# ON THE MULTI-BUBBLE BLOW-UP SOLUTIONS TO ROUGH NONLINEAR SCHRÖDINGER EQUATIONS 

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#### Abstract

We are concerned with the multi-bubble blow-up solutions to rough nonlinear Schrödinger equations in the focusing mass-critical case. In both dimensions one and two, we construct the finite time multi-bubble solutions, which concentrate at $K$ distinct points, $1 \leq K<\infty$, and behave asymptotically like a sum of pseudo-conformal blow-up solutions in the pseudo-conformal space $\Sigma$ near the blow-up time. The upper bound of the asymptotic behavior is closely related to the flatness of noise at blow-up points. Moreover, we prove the conditional uniqueness of multi-bubble solutions in the case where the asymptotic behavior in the energy space $H^{1}$ is of the order $(T-t)^{3+\zeta}, \zeta>0$. These results are also obtained for nonlinear Schrödinger equations with lower order perturbations, particularly, in the absence of the classical pseudo-conformal symmetry and the conversation law of energy. The existence results are applicable to the canonical deterministic nonlinear Schrödinger equation and complement the previous work [43]. The conditional uniqueness results are new in both the stochastic and deterministic case.


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## 1. Introduction

This work is devoted to the existence and uniqueness of blow-up solutions at multiple points to the rough nonlinear Schrödinger equations in the focusing mass-critical case. Precisely, we consider

$$
\begin{align*}
& i d X=-\Delta X d t-|X|^{\frac{4}{d}} X d t-i \mu X d t+i X d W(t)  \tag{1.1}\\
& X(0)=X_{0} \in H^{1}\left(\mathbb{R}^{d}\right)
\end{align*}
$$

Here, $W$ is the Wiener process of the form

$$
W(t, x)=\sum_{k=1}^{N} i \phi_{k}(x) B_{k}(t), \quad x \in \mathbb{R}^{d}, t \geq 0
$$

where $\phi_{k} \in C_{b}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}\right), B_{k}$ are the standard $N$-dimensional real valued Brownian motions on a stochastic basis $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}, \mathbb{P}\right), 1 \leq k \leq N$, and $\mu=\frac{1}{2} \sum_{k=1}^{N} \phi_{k}^{2}$. The last term $X d W(t)$ in (1.1) is taken in the sense of controlled rough path (see Definition 2.1 below). In particular, the rough integration coincides with the usual Itô integration if the corresponding processes are $\left\{\mathscr{F}_{t}\right\}$-adapted (see [27, Chapter 5]). For simplicity we assume that $N<\infty$, but the arguments below can be also applied to the case when $N=\infty$ under suitable summability conditions.

The noise here is mainly considered of conservative type, i.e., $\operatorname{Re} W=0$. In this case, the quantum system evolves on the unit ball if the initial state is normalized $\left\|X_{0}\right\|_{L^{2}}=1$ and thus verifies the conservation of probability.

One significant model of nonlinear Schrödinger equations with noise arises from the molecular aggregates with thermal fluctuations, where the multiplicative noise corresponds to scattering of exciton by phonons, due to thermal vibrations of the molecules. The noise effect on the coherence of the ground state solitary solution was studied in [1, 2] for the two dimensional case with the critical cubic nonlinearity. The influence of noise on collapse in the one dimensional case with quintic nonlinearity was studied in [56].

Another important application is related to the open quantum systems [9, in which the stochastic perturbation $i X \phi_{k} d B_{k}$ represents a stochastic continuous measurement via the pointwise quantum observable $R_{k}(X)=X \phi_{k}$, while $B_{k}$ represents the output of continuous measurement, $1 \leq k \leq N$. We refer to [9, Chapter 2] for more physical interpretations. See also [59 for other physical applications of Schrödinger equations.

The local well-posedness of equation (1.1) is quite well known when the stochastic integration is taken in the sense of Itô or rough path. See, e.g., [5, 10, 18, 58].

However, the situation becomes much more delicate for the large time behavior of solutions. As a matter of fact, solutions may formalize singularities in finite time in the focusing mass-(super)critical case.

When the input noise is of conservative type, it was first proved by de Bouard and Debussche [17, 19 that the noise has the effect to accelerate blow-up with positive probability in the mass-supercritical case (i.e., the exponent of nonlinearity is in the region $\left.\left(1+\frac{4}{d}, 1+\frac{4}{(d-2)_{+}}\right)\right)$. When the noise is of non-conservative type, the explosion, however, can be prevented with high probability as long as the strengthen of noise is large enough, which reflects the damped effect of non-conservative noise ( 7 ]). We also refer to [50] for the global well-posedness below the threshold in the mass-(super)critical case.

Moreover, many numerical simulations have been made to investigate the blow-up phenomena in the stochastic case. It was observed in [20, 21, 22] that the colored multiplicative noise has the effect to delay blow-up, while the white noise may even prevent blow-up. Such phenomena have been also confirmed by the recent numerical results in
[51, 52]. The noise effects on the energy, global well-posedness and blow-up profiles are also studied in [51, 52], which partially confirm the conjecture that, in the mass-critical case the stable blow-up solutions with slightly supercritical mass shall have the $\log \log$ blow-up rate, while in the mass-supercritical case the blow-up rate is of a different polynomial type.

Recently, the minimal mass blow-up solutions have been constrcuted by the authors in [58], and it is shown that the mass of the ground state characterizes the threshold of global well-posedness and blow-up in the stochastic case. The log-log blow-up solutions have been also constructed in [26] in the stochastic case. We also would like to refer to the recent work [25] for the $\log$-log blow-up solutions with $L^{2}$-regularity randomized initial data.

In this paper, we are mainly interested in the blow-up dynamics in the large mass regime, particulary, the existence and uniqueness of multi-bubble blow-up solutions in the stochastic case. It should be mentioned that, the presence of noise destroys the symmetries and several conservation laws, which makes it rather difficult to obtain the quantitative descriptions of blow-up dynamics.

Before stating the main results, let us first review the existing results for the deterministic nonlinear Schrödinger equation (NLS)

$$
\left\{\begin{array}{l}
i \partial_{t} u+\Delta u+|u|^{\frac{4}{d}} u=0  \tag{1.2}\\
u(0)=u_{0} \in H^{1}\left(\mathbb{R}^{d}\right),
\end{array}\right.
$$

Equation (1.2) admits a number of symmetries and conservation laws. It is invariant under the translation, scaling, phase rotation and Galilean transform, i.e., if $u$ solves (1.2), then so does

$$
\begin{equation*}
v(t, x)=\lambda_{0}^{-\frac{d}{2}} u\left(\frac{t-t_{0}}{\lambda_{0}^{2}}, \frac{x-x_{0}}{\lambda_{0}}-\frac{\beta_{0}\left(t-t_{0}\right)}{\lambda_{0}}\right) e^{i \frac{\beta_{0}}{2} \cdot\left(x-x_{0}\right)-i \frac{\left|\beta_{0}\right|^{2}}{4}\left(t-t_{0}\right)+i \theta_{0}}, \tag{1.3}
\end{equation*}
$$

with $v\left(t_{0}, x\right)=\lambda_{0}^{-\frac{d}{2}} u_{0}\left(\frac{x-x_{0}}{\lambda_{0}}\right) e^{i \frac{\beta_{0}}{2} \cdot\left(x-x_{0}\right)+i \theta_{0}}$, where $\left(\lambda_{0}, \beta_{0}, \theta_{0}\right) \in \mathbb{R}^{+} \times \mathbb{R}^{d} \times \mathbb{R}, x_{0} \in \mathbb{R}^{d}, t_{0} \in$ $\mathbb{R}$. In particular, the $L^{2}$-norm of solutions is preserved under the symmetries above, and thus (1.2) is called the mass-critical equation. Another symmetry, particularly important in the blow-up analysis, is related to the pseudo-conformal transformation in the pseudoconformal space $\Sigma:=\left\{u \in H^{1}\left(\mathbb{R}^{d}\right),\|x u\|_{L^{2}\left(\mathbb{R}^{d}\right)}<\infty\right\}$,

$$
\begin{equation*}
(-t)^{-\frac{d}{2}} u\left(\frac{1}{-t}, \frac{x}{-t}\right) e^{-i \frac{|x|^{2}}{4 t}}, \quad t \neq 0 . \tag{1.4}
\end{equation*}
$$

The conservation laws related to (1.2) contain

$$
\begin{align*}
& \text { Mass : } \quad M(u)(t):=\int_{\mathbb{R}^{d}}|u(t)|^{2} d x=M\left(u_{0}\right) .  \tag{1.5}\\
& \text { Energy : } \quad E(u)(t):=\frac{1}{2} \int_{\mathbb{R}^{d}}|\nabla u(t)|^{2} d x-\frac{d}{2 d+4} \int_{\mathbb{R}^{d}}|u(t)|^{2+\frac{4}{d}} d x=E\left(u_{0}\right) .  \tag{1.6}\\
& \text { Momentum : } \operatorname{Mom}(u):=\operatorname{Im} \int_{\mathbb{R}^{d}} \nabla u \bar{u} d x=\operatorname{Mom}\left(u_{0}\right) . \tag{1.7}
\end{align*}
$$

An important role here is played by the ground state $Q$, which is the unique positive spherically symmetric solution to the elliptic equation

$$
\begin{equation*}
\Delta Q-Q+Q^{1+\frac{4}{d}}=0 \tag{1.8}
\end{equation*}
$$

It is well known that (see, e.g., [62]), the mass of the ground state characterizes the threshold for the global well-posedness and blow-up of solutions to NLS. More precisely,
solutions to (1.2) exist globally if the initial data have the subcritical mass, i.e., $\left\|u_{0}\right\|_{L^{2}}<$ $\|Q\|_{L^{2}}$, while solutions may formolize singularities in finite time in the critical mass case, i.e., $\left\|u_{0}\right\|_{L^{2}}=\|Q\|_{L^{2}}$. In particular, by virtue of the pseudo-conformal transformation (1.4), one may construct the so-called pseudo-conformal blow-up solutions

$$
\begin{equation*}
S_{T}(t, x)=(\omega(T-t))^{-\frac{d}{2}} Q\left(\frac{x-\alpha}{\omega(T-t)}\right) e^{-\frac{i}{4} \frac{|x-\alpha|^{2}}{T-t}+\frac{i}{\omega^{2}(T-t)}+i \theta}, \quad T \in \mathbb{R} \tag{1.9}
\end{equation*}
$$

Note that, $\left\|S_{T}\right\|_{L^{2}}=\|Q\|_{L^{2}}$, and $S_{T}$ blows up at time $T$ with the blow-up rate $\sim(T-t)^{-1}$. Thus, $S_{T}$ is the minimal mass blow-up solution.

When the mass of initial data is slightly above $\|Q\|_{L^{2}}$, two different blow-up scenarios have been observed: the pseudo-conformal blow-up rate $\sim(T-t)^{-1}$, and the log-log blow-up rate $\sim \sqrt{\frac{\log |\log (T-t)|}{T-t}}$. For the blow-up and classification results in this case we refer to [11, 47, 48, 53] and the references therein.

For even larger mass of initial data, the complete characterization of the formation of singularity is still an open problem. It is conjectured by Merle and Raphaël [47 that every $H^{1}$ blow-up solution can be decomposed into a singular part and a $L^{2}$ residual, and the singular part expands asymptotically as multiple bubbles concentrating at a finite number of points. This conjecture is known as the blow-up version of the soliton resolution conjecture.

Thus, an important step to understand the singularity formulation in the large mass regime is to construct multi-bubble blow-up solutions.

In the pioneering work [43], Merle initiated the construction of blow-up solutions concentrating at arbitrary $K(<\infty)$ distinct points, which behave asymptotically like a sum of $K$ pseudo-conformal blow-up solutions and thus have the pseudo-conformal blow-up rate. The multi-bubble solutions with the log-log blow-up rate have been constructed by Fan [24]. Moreover, Martel and Raphëal [42] constructed blow-up solutions with multiple bubbles concentrating at exactly the same point. Bubbling phenomena have been also exhibited in various other settings. See, e.g., 32 for the energy-critical NLS, and 57 for the nonlinear Schrödinger system. We also would like to refer to [15, 41] for the generalized Korteweg-de Vries equations (gKDV), [33, 34, 39] for the wave maps, and [16, 44 ] for the nonlinear heat equations.

