *and* v.

Proof. The first two statements (5.5) and (5.6) are proved for instance in [27, Proposition 3.6] and [27, Proposition 3.7].

To prove the remaining statement, let $|\alpha|+|\gamma|=q_0$ and $|\beta|+|\kappa|=p$. We then start by estimating the same quantity in two different ways. First,

$$\begin{aligned}\|(\partial_x^\alpha\partial_y^\beta u)(\partial_x^\gamma\partial_y^\kappa v)\|_{H^{0,0}}^2 &= \int |\left(\partial_x^\alpha\partial_y^\beta u(x,y)\right)\left(\partial_x^\gamma\partial_y^\kappa v(x,y)\right)|^2 dxdy\\ &\le \sup_{(x,y)\in\Omega\times Y}|\partial_x^\alpha\partial_y^\beta u(x,y)|^2\int|\partial_x^\gamma\partial_y^\kappa v(x,y)|^2dxdy\\ &\le C\|\partial_x^\alpha\partial_y^\beta u\|_{H^{2,2}}^2\|\partial_x^\gamma\partial_y^\kappa v\|_{H^{0,0}}^2\le C\|u\|_{H^{|\alpha|+2,p+2}}^2\|v\|_{H^{q_0,p}}^2. \end{aligned} \tag{5.9}$$

Second,

$$\begin{aligned}\|(\partial_x^\alpha\partial_y^\beta u)(\partial_x^\gamma\partial_y^\kappa v)\|_{H^{0,0}}^2 &\le \int\sup_{y\in\Omega}|\partial_x^\alpha\partial_y^\beta u(x,y)|^2\sup_{x\in Y}|\partial_x^\gamma\partial_y^\kappa v(x,y)|^2dxdy\\ &\le C\int\|\partial_x^\alpha\partial_y^\beta u(x,\cdot)\|_{H^2(Y)}^2\|\partial_x^\gamma\partial_y^\kappa v(\cdot,y)\|_{H^2(\Omega)}^2dxdy\\ &= C\|\partial_x^\alpha\partial_y^\beta u\|_{H^{0,2}}^2\|\partial_x^\gamma\partial_y^\kappa v\|_{H^{2,0}}^2\le C\|u\|_{H^{q_0,p+2}}^2\|v\|_{H^{|\gamma|+2,p}}^2. \end{aligned} \tag{5.10}$$

We then consider the case when $q_1+q_2\ge\min(q_0+3,5)$. Suppose $q_1\le q_2$ and assume that $|\alpha|\le q_1-2$. Then it follows from (5.9) that

$$\|(\partial_x^\alpha\partial_y^\beta u)(\partial_x^\gamma\partial_y^\kappa v)\|_{H^{0,0}}\le C\|u\|_{H^{q_1,p+2}}\|v\|_{H^{q_2,p}}. \tag{5.11}$$

If, on the other hand, $|\alpha|\ge q_1-1$, then when $q_0\le 2$,

$$q_2\ge q_0+3-q_1\ge q_0+2-\max(0,q_1-1)\ge q_0+2-|\alpha|=|\gamma|+2,$$

while if $q_0\ge 3$,

$$q_2\ge q_0\ge 3\ge q_1-|\alpha|+2\ge q_0-|\alpha|+2=|\gamma|+2.$$

By (5.10), this shows that (5.11) holds also for for $|\alpha|\ge q_1-1$. When $q_2\le q_1$ we get the same result upon switching the cases and using (5.9) for $|\gamma|\ge q_2-1$ and (5.10) for $|\gamma|\le q_2-2$. Finally, (5.11) follows directly from (5.9) in the case when $q_1\ge q_0+2$.

From the estimates (5.11) we finally have

$$\begin{aligned}\|uv\|_{H^{q,p}}^2 &\le \sum_{\substack{|\alpha+\gamma|\le q\\|\beta+\kappa|\le p}}\binom{\alpha+\gamma}{\alpha}\binom{\beta+\kappa}{\beta}\|(\partial_x^\alpha\partial_y^\beta u)(\partial_x^\gamma\partial_y^\kappa v)\|_{H^{0,0}}^2\\ &\le C\sum_{j=0}^{q}\sum_{k=0}^{p}\|u\|_{H^{q_1,p+2}}^2\|v\|_{H^{q_2,p}}^2\le C\|u\|_{H^{q_1,p+2}}^2\|v\|_{H^{q_2,p}}^2. \end{aligned}$$

This proves the lemma. ∎

The next two results, regarding the cross product of vector-valued functions, are consequences of Lemma 5.3.

LEMMA 5.4. *Suppose* $\partial_t^\ell \mathbf{u}_m, \partial_t^\ell \mathbf{v}_m \in H^{r-m-2\ell,\infty}(\Omega;Y)$ *for* $0 \le 2\ell \le 2k \le r-j$ *and* $0 \le m \le j$. *Then* $\partial_t^k(\mathbf{u}_m \times \mathbf{v}_{m'}) \in H^{r-j-2k,\infty}(\Omega;Y)$ *when* $m+m' \le j+2$, *and for all* $p \ge 0$,

$$\|\partial_t^k(\mathbf{u}_m \times \mathbf{v}_{m'})\|_{H^{r-j-2k,p}} \le C \sum_{\ell=0}^{k} \|\partial_t^{k-\ell}\mathbf{u}_m\|_{H^{r-m-2k+2\ell,p+2}} \|\partial_t^\ell \mathbf{v}_{m'}\|_{H^{r-m'-2\ell,p}},$$

where C *is independent of* $\mathbf{u}_m$ *and* $\mathbf{v}_{m'}$.

Proof. By (5.7) in Lemma 5.3, where we choose $q_0 = r-j-2k$, $q_1 = r-m-2k+2\ell$ and $q_2 = r-m'-2\ell$ for $0 \le \ell \le k$, we get

$$\begin{aligned}\|\partial_t^k(\mathbf{u}_m \times \mathbf{v}_{m'})\|_{H^{r-j-2k,p}} &\le C \sum_{\ell=0}^{k} \|(\partial_t^{k-\ell}\mathbf{u}_m) \times (\partial_t^\ell \mathbf{v}_{m'})\|_{H^{r-j-2k,p}} \\ &\le C \sum_{\ell=0}^{k} \|\partial_t^{k-\ell}\mathbf{u}_m\|_{H^{r-m-2k+2\ell,p+2}} \|\partial_t^\ell \mathbf{v}_{m'}\|_{H^{r-m'-2\ell,p}}.\end{aligned}$$

It is indeed valid to use Lemma 5.3 since $q_0 = q_1 - (j-m) - 2\ell = q_2 - (j-m') - 2(k-\ell) \le \min(q_1, q_2)$ and

$$q_1 + q_2 = q_0 + r + j - (m+m') \ge q_0 + r - 2 \ge q_0 + 3,$$

satisfying the left condition in (5.8). The proof is complete. ∎

As a consequence of this lemma, we get estimates for the time derivatives of precession and damping term in the Landau-Lifshitz equation by taking $\mathbf{m}_0$, the solution to the homogenized equation (3.2), as one of the functions in Lemma 5.4.

COROLLARY 5.1. *Suppose that* $\mathbf{m}_0$ *satisfies (A5). For* $0 \le 2\ell \le 2k \le q \le r$ *and* $\partial_t^\ell \mathbf{f}(\cdot,\cdot,t) \in H^{q-2\ell,p}$ *when* $0 \le t \le T$, *we have for all* $p \ge 0$ *and* $0 \le t \le T$

$$\|\partial_t^k(\mathbf{m}_0 \times \mathbf{f})\|_{H^{q-2k,p}} \le C \sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{f}\|_{H^{q-2\ell,p}},$$

$$\|\partial_t^k(\mathbf{m}_0 \times \mathbf{m}_0 \times \mathbf{f})\|_{H^{q-2k,p}} \le C \sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{f}\|_{H^{q-2\ell,p}},$$

where C *is independent of* $\mathbf{f}$ *and* t.

Proof. The first inequality is obtained by taking $\mathbf{u}_m = \mathbf{m}_0$, $\mathbf{v}_{m'} = \mathbf{f}$, $q = r-j$, $m = 0$ and $m' = r-q$ in Lemma 5.4, which is a valid choice due to (A5). The triple product case then follows since

$$\begin{aligned}\|\partial_t^k(\mathbf{m}_0 \times \mathbf{m}_0 \times \mathbf{f})\|_{H^{q-2k,p}} &\le C \sum_{\ell=0}^{k} \|\partial_t^{k-\ell}\mathbf{m}_0\|_{H^{r-2k+2\ell,p}} \|\partial_t^\ell(\mathbf{m}_0 \times \mathbf{f})\|_{H^{q-2\ell,p}} \\ &\le C \sum_{\ell=0}^{k} \|\partial_t^\ell(\mathbf{m}_0 \times \mathbf{f})\|_{H^{q-2\ell,p}}.\end{aligned}$$

∎

Finally, we consider the product of two functions with a maximum norm bound given for one of them. Then the following bilinear estimate holds.

LEMMA 5.5. *Suppose* $f\in H^q(\Omega)$ *and* $g\in W^{q,\infty}(\Omega)$. *Then*

$$\|fg\|_{H^q}\leq C\sum_{j=0}^{q}\|g\|_{W^{j,\infty}}\|f\|_{H^{q-j}},\qquad \|fg\|_{H^q_\varepsilon}\leq C\sum_{j=0}^{q}\varepsilon^j\|g\|_{W^{j,\infty}}\|f\|_{H^{q-j}_\varepsilon}. \tag{5.12}$$

In particular, consider $h\in C^\infty(Y)$ *and let* $h^\varepsilon=h(x/\varepsilon)$, *then it holds for* $0\leq j\leq q$ *that*

$$\|h^\varepsilon f\|_{H^j}\leq C\frac{1}{\varepsilon^j}\|h\|_{W^{j,\infty}}\|f\|_{H^j_\varepsilon},\qquad \|h^\varepsilon f\|_{H^j_\varepsilon}\leq C\|h\|_{W^{j,\infty}}\|f\|_{H^j_\varepsilon}. \tag{5.13}$$

In all cases, the constant C *is independent of* ε.

Proof. Consider first the $\|\cdot\|_{H^q}$-norm of the product. It holds that

$$\begin{aligned}\|fg\|^2_{H^q}&=\sum_{\substack{|\alpha|\leq q\\ \gamma\leq\alpha}}\binom{\alpha}{\gamma}\int|\partial^\gamma g\partial^{\alpha-\gamma}f|^2dx\\ &\leq C\sum_{\substack{|\alpha|\leq q\\ \gamma\leq\alpha}}\sup|\partial^\gamma g|^2\int|\partial^{\alpha-\gamma}f|^2dx\leq C\sum_{j=0}^{q}\|g\|^2_{W^{j,\infty}}\|f\|^2_{H^{q-j}},\end{aligned}$$

which shows the first statement. Consequently, we find

$$\begin{aligned}\|fg\|_{H^q_\varepsilon}&=\sum_{j=0}^{q}\varepsilon^j\|fg\|_{H^j}\leq C\sum_{j=0}^{q}\sum_{i=0}^{j}\varepsilon^j\|g\|_{W^{i,\infty}}\|f\|_{H^{j-i}}\leq C\sum_{j=0}^{q}\sum_{i=0}^{j}\varepsilon^i\|g\|_{W^{i,\infty}}\varepsilon^{j-i}\|f\|_{H^{j-i}}\\ &=C\sum_{i=0}^{q}\sum_{j=0}^{q-i}\varepsilon^i\|g\|_{W^{i,\infty}}\varepsilon^j\|f\|_{H^j}=C\sum_{i=0}^{q}\varepsilon^i\|g\|_{W^{i,\infty}}\|f\|_{H^{q-i}_\varepsilon}.\end{aligned}$$

When given $h\in C^\infty(Y)$, we one can bound

$$\|h^\varepsilon\|_{W^{k,\infty}}\leq\frac{\|h\|_{W^{k,\infty}}}{\varepsilon^k},\qquad k\geq 0,$$

hence the $\|\cdot\|_{H^j}$-estimate in (5.13) follows from the $\|\cdot\|_{H^q}$-estimate in (5.12),

$$\|h^\varepsilon f\|_{H^q}\leq C\sum_{j=0}^{q}\|h^\varepsilon\|_{W^{q-j,\infty}}\|f\|_{H^j}\leq C\sum_{j=0}^{q}\frac{\|h\|_{W^{q-j,\infty}}}{\varepsilon^{q-j}}\|f\|_{H^j}=\frac{C}{\varepsilon^q}\|h\|_{W^{q,\infty}}\|f\|_{H^q_\varepsilon}.$$

The $\|\cdot\|_{H^q_\varepsilon}$-estimate then is a direct consequence of (5.3). □

5.3. Norms involving the linear operator L. Consider now $a^\varepsilon(x)=a(x/\varepsilon)$ such that (A1) holds and let $L=\nabla\cdot(a^\varepsilon\nabla)$, which is the setup we consider in the rest of this paper. We then show two results, allowing us to switch between H^q_ε-norms and L^2-norms involving L. First we can estimate L^pu in terms of ∇u.

LEMMA 5.6. *Suppose* $u\in H^r(\Omega)$ *and* $a\in C^\infty(\Omega)$. *Then it holds for* $2\leq 2k\leq r-1-\ell$ *and* $0\leq q\leq r-2k$

$$\|L^ku\|_{H^q}\leq C\frac{1}{\varepsilon^{q+2k-1}}\|\nabla u\|_{H^{q+2k-1}_\varepsilon}, \tag{5.14}$$

where the constant C is independent of ε.

Proof. Let β be a multi-index with $|\beta|\le 2k$. Since $a\in C^\infty(Y)$, there exist functions $c_\beta(y)\in C^\infty(Y)$, which are either zero or consist of a product of $\partial^\gamma a(y), |\gamma|\le|\beta|$, such that

$$L^k u=\sum_{1\le|\beta|\le 2k}\frac{1}{\varepsilon^{2k-|\beta|}}c_\beta^\varepsilon\partial^\beta u,$$

where $c_\beta^\varepsilon=c_\beta(x/\varepsilon)$. It thus follows by (5.13) and (5.4) in Lemma 5.2 that

$$\begin{aligned}\|L^k u\|_{H^q}&\le C\sum_{1\le|\beta|\le 2k}\frac{1}{\varepsilon^{q+2k-|\beta|}}\left\|\partial^\beta u\right\|_{H^q_\varepsilon}\le C\frac{1}{\varepsilon^{q+2k-1}}\sum_{0\le|\nu|\le 2k-1}\varepsilon^{|\nu|}\|\partial^\nu\nabla u\|_{H^q_\varepsilon}\\&\le C\frac{1}{\varepsilon^{q+2k-1}}\|\nabla u\|_{H^{q+2k-1}_\varepsilon}.\end{aligned}$$

□

Second, we have the following multiscale version of elliptic regularity. (Note that standard elliptic regularity estimates have constants that depend on ε.)

LEMMA 5.7. *Suppose $u\in H^q(Y)$ with $q\ge 2$ and $0<\varepsilon\le 1$. Then*

$$\|u\|_{H^q}\le C\left(\|u\|_{L^2}+\frac{1}{\varepsilon^{q-1}}\|\nabla u\|_{H^{q-2}_\varepsilon}+\begin{cases}\|L^p u\|_{L^2} & q=2p,\\ \|L^p u\|_{H^1} & q=2p+1,\end{cases}\right). \tag{5.15}$$

Moreover, let $\ell\in\{0,1\}$, then it holds for $0\le 2k\le q-1-\ell$ that

$$\|u\|_{H^{2k+1+\ell}_\varepsilon}\le C\begin{cases}\varepsilon^{2k+1}\|\sqrt{a^\varepsilon}\nabla L^k u\|_{L^2}+\|u\|_{H^{2k}_\varepsilon}, & \ell=0,\\ \varepsilon^{2k+2}\|L^{k+1}u\|_{L^2}+\|u\|_{H^{2k+1}_\varepsilon}, & \ell=1,\end{cases} \tag{5.16}$$

where the constant C is independent of ε.

Proof. To show (5.15) we first prove that given a multi-index σ with $2\le|\sigma|\le q$,

$$\|\partial^\sigma u\|_{L^2}\le C\left(\frac{1}{\varepsilon^{|\sigma|-1}}\|\nabla u\|_{H^{|\sigma|-2}_\varepsilon}+\begin{cases}\|L^p u\|_{L^2} & |\sigma|=2p,\\ \|L^p u\|_{H^1} & |\sigma|=2p+1,\end{cases}\right). \tag{5.17}$$

We start by proving this for $p=1$ and $|\sigma|=2$. Then we have, with $u_k:=\partial_{x_k}u$,

$$\begin{aligned}\|D^2u\|_{L^2}^2&=:\sum_{|\sigma|=2}\|\partial^\sigma u\|_{L^2}^2=\sum_{k=1}^n\int_\Omega|\nabla u_k|^2dx\le C\sum_{k=1}^n\int_\Omega a^\varepsilon|\nabla u_k|^2dx=-\sum_{k=1}^n\int_\Omega u_kLu_kdx\\&=\int_\Omega Lu\sum_{k=1}^n\partial_{x_k}^2udx-\sum_{k=1}^n\int_\Omega u_k[Lu_k-\partial_{x_k}(Lu)]dx\\&=\int_\Omega Lu\Delta udx+\sum_{k=1}^n\int_\Omega(\nabla u_k)\cdot[a^\varepsilon\nabla u_k-\partial_{x_k}a^\varepsilon\nabla u]dx\\&=\int_\Omega Lu\Delta udx-\sum_{k=1}^n\int_\Omega\partial_{x_k}a^\varepsilon\nabla u_k\cdot\nabla udx.\end{aligned}$$

Application of Cauchy-Schwarz and Young's inequality with a constant hence yields

$$\|D^2u\|_{L^2}^2\le\frac{\gamma}{2}\|\Delta u\|_{L^2}^2+\frac{1}{2\gamma}\|Lu\|_{L^2}^2+\sum_{k=1}^n\frac{\gamma}{2}\|\nabla u_k\|_{L^2}^2+\sum_{k=1}^n\frac{1}{2\gamma}\frac{1}{\varepsilon^2}\|a\|_{W^{1,\infty}}^2\|\nabla u\|_{L^2}^2$$

$$\leq \gamma \|D^2 u\|_{L^2}^2 + \frac{1}{2\gamma}\|Lu\|_{L^2}^2 + \frac{n}{2\gamma\varepsilon^2}\|a\|_{W^{1,\infty}}^2 \|\nabla u\|_{L^2}^2,$$

for any constant $\gamma > 0$. Thus, by taking γ small enough we get

$$\|D^2 u\|_{L^2}^2 \leq C\left(\|Lu\|_{L^2}^2 + \frac{1}{\varepsilon^2}\|\nabla u\|_{L^2}^2\right),$$

from which (5.17) for $|\sigma| = 2$ follows since $\varepsilon \leq 1$. Next, we assume that (5.17) holds for $2 \leq |\sigma| \leq 2p$. Given another multi-index α, we then obtain upon applying (5.17) for $|\sigma| = 2p$ and Lemma 5.2, that

$$\begin{aligned}\|\partial^{\sigma+\alpha} u\|_{L^2} &\leq C\left(\|L^p \partial^\alpha u\|_{L^2} + \frac{1}{\varepsilon^{2p-1}}\|\partial^\alpha \nabla u\|_{H_\varepsilon^{2p-2}}\right)\\ &\leq C\left(\|\partial^\alpha L^p u\|_{L^2} + \|\partial^\alpha L^p u - L^p \partial^\alpha u\|_{L^2} + \frac{1}{\varepsilon^{2p+|\alpha|-1}}\|\nabla u\|_{H_\varepsilon^{2p+|\alpha|-2}}\right).\end{aligned}$$

Expressing $L^p u$ involving some smooth functions $c_\beta^\varepsilon(x) = c_\beta(x/\varepsilon)$, as in the proof of Lemma 5.6, we can write

$$\partial^\alpha L^p u = \sum_{\substack{1 \leq |\beta| \leq 2p\\ 0 \leq \gamma \leq \alpha}} \binom{\alpha}{\gamma} \frac{1}{\varepsilon^{2p-|\beta|+|\gamma|}} (\partial^\gamma c_\beta^\varepsilon)\partial^{\beta+\alpha-\gamma} u.$$

Therefore, it holds that

$$\|\partial^\alpha L^p u - L^p \partial^\alpha u\|_{L^2} \leq C \sum_{\substack{1 \leq |\beta| \leq 2p\\ 1 \leq |\gamma| \leq |\alpha|}} \frac{1}{\varepsilon^{2p-|\beta|+|\gamma|}}\|\partial^{\beta+\alpha-\gamma} u\|_{L^2} \leq \frac{C}{\varepsilon^{2p+|\alpha|-1}}\|\nabla u\|_{H_\varepsilon^{2p+|\alpha|-2}},$$

and thus we have in total

$$\|\partial^{\sigma+\alpha} u\|_{L^2} \leq C\left(\|\partial^\alpha L^p u\|_{L^2} + \frac{1}{\varepsilon^{2p+|\alpha|-1}}\|\nabla u\|_{H_\varepsilon^{2p+|\alpha|-2}}\right).$$

When $|\alpha| = 1$ we then get (5.17) with $|\sigma| = 2p+1$ by noting that

$$\|\partial^\alpha L^p u\|_{L^2} \leq C\|L^p u\|_{H^1}.$$

On the other hand, when $|\alpha| = 2$, we get with one more application of (5.17) and Lemma 5.6,

$$\|\partial^\alpha L^p u\|_{L^2} \leq C\left(\|L^{p+1} u\|_{L^2} + \frac{1}{\varepsilon}\|\nabla L^p u\|_{L^2}\right) \leq C\left(\|L^{p+1} u\|_{L^2} + \frac{1}{\varepsilon^{2p+1}}\|\nabla u\|_{H_\varepsilon^{2p}}\right).$$

This completes the induction step and proves (5.17). To finally prove (5.15) we use (5.17) together with Lemma 5.6, and note that for $2 \leq |\sigma| \leq q-1$,

$$\|\partial^\sigma u\|_{L^2} \leq C\left(\frac{1}{\varepsilon^{q-2}}\|\nabla u\|_{H_\varepsilon^{q-3}} + \begin{cases}\frac{1}{\varepsilon^{2p-1}}\|\nabla u\|_{H_\varepsilon^{2p-1}}, & |\sigma| = 2p,\\ \frac{1}{\varepsilon^{2p}}\|\nabla u\|_{H_\varepsilon^{2p}}, & |\sigma| = 2p+1,\end{cases}\right) \leq \frac{C}{\varepsilon^{q-2}}\|\nabla u\|_{H_\varepsilon^{q-2}},$$

which clearly also holds for $|\sigma| = 1$. Hence,

$$\|u\|_{H^q} \leq C\sum_{|\sigma|=0}^{q}\|\partial^\sigma u\|_{L^2} \leq C\left(\|u\|_{L^2} + \frac{1}{\varepsilon^{q-2}}\|\nabla u\|_{H_\varepsilon^{q-2}} + \sum_{|\sigma|=q}\|\partial^\sigma u\|_{L^2}\right) \tag{5.18}$$

which together with (5.17) gives (5.15).

To finally prove (5.16), we consider odd and even indices in the sum in $\|\cdot\|_{H_\varepsilon^{2k+1+\ell}}$ separately and use elliptic regularity as given by (5.15), which results in

$$
\begin{aligned}
\|u\|_{H_\varepsilon^{2k+1+\ell}} &= \sum_{j=0}^{2k+1+\ell} \varepsilon^j \|u\|_{H^j} = \sum_{j=0}^{k+\ell} \varepsilon^{2j}\|u\|_{H^{2j}} + \sum_{j=0}^{k} \varepsilon^{2j+1}\|u\|_{H^{2j+1}} \\
&\le C\left(\sum_{j=0}^{k+\ell}\varepsilon^{2j}\|L^j u\|_{L^2} + \sum_{j=0}^{k}\varepsilon^{2j+1}\|L^j u\|_{H^1} + \varepsilon\|\nabla u\|_{H_\varepsilon^{2(k+\ell)-2}} + \varepsilon\|\nabla u\|_{H_\varepsilon^{2k-1}}\right) \\
&\le C\left(\sum_{j=0}^{k+\ell}\varepsilon^{2j}\|L^j u\|_{L^2} + \sum_{j=0}^{k}\varepsilon^{2j+1}\|\nabla L^j u\|_{L^2} + \varepsilon\|\nabla u\|_{H_\varepsilon^{2k-1+\ell}}\right).
\end{aligned}
$$

Application of Lemma 5.6 to all but the highest order terms in each sum together with Lemma 5.2 then yields

$$
\begin{aligned}
\|u\|_{H_\varepsilon^{2k+1+\ell}} &\le C\left(\varepsilon^{2(k+\ell)}\|L^{k+\ell}u\|_{L^2} + \varepsilon^{2k+1}\|\nabla L^k u\|_{L^2} + \varepsilon\|\nabla u\|_{H_\varepsilon^{2k+\ell-1}}\right) \\
&\le C\begin{cases} \varepsilon^{2k+1}\|\nabla L^k u\|_{L^2} + \|u\|_{H_\varepsilon^{2k}}, & \ell = 0, \\ \varepsilon^{2k+2}\|L^{k+1}u\|_{L^2} + \|u\|_{H_\varepsilon^{2k+1}}, & \ell = 1. \end{cases}
\end{aligned}
$$

Using the fact that $a_{\min} \le a \le a_{\max}$ we then obtain the result in the lemma. □

5.4. Application of $\mathcal{L}$ to a cross product. The next lemma is based on ideas from [24] but has to be significantly adapted for the problem considered here. We consider ε-dependent functions $\mathbf{u}$ and $\mathbf{f}$, where we assume that $\mathbf{f} \in W^{q+2k-1,\infty}(\Omega)$ such that its $\|\cdot\|_{W^j}$ norm is bounded in terms of ε. We show that when applying the operator $\mathcal{L}^k$ to the cross product of either $\mathbf{u}$ or $\mathbf{f}$ and $\mathcal{L}\mathbf{u}$, one can factor out the highest order term and obtains a remainder term that is bounded in terms of the $\|\cdot\|_{H_\varepsilon^{q+2k}}$-norm of the gradient of $\mathbf{u}$. Again we assume that (A1) is true.

LEMMA 5.8. *Given $k \ge 0, q \ge 0$, suppose $\mathbf{u} \in H^{q+2k+1}(\Omega)$ and $\mathbf{f} \in W^{q+2k-1,\infty}(\Omega)$ such that*

$$
\|\boldsymbol{\nabla}\mathbf{u}\|_{L^\infty} \le M, \qquad \|\mathbf{f}\|_{W^{j,\infty}} \le \tilde{M}\left(1+\varepsilon^{1-j}\right), \qquad 0 \le j \le q+2k-1, \tag{5.19}
$$

for constants M and $\tilde{M}$ independent of ε. Then it holds for $\mathbf{w} \in \{\mathbf{u},\mathbf{f}\}$ that

$$
\mathcal{L}^k(\mathbf{w}\times\mathcal{L}\mathbf{u}) = \mathbf{w}\times\mathcal{L}^{k+1}\mathbf{u} + \mathbf{R}_{k,\mathbf{w}}, \qquad \text{where} \qquad \|\mathbf{R}_{k,\mathbf{w}}\|_{H^q} \le C\frac{1}{\varepsilon^{q+2k}}\|\boldsymbol{\nabla}\mathbf{u}\|_{H_\varepsilon^{q+2k}},
$$

for a constant C independent of ε.