However, one major challenge in the stochastic case is, that the symmetries and several conservation laws are destroyed, because of the presence of noise. Equation (1.1) is no longer invariant under the pseudo-conformal symmetry, which, however, is the key ingredient in the classification of minimal mass blow-up solutions to NLS in 46. Moreover, the failure of the conservation law of energy creates a major problem to understand the global behavior in the stochastic case, which motivates the recent numerical tracking of the energy in the works [51, 52]. Another important qualitative difference is, that the perturbation order of profiles is of merely polynomial type in the stochastic case, which makes it rather intricate to decouple different bubbles, particularly, the remainders in the corresponding geometrical decomposition. This is different from the previous construction of multi-bubble solutions in [43] for NLS, where the interactions are exponentially small in time.

In the present work, in both dimensions one and two, we are able to construct the multiple bubble blow-up solutions concentrating at $K$ distinct points to the rough nonlinear Schrödinger equation (1.1), $1 \leq K<\infty$. The constructed multi-bubble solutions behave asymptotically like a sum of pseudo-conformal blow-up solutions in the pseudoconformal space $\Sigma$ near the blow-up time and so have the pseudo-conformal blow-up rate $\sim(T-t)^{-1}$. The upper bound of the approximation is also obtained and, interestingly,
is closely related to that of the flatness of noise at blow-up points. As a matter of fact, if the noise is more flat at the blow-up points, the approximation can be even taken in the more regular space $H^{\frac{3}{2}}$, and the perturbation orders of the corresponding geometrical parameters and the remainder can be also improved.

Another novelty of this work is concerned with the uniqueness of multi-bubble solutions. The uniqueness issue is of significant importance in the classification of blow-up solutions to dispersive equations. In the remarkable paper [46], Merle obtained the uniqueness of minimal mass blow-up solutions to NLS, which states that the pseudo-conformal blowup solution is indeed the unique minimal mass blow-up solution up to the symmetries of NLS. Such strong rigidity results were also obtained by Raphaël and Szeftel [55] for the inhomogeneous nonlinear Schrödinger equation, and by Martel, Merle, Raphaël 41 for the mass-critical gKdV equation. We also refer to [49] for the conditional uniqueness result for the Bourgain-Wang solutions, and [36] for the Chern-Simons-Schrödinger equation.

However, to the best of our knowledge, there are very few results on the uniqueness of multi-bubble blow-up solutions to dispersive equations.

We prove that, two multi-bubble blow-up solutions to equation (1.1) are indeed the same if they have the same asymptotic blow-up profile within the order $(T-t)^{3+\zeta}, \zeta>0$, in the energy space $H^{1}$. Hence, in this asymptotic regime, the $H^{1}$ multi-bubble blow-up solution is exactly the above constructed solution and so lies in the more regular pseudoconformal space $\Sigma$. This conditional uniqueness result of multi-bubbles solutions can be also viewed as similar to the local uniqueness results in the elliptic setting, see, e.g., [12, 13, 23].
The existence and conditional uniqueness results are also obtained for a class of nonlinear Schrödinger equations with lower order perturbations (see equations (2.11) and (2.17) below), particularly, in the absence of the pseudo-conformal symmetry and the conservation of energy.

In particular, the obtained results are applicable to the single bubble case and give the existence and conditional uniqueness of minimal mass blow-up solutions for both the stochastic equation (1.1) and the deterministic equation (2.17).

We would like to mention that, the existence result is also applicable to the canonical deterministic NLS. The positive frequencies $\left\{\omega_{j}\right\}_{j=1}^{K}$ in the construction here are allowed to be arbitrarily small, and the asymptotic behavior can be taken in the pseudoconformal space $\Sigma$ instead of the space $L^{2+\frac{4}{d}}$, which complement the previous results in [43]. Furthermore, the conditional uniqueness results are new in both the stochastic and deterministic case.

The strategy of proof is mainly based on the modulation method developed in [55] for the minimal mass blow-up solutions to the inhomogeneous nonlinear Schrödinger equation. See also the recent work [58] in the stochastic setting. One major difference here is, that the study of multi-bubble solutions requires a delicate localization procedure. A great effort is dedicated to the decoupling of different bubbles. Particularly, because of the low polynomial type perturbation orders, the decoupling of the remainders in the geometrical decomposition is quite delicate. Moreover, a new generalized energy is introduced here, it incorporates the localized functions in an appropriate way such that different bubbles can be decoupled and, simultaneously, the key monotonicity property keeps still preserved. Let us also mention that, the proof of the conditional uniqueness result requires an iterated argument, which is also different from the single bubble case.

We expect the arguments developed here would be also of interest in the further understanding of multi-bubble solutions of other dispersive equations.

Notations. For any $x=\left(x_{1}, \cdots, x_{d}\right) \in \mathbb{R}^{d}$ and any multi-index $\nu=\left(\nu_{1}, \cdots, \nu_{d}\right)$, let $|\nu|=\sum_{j=1}^{d} \nu_{j},\langle x\rangle=\left(1+|x|^{2}\right)^{1 / 2}, \partial_{x}^{\nu}=\partial_{x_{1}}^{\nu_{1}} \cdots \partial_{x_{d}}^{\nu_{d}}$, and $\langle\nabla\rangle=(I-\Delta)^{1 / 2}$.

We use the standard Sobolev spaces $H^{s, p}\left(\mathbb{R}^{d}\right)$, $s \in \mathbb{R}, 1 \leq p \leq \infty$. In particular, $L^{p}:=H^{0, p}\left(\mathbb{R}^{d}\right)$ is the space of $p$-integrable (complex-valued) functions, $L^{2}$ denotes the Hilbert space endowed with the scalar product $\langle v, w\rangle=\int_{\mathbb{R}^{d}} v(x) \bar{w}(x) d x$, and $H^{s}:=H^{s, 2}$. Let $\Sigma$ denote the pseudo-conformal space, i.e., $\Sigma:=\left\{u \in H^{1},|x| u \in L^{2}\right\}$. As usual, if $B$ is a Banach space, $L^{q}(0, T ; B)$ means the space of all integrable $B$-valued functions $f:(0, T) \rightarrow B$ with the norm $\|\cdot\|_{L^{q}(0, T ; B)}$, and $C([0, T] ; B)$ denotes the space of all $B$-valued continuous functions on $[0, T]$ with the sup norm over $t$. The local smoothing spaces is defined by $L^{2}\left(I ; H_{\beta}^{\alpha}\right)=\left\{u \in \mathscr{S}^{\prime}: \int_{I} \int\langle x\rangle^{2 \beta}\left|\langle\nabla\rangle^{\alpha} u(t, x)\right|^{2} d x d t<\infty\right\}, \alpha, \beta \in \mathbb{R}$.

Throughout this paper, the positive constants $C$ and $\delta$ may change from line to line.

## 2. Formulation of main results

2.1. Main results. To begin with, let us first present the precise definition of solutions to (1.1), in which the noise term is taken in the sense of the controlled rough path. For more details on the theory of rough paths, see [27, 28].

Definition 2.1. We say that $X$ is a solution to (1.1) on $\left[0, \tau^{*}\right)$, where $\tau^{*} \in(0, \infty]$ is a random variable, if $\mathbb{P}$-a.s. for any $\varphi \in C_{c}^{\infty}, t \mapsto\langle X(t), \varphi\rangle$ is continuous on $\left[0, \tau^{*}\right)$ and for any $0<s<t<\tau^{*}$,

$$
\left.\langle X(t)-X(s), \varphi\rangle-\int_{s}^{t}\langle i X, \Delta \varphi\rangle+\left.\langle i| X\right|^{\frac{4}{d}} X, \varphi\right\rangle-\langle\mu X, \varphi\rangle d r=\sum_{k=1}^{N} \int_{s}^{t}\left\langle i \phi_{k} X, \varphi\right\rangle d B_{k}(r) .
$$

Here, the integral $\int_{s}^{t}\left\langle i \phi_{k} X, \varphi\right\rangle d B_{k}(r)$ is taken in the sense of controlled rough path with respect to the rough paths $(B, \mathbb{B})$, where $\mathbb{B}=\left(\mathbb{B}_{j k}\right), \mathbb{B}_{j k, s t}:=\int_{s}^{t} \delta B_{j, s r} d B_{k}(r)$ with the integration taken in the sense of Itô. That is, $\left\langle i \phi_{k} X, \varphi\right\rangle \in C^{\alpha}([s, t])$,

$$
\begin{equation*}
\delta\left(\left\langle i \phi_{k} X, \varphi\right\rangle\right)_{s t}=-\sum_{j=1}^{N}\left\langle\phi_{j} \phi_{k} X(s), \varphi\right\rangle \delta B_{j, s t}+\delta R_{k, s t}, \tag{2.1}
\end{equation*}
$$

and $\left\|\left\langle\phi_{j} \phi_{k} X, \varphi\right\rangle\right\|_{\alpha,[s, t]}<\infty, \quad\left\|R_{k}\right\|_{2 \alpha,[s, t]}<\infty$.
We mention that, because of the backward propagation procedure in the construction below, the solution to (1.1) is no longer adapted to the filtration $\left\{\mathscr{F}_{t}\right\}$. Hence, the stochastic integration in (1.1) should be interpreted in the sense of the controlled rough path, instead of the Itô sense.

The theory of (controlled) rough paths, and the recent development of the theory of regularity structures [30] and the para-controlled calculus 29] have led to significant progress in solving singular parabolic stochastic partial differential equations with white noises. We refer the interested readers to the monograph [27] and the references therein for more details on these topics.

Throughout this paper we assume that
(A0) (Asymptotical flatness) For any multi-index $\nu \neq 0$ and $1 \leq k \leq N$,

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty}\langle x\rangle^{2}\left|\partial_{x}^{\nu} \phi_{k}(x)\right|=0 \tag{2.2}
\end{equation*}
$$

(A1) (Flatness at blow-up points) There exists $\nu_{*} \in \mathbb{N}$ such that for every $1 \leq k \leq N$ and $1 \leq j \leq K$,

$$
\begin{equation*}
\partial_{x}^{\nu} \phi_{k}\left(x_{j}\right)=0, \quad \forall 0 \leq|\nu| \leq \nu_{*} . \tag{2.3}
\end{equation*}
$$

Remark 2.2. The asymptotical flatness condition guarantees the local well-posedness of equation (1.1) (see [4, 5), while the flatness condition at blow-up points is mainly for the construction of multi-bubble solutions. More interestingly, the order $\nu_{*}$ is closely related to that of the asymptotic behavior of solutions near the blow-up time. See (2.4) and (2.15) below.

For the frequencies $\omega_{j}>0$ and the blow-up points $x_{j} \in \mathbb{R}^{d}, 1 \leq j \leq K$, we mainly consider two cases below:

Case (I). $\left\{x_{j}\right\}_{j=1}^{K}$ are arbitrary distinct points in $\mathbb{R}^{d}$, and $\left\{\omega_{j}\right\}_{j=1}^{K}\left(\subseteq \mathbb{R}^{+}\right)$satisfy that for some $\omega>0,\left|\omega_{j}-\omega\right| \leq \varepsilon$ for any $1 \leq j \leq K$, where $\varepsilon>0$;

Case (II). $\left\{\omega_{j}\right\}_{j=1}^{K}$ are arbitrary points in $\mathbb{R}^{+}$, and $\left\{x_{j}\right\}_{j=1}^{K}\left(\subseteq \mathbb{R}^{d}\right)$ satisfy that $\mid x_{j}-$ $x_{l} \mid \geq \varepsilon^{-1}$ for any $1 \leq j \neq l \leq K$, where $\varepsilon>0$.

The existence of multiple bubble solutions is formulated in Theorem 2.3 below.
Theorem 2.3. (Existence) Consider $d=1,2$. Assume (A0) and (A1) with $\nu_{*} \geq 5$. For every $1 \leq K<\infty$, let $\left\{\vartheta_{j}\right\} \subseteq \mathbb{R},\left\{x_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{d}$ be distinct points, $\left\{\omega_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{+}$, satisfying Case (I) or Case (II).

Then, for $\mathbb{P}-$ a.e. $\omega$ there exists $\varepsilon^{*}(\omega)>0$ sufficiently small such that for any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, there exists $\tau^{*}>0$ small enough such that for any $T \in\left(0, \tau^{*}(\omega)\right]$, there exist $X_{0}(\omega) \in \Sigma$ and a corresponding blow-up solution $X(\omega)$ to (1.1), satisfying that for some $C>0$, $\zeta \in(0,1)$,

$$
\begin{equation*}
\left\|e^{-W(t, \omega)} X(t, \omega)-\sum_{j=1}^{K} S_{j}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{1}{2}\left(\nu_{*}-5\right)+\zeta}, \quad t \in[0, T), \tag{2.4}
\end{equation*}
$$

where $S_{j}, 1 \leq j \leq K$, are the pseudo-conformal blow-up solutions

$$
\begin{equation*}
S_{j}(t, x)=\left(\omega_{j}(T-t)\right)^{-\frac{d}{2}} Q\left(\frac{x-x_{j}}{\omega_{j}(T-t)}\right) e^{-\frac{i}{4} \frac{\left|x-x_{j}\right|^{2}}{T-t}+\frac{i}{\omega_{j}^{2}(T-t)}+i \vartheta_{j}}, \quad t \in(0, T) \tag{2.5}
\end{equation*}
$$

Remark 2.4. The asymptotic behavior (2.4) yields that the blow-up solution concentrates at the given $K$ points, i.e.,

$$
\begin{equation*}
|X(t, \omega)|^{2} \rightharpoonup \sum_{j=1}^{K}\|Q\|_{L^{2}}^{2} \delta_{x=x_{j}}, \quad \text { as } t \rightarrow T \tag{2.6}
\end{equation*}
$$

In particular, $\|X(t, \omega)\|_{L^{2}}=K\|Q\|_{L^{2}}$. Hence the multi-bubble solutions are constructed in the large mass regime, which is different from the minimal mass case in 58. Moreover, the asymptotic can be taken in the pseudo-conformal space $\Sigma$, which improves the $H^{1}$ approximation result in [58].

Remark 2.5. The estimate (2.4) also shows that the order of approximation can be improved if the noise is more flat at the blow-up points. In the case $\nu_{*} \geq 6$, the approximation (2.4) can be even taken in the more regular space $H^{\frac{3}{2}}$ (see Proposition 7.1 below). One can also improve the perturbation orders of the geometrical parameters and the remainder with more flat noise, see estimates (5.1)-(5.4) and Theorem 6.1 below. Let us also mention that, such asymptotic behavior (2.4) is exhibited only after the stochastic solutions are rescaled by the random transformation $e^{-W}$.

Remark 2.6. The multi-bubble blow-up solutions were first constructed by Merle in the pioneering work [43] for NLS in any dimensions, the main blow-up profile in [43] is built on any functions that decay exponentially, while the frequencies $\left\{\omega_{j}\right\}_{j=1}^{K}$ are required to have a uniform positive lower bound and the asymptotic behavior is taken in the space $L^{2+\frac{4}{d}}$. In Theorem [2.3, the multi-bubble solutions are constructed in dimensions one and two and the blow-up profile is built on the ground state, because the corresponding linearized operators are used in the construction. The gain here is that, in Case (I) the frequencies are allowed to be arbitrarily small, and in (2.4) the approximation can be taken in the energy space $H^{1}$, which is important in the proof of uniqueness result below.

Our next main result is concerned with the conditional uniqueness of multi-bubble solutions, which is the content of Theorem 2.7 below.

Theorem 2.7. (Conditional uniqueness) Consider $d=1,2$. Assume (A0) and (A1) with $\nu_{*} \geq 5$. For any $1 \leq K<\infty$, let $\left\{\vartheta_{j}\right\} \subseteq \mathbb{R},\left\{x_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{d}$ be distinct points, $\left\{\omega_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{+}$, satisfying Case (I) or Case (II).

Then, for $\mathbb{P}$-a.e. $\omega$ there exists $\varepsilon^{*}(\omega)>0$ sufficiently small such that for any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, there exists $\tau^{*}>0$ small enough such that for any $T \in\left(0, \tau^{*}(\omega)\right]$, there exists a unique blow-up solution $X(\omega)$ to (1.1) satisfying

$$
\begin{equation*}
\left\|e^{-W(t, \omega)} X(t, \omega)-\sum_{j=1}^{K} S_{j}(t)\right\|_{H^{1}} \leq C(T-t)^{3+\zeta}, \quad t \in[0, T) \tag{2.7}
\end{equation*}
$$

where $C>0, \zeta \in(0,1)$, and $S_{j}$ are the pseudo-conformal blow-up solutions as in (2.5), $1 \leq j \leq K$.
Remark 2.8. Theorem 2.7 states that two multi-bubble solutions are the same if they both have the asymptotic behavior (2.7) in the energy space $H^{1}$. Moreover, it also yields that any $H^{1}$ solution satisfying (2.7) is exactly the constructed solution in Theorem 2.3, which lies in the more regular $\Sigma$ space.
Remark 2.9. The conditional uniqueness result also holds for the (deterministic) nonlinear Schrödinger equations with lower order perturbations (see Theorem 2.15 and Remark 2.16 below), which include the canonical NLS equation. Let us also mention that, these conditional uniqueness results are new in both the stochastic and deterministic case.

In particular, in the special single bubble case (i.e., $K=1$ ), we have the following existence and conditional uniqueness of minimal mass blow-up solutions.
Theorem 2.10. Consider $d=1,2$. Assume (A0) and (A1) with $\nu_{*} \geq 5$. Let $x_{*}, \omega_{*}, \vartheta_{*}$ be any given points. Then, for $\mathbb{P}$-a.e. $\omega$ there exists $\tau^{*}(\omega)>0$ sufficiently small, such that for any $T \in\left(0, \tau^{*}(\omega)\right]$ there exists a minimal mass blow-up solution $X(\omega)$ to (1.1) satisfying that

$$
\begin{equation*}
\left\|e^{-W(t, \omega)} X(t, \omega)-S_{*}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{1}{2}\left(\nu_{*}-5\right)+\zeta}, \quad t \in[0, T) \tag{2.8}
\end{equation*}
$$

where $C>0, \zeta \in(0,1)$, and $S_{*}$ is as in (1.9) with $x_{*}, \omega_{*}, \vartheta_{*}$ replacing $\alpha_{j}, \omega_{j}, \vartheta_{j}$, respectively.

Moreover, in the case where $\nu_{*} \geq 11$, there exists a unique minimal mass blow-up solution $X(\omega)$ to (1.1) satisfying that

$$
\begin{equation*}
\left\|e^{-W(t, \omega)} X(t, \omega)-S_{*}(t)\right\|_{H^{1}} \leq C(T-t)^{3+\zeta}, \quad t \in[0, T) \tag{2.9}
\end{equation*}
$$

Remark 2.11. The existence of minimal mass blow-up solutions are proved in the recent work [58], but with the asymptotic (2.8) taken in the $H^{1}$ space. The conditional uniqueness
result is new in the stochastic case. It should be mentioned that, the strong uniqueness of minimal mass blow-up solutions to NLS in the deterministic case was first obtained by Merle in the remarkable paper [46]. Such strong rigidity results have been also obtained for the inhomogeneous NLS equations [55] and for the gKdV equations 41]. For the stochastic equation (1.1) the strong uniqueness of minimal mass blow-up solutions is at present still unclear, due to the lack of the conservation law of energy.

Equation (1.1) is indeed closely related to the nonlinear Schrödinger equations with lower order perturbations. More precisely, by virtue of the rescaling transformation

$$
\begin{equation*}
u:=e^{-W} X, \tag{2.10}
\end{equation*}
$$

we may reduce equation (1.1) to the random equation below

$$
\begin{align*}
& i \partial_{t} u+\Delta u+|u|^{\frac{4}{d}} u+b \cdot \nabla u+c u=0  \tag{2.11}\\
& u(0)=u_{0}
\end{align*}
$$

where $b$ and $c$ are the coefficients of lower order perturbations

$$
\begin{align*}
b(t, x) & =2 \nabla W(t, x)=2 i \sum_{k=1}^{N} \nabla \phi_{k}(x) B_{k}(t),  \tag{2.12}\\
c(t, x) & =\sum_{j=1}^{d}\left(\partial_{j} W(t, x)\right)^{2}+\Delta W(t, x) \\
& =-\sum_{j=1}^{d}\left(\sum_{k=1}^{N} \partial_{j} \phi_{k}(x) B_{k}(t)\right)^{2}+i \sum_{k=1}^{N} \Delta \phi_{k}(x) B_{k}(t) . \tag{2.13}
\end{align*}
$$

The key equivalent result has been proved in the recent work [58], based on a delicate analysis of the temporal regularities. Let us mention that, such transformation is known as the Doss-Sussman transformation in finite dimensional case, and proves to be also very robust in the infinite dimensional spaces. One main advantage is that, from the viewpoint of analysis, it enables one to treat equation (1.1) as a random dynamic system outside a uniform probability null set, and thus to perform the sharp path-by-path analysis of stochastic solutions, which is in general not easy by standard stochastic analysis. Furthermore, the rescaling approach also reveals the structure of stochastic equations, which becomes more apparent in the reduced nonlinear Schrödinger equations with lower order perturbations. See, for instance, the stochastic logarithmic Schrödinger equations in [6], the damped effect of non-conservative noise in [7], and the scattering behavior in the stochastic setting in [31]. See also the existence and geometrical characterization of optimal controllers in [8, 65], related to the Ekeland's variational principle and the theory of $U^{p}-V^{p}$ spaces.

The solutions to equation (2.11) are defined below.
Definition 2.12. We say that $u$ is a solution to (2.11) on $\left[0, \tau^{*}\right)$, where $\tau^{*} \in(0, \infty]$ is a random variable, if $\mathbb{P}-$ a.s. $u \in C\left(\left[0, \tau^{*}\right) ; H^{1}\right),|u|^{\frac{4}{d}} u \in L^{1}\left(0, \tau^{*} ; H^{-1}\right)$, and $u$ satisfies

$$
\begin{equation*}
u(t)=u(0)+\int_{0}^{t} i e^{-W(s)} \Delta\left(e^{W(s)} u(s)\right)+i|u(s)|^{\frac{4}{d}} u(s) d s, \quad t \in\left[0, \tau^{*}\right) \tag{2.14}
\end{equation*}
$$

as an equation in $H^{-1}$.
The key equivalent relationship between equations (1.1) and (2.11) is stated in Theorem 2.13 below.

Theorem 2.13. ([58, Theorem 2.10]) (i). Let $u$ be the solution to (2.11) on $\left[0, \tau^{*}\right)$ with $u(0)=u_{0} \in H^{1}$ in the sense of Definition [2.12, where $\tau^{*} \in(0, \infty]$ is a random variable. Then, $\mathbb{P}$-a.s., $X:=e^{W} u$ is the solution to equation (1.1) on $\left[0, \tau^{*}\right)$ with $X(0)=u_{0}$ in the sense of Definition 2.1.
(ii). Let $X$ be the solution to equation (1.1) on $\left[0, \tau^{*}\right)$ with $X(0)=X_{0} \in H^{1}$ in the sense of Definition [2.1, satisfying that $\mathbb{P}$-a.s. $\|X\|_{C\left([0, T] ; H^{1}\right)}+\|X\|_{L^{2}\left(0, T ; H_{-1}^{\frac{3}{2}}\right)}<\infty$, $T \in\left(0, \tau^{*}\right)$, and

$$
\left\|e^{-i t \Delta} e^{-W(t)} X(t)-e^{-i s \Delta} e^{-W(s)} X(s)\right\|_{L^{2}} \leq C(t)(t-s), \quad \forall 0 \leq s<t<\tau^{*}
$$

Then, $u:=e^{-W} X$ solves equation (2.11) on $\left[0, \tau^{*}\right)$ with $u(0)=X_{0}$ in the sense of Definition 2.12.

Hence, by virtue of Theorem [2.13, the proof of Theorems 2.3 and 2.7 is now reduced to that of Theorems 2.14 and 2.15 below corresponding to the equation (2.11).
Theorem 2.14. (Existence) Consider $d=1,2$. Assume (A0) and (A1) with $\nu_{*} \geq 5$. For any $1 \leq K<\infty$, let $\left\{\vartheta_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R},\left\{x_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{d}$ be distinct points, and $\left\{\omega_{j}\right\}_{j=1}^{K} \subseteq \mathbb{R}^{+}$, satisfying either Case (I) or Case (II).