Proof. When $k=0$, the claim in the lemma is trivially true with $\mathbf{R}_{0,\mathbf{w}} = 0$. Let $\mathbf{w}$ be either $\mathbf{u}$ or $\mathbf{f}$, then it holds for $k > 0$ that

$$
\begin{aligned}
\mathcal{L}^k(\mathbf{w}\times\mathcal{L}\mathbf{u}) &= \mathcal{L}^{k-1}(\mathbf{w}\times\mathcal{L}^2\mathbf{u} + \mathcal{L}\mathbf{w}\times\mathcal{L}\mathbf{u} + 2a\sum_{j=1}^n \partial_{x_j}\mathbf{w}\times\partial_{x_j}\mathcal{L}\mathbf{u}) \\
&= \mathbf{w}\times\mathcal{L}^{k+1}\mathbf{u} + \sum_{\ell=1}^{k}\mathcal{L}^{k-\ell}\left(\mathcal{L}\mathbf{w}\times\mathcal{L}^\ell\mathbf{u} + 2a\sum_{j=1}^n \partial_{x_j}\mathbf{w}\times\partial_{x_j}\mathcal{L}^\ell\mathbf{u}\right),
\end{aligned}
$$

which implies that $\mathbf{R}_{k,\mathbf{w}}$ in the lemma is given by

$$\mathbf{R}_k =: \sum_{\ell=1}^{k} \mathcal{L}^{k-\ell}\mathbf{r}_\ell(\mathbf{w}) \quad \text{and} \quad \mathbf{r}_\ell(\mathbf{w}) := \mathcal{L}\mathbf{w}\times\mathcal{L}^\ell\mathbf{u} + 2a\sum_{j=1}^{n}\partial_{x_j}\mathbf{w}\times\partial_{x_j}\mathcal{L}^\ell\mathbf{u}$$

In the following, we obtain bounds for $\|\mathbf{R}_{k,\mathbf{w}}\|_{H^q}$, first for $\mathbf{w}=\mathbf{u}$ and then later for $\mathbf{w}=\mathbf{f}$. For the first estimate, we use the fact that according to assumption (A1), there exist functions $c_{\beta,\gamma}(y)\in C^\infty(\Omega)$, similar to the ones in the proof of Lemma 5.6, which might also be zero, such that

$$\mathcal{L}\mathbf{u}\times\mathcal{L}^\ell\mathbf{u} + 2a\sum_{j=1}^{n}\partial_{x_j}\mathbf{u}\times\partial_{x_j}\mathcal{L}^\ell\mathbf{u} = \frac{1}{\varepsilon^{2\ell}}\sum_{\substack{1\le|\beta|,1\le|\gamma|\\|\beta+\gamma|\le 2+2\ell}} c_{\beta,\gamma}\left(\frac{x}{\varepsilon}\right)\varepsilon^{|\beta|+|\gamma|-2}\left(\partial^\gamma\mathbf{u}\times\partial^\beta\mathbf{u}\right).$$

Furthermore, it is a consequence of the interpolation inequality (5.5) that given multi-indices β and γ with $|\beta|\ge 1$, $|\gamma|\ge 1$,

$$\|\partial^\gamma\mathbf{u}\times\partial^\beta\mathbf{u}\|_{H^j} \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^{j+|\beta|+|\gamma|-2}}, \qquad 0\le j\le q,$$

wherefore we find proceeding as in the proof of Lemma 5.2 that

$$\begin{aligned}\varepsilon^{|\beta|+|\gamma|-2}\|\partial^\gamma\mathbf{u}\times\partial^\beta\mathbf{u}\|_{H^q_\varepsilon} &\le C\|\nabla\mathbf{u}\|_{L^\infty}\sum_{j=0}^{q}\varepsilon^{j+|\beta|+|\gamma|-2}\|\nabla\mathbf{u}\|_{H^{j+|\beta|+|\gamma|-2}}\\ &\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q+|\beta|+|\gamma|-2}}.\end{aligned} \tag{5.20}$$

Therefore, it follows by (5.13) in Lemma 5.5 and (5.20) that

$$\|\mathbf{r}_\ell(\mathbf{u})\|_{H^q} \le \frac{C}{\varepsilon^{q+2\ell}}\sum_{\substack{1\le|\beta|,1\le|\gamma|\\|\beta+\gamma|\le 2+2\ell}}\varepsilon^{|\beta|+|\gamma|-2}\|\partial^\gamma\mathbf{u}\times\partial^\beta\mathbf{u}\|_{H^q_\varepsilon} \le \frac{C}{\varepsilon^{q+2\ell}}\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q+2\ell}},$$

and we obtain using Lemma 5.6 and (5.3) in Lemma 5.2 that

$$\left\|\mathcal{L}^{k-\ell}\mathbf{r}_\ell(\mathbf{u})\right\|_{H^q} \le \frac{C}{\varepsilon^{q+2k-2\ell}}\|\mathbf{r}_\ell(\mathbf{u})\|_{H_\varepsilon^{q+2k-2\ell}} \le \frac{C}{\varepsilon^{q+2k}}\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q+2k}}.$$

This shows that the norm of $\mathbf{R}_{k,\mathbf{u}}$ can be bounded as stated in the lemma.

In case of $\mathbf{w}=\mathbf{f}$, the estimate is based on (5.12) in Lemma 5.5 and the fact that (5.19) holds for $\mathbf{f}$. When applying Lemma 5.6 and using these bounds, we find that given $q'=q+2k-2\ell$ and a multi-index γ with $|\gamma|=1$,

$$\|\mathcal{L}\mathbf{f}\times\mathcal{L}^\ell\mathbf{u}\|_{H_\varepsilon^{q'}} \le C\sum_{j=0}^{q'}\varepsilon^j\|\mathcal{L}\mathbf{f}\|_{W^{j,\infty}}\|\mathcal{L}^\ell\mathbf{u}\|_{H_\varepsilon^{q'-j}} \le \frac{C}{\varepsilon}\|\mathcal{L}^\ell\mathbf{u}\|_{H_\varepsilon^{q'}} \le C\varepsilon^{-2\ell}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q'+2\ell-1}},$$

$$\|\partial^\gamma\mathbf{f}\times\partial^\gamma\mathcal{L}^\ell\mathbf{u}\|_{H_\varepsilon^{q'}} \le C\sum_{j=0}^{q'}\varepsilon^j\|\partial^\gamma\mathbf{f}\|_{W^{j,\infty}}\|\partial^\gamma\mathcal{L}^\ell\mathbf{u}\|_{H^{q'-j}} \le C\varepsilon^{-2\ell}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q'+2\ell}}.$$

Hence, it holds that

$$\|\mathcal{L}^{k-\ell}(\mathcal{L}\mathbf{f}\times\mathcal{L}^\ell\mathbf{u})\|_{H^q} \le C\frac{1}{\varepsilon^{q+2k-2\ell}}\|\mathcal{L}\mathbf{f}\times\mathcal{L}^\ell\mathbf{u}\|_{H_\varepsilon^{q+2k-2\ell}} \le C\frac{1}{\varepsilon^{q+2k}}\|\nabla\mathbf{u}\|_{H_\varepsilon^{q+2k-1}},$$

as well as

$$\left\|\mathcal{L}^{k-\ell}\left(a\sum_{i=1}^{n}\partial_{x_i}\mathbf{f}\times\partial_{x_i}\mathcal{L}^{\ell}\mathbf{u}\right)\right\|_{H^q}\leq C\frac{1}{\varepsilon^{q+2k-2\ell}}\sum_{|\gamma|=1}\|\partial^{\gamma}\mathbf{f}\times\partial^{\gamma}\mathcal{L}^{\ell}\mathbf{u}\|_{H_\varepsilon^{q+2k-2\ell}}$$
$$\leq C\frac{1}{\varepsilon^{q+2k}}\|\boldsymbol{\nabla}\mathbf{u}\|_{H_\varepsilon^{q+2k}}.$$

Thus, $\mathbf{R}_{k,\mathbf{f}}$ can be bounded in the same way as $\mathbf{R}_{k,\mathbf{u}}$. This completes the proof. □

6. Stability estimate

In this section, we derive a stability estimate for the error introduced when approximating $\mathbf{m}^\varepsilon$ satisfying the Landau-Lifshitz equation, (3.1), by $\tilde{\mathbf{m}}^\varepsilon$ that satisfies a perturbed version of the equation,

$$\partial_t\tilde{\mathbf{m}}^\varepsilon=-\tilde{\mathbf{m}}^\varepsilon\times\mathcal{L}\tilde{\mathbf{m}}^\varepsilon-\alpha\tilde{\mathbf{m}}^\varepsilon\times\tilde{\mathbf{m}}^\varepsilon\times\mathcal{L}\tilde{\mathbf{m}}^\varepsilon-\boldsymbol{\eta}^\varepsilon,\quad 0\leq t\leq T^\varepsilon, \tag{6.1}$$

where we recall that $T^\varepsilon=\varepsilon^\sigma T$ some some $\sigma\in[0,2]$. In particular, we suppose that the assumptions (A1)-(A4) hold and that initially, $\tilde{\mathbf{m}}^\varepsilon(x,0)=\mathbf{m}^\varepsilon(x,0)$. Moreover, we assume that $\tilde{\mathbf{m}}^\varepsilon\in C([0,T^\varepsilon];W^{q+1,\infty}(\Omega)$ and that there is a constant $\tilde{M}$ such that

$$\|\tilde{\mathbf{m}}^\varepsilon(\cdot,t)\|_{W^{k,\infty}}\leq\tilde{M}\left(1+\frac{1}{\varepsilon^{k-1}}\right),\qquad 0\leq k\leq q+1, \tag{6.2}$$

for $0\leq t\leq T^\varepsilon$, uniformly in ε. Note that this assumption is chosen such that it fits with the estimates that will be shown in Section 7. We can then prove the following stability estimate for the difference between $\mathbf{m}^\varepsilon$ and $\tilde{\mathbf{m}}^\varepsilon$.

Theorem 6.1. *Assume (A1) - (A4) hold and let $q\leq s$ as given in (A4). Suppose $\tilde{\mathbf{m}}^\varepsilon\in C^1([0,T^\varepsilon];W^{q+1,\infty}(\Omega))$ is the solution to (6.1) such that (6.2) holds and $\boldsymbol{\eta}^\varepsilon(\cdot,t)\in H^q(\Omega)$ for $0\leq t\leq T^\varepsilon$. Then there is a constant C independent of ε but dependent on T and a, such that the error $\mathbf{e}:=\mathbf{m}^\varepsilon-\tilde{\mathbf{m}}^\varepsilon$ satisfies,*

$$\|\mathbf{e}(\cdot,t)\|_{H^q}^2\leq Ct\sup_{0\leq\zeta\leq t}\frac{1}{\varepsilon^{2q}}\left(\|\boldsymbol{\eta}^\varepsilon(\cdot,\zeta)\|_{H_\varepsilon^q}^2+\|\nabla|\tilde{\mathbf{m}}^\varepsilon(\cdot,\zeta)|^2\|_{H_\varepsilon^q}^2\right),\qquad 0\leq t\leq T^\varepsilon. \tag{6.3}$$

To prove this theorem, we first derive a differential equation for $\mathbf{e}$. Then an estimate for $\|\mathbf{e}\|_{L^2}$ is shown, since the proof in that case is somewhat different then for higher order norms. Finally, we complete the section by using induction to show that (6.3) holds for general q. Note that these proofs are based on ideas from [24]. For better readability, we drop the superscript ε for $\mathbf{m}$ and $\boldsymbol{\eta}$ in the rest of this section, keeping in mind that they are ε-dependent. However, we keep the notation a^ε to stress that the constants in the estimates depend on norms of a, but not a^ε.

To obtain a differential equation for $\mathbf{e}:=\mathbf{m}-\tilde{\mathbf{m}}$, let $\mathbf{m}$ and $\tilde{\mathbf{m}}$ satisfy (3.1) and (6.1), respectively. Then $\mathbf{e}$ is the solution to

$$\partial_t\mathbf{e}=\mathbf{D}_1+\alpha(\mathcal{L}\mathbf{e}+\mathbf{D}_2+\mathbf{D}_3)+\boldsymbol{\eta}, \tag{6.4}$$

where $\mathbf{D}_1$ is the difference between the precession terms in (3.1) and (6.1),

$$\mathbf{D}_1:=-\mathbf{m}\times\mathcal{L}\mathbf{m}+\tilde{\mathbf{m}}\times\mathcal{L}\tilde{\mathbf{m}}=-\mathbf{m}\times\mathcal{L}\mathbf{e}-\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}}, \tag{6.5}$$

and $\mathbf{D}_2$ and $\mathbf{D}_3$ arise when taking the difference of the damping terms,

$$-\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m}+\tilde{\mathbf{m}}\times\tilde{\mathbf{m}}\times\mathcal{L}\tilde{\mathbf{m}}=\mathcal{L}\mathbf{m}|\mathbf{m}|^2-\mathcal{L}\tilde{\mathbf{m}}|\tilde{\mathbf{m}}|^2+a^\varepsilon\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|^2$$

$$-a^\varepsilon\tilde{\mathbf{m}}|\nabla\tilde{\mathbf{m}}|^2+\nabla\cdot(a^\varepsilon\tilde{\mathbf{m}}\cdot\nabla\tilde{\mathbf{m}})\tilde{\mathbf{m}}=\mathcal{L}\mathbf{e}+\mathbf{D}_2+\mathbf{D}_3\,,$$

where

$$\mathbf{D}_2:=(\mathbf{e}\cdot(\mathbf{m}+\tilde{\mathbf{m}}))\mathcal{L}\tilde{\mathbf{m}}+a^\varepsilon\mathbf{e}|\nabla\mathbf{m}|^2+a^\varepsilon\tilde{\mathbf{m}}(\nabla\mathbf{e}:\nabla(\mathbf{m}+\tilde{\mathbf{m}})),\tag{6.6a}$$

$$\mathbf{D}_3:=\nabla\cdot(a^\varepsilon\tilde{\mathbf{m}}\cdot\nabla\tilde{\mathbf{m}})\tilde{\mathbf{m}}=\frac{1}{2}L|\tilde{\mathbf{m}}|^2\tilde{\mathbf{m}}.\tag{6.6b}$$

Note that by assumption, $|\mathbf{m}|^2=1$, constant in time and space, but $|\tilde{\mathbf{m}}|^2$ is not constant, therefore the remainder term involving only $\tilde{\mathbf{m}}$, $\mathbf{D}_3$, does not vanish.

6.1. **L^2-estimate.** To obtain an estimate for the change in the norm of the error $\mathbf{e}$, we multiply (6.4) by $\mathbf{e}$ and integrate in space, which yields

$$\begin{aligned}\frac{1}{2}\partial_t\|\mathbf{e}\|_{L^2}^2&=\int_\Omega\mathbf{e}\cdot\partial_t\mathbf{e}=\int_\Omega\mathbf{e}\cdot\mathbf{D}_1dx+\alpha\int_\Omega\mathbf{e}\cdot(\mathcal{L}\mathbf{e}+\mathbf{D}_2+\mathbf{D}_3)dx+\int_\Omega\mathbf{e}\cdot\boldsymbol{\eta}dx\\&=\int_\Omega\mathbf{e}\cdot\mathbf{D}_1dx-\alpha\int_\Omega a^\varepsilon\nabla\mathbf{e}:\nabla\mathbf{e}dx+\alpha\int_\Omega\mathbf{e}\cdot(\mathbf{D}_2+\mathbf{D}_3)dx+\int_\Omega\mathbf{e}\cdot\boldsymbol{\eta}dx\,.\end{aligned}$$

It thus holds that

$$\frac{1}{2}\partial_t\|\mathbf{e}\|_{L^2}^2+\alpha\|\sqrt{a^\varepsilon}\nabla\mathbf{e}\|_{L^2}^2=\mathbf{I}_1+\alpha(\mathbf{I}_2+\mathbf{I}_3)+\int_\Omega\mathbf{e}\cdot\boldsymbol{\eta}dx,\tag{6.7}$$

where we define for the sake of notation,

$$\mathbf{I}_k:=\int_\Omega\mathbf{e}\cdot\mathbf{D}_kdx,\qquad k=1,2,3.$$

Our goal in the following then is to derive bounds for the integrals $\mathbf{I}_k$ that only depend on the L^2-norms of $\mathbf{e}$ and $\sqrt{a^\varepsilon}\nabla\mathbf{e}$, multiplied by a suitable constant that we can choose such that the terms involving $\sqrt{a^\varepsilon}\nabla\mathbf{e}$ on the left- and right-hand side cancel. This makes it possible to use Grönwall's inequality to obtain (6.3) for $q=0$. Using the fact that the cross product of a vector by itself is zero, $\mathbf{D}_1$ can be rewritten as

$$\mathbf{D}_1=-\tilde{\mathbf{m}}\times\mathcal{L}\mathbf{e}-\mathbf{e}\times\mathcal{L}\mathbf{m}=-\nabla\cdot(\mathbf{e}\times a^\varepsilon\nabla\mathbf{m}+\tilde{\mathbf{m}}\times a^\varepsilon\nabla\mathbf{e})\,.\tag{6.8}$$

Applying integration by parts and the identity (2.3), we then find that due to orthogonality,

$$\mathbf{I}_1=-\int_\Omega\mathbf{e}\cdot[\nabla\cdot(\mathbf{e}\times a^\varepsilon\nabla\mathbf{m}+\tilde{\mathbf{m}}\times a^\varepsilon\nabla\mathbf{e})]\,dx=\int_\Omega a^\varepsilon\nabla\mathbf{e}:(\mathbf{e}\times\nabla\mathbf{m})dx\,.$$

Therefore one can bound the first integral as

$$|\mathbf{I}_1|\le\|\sqrt{a^\varepsilon}\nabla\mathbf{e}\|_{L^2}\|\mathbf{e}\|_{L^2}\|\sqrt{a^\varepsilon}\nabla\mathbf{m}\|_\infty\le\frac{\gamma}{2}\|\sqrt{a^\varepsilon}\nabla\mathbf{e}\|_{L^2}^2+\frac{a_{\max}M^2}{2\gamma}\|\mathbf{e}\|_{L^2}^2\,.\tag{6.9}$$

For the second integral, we have according to the definition of $\mathbf{D}_2$, (6.6a),

$$\begin{aligned}\mathbf{I}_2=&\int_\Omega a^\varepsilon|\mathbf{e}|^2|\nabla\mathbf{m}|^2dx+\int_\Omega a^\varepsilon(\mathbf{e}\cdot\tilde{\mathbf{m}})(\nabla\mathbf{e}:\nabla(\mathbf{m}+\tilde{\mathbf{m}}))dx\\&-\int_\Omega a^\varepsilon\nabla(\mathbf{e}(\mathbf{e}\cdot(\mathbf{m}+\tilde{\mathbf{m}}))):\nabla\tilde{\mathbf{m}}dx,\end{aligned}$$

where we used integration by parts on the last term. Applying Cauchy-Schwarz to these integrals yields

$$\left|\int_\Omega a^\varepsilon(\mathbf{e}\cdot\tilde{\mathbf{m}})(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}(\mathbf{m}+\tilde{\mathbf{m}}))dx\right| \le \|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{e}\|_{L^2}\|\mathbf{e}\|_{L^2}\|\sqrt{a^\varepsilon}\tilde{\mathbf{m}}\|_\infty\|\boldsymbol{\nabla}(\mathbf{m}+\tilde{\mathbf{m}})\|_\infty$$

and similarly,

$$\begin{aligned}&\left|\int_\Omega a^\varepsilon\boldsymbol{\nabla}(\mathbf{e}(\mathbf{e}\cdot(\mathbf{m}+\tilde{\mathbf{m}}))):\boldsymbol{\nabla}\tilde{\mathbf{m}}dx\right|\\ &\qquad\le\sqrt{a_{\max}}\|\tilde{\mathbf{m}}\|_{L^\infty}\|\mathbf{m}+\tilde{\mathbf{m}}\|_{W^{1,\infty}}\left(\|\mathbf{e}\|_{L^2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{e}\|_{L^2}+\|\mathbf{e}\|_{L^2}^2\right).\end{aligned}$$

From Young's inequality with a constant together with the bounds (6.2) and using assumption (A4), we thus obtain for $t\in[0,T^\varepsilon]$,

$$|\mathbf{I}_2|\le\frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{e}\|_{L^2}^2+a_{\max}\left(\frac{1}{2\gamma}+1\right)(M^2\tilde{M}^2+\tilde{M}^4)\|\mathbf{e}\|_{L^2}^2,\qquad\text{for all }\gamma>0.\tag{6.10}$$

In order to derive a bound for $\mathbf{I}_3$, note first that since $\mathbf{m}\cdot\boldsymbol{\nabla}\mathbf{m}=\mathbf{0}$, it holds that

$$\nabla(\mathbf{e}\cdot\tilde{\mathbf{m}})=(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla}\mathbf{e}-\mathbf{m}\cdot\boldsymbol{\nabla}\mathbf{e}-\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla}\tilde{\mathbf{m}})^T=-(\mathbf{e}\cdot\boldsymbol{\nabla}\mathbf{e})^T-\frac{1}{2}\nabla|\tilde{\mathbf{m}}|^2,$$

which implies that

$$\begin{aligned}\mathbf{I}_3&=\frac{1}{2}\int_\Omega(\mathbf{e}\cdot\tilde{\mathbf{m}})L|\tilde{\mathbf{m}}|^2dx=-\frac{1}{2}\int_\Omega a^\varepsilon\nabla(\mathbf{e}\cdot\tilde{\mathbf{m}})\cdot\nabla|\tilde{\mathbf{m}}|^2dx\\&=\frac{1}{2}\int_\Omega a^\varepsilon(\mathbf{e}\cdot\boldsymbol{\nabla}\mathbf{e})^T\cdot\nabla|\tilde{\mathbf{m}}|^2dx+\frac{1}{4}\|\sqrt{a^\varepsilon}\nabla|\tilde{\mathbf{m}}|^2\|_{L^2}^2.\end{aligned}$$

It then follows that for any $\gamma>0$,

$$|\mathbf{I}_3|\le\frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{e}\|_{L^2}^2+Ca_{\max}\left(\frac{1}{2\gamma}\tilde{M}^2\|\mathbf{e}\|_{L^2}^2+\|\nabla|\tilde{\mathbf{m}}|^2\|_{L^2}^2\right).\tag{6.11}$$

The last integral in (6.7) can be directly bounded using Cauchy-Schwarz and Young,

$$\int_\Omega\mathbf{e}\cdot\boldsymbol{\eta}\,dx\le C(\|\mathbf{e}\|_{L^2}^2+\|\boldsymbol{\eta}\|_{L^2}^2).\tag{6.12}$$

Putting (6.9), (6.10) and (6.11) into (6.7) then yields, upon choosing γ sufficiently small,

$$\partial_t\|\mathbf{e}\|_{L^2}^2\le C\left(\frac{M^2}{\gamma}\|\mathbf{e}\|_{L^2}^2+\|\nabla|\tilde{\mathbf{m}}|^2\|_{L^2}^2+\|\boldsymbol{\eta}\|_{L^2}^2\right),\qquad 0\le t\le T^\varepsilon,$$

for some C independent of ε and t. As $\mathbf{e}(0)=0$, it follows by Grönwall's inequality that

$$\|\mathbf{e}(\cdot,t)\|_{L^2}^2\le ce^{C(M^2/\gamma)T^\varepsilon}\int_0^t\|\boldsymbol{\eta}(\cdot,s)\|_{L^2}^2+\|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|_{L^2}^2ds,\qquad 0\le t\le T^\varepsilon,\tag{6.13}$$

where the prefactor can be taken independent of ε as $T^\varepsilon\le T$. This proves the estimate in Theorem 6.1 for $q=0$.

6.2. Higher-order estimates. In this section, we show estimates for $\|\mathbf{e}\|_{H^q}$, $q>0$ to complete the proof of Theorem 6.1. The general structure of these estimates is similar to the L^2-estimate. However, we include an induction argument to obtain the final result. Furthermore, bounds for the H^q-norms of $\mathbf{D}_2$ are required to complete the proof. These are given in the following lemma.