Then, for $\mathbb{P}$-a.e. $\omega$ there exists $\varepsilon^{*}(\omega)>0$ sufficiently small such that for any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, there exists $\tau^{*}>0$ small enough such that for any $T \in\left(0, \tau^{*}(\omega)\right]$, there exist $u_{0}(\omega) \in \Sigma$ and a corresponding blow-up solution $u(\omega)$ to (2.11) such that

$$
\begin{equation*}
\left\|u(t, \omega)-\sum_{j=1}^{K} S_{j}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{1}{2}\left(\nu_{*}-5\right)+\zeta}, \quad t \in[0, T) \tag{2.15}
\end{equation*}
$$

where $C>0, \zeta \in(0,1)$, and $S_{j}$ are the pseudo-conformal blow-up solutions given by (2.5), $1 \leq j \leq K$.

Theorem 2.15. (Conditional uniqueness) Consider the situations as in Theorem 2.14. Then, for $\mathbb{P}-$ a.e. $\omega$ there exists $\varepsilon^{*}(\omega)>0$ sufficiently small such that for any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, there exists $\tau^{*}>0$ small enough such that for any $T \in\left(0, \tau^{*}(\omega)\right]$, there exists a unique blow-up solution $u(\omega)$ to (2.11) satisfying

$$
\begin{equation*}
\left\|u(t, \omega)-\sum_{j=1}^{K} S_{j}(t)\right\|_{H^{1}} \leq C(T-t)^{3+\zeta}, \quad t \in[0, T) \tag{2.16}
\end{equation*}
$$

where $C>0, \zeta \in(0,1)$, and $S_{j}$ are as in (2.5), $1 \leq j \leq K$.
Remark 2.16. The existence and conditional uniqueness results of multi-bubbles also hold for the deterministic nonlinear Schrödinger equation with lower order perturbations if the Brownian motions $\left\{B_{k}\right\}$ in (2.11) are replaced by any deterministic continuous functions, namely,

$$
\begin{equation*}
i \partial_{t} v+\Delta v+|v|^{\frac{4}{d}} v+a_{1} \cdot \nabla u+a_{2} u=0 \tag{2.17}
\end{equation*}
$$

where

$$
a_{1}(t, x)=2 i \sum_{k=1}^{N} \nabla \phi_{k}(x) h_{k}(t), \quad a_{2}(t, x)=-\sum_{j=1}^{d}\left(\sum_{k=1}^{N} \partial_{j} \phi_{k}(x) h_{k}(t)\right)^{2}+i \sum_{k=1}^{N} \Delta \phi_{k}(x) h_{k}(t),
$$

$\phi_{k}$ satisfy Assumptions $(A 0)$ and $(A 1)$ and $h_{k} \in C\left(\mathbb{R}^{+} ; \mathbb{R}\right), 1 \leq k \leq K$. In particular, these results are applicable to the canonical NLS equation, in which $a_{1}, a_{2} \equiv 0$. Note that, the standard pseudo-conformal symmetry and the conservation law of energy are also destroyed in equation (2.17).
2.2. Strategy of proof. By virtue of the equivalent result Theorem [2.13, we shall mainly focus on the proof of Theorems 2.14 and 2.15, namely, the existence and uniqueness of multi-bubble blow-up solutions to nonlinear Schrödinger equations with lower order perturbations (2.11).

As mentioned above, the absence of specific symmetries and the conservation law of energy makes the blow-up analysis rather intricate. A robust modulation method was developed by Raphaël and Szeftel [55] for the existence and uniqueness of minimal mass blow-up solutions to inhomogeneous nonlinear Schrödinger equations, which is a canonical model proposed by Merle [45] to break the pseudo-conformal symmetry. This modulation method has been recently applied in [58] to construct minimal mass blow-up solutions for both equations (1.1) and (2.11). The main strategy consists of geometrical decompositions, a bootstrap device and backward propagation from the singularity.

Here, we use and extend the modulation method to address the multi-bubble problem. More specifically, we first decompose the solution to (2.11) into a main blow-up profile and a remainder

$$
u(t, x)=\sum_{j=1}^{K} \lambda_{j}^{-\frac{d}{2}} Q_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}}+R(t, x), \text { with } Q_{j}(t, y)=Q(y) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}
$$

where $Q_{j}$ and $R$ satisfy the orthogonality conditions in (4.5) below, which are related to the null space of the linearized operators around the ground state and ensure the uniqueness of this decomposition. Such geometrical decomposition enables us to reduce the blow-up analysis into those of the five finite-dimensional geometrical parameters $\left(\lambda_{j}, \alpha_{j}, \beta_{j}, \gamma_{j}, \theta_{j}\right)$ and the infinite-dimensional remainder $R$. As a first consequence, the estimate of the modulation equations is derived, which indeed captures the dynamics of the geometrical parameters. This part is contained in Section 4 ,

The key unform estimates of the geometrical parameters and the remainder are obtained by using a bootstrap device and the propagation backward from the singularity. The main efforts here are dedicated to the analysis of the localized mass, the energy and a new generalized functional.

It should be mentioned that, unlike the single bubble case in [55, 58, the growth in the unstable direction $Q_{j}$ is analyzed via a localized mass, instead of the usual whole mass. Moreover, we introduce a new generlized energy (5.28) adapted to the multi-bubble setting. It incorporates the localized functions in an appropriate way, such that different bubbles can be decoupled while the key monotonicity property keeps preserved. One delicate problem here lies in the decoupling of the remainders, due to the corresponding low polynomial type perturbation orders, and, actually, extra decays have to be explored from the test functions. Furthermore, a refined estimate of the modulation parameter $\beta$ is derived from the coercivity of the energy, of which the proof requires a careful treatment to balance the localized functions and the test functions involved in the localized coercivity of the linearized operators. These constitute the main part of Section 5 .

Then in Section 6, the construction of the multi-bubble blow-up solutions follows from a compactness argument, based on the uniform estimates and integrating the flow backward from the blow-up time. Let us mention that, the uniform estimates can be also obtained in the pseudo-conformal space $\Sigma$, which improves the previous $H^{1}$-estimate in 58] and also simplifies the compactness argument.

Concerning the uniqueness part in Section 7, the key idea again relies on the monotonicity formula of the generalized energy adapted to the difference of two multi-bubble solutions, and is to show that the implied a priori bound of the difference is exactly zero.

More precisely, via the generalized energy (7.41) below, we obtain the estimate (see Theorem 7.6 below)

$$
\sup _{t \leq s<T} D(s) \leq C\left(\sum_{j=1}^{K} \sup _{t \leq s<T} \frac{\operatorname{Scal}_{j}(s)}{\lambda_{j}^{2}(s)}+\int_{t}^{T} \sum_{j=1}^{K} \frac{\operatorname{Scal}_{j}(s)}{\lambda_{j}^{3}(s)}+\varepsilon^{*} \frac{D(s)}{T-s} d s\right) .
$$

where $D(t):=\|\nabla w(t)\|_{L^{2}}^{2}+\sum_{j=1}^{K} \lambda_{j}^{-2}\left\|w_{j}(t)\right\|_{L^{2}}^{2}$ is defined on the difference $w, w_{j}=w \Phi_{j}$ with the localized function $\Phi_{j}$, and $S_{c a l}$ denotes the scalar products of $w_{j}$ and the unstable directions in the null space. This step requires a careful analysis of the differences between nonlinearities.

The next step is to control the unstable growth generated by the null space, that is, we prove that (see Theorem 7.7 below), for some $\zeta \in(0,1)$,

$$
\operatorname{Scal}_{j}(t) \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} D(s) .
$$

For this purpose, a new renormalized variable is introduced. It satisfies a neat formulation of equation and enables us to obtain the estimates of the scalar products in $S c a l_{j}$ in a simplified diagonalized form (see Proposition 7.12 below).

At this stage, by virtue of the two estimates above, we obtain the estimate of $D(t)$ in a closed form. It should be mentioned that, because of the localization functions in the multi-bubble case, an extra error $\varepsilon^{*} \frac{D(t)}{T-t}$ is also involved here. This requires an iterated argument to show that $D(t)$ is exactly zero, which is different from the single bubble case in 555.

The remainder of this paper is structured as follows. In Section 3 we first present some preliminaries including the localization functions, the coercivity of linearized operator, and the Taylor expansion in the complex situation. Then, Sections 4 , 5 and 6 are mainly devoted to the proof of the existence of blow-up solutions at multiple points. In Section 7 we prove the uniqueness of blow-up solutions. Finally, some technical proofs are postponed to the Appendix, i.e., Section 8 ,

## 3. Preliminaries

3.1. Localization. We shall use the localization functions in order to construct the blowup profiles concentrating at distinct blow-up points.

For this purpose, we note that, because equation (2.11) is invariant under orthogonal transforms, we may take an orthonormal basis $\left\{\mathbf{v}_{\mathbf{j}}\right\}_{j=1}^{d}$ of $\mathbb{R}^{d}$, such that $\left(x_{j}-x_{l}\right) \cdot \mathbf{v}_{\mathbf{1}} \neq 0$ for any $1 \leq j \neq l \leq K$. Hence, we may assume that $x_{1} \cdot \mathbf{v}_{\mathbf{1}}<x_{2} \cdot \mathbf{v}_{\mathbf{1}}<\cdots<x_{K} \cdot \mathbf{v}_{\mathbf{1}}$. Then,

$$
\begin{equation*}
\sigma:=\frac{1}{12} \min _{1 \leq j \leq K-1}\left\{\left(x_{j+1}-x_{j}\right) \cdot \mathbf{v}_{\mathbf{1}}\right\}>0 . \tag{3.1}
\end{equation*}
$$

Let $\Phi(x)$ be a smooth function on $\mathbb{R}^{d}$ such that $0 \leq \Phi(x) \leq 1,|\nabla \Phi(x)| \leq C \sigma^{-1}$, $\Phi(x)=1$ for $x \cdot \mathbf{v}_{\mathbf{1}} \leq 4 \sigma$ and $\Phi(x)=0$ for $x \cdot \mathbf{v}_{\mathbf{1}} \geq 8 \sigma$. The localization functions $\Phi_{j}$ are defined by namely,

$$
\begin{align*}
& \Phi_{1}(x):=\Phi\left(x-x_{1}\right), \quad \Phi_{K}(x):=1-\Phi\left(x-x_{K-1}\right), \\
& \Phi_{j}(x):=\Phi\left(x-x_{j}\right)-\Phi\left(x-x_{j-1}\right), \quad 2 \leq j \leq K-1 . \tag{3.2}
\end{align*}
$$

In particular, we have the partition of unity $1=\sum_{j=1}^{K} \Phi_{j}$.
Lemma 3.1 below enables us to decouple different blow-up profiles and will be used frequently throughout this paper.

Lemma 3.1. (Interaction estimates) Let $0<t^{*}<T_{*}<T<\infty$. For every $1 \leq j \leq K$, set

$$
\begin{equation*}
G_{j}(t, x):=\lambda_{j}^{-\frac{d}{2}} g_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}}, \text { with } g_{j}(t, y):=g(y) e^{i\left(\beta_{j}(t) \cdot y-\frac{1}{4} \gamma_{j}(t)|y|^{2}\right)} \tag{3.3}
\end{equation*}
$$

where $g \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ decays exponentially fast at infinity

$$
\left|\partial^{\nu} g(y)\right| \leq C e^{-\delta|y|}, \quad|\nu| \leq 2
$$

with $C, \delta>0$, for $1 \leq j \leq K, \mathcal{P}_{j}:=\left(\lambda_{j}, \alpha_{j}, \beta_{j}, \gamma_{j}, \theta_{j}\right) \in C\left(\left[t^{*}, T_{*}\right] ; \mathbb{R}^{2 d+3}\right)$ satisfies

$$
\begin{equation*}
\left|\left(\alpha_{j}(t)-x_{j}\right) \cdot \mathbf{v}_{\mathbf{1}}\right| \leq \sigma,\left|x_{j}-\alpha_{j}(t)\right| \leq 1, \frac{1}{2} \leq \frac{\lambda_{j}(t)}{\left|\omega_{j}(T-t)\right|} \leq 2, \quad t \in\left[t^{*}, T_{*}\right] \tag{3.4}
\end{equation*}
$$

and $\left|\beta_{j}\right|+\left|\gamma_{j}\right| \leq 1$,

$$
\begin{equation*}
C T\left(1+\max _{1 \leq j \leq K}\left|x_{j}\right|\right) \leq 1 \tag{3.5}
\end{equation*}
$$

where $C$ is sufficiently large but independent of $T$. Then, there exists $\delta>0$ such that for any $1 \leq j \neq l \leq K, m \in \mathbb{N}$, and for any multi-index $\nu$ with $|\nu| \leq 2$,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|x-\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{j}\right|^{m}\left|G_{j}(t)\right| d x \leq C e^{-\frac{\delta}{T-t}}, \quad t \in\left[t^{*}, T_{*}\right] . \tag{3.6}
\end{equation*}
$$

Moreover, for any $h \in L^{1}$ or $L^{2}, 1 \leq j \neq l \leq K, m, n \in \mathbb{N}$, and for any multi-index $\nu$ with $|\nu| \leq 2$,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|x-\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{j}\right|^{m}|h| \Phi_{j} d x \leq C e^{-\frac{\delta}{T-t}} \min \left\{\|h\|_{L^{1}},\|h\|_{L^{2}}\right\}, \quad t \in\left[t^{*}, T_{*}\right] . \tag{3.7}
\end{equation*}
$$

The proof is postponed to the Appendix for simplicity. Lemma 3.1 actually shows that the interactions between $\left\{U_{j}\right\}$ and other profiles are very weak, mainly due to the exponential decay of the ground state.
3.2. Coercivity of linearized operators. We denote $Q$ the ground state that solves the soliton equation (1.8). It follows from [14, Theorem 8.1.1] that $Q$ is smooth and decays at infinity exponentially fast, i.e., there exist $C, \delta>0$ such that for any multi-index $|\nu| \leq 3$,

$$
\begin{equation*}
\left|\partial_{x}^{\nu} Q(x)\right| \leq C e^{-\delta|x|}, \quad x \in \mathbb{R}^{d} \tag{3.8}
\end{equation*}
$$

Let $L=\left(L_{+}, L_{-}\right)$be the linearized operator around the ground state, defined by

$$
\begin{equation*}
L_{+}:=-\Delta+I-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}}, \quad L_{-}:=-\Delta+I-Q^{\frac{4}{d}} . \tag{3.9}
\end{equation*}
$$

The generalized null space of operator $L$ is spanned by $\left\{Q, x Q,|x|^{2} Q, \nabla Q, \Lambda Q, \rho\right\}$, where $\Lambda:=\frac{d}{2} I+x \cdot \nabla$, and $\rho$ is the unique $H^{1}$ spherically symmetric solution to the equation

$$
\begin{equation*}
L_{+} \rho=-|x|^{2} Q \tag{3.10}
\end{equation*}
$$

which satisfies the exponential decay property (see, e.g., [38, 42]), i.e., for some $C, \delta>0$,

$$
|\rho(x)|+|\nabla \rho(x)| \leq C e^{-\delta|x|}
$$

Moreover, we have (see, e.g., [63, (B.1), (B.10), (B.15)])

$$
\begin{array}{ll}
L_{+} \nabla Q=0, \quad L_{+} \Lambda Q=-2 Q, & L_{+} \rho=-|x|^{2} Q \\
L_{-} Q=0, & L_{-} x Q=-2 \nabla Q,  \tag{3.11}\\
L_{-}|x|^{2} Q=-4 \Lambda Q
\end{array}
$$

For any complex valued $H^{1}$ function $f=f_{1}+i f_{2}$ in terms of the real and imaginary parts, we set

$$
\begin{equation*}
(L f, f):=\int f_{1} L_{+} f_{1} d x+\int f_{2} L_{-} f_{2} d x \tag{3.12}
\end{equation*}
$$

Let $\mathcal{K}$ denote the set of all complex valued $H^{1}$ functions $f=f_{1}+i f_{2}$ satisfying the orthogonality conditions below

$$
\begin{align*}
& \int Q f_{1} d x=0, \quad \int x Q f_{1} d x=0, \quad \int|x|^{2} Q f_{1} d x=0 \\
& \int \nabla Q f_{2} d x=0, \quad \int \Lambda Q f_{2} d x=0, \quad \int \rho f_{2} d x=0 \tag{3.13}
\end{align*}
$$

The coercivity property below is crucial in the proof of main results.
Lemma 3.2. (63, Theorem 2.5]) There exists $C>0$ such that

$$
\begin{equation*}
(L f, f) \geq C\|f\|_{H^{1}}^{2}, \quad \forall f \in \mathcal{K} \tag{3.14}
\end{equation*}
$$

We define the scalar products along all the unstable directions in the null space

$$
\begin{equation*}
\left.S \operatorname{cal}(f)=\left\langle f_{1}, Q\right\rangle^{2}+\left\langle f_{1}, x Q\right\rangle^{2}+\left.\left\langle f_{1},\right| x\right|^{2} Q\right\rangle^{2}+\left\langle f_{2}, \nabla Q\right\rangle^{2}+\left\langle f_{2}, \Lambda Q\right\rangle^{2}+\left\langle f_{2}, \rho\right\rangle^{2} \tag{3.15}
\end{equation*}
$$

where $f=f_{1}+i f_{2} \in H^{1}$. As a consequence of Lemma 3.2 we have
Corollary 3.3. ([58, Corollary 3.2]) There exist positive constants $C_{1}, C_{2}>0$, such that

$$
\begin{equation*}
(L f, f) \geq C_{1}\|f\|_{H^{1}}^{2}-C_{2} S \operatorname{cal}(f), \quad \forall f \in H^{1}, \tag{3.16}
\end{equation*}
$$

where $f_{1}$ and $f_{2}$ are the real and imaginary parts of $f$, respectively.
Corollary 3.4. (Localized coercivity) Let $\phi$ be a positive smooth radial function on $\mathbb{R}^{d}$, such that $\phi(x)=1$ for $|x| \leq 1, \phi(x)=e^{-|x|}$ for $|x| \geq 2,0<\phi \leq 1$, and $\left|\frac{\nabla \phi}{\phi}\right| \leq C$ for some $C>0$. Set $\phi_{A}(x):=\phi\left(\frac{x}{A}\right), A>0$. Then, for A large enough we have
$\int\left(|f|^{2}+|\nabla f|^{2}\right) \phi_{A}-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} f_{1}^{2}-Q^{\frac{4}{d}} f_{2}^{2} d x \geq C_{1} \int\left(|\nabla f|^{2}+|f|^{2}\right) \phi_{A} d x-C_{2} \operatorname{Scal}(f)$,
where $C_{1}, C_{2}>0$, and $f_{1}, f_{2}$ are the real and imaginary parts of $f$, respectively.
The proof of Corollary 3.4 is similar to that of [58, Corollary 3.3] and is postponed to the Appendix for simplicity.
3.3. Expansion of the nonlinearity. We shall use the notations that, for any continuous differentiable function $g: \mathbb{C} \rightarrow \mathbb{C}$ and for any $v, R \in \mathbb{C}$,

$$
\begin{align*}
g^{\prime}(v, R) \cdot R:= & R \int_{0}^{1} \partial_{z} g(v+s R) d s+\bar{R} \int_{0}^{1} \partial_{\bar{z}} g(v+s R) d s,  \tag{3.18}\\
g^{\prime \prime}(v, R) \cdot R^{2}:= & R^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{z z} g(v+s t R) d s d t+2|R|^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{z \bar{z}} g(v+s t R) d s d t \\
& +\bar{R}^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{\bar{z} \bar{z}} g(v+s t R) d s d t, \tag{3.19}
\end{align*}
$$

where $\partial_{z} g$ and $\partial_{\bar{z}} g$ are the usual complex derivatives $\partial_{z} g=\frac{1}{2}\left(\partial_{x} g-i \partial_{y} g\right), \partial_{\bar{z}} g=\frac{1}{2}\left(\partial_{x} g+\right.$ $i \partial_{y} g$ ), respectively. Then, one has (see, e.g., [35, (3.10)])

$$
\begin{equation*}
g(v+R)=g(v)+g^{\prime}(v, R) \cdot R \tag{3.20}
\end{equation*}
$$

Moreover, if $\partial_{z} g$ and $\partial_{\bar{z}} g$ are also continuously differentiable, we may expand $g$ up to the second order

$$
\begin{equation*}
g(v+R)=g(v)+\partial_{z} g(v) R+\partial_{\bar{z}} g(v) \bar{R}+g^{\prime \prime}(v, R) \cdot R^{2} . \tag{3.21}
\end{equation*}
$$

In particular, for the complex function $f(z)=|z|^{\frac{4}{d}} z$ with $d=1,2$, we have

$$
\begin{align*}
\left|f^{\prime}(v, R) \cdot R\right| & \leq C\left(|v|^{\frac{4}{d}}+|R|^{\frac{4}{d}}\right)|R|  \tag{3.22}\\
\left|f^{\prime \prime}(v, R) \cdot R^{2}\right| & \leq C\left(|v|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R|^{2} \tag{3.23}
\end{align*}
$$

It would be also useful to use the expansion, for $f(z)=|z|^{\frac{4}{d}} z$ with $d=1,2$,

$$
\begin{equation*}
f(v+R)=f(v)+f^{\prime}(v) \cdot R+f^{\prime \prime}(v) \cdot R^{2}+\mathcal{O}\left(\sum_{k=3}^{1+\frac{4}{d}}|v|^{1+\frac{4}{d}-k}|R|^{k}\right), \tag{3.24}
\end{equation*}
$$

where

$$
\begin{align*}
f^{\prime}(v) \cdot R & :=\partial_{z} f(v) R+\partial_{\bar{z}} f(v) \bar{R}=\left(1+\frac{2}{d}\right)|v|^{\frac{4}{d}} R+\frac{2}{d}|v|^{\frac{4}{d}-2} v^{2} \bar{R},  \tag{3.25}\\
f^{\prime \prime}(v) \cdot R^{2} & :=\frac{1}{2} \partial_{z z} f(v) R^{2}+\partial_{z \bar{z}} f(v)|R|^{2}+\frac{1}{2} \partial_{\bar{z} \bar{z}} f(v) \bar{R}^{2} \\
& =\frac{1}{d}\left(1+\frac{2}{d}\right)|v|^{\frac{4}{d}-2} \bar{v} R^{2}+\frac{2}{d}\left(1+\frac{2}{d}\right)|v|^{\frac{4}{d}-2} v|R|^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)|v|^{\frac{4}{d}-4} v^{3} \bar{R}^{2} . \tag{3.26}
\end{align*}
$$

Similarly, for $F(z):=\frac{d}{2 d+4}|z|^{2+\frac{4}{d}}$ with $d=1,2$, we have the expansion

$$
F(u)=F(v)+\frac{1}{2}|v|^{\frac{4}{d}} \bar{v} R+\frac{1}{2}|v|^{\frac{4}{d}} v \bar{R}
$$

$$
\begin{equation*}
+\frac{1}{2 d}|v|^{\frac{4}{d}-2} \bar{v}^{2} R^{2}+\frac{1}{2}\left(1+\frac{2}{d}\right)|v|^{\frac{4}{d}}|R|^{2}+\frac{1}{2 d}|v|^{\frac{4}{d}-2} v^{2} \bar{R}^{2}+\mathcal{O}\left(\sum_{k=3}^{2+\frac{4}{d}}|v|^{2+\frac{4}{d}-k}|R|^{k}\right) \tag{3.27}
\end{equation*}
$$

In most cases in this paper, the high order terms in the expansion of nonlinearity can be controlled by the Gagliardo-Nirenberg inequality contained in Lemma 3.5 below.
Lemma 3.5. ([14, Theorem 1.3.7]) Let $d \geq 1$ and $2 \leq p<\infty$. Then, there exists $C>0$ such that

$$
\begin{equation*}
\|f\|_{L^{p}} \leq C\|f\|_{L^{2}}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\nabla f\|_{L^{2}}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}, \quad \forall f \in H^{1} \tag{3.28}
\end{equation*}
$$

In particular, for any $1<p<\infty$,

$$
\begin{equation*}
\|f\|_{L^{p}} \leq C\|f\|_{H^{1}}, \quad \forall f \in L^{p} \tag{3.29}
\end{equation*}
$$