LEMMA 6.1. *Let $\mathbf{D}_2$ be given by* (6.6a) *and suppose that $\mathbf{e}\in H^{q+1}(\Omega)$ and that there is a constant C independent of ε such that $\|\mathbf{e}\|_\infty \le C$ and $\|\boldsymbol{\nabla}\mathbf{e}\|_\infty \le C$. Then it holds that*

$$\|\mathbf{D}_2\|_{H^q} \le \frac{1}{\varepsilon^{q+1}}\|\mathbf{e}\|_{H^{q+1}_\varepsilon}.$$

Proof. First, note that we can use (5.6) to bound for $\ell \le q$,

$$\| |\mathbf{e}|^2 \|_{H^\ell} \le C\|\mathbf{e}\|_{L^\infty}\|\mathbf{e}\|_{H^\ell}, \qquad \| |\boldsymbol{\nabla}\mathbf{e}|^2 \|_{H^\ell} \le C\|\boldsymbol{\nabla}\mathbf{e}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{e}\|_{H^\ell}. \tag{6.14}$$

Using the fact that $\mathbf{m}=\mathbf{e}+\tilde{\mathbf{m}}$, we can moreover show that

$$\begin{aligned}\mathbf{D}_2 = &\mathcal{L}\tilde{\mathbf{m}}(|\mathbf{e}|^2+2(\mathbf{e}\cdot\tilde{\mathbf{m}})) + a^\varepsilon\mathbf{e}(|\boldsymbol{\nabla}\tilde{\mathbf{m}}|^2+|\boldsymbol{\nabla}\mathbf{e}|^2)\\ &+ a^\varepsilon\tilde{\mathbf{m}}(|\boldsymbol{\nabla}\mathbf{e}|^2+2(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}\tilde{\mathbf{m}})) + 2a^\varepsilon\mathbf{e}(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}\tilde{\mathbf{m}}),\end{aligned}$$

where the last term satisfies

$$|a^\varepsilon\mathbf{e}(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}\tilde{\mathbf{m}})| \le C|a^\varepsilon\mathbf{e}|\boldsymbol{\nabla}\mathbf{e}|^2 + a^\varepsilon\mathbf{e}|\boldsymbol{\nabla}\tilde{\mathbf{m}}|^2|.$$

Thus, it holds according to (5.13) in Lemma 5.5 that

$$\|\mathbf{D}_2\|_{H^q} \le (\|\mathbf{D}_{21}\|_{H^q} + \|a^\varepsilon\mathbf{D}_{22}\|_{H^q}) \le C\left(\|\mathbf{D}_{21}\|_{H^q} + \frac{1}{\varepsilon^q}\|\mathbf{D}_{22}\|_{H^q_\varepsilon}\right), \tag{6.15}$$

where we let

$$\begin{aligned}\mathbf{D}_{21} &:= \mathcal{L}\tilde{\mathbf{m}}|\mathbf{e}|^2 + \mathcal{L}\tilde{\mathbf{m}}(\mathbf{e}\cdot\tilde{\mathbf{m}}),\\ \mathbf{D}_{22} &:= \mathbf{e}|\boldsymbol{\nabla}\tilde{\mathbf{m}}|^2 + \mathbf{e}|\boldsymbol{\nabla}\mathbf{e}|^2 + \tilde{\mathbf{m}}|\boldsymbol{\nabla}\mathbf{e}|^2 + \tilde{\mathbf{m}}(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}\tilde{\mathbf{m}}).\end{aligned}$$

For the norms of the terms involved in $\mathbf{D}_{21}$, it holds by Lemma 5.5 and (6.14) that

$$\begin{aligned}\|\mathcal{L}\tilde{\mathbf{m}}|\mathbf{e}|^2\|_{H^q} &\le C\sum_{j=0}^{q}\|\mathcal{L}\tilde{\mathbf{m}}\|_{W^{q-j,\infty}}\||\mathbf{e}|^2\|_{H^j} \le C\sum_{j=0}^{q}\|\mathcal{L}\tilde{\mathbf{m}}\|_{W^{q-j,\infty}}\|\mathbf{e}\|_\infty\|\mathbf{e}\|_{H^j},\\ \|\mathcal{L}\tilde{\mathbf{m}}(\mathbf{e}\cdot\tilde{\mathbf{m}})\|_{H^q} &\le C\sum_{j=0}^{q}\sum_{i=0}^{j}\|\mathcal{L}\tilde{\mathbf{m}}\|_{W^{q-j,\infty}}\|\tilde{\mathbf{m}}\|_{W^{j-i,\infty}}\|\mathbf{e}\|_{H^i},\end{aligned}$$

which together with the assumption on the boundedness of $\tilde{\mathbf{m}}$, (6.2), implies that

$$\|\mathbf{D}_{21}\|_{H^q} \le C\sum_{j=0}^{q}\left(\frac{1}{\varepsilon^{q-j+1}}\|\mathbf{e}\|_{H^j} + \sum_{i=0}^{j-1}\frac{1}{\varepsilon^{q-i}}\|\mathbf{e}\|_{H^i}\right) \le C\frac{1}{\varepsilon^{q+1}}\|\mathbf{e}\|_{H^q_\varepsilon}. \tag{6.16}$$

Again using Lemma 5.5 and (6.14), we can furthermore show that the norms involved in $\mathbf{D}_{22}$ satisfy

$$\|\mathbf{e}|\boldsymbol{\nabla}\tilde{\mathbf{m}}|^2\|_{H^q} \le C\sum_{j=0}^{q}\sum_{i=0}^{j}\|\boldsymbol{\nabla}\tilde{\mathbf{m}}\|_{W^{q-j-i,\infty}}\|\boldsymbol{\nabla}\tilde{\mathbf{m}}\|_{W^{i,\infty}}\|\mathbf{e}\|_{H^j},$$

$$\|\tilde{\mathbf{m}}|\boldsymbol{\nabla}\mathbf{e}|^2\|_{H^q} \leq C\sum_{j=0}^{q}\|\tilde{\mathbf{m}}\|_{W^{q-j,\infty}}\||\boldsymbol{\nabla}\mathbf{e}|^2\|_{H^j} \leq C\sum_{j=0}^{q}\|\tilde{\mathbf{m}}\|_{W^{q-j,\infty}}\|\boldsymbol{\nabla}\mathbf{e}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{e}\|_{H^j},$$

$$\|\tilde{\mathbf{m}}(\boldsymbol{\nabla}\mathbf{e}:\boldsymbol{\nabla}\tilde{\mathbf{m}})\|_{H^q} \leq \sum_{j=0}^{q}\sum_{i=0}^{j}\|\tilde{\mathbf{m}}\|_{W^{q-j},\infty}\|\boldsymbol{\nabla}\tilde{\mathbf{m}}\|_{W^{j-i,\infty}}\|\boldsymbol{\nabla}\mathbf{e}\|_{H^i},$$

and finally, as shown in [24], we have as a consequence of (5.6) and the boundedness of the gradients of $\mathbf{m}$ and $\tilde{\mathbf{m}}$ that

$$\|\mathbf{e}|\boldsymbol{\nabla}\mathbf{e}|^2\|_{H^q} \leq C\left(\|\mathbf{e}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{e}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{e}\|_{H^q} + \|\boldsymbol{\nabla}\mathbf{e}\|_{L^\infty}^2\|\boldsymbol{\nabla}\mathbf{e}\|_{H^{q-1}}\right) \leq C\|\boldsymbol{\nabla}\mathbf{e}\|_{H^q}.$$

Applying the assumption (6.2), we thus get

$$\begin{aligned}
\|\mathbf{D}_{22}\|_{H_j} &\leq C\left(\sum_{i=0}^{j}\frac{1}{\varepsilon^{j-i}}\|\mathbf{e}\|_{H^i} + \sum_{i=0}^{j}\frac{1}{\varepsilon^{\max(0,j-i-1)}}\|\mathbf{e}\|_{H^{i+1}} + \sum_{i=0}^{j}\frac{1}{\varepsilon^{j-i}}\|\mathbf{e}\|_{H^{i+1}} + \|\mathbf{e}\|_{H^{j+1}}\right)\\
&\leq C\left(\sum_{i=0}^{j}\frac{1}{\varepsilon^{j-i}}\|\mathbf{e}\|_{H^i} + \|\mathbf{e}\|_{H^{j+1}}\right) \leq C\frac{1}{\varepsilon^{j+1}}\|\mathbf{e}\|_{H_\varepsilon^{j+1}}. \qquad (6.17)
\end{aligned}$$

In total, the combination of (6.16) and (6.17) with (6.15) and application of (5.3) in Lemma 5.2 results in

$$\|\mathbf{D}_2\|_{H^q} \leq C\left(\frac{1}{\varepsilon^{q+1}}\|\mathbf{e}\|_{H_\varepsilon^q} + \frac{1}{\varepsilon^{q+1}}\|\mathbf{e}\|_{H_\varepsilon^{q+1}}\right) \leq C\frac{1}{\varepsilon^{q+1}}\|\mathbf{e}\|_{H_\varepsilon^{q+1}}.$$

This completes the proof. □

To continue with the proof of Theorem 6.1, consider now $\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}$ with $k \geq 0$. Based on (6.4), we find using integration by parts that

$$\begin{aligned}
\frac{1}{2}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 &= \int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}:\boldsymbol{\nabla}\mathcal{L}^k\partial_t\mathbf{e}dx = -\int_\Omega(\mathcal{L}^{k+1}\mathbf{e})\cdot\mathcal{L}^k\partial_t\mathbf{e}dx \qquad (6.18)\\
&= -\int_\Omega(\mathcal{L}^{k+1}\mathbf{e})\cdot\mathcal{L}^k\mathbf{D}_1dx - \alpha\int_\Omega(\mathcal{L}^{k+1}\mathbf{e})\cdot\mathcal{L}^k(\mathcal{L}\mathbf{e}+(\mathbf{D}_2+\mathbf{D}_3))dx\\
&\quad + \int_\Omega(a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e})\cdot\boldsymbol{\nabla}\mathcal{L}^k\eta dx.
\end{aligned}$$

Similarly, we obtain for $k > 0$ that

$$\begin{aligned}
\frac{1}{2}\partial_t\|\mathcal{L}^k\mathbf{e}\|_{L^2}^2 = &-\int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}:\boldsymbol{\nabla}\mathcal{L}^{k-1}\mathbf{D}_1dx - \alpha\int_\Omega(a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}):\boldsymbol{\nabla}\mathcal{L}^{k-1}(\mathcal{L}\mathbf{e}+(\mathbf{D}_2+\mathbf{D}_3)dx\\
&+\int_\Omega(\mathcal{L}^k\mathbf{e})\cdot\mathcal{L}^k\eta dx.
\end{aligned}$$

It thus holds that for $k \geq 0$,

$$\frac{1}{2}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 + \alpha\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 = -\mathbf{J}_{1,k} - \alpha(\mathbf{J}_{2,k}+\mathbf{J}_{3,k}) + \int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathbf{e}\cdot\boldsymbol{\nabla}\eta dx, \quad (6.19)$$

$$\frac{1}{2}\partial_t\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \alpha\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 = -\mathbf{K}_{1,k} - \alpha(\mathbf{K}_{2,k}+\mathbf{K}_{3,k}) + \int_\Omega \mathcal{L}\mathbf{e}\cdot\mathcal{L}\eta dx, \quad (6.20)$$

where

$$\mathbf{J}_{j,k} := \int_\Omega \mathcal{L}^{k+1}\mathbf{e}\cdot\mathcal{L}^k\mathbf{D}_jdx, \quad \mathbf{K}_{j,k} := \int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}:\boldsymbol{\nabla}\mathcal{L}^k\mathbf{D}_jdx.$$

We now derive bounds for these integrals. In general, the estimates for the $\mathbf{J}_{j,k}$ and $\mathbf{K}_{j,k}$ integrals are very similar to each other and only differ regarding details. We therefore focus mostly on the $\mathbf{J}_{j,k}$ estimates.

To bound the first terms, $\mathbf{J}_{1,k}$ and $\mathbf{K}_{1,k}$, one can use the fact that by Lemma 5.8,

$$\mathcal{L}^k\mathbf{D}_1 = \mathcal{L}^k(\mathbf{e}\times\mathcal{L}\mathbf{e}+\tilde{\mathbf{m}}\times\mathcal{L}\mathbf{e}+\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}}) = \mathbf{m}\times\mathcal{L}^{k+1}\mathbf{e}+\mathbf{R}_{k,\mathbf{e}}+\mathbf{R}_{k,\tilde{\mathbf{m}}}+\mathcal{L}^k(\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}}).$$

The highest order term here, $\mathbf{m}\times\tilde{\mathcal{L}}^{k+1}\mathbf{e}$, cancels in the integral in $\mathbf{J}_{1,k}$ due to orthogonality. Consequently, application of Cauchy-Schwarz and Young's inequality yields

$$\begin{aligned}|\mathbf{J}_{1,k}| &= \left|\int_\Omega \mathcal{L}^{k+1}\mathbf{e}\cdot(\mathbf{R}_{k,\mathbf{e}}+\mathbf{R}_{k,\tilde{\mathbf{m}}}+\mathcal{L}^k(\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}}))dx\right| \\ &\le \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2+\frac{1}{2\gamma}\left(\|\mathbf{R}_{k,\mathbf{e}}\|_{L^2}^2+\|\mathbf{R}_{k,\tilde{\mathbf{m}}}\|_{L^2}^2+\|\mathcal{L}^k(\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}})\|_{L^2}^2\right).\end{aligned}$$

Making use of Lemma 5.6, Lemma 5.5 and the assumption (6.2), the latter norm can be bounded as

$$\|\mathcal{L}^k(\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}})\|_{L^2} \le C\frac{1}{\varepsilon^{2k}}\sum_{i=0}^{2k}\varepsilon^i\|\mathcal{L}\tilde{\mathbf{m}}\|_{W^{i,\infty}}\|\mathbf{e}\|_{H_\varepsilon^{2k-i}} \le C\frac{1}{\varepsilon^{2k+1}}\|\mathbf{e}\|_{H_\varepsilon^{2k}}.$$

Together with the bounds for $\|\mathbf{R}_{k,\mathbf{u}}\|_{L^2}$ according to Lemma 5.8, we thus get

$$|\mathbf{J}_{1,k}| \le \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2+\frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+1)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+1}}^2. \tag{6.21}$$

We obtain an according estimate for $\mathbf{K}_{1,k}$ by taking the gradient of $\mathcal{L}^k\mathbf{D}_1$ and proceeding in the same way as for $\mathbf{J}_{1,k}$. However, we have to consider that

$$\boldsymbol{\nabla}(\mathbf{m}\times\mathcal{L}^{k+1}\mathbf{e}) = \mathbf{m}\times\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}+\boldsymbol{\nabla}\mathbf{m}\times\mathcal{L}^{k+1}\mathbf{e},$$

where only the first term on the right-hand side cancels due to orthogonality in $\mathbf{K}_{1,k}$. To bound the L^2-norm of the second term, we use the fact that by assumption (A4) we have an infinity bound on $\boldsymbol{\nabla}\mathbf{m}$, making it possible to remove it form the norm. The remaining term can be bounded using Lemma 5.6. In total, this results in

$$\begin{aligned}|\mathbf{K}_{1,k}| &\le \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 \\ &\quad+\frac{1}{2\gamma}\left(\|\boldsymbol{\nabla}\mathbf{m}\times\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2+\|\mathbf{R}_{k,\mathbf{e}}\|_{H^1}^2+\|\mathbf{R}_{k,\tilde{\mathbf{m}}}\|_{H^1}^2+\|\boldsymbol{\nabla}\mathcal{L}^k(\mathbf{e}\times\mathcal{L}\tilde{\mathbf{m}})\|_{L^2}^2\right) \\ &\le \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2+\frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+2)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+2}}^2.\end{aligned} \tag{6.22}$$

For the second kind of integrals, $\mathbf{J}_{2,k}$ and $\mathbf{K}_{2,k}$, application of Cauchy-Schwarz and Young's inequality, yields directly that for all constants $\gamma>0$,

$$|\mathbf{J}_{2,k}| \le \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2+\frac{1}{2\gamma}\|\mathcal{L}^k\mathbf{D}_2\|_{L^2}^2.$$

Using Lemma 5.6 together with Lemma 6.1 to go from the norm of $\mathcal{L}^k\mathbf{D}_2$ to an estimate in terms of $\mathbf{e}$ then gives

$$\|\mathcal{L}^k\mathbf{D}_2\|_{L^2} \le C\sum_{j=1}^{2k}\frac{1}{\varepsilon^{2k-j}}\|\mathbf{D}_2\|_{H^j} \le C\frac{1}{\varepsilon^{2k+1}}\sum_{j=1}^{2k}\|\mathbf{e}\|_{H_\varepsilon^{j+1}} \le C\frac{1}{\varepsilon^{2k+1}}\|\mathbf{e}\|_{H_\varepsilon^{2k+1}},$$

and it follows that

$$|\mathbf{J}_{2,k}| \leq \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+1)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+1}}^2 \tag{6.23}$$

and for $\mathbf{K}_{2,k}$ we obtain similarly,

$$|\mathbf{K}_{2,k}| \leq \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+2)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+2}}^2. \tag{6.24}$$

Application of (5.16) in Lemma 5.7 to the right-hand side in the estimates (6.21) and (6.23) then results in

$$\begin{aligned}|\mathbf{J}_{1,k}| + \alpha|\mathbf{J}_{2,k}| &\leq c\frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+1)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+1}}^2 \\ &\leq c\frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\left(\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 + \frac{1}{\varepsilon^{2(2k+1)}}\|\mathbf{e}\|_{H_\varepsilon^{2k}}^2\right).\end{aligned}$$

Correspondingly, we find based on (5.16), (6.22) and (6.24) that

$$|\mathbf{K}_{1,k}| + \alpha|\mathbf{K}_{2,k}| \leq c\frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\left(\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{1}{\varepsilon^{2(2k+2)}}\|\mathbf{e}\|_{H_\varepsilon^{2k+1}}^2\right).$$

To do the estimates for $\mathbf{J}_{3,k}$ and $\mathbf{K}_{3,k}$, note that it follows by Lemma 5.6, Lemma 5.5 and (6.2), that

$$\begin{aligned}\|\mathcal{L}^k(L|\tilde{\mathbf{m}}|^2\tilde{\mathbf{m}})\|_{H^q} &\leq C\frac{1}{\varepsilon^{q+2k}}\|L|\tilde{\mathbf{m}}|^2\tilde{\mathbf{m}}\|_{H_\varepsilon^{q+2k}} \leq C\frac{1}{\varepsilon^{q+2k}}\sum_{j=0}^{q+2k}\varepsilon^j\|\tilde{\mathbf{m}}\|_{W^{j,\infty}}\|L|\tilde{\mathbf{m}}|^2\|_{H_\varepsilon^{q+2k-j}} \\ &\leq C\frac{1}{\varepsilon^{q+2k}}\|L|\tilde{\mathbf{m}}|^2\|_{H_\varepsilon^{q+2k}} \leq C\frac{1}{\varepsilon^{q+2k+1}}\|\nabla|\tilde{\mathbf{m}}|^2\|_{H_\varepsilon^{q+2k+1}}.\end{aligned}$$

Hence, we find for $\mathbf{J}_{3,k}$ and $\mathbf{K}_{3,k}$ that

$$|\mathbf{J}_{3,k}| = \left|\frac{1}{2}\int_\Omega \mathcal{L}^{k+1}\mathbf{e}\cdot\mathcal{L}^k(\tilde{\mathbf{m}}L|\tilde{\mathbf{m}}|^2)dx\right| \leq \frac{\gamma}{4}\|\mathcal{L}^{k+1}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+1)}}\|\nabla|\tilde{\mathbf{m}}|^2\|_{H_\varepsilon^{2k+1}}^2, \tag{6.25a}$$

$$|\mathbf{K}_{3,k}| \leq \frac{\gamma}{4}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+2)}}\|\nabla|\tilde{\mathbf{m}}|^2\|_{H_\varepsilon^{2k+2}}^2. \tag{6.25b}$$

The remaining integrals in (6.19) and (6.20), involving $\boldsymbol{\eta}$, are bounded using Cauchy-Schwarz and Young in the same way as in (6.12), which together with Lemma 5.2 and Lemma 5.6 results in

$$\begin{aligned}\left|\int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}:\boldsymbol{\nabla}\mathcal{L}^k\boldsymbol{\eta}dx\right| &\leq \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\|\boldsymbol{\nabla}\mathcal{L}^k\boldsymbol{\eta}\|_{L^2}^2 \\ &\leq \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 + \frac{C}{2\gamma}\frac{1}{\varepsilon^{2(2k+1)}}\|\boldsymbol{\eta}\|_{H_\varepsilon^{2k+1}}^2,\end{aligned}$$

and correspondingly for (6.20). Thus, it holds in total that when choosing γ small enough in the estimates for $\mathbf{J}_{1,k}$, $\mathbf{J}_{2,k}$ and $\mathbf{J}_{3,k}$, we get from (6.19)

$$\frac{1}{2}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 \leq C\left(\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}\|_{L^2}^2 + \mathbf{J}_{R,k}(t)\right), \qquad 0 \leq t \leq T^\varepsilon,$$

for some C independent of ε and t, and where $\mathbf{J}_{R,k}$ only depends on lower order norms of $\mathbf{e}$ as well as terms involving $\boldsymbol{\eta}$ and $\tilde{\mathbf{m}}$,

$$\mathbf{J}_{R,k}(t) := \frac{1}{\varepsilon^{2(2k+1)}}\left(\|\mathbf{e}\|^2_{H^{2k}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}|^2\|^2_{H^{2k+1}_\varepsilon} + \|\boldsymbol{\eta}\|^2_{H^{2k+1}_\varepsilon}\right).$$

Assume now that (6.3) holds up to $q=2k$. This is true for $k=0$ according to the estimate in the previous section, (6.13). Then

$$\|\mathbf{e}\|^2_{H^{2k}_\varepsilon} = C\sum_{j=0}^{2k}\varepsilon^{2j}\|\mathbf{e}\|^2_{H^j} \le Ct\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^{2k}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^{2k}_\varepsilon}\right)$$

and thus

$$\mathbf{J}_{R,k}(t) \le \frac{C(t+1)}{\varepsilon^{2(2k+1)}}\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^{2k+1}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^{2k+1}_\varepsilon}\right).$$

Since moreover $\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}(0,x)=0$, we have by Grönwall's inequality, that for $0\le t\le T^\varepsilon$,

$$||\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{e}(\cdot,t)||^2_{L^2} \le C\int_0^t \frac{Ct+1}{\varepsilon^{2(2k+1)}}\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^{2k+1}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^{2k+1}_\varepsilon}\right)ds.$$

where, as in (6.13), the prefactor is independent of ε. It then holds for $k=0$ that

$$\|\mathbf{e}(\cdot,t)\|^2_{H^1} \le C\left(\|\mathbf{e}\|^2_{L^2} + \|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{e}\|^2_{L^2}\right) \le \frac{Ct}{\varepsilon^2}\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^1_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^1_\varepsilon}\right),$$

while it follows by elliptic regularity, (5.15), that for $k\ge 1$.

$$\begin{aligned}\|\mathbf{e}(\cdot,t)\|^2_{H^{2k+1}} &\le C\left(\|\mathbf{e}\|^2_{L^2} + \frac{1}{\varepsilon^{2(2k+1)}}\|\mathbf{e}\|^2_{H^{2k}_\varepsilon} + \|\sqrt{a^\varepsilon}\boldsymbol{\nabla}L^k\mathbf{e}\|^2_{L^2}\right)\\ &\le Ct\frac{1}{\varepsilon^{2(2k+1)}}\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^{2k+1}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^{2k+1}_\varepsilon}\right),\end{aligned}$$

which shows the estimate in Theorem 6.1 for odd q given that it holds up to $q-1$. Finally, we obtain in the same way when combining the estimates (6.22), (6.24), (6.25b), for $\mathbf{K}_{1,k}$, $\mathbf{K}_{2,k}$ and $\mathbf{K}_{3,k}$, with (6.20), and again using ellitpic regularity (5.15) and applying Grönwall's inequality, that for $0\le t\le T^\varepsilon$,

$$\begin{aligned}\|\mathbf{e}\|^2_{H^{2k+2}} &\le C\left(\|\mathbf{e}\|^2_{L^2} + \frac{1}{\varepsilon^{2(2k+2)}}\|\mathbf{e}\|^2_{H^{2k+1}_\varepsilon} + \|\mathcal{L}^{k+1}\mathbf{e}\|^2_{L^2}\right)\\ &\le Ct\frac{1}{\varepsilon^{2(2k+2)}}\sup_{0\le s\le t}\left(\|\boldsymbol{\eta}(\cdot,s)\|^2_{H^{2k+2}_\varepsilon} + \|\nabla|\tilde{\mathbf{m}}(\cdot,s)|^2\|^2_{H^{2k+2}_\varepsilon}\right),\end{aligned}$$

which shows the estimate in Theorem 6.1 for even $q>0$ given that it holds for $q-1$. This completes the proof.

7. Estimates of homogenized solution and correction terms

In this section, we provide estimates for the norms of the correction terms $\mathbf{m}_j$, $j\ge 1$. To obtain these, we use a theorem for general linear equations of the form as (4.7), which is presented in the next subsection. We moreover derive bounds for the remaining quantities involved in the stability estimate Theorem 6.1.

7.1. Linear equation. First, we consider solutions $\mathbf{m}$ to the inhomogeneous linear equation

$$\partial_\tau \mathbf{m}(x,y,t,\tau) = \mathscr{L}\mathbf{m}(x,y,t,\tau) + \mathbf{F}(x,y,t,\tau), \tag{7.1a}$$

$$\mathbf{m}(x,y,t,0) = \mathbf{g}(x,y,t), \tag{7.1b}$$

with periodic boundary conditions in y and up to some fixed final time $T>0$. The linear operator $\mathscr{L}$ is defined as in (4.8). It depends on the material coefficient a and on the solution of the homogenized equation $\mathbf{m}_0$.

We note that since $\mathscr{L}$ has a non-trivial null space and $\alpha>0$ this is a degenerate parabolic equation in (y,τ). In the following, it will be beneficial to split the solution $\mathbf{m}$, initial data $\mathbf{g}$ and forcing $\mathbf{F}$ in (7.1) into a part that lies in the null-space, and a part that is orthogonal to it. To this means, we introduce the matrix $\mathbf{M}$ corresponding to the orthogonal projection onto $\mathbf{m}_0$ and the averaging operator $\mathcal{A}$,

$$\mathbf{M}(x,t) := \mathbf{m}_0\mathbf{m}_0^T, \qquad \mathcal{A}\mathbf{m} := \int_Y \mathbf{m}(x,y,t,\tau)dy. \tag{7.2}$$

and then define projections

$$\mathcal{P}\mathbf{m} := (\mathbf{I}-\mathbf{M})(\mathbf{I}-\mathcal{A})\mathbf{m} \qquad \text{and} \qquad \mathcal{Q}\mathbf{m} := \mathbf{M}\mathbf{m} + (\mathbf{I}-\mathbf{M})\mathcal{A}\mathbf{m}, \tag{7.3}$$

which means that $\mathcal{Q} = \mathbf{I} - \mathcal{P}$. According to this definition, $\mathcal{P}\mathbf{m}$ is orthogonal to $\mathbf{m}_0$ and has zero average in y, while $\mathcal{Q}\mathbf{m}$ consists of the average of $\mathbf{m}$ and the contribution to $\mathbf{m} - \mathcal{A}\mathbf{m}$ that is parallel to $\mathbf{m}_0$. In particular, $\mathcal{Q}$ is a projection onto the null-space of $\mathscr{L}$. Note that $\mathcal{P}$ and $\mathcal{Q}$ depend on (x,t), but not on (y,τ). Then we have the following theorem about the size of the two parts of the solution

THEOREM 7.1. *Assume (A1), (A3) and (A5) hold. If $\partial_t^\ell \mathbf{F}(\cdot,\cdot,t,\cdot) \in C(\mathbb{R}^+; H^{q-2\ell,\infty})$ and $\partial_t^\ell \mathbf{g}(\cdot,\cdot,t) \in H^{q-2\ell,\infty}$ for $0 \le 2\ell \le 2k \le q \le r$ and $0 \le t \le T$, then $\partial_t^k \mathbf{m}(\cdot,\cdot,t,\cdot) \in C(\mathbb{R}^+; H^{q-2k,\infty})$ when $t \in [0,T]$ and for each integer $p \ge 0$, there are constants C and $\gamma > 0$, independent of $\tau \ge 0$, $t \in [0,T]$, $\mathbf{F}$ and $\mathbf{g}$, such that*

$$\|\partial_t^k \mathcal{P}\mathbf{m}(\cdot,\cdot,t,\tau)\|_{H^{q-2k,p}} \le C \sum_{\ell=0}^{k} \Big(e^{-\gamma\tau} \|\partial_t^\ell \mathcal{P}\mathbf{g}(\cdot,\cdot,t)\|_{H^{q-2\ell,p}} + \int_0^\tau e^{-\gamma(\tau-s)} \|\partial_t^\ell \mathcal{P}\mathbf{F}(\cdot,\cdot,t,s)\|_{H^{q-2\ell,p}} ds \Big), \tag{7.4}$$

$$\|\partial_t^k \mathcal{Q}\mathbf{m}(\cdot,\cdot,t,\tau)\|_{H^{q-2k,p}} \le \|\partial_t^k \mathcal{Q}\mathbf{g}(\cdot,\cdot,t)\|_{H^{q-2k,p}} + \int_0^\tau \|\partial_t^k \mathcal{Q}\mathbf{F}(\cdot,\cdot,t,s)\|_{H^{q-2k,p}} ds. \tag{7.5}$$

This is proved in Appendix A . The proof uses standard energy estimates in which the precise growth rates of the different solution parts are carefully analyzed. Note that, since τ represents the fast scale, sharp bounds on the growth in τ are necessary.

In Appendix A we also prove a few properties of $\mathcal{P}(\cdot,t)$ and $\mathcal{Q}(\cdot,t)$, in particular that they are bounded operators on $H^{q,p}$ for $0 \le q \le r$ and $p \ge 0$, uniformly in $t \in [0,T]$. The following lemma gives the more general result.

LEMMA 7.1. *Assume (A5) holds. Suppose $\partial_t^\ell \mathbf{v}(\cdot,\cdot,t) \in H^{q-2\ell,p}$ for $0 \le 2\ell \le 2k \le q \le r$ and $p \ge 0$. Then*

$$\|\partial_t^k \mathcal{P}\mathbf{v}(\cdot,\cdot,t)\|_{H^{q-2k,p}} \le C \sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{v}(\cdot,\cdot,t)\|_{H^{q-2\ell,p}}, \tag{7.6a}$$

$$\|\partial_t^k \mathcal{Q}\mathbf{v}(\cdot,\cdot,t)\|_{H^{q-2k,p}} \le C\sum_{\ell=0}^{k}\|\partial_t^\ell \mathbf{v}(\cdot,\cdot,t)\|_{H^{q-2\ell,p}}, \tag{7.6b}$$

for $0\le t\le T$, where C is independent of $\mathbf{v}$ and t.

Note that this lemma shows that the projected initial data functions $\mathcal{P}\mathbf{g},\mathcal{Q}\mathbf{g}\in H^{q,\infty}(\Omega)$ if the unprojected function $\mathbf{g}\in H^{q,\infty}(\Omega)$ and similar for the forcing function $\mathbf{F}$ and the time derivatives of the functions. This justifies why we only ask for smoothness of the unprojected functions in Theorem 7.1.