We also have the product rule below.
Lemma 3.6. (Product rule [61, p.105, Proposition 1.1]) For any $s>0$,

$$
\begin{equation*}
\|u v\|_{H^{s, p}} \leq C\left(\|u\|_{L^{q_{1}}}\|v\|_{H^{s, q_{2}}}+\|v\|_{L^{r_{1}}}\|u\|_{H^{s, r_{2}}}\right) \tag{3.30}
\end{equation*}
$$

where $\frac{1}{p}=\frac{1}{q_{1}}+\frac{1}{q_{2}}=\frac{1}{r_{1}}+\frac{1}{r_{2}}, q_{1}, r_{1} \in(1, \infty], q_{2}, r_{2} \in(1, \infty)$.
As a consequence we have
Lemma 3.7. For any complex functions $f, g, h$ and for any $l, m, n \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\|f^{l} g^{m} h^{n}\right\|_{L^{p}} \leq C\|f\|_{H^{1}}^{l}\|g\|_{H^{1}}^{m}\|h\|_{H^{1}}^{n} \tag{3.31}
\end{equation*}
$$

Moreover, we also have

$$
\begin{equation*}
\left\|f^{l} g^{m} h^{n}\right\|_{H^{\frac{1}{2}}} \leq \underset{15}{C \| f}\left\|_{H^{1}}^{l}\right\| g\left\|_{H^{1}}^{m}\right\| h \|_{H^{1}}^{n} . \tag{3.32}
\end{equation*}
$$

Proof. (3.31) follows from Hölder's inequality and (3.29). Regarding (3.32), by the product rule,

$$
\begin{aligned}
&\left\|f^{l} g^{m} h^{n}\right\|_{\dot{H}^{\frac{1}{2}}} \leq C\left(\|f\|_{H^{\frac{1}{2}, p_{1}}}\|f\|_{L^{p_{12}}}^{l-1}\|g\|_{L^{q_{1}}}^{m}\|h\|_{L^{r_{1}}}^{n}+\|f\|_{L^{p_{2}}}^{l}\|g\|_{H^{\frac{1}{2}, q_{21}}}\|g\|_{L^{q_{22}}}^{m-1}\|h\|_{L^{r_{2}}}^{n}\right. \\
&+\|f\|_{L^{p_{2}}}^{l}\|g\|_{L^{q_{3}}}^{m}\|h\|_{H^{\frac{1}{2}, r_{31}}}^{n}\|h\|_{L^{r_{32}}}^{n-1},
\end{aligned}
$$

where $\frac{1}{2}=\frac{1}{p_{11}}+\frac{l-1}{p_{12}}+\frac{m}{q_{1}}+\frac{n}{r_{1}}=\frac{l}{p_{2}}+\frac{1}{q_{21}}+\frac{m-1}{q_{22}}+\frac{n}{r_{2}}=\frac{l}{p_{2}}+\frac{m}{q_{3}}+\frac{1}{r_{31}}+\frac{n-1}{r_{32}}$. We then take $p_{11}, q_{21}, r_{31}$ close to 2 such that $H^{1}$ is imbedded into the Sobolev spaces $H^{\frac{1}{2}, p_{11}}, H^{\frac{1}{2}, q_{21}}$ and $H^{\frac{1}{2}, r_{31}}$. Then, taking into account (3.29) we obtain (3.32) and finish the proof.

## 4. Geometrical decomposition and modulation equations

4.1. Geometrical decomposition. For each $1 \leq j \leq K$, define the modulation parameters by $\mathcal{P}_{j}:=\left(\lambda_{j}, \alpha_{j}, \beta_{j}, \gamma_{j}, \theta_{j}\right) \in \mathbb{R}^{2 d+3}$, where $\lambda_{j} \in \mathbb{R}, \alpha_{j} \in \mathbb{R}^{d}, \beta_{j} \in \mathbb{R}^{d}, \gamma_{j} \in \mathbb{R}$ and $\theta_{j} \in \mathbb{R}$, and set $P_{j}:=\left|\lambda_{j}\right|+\left|\alpha_{j}-x_{j}\right|+\left|\beta_{j}\right|+\left|\gamma_{j}\right|$, where $x_{j}$ are the given blow-up points, $1 \leq j \leq K$.

We also set $\mathcal{P}:=\left(\mathcal{P}_{1}, \cdots, \mathcal{P}_{K}\right) \in \mathbb{R}^{(2 d+3) K}, P:=\sum_{j=1}^{K} P_{j}$. Similarly, let $\lambda:=$ $\left(\lambda_{1}, \cdots, \lambda_{K}\right) \in \mathbb{R}^{K}$ and $|\lambda|:=\sum_{j=1}^{K}\left|\lambda_{j}\right|$. Similar notations are also used for the remaining parameters.
Proposition 4.1. (Geometrical decomposition) Assume that $u \in C\left(\left[\widetilde{t}, T_{*}\right] ; H^{1}\right)$ for some $\widetilde{t} \in\left[0, T_{*}\right)$ and $u\left(T_{*}\right)=S_{T}\left(T_{*}\right)$. Then, for $T_{*}$ sufficiently close to $T$, there exist $t^{*}<T_{*}$ and unique modulation parameters $\mathcal{P} \in C^{1}\left(\left(t^{*}, T_{*}\right) ; \mathbb{R}^{(2 d+3) K}\right)$, such that $u$ can be geometrically decomposed into the main blow-up profile and the remainder

$$
\begin{equation*}
u(t, x)=U(t, x)+R(t, x), \quad t \in\left[t^{*}, T_{*}\right], x \in \mathbb{R}^{d} \tag{4.1}
\end{equation*}
$$

where the main blow-up profile

$$
\begin{equation*}
U(t, x)=\sum_{j=1}^{K} U_{j}(t, x) \tag{4.2}
\end{equation*}
$$

with

$$
\begin{equation*}
U_{j}(t, x)=\lambda_{j}^{-\frac{d}{2}} Q_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}}, \quad Q_{j}(t, y)=Q(y) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)} \tag{4.3}
\end{equation*}
$$

and $R\left(T_{*}\right)=0$, the modulation parameters satisfy

$$
\begin{equation*}
\mathcal{P}_{j}\left(T_{*}\right)=\left(\omega_{j}\left(T-T_{*}\right), x_{j}, 0, \omega_{j}^{2}\left(T-T_{*}\right), \omega_{j}^{-2}\left(T-T_{*}\right)^{-1}+\vartheta_{j}\right) . \tag{4.4}
\end{equation*}
$$

Moreover, for each $1 \leq j \leq K$, the following orthogonality conditions hold on $\left[t^{*}, T_{*}\right]$ :

$$
\begin{align*}
& \operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j}(t) \bar{R}(t) d x=0, \quad \operatorname{Re} \int\left|x-\alpha_{j}\right|^{2} U_{j}(t) \bar{R}(t) d x=0 \\
& \operatorname{Im} \int \nabla U_{j}(t) \bar{R}(t) d x=0, \quad \operatorname{Im} \int \Lambda U_{j}(t) \bar{R}(t) d x=0, \quad \operatorname{Im} \int \varrho_{j}(t) \bar{R}(t) d x=0 \tag{4.5}
\end{align*}
$$

where

$$
\begin{equation*}
\varrho_{j}(t, x)=\lambda_{j}^{-\frac{d}{2}} \widetilde{\rho}_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{n}}\right) e^{i \theta_{j}} \text { with } \widetilde{\rho}_{j}(t, y):=\rho(y)^{i\left(\beta_{j}(t) \cdot y-\frac{1}{4} \gamma_{j}(t)|y|^{2}\right)} \tag{4.6}
\end{equation*}
$$

and $\rho$ is given by (3.10).
Remark 4.2. Proposition 4.1 is actually a local version of the geometrical decomposition as $t^{*}$ may depend on $T_{*}$. However, as we shall see later, by virtue of the bootstrap estimates in Theorem 5.1 below we indeed have the global geometrical decomposition on the time interval $\left[0, T_{*}\right] \subseteq[0, T)$ if $T$ is sufficiently small. See Theorem 6.1 below.

The proof of Proposition 4.1 is quite similar to that of [58, Proposition 4.1] and is mainly based on the implicit function theorem. We mention that, the computations of the Jacobian of transformation also include the interactions between different profiles $\left\{U_{j}\right\}_{j=1}^{K}$ which, however, by Lemma 3.1, only contribute exponentially small errors due to the exponential decay of the ground state. Thus the Jacobian is still non-zero. The details are omitted here for simplicity.
4.2. Modulation equations. Let $\dot{g}:=\frac{d}{d t} g$ for any $C^{1}$ function $g$. For each $1 \leq j \leq K$, define the vector of modulation equations by

$$
\begin{equation*}
\operatorname{Mod}_{j}:=\left|\lambda_{j} \dot{\lambda}_{j}+\gamma_{j}\right|+\left|\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right|+\left|\lambda_{j} \dot{\alpha}_{n, j}-2 \beta_{j}\right|+\left|\lambda_{j}^{2} \dot{\beta}_{j}+\gamma_{j} \beta_{j}\right|+\left|\lambda_{j}^{2} \dot{\theta}_{j}-1-\left|\beta_{j}\right|^{2}\right|, \tag{4.7}
\end{equation*}
$$

and set $\operatorname{Mod}:=\sum_{j=1}^{K} \operatorname{Mod}_{j}$.
The main result of this subsection is formulated in Proposition 4.3 below.
Proposition 4.3. Assume that $u$ has the geometrical decomposition on $\left[t^{*}, T_{*}\right] \subseteq[0, T)$ as in (4.1) with the modulation parameters $\mathcal{P}=(\lambda, \alpha, \beta, \gamma, \theta) \in \mathbb{R}^{(2 d+3) K}$, and

$$
C_{1}(T-t) \leq|\lambda(t)| \leq C_{2}(T-t), \quad t \in\left[t^{*}, T_{*}\right]
$$

where $C_{1}, C_{2}>0$. Then, for $T$ small enough and for $t^{*}$ close to $T_{*}$, we have for any $t \in\left[t^{*}, T_{*}\right]$,
$\operatorname{Mod}(t) \leq C\left(\sum_{j=1}^{K}\left|\operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle\right|+P^{2}(t)\|R(t)\|_{L^{2}}+\|R(t)\|_{L^{2}}^{2}+\|R(t)\|_{H^{1}}^{3}+P^{\nu_{*}+1}(t)+e^{-\frac{\delta}{T-t}}\right)$,
where $C>0$ and $\nu_{*}$ is the index of flatness in Assumption (A1).
The proof relies mainly on the analysis of the equation of remainder $R$, the almost orthogonality of profiles $U_{j}$ and $R_{j}$, and the decoupling of different blow-up profiles $U_{j}$ and $U_{l}, j \neq l$.

To be precise, we use the partition of unity $1=\sum_{j=1}^{K} \Phi_{j}$ to get

$$
\begin{equation*}
R=\sum_{j=1}^{K} R_{j}, \text { with } \quad R_{j}:=R \Phi_{j} . \tag{4.9}
\end{equation*}
$$

Define the renormalized remainder $\varepsilon_{j}$ by

$$
\begin{equation*}
R_{j}(t, x)=\lambda_{j}^{-\frac{d}{2}} \varepsilon_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}} . \tag{4.10}
\end{equation*}
$$

Then, by (2.11) and (4.1), the remainder $R$ satisfies the equation

$$
\begin{align*}
& i \partial_{t} R+\sum_{l=1}^{K}\left(\Delta R_{l}+\left(1+\frac{2}{d}\right)\left|U_{l}\right|^{\frac{4}{d}} R_{l}+\frac{2}{d}\left|U_{l}\right|^{\frac{4}{d}-2} U_{l}^{2} \overline{R_{l}}+i \partial_{t} U_{l}+\Delta U_{l}+\left|U_{l}\right|^{\frac{4}{d}} U_{l}\right) \\
= & -H_{1}-H_{2}-f^{\prime \prime}(U, R) \cdot R^{2}-\sum_{l=1}^{K}\left(b \cdot \nabla\left(U_{l}+R_{l}\right)+c\left(U_{l}+R_{l}\right)\right) \tag{4.11}
\end{align*}
$$

Here, $H_{1}, H_{2}$ contain the interactions between different blow-up profiles

$$
\begin{align*}
& H_{1}:=\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}} R+\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}-\sum_{l=1}^{K}\left(\left(1+\frac{2}{d}\right)\left|U_{l}\right|^{\frac{4}{d}} R_{l}+\frac{2}{d}\left|U_{l}\right|^{\frac{4}{d}-2} U_{l}^{2} \overline{R_{l}}\right),  \tag{4.12}\\
& H_{2}:=|U|^{\frac{4}{d}} U-\sum_{l=1}^{K}\left|U_{l}\right|^{\frac{4}{d}} U_{l} \tag{4.13}
\end{align*}
$$

and $f^{\prime \prime}(U, R) \cdot R^{2}$ is defined as in (3.19) with $f, U$ replacing $g$ and $v$, respectively.
Using (4.3) we have

$$
\begin{align*}
& i \partial_{t} U_{j}+\Delta U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}=\frac{e^{i \theta_{j}}}{\lambda_{j}^{2+\frac{d}{2}}}\left\{-\left(\lambda_{j}^{2} \dot{\theta}_{j}-1-\left|\beta_{j}\right|^{2}\right) Q_{j}-\left(\lambda_{j}^{2} \dot{\beta}_{j}+\gamma_{j} \beta_{j}\right) \cdot y Q_{j}\right. \\
& \left..14) \quad+\frac{1}{4}\left(\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right)|y|^{2} Q_{j}-i\left(\lambda_{j} \dot{\alpha}_{j}-2 \beta_{j}\right) \cdot \nabla Q_{j}-i\left(\lambda_{j} \dot{\lambda}_{j}+\gamma_{j}\right) \Lambda Q_{j}\right\}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) . \tag{4.14}
\end{align*}
$$

The important fact here is, that the modulation equations show up on the right-hand side of (4.14) as the coefficients of the five directions in the generalized null space of the operator $L$ defined in (3.9). This enables us to extract each modulation equation by applying the almost orthogonality in Lemma 4.4 below, which in turn follows from Lemma 3.1 and the orthogonality conditions in (4.5).

Lemma 4.4. (Almost orthogonality) Let $t^{*}$ be as in Proposition 4.3. Then, for $t^{*}$ close to $T$, there exists $\delta>0$ such that for any $1 \leq j \leq K$, it holds on $\left[t^{*}, T\right]$ that

$$
\begin{align*}
& \left|\operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j} \overline{R_{j}} d x\right|+\left|\operatorname{Re} \int\right| x-\left.\alpha_{j}\right|^{2} U_{j} \overline{R_{j}} d x \left\lvert\, \leq C e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right. \\
& \left|\operatorname{Im} \int \nabla U_{j} \overline{R_{j}} d x\right|+\left|\operatorname{Im} \int \Lambda U_{j} \overline{R_{j}} d x\right|+\left|\operatorname{Im} \int \varrho_{j} \overline{R_{j}} d x\right| \leq C e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}} \tag{4.15}
\end{align*}
$$

Proof. By the orthogonality condition (4.5),

$$
\begin{equation*}
\operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j}(t) \overline{R_{j}}(t) d x=-\sum_{l \neq j} \operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j}(t) \overline{R_{l}}(t) d x \tag{4.16}
\end{equation*}
$$

which along with Lemma 3.1 yields immediately that for some $\delta>0$,

$$
\begin{equation*}
\left|\operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j}(t) \overline{R_{j}}(t) d x\right| \leq C e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}} \tag{4.17}
\end{equation*}
$$

The remaining four estimates in (4.15) can be proved similarly.
We are now ready to prove Proposition 4.3,
Proof of Proposition 4.3. The proof is similar to that of [58, Proposition 4.3].
Below, we take the modulation equation $\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}$, corresponding to the direction $\Lambda U_{j}$, for an example to illustrate the main arguments and to show that the scalar $\operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle$ in the unstable direction $U_{j}$ is also required to bound the modulation equation.

Taking the inner product of (4.11) with $\Lambda U_{j}$ and then taking the real part we get

$$
\begin{aligned}
& \left.-\operatorname{Im}\left\langle\partial_{t} R, \Lambda U_{j}\right\rangle+\left.\operatorname{Re}\left\langle\Delta R_{j}+\left(1+\frac{2}{d}\right)\right| U_{j}\right|^{\frac{4}{d}} R_{j}+\frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}^{2} \overline{R_{j}}, \Lambda U_{j}\right\rangle \\
& \left.+\left.\operatorname{Re}\left\langle\partial_{t} U_{j}+\Delta U_{j}+\right| U_{j}\right|^{\frac{4}{d}} U_{j}, \Lambda U_{j}\right\rangle \\
= & -\operatorname{Re}\left\langle\sum_{l \neq j}\left(\Delta R_{l}+\left(1+\frac{2}{d}\right)\left|U_{l}\right|^{\frac{4}{d}} R_{l}+\frac{2}{d}\left|U_{l}\right|^{\frac{4}{d}-2} U_{l}^{2} \overline{R_{l}}\right)+H_{1}, \Lambda U_{j}\right\rangle \\
& -\operatorname{Re}\left\langle\sum_{l \neq j}\left(i \partial_{t} U_{l}+\Delta U_{l}+\left|U_{l}\right|^{\frac{4}{d}} U_{l}\right)+H_{2}, \Lambda U_{j}\right\rangle \\
& -\operatorname{Re}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, \Lambda U_{j}\right\rangle \\
& -\operatorname{Re}\left\langle\sum_{l=1}^{K}\left(b \cdot \nabla\left(U_{l}+R_{l}\right)+c\left(U_{l}+R_{l}\right)\right), \Lambda U_{j}\right\rangle,
\end{aligned}
$$

where $H_{1}$ an $H_{2}$ are given by (4.12) and (4.13), respectively.
As we shall see below that, the right-hand side of equation (4.18) merely contribute negligibly small errors.

Actually, we may take $t^{*}$ close to $T_{*}$ such that (3.4) and thus Lemma 3.1 hold. By Lemma 3.1, the interactions between different profiles are exponentially small, and thus we infer that for some $\delta>0$,

$$
\begin{equation*}
\left|\left\langle\sum_{l \neq j}\left(\Delta R_{l}+\left(1+\frac{2}{d}\right)\left|U_{l}\right|^{\frac{4}{d}} R_{l}+\frac{2}{d}\left|U_{l}\right|^{\frac{4}{d}-2} U_{l}^{2} \overline{R_{l}}\right)+H_{1}, \Lambda U_{j}\right\rangle\right| \leq C|\lambda|^{-2} e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}, \tag{4.19}
\end{equation*}
$$

Similarly, by (4.14),

$$
\begin{equation*}
\left|\left\langle\sum_{l \neq j}\left(i \partial_{t} U_{l}+\Delta U_{l}+\left|U_{l}\right|^{\frac{4}{d}} U_{l}\right)+H_{2}, \Lambda U_{j}\right\rangle\right| \leq C|\lambda|^{-2} e^{-\frac{\delta}{T-t}}(1+\text { Mod }) \tag{4.20}
\end{equation*}
$$

For the remainder $f^{\prime \prime}(U, R) \cdot R^{2}$ containing high order terms of $R$, using (3.23) with $U$ replacing $v$, (3.29) and $\|R\|_{H^{1}} \leq 1$, we get

$$
\begin{equation*}
\left|\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, \Lambda U_{j}\right\rangle\right| \leq C|\lambda|^{-2}\left(\|R\|_{L^{2}}^{2}+\|R\|_{H^{1}}^{3}\right) \tag{4.21}
\end{equation*}
$$

Regarding the last term involving $b$ and $c$ on the right-hand side of (4.18), using Lemma 3.1 and (4.3), (4.10) to rewrite it in the renormalized variables we have

$$
\begin{align*}
& \operatorname{Re}\left\langle\sum_{l=1}^{K}\left(b \cdot \nabla\left(U_{l}+R_{l}\right)+c\left(U_{l}+R_{l}\right)\right), \Lambda U_{j}\right\rangle \\
= & \operatorname{Re}\left\langle\lambda_{j}^{-1} \widetilde{b} \cdot \nabla\left(Q_{j}+\varepsilon_{j}\right)+\widetilde{c}\left(Q_{j}+\varepsilon_{j}\right), \Lambda Q_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\left(1+\|R\|_{L^{2}}\right)\right), \tag{4.22}
\end{align*}
$$

where $\widetilde{b}(y):=b\left(\lambda_{j} y+\alpha_{j}\right)$ and $\widetilde{c}(y):=c\left(\lambda_{j} y+\alpha_{j}\right), y \in \mathbb{R}^{d}$. Then, using (2.12), (2.13) and integrating by parts formula we get

$$
\begin{aligned}
& \operatorname{Re}\left\langle\lambda_{j}^{-1} \widetilde{b} \cdot \nabla\left(Q_{j}+\varepsilon_{j}\right)+\widetilde{c}\left(Q_{j}+\varepsilon_{j}\right), \Lambda Q_{j}\right\rangle \\
= & 2 \operatorname{Im} \sum_{k=1}^{N} B_{k} \int \Delta \widetilde{\phi_{k}}\left(Q_{j}+\varepsilon_{j}\right) \overline{\Lambda Q_{j}} d y+2 \lambda_{j}^{-1} \operatorname{Im} \sum_{k=1}^{N} B_{k} \int\left(Q_{j}+\varepsilon_{j}\right) \nabla \widetilde{\phi_{k}} \cdot \nabla\left(\overline{\Lambda Q_{j}}\right) d y \\
& -\sum_{l=1}^{d} \operatorname{Re} \int\left(\sum_{k=1}^{N} \partial_{l} \widetilde{\phi_{k}} B_{k}\right)^{2}\left(Q_{j}+\varepsilon_{j}\right) \overline{\Lambda Q_{j}} d y-\operatorname{Im} \sum_{k=1}^{N} \int \widetilde{\phi_{k}} B_{k}\left(Q_{j}+\varepsilon_{j}\right) \overline{\Lambda Q_{j}} d y,
\end{aligned}
$$

where $\partial^{\nu} \widetilde{\phi_{k}}(y):=\left(\partial^{\nu} \phi_{k}\right)\left(\lambda_{j} y+\alpha_{j}\right),|\nu| \leq 2$. Note that, by the flatness condition (2.3) and the fact that $\partial_{y}^{\nu} \phi_{k} \in L^{\infty}$ for any multi-index $\nu$,

$$
\begin{equation*}
\left|\partial_{y}^{\nu} \widetilde{\phi_{k}}(y)\right| \leq C\left(\lambda_{j} y+\alpha_{j}-x_{j}\right)^{\nu_{*}+1-|\nu|} \leq C P^{\nu_{*}+1-|\nu|}\langle y\rangle^{\nu_{*}+1}, \quad 0 \leq|\nu| \leq \nu_{*} . \tag{4.23}
\end{equation*}
$$

This yields that

$$
\begin{equation*}
\left|\operatorname{Re}\left\langle\lambda_{j}^{-1} \tilde{b} \cdot \nabla\left(Q_{j}+\varepsilon_{j}\right)+\tilde{c}\left(Q_{j}+\varepsilon_{j}\right), \Lambda Q_{j}\right\rangle\right| \leq C|\lambda|^{-2} P^{\nu_{*}+1}\left(1+\|R\|_{L^{2}}\right) \tag{4.24}
\end{equation*}
$$

Thus, we conclude from estimates (4.19)-(4.24) that
R.H.S. of (4.18) $\leq C|\lambda|^{-2}\left(\left(e^{-\frac{\delta}{T-t}}+P^{\nu_{*}+1}\right)\left(1+\|R\|_{L^{2}}\right)+e^{-\frac{\delta}{T-t}} \operatorname{Mod}+\|R\|_{L^{2}}^{2}+\|R\|_{H^{1}}^{3}\right)$.