7.2. Correction terms. We now apply Theorem 7.1 to the correctors $\mathbf{m}_j$ in the asymptotic expansion (4.1) in order to obtain estimates for their norms. Here and throughout the rest of this section we suppose that the assumptions (A1)-(A5) are true. We recall from (4.7) that the correction terms satisfy linear PDEs,

$$\begin{aligned}\partial_\tau \mathbf{m}_j(x,y,t,\tau) &= \mathscr{L}\mathbf{m}_j(x,y,t,\tau)+\mathbf{F}_j(x,y,t,\tau),\\ \mathbf{m}_j(x,y,t,0)&=0,\end{aligned} \tag{7.7}$$

where $\mathbf{F}_j$ is defined in (4.9). Additionally, we consider the term $\mathbf{v}$ in the definition of $\mathbf{m}_1$ (4.12) to get a better understanding of the behavior of $\mathbf{m}_1$. As given in (4.15), $\mathbf{v}$ satisfies

$$\begin{aligned}\partial_\tau \mathbf{v}(x,y,t,\tau) &= \mathscr{L}\mathbf{v}(x,y,t,\tau),\\ \mathbf{v}(x,y,t,0)&=-\boldsymbol{\nabla}_x\mathbf{m}_0(x,t)\boldsymbol{\chi}(y),\end{aligned} \tag{7.8}$$

where $\boldsymbol{\chi}$ is the solution to the cell problem, (4.13). We then obtain the following result.
THEOREM 7.2. *For $0\le t\le T$ and $0\le 2k\le r-j$ we have*

$$\partial_t^k\mathbf{m}_j(\cdot,\cdot,t,\cdot)\in C(\mathbb{R}^+;H^{r-j-2k,\infty}), \qquad \partial_t^k\mathbf{v}(\cdot,\cdot,t,\cdot)\in C(\mathbb{R}^+,H^{r-1-2k,\infty}), \tag{7.9}$$

and there are constants $\gamma>0$ and C independent of $\varepsilon,\tau\ge 0$ and $0\le t\le T$ such that when $p\ge 0$,

$$\|\partial_t^k\mathbf{v}(\cdot,\cdot,t,\tau)\|_{H^{r-1-2k,p}} \le Ce^{-\gamma\tau}, \tag{7.10a}$$

$$\|\partial_t^k\mathcal{P}\mathbf{m}_j(\cdot,\cdot,t,\tau)\|_{H^{r-j-2k,p}} \le C, \tag{7.10b}$$

$$\|\partial_t^k\mathcal{Q}\mathbf{m}_j(\cdot,\cdot,t,\tau)\|_{H^{r-j-2k,p}} \le C\left(1+\tau^{\max(0,j-2)}\right), \tag{7.10c}$$

$$\|\partial_t^k\mathbf{m}_j(\cdot,\cdot,t,\tau)\|_{H^{r-j-2k,p}} \le C\left(1+\tau^{\max(0,j-2)}\right). \tag{7.10d}$$

Moreover, it holds for $\tau\ge 0$ and $0\le t\le T$ that

$$\mathbf{m}_1\perp\mathbf{m}_0, \qquad \mathbf{v}\perp\mathbf{m}_0, \qquad \mathcal{P}\mathbf{m}_1=\mathbf{m}_1, \qquad \mathcal{P}\mathbf{v}=\mathbf{v}. \tag{7.11}$$

This theorem entails in particular the following.

- The first corrector $\mathbf{m}_1$ has zero average, is orthogonal to $\mathbf{m}_0$ and stays bounded for all $\tau\ge 0$.
- As the first one, the second corrector $\mathbf{m}_2$ is uniformly bounded in τ, but it is neither orthogonal nor parallel to $\mathbf{m}_0$.
- Higher order correctors are not bounded in τ but grow algebraically.

To prove the theorem, we use induction. We consider first the base cases, norms of $\mathbf{v}$ and $\mathbf{m}_1$ and their time derivatives, which in turn makes it possible to bound the norms of $\partial_t^k\mathcal{Q}\mathbf{F}_2$. Then we provide a utility lemma giving estimates for the quantities involved in higher order $\mathbf{F}_j$, $j\ge 3$. Finally, we conclude the proof with an induction step showing (7.10) for general j.

7.2.1. $\mathbf{m}_1$ and $\mathbf{v}$ estimates. To begin with, we show (7.11). For $\mathbf{m}_1$, the forcing term $\mathbf{F}_1$ only depends on $\mathbf{m}_0$. In fact, as $\mathbf{Z}_0=\mathcal{L}_1\mathbf{m}_0=\nabla_x\mathbf{m}_0\nabla_y a$ the expression for $\mathbf{F}_1$ in (4.10) can be written as

$$\mathbf{F}_1(x,y,t,\tau)=-\mathbf{m}_0(x,t)\times[\nabla_x\mathbf{m}_0(x,t)+\alpha\mathbf{m}_0(x,t)\times\nabla_x\mathbf{m}_0(x,t)]\nabla_y a(y),$$

which shows that $\mathbf{F}_1$ is orthogonal to $\mathbf{m}_0$. Moreover, since the averaging operator $\mathcal{A}$ commutes with differentiation in y,

$$\mathcal{A}\mathbf{F}_1=-\mathbf{m}_0(x,t)\times[\nabla_x\mathbf{m}_0(x,t)+\alpha\mathbf{m}_0(x,t)\times\nabla_x\mathbf{m}_0(x,t)]\mathcal{A}\nabla_y a(y)=0,$$

and consequently $\mathcal{Q}\mathbf{F}_1=0$ and $\mathcal{P}\mathbf{F}_1=\mathbf{F}_1$. For $\mathbf{v}$ it holds at the initial time $\tau=0$,

$$\mathbf{g}(x,y,t)=\mathbf{v}(x,y,t,0)=-\nabla_x\mathbf{m}_0(x,t)\boldsymbol{\chi}(y).$$

Since we choose $\boldsymbol{\chi}$ such that $\mathcal{A}\boldsymbol{\chi}=0$ and due to the fact that $\mathbf{m}_0$ is orthogonal to its gradient, $\mathbf{m}_0\cdot\boldsymbol{\nabla}_x\mathbf{m}_0=\nabla|\mathbf{m}_0|^2/2=0$, we have $\mathcal{Q}\mathbf{g}=0$ and $\mathcal{P}\mathbf{g}=\mathbf{g}$. It thus follows from Theorem 7.1 that $\mathcal{Q}\mathbf{m}_1=\mathcal{Q}\mathbf{v}=0$ and consequently for all $\tau\geq 0$,

$$\mathcal{P}\mathbf{m}_1=\mathbf{m}_1,\qquad \mathcal{P}\mathbf{v}=\mathbf{v}.$$

Hence, (7.11) holds. Next, we consider the norms of $\partial_t^k\mathbf{v}$ and $\partial_t^k\mathbf{m}_1$. Corollary 5.1 shows that for $0\leq\ell\leq k$,

$$\|\partial_t^\ell\mathbf{F}_1\|_{H^{r-1-2\ell,p}}\leq C\sum_{s=0}^{\ell}\|\nabla_x\partial_t^s\mathbf{m}_0\nabla_y a\|_{H^{r-1-2s,p}}\leq C\sum_{s=0}^{\ell}\|\partial_t^s\mathbf{m}_0\|_{H^{r-2s,p}}\|a\|_{H^{p+1}}\leq C,$$

and similarly,

$$\|\partial_t^\ell\mathbf{g}\|_{H^{r-1-2\ell,p}}\leq\|\nabla_x\partial_t^\ell\mathbf{m}_0\|_{H^{r-1-2\ell}}\|\boldsymbol{\chi}\|_{H^p}\leq C\|\partial_t^\ell\mathbf{m}_0\|_{H^{r-2\ell}}\leq C.$$

Since $\mathbf{F}_1$ and $\mathbf{g}$ coincide with their $\mathcal{P}$-projections, we have

$$\|\partial_t^\ell\mathcal{P}\mathbf{F}_1\|_{H^{r-1-2\ell,p}}=\|\partial_t^\ell\mathbf{F}_1\|_{H^{r-1-2\ell,p}},\quad \|\partial_t^\ell\mathcal{P}\mathbf{g}\|_{H^{r-1-2\ell,p}}=\|\partial_t^\ell\mathbf{g}\|_{H^{r-1-2\ell,p}},$$

and thus obtain from Theorem 7.1 that

$$\|\partial_t^k\mathbf{v}\|_{H^{r-1-2k,p}}=\|\partial_t^k\mathcal{P}\mathbf{v}\|_{H^{r-1-2k,p}}\leq C\sum_{\ell=0}^{k}e^{-\gamma\tau}\|\partial_t^\ell\mathcal{P}\mathbf{g}\|_{H^{r-1-2\ell,p}}\leq Ce^{-\gamma\tau},\tag{7.12}$$

$$\begin{aligned}\|\partial_t^k\mathbf{m}_1\|_{H^{r-1-2k,p}}=\|\partial_t^k\mathcal{P}\mathbf{m}_1\|_{H^{r-1-2k,p}}&\leq C\sum_{\ell=0}^{k}\int_0^\tau e^{-\gamma(\tau-s)}\|\partial_t^\ell\mathcal{P}\mathbf{F}_1\|_{H^{r-1-2\ell,q}}ds\\&\leq C\int_0^\tau e^{-\gamma(\tau-s)}ds\leq C.\end{aligned}\tag{7.13}$$

This shows (7.10) for $\mathbf{v}$ and the first corrector $\mathbf{m}_1$.

Consider now $\mathbf{F}_2$ as defined in (4.17), which consists only of quantities involving $\mathbf{v}$ and $\mathbf{m}_1$. Combining (4.21) with the definition of the homogenized solution (4.23) shows that the average of $\mathbf{F}_2$ is $\mathcal{A}\mathbf{F}_2=-(\mathbf{E}_1+\mathbf{E}_2)$, with $\mathbf{E}_1$ and $\mathbf{E}_2$ given in (4.20) and (4.22),

$$\mathbf{E}_1=\mathcal{A}(\mathbf{R}_1+\alpha\mathbf{S}_1),\qquad \mathbf{E}_2=\mathbf{m}_0\times\mathcal{A}(\mathcal{L}_1\mathbf{v})+\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{A}(\mathcal{L}_1\mathbf{v}),$$

where $\mathbf{R}_1 = \mathbf{m}_1 \times \mathcal{L}_2\mathbf{v}$ and $\mathbf{S}_1 = \mathbf{m}_1 \times \mathbf{m}_0 \times \mathcal{L}_2\mathbf{v}$, defined according to (4.3) and (4.4), are parallel to $\mathbf{m}_0$ due to the orthogonality given in (7.11). This implies that $\mathbf{E}_1$ is parallel to $\mathbf{m}_0$ as well, while $\mathbf{E}_2$ is orthogonal to $\mathbf{m}_0$ as it is of the form $\mathbf{m}_0 \times \cdot$. Hence, it holds that

$$(\mathbf{I}-\mathbf{M})(\mathbf{E}_1+\mathbf{E}_2) = \mathbf{E}_2.$$

Again using the fact that terms of the form $\mathbf{m}_0 \times \cdot$, as well as $\partial_t \mathbf{m}_0$, are orthogonal to $\mathbf{m}_0$, we thus can show that application of the operator $\mathcal{Q}$ to $\mathbf{F}_2$ yields

$$\mathcal{Q}\mathbf{F}_2 = \mathbf{M}\mathbf{F}_2 + (\mathbf{I}-\mathbf{M})\mathcal{A}\mathbf{F}_2 = -\mathbf{M}(\mathbf{R}_1+\alpha\mathbf{S}_1) - (\mathbf{I}-\mathbf{M})(\mathbf{E}_1+\mathbf{E}_2), = -\mathbf{R}_1 - \alpha\mathbf{S}_1 - \mathbf{E}_2.$$

Using Lemma 5.4, with $j=2$ and $m=m'=1$ as $\partial_t^k \mathbf{m}_1, \partial_t^k \mathbf{v} \in H^{r-1-2k,p}$, therefore yields together with (7.12) and (7.13),

$$\begin{aligned}
\|\partial_t^k \mathbf{R}_1\|_{H^{r-2-2k,p}} &\leq C\sum_{\ell=0}^{k} \|\partial_t^{k-\ell}\mathbf{m}_1\|_{H^{r-1-2k+2\ell,p+2}} \|\partial_t^\ell \mathcal{L}_2\mathbf{v}\|_{H^{r-1-2\ell,p}} \\
&\leq C\sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{v}\|_{H^{r-1-2\ell,p+2}} \leq Ce^{-\gamma\tau}, \\
\|\partial_t^k \mathbf{S}_1\|_{H^{r-2-2k,p}} &\leq C\sum_{\ell=0}^{k} \|\partial_t^\ell(\mathbf{m}_0 \times \mathcal{L}_2\mathbf{v})\|_{H^{r-1-2\ell,p}} \leq C\sum_{\ell=0}^{k} \|\partial_t^\ell \mathcal{L}_2\mathbf{v}\|_{H^{r-1-2\ell,p}} \\
&\leq C\sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{v}\|_{H^{r-1-2\ell,p+2}} \leq Ce^{-\gamma\tau}.
\end{aligned}$$

Finally, using Corollary 5.1 with $\mathbf{f} = \mathcal{A}\mathcal{L}_1\mathbf{v}$ gives

$$\|\partial_t^k \mathbf{E}_2\|_{H^{r-2-2k,p}} \leq C\sum_{\ell=0}^{k} \|\partial_t^\ell \mathcal{A}\mathcal{L}_1\mathbf{v}\|_{H^{r-2-2\ell,p}} \leq C\sum_{\ell=0}^{k} \|\partial_t^\ell \mathbf{v}\|_{H^{r-1-2\ell,p+1}} \leq Ce^{-\gamma\tau}.$$

We thus conclude that

$$\|\partial_t^k \mathcal{Q}\mathbf{F}_2\|_{H^{r-j-2k,p}} \leq Ce^{-\gamma\tau}. \tag{7.14}$$

7.2.2. Higher order $\mathbf{m}_j$-estimate. We now consider $\mathbf{m}_j$ with $j \geq 2$. First, note that due to assumption (A1), it holds in general for $\mathbf{m}_j$, $j \geq 0$, that

$$\|\mathcal{L}_k \mathbf{m}_j\|_{H^{q,p}} \leq C\|\mathbf{m}_j\|_{H^{q+2-k,p+k}}, \qquad k=0,1,2,$$

for $p,q \geq 0$, whenever the norms are bounded. This can be used to prove a lemma providing upper bounds for norms of all the intermediate quantities involved in the forcing term $\mathbf{F}_j$ in (7.7).

Lemma 7.2. *Suppose that (7.9) and (7.10) hold for $1 \leq j \leq J$, $0 \leq 2k \leq r-j$ and $0 \leq t \leq T$. Then, for $p \geq 0$,*

$$\|\partial_t^k \mathbf{Z}_j\|_{H^{r-j-1-2k,p}} \leq C\left(1+\tau^{\max(0,j-2)}\right), \quad 0 \leq j \leq J, \quad 0 \leq 2k \leq r-j-1, \tag{7.15}$$

$$\|\partial_t^k \mathbf{X}_j\|_{H^{r-j-2k,p}} \leq C\left(1+\tau^{\max(0,j-2)}\right), \quad 1 \leq j \leq J, \quad 0 \leq 2k \leq r-j, \tag{7.16}$$

where $\mathbf{X}_j$ *is any of* $\mathbf{R}_j$, $\mathbf{S}_j$, $\mathbf{T}_j$ *and* $\mathbf{V}_j$ *as defined in* (4.2), (4.3) *and* (4.4) *and the constant* C *is independent of* t *and* τ.

Proof. Let $1\le j\le J$, $p\ge 0$ and $0\le 2k\le r-j-1$. For $\mathbf{Z}_j$ we have

$$\begin{aligned}\|\partial_t^k\mathbf{Z}_j\|_{H^{r-j-1-2k,p}} &\le \|\partial_t^k\mathcal{L}_0\mathbf{m}_{j-1}\|_{H^{r-j-1-2k,p}}+\|\partial_t^k\mathcal{L}_1\mathbf{m}_j\|_{H^{r-j-1-2k,p}}\\ &\le C(\|\partial_t^k\mathbf{m}_{j-1}\|_{H^{r-j+1-2k,p}}+\|\partial_t^k\mathbf{m}_j\|_{H^{r-j-2k,p+1}})\\ &\le C(1+\tau^{\max(0,j-3)}+\tau^{\max(0,j-2)})\le C\left(1+\tau^{\max(0,j-2)}\right).\end{aligned}$$

Since by definition, $\mathbf{m}_{-1}\equiv 0$, the result still holds true for $j=0$. For $\mathbf{V}_j$ we then have accordingly, when $0\le 2k\le r-j$,

$$\begin{aligned}\|\partial_t^k\mathbf{V}_j\|_{H^{r-j-2k,p}} &\le \|\partial_t^k\mathcal{L}_2\mathbf{m}_j\|_{H^{r-j-2k,p}}+\|\partial_t^k\mathbf{Z}_{j-1}\|_{H^{r-j-2k,p}}\\ &\le C\|\partial_t^k\mathbf{m}_j\|_{H^{r-j-2k,p+2}}+\|\partial_t^k\mathbf{Z}_{j-1}\|_{H^{r-j-2k,p}}\\ &\le C(1+\tau^{\max(0,j-2)}+\tau^{\max(0,j-3)})\le C\left(1+\tau^{\max(0,j-2)}\right).\end{aligned}$$

This shows Lemma 5.4 for $\mathbf{X}_j=\mathbf{V}_j$. Suppose now that that $\mathbf{Y}_m$ satisfies (7.16) for $1\le m\le j$ and that $m'\in\{m-1,m\}$. Since $j-m'+m\le j+1$, we then find by Lemma 5.4 that

$$\begin{aligned}\left\|\partial_t^k(\mathbf{m}_{j-m'}\times\mathbf{Y}_m)\right\|_{H^{r-j-2k,p}} &\le C\sum_{\ell=0}^{k}\left\|\partial_t^{k-\ell}\mathbf{m}_{j-m'}\right\|_{H^{r-j+m'-2k+2\ell,p+2}}\left\|\partial_t^{\ell}\mathbf{Y}_m\right\|_{H^{r-m-2\ell,p}}\\ &\le C(1+\tau^{\max(0,j-m'-2)})(1+\tau^{\max(0,m-2)})\\ &\le C(1+\tau^{\max(0,j-m'-2,m-2,j+m-m'-4)})\\ &\le C(1+\tau^{\max(0,j-2,j-3,j-4)})\le C(1+\tau^{\max(0,j-2)}).\end{aligned}$$

When using this result for $\mathbf{Y}_m=\mathbf{V}_m$ and $m'=m$, we get

$$\|\partial_t^k\mathbf{T}_j\|_{H^{r-j-2k,p}}\le\sum_{m=1}^{j}\left\|\partial_t^k(\mathbf{m}_{j-m}\times\mathbf{V}_m)\right\|_{H^{r-j-2k,p}}\le C\left(1+\tau^{\max(0,j-2)}\right).$$

Similarly, when $m'=m-1$, we find by choosing $\mathbf{Y}_m$ to be $\mathbf{V}_m$ and $\mathbf{T}_m$ respectively,

$$\begin{aligned}\|\partial_t^k\mathbf{R}_j\|_{H^{r-j-2k,p}} &\le\sum_{m=1}^{j}\left\|\partial_t^k(\mathbf{m}_{j+1-m}\times\mathbf{V}_m)\right\|_{H^{r-j-2k,p}}\le C\left(1+\tau^{\max(0,j-2)}\right),\\ \|\partial_t^k\mathbf{S}_j\|_{H^{r-j-2k,p}} &\le\sum_{m=1}^{j}\left\|\partial_t^k(\mathbf{m}_{j+1-m}\times\mathbf{T}_m)\right\|_{H^{r-j-2k,p}}\le C\left(1+\tau^{\max(0,j-2)}\right).\end{aligned}$$

This proves the lemma. □

We now have the tools necessary to conclude the induction step for Theorem 7.2. For $j=1$, we have already shown (7.9) and (7.10d). Assume now that they hold up to some j with $1\le j\le r-2k-1$. We then show in the following that they also hold for $j+1\le r-2k$. To this means, suppose $0\le 2k\le r-j-1=:q'$ and $p\ge 0$. From the definition of $\mathbf{F}_j$ according to (4.9) it follows using Corollary 5.1 and Lemma 7.2 that

$$\begin{aligned}\|\partial_t^k\mathbf{F}_{j+1}\|_{H^{q'-2k,p}} &\le\|\partial_t^{k+1}\mathbf{m}_{j-1}\|_{H^{q'-2k,p}}+\|\partial_t^k\mathbf{R}_j\|_{H^{q'-2k,p}}+\|\partial_t^k(\mathbf{m}_0\times\mathbf{Z}_j)\|_{H^{q'-2k,p}}\\ &\quad+\alpha\left(\|\partial_t^k(\mathbf{m}_0\times\mathbf{R}_j)\|_{H^{q'-2k,p}}+\|\partial_t^k(\mathbf{m}_0\times\mathbf{m}_0\times\mathbf{Z}_j)\|_{H^{q'-2k,p}}+\left\|\partial_t^k\mathbf{S}_j\right\|_{H^{q'-2k,p}}\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{\ell=0}^{k}\left(\|\partial_t^\ell \mathbf{Z}_j\|_{H^{q'-2\ell,p}}+\|\partial_t^\ell \mathbf{R}_j\|_{H^{q'-2\ell,p}}\right)+\alpha\left\|\partial_t^k \mathbf{S}_j\right\|_{H^{q'-2k,p}}+\|\partial_t^{k+1}\mathbf{m}_{j-1}\|_{H^{q'-2k,p}}\\
&\le C\left(1+\tau^{\max(0,j-2)}+\tau^{\max(0,j-3)}\right)\le C\left(1+\tau^{\max(0,j-2)}\right).
\end{aligned}
$$

By (7.6) the same estimate holds for $\partial_t^k\mathcal{P}\mathbf{F}_{j+1}$ and $\partial_t^k\mathcal{Q}\mathbf{F}_{j+1}$. However, for the latter we have due to (7.14),

$$
\|\partial_t^k\mathcal{Q}\mathbf{F}_{j+1}\|_{H^{r-j-1-2k,p}}\le C\begin{cases}e^{-\gamma\tau}, & j=1,\\ 1+\tau^{j-2}, & j\ge 2.\end{cases}
$$

The estimate (7.10b) with $j+1$ then follows from Theorem 7.1 as

$$
\begin{aligned}
\|\partial_t^k\mathcal{P}\mathbf{m}_{j+1}\|_{H^{r-j-1-2k,p}}&\le C\sum_{\ell=0}^{k}\int_0^\tau e^{-\gamma(\tau-s)}\|\partial_t^\ell\mathcal{P}\mathbf{F}_{j+1}\|_{H^{r-j-1-2\ell,p}}ds\\
&\le C\int_0^\tau e^{-\gamma(\tau-s)}(1+\tau^{\max(0,j-2)})ds\le C,
\end{aligned}
$$

and accordingly,

$$
\begin{aligned}
\|\partial_t^k\mathcal{Q}\mathbf{m}_{j+1}\|_{H^{r-j-1-2k,p}}&\le C\int_0^\tau \|\partial_t^k\mathcal{Q}\mathbf{F}_{j+1}\|_{H^{r-j-1-2k,p}}ds\le C\int_0^\tau \begin{cases}e^{-\gamma\tau}, & j=1,\\ 1+\tau^{j-2}, & j\ge 2,\end{cases} ds\\
&\le C\begin{cases}1, & j=1,\\ 1+\tau^{j-1}, & j\ge 2,\end{cases}\le C(1+\tau^{\max(0,j-1)}),
\end{aligned}
$$

which yields (7.10c) with $j+1$. Finally, (7.10d) is obtained using the triangle inequality. This concludes the induction step and the proof of Theorem 7.2.

7.3. Approximations $\tilde{\mathbf{m}}_J$ and $\tilde{\mathbf{m}}_J^\varepsilon$. In this section we consider the approximation $\tilde{\mathbf{m}}_J^\varepsilon$ to $\tilde{\mathbf{m}}$ as defined in (3.3) and correspondingly,

$$
\tilde{\mathbf{m}}_J(x,y,t,\tau;\varepsilon)=\mathbf{m}_0(x,t)+\sum_{j=1}^{J}\varepsilon^j\mathbf{m}_j(x,y,t,\tau),\qquad \tilde{\mathbf{m}}_J^\varepsilon(x,t)=\tilde{\mathbf{m}}_J(x,x/\varepsilon,t,t/\varepsilon^2).
$$

We are interested in different aspects of $\tilde{\mathbf{m}}_J^\varepsilon$ and $\tilde{\mathbf{m}}_J$ up to time T^ε as given in (3.4), (3.7) , here repeated for convenience of the reader,

$$
T^\varepsilon:=\varepsilon^\sigma T \qquad \text{with} \qquad \begin{cases}0\le\sigma\le 2, & J\le 2,\\ 1-\frac{1}{J-2}\le\sigma\le 2, & J\ge 3.\end{cases} \tag{7.17}
$$

Up to this final time, we have

$$
1+\tau\le C(1+\varepsilon^{-(2-\sigma)})\le C\varepsilon^{-(2-\sigma)}.
$$

As a consequence, we can simplify the estimate (7.10d) for final time T^ε. Under the assumptions in Theorem 7.2, it holds for $0\le t\le T^\varepsilon$, $0\le\tau\le T^\varepsilon/\varepsilon^2$, that

$$
\|\partial_t^k\mathbf{m}_j(\cdot,\cdot,t,\tau)\|_{H^{r-j-2k,p}}\le C\varepsilon^{-(2-\sigma)\max(0,j-2)},\qquad 0\le 2k\le r-j. \tag{7.18}
$$

7.3.1. Norms of approximations. We start by estimating the approximations $\tilde{\mathbf{m}}_J$ and $\tilde{\mathbf{m}}_J^\varepsilon$ in different Sobolev norms. We obtain the following theorem.
THEOREM 7.3. *For* $0 \le t \le T$,

$$\tilde{\mathbf{m}}_J(\cdot,\cdot,t,\cdot;\varepsilon) \in C(\mathbb{R}^+; H^{r-J,\infty}(\Omega,\mathbb{R}^3)), \qquad \tilde{\mathbf{m}}_J^\varepsilon(\cdot,t) \in H^{r-J}(\Omega). \tag{7.19}$$

Moreover, consider T^ε *as given in* (7.17)*, then for any* $p \ge 0$

$$\|\tilde{\mathbf{m}}_J(\cdot,\cdot,t,\tau;\varepsilon)\|_{H^{r-J,p}} \le C, \qquad 0 \le t \le T^\varepsilon, \quad 0 \le \tau \le \frac{T^\varepsilon}{\varepsilon^2}. \tag{7.20}$$

Additionally, for $0 \le q \le r-J$ *and* $0 \le q' \le r-J-2$

$$\|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^q} \le C\varepsilon^{\min(0,1-q)}, \qquad \|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{W^{q',\infty}} \le C\varepsilon^{\min(0,1-q')}, \quad 0 \le t \le T^\varepsilon. \tag{7.21}$$

All constants denoted C *are independent of* τ, t *and* ε, *but depend on* T.

Proof. The simplification (7.18) of the estimate in Theorem 7.2 gives for fixed t, τ as in (7.20) and $0 \le j \le J$,

$$\varepsilon^j \|\mathbf{m}_j\|_{H^{r-j,p}} \le C\varepsilon^{j-(2-\sigma)\max(0,j-2)} = C\begin{cases} \varepsilon^j, & 0 \le j \le 2, \\ \varepsilon^{j-(2-\sigma)(j-2)} & 3 \le j \le J, \end{cases} \le C \begin{cases} 1, & j=0, \\ \varepsilon, & j \ge 1, \end{cases}$$

where we used for the second case that

$$j-(2-\sigma)(j-2) = 2+(j-2)(\sigma-1) \ge 2 - \frac{j-2}{J-2} \ge 2 - \frac{J-2}{J-2} = 1.$$

This shows (7.20), as

$$\|\tilde{\mathbf{m}}_J\|_{H^{r-J,p}} \le \sum_{j=0}^J \varepsilon^j \|\mathbf{m}_j\|_{H^{r-J,p}} \le \sum_{j=0}^J \varepsilon^j \|\mathbf{m}_j\|_{H^{r-j,p}} \le C.$$

For the second statement, we use Lemma 5.1 and the fact that $\mathbf{m}_0$ does not depend on y, which yields

$$\begin{aligned}\|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^q} &\le \|\mathbf{m}_0(x,t)\|_{H^q} + \sum_{j=1}^J \varepsilon^j \|\mathbf{m}_j(\cdot,\cdot/\varepsilon,t,t/\varepsilon^2)\|_{H^q} \\ &\le \|\mathbf{m}_0(x,t)\|_{H^q} + C\sum_{j=1}^J \varepsilon^{j-q} \|\mathbf{m}_j(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{q,q+2}}.\end{aligned}$$

Proceeding similarly to before, we then get for $j \ge 1$,

$$\|\mathbf{m}_j\|_{H^{q,q+2}} \le \|\mathbf{m}_j\|_{H^{r-J,q+2}} \le \|\mathbf{m}_j\|_{H^{r-j,q+2}} \le C\varepsilon^{(2-\sigma)\max(0,j-2)} \le C\varepsilon^{1-j}.$$

Therefore,

$$\|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{H^q} \le C(1+\varepsilon^{1-q}).$$

Finally, by Lemma 5.1,

$$\|\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)\|_{W^{q',\infty}} \le \|\mathbf{m}_0(x,t)\|_{W^{q',\infty}} + C\sum_{j=1}^J \varepsilon^{j-q'} \|\mathbf{m}_j(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{q'+2,q'+2}},$$

where

$$\|\mathbf{m}_j\|_{H^{q'+2,q'+2}} \le \|\mathbf{m}_j\|_{H^{r-J,q'+2}},$$

from which (7.21) follows in the same way as above. This completes the proof.
∎

7.3.2. Residual. The truncated approximation $\tilde{\mathbf{m}}_J^\varepsilon$ satisfies the differential equation (3.1) for the original $\mathbf{m}^\varepsilon$ only up to a certain residual $\boldsymbol{\eta}_J^\varepsilon$. In the following, we derive an expression for this residual $\boldsymbol{\eta}_J^\varepsilon$ that is then used to obtain a bound for its $\|\cdot\|_{H^q}$-norm.

THEOREM 7.4. *Let the residual $\boldsymbol{\eta}_J^\varepsilon$ be defined as*

$$\boldsymbol{\eta}_J^\varepsilon := \partial_t \tilde{\mathbf{m}}_J^\varepsilon + \tilde{\mathbf{m}}_J^\varepsilon \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon + \alpha \tilde{\mathbf{m}}_J^\varepsilon \times \tilde{\mathbf{m}}_J^\varepsilon \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon \tag{7.22}$$

and suppose $2 \le J \le r-2$ and $0 \le t \le T^\varepsilon$ with T^ε as in (7.17). Then for $0 \le q \le r-J-2$,

$$\|\boldsymbol{\eta}_J^\varepsilon(\cdot,t)\|_{H^q} \le C\varepsilon^{1+(\sigma-1)(J-2)-q}, \qquad \|\boldsymbol{\eta}_J^\varepsilon(\cdot,t)\|_{H^q_\varepsilon} \le C\varepsilon^{1+(\sigma-1)(J-2)}.$$

The constant C is independent of t and ε, but depends on T.

Proof. Using the notation given in (4.2) and (4.3), we find along the same steps as in Section 4 that the expression corresponding to (4.5) for the truncated expansion $\tilde{\mathbf{m}}_J^\varepsilon$ becomes

$$\mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon = \sum_{j=0}^J \varepsilon^{j-2}\mathcal{L}_2\mathbf{m}_j + \sum_{j=0}^J \varepsilon^{j-1}\mathcal{L}_1\mathbf{m}_j + \sum_{j=0}^J \varepsilon^{j}\mathcal{L}_0\mathbf{m}_j = \sum_{j=1}^J \varepsilon^{j-2}\mathbf{V}_j + \varepsilon^{J-1}\boldsymbol{\mu}_1,$$

where

$$\boldsymbol{\mu}_1 := \mathcal{L}_1\mathbf{m}_J + \mathcal{L}_0\mathbf{m}_{J-1} + \varepsilon\mathcal{L}_0\mathbf{m}_J .$$

To obtain an expanded expression for the precession term, we then take the cross product of $\tilde{\mathbf{m}}_J^\varepsilon$ with the expanded expression for $\mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon$ which results in

$$\tilde{\mathbf{m}}_J^\varepsilon \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon = \sum_{j=0}^J \varepsilon^j \mathbf{m}_j \times \left(\sum_{k=1}^J \varepsilon^{k-2}\mathbf{V}_k + \varepsilon^{J-1}\boldsymbol{\mu}_1\right) = \sum_{j=1}^J \varepsilon^{j-2}\mathbf{T}_j + \varepsilon^{J-1}\boldsymbol{\eta}_1, \tag{7.23}$$

where

$$\boldsymbol{\eta}_1 := \boldsymbol{\mu}_2 + \tilde{\mathbf{m}}_J \times \boldsymbol{\mu}_1 \qquad \text{and} \qquad \boldsymbol{\mu}_2 := \sum_{j=0}^{J-1} \varepsilon^j \sum_{k=j+1}^J \mathbf{m}_{J+1+j-k} \times \mathbf{V}_k .$$

Taking one more cross product by $\tilde{\mathbf{m}}^\varepsilon$ yields an expanded form of the damping term,

$$\tilde{\mathbf{m}}_J^\varepsilon \times \tilde{\mathbf{m}}_J^\varepsilon \times \mathcal{L}\tilde{\mathbf{m}}_J^\varepsilon = \sum_{j=1}^J \varepsilon^{j-2} \sum_{k=1}^j \mathbf{m}_{j-k} \times \mathbf{T}_k + \varepsilon^{J-1}\boldsymbol{\eta}_2, \tag{7.24}$$

where

$$\boldsymbol{\eta}_2 := \boldsymbol{\mu}_3 + \tilde{\mathbf{m}}_J \times \boldsymbol{\eta}_1 \qquad \text{and} \qquad \boldsymbol{\mu}_3 := \sum_{j=0}^{J-1} \varepsilon^j \sum_{k=j+1}^J \mathbf{m}_{J+1+j-k} \times \mathbf{T}_k .$$

Moreover, it holds for the time derivative of $\tilde{\mathbf{m}}_J^\varepsilon$ that

$$\partial_t \tilde{\mathbf{m}}_J^\varepsilon = \sum_{j=0}^J (\varepsilon^j \partial_t \mathbf{m}_j + \varepsilon^{j-2}\partial_\tau \mathbf{m}_j) = \sum_{j=0}^J \varepsilon^{j-2}(\partial_t \mathbf{m}_{j-2} + \partial_\tau \mathbf{m}_j) + \varepsilon^{J-1}\partial_t \mathbf{m}_{J-1} + \varepsilon^J \partial_t \mathbf{m}_J . \tag{7.25}$$

Putting the expanded expressions as given in (7.23), (7.24) and (7.25) into the definition of $\boldsymbol{\eta}^\varepsilon$ that is given by the differential equation, (7.22), yields together with (4.7) that

$$\boldsymbol{\eta}_J^\varepsilon(x,t) = \boldsymbol{\eta}_J(x,x/\varepsilon,t,t/\varepsilon^2), \quad \text{where} \quad \boldsymbol{\eta}_J = \varepsilon^{J-1}(\partial_t \mathbf{m}_{J-1} + \varepsilon\partial_t\mathbf{m}_J + \boldsymbol{\eta}_1 + \alpha\boldsymbol{\eta}_2). \tag{7.26}$$

This implies that in order to get a bound for the H^q-norm in space of $\boldsymbol{\eta}^\varepsilon$ we have to consider both x and y in the expanded form. By Lemma 5.1 it holds that

$$\|\boldsymbol{\eta}_J^\varepsilon(\cdot,t)\|_{H^q} \le \frac{C}{\varepsilon^q}\|\boldsymbol{\eta}_J(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{q,q+2}}. \tag{7.27}$$

Using the explicit form of $\boldsymbol{\eta}_J$ given in (7.26), one can obtain an upper bound on the norm of $\boldsymbol{\eta}_J$.To begin with, let $q' := r - J - 2$, then we have

$$\|\boldsymbol{\eta}_J(\cdot,\cdot,t,\tau)\|_{H^{q',p}} \le C\varepsilon^{J-1}\left(\|\partial_t\mathbf{m}_{J-1}\|_{H^{q',p}} + \varepsilon\|\partial_t\mathbf{m}_J\|_{H^{q',p}} + \|\boldsymbol{\eta}_1\|_{H^{q',p}} + \|\boldsymbol{\eta}_2\|_{H^{q',p}}\right).$$

For the first two terms we get from (7.18), as $J \ge 2$,

$$\begin{aligned}
\varepsilon^{J-1}\|\partial_t\mathbf{m}_{J-1}\|_{H^{q',p}} + \varepsilon^J\|\partial_t\mathbf{m}_J\|_{H^{q',p}} &\le C\varepsilon^{J-1+(\sigma-2)\max(0,J-3)} + C\varepsilon^{J+(\sigma-2)(J-2)} \\
&= C\varepsilon^{1+(\sigma-1)(J-2)}\left(\varepsilon^{(2-\sigma)(J-2-\max(0,J-3))} + \varepsilon\right) \\
&\le C\varepsilon^{1+(\sigma-1)(J-2)}.
\end{aligned}$$

Note that by the assumptions on J and σ the exponent for ε here is positive. To get an estimate for the norms of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, consider first the norms of the perturbation terms $\boldsymbol{\mu}_i$, $i = 1,2,3$, individually. By (7.18) and since $J \ge 2$, it holds that

$$\begin{aligned}
\|\boldsymbol{\mu}_1\|_{H^{r-J-2,p}} &\le C\left(\|\mathbf{m}_J\|_{H^{r-J-1,p+1}} + \|\mathbf{m}_{J-1}\|_{H^{r-J,p}} + \varepsilon\|\mathbf{m}_J\|_{H^{r-J,p}}\right) \\
&\le C\left(\|\mathbf{m}_J\|_{H^{r-J,p+1}} + \|\mathbf{m}_{J-1}\|_{H^{r-J+1,p}} + \varepsilon\|\mathbf{m}_J\|_{H^{r-J,p}}\right) \\
&\le C(\varepsilon^{-(2-\sigma)\max(0,J-3)} + (1+\varepsilon)\varepsilon^{-(2-\sigma)(J-2)}) \le C\varepsilon^{-(2-\sigma)(J-2)},
\end{aligned}$$

and therefore we can bound $\varepsilon^{J-1}\|\boldsymbol{\mu}_1\|_{H^{r-J-2,p}}$ in the same way as the terms above,

$$\varepsilon^{J-1}\|\boldsymbol{\mu}_1\|_{H^{r-J-2,p}} \le C\varepsilon^{J-1-(2-\sigma)(J-2)} = C\varepsilon^{1+(\sigma-1)(J-2)}.$$

Consider now the cross-products $\mathbf{m}_{J+1+j-k} \times \mathbf{V}_k$ when $j+1 \le k \le J$ and $0 \le j \le J-1$, which appear in the definition of $\boldsymbol{\mu}_2$. By Lemma 7.2 we have that

$$\|\mathbf{V}_k\|_{H^{r-k,p}} \le C(1+\tau^{\max(0,k-2)}) \le C\varepsilon^{-(2-\sigma)\max(0,k-2)}.$$

We then use (5.7) in Lemma 5.3 with $q_0 = r - J - 1$, $q_1 = r - J - 1 - j + k$ and $q_2 = r - k$ for the cross-product. This choice is valid since $q_0 \le \min(q_1, q_2)$ and

$$q_1 + q_2 = r - J - 1 - j + r = q_0 + r - j \ge q_0 + J + 2 - j \ge q_0 + 3,$$

which satisfies the left condition in (5.8). Together with (7.18), we thus get

$$\begin{aligned}
\|\mathbf{m}_{J+1+j-k} \times \mathbf{V}_k\|_{H^{r-J-1,p}} &\le C\|\mathbf{m}_{J+1+j-k}\|_{H^{r-J-1-j+k,p+2}}\|\mathbf{V}_k\|_{H^{r-k,p}} \\
&\le C\varepsilon^{-(2-\sigma)\max(0,J+j-k-1)}\varepsilon^{-(2-\sigma)\max(0,k-2)} \\
&\le C\varepsilon^{-(2-\sigma)\max(0,J-2,J+j-3)}.
\end{aligned}$$

Exploiting the fact that

$$-(2-\sigma)\max(0,J-2,J+j-3) = -(2-\sigma)(J-2+\max(0,j-1)),$$

we hence find for the norm of $\boldsymbol{\mu}_2$ that

$$\begin{aligned}\|\boldsymbol{\mu}_2\|_{H^{r-J-1,p}} &\le C\sum_{j=0}^{J-1}\sum_{k=j+1}^{J}\varepsilon^j\|\mathbf{m}_{J+1+j-k}\times\mathbf{V}_k\|_{H^{r-J-1,p}}\\ &= C\varepsilon^{-(2-\sigma)(J-2)}\sum_{j=0}^{J-1}\varepsilon^{j-(2-\sigma)\max(0,(j-1))},\end{aligned}$$

and therefore obtain

$$\varepsilon^{J-1}\|\boldsymbol{\mu}_2\|_{H^{r-J-1,p}} \le C\varepsilon^{1+(\sigma-1)(J-2)}\left(1+\sum_{j=1}^{J-1}\varepsilon^{1+(\sigma-1)(j-1)}\right) \le C\varepsilon^{1+(\sigma-1)(J-2)},$$

where the last step is valid since, for $J\ge 3$,

$$1+(\sigma-1)(j-1)\ge 1-\frac{j-1}{J-2}\ge 1-\frac{J-2}{J-2}=0.$$

We get the same estimate for $\boldsymbol{\mu}_3$ upon considering instead $\mathbf{m}_{J+1+j-k}\times\mathbf{T}_k$. Finally, note that multiplication by $\tilde{\mathbf{m}}_J$ does not affect the results. We can therefore use Lemma 5.3 with the right condition in (5.8) together with (7.20) in Theorem 7.3, which yields

$$\|\tilde{\mathbf{m}}_J\times\boldsymbol{\mu}_1\|_{H^{r-J-2,p}} \le C\|\tilde{\mathbf{m}}_J\|_{H^{r-J,p+2}}\|\boldsymbol{\mu}_1\|_{H^{r-J-2,p}} \le C\|\boldsymbol{\mu}_1\|_{H^{r-J-2,p}},$$

and thus

$$\begin{aligned}\varepsilon^{J-1}\|\boldsymbol{\eta}_1\|_{H^{r-J-2,p}} &\le \varepsilon^{J-1}\left(\|\boldsymbol{\mu}_2\|_{H^{r-J-2,p}}+\|\tilde{\mathbf{m}}_J\times\boldsymbol{\mu}_1\|_{H^{r-J-2,p}}\right)\\ &\le \varepsilon^{J-1}\left(\|\boldsymbol{\mu}_2\|_{H^{r-J-1,p}}+\|\tilde{\mathbf{m}}_J\times\boldsymbol{\mu}_1\|_{H^{r-J-2,p}}\right)\le C\varepsilon^{1+(\sigma-1)(J-2)}.\end{aligned}$$

For the remaining terms we proceed similarly.

The $\|\cdot\|_{H^q_\varepsilon}$-norm estimate follows immediately from the $\|\cdot\|_{H^q}$-estimate using (5.3) in Lemma 5.2. □

7.3.3. Length variation. While by assumption (A2), $|\mathbf{m}^\varepsilon|\equiv 1$ in space and constant in time due to the norm preservation property of the Landau-Lifshitz equation, (3.5), the norm of the approximation $\tilde{\mathbf{m}}^\varepsilon_J$ is not constant in time since it does not satisfy (3.1) exactly. We now consider the length of $\tilde{\mathbf{m}}^\varepsilon_J$ and obtain an upper bound for its deviation from one, the length of $\mathbf{m}^\varepsilon$.

LEMMA 7.3. *Suppose $2\le J\le r-2$ and let T^ε be defined as in (7.17). Then for $0\le t\le T^\varepsilon$ and $0\le q\le r-J-1$,*

$$\||\tilde{\mathbf{m}}^\varepsilon_J(\cdot,t)|^2-1\|_{H^q}\le C\varepsilon^{3+(\sigma-1)(J-2)-q}, \qquad \||\tilde{\mathbf{m}}^\varepsilon_J(\cdot,t)|^2-1\|_{H^q_\varepsilon}\le C\varepsilon^{3+(\sigma-1)(J-2)},$$

where the constant C is independent of t and ε, but depends on T.

This lemma implies by (5.4) in Lemma 5.2 that for $0\le q\le r-J-2$ and $0\le t\le T^\varepsilon$,

$$\|\nabla|\mathbf{m}^\varepsilon_J|^2\|_{H^q_\varepsilon}=\|\nabla(|\mathbf{m}^\varepsilon_J|^2-1)\|_{H^q_\varepsilon}\le\varepsilon^{-1}\||\mathbf{m}^\varepsilon_J|^2-1\|_{H^{q+1}_\varepsilon}\le C\varepsilon^{2+(\sigma-1)(J-2)}. \tag{7.28}$$

Proof. We note first that since $\tilde{\mathbf{m}}^\varepsilon_J$ satsifies (7.22),

$$\boldsymbol{\eta}^\varepsilon_J\cdot\tilde{\mathbf{m}}^\varepsilon_J=\partial_t\tilde{\mathbf{m}}^\varepsilon_J\cdot\tilde{\mathbf{m}}^\varepsilon_J,$$

which together with (7.26) implies that

$$\partial_t |\tilde{\mathbf{m}}_J^\varepsilon|^2 = 2\tilde{\mathbf{m}}_J^\varepsilon \cdot \boldsymbol{\eta}_J^\varepsilon = 2\varepsilon^{J-1}\tilde{\mathbf{m}}_J^\varepsilon \cdot (\partial_t \mathbf{m}_{J-1} + \varepsilon \partial_t \mathbf{m}_J + \boldsymbol{\eta}_1 + \sigma \boldsymbol{\eta}_2)\big|_{y=x/\varepsilon, \tau=t/\varepsilon^2}$$
$$=: \sum_{j=J-1}^{J'} \varepsilon^j d_j(x, x/\varepsilon, t, t/\varepsilon^2), \tag{7.29}$$

for some functions d_j and integer J'. On the other hand, we can expand $|\tilde{\mathbf{m}}_J^\varepsilon|^2$ as

$$|\tilde{\mathbf{m}}_J^\varepsilon(x,t)|^2 = \left| \mathbf{m}_0(x,t) + \sum_{j=1}^{J} \varepsilon^j \mathbf{m}_j(x, x/\varepsilon, t, t/\varepsilon^2) \right|^2 =: \sum_{j=0}^{2J} \varepsilon^j c_j(x, x/\varepsilon, t, t/\varepsilon^2), \tag{7.30}$$

where

$$c_j = \sum_{k=\max(0, j-J)}^{\min(j,J)} \mathbf{m}_k \cdot \mathbf{m}_{j-k}\,.$$

In particular, $c_0 = |\mathbf{m}_0|^2 \equiv 1$ and $c_1 = 2\mathbf{m}_0 \cdot \mathbf{m}_1 \equiv 0$ due to the orthogonality of $\mathbf{m}_0$ and $\mathbf{m}_1$ shown in (7.11) in Theorem 7.2. By (7.29), the full time derivative of the first $J-2$ terms vanishes, since

$$\left(\frac{\partial}{\partial t} + \varepsilon^{-2}\frac{\partial}{\partial \tau}\right) \sum_{j=0}^{2J} \varepsilon^j c_j = \sum_{j=J-1}^{J'} \varepsilon^j d_j.$$

As this identity is valid for all ε, it holds that

$$\partial_t c_j + \partial_\tau c_{j+2} = 0, \qquad j = 0, \ldots, J-2.$$

We claim that this implies that $c_j \equiv 0$ for $j = 1, \ldots J$. For $j = 1$ this is true due to (7.11) in Theorem 7.2 as shown above. Assume now that the claim holds up to $j \le J-1$. Then $j-1 \le J-2$, and we thus have

$$\partial_\tau c_{j+1} = -\partial_t c_{j-1} = 0,$$

which is true also for $j = 1$ since $\partial_t c_0 = 0$ as $c_0 = |\mathbf{m}_0|^2 \equiv 1$ for all time by (3.5). Moreover, at time $\tau = 0$, $c_j(x, y, t, 0) = 0$ for $j \ge 1$ and all $t \ge 0$, since this is true for the correctors $\mathbf{m}_j$. Hence, $c_{j+1} \equiv 0$. By induction we thus obtain

$$|\tilde{\mathbf{m}}_J^\varepsilon(x,t)|^2 = 1 + \varepsilon^{J+1} \sum_{j=0}^{J-1} \varepsilon^j \tilde{c}_j(x, x/\varepsilon, t, t/\varepsilon^2), \qquad \tilde{c}_j = c_{j+J+1} = \sum_{k=j+1}^{J} \mathbf{m}_k \cdot \mathbf{m}_{j+J+1-k}\,.$$

Using Lemma 5.1 it then follows that

$$\| |\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)|^2 - 1 \|_{H^q} \le \varepsilon^{J+1-q} \sum_{j=0}^{J-1} \varepsilon^j \|\tilde{c}_j(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{q,q+2}}.$$

We have left to estimate $\tilde{c}_j$ and note that it is of the same type as the terms in the sum definining $\boldsymbol{\mu}_2$ in the proof of Theorem 7.4. Therefore, with the same steps as in that proof, we obtain

$$\varepsilon^{J-1} \sum_{j=0}^{J-1} \varepsilon^j \|\tilde{c}_j(\cdot,\cdot,t,t/\varepsilon^2)\|_{H^{r-J-1,p}} \le C\varepsilon^{1+(\sigma-1)(J-2)}.$$

This finally gives

$$\||\tilde{\mathbf{m}}_J^\varepsilon(\cdot,t)|^2-1\|_{H^q}\le C\varepsilon^{3+(\sigma-1)(J-2)-q},$$

for $0\le q\le r-J-1$ and the corresponding $\|\cdot\|_{H^q_\varepsilon}$-norm estimate follows by (5.3) in Lemma 5.2. □

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Appendix A. Estimates linear equation.

In this section, we consider solutions $\mathbf{m}$ to the inhomogeneous linear PDE given in (7.1) repeated here for convenience,

$$\partial_\tau \mathbf{m}(x,y,t,\tau) = \mathscr{L}\mathbf{m}(x,y,t,\tau) + \mathbf{F}(x,y,t,\tau), \tag{A.1a}$$
$$\mathbf{m}(x,y,t,0) = \mathbf{g}(x,y,t). \tag{A.1b}$$

The PDE has periodic boundary conditions in y and up to some fixed final time $T>0$. The linear operator $\mathscr{L}$ is defined as in (4.8) as

$$\mathscr{L}\mathbf{m}_j := -\mathbf{m}_0 \times \mathcal{L}_2\mathbf{m}_j - \alpha\mathbf{m}_0 \times \mathbf{m}_0 \times \mathcal{L}_2\mathbf{m}_j, \qquad \text{where} \qquad \mathcal{L}_2 := \boldsymbol{\nabla}_y \cdot (a(y)\boldsymbol{\nabla}_y).$$

It depends on the material coefficient a and on the solution of the homogenized equation $\mathbf{m}_0$. We assume (A1), (A3) and (A5) hold.

Since $\mathscr{L}$ has a non-trivial null space and $\alpha>0$ this is a degenerate parabolic equation in (y,τ). It can also been viewed as a mixed parabolic hyperbolic system where the hyperbolic part has zero advection. As $\mathbf{m}_0$ is independent of (y,τ), standard theory ensures the existence of a unique smooth solution for all time

$$\mathbf{m}(x,\cdot,t,\cdot) \in C^1(\mathbb{R}^+; H^\infty(Y)),$$

to (7.1), if

$$\mathbf{F}(x,\cdot,t,\cdot) \in C(\mathbb{R}^+; H^\infty(Y)), \qquad \mathbf{g}(x,\cdot,t) \in H^\infty(Y).$$

See e.g. [22, Chapter 6].

In this appendix we prove Theorem 7.1 where we obtain energy estimates for $\mathbf{m}$ and its derivatives with respect to x, y and t. The energy method can easily be used to show that these derivatives grow at most exponentially fast in τ. However, since τ represents that fast scale, we need sharper bounds on the growth. The energy method must therefore be applied with more care. Below we prove energy estimates that show the precise growth rate in τ.

As in Section 7.1 we use the projections $\mathcal{P}$ and $\mathcal{Q}$ to split the solution into a part that gets damped away with time ($\mathcal{P}\mathbf{m}$) and a part that is invariant ($\mathcal{Q}\mathbf{m}$). We write again the definitions of the projections

$$\mathbf{M}(x,t) := \mathbf{m}_0\mathbf{m}_0^T, \qquad \mathcal{A}\mathbf{m} := \int_Y \mathbf{m}(x,y,t,\tau)dy, \tag{A.2}$$
$$\mathcal{P}\mathbf{m} := (\mathbf{I}-\mathbf{M})(\mathbf{I}-\mathcal{A})\mathbf{m} \qquad \mathcal{Q}\mathbf{m} := \mathbf{M}\mathbf{m} + (\mathbf{I}-\mathbf{M})\mathcal{A}\mathbf{m}. \tag{A.3}$$

It is easy to check that $\mathcal{A}^2 = \mathcal{A}$, the average of the average is equal to the average, and $\mathbf{M}^2 = \mathbf{m}_0\mathbf{m}_0^T\mathbf{m}_0\mathbf{m}_0^T = \mathbf{m}_0|\mathbf{m}_0|^2\mathbf{m}_0^T = \mathbf{M}$, from which it directly follows that $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{Q}^2 = \mathcal{Q}$.

To prove the theorem, we proceed as follows: in Lemma A.3 we show that the first statement in Theorem 7.1 holds for $p=q=0$ when $\mathbf{m}$ is independent of t. In Lemma A.4 we extend this to general p and in Lemma A.5 to general q. We then complete the proof by also taking the t-dependence of $\mathbf{m}$ into account.

A.1. Projections. We first take a closer look at the properties of the projections $\mathcal{P}$ and $\mathcal{Q}$ as defined by (A.3) and (A.2). We note that they depend on (x,t) but we mostly suppress this dependence in the notation.

We start with a lemma showing that $\mathbf{M}$ belongs to the same space as $\mathbf{m}_0$, and that $\mathcal{A}$ is bounded on $H^{q,p}$.

LEMMA A.1. *If (A5) holds, then*

$$\partial_t^k \mathbf{M} \in C(0,T;H^{r-2k}(\Omega)), \qquad 0 \le 2k \le r. \tag{A.4}$$

The averaging operator $\mathcal{A}$ is bounded on $H^{q,p}$ for all $q,p \ge 0$,

$$||\mathcal{A}\mathbf{v}||_{H^{q,p}} \le ||\mathbf{v}||_{H^{q,0}} \le ||\mathbf{v}||_{H^{q,p}}, \qquad \forall \mathbf{v} \in H^{q,p}. \tag{A.5}$$

Proof. Since $\partial_t^k \mathbf{m}_0 \in C(0,T;H^{r-2k}(\Omega))$ with $r \ge 5$ by (A5) it follows from Lemma 5.3 with $q_0 = r-2k, q_1 = r-2l \ge q_0$ and $q_2 = r-2k+2l \ge q_0$ which is valid by (5.8) since $q_1 + q_2 = r + q_0 \ge 5$, that

$$||\partial_t^k \mathbf{M}(\cdot,t)||_{H^{r-2k}} \le C \sum_{\ell=0}^{k} ||\partial_t^\ell \mathbf{m}_0(\cdot,t)||_{H^{r-2\ell}} ||\partial_t^{k-\ell} \mathbf{m}_0(\cdot,t)||_{H^{r-2k+2\ell}} \le C.$$

This shows (A.4). Next, we get (A.5) since

$$\begin{aligned} ||\mathcal{A}\mathbf{v}||_{H^{q,p}}^2 &= \sum_{|\beta|\le q, |\gamma| \le p} \int_\Omega \int_Y \left| \partial_x^\beta \partial_y^\gamma \int_Y \mathbf{v}(x,y) dy \right|^2 dx dy = \sum_{|\beta|\le q} \int_\Omega \left| \int_Y \partial_x^\beta \mathbf{v}(x,y) dy \right|^2 dx \\ &\le \sum_{|\beta|\le q} \int_\Omega \int_Y \left| \partial_x^\beta \mathbf{v}(x,y) \right|^2 dx dy = \|\mathbf{v}||_{H^{q,0}}^2. \end{aligned}$$

□

Using this lemma we can now prove Lemma 7.1 and the boundedness of the projections in $H^{q,p}$. We use (A.5) and Lemma 5.3 with $q_0 = q-2k, q_1 = r-2k+2\ell \ge q_0, q_2 = q-2\ell \ge q_0$ for $\ell \le k$, which again is valid by (5.8) since $q_1 + q_2 = q_0 + r$. This gives

$$\begin{aligned} ||\partial_t^k \mathcal{P}\mathbf{v}||_{H^{q-2k,p}} &= ||(\mathbf{I}-\mathcal{A})\partial_t^k (\mathbf{I}-\mathbf{M})\mathbf{v}||_{H^{q-2k,p}} \le C ||\partial_t^k (\mathbf{I}-\mathbf{M})\mathbf{v}||_{H^{q-2k,p}} \\ &\le ||\partial_t^k \mathbf{v}||_{H^{q-2k,p}} + C \sum_{\ell=0}^{k} ||\partial_t^{k-\ell} \mathbf{M}(\cdot,t)||_{H^{r-2k+2\ell}} ||\partial_t^\ell \mathbf{v}(\cdot,t)||_{H^{q-2\ell,p}} \\ &\le C \sum_{\ell=0}^{k} ||\partial_t^\ell \mathbf{v}(\cdot,t)||_{H^{q-2\ell,p}}. \end{aligned}$$

The $\mathcal{Q}$ part is given directly as $\mathcal{Q} = \mathbf{I} - \mathcal{P}$. Lemma 7.1 is proved.

Further properties of the projections $\mathcal{P}$ and $\mathcal{Q}$, in particular regarding commutations with derivatives, are given in the next lemma.

LEMMA A.2. *Let $\mathcal{A}$ be the averaging operator defined by* (A.2) *and $\mathcal{P}$ and $\mathcal{Q}$ be given by* (A.3)*. On the domain $H^1(Y)$, they have the following properties:*

1. *The derivative of the average and the average of the derivative are zero,*

$$\partial_y \mathcal{A} = \mathcal{A} \partial_y = 0, \tag{A.6}$$

and

$$\mathcal{P}\mathcal{A} = \mathcal{A}\mathcal{P} = 0, \qquad \mathcal{Q}\mathcal{A} = \mathcal{A}\mathcal{Q} = \mathcal{A}. \tag{A.7}$$

2. $\mathcal{P}$ and $\mathcal{Q}$ commute with ∂_y,

$$\partial_y \mathcal{P} = \mathcal{P} \partial_y , \qquad \partial_y \mathcal{Q} = \mathcal{Q} \partial_y , \tag{A.8}$$

and consequently for $\mathbf{m} \in H^2(Y)$,

$$\mathcal{P}\mathscr{L}\mathbf{m} = \mathscr{L}\mathcal{P}\mathbf{m} = \mathscr{L}\mathbf{m}, \qquad \mathcal{P}\mathcal{L}_2\mathbf{m} = \mathcal{L}_2\mathcal{P}\mathbf{m}, \tag{A.9}$$

$$\mathcal{Q}\mathscr{L}\mathbf{m} = \mathscr{L}\mathcal{Q}\mathbf{m} = \mathbf{0}, \qquad \mathcal{Q}\mathcal{L}_2\mathbf{m} = \mathcal{L}_2\mathcal{Q}\mathbf{m}. \tag{A.10}$$

3. Applying one of the projections $\mathcal{P}$ and $\mathcal{Q}$ to the x-derivative of the other results in an expression involving only the x-derivative of $\mathbf{M}$ and $\mathcal{P}$,

$$\mathcal{Q}\partial_x \mathcal{P} = -(\partial_x \mathbf{M})\mathcal{P} \qquad \text{and} \qquad \mathcal{P}\partial_x \mathcal{Q} = \mathcal{P}(\partial_x \mathbf{M}).$$

Proof. We show the points in turn:

1. To begin with, note that when $\mathbf{m} \in H^1(Y)$ it is Y-periodic in y, and it follows that $\mathcal{A}\partial_y \mathbf{m} = 0$. Moreover, $\partial_y \mathcal{A}\mathbf{m} = 0$ as well, as $\mathcal{A}\mathbf{m}$ is constant with respect to y. Furthermore,

$$\begin{aligned}\mathcal{A}\mathcal{P}\mathbf{m} &= (\mathbf{I}-\mathbf{M})\mathcal{A}(\mathbf{I}-\mathcal{A})\mathbf{m} = (\mathbf{I}-\mathbf{M})(\mathbf{I}-\mathcal{A})\mathcal{A}\mathbf{m} = \mathcal{P}\mathcal{A}\mathbf{m} \\ &= (\mathbf{I}-\mathbf{M})(\mathcal{A}-\mathcal{A}^2)\mathbf{m} = 0,\end{aligned}$$

since $(\mathbf{I}-\mathbf{M})$ is independent of y. Similarly,

$$\mathcal{Q}\mathcal{A}\mathbf{m} = \mathbf{M}\mathcal{A}\mathbf{m} + (\mathbf{I}-\mathbf{M})\mathcal{A}^2\mathbf{m} = \mathcal{A}\mathbf{m}$$

and $\mathcal{A}\mathcal{Q}\mathbf{m} = \mathcal{A}\mathbf{m}$ follows in the same way.

2. To prove the second claim, we use the fact that $\mathbf{M}$ is independent of y. It therefore follows by (A.6) that $\mathcal{P}$ commutes with derivatives in y and with $\mathcal{L}_2$,

$$\begin{aligned}\mathcal{P}\partial_y\mathbf{m} &= (\mathbf{I}-\mathbf{M})\partial_y\mathbf{m} - (\mathbf{I}-\mathbf{M})\mathcal{A}\partial_y\mathbf{m} = (\mathbf{I}-\mathbf{M})\partial_y\mathbf{m} \\ &= \partial_y(\mathbf{I}-\mathbf{M})(\mathbf{I}-\mathcal{A})\mathbf{m} = \partial_y\mathcal{P}\mathbf{m},\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}\mathcal{L}_2\mathbf{m} &= (\mathbf{I}-\mathbf{M})\mathcal{L}_2\mathbf{m} - (\mathbf{I}-\mathbf{M})\mathcal{A}\mathcal{L}_2\mathbf{m} = (\mathbf{I}-\mathbf{M})\mathcal{L}_2\mathbf{m} \\ &= (\mathbf{I}-\mathbf{M})\mathcal{L}_2(\mathbf{I}-\mathcal{A})\mathbf{m} = \mathcal{L}_2\mathcal{P}\mathbf{m}.\end{aligned}$$

The same holds for $\mathcal{Q}$ since $\mathcal{Q} = \mathbf{I} - \mathcal{P}$. Moreover, for any vector $\mathbf{u} \in \mathbb{R}^3$, the definition of $\mathbf{M}$ implies that

$$\mathbf{m}_0 \times \mathbf{M}\mathbf{u} = 0, \qquad \mathbf{M}(\mathbf{m}_0 \times \mathbf{u}) = 0.$$

This property combined with the commutation above entails that

$$\mathbf{m}_0 \times \mathcal{L}_2\mathcal{P}\mathbf{m} = \mathbf{m}_0 \times \mathcal{P}\mathcal{L}_2\mathbf{m} = \mathbf{m}_0 \times (\mathbf{I}-\mathbf{M})(\mathbf{I}-\mathcal{A})\mathcal{L}_2\mathbf{m} = \mathbf{m}_0 \times \mathcal{L}_2\mathbf{m},$$

and

$$\mathcal{P}(\mathbf{m}_0 \times \mathcal{L}_2\mathbf{m}) = (\mathbf{I}-\mathbf{M})\mathbf{m}_0 \times (\mathbf{I}-\mathcal{A})\mathcal{L}_2\mathbf{m} = \mathbf{m}_0 \times \mathcal{L}_2\mathbf{m}.$$

A similar argument for the damping term implies that $\mathcal{P}$ and $\mathscr{L}$ commute as stated in (A.9). As $\mathcal{Q} = \mathbf{I} - \mathcal{P}$, it then follows directly that $\mathcal{Q}\mathscr{L}\mathbf{m} = \mathscr{L}\mathcal{Q}\mathbf{m} = 0$.

3. To show the last statement, note that

$$\mathbf{M}(\partial_x \mathbf{M}) = \partial_x \mathbf{M}^2 - (\partial_x \mathbf{M})\mathbf{M} = \partial_x \mathbf{M}(\mathbf{I} - \mathbf{M}). \tag{A.11}$$

As $\mathcal{A}$ is independent of x, (A.11) yields together with (A.7) and $\mathcal{Q}\mathcal{P}\mathbf{m} = 0$, that

$$\mathcal{Q}\partial_x(\mathcal{P}\mathbf{m}) = \mathcal{Q}[\mathcal{P}\partial_x \mathbf{m} - (\partial_x \mathbf{M})(\mathbf{I} - \mathcal{A})\mathbf{m}] = -\mathbf{M}(\partial_x \mathbf{M})(\mathbf{I} - \mathcal{A})\mathbf{m} = -(\partial_x \mathbf{M})\mathcal{P}\mathbf{m}$$

and, since $\mathcal{P} = \mathcal{P}(\mathbf{I} - \mathcal{A})$ by (A.7),

$$\mathcal{P}\partial_x(\mathcal{Q}\mathbf{m}) = \mathcal{P}[\mathcal{Q}\partial_x \mathbf{m} + (\partial_x \mathcal{Q})\mathbf{m}] = \mathcal{P}(\partial_x \mathbf{M})(\mathbf{I} - \mathcal{A})\mathbf{m} = \mathcal{P}(\partial_x \mathbf{M})\mathbf{m}.$$

□

A.2. Estimates for $\mathcal{Q}$ part. Suppose $\mathbf{m}$ satsifies (A.1). By (A.9) in Lemma A.2 we then have for $\mathcal{Q}\mathbf{m}$,

$$\partial_\tau(\mathcal{Q}\mathbf{m}) = \mathcal{Q}\mathscr{L}\mathbf{m} + \mathcal{Q}\mathbf{F} = \mathcal{Q}\mathbf{F}, \qquad \mathcal{Q}\mathbf{m}(x,y,t,0) = \mathcal{Q}\mathbf{g}(x,y,t).$$

Hence,

$$\mathcal{Q}\mathbf{m}(x,y,t,\tau) = \mathcal{Q}\mathbf{m}(x,y,t,0) + \int_0^\tau \mathcal{Q}\mathbf{F}(x,y,t,s)ds = \mathcal{Q}\mathbf{g}(x,y,t) + \int_0^\tau \mathcal{Q}\mathbf{F}(x,y,t,s)ds.$$

Therefore, (7.5) follows directly.

A.3. Estimates for $\mathcal{P}$ part: y-derivatives. In this section we consider $\mathcal{P}\mathbf{m}$ assuming that $\mathbf{F}$ and $\mathbf{g}$ only depend on the fast variables. We start by proving L^2-norm energy estimates for $\mathcal{P}\mathbf{m}(y,\tau)$ and its gradient in this setting. Those results can subsequently be used to prove estimates for higher derivatives. The proof of the lemma involves the following two variations of the Poincaré-inequality. Let $\mathbf{u}_p$ be an arbitrary, Y-periodic function in $H^p(Y;\mathbb{R}^3)$. Then the we have

$$\|\mathbf{u}_1 - \bar{\mathbf{u}}_1\|_{L^2} \le C_P \|\boldsymbol{\nabla}\mathbf{u}_1\|_{L^2}, \qquad \|\boldsymbol{\nabla}\mathbf{u}_2\|_{L^2} \le \frac{C_P}{a_{\min}} \|\mathcal{L}_2 \mathbf{u}_2\|_{L^2}, \tag{A.12}$$

where $\bar{\mathbf{u}}_1$ is the average of $\mathbf{u}_1$ over Y, $a_{\min} = \inf_{y\in Y} a(y)$ and C_P only depends on Y. The left statement in (A.12) is the standard Poincaré inequality. The right one follows since due to periodicity,

$$\begin{aligned}\|\boldsymbol{\nabla}\mathbf{u}_2\|_{L^2}^2 &\le \frac{1}{a_{\min}} \int_Y a\boldsymbol{\nabla}\mathbf{u}_2 : \boldsymbol{\nabla}\mathbf{u}_2 dx = -\frac{1}{a_{\min}} \int_Y \mathbf{u}_2 \cdot \mathcal{L}_2 \mathbf{u}_2 dx \\ &= -\frac{1}{a_{\min}} \int_Y (\mathbf{u}_2 - \bar{\mathbf{u}}_2) \cdot \mathcal{L}_2 \mathbf{u}_2 dx \le \frac{1}{a_{\min}} \|\mathbf{u}_2 - \bar{\mathbf{u}}_2\|_{L^2} \|\mathcal{L}_2 \mathbf{u}_2\|_{L^2},\end{aligned}$$

which, upon application of the left Poincaré inequality, yields the right statement.
Lemma A.3. *Assume that $\mathbf{m} \in C^1(\mathbb{R}^+; H^\infty(Y))$ is a solution to*

$$\partial_\tau \mathbf{m}(y,\tau) = \mathscr{L}\mathbf{m}(y,\tau) + \mathbf{F}(y,\tau), \tag{A.13a}$$
$$\mathbf{m}(y,0) = \mathbf{g}(y), \tag{A.13b}$$

where $\mathbf{F} \in C(\mathbb{R}^+; H^\infty(Y))$ and $\mathbf{g} \in H^\infty(Y)$. Then it holds for some constant $\gamma > 0$ that

$$\|\mathcal{P}\mathbf{m}(\cdot,\tau)\|_{L^2} \le e^{-\gamma\tau} \|\mathcal{P}\mathbf{g}\|_{L^2} + \int_0^\tau e^{-\gamma(\tau - s)} \|\mathcal{P}\mathbf{F}(\cdot,s)\|_{L^2} ds. \tag{A.14}$$

Moreover, there is a constant C and $\tilde\gamma \geq \gamma$ such that

$$\|\boldsymbol{\nabla}_y \mathcal{P}\mathbf{m}(\cdot,\tau)\|_{L^2} \leq C\left(e^{-\tilde\gamma\tau}\|\boldsymbol{\nabla}_y\mathcal{P}\mathbf{g}\|_{L^2} + \int_0^\tau e^{-\tilde\gamma(\tau-s)}\|\boldsymbol{\nabla}_y\mathcal{P}\mathbf{F}(\cdot,s)\|_{L^2}ds\right). \tag{A.15}$$

The constants C, γ and $\tilde\gamma$ are independent of τ, x, t, $\mathbf{F}$ and $\mathbf{g}$.

Proof. Suppose first that $\mathbf{F}\equiv 0$. From (A.9) in Lemma A.2 it follows that $\mathbf{m}_\perp := \mathcal{P}\mathbf{m}$ then satisfies

$$\begin{aligned} \partial_\tau \mathbf{m}_\perp(y,\tau) &= \mathscr{L}\mathbf{m}_\perp(y,\tau), \\ \mathbf{m}_\perp(y,0) &= \mathcal{P}\mathbf{g}(y). \end{aligned} \tag{A.16}$$

Multiplication of (A.16) by $\mathbf{m}$ and integration over Y yields, due to the triple product identites (2.1) and (2.2) and $\mathbf{m}_0$ being independent of y,

$$\begin{aligned} \int_Y \mathbf{m}_\perp\cdot\partial_\tau\mathbf{m}_\perp dy = \mathbf{m}_0\cdot\int_Y \mathbf{m}_\perp\times\mathcal{L}_2\mathbf{m}_\perp dy - \alpha\int \mathcal{L}_2(\mathbf{m}_0\cdot\mathbf{m}_\perp)(\mathbf{m}_0\cdot\mathbf{m}_\perp)dy \\ +\alpha\int \mathbf{m}_\perp\cdot\mathcal{L}_2\mathbf{m}_\perp dy. \end{aligned}$$

The first two terms here become zero, due to the integral identity (2.4) and since $\mathbf{m}_\perp$ is orthogonal to $\mathbf{m}_0$, $\mathbf{m}_\perp\cdot\mathbf{m}_0 = 0$. Using integration by parts and the Poincaré inequality (A.12) then gives

$$\frac{1}{2}\partial_\tau\|\mathbf{m}_\perp\|^2_{L^2(Y)} = -\alpha\int a\nabla_y\mathbf{m}:\nabla_y\mathbf{m}dy \leq -\alpha a_{\min}\|\nabla_y\mathbf{m}_\perp\|^2_{L^2(Y)} \leq -\gamma\|\mathbf{m}_\perp\|^2_{L^2(Y)},$$

where $\gamma = \alpha a_{\min}/C_P^2$. Here we also used the fact that the average of $\mathbf{m}_\perp$ is zero. Via the Grönwall inequality we then get

$$\|S^\tau\mathcal{P}\mathbf{g}\|^2_{L^2(Y)} = \|\mathbf{m}_\perp(\cdot,\tau)\|^2_{L^2(Y)} \leq e^{-2\gamma\tau}\|\mathbf{m}_\perp(\cdot,0)\|^2_{L^2(Y)} = e^{-2\gamma\tau}\|\mathcal{P}\mathbf{g}\|^2_{L^2(Y)}, \tag{A.17}$$

where S^τ is the solution operator of (A.16).

To show the second statement, (A.15), we can differentiate (A.16) and get

$$\begin{aligned} \partial_\tau\boldsymbol{\nabla}_y\mathbf{m}_\perp(y,\tau) &= \boldsymbol{\nabla}_y\mathscr{L}\mathbf{m}(y,\tau), \\ \boldsymbol{\nabla}_y\mathbf{m}_\perp(y,0) &= \boldsymbol{\nabla}_y\mathcal{P}\mathbf{g}(y). \end{aligned}$$

We multiply this by $a\boldsymbol{\nabla}_y\mathbf{m}_\perp$ and integrate over Y. Using integration by parts as in (2.4) as well as (2.1) and (2.2), then yields

$$\begin{aligned} \frac{1}{2}\partial_\tau\|a^{1/2}\boldsymbol{\nabla}_y\mathbf{m}_\perp\|^2_{L^2} &= \int_Y \mathcal{L}_2\mathbf{m}_\perp\cdot(\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_\perp)dy + \alpha\int_Y \mathcal{L}_2\mathbf{m}_\perp\cdot(\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{m}_\perp)dy \\ &= \int_Y \mathbf{m}_0\cdot(\mathcal{L}_2\mathbf{m}_\perp\times\mathcal{L}_2\mathbf{m}_\perp)dy + \alpha\int_Y (\mathbf{m}_0\cdot\mathcal{L}_2\mathbf{m}_\perp)^2 - |\mathcal{L}_2\mathbf{m}_\perp|^2 dy \\ &= -\alpha\|\mathcal{L}_2\mathbf{m}_\perp\|^2_{L^2(Y)}, \end{aligned}$$

due to the orthogonality of $\mathbf{m}_\perp$ and $\mathbf{m}_0$. We can now use the second version of the Poincaré inequality in (A.12) to show that

$$\frac{1}{2}\partial_\tau\|a^{1/2}\boldsymbol{\nabla}_y\mathbf{m}_\perp\|^2 \leq -\frac{\alpha a_{\min}^2}{C_P^2}\|\boldsymbol{\nabla}_y\mathbf{m}_\perp\|^2 \leq -\frac{\alpha a_{\min}^2}{C_P^2 a_{\max}}\|a^{1/2}\boldsymbol{\nabla}_y\mathbf{m}_\perp\|^2$$

and obtain by Grönwall's inequality,

$$\|a^{1/2}\boldsymbol{\nabla}_y\mathbf{m}_\perp\|^2 \le e^{-2\tilde\gamma\tau}\|a^{1/2}\mathcal{P}\boldsymbol{\nabla}_y\mathbf{g}\|^2, \qquad \tilde\gamma = \frac{a_{\min}}{a_{\max}}\gamma.$$

The resulting estimate for the gradient of $\mathbf{m}_\perp$ is then

$$\|\boldsymbol{\nabla}_y S^\tau\mathcal{P}\mathbf{g}\|_{L^2} = \|\boldsymbol{\nabla}_y\mathbf{m}_\perp(\cdot,\tau)\|_{L^2} \le \sqrt{\frac{a_{\max}}{a_{\min}}}e^{-\tilde\gamma\tau}\|\boldsymbol{\nabla}_y\mathcal{P}\mathbf{g}\|_{L^2}. \tag{A.18}$$

The estimates (A.14) and (A.15) now follow from (A.17), (A.18) and Duhamel's principle

$$\mathbf{m}(y,\tau) = S^\tau\mathbf{g} + \int_0^\tau S^{\tau-s}\mathbf{F}(y,s)ds.$$

This concludes the proof. □

We now continue to show the estimates for an arbitrary number of y-derivatives of $\mathbf{m}$, from which the following H^p-norm estimate follows.

Lemma A.4. *Assume that $\mathbf{m}\in C^1(\mathbb{R}^+;H^\infty(Y))$ is a solution to*

$$\partial_\tau\mathbf{m}(y,\tau) = \mathscr{L}\mathbf{m}(y,\tau) + \mathbf{F}(y,\tau), \tag{A.19a}$$

$$\mathbf{m}(y,0) = \mathbf{g}(y), \tag{A.19b}$$

where $\mathbf{F}\in C(\mathbb{R}^+;H^\infty(Y))$ and $\mathbf{g}\in H^\infty(Y)$. Then, for each integer $p\ge 0$, there are constants C and $\gamma>0$, independent of τ, x, t, $\mathbf{F}$ and $\mathbf{g}$, such that

$$\|\mathcal{P}\mathbf{m}(\cdot,\tau)\|_{H^p} \le C\left(e^{-\gamma\tau}\|\mathcal{P}\mathbf{g}\|_{H^p} + \int_0^\tau e^{-\gamma(\tau-s)}\|\mathcal{P}\mathbf{F}(\cdot,s)\|_{H^p}ds\right). \tag{A.20}$$

Proof. Suppose $p=2k+\ell$ with $\ell\in\{0,1\}$ and let $\mathbf{u} := \mathcal{L}_2^k\mathbf{m}$. Then $\mathbf{u}$ satisfies a differential equation with the same structure as in (A.19),

$$\partial_\tau\mathbf{u} = \mathscr{L}\mathbf{u} + \mathcal{L}_2^k\mathbf{F}, \tag{A.21a}$$

$$\mathbf{u}(y,0) = \mathcal{L}_2^k\mathbf{g}(y). \tag{A.21b}$$

One therefore obtains from Lemma A.3 that

$$\|\mathcal{P}\mathcal{L}_2^k\mathbf{m}(\cdot,\tau)\|_{L^2} = \|\mathcal{P}\mathbf{u}(\cdot,\tau)\|_{L^2} \le e^{-\gamma\tau}\|\mathcal{P}\mathcal{L}_2^k\mathbf{g}\|_{L^2} + \int_0^\tau e^{-\gamma(\tau-s)}\|\mathcal{P}\mathcal{L}_2^k\mathbf{F}(\cdot,s)\|_{L^2}ds. \tag{A.22}$$

By (A.9) in Lemma A.2 we know that $\mathcal{P}$ and $\mathcal{L}_2^k$ commute, which together with (A2), $a\in C^\infty$, entails that there is a constant C such that for any $\mathbf{v}\in H^{2k}(Y)$,

$$\|\mathcal{P}\mathcal{L}_2^k\mathbf{v}\|_{L^2} = \|\mathcal{L}_2^k\mathcal{P}\mathbf{v}\|_{L^2} \le C\|\mathcal{P}\mathbf{v}\|_{H^{2k}}.$$

Hence, the right-hand side in (A.22) is bounded by the right-hand side in (A.20). By elliptic regularity, it holds for an arbitrary function $\mathbf{u}_p$ in $H^p(Y;\mathbb{R}^3)$ that

$$\|\mathbf{u}_{2p}\|_{H^{2p}} \le C\sum_{j=0}^p\|\mathcal{L}_2^j\mathbf{u}_{2p}\|_{L^2}, \qquad \|\mathbf{u}_{2p+1}\|_{H^{2p+1}} \le C\Big(\|\mathcal{L}_2^p\mathbf{u}_{2p+1}\|_{H^1} + \sum_{j=0}^{p-1}\|\mathcal{L}_2^j\mathbf{u}_{2p+1}\|_{L^2}\Big), \tag{A.23}$$

where C are constants independent of the functions $\mathbf{u}_{2p}$ and $\mathbf{u}_{2p+1}$.

For even p (when $\ell=0$), it then follows from the elliptic regularity (A.23), that

$$\|\mathcal{P}\mathbf{m}(\cdot,\tau)\|_{H^{2k}} \le C\sum_{j=0}^{k}\|\mathcal{L}_2^j\mathcal{P}\mathbf{m}(\cdot,\tau)\|_{L^2},$$

and consequently, we obtain (A.20). For odd p, when $\ell=1$, we use (A.15) in Lemma A.3 applied to (A.21), which yields

$$\begin{aligned}\|\boldsymbol{\nabla}_y\mathcal{P}\mathcal{L}_2^k\mathbf{m}(\cdot,\tau)\|_{L^2} &= \|\boldsymbol{\nabla}_y\mathcal{P}\mathbf{u}(\cdot,\tau)\|_{L^2}\\ &\le C\left(e^{-\tilde{\gamma}\tau}\|\boldsymbol{\nabla}_y\mathcal{P}\mathcal{L}_2^k\mathbf{g}\|_{L^2}+\int_0^\tau e^{-\tilde{\gamma}(\tau-s)}\|\boldsymbol{\nabla}_y\mathcal{P}\mathcal{L}_2^k\mathbf{F}(\cdot,s)\|_{L^2}ds\right).\end{aligned}$$

This gives the extra H^1-norm needed to use the second variant of elliptic regularity in (A.23). Combined with the fact that $\|\mathcal{P}\boldsymbol{\nabla}\mathcal{L}_2^k\mathbf{v}\|_{L^2}\le C\|\mathcal{P}\mathbf{v}\|_{H^{2k+1}}$ for $\mathbf{v}\in H^{2k+1}(Y)$ we obtain (A.20) also for odd p. □

A.4. Estimate for $\mathcal{P}$ part: x-derivatives. To obtain an estimate for the norms of x-derivatives of $\mathbf{m}$ satsfying (A.1), we again aim to split the function we want to estimate. In the process, the order in which partial derivatives are taken matters. To handle this, let $\beta=[\beta_1,\beta_2,\dots,\beta_n]$ be an *ordered* multi-index with n elements. Each of these elements β_j, $j=1,...,n$, specifies one of the coordinate directions. The length of any ordered multi-index is the number of elements it contains, $|\beta|=n$. We call η an ordered subset of β, denoted $\eta\subset\beta$, if it contains any selection of elements in β where the order of elements is not changed compared to their order in β. Each η is again an ordered multi-index.

Moreover, we want to be able to extend a given ordered multi-index with additional coordinate directions. This is denoted as follows: if β is as above and we add an additional coordinate direction k, then we write

$$\bar{\beta}=[\beta;k]:=[\beta_1,\beta_2,...,\beta_n,k], \qquad |\bar{\beta}|=n+1.$$

When using the ordered multi-index for denoting partial derivatives we define for $\mathbf{f}\in H^{|\beta|}(\Omega)$,

$$\partial_x^\beta\mathbf{f}:=\partial_{x_{\beta_n}}\partial_{x_{\beta_{n-1}}}\cdots\,\partial_{x_{\beta_1}}\mathbf{f}.$$

We also introduce the operator D, where the partial derivatives are interlaced with the projection $\mathcal{P}$,

$$D^\beta\mathbf{f}:=\mathcal{P}\partial_{x_{\beta_n}}\mathcal{P}\partial_{x_{\beta_{n-1}}}\cdots\,\mathcal{P}\partial_{x_{\beta_1}}\mathcal{P}\mathbf{f}, \qquad D^0\mathbf{f}:=\mathcal{P}\mathbf{f}.$$

This operator is useful since if $\mathbf{m}$ satisfies (A.1) then $D^\beta\mathbf{m}$ satisfies the same PDE where $\mathbf{F}$ and $\mathbf{g}$ are replaced by $D^\beta\mathbf{F}$ and $D^\beta\mathbf{g}$. We can then proceed to estimate $D^\beta\mathbf{m}$ using Lemma A.4. We get the following lemma.

LEMMA A.5. *Let β be an ordered multi-index. Suppose $D^\beta\mathbf{F}(x,\cdot,t,\cdot)\in C(\mathbb{R}^+;H^\infty(Y))$ and $D^\beta\mathbf{g}(x,\cdot,t)\in H^\infty(Y)$. Then, for each integer $p\ge 0$, there are constants C and $\gamma>0$, independent of τ, x, t, $\mathbf{F}$ and $\mathbf{g}$, such that*

$$\|D^\beta\mathbf{m}(x,\cdot,t,\tau)\|_{H^p(Y)} \le C\Big(e^{-\gamma\tau}\|D^\beta\mathbf{g}(x,\cdot,t)\|_{H^p(Y)}$$

$$+\int_0^\tau e^{-\gamma(\tau-s)}\|D^\beta \mathbf{F}(x,\cdot,t,s)\|_{H^p(Y)}ds\Bigg).$$

Proof. We first show that $\mathbf{w}_\beta(x,y,t,\tau) := D^\beta\mathbf{m}(x,y,t,\tau)$ satisfies

$$\partial_\tau \mathbf{w}_\beta(x,y,t,\tau) = \mathscr{L}\mathbf{w}_\beta(x,y,t,\tau) + D^\beta\mathbf{F}(x,y,t,\tau), \tag{A.24a}$$
$$\mathbf{w}_\beta(x,y,t,0) = D^\beta\mathbf{g}(x,y,t), \tag{A.24b}$$

given that $\mathbf{m}$ satisfies (A.1). We use induction and consider first the case $\beta=0$. Then based on Lemma A.2, we have

$$\partial_\tau\mathbf{w}_0 = \partial_\tau\mathcal{P}\mathbf{m} = \mathscr{L}\mathcal{P}\mathbf{m} + \mathcal{P}\mathbf{F} = \mathscr{L}\mathbf{w}_0 + \mathcal{P}\mathbf{F}.$$

In general, assume that (A.24) holds for some β and consider $\bar\beta := [\beta;k]$. As $\mathbf{w}_{\bar\beta} = \mathcal{P}\partial_{x_k}\mathbf{w}_\beta$, it satisfies

$$\begin{aligned}\partial_\tau\mathbf{w}_{\bar\beta} &= \mathcal{P}\partial_{x_k}\partial_\tau\mathbf{w}_\beta = \mathcal{P}\partial_{x_k}(\mathscr{L}\mathbf{w}_\beta + D^\beta\mathbf{F})\\ &= \mathscr{L}\mathcal{P}\partial_{x_k}\mathbf{w}_\beta - \mathcal{P}(\partial_{x_k}\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta)\\ &\qquad + \alpha\mathcal{P}(\partial_{x_k}\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta + \mathbf{m}_0\times\partial_{x_k}\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta) + D^{\bar\beta}\mathbf{F},\end{aligned}$$

as $\mathscr{L}$ and $\mathcal{P}$ commute, as shown in Lemma A.2. Since $\mathbf{m}_0\perp\partial_{x_k}\mathbf{m}_0$ for any k given that $|\mathbf{m}_0|\equiv 1$, and by the definition of $\mathcal{P}$, $\mathbf{m}_0\perp\mathcal{L}_2\mathbf{w}_\beta$, we can conclude that $\partial_{x_k}\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta = \kappa\mathbf{m}_0$ for some $\kappa\in\mathbb{R}$ and thus $\mathcal{P}(\partial_{x_k}\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta) = \kappa\mathcal{P}\mathbf{m}_0 = 0$. Moreover,

$$\mathbf{m}_0\times\partial_{x_k}\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta = \kappa\mathbf{m}_0\times\mathbf{m}_0 = 0,$$

and by the triple product identity (2.2),

$$\mathcal{P}(\partial_{x_k}\mathbf{m}_0\times\mathbf{m}_0\times\mathcal{L}_2\mathbf{w}_\beta) = \mathcal{P}(\partial_{x_k}\mathbf{m}_0\cdot\mathcal{L}_2\mathbf{w}_\beta)\mathbf{m}_0 - \mathcal{P}(\mathbf{m}_0\cdot\partial_{x_k}\mathbf{m}_0)\mathcal{L}_2\mathbf{w}_\beta = 0.$$

This proves the claim that $\mathbf{w}_\beta$ satisfies (A.24) by induction.

Since $\mathbf{w}_\beta$ satisfies (A.24) and $\mathcal{Q}\mathbf{w}_\beta = 0$, it follows from Lemma A.4 that

$$\|\mathbf{w}_\beta\|_{H^p(Y)} \le C\left(e^{-\gamma\tau}\|D^\beta\mathbf{g}\|_{H^p(Y)} + \int_0^\tau e^{-\gamma(\tau-s)}\|D^\beta\mathbf{F}\|_{H^p(Y)}ds\right).$$

This concludes the proof. □

For the final estimate of $\mathbf{m}$ we also need to exploit the fact that the usual Sobolev norms of $\mathcal{P}\mathbf{m}$ are equivalent to Sobolev norms based on D, as follows.

Lemma A.6. *Assume* (3.6) *and let* $0\le q\le r$. *Then for all* $\mathbf{f}\in H^q(\Omega;\mathbb{R}^3)$ *and* $0\le t\le T$,

$$C_0||\mathcal{P}\mathbf{f}||^2_{H^q(\Omega)} \le \sum_{|\beta|\le q}||D^\beta\mathbf{f}||^2_{L^2(\Omega)} \le C_1||\mathcal{P}\mathbf{f}||^2_{H^q(\Omega)},$$

where C_0 *and* C_1 *are independent of* t.

Proof. In the first step we prove that for each ordered multi-index β with $|\beta|\le q\le r$ there are $\mathbf{c}_{\eta,\beta}\in C(0,T;H^{r-|\beta|+|\eta|}(\Omega))$ such that

$$\partial_x^\beta\mathcal{P}\mathbf{f} = D^\beta\mathbf{f} + \sum_{\substack{\eta\subset\beta\\|\eta|<|\beta|}}\mathbf{c}_{\eta,\beta}D^\eta\mathbf{f}. \tag{A.25}$$

We again use induction. The statement is trivially true for $\beta = 0$. Assume now that (A.25) holds for all β with $0 \leq |\beta| < q$ and that the corresponding coefficient functions $\mathbf{c}_{\eta,\beta} \in C(0,T;H^{r-|\beta|+|\eta|}(\Omega))$. Let $|\beta| = q' < q$ and consider the ordered multi-index $\bar{\beta} = [\beta; j]$. From (A.25) we then get, using Lemma A.2, that

$$\begin{aligned}
\partial_x^{\bar{\beta}} \mathcal{P}\mathbf{f} = \partial_{x_j} \left(D^\beta \mathbf{f} + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} \mathbf{c}_{\eta,\beta} D^\eta \mathbf{f} \right) &= \mathcal{P}\partial_{x_j} D^\beta \mathbf{f} + \mathcal{Q}\partial_{x_j} D^\beta \mathbf{f} + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} (\partial_{x_j} \mathbf{c}_{\eta,\beta}) D^\eta \mathbf{f} \\
&\quad + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} \mathbf{c}_{\eta,\beta} \mathcal{P}\partial_{x_j} D^\eta \mathbf{f} + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} \mathbf{c}_{\eta,\beta} \mathcal{Q}\partial_{x_j} D^\eta \mathbf{f} \\
&= D^{\bar{\beta}} \mathbf{f} - (\partial_{x_j} \mathbf{M}) D^\beta \mathbf{f} + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} (\partial_{x_j} \mathbf{c}_{\eta,\beta}) D^\eta \mathbf{f} \\
&\quad + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} \mathbf{c}_{\eta,\beta} D^{[\eta;j]} \mathbf{f} - \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} \mathbf{c}_{\eta,\beta} (\partial_{x_j} \mathbf{M}) D^\eta \mathbf{f}.
\end{aligned}$$

We note that all terms to the right of $D^{\bar{\beta}}\mathbf{f}$ involve ordered multi-indices of length at most $|\beta|$, which are subsets of $\bar{\beta}$. Moreover, the order of their elements is the same as their order in $\bar{\beta}$. Thus, $\partial_x^{\bar{\beta}} \mathcal{P}\mathbf{f}$ satisfies (A.25) for some coefficient functions $\mathbf{c}_{\eta,\bar{\beta}}$.

Finally, we need to show that the coefficient functions belong to $C(0,T;H^{r-|\bar{\beta}|+|\eta|}(\Omega))$. This is the case since

$$\begin{aligned}
\partial_{x_j} \mathbf{M} &\in C(0,T;H^{r-1}(\Omega)) = C(0,T;H^{r-|\bar{\beta}|+|\beta|}(\Omega)), \\
\partial_{x_j} \mathbf{c}_{\eta,\beta} &\in C(0,T;H^{r-|\beta|+|\eta|-1}(\Omega)) = C(0,T;H^{r-|\bar{\beta}|+|\eta|}(\Omega)), \\
\mathbf{c}_{\eta,\beta} &\in C(0,T;H^{r-|\beta|+|\eta|}(\Omega)) = C(0,T;H^{r-|\bar{\beta}|+|[\eta;j]|}(\Omega)).
\end{aligned}$$

For the last term we use Lemma 5.3, with $q_0 = r - |\bar{\beta}| + |\eta|$, $q_1 = r - |\beta| + |\eta|$ and $q_2 = r - 1$. This is a valid choice since $|\beta| < |\bar{\beta}|$, which implies that $q_0 \leq \min(q_1, q_2)$ and $q_1 + q_2 = q_0 + |\bar{\beta}| - |\beta| + r - 1 \geq q_0 + r$. We get

$$||\mathbf{c}_{\eta,\beta}(\cdot,t)\partial_{x_j}\mathbf{M}(\cdot,t)||_{H^{r-|\bar{\beta}|+|\eta|}} \leq C||\mathbf{c}_{\eta,\beta}(\cdot,t)||_{H^{r-|\beta|+|\eta|}} ||\partial_{x_j}\mathbf{M}(\cdot,t)||_{H^{r-1}} \leq C.$$

This proves the first claim.

We next note that the boundedness of $\mathcal{P}$ in (7.6) implies that for $\eta = [j]$ and $q' \leq q$,

$$||D^\eta \mathbf{f}||^2_{H^{q'-1}} = ||\mathcal{P}\partial_{x_j}\mathcal{P}\mathbf{f}||_{H^{q'-1}} \leq C||\partial_{x_j}\mathcal{P}\mathbf{f}||_{H^{q'-1}} \leq C||\mathcal{P}\mathbf{f}||^2_{H^{q'}}.$$

By induction it follows that for general $|\eta| \leq q' \leq q$,

$$||D^\eta \mathbf{f}||_{H^{q'-|\eta|}} \leq C||\mathcal{P}\mathbf{f}||_{H^{q'}}. \tag{A.26}$$

This shows the right inequality in the lemma upon taking $q' = |\eta|$.

Furthermore, from (A.25) and (A.26) we get

$$||\mathcal{P}\mathbf{f}||^2_{H^q} = \sum_{|\beta| \leq q} ||\partial_x^\beta \mathcal{P}\mathbf{f}||^2_{L^2} \leq C \sum_{|\beta| \leq q} \left(||D^\beta \mathbf{f}||^2_{L^2} + \sum_{\substack{\eta \subset \beta \\ |\eta| < |\beta|}} ||\mathbf{c}_{\eta,\beta} D^\eta \mathbf{f}||^2_{L^2} \right)$$

$$\leq C\sum_{|\beta|\leq q}||D^\beta \mathbf{f}||_{L^2}^2 + C\sum_{|\beta|\leq q}\sum_{\substack{\eta\subset\beta\\|\eta|<|\beta|}}||\mathbf{c}_{\eta,\beta}||^2_{H^{r-|\beta|+|\eta|}}||D^\eta\mathbf{f}||^2_{H^{|\beta|-|\eta|-1}}$$

$$\leq C\sum_{|\beta|\leq q}||D^\beta \mathbf{f}||_{L^2}^2 + C\sum_{|\beta|\leq q}||\mathcal{P}\mathbf{f}||^2_{H^{|\beta|-1}} \leq C\sum_{|\beta|\leq q}||D^\beta \mathbf{f}||_{L^2}^2 + C||\mathcal{P}\mathbf{f}||^2_{H^{q-1}}.$$

To bound the L^2-norm of $\mathbf{c}_{\eta,\beta}D^\eta\mathbf{f}$ we used Lemma 5.3 with $q_0=0$, $q_1=r-|\beta|+|\eta|\geq 0$ and $q_2=|\beta|-|\eta|-1\geq 0$, which is valid since $q_1+q_2=r-1\geq q_0+4$. The left inequality in the lemma now follows by induction. □

We have left to put everything together to show Theorem 7.1 for $k=0$, which is the estimate

$$\|\mathcal{P}\mathbf{m}(\cdot,\cdot,t,\tau)\|_{H^{q,p}} \leq C\left(e^{-\gamma\tau}\|\mathcal{P}\mathbf{g}(\cdot,\cdot,t)\|_{H^{q,p}} + \int_0^\tau e^{-\gamma(\tau-s)}\|\mathcal{P}\mathbf{F}(\cdot,\cdot,t,s)\|_{H^{q,p}}ds\right). \tag{A.27}$$

First we note that as a consequence of Lemma A.6, it holds for $\mathbf{f}\in H^{q,p}$, when $0\leq q\leq r$ and $p\geq 0$, that

$$C_0||\mathcal{P}\mathbf{f}||^2_{H^{q,p}} \leq \sum_{|\beta|\leq q}||D^\beta\mathbf{f}||^2_{H^{0,p}} \leq C_1||\mathcal{P}\mathbf{f}||^2_{H^{q,p}}. \tag{A.28}$$

This is true since

$$C_0||\mathcal{P}\mathbf{f}||^2_{H^{q,p}} = C_0\sum_{|\alpha|\leq p}\int_Y||\mathcal{P}\partial_y^\alpha\mathbf{f}(\cdot,y)||^2_{H^q}dy \leq \sum_{|\alpha|\leq p}\sum_{|\beta|\leq q}\int_Y||D^\beta\mathcal{P}\partial_y^\alpha\mathbf{f}(\cdot,y)||^2_{L^2}dy$$
$$= \sum_{|\beta|\leq q}||D^\beta\mathbf{f}||^2_{H^{0,p}},$$

where we used the fact that $D\partial_y^\alpha = \partial_y^\alpha D$. The second inequality follows in the same way. Then by Lemma A.5,

$$||\mathcal{P}\mathbf{m}(\cdot,\cdot,t,\tau)||^2_{H^{q,p}} \leq C\sum_{|\beta|\leq q}||D^\beta\mathbf{m}(\cdot,\cdot,t,\tau)||^2_{H^{0,p}} = C\sum_{|\beta|\leq q}\int_\Omega||D^\beta\mathbf{m}(x,\cdot,t,\tau)||^2_{H^p}dx$$

$$\leq C\sum_{|\beta|\leq q}\int_\Omega e^{-2\gamma\tau}\|D^\beta\mathbf{g}(x,\cdot,t)\|^2_{H^p} + C\sum_{|\beta|\leq q}\int_\Omega\left(\int_0^\tau e^{-\gamma(\tau-s)}\|D^\beta\mathbf{F}(x,\cdot,t,s)\|_{H^p}ds\right)^2dx.$$

For the first term we have from (A.28),

$$\sum_{|\beta|\leq q}\int_\Omega e^{-2\gamma\tau}\|D^\beta\mathbf{g}(x,\cdot,t)\|^2_{H^p}dx = e^{-2\gamma\tau}\sum_{|\beta|\leq q}\|D^\beta\mathbf{g}(\cdot,\cdot,t)\|^2_{H^{0,p}} \leq Ce^{-2\gamma\tau}\|\mathcal{P}\mathbf{g}(\cdot,\cdot,t)\|^2_{H^{q,p}}.$$

For the second term we use the general estimates

$$\sum_n||f_n||^2_{L^1} \leq \left\|\left(\sum_n|f_n(x)|^2\right)^{1/2}\right\|^2_{L^1}, \qquad \int||f(x,\cdot)||^2_{L^1}dx \leq \left(\int||f(\cdot,y)||_{L^2}dy\right)^2,$$

which, again together with (A.28), give

$$\sum_{|\beta|\leq q}\int_\Omega\left(\int_0^\tau e^{-\gamma(\tau-s)}\|D^\beta\mathbf{F}(x,\cdot,t,s)\|_{H^p}ds\right)^2dx$$

$$
\begin{aligned}
&\leq \int_\Omega \left(\int_0^\tau e^{-\gamma(\tau-s)} \left(\sum_{|\beta|\leq q} \|D^\beta \mathbf{F}(x,\cdot,t,s)\|_{H^p}^2 \right)^{1/2} ds \right)^2 dx \\
&\leq \left(\int_0^\tau e^{-\gamma(\tau-s)} \left(\sum_{|\beta|\leq q} \|D^\beta \mathbf{F}(\cdot,\cdot,t,s)\|_{H^{0,p}}^2 \right)^{1/2} ds \right)^2 \\
&\leq C \left(\int_0^\tau e^{-\gamma(\tau-s)} \|\mathcal{P}\mathbf{F}(\cdot,\cdot,t,s)\|_{H^{q,p}}^2 ds \right)^2 .
\end{aligned}
$$

This finally shows (A.27).

A.5. Estimate for $\mathcal{P}$ part: time derivative. In this final part of the proof of Theorem 7.1, an estimate for $\|\partial_t^k \mathcal{P}\mathbf{m}(\cdot,\cdot,t,\tau)\|_{H^{q,p}}$ is derived. We proceed in a similar way as in the previous section and define

$$
D_t^k = (\mathcal{P}\partial_t)^k \mathcal{P}, \qquad k \geq 0,
$$

and let $\mathbf{w}_k = D_t^k \mathbf{m}$. Then

$$
\begin{aligned}
\partial_\tau \mathbf{w}_k(x,y,t,\tau) &= \mathscr{L}\mathbf{w}_k(x,y,t,\tau) + D_t^k \mathbf{F}(x,y,t,\tau), \\
\mathbf{w}_k(x,y,t,0) &= D_t^k \mathbf{g}(x,y,t).
\end{aligned}
$$

This can be shown by an argument similar to the one in the proof of Lemma A.5. Since $\mathcal{Q}\mathbf{w} = 0$, (A.27) then gives

$$
\|\mathbf{w}_k(\cdot,\cdot,t,\tau)\|_{H^{q,p}} \leq C \left(e^{-\gamma\tau} \|D_t^k \mathbf{g}(\cdot,\cdot)\|_{H^{q,p}} + \int_0^\tau e^{-\gamma(\tau-s)} \|D_t^k \mathbf{F}(\cdot,\cdot,t,\tau)\|_{H^{q,p}} ds \right). \tag{A.29}
$$

We also need the following lemma, which corresponds to Lemma A.6 in the previous section.

LEMMA A.7. *Assume (3.6) and suppose $\partial_t^\ell \mathbf{f} \in H^{q-2\ell}(\Omega;\mathbb{R}^3)$ for $0 \leq 2\ell \leq 2k \leq q \leq r$ and $p \geq 0$. Then*

$$
C_0 \sum_{\ell=0}^k ||\partial_t^\ell \mathcal{P}\mathbf{f}||_{H^{q-2\ell,p}} \leq \sum_{\ell=0}^k ||D_t^\ell \mathbf{f}||_{H^{q-2\ell,p}} \leq C_1 \sum_{\ell=0}^k ||\partial_t^\ell \mathcal{P}\mathbf{f}||_{H^{q-2\ell,p}}, \qquad t \in [0,T],
$$

where C_0 and C_1 are independent of t.

Proof. In the first step we prove that for each k with $0 \leq q \leq r-2k$ there are functions $\mathbf{c}_{\ell,k}$ satisfying

$$
\partial_t^s \mathbf{c}_{\ell,k} \in C(0,T;H^{r-2k+2\ell-2s}(\Omega)), \qquad r-2k+2\ell-2s \geq 0,
$$

for which

$$
\partial_t^k \mathcal{P}\mathbf{f} = D_t^k \mathbf{f} + \sum_{\ell=0}^{k-1} \mathbf{c}_{\ell,k} D_t^\ell \mathbf{f}. \tag{A.30}
$$

We use induction. The statement is trivially true for $k=0$. Assume that (A.30) holds upto k and that $2(k+1)\le r$. From (A.30) we then get

$$
\begin{aligned}
\partial_t^{k+1}\mathcal{P}\mathbf{f} &= \partial_t\left(D_t^k\mathbf{f}+\sum_{\ell=0}^{k-1}\mathbf{c}_{\ell,k}D_t^\ell\mathbf{f}\right)\\
&= \mathcal{P}\partial_t D_t^k\mathbf{f}+\mathcal{Q}\partial_t D_t^k\mathbf{f}+\sum_{\ell=0}^{k-1}(\partial_t\mathbf{c}_{\ell,k})D_t^\ell\mathbf{f}+\sum_{\ell=0}^{k-1}\mathbf{c}_{\ell,k}\mathcal{P}\partial_t D_t^\ell\mathbf{f}+\sum_{\ell=0}^{k-1}\mathbf{c}_{\ell,k}\mathcal{Q}\partial_t D_t^\ell\mathbf{f}\\
&= D_t^{k+1}\mathbf{f}-(\partial_t\mathbf{M})D_t^k\mathbf{f}+\sum_{\ell=0}^{k-1}(\partial_t\mathbf{c}_{\ell,k})D_t^\ell\mathbf{f}+\sum_{\ell=0}^{k-1}\mathbf{c}_{\ell,k}D^{\ell+1}\mathbf{f}-\sum_{\ell=0}^{k-1}\mathbf{c}_{\ell,k}(\partial_t\mathbf{M})D_t^\ell\mathbf{f}.
\end{aligned}
$$

For the coefficient functions we have by Lemma A.1

$$
\begin{aligned}
\partial_t^{s+1}\mathbf{M} &\in C(0,T;H^{r-2-2s}(\Omega))=C(0,T;H^{r-2(k+1)+2k-2s}(\Omega)),\\
\partial_t^{s+1}\mathbf{c}_{\ell,k} &\in C(0,T;H^{r-2k+2\ell-2(s+1)}(\Omega))=C(0,T;H^{r-2(k+1)+2\ell-2s}(\Omega)),\\
\partial_t^{s}\mathbf{c}_{\ell,k} &\in C(0,T;H^{r-2k+2\ell-2s}(\Omega))=C(0,T;H^{r-2(k+1)+2(\ell+1)-2s}(\Omega)),
\end{aligned}
$$

and by Lemma 5.3,

$$
\begin{aligned}
||\partial_t^s(\mathbf{c}_{\ell,k}(\cdot,t)\partial_t\mathbf{M}(\cdot,t))||_{H^{q_0}} &\le C\sum_{\eta=0}^{s}\left\|[\partial_t^\eta\mathbf{c}_{\ell,k}(\cdot,t)][\partial_t^{s-\eta+1}\mathbf{M}(\cdot,t)]\right\|_{H^{q_0}}\\
&\le C\sum_{\eta=0}^{s}\|\partial_t^\eta\mathbf{c}_{\ell,k}(\cdot,t)\|_{H^{q_{1,\eta}}}\left\|\partial_t^{s-\eta+1}\mathbf{M}(\cdot,t)\right\|_{H^{q_{2,\eta}}},
\end{aligned}
$$

where

$$
q_0=r-2(k+1)+2\ell-2s,\qquad q_{1,\eta}=r-2k+2\ell-2\eta,\qquad q_{2,\eta}=r-2s+2\eta-2.
$$

To check that Lemma 5.3 can indeed be used we note that

$$
q_0=q_{1,\eta}-2(s-\eta+1)=q_{2,\eta}-2(\eta+k-\ell)\le\min(q_{1,\eta},q_{2,\eta}),\quad\text{and}\quad q_1+q_2=q_0+r.
$$

This proves (A.30). We next show that for $0\le\ell\le k$,

$$
||D_t^\ell\mathbf{f}||_{H^{q-2\ell,p}}\le C\sum_{j=0}^{\ell}||\partial_t^j\mathcal{P}\mathbf{f}||_{H^{q-2j,p}}. \tag{A.31}
$$

This is true for $\ell=0$, and if true for $\ell<k$, then by (7.6),

$$
\begin{aligned}
||D_t^{\ell+1}\mathbf{f}||_{H^{q-2-2\ell,p}} &\le C\sum_{j=0}^{\ell}||\partial_t^j D_t\mathbf{f}||_{H^{q-2-2j,p}}=C\sum_{j=0}^{\ell}||\partial_t^j\mathcal{P}\partial_t\mathcal{P}\mathbf{f}||_{H^{q-2-2j,p}}\\
&\le C\sum_{j=0}^{\ell}\sum_{s=0}^{j}||\partial_t^{s+1}\mathcal{P}\mathbf{f}||_{H^{q-2-2s,p}}\le C\sum_{j=0}^{\ell+1}||\partial_t^j\mathcal{P}\mathbf{f}||_{H^{q-2j,p}}.
\end{aligned}
$$

Hence, (A.31) follows by induction. This shows the right inequality in the lemma upon summing over ℓ. Next, (A.31) and (A.30) give us the following estimate for $0\le 2k\le q$,

$$
||\partial_t^k\mathcal{P}\mathbf{f}||_{H^{q-2k,p}}\le||D_t^k\mathbf{f}||_{H^{q-2k,p}}+\sum_{\ell=0}^{k-1}||\mathbf{c}_{\ell,k}||_{H^{r-2k+2\ell}}||D_t^\ell\mathbf{f}||_{H^{q-2\ell,p}}
$$

$$
\begin{aligned}
&\le ||D_t^k \mathbf{f}||_{H^{q-2k,p}} + \sum_{\ell=0}^{k-1} ||D_t^\ell \mathbf{f}||_{H^{q-2\ell,p}} \\
&\le ||D_t^k \mathbf{f}||_{H^{q-2k,p}} + C\sum_{\ell=0}^{k-1} ||\partial_t^\ell \mathcal{P}\mathbf{f}||_{H^{q-2\ell,p}},
\end{aligned}
$$

where Lemma 5.3 was used again, this time with $q_0 = q-2k$, $q_1 = r-2k+2l \ge q_0$, $q_2 = q-2l \ge q_0$ for $l<k$, and $q_1+q_2 = r+q_0$. The left inequality in the lemma follows by induction. □

Two applications of Lemma A.7 and (A.29) now lead to

$$
\begin{aligned}
\|\partial_t^k \mathcal{P}\mathbf{m}(\cdot,\cdot,t,\tau)\|_{H^{q-2k,p}} &\le C\sum_{\ell=0}^{k} \|\mathbf{w}_\ell\|_{H^{q-2\ell,p}} \\
&\le C\sum_{\ell=0}^{k}\left(e^{-\gamma\tau}\|D_t^\ell \mathbf{g}\|_{H^{q-2\ell,p}} + \int_0^\tau e^{-\gamma(\tau-s)}\|D_t^\ell \mathbf{F}\|_{H^{q-2\ell,p}} ds\right) \\
&\le C\sum_{\ell=0}^{k}\left(e^{-\gamma\tau}\|\partial_t^\ell \mathcal{P}\mathbf{g}\|_{H^{q-2\ell,p}} + \int_0^\tau e^{-\gamma(\tau-s)}\|\partial_t^\ell \mathcal{P}\mathbf{F}\|_{H^{q-2\ell,p}} ds\right).
\end{aligned}
$$

Theorem 7.1 is proved.

Appendix B. Energy estimates nonlinear equation.

In this appendix, we derive energy estimates for the Landau-Lifshitz equation with a material coefficient. We consider first a matrix-valued, constant coefficient, which is relevant for the homogenized equation (3.2), and then the case of a highly oscillatory, scalar coefficient, as in (3.1). These estimates are generalizations of an estimate in [24] and can be seen as a step on the way to prove existence of solutions to the Landau-Lifshitz equation with material coefficient.

B.1. Constant matrix-valued coefficient. Here we derive a priori estimates for the solution of the Landau–Lifshitz Gilbert equation when

$$
\mathbf{H}(\mathbf{m}) = \mathcal{L}\mathbf{m}, \qquad Lu = \nabla\cdot(\mathbf{A}\nabla u), \qquad \mathcal{L}\mathbf{m} = \left[Lm^{(1)}, Lm^{(2)}, Lm^{(3)}\right]^T,
$$

with $\mathbf{A}$ being a constant, symmetric positive definite matrix. To obtain the estimates we assume that an L^∞ bound on the gradient $\boldsymbol{\nabla}\mathbf{m}$ is given.

To simplify notation in this section, we define a norm on $\mathbb{R}^{3\times n}$, weighted by the coefficient matrix $\mathbf{A}\in\mathbb{R}^{n\times n}$. For a matrix-valued function $\mathbf{B}\in\mathbb{R}^{3\times n}$, let

$$
|\mathbf{B}|_A^2 := \mathbf{B}:(\mathbf{B}\mathbf{A}),
$$

and the corresponding L^2-norm on $\Omega\to\mathbb{R}^{3\times n}$ is given by

$$
\|\mathbf{B}\|_{0,A}^2 := \int_\Omega |\mathbf{B}(x)|_A^2 dx = \int_\Omega \mathbf{B}:(\mathbf{B}\mathbf{A})dx.
$$

We furthermore consider several identities that are useful in the following proof. First, note that since the cross product of a vector by itself is zero, the cross product of a vector $\mathbf{v}$ and $\mathcal{L}\mathbf{v}$ can be rewritten as

$$
\mathbf{v}\times\mathcal{L}\mathbf{v} = \nabla\cdot(\mathbf{v}\times(\nabla\mathbf{v}\mathbf{A})). \tag{B.1}
$$

For the special case of a vector-function $\mathbf{v}$ with constant length throughout the domain, $|\mathbf{v}(x)| \equiv \text{const}$, the dot product between $\mathbf{v}$ and its gradient is zero, $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}=\mathbf{0}$. Then it follows from the vector triple produce identity (2.2) that

$$-\mathbf{v}\times\mathbf{v}\times\mathcal{L}\mathbf{v}=|\mathbf{v}|^2\mathcal{L}\mathbf{v}+(\boldsymbol{\nabla}\mathbf{v}:\boldsymbol{\nabla}\mathbf{A})\mathbf{v}. \tag{B.2}$$

Moreover, (2.1) implies that

$$\mathbf{B}:(\mathbf{v}\times\mathbf{B})\mathbf{A}=\mathbf{0}. \tag{B.3}$$

We can then prove the following energy estimate.

Theorem B.1. *Suppose* $\mathbf{m}\in\mathcal{C}^0((0,T);H^q(\mathbb{R}^n))$ *is a solution of*

$$\partial_t\mathbf{m}=-\mathbf{m}\times\mathcal{L}\mathbf{m}-\alpha\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m}, \tag{B.4}$$

where $0<\alpha\le 1$ *and* $L=\nabla\cdot(\mathbf{A}\nabla)$ *with a constant, symmetric positive definite, matrix* $\mathbf{A}$. *Given any integer* $2\le\sigma\le q$ *and* $|\mathbf{m}(x,0)|\equiv 1$, *there exists a constant* $c>0$ *such that*

$$\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H^{\sigma-1}}^2\le e^{C(t)}\|\boldsymbol{\nabla}\mathbf{m}(0,\cdot)\|_{H^{\sigma-1}}^2, \qquad C(t)=c\int_0^t\left(\alpha+\frac{1}{\alpha}\|\boldsymbol{\nabla}\mathbf{m}(s,\cdot)\|_{L^\infty}^2\right)ds. \tag{B.5}$$

The constant c *is independent of* $\mathbf{m}(x,0)$, α *and* t, *but depends on* σ *and* $\mathbf{A}$.

Proof. For any given multi-index $1\le|\beta|\le\sigma$, let

$$\partial^\beta(\mathbf{m}\times\boldsymbol{\nabla}\mathbf{m})=\mathbf{m}\times\partial^\beta\boldsymbol{\nabla}\mathbf{m}+\mathbf{R}, \tag{B.6}$$

where $\mathbf{R}$ is given by

$$\mathbf{R}=\sum_{0<\nu\le\beta}\binom{\beta}{\nu}\partial^\nu\mathbf{m}\times\partial^{\beta-\nu}\boldsymbol{\nabla}\mathbf{m}.$$

Writing $\nu=\nu_0+\nu_1$ with $|\nu_0|=1$ and using (5.5) as in [24], we obtain the following bound,

$$\begin{aligned}\|\mathbf{R}\|_{L^2}&\le c\sum_{\substack{0\le\nu_1\le\beta-\nu_0\\|\nu_0|=1}}\|\partial^{\nu_0}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{|\beta|-1}}+\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\partial^{\nu_0}\mathbf{m}\|_{H^{|\beta|-1}}\\&\le c\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}.\end{aligned} \tag{B.7}$$

When applying ∂^β to (B.4) and multiplying by $\partial^\beta\mathbf{m}$, one gets

$$\partial^\beta\mathbf{m}\cdot\partial_t\partial^\beta\mathbf{m}=-\partial^\beta\mathbf{m}\cdot\partial^\beta(\mathbf{m}\times\mathcal{L}\mathbf{m}+\alpha\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m}).$$

Integration over Ω and application of the vector identities (B.1) and (B.2) then gives

$$\begin{aligned}\frac{1}{2}\partial_t\|\partial^\beta\mathbf{m}\|_{L^2}^2=&\int_\Omega\partial^\beta\boldsymbol{\nabla}\mathbf{m}:\partial^\beta(\mathbf{m}\times\boldsymbol{\nabla}\mathbf{m}\mathbf{A})dx-\alpha\int_\Omega\partial^\beta\boldsymbol{\nabla}\mathbf{m}:\partial^\beta\boldsymbol{\nabla}\mathbf{m}\mathbf{A}dx\\&+\alpha\int_\Omega\partial^\beta\mathbf{m}\cdot\partial^\beta(|\boldsymbol{\nabla}\mathbf{m}|_A^2\mathbf{m})dx.\end{aligned}$$

Together with (B.6) and the fact that $\mathbf{m}\times(\boldsymbol{\nabla}\mathbf{m}\mathbf{A})=(\mathbf{m}\times\boldsymbol{\nabla}\mathbf{m})\mathbf{A}$, we obtain due to the orthogonality (B.3) that

$$\frac{1}{2}\partial_t\|\partial^\beta\mathbf{m}\|_{L^2}^2=-\alpha\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2+\int_\Omega\partial^\beta\boldsymbol{\nabla}\mathbf{m}:(\mathbf{m}\times\partial^\beta\boldsymbol{\nabla}\mathbf{m}+\mathbf{R})\mathbf{A}dx$$

$$
\begin{aligned}
&+\alpha\int_\Omega \partial^\beta \mathbf{m}\cdot\partial^\beta(|\boldsymbol{\nabla}\mathbf{m}|_A^2\mathbf{m})dx\\
&=-\alpha\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2+\int_\Omega \partial^\beta\boldsymbol{\nabla}\mathbf{m}:\mathbf{R}A dx+\alpha\int_\Omega \partial^\beta\mathbf{m}\cdot\partial^\beta(|\boldsymbol{\nabla}\mathbf{m}|_A^2\mathbf{m})dx.
\end{aligned}
$$

Application of Cauchy-Schwarz then yields

$$
\frac{1}{2}\partial_t\|\partial^\beta\mathbf{m}\|_{L^2}^2+\alpha\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2\le\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{L^2}\|\mathbf{R}\|_{L^2}+\alpha\|\partial^\beta\mathbf{m}\|_{L^2}\|\partial^\beta(|\boldsymbol{\nabla}\mathbf{m}|_A^2\mathbf{m})\|_{L^2}. \tag{B.8}
$$

An estimate for $\|\partial^\beta(\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|_A^2)\|_{L^2}$ can be achieved similarly to the estimate for $\|\mathbf{R}\|_{L^2}$. We rewrite the derivative of the product as a sum,

$$
\partial^\beta(\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|_A^2)=\mathbf{m}\partial^\beta(|\boldsymbol{\nabla}\mathbf{m}|_A^2)+\sum_{0<\nu\le\beta}\binom{\beta}{\nu}\partial^\nu\mathbf{m}\,\partial^{\beta-\nu}|\boldsymbol{\nabla}\mathbf{m}|_A^2.
$$

Since $|\mathbf{m}|\equiv 1$, the L^2-norm of the first term here can be bounded in terms of $\|\,|\boldsymbol{\nabla}\mathbf{m}|_A^2\|_{H^\sigma}$, and the norm of second term can be estimated in the same way as $\mathbf{R}$, which results in a bound in terms of $\|\,|\boldsymbol{\nabla}\mathbf{m}|_A^2\|_{H^{\sigma-1}}$. To estimate the norm of $|\boldsymbol{\nabla}\mathbf{m}|_A^2$ that appears in these two bounds we can use (5.6) to obtain

$$
\|\,|\boldsymbol{\nabla}\mathbf{m}|_A^2\|_{H^\ell}\le C\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\ell},\qquad 0\le\ell\le q-1. \tag{B.9}
$$

Hence we can bound

$$
\begin{aligned}
\|\partial^\beta(\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|_A^2)\|_{L^2}&\le C\left(\|\mathbf{m}\|_{L^\infty}\|\,|\boldsymbol{\nabla}\mathbf{m}|_A^2\|_{H^\sigma}+\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\,|\boldsymbol{\nabla}\mathbf{m}|_A^2\|_{H^{\sigma-1}}\right)\\
&\le C\left(\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}+\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}\right).
\end{aligned} \tag{B.10}
$$

Applying the bounds (B.7) and (B.10) to the right-hand side of (B.8) results in

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\partial^\beta\mathbf{m}\|_{L^2}^2+\alpha\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2\\
&\qquad\le C\left(\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}+\alpha\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2\right),
\end{aligned}
$$

for $1\le|\beta|\le\sigma$. In order to obtain an estimate for $\partial_t\|\boldsymbol{\nabla}\mathbf{m}\|_{L^2}^2$, we now sum over all the multi-indices $1\le|\beta|\le\sigma$ as well as the zeroth order term $\|\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2$, which yields

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2+A_{\min}\alpha\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}^2\\
&\qquad\le\sum_{|\beta|=1}^{\sigma}\left(\frac{1}{2}\partial_t\|\partial^\beta\mathbf{m}\|_{L^2}^2+\alpha\|\partial^\beta\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2\right)+\alpha\|\boldsymbol{\nabla}\mathbf{m}\|_{0,A}^2\\
&\qquad\le c_0\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}+c_1\alpha\left(\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2+1\right)\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2.
\end{aligned}
$$

Here $A_{\min}$ is the lower bound for $\mathbf{A}$, satisfying $\mathbf{x}^T\mathbf{A}\mathbf{x}\ge A_{\min}|\mathbf{x}|^2$ for all $\mathbf{x}\in\mathbb{R}^n$. We next apply Young's inequality, for any $\gamma>0$

$$
\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}\le\frac{1}{2\gamma}\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2+\frac{\gamma}{2}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}^2,
$$

and, upon choosing $\gamma=A_{\min}\alpha/c_0$, we get

$$
\frac{1}{2}\partial_t\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2+\frac{1}{2}A_{\min}\alpha\|\boldsymbol{\nabla}\mathbf{m}\|_{H^\sigma}^2\le c\left(\alpha+\frac{1}{\alpha}\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\right)\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{\sigma-1}}^2.
$$

The statement (B.5) then follows from Grönwall's inequality. $\square$

B.2. Variable scalar coefficient. Given $L=\boldsymbol{\nabla}\cdot(a^\varepsilon\boldsymbol{\nabla})$, where $a^\varepsilon=a(x/\varepsilon)$ is a scalar, variable coefficient satisfying the assumption (A1),one can proceed in a similar way as in the previous section to obtain an energy estimate. This setup corresponds to (3.1) and we can therefore use the lemmas derived in Section 5.

Theorem B.2. *Consider $\mathbf{m}\in C^1(0,T;H^q(\Omega))$ satisfying (3.1) under the assumptions $(A1)-(A3)$. Then it holds that*

$$\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{L^2}\le C\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{L^2}, \qquad 0\le t\le T. \tag{B.11}$$

If there is a constant M independent of ε such that $\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_\infty\le M$ for $0\le t\le T$, then it is moreover true that

$$\|\mathbf{m}(\cdot,t)\|_{H^q}\le C\left(\|\mathbf{m}(\cdot,0)\|_{L^2}+\frac{1}{\varepsilon^{q-1}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H^{q-1}_\varepsilon}\right)\le C\left(1+\varepsilon^{1-q}\right), \tag{B.12}$$

where the constant C is independent of ε and t but depends on T and M.

Proof. To prove this lemma, we first show by induction that for $0\le j\le q-1$ and $0\le t\le T$,

$$\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H^j}\le C\frac{1}{\varepsilon^j}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H^j_\varepsilon}, \tag{B.13}$$

which by the definition of $\|\cdot\|_{H^j_\varepsilon}$ entails that

$$\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H^j_\varepsilon}\le C\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H^j_\varepsilon}. \tag{B.14}$$

To begin with the proof, consider the L^2-norm of $\boldsymbol{\nabla}\mathbf{m}$. By the identity (2.2) it holds that

$$\begin{aligned}
\frac{1}{2}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathbf{m}\|^2 &= \int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathbf{m}:\boldsymbol{\nabla}\partial_t\mathbf{m}dx=-\int_\Omega \mathcal{L}\mathbf{m}\cdot\partial_t\mathbf{m}dx\\
&=\int_\Omega \mathcal{L}\mathbf{m}\cdot(\mathbf{m}\times\mathcal{L}\mathbf{m}+\alpha\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m})=\alpha\int_\Omega\mathcal{L}\mathbf{m}\cdot(\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m})\\
&=\alpha\int_\Omega|\mathcal{L}\mathbf{m}\cdot\mathbf{m}|^2-|\mathcal{L}\mathbf{m}|^2dx\le 0,
\end{aligned}$$

as $|\mathcal{L}\mathbf{m}\cdot\mathbf{m}|^2\le|\mathcal{L}\mathbf{m}|^2|\mathbf{m}|^2=|\mathcal{L}\mathbf{m}|^2$. Due to the boundedness of $a(x)$, (A1), this entails

$$\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{L^2}^2\le C\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{L^2}^2, \tag{B.15}$$

which shows (B.13) for $q=0$. Next, we proceed similarly as in Section 6 and consider $\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{m}\|_{L^2}$ and $\|\mathcal{L}^k\mathbf{m}\|_{L^2}$ for $k\ge 1$. Using the identity

$$-\mathbf{m}\times\mathbf{m}\times\mathcal{L}\mathbf{m}=\mathcal{L}\mathbf{m}+a^\varepsilon|\boldsymbol{\nabla}\mathbf{m}|^2,$$

which corresponds to (B.2), and applying integration by parts, we get

$$\begin{aligned}
\frac{1}{2}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{m}\|_{L^2}^2&=-\int_\Omega a^\varepsilon\boldsymbol{\nabla}\mathcal{L}^k\mathbf{m}:\boldsymbol{\nabla}\mathcal{L}^k(\mathbf{m}\times\mathcal{L}\mathbf{m}-\alpha\mathcal{L}\mathbf{m}+\alpha a^\varepsilon\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|^2)dx\\
&=\int_\Omega\mathcal{L}^{k+1}\mathbf{m}\cdot\mathcal{L}^k(\mathbf{m}\times\mathcal{L}\mathbf{m}+\alpha a^\varepsilon\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|^2)dx-\alpha\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2
\end{aligned}$$

and similarly,

$$\frac{1}{2}\partial_t\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2=-\int_\Omega\mathcal{L}^{k+1}\mathbf{m}\cdot\mathcal{L}^{k+1}(\mathbf{m}\times\mathcal{L}\mathbf{m}-\alpha\mathcal{L}\mathbf{m}+\alpha a^\varepsilon\mathbf{m}|\boldsymbol{\nabla}\mathbf{m}|^2)dx$$

$$
\begin{aligned}
&= \int_\Omega a^\varepsilon \nabla \mathcal{L}^{k+1}\mathbf{m} : \nabla \mathcal{L}^k (\mathbf{m}\times\mathcal{L}\mathbf{m} + \alpha a^\varepsilon \mathbf{m}|\nabla \mathbf{m}|^2)dx \\
&\qquad - \alpha \|\sqrt{a^\varepsilon}\nabla \mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 .
\end{aligned}
$$

We then use Lemma 5.8 to rewrite

$$
\mathcal{L}^k(\mathbf{m}\times\mathcal{L}\mathbf{m}) = \mathbf{m}\times\mathcal{L}^{k+1}\mathbf{m} + \mathbf{R}_k,
$$

where the highest order terms cancel in the integral due to orthogonality. Application of Cauchy-Schwarz and Young's inequality thus gives that for any $\gamma > 0$,

$$
\begin{aligned}
\int_\Omega \mathcal{L}^{k+1}\mathbf{m}\cdot\mathcal{L}^k(\mathbf{m}\times\mathcal{L}\mathbf{m})dx &\le \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 + \frac{1}{2\gamma}\|\mathbf{R}_k\|_{L^2}^2 \\
\int_\Omega a^\varepsilon \nabla\mathcal{L}^{k+1}\mathbf{m} : \nabla\mathcal{L}^k(\mathbf{m}\times\mathcal{L}\mathbf{m})dx &\le \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\nabla\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 \\
&\quad + \frac{C}{2\gamma}(\|\nabla\mathbf{m}\|_{L^\infty}^2\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 + \|\nabla\mathbf{R}_k\|_{L^2}^2).
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
\int_\Omega \mathcal{L}^{k+1}\mathbf{m}\cdot\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)dx &\le \frac{\gamma}{2}\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 + \frac{1}{2\gamma}\|\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)\|_{L^2}^2, \\
\int_\Omega a^\varepsilon\nabla\mathcal{L}^{k+1}\mathbf{m} : \nabla\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)dx &\le \frac{\gamma}{2}\|\sqrt{a^\varepsilon}\nabla\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 + \frac{C}{2\gamma}\|\nabla\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)\|_{L^2}^2.
\end{aligned}
$$

Choosing γ sufficiently small and making use of the assumed bound on $\|\nabla\mathbf{m}\|_{L^\infty}$, it thus follows that

$$
\partial_t\|\sqrt{a^\varepsilon}\nabla\mathcal{L}^k\mathbf{m}\|_{L^2}^2 \le \frac{C}{2\gamma}\left(\|\mathbf{R}_k\|_{L^2}^2 + \|\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)\|_{L^2}^2\right), \tag{B.16a}
$$

$$
\partial_t\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 \le \frac{C}{2\gamma}\left(M^2\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 + \|\nabla\mathbf{R}_k\|_{L^2}^2 + \|\nabla\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)\|_{L^2}^2\right). \tag{B.16b}
$$

By (B.9), we obtain using Lemma 5.6 and Lemma 5.5, that

$$
\|\mathcal{L}^k(a^\varepsilon\mathbf{m}|\nabla\mathbf{m}|^2)\|_{L^2} \le C\frac{1}{\varepsilon^{2k}}\|\mathbf{m}|\nabla\mathbf{m}|^2\|_{H_\varepsilon^{2k}} \le C\frac{1}{\varepsilon^{2k}}\|\nabla\mathbf{m}\|_{L^\infty}\|\nabla\mathbf{m}\|_{H_\varepsilon^{2k}}.
$$

According to Lemma 5.8, the L^2-norm of $\mathbf{R}_k$ can be bounded in the same way.

Next, we consider slight variations of the elliptic regularity estimates (5.15) and (5.16) in Lemma 5.7. For $\ell\in\{0,1\}$ and $0 < 2k+\ell \le q-1$, it holds that

$$
\|\nabla\mathbf{m}\|_{H^{2k+\ell}} \le C\left(\frac{1}{\varepsilon^{2k+\ell}}\|\nabla\mathbf{m}\|_{H_\varepsilon^{2k+\ell-1}} + \begin{cases}\|\nabla\mathcal{L}^k\mathbf{m}\|_{L^2}, & \ell=0,\\ \|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}, & \ell=1,\end{cases}\right), \tag{B.17}
$$

which in turn implies that

$$
\|\nabla\mathbf{m}\|_{H_\varepsilon^{2k+\ell}} \le C\left(\|\nabla\mathbf{m}\|_{H_\varepsilon^{2k+\ell-1}} + \begin{cases}\varepsilon^{2k}\|\sqrt{a^\varepsilon}\nabla\mathcal{L}^k\mathbf{m}\|_{L^2}, & \ell=0,\\ \varepsilon^{2k+1}\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}, & \ell=1\end{cases}\right).
$$

This can be proved proceeding in the same way as for (5.15) and (5.16). Consequently, the right-hand side in (B.16) can be bounded such that we get

$$\begin{aligned}\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^k\mathbf{m}\|_{L^2}^2 &\le \frac{C}{\gamma\varepsilon^{2(2k)}}\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\|\boldsymbol{\nabla}\mathbf{m}\|_{H_\varepsilon^{2k}}^2\\ &\le \frac{CM^2}{\gamma}\left(\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^j\mathbf{m}\|_{L^2}^2+\frac{1}{\varepsilon^{2(2k)}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H_\varepsilon^{2k-1}}^2\right)\end{aligned}$$

and similarly,

$$\begin{aligned}\partial_t\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2 &\le \frac{C}{\gamma}\|\boldsymbol{\nabla}\mathbf{m}\|_{L^\infty}^2\left(\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}+\frac{1}{\varepsilon^{2k+1}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{2k+1}}\right)^2\\ &\le \frac{CM^2}{\gamma}\left(\|\mathcal{L}^{k+1}\mathbf{m}\|_{L^2}^2+\frac{1}{\varepsilon^{2(2k+1)}}\|\boldsymbol{\nabla}\mathbf{m}\|_{H_\varepsilon^{2k}}^2\right).\end{aligned}$$

Assume now that (B.13) holds for $0\le j\le 2k$, which we already showed for $k=0$. Then it follows from the above estimates that

$$\partial_t\|\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}^2 \le \frac{CM^2}{\gamma}\left(\|\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}^2+\frac{1}{\varepsilon^{2(2k+1)}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H_\varepsilon^{2k}}^2\right).$$

and we obtain using Grönwall's inequality and Lemma 5.6 that

$$\begin{aligned}\|\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}^2 &\le e^{C(M^2/\gamma)t}\left(\|\mathcal{L}^{k+1}\mathbf{m}(\cdot,0)\|_{L^2}^2+Ct\frac{1}{\varepsilon^{2(2k+1)}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H_\varepsilon^{2k}}^2\right)\\ &\le C\frac{1}{\varepsilon^{2(2k+1)}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H_\varepsilon^{2k+1}}^2.\end{aligned}$$

Due to elliptic regularity as given in (B.17) and (B.14), we then get

$$\begin{aligned}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H^{2k+1}} &\le C\left(\frac{1}{\varepsilon^{2k+1}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H_\varepsilon^{2k}}+\|\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}\right)\\ &\le \frac{C}{\varepsilon^{2k+1}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H_\varepsilon^{2k+1}}.\end{aligned}$$

Moreover, it holds similarly that

$$\partial_t\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}^2 \le C\frac{M^2}{\gamma}\left(\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}^2+\frac{1}{\varepsilon^{2(2k+2)}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H^{2k+1}}^2\right),$$

from which it follows in the same way as above that

$$\begin{aligned}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H^{2k+2}} &\le C\left(\frac{1}{\varepsilon^{2k+2}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,t)\|_{H_\varepsilon^{2k+1}}+\|\sqrt{a^\varepsilon}\boldsymbol{\nabla}\mathcal{L}^{k+1}\mathbf{m}(\cdot,t)\|_{L^2}\right)\\ &\le C\frac{1}{\varepsilon^{2k+2}}\|\boldsymbol{\nabla}\mathbf{m}(\cdot,0)\|_{H_\varepsilon^{2k+2}},\end{aligned}$$

which proves the claim (B.13) by induction. To complete the proof of Theorem B.2, note that

$$\begin{aligned}\|\mathbf{m}\|_{H^q} &\le C\sum_{|\sigma|=0}^{q}\|\partial^\sigma\mathbf{m}\|_{L^2}\le C\left(\|\mathbf{m}\|_{L^2}+\sum_{|\sigma|=0}^{q-1}\|\partial^\sigma\boldsymbol{\nabla}\mathbf{m}\|_{L^2}\right)\\ &\le C\left(\|\mathbf{m}\|_{L^2}+\|\boldsymbol{\nabla}\mathbf{m}\|_{H^{q-1}}\right).\end{aligned}$$

Hence, (B.12) follows from (B.13) and the fact that $\|\mathbf{m}(\cdot,t)\|_{L^2}=\|\mathbf{m}(\cdot,0)\|_{L^2}$ due to the norm preservation property of the Landau-Lifshitz equation as shown in (A2).

□