Regarding the left-hand side, by the orthogonality condition (4.5), (4.14) and Lemma 3.1.

$$
\begin{align*}
\operatorname{Im}\left\langle\partial_{t} R, \Lambda U_{j}\right\rangle & =\operatorname{Im}\left\langle\Lambda R, \partial_{t} U_{j}\right\rangle=\operatorname{Im}\left\langle\Lambda R_{j}, \partial_{t} U_{j}\right\rangle+\sum_{l \neq j} \operatorname{Im}\left\langle\Lambda R_{l}, \partial_{t} U_{j}\right\rangle \\
& =\operatorname{Im}\left\langle\Lambda R_{j}, \partial_{t} U_{j}\right\rangle+|\lambda|^{-2} \mathcal{O}\left(\operatorname{Mod}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}} \tag{4.26}
\end{align*}
$$

Then, using the identity (4.14) and the renormalized variables $Q_{j}, \varepsilon_{j}$ in (4.3) and (4.10), respectively, we get

$$
\begin{align*}
\lambda_{j}^{2} \operatorname{Im}\left\langle\Lambda R_{j}, \partial_{t} U_{j}\right\rangle= & \left.-\left.\operatorname{Re}\left\langle\Lambda \varepsilon_{j}, \Delta Q_{j}+\right| Q_{j}\right|^{\frac{4}{d}} Q_{j}\right\rangle+\mathcal{O}\left(\operatorname{Mod}\|R\|_{L^{2}}\right) \\
= & \operatorname{Re}\left\langle\varepsilon_{j}, \Lambda Q_{j}\right\rangle+\gamma_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \Lambda Q_{j}\right\rangle-2 \beta_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \nabla Q_{j}\right\rangle \\
& +\mathcal{O}\left(\left(\operatorname{Mod}+P^{2}\right)\|R\|_{L^{2}}\right), \tag{4.27}
\end{align*}
$$

where in the last step we used the almost orthogonality (4.15) and the identity

$$
\begin{equation*}
\Delta Q_{j}-Q_{j}+\left|Q_{j}\right|^{\frac{4}{d}} Q_{j}=\left|\beta_{j}-\frac{\gamma_{j}}{2}\right|^{2} Q_{j}-i \gamma_{j} \Lambda Q_{j}+2 i \beta_{j} \cdot \nabla Q_{j} . \tag{4.28}
\end{equation*}
$$

Furthermore, using (4.14), the identities

$$
\begin{align*}
& \Lambda Q_{j}=\left(\Lambda Q+i\left(\beta_{j} \cdot y-\frac{1}{2} \gamma_{j}|y|^{2}\right) Q\right) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}  \tag{4.29}\\
& \nabla Q_{j}=\left(\nabla Q+i\left(\beta_{j}-\frac{1}{2} \gamma_{j} y\right) Q\right) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)} \tag{4.30}
\end{align*}
$$

and $\left.\left.\langle\Lambda Q| y\right|^{2} Q,\right\rangle=-\|y Q\|_{L^{2}}^{2}$ we have

$$
\begin{equation*}
\left.\left.\lambda_{j}^{2} \operatorname{Re}\left\langle i \partial_{t} U_{j}+\Delta U_{j}+\right| U_{j}\right|^{\frac{4}{d}} U_{j}, \Lambda U_{j}\right\rangle=-\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right)+\mathcal{O}\left(M o d\left|\beta_{j}\right|\right) \tag{4.31}
\end{equation*}
$$

Thus, plugging (4.26), (4.27) and (4.31) into (4.18) and rearranging the terms according to the orders of the renormalized viarable $\varepsilon_{j}$ we get

$$
\begin{align*}
& \lambda_{j}^{2} \times(\text { L.H.S. of (4.18) })  \tag{4.32}\\
= & \left.-\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right)+\left.\operatorname{Re}\left\langle\Delta \varepsilon_{j}-\varepsilon_{j}+\left(1+\frac{2}{d}\right)\right| Q_{j}\right|^{\frac{4}{d} \varepsilon_{j}}+\frac{2}{d}\left|Q_{j}\right|^{\frac{4}{d}-2} Q_{j}^{2} \overline{\varepsilon_{j}}, \Lambda Q_{j}\right\rangle \\
& -\gamma_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \Lambda Q_{j}\right\rangle+2 \beta_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \nabla Q_{j}\right\rangle+\mathcal{O}\left(\left(\operatorname{Mod}+P^{2}\right)\|R\|_{L^{2}}+M o d\left|\beta_{j}\right|\right) .
\end{align*}
$$

By (3.9), straightforward computations show that, if $\varepsilon_{j}=\varepsilon_{j, 1}+i \varepsilon_{j, 2}$,

$$
\begin{align*}
& \left.\left.\operatorname{Re}\left\langle\Delta \varepsilon_{j}-\varepsilon_{j}+\left(1+\frac{2}{d}\right)\right| Q_{j}\right|^{\frac{4}{d}} \varepsilon_{j}+\frac{2}{d}\left|Q_{j}\right|^{\frac{4}{d}-2} Q_{j}^{2} \overline{j_{j}}, \Lambda Q_{j}\right\rangle \\
= & -\left\langle L_{+} \varepsilon_{j, 1}, \Lambda Q\right\rangle-\left\langle L_{+} \varepsilon_{j, 2},\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{4}|y|^{2}\right) \Lambda Q\right\rangle \\
& -\left\langle L_{-} \varepsilon_{j, 2},\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{2}|y|^{2}\right) Q\right\rangle+\mathcal{O}\left(P^{2}\|R\|_{L^{2}}\right) . \tag{4.33}
\end{align*}
$$

Note that

$$
\begin{aligned}
& L_{+}\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{4}|y|^{2}\right) \Lambda Q=\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{4}|y|^{2}\right) L_{+} \Lambda Q+\gamma_{j} \Lambda^{2} Q-2 \beta_{j} \cdot \nabla \Lambda Q \\
& L_{-}\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{2}|y|^{2}\right) Q=\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{2}|y|^{2}\right) L_{-} Q+2 \gamma_{j} \Lambda Q-2 \beta_{j} \cdot \nabla Q
\end{aligned}
$$

Taking into account the self-adjointness of $L_{ \pm}$and $L_{+} \Lambda Q=-2 Q, L_{-} Q=0$ we get

$$
\begin{align*}
& \left.\left.\operatorname{Re}\left\langle\Delta \varepsilon_{j}-\varepsilon_{j}+\left(1+\frac{2}{d}\right)\right| Q_{j}\right|^{\frac{4}{d}} \varepsilon_{j}+\frac{2}{d}\left|Q_{j}\right|^{\frac{4}{d}-2} Q_{j}^{2} \overline{\varepsilon_{j}}, \Lambda Q_{j}\right\rangle \\
= & 2\left\langle\varepsilon_{j, 1}, Q\right\rangle+2\left\langle\varepsilon_{j, 2},\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{4}|y|^{2}\right) Q\right\rangle \\
& +\gamma_{j}\left\langle\Lambda \varepsilon_{j, 2}, \Lambda Q\right\rangle-2 \beta_{j}\left\langle\nabla \varepsilon_{j, 2}, \Lambda Q\right\rangle-2 \gamma_{j}\left\langle\varepsilon_{j, 2}, \Lambda Q\right\rangle+2 \beta_{j}\left\langle\varepsilon_{j, 2}, \nabla Q\right\rangle . \tag{4.34}
\end{align*}
$$

Moreover, we see that

$$
\begin{equation*}
-\gamma_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \Lambda Q_{j}\right\rangle=-\gamma_{j}\left\langle\Lambda \varepsilon_{j, 2}, \Lambda Q\right\rangle+\mathcal{O}\left(P^{2}\|R\|_{L^{2}}\right) \tag{4.35}
\end{equation*}
$$

and by the almost orthogonality (4.15),

$$
\operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \nabla Q_{j}\right\rangle=\operatorname{Im}\left\langle\nabla \varepsilon_{j}, \Lambda Q_{j}\right\rangle+\operatorname{Im}\left\langle\varepsilon_{j}, \nabla Q_{j}\right\rangle=\operatorname{Im}\left\langle\nabla \varepsilon_{j}, \Lambda Q_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right)
$$

which yields that

$$
\begin{align*}
2 \beta_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \nabla Q_{j}\right\rangle & =2 \beta_{j} \operatorname{Im}\left\langle\nabla \varepsilon_{j}, \Lambda Q_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right) \\
& =2 \beta_{j}\left\langle\nabla \varepsilon_{j, 2}, \Lambda Q\right\rangle+\mathcal{O}\left(\left(P^{2}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}}\right) \tag{4.36}
\end{align*}
$$

Thus, we conclude from (4.34), (4.35), (4.36) and the almost orthogonality (4.15) that

$$
\begin{align*}
& \left.\left.\operatorname{Re}\left\langle\Delta \varepsilon_{j}-\varepsilon_{j}+\left(1+\frac{2}{d}\right)\right| Q_{j}\right|^{\frac{4}{d}} \varepsilon_{j}+\frac{2}{d}\left|Q_{j}\right|^{\frac{4}{d}-2} Q_{j}^{2} \overline{\varepsilon_{j}}, \Lambda Q_{j}\right\rangle \\
& -\gamma_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \Lambda Q_{j}\right\rangle+2 \beta_{j} \operatorname{Im}\left\langle\Lambda \varepsilon_{j}, \nabla Q_{j}\right\rangle \\
= & 2\left\langle\varepsilon_{j, 1}, Q\right\rangle+2\left\langle\varepsilon_{j, 2},\left(\beta_{j} \cdot y-\frac{\gamma_{j}}{4}|y|^{2}\right) Q\right\rangle-2 \gamma_{j}\left\langle\varepsilon_{j, 2}, \Lambda Q\right\rangle+2 \beta_{j}\left\langle\varepsilon_{j, 2}, \nabla Q\right\rangle \\
& +\mathcal{O}\left(\left(P^{2}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}}\right) \\
= & 2 \operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle+\mathcal{O}\left(\left(P^{2}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}}\right) . \tag{4.37}
\end{align*}
$$

This along with (4.32) yields that

$$
\begin{align*}
\lambda_{j}^{2} \times(\text { L.H.S. of (4.18) })= & -\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right)+2 \operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle \\
& +\mathcal{O}\left(\left(\operatorname{Mod}+P^{2}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}}+\operatorname{Mod}\left|\beta_{j}\right|\right) \tag{4.38}
\end{align*}
$$

Then, combining (4.25) and (4.38) we obtain that for each $1 \leq j \leq K$,
$\left|\lambda_{j}^{2} \dot{\gamma}_{j}+\gamma_{j}^{2}\right| \leq C\left(\operatorname{Mod}\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right)+\left|\operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle\right|+\left(e^{-\frac{\delta}{T-t}}+P^{\nu_{*}+1}\right)\left(1+\|R\|_{L^{2}}\right)\right.$

$$
\begin{equation*}
\left.+P^{2}\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}+\|R\|_{H^{1}}^{3}\right) \tag{4.39}
\end{equation*}
$$

Similar arguments apply also to the remaining four modulation equations. Actually, taking the inner products of equation (4.11) with $i\left(x-\alpha_{j}\right) U_{j}, i\left|x-\alpha_{j}\right|^{2} U_{j}, \nabla U_{j}, \varrho_{j}$, respectively, then taking the real parts and using analogous arguments as above, we can obtain the same bounds for $\left|\lambda_{j} \dot{\alpha}_{j}-2 \beta_{j}\right|,\left|\lambda_{j} \dot{\lambda}_{j}+\gamma_{j}\right|,\left|\lambda_{j}^{2} \dot{\beta}_{j}+\beta_{j} \gamma_{j}\right|$ and $\left|\lambda_{j}^{2} \dot{\theta}_{j}-1-\left|\beta_{j}\right|^{2}\right|$, respectively. We then get
$\operatorname{Mod}_{j}(t) \leq C\left(\operatorname{Mod}\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right)+\sum_{j=1}^{K}\left|\operatorname{Re}\left\langle R_{j}, U_{j}\right\rangle\right|+\left(e^{-\frac{\delta}{T-t}}+P^{\nu_{*}+1}\right)\left(1+\|R\|_{L^{2}}\right)\right.$

$$
\begin{equation*}
\left.+P^{2}\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}+\|R\|_{H^{1}}^{3}\right) . \tag{4.40}
\end{equation*}
$$

Therefore, taking $T$ possibly even smaller such that

$$
(1+C)\left(P+\sup _{t^{*} \leq t \leq T_{*}}\|R(t)\|_{H^{1}}+e^{-\frac{\delta}{T}}\right) \leq \frac{1}{2}
$$

and then summing over $j$ we obtain (4.8) and finish the proof.

## 5. Bootstrap estimates

This section is mainly devoted to the bootstrap type estimates of the remainder $R$ and the modulation parameters $\mathcal{P}$, which are the key ingredients in the construction of multibubble blow-up solutions in Section 6 later. The main result is formulated in Theorem 5.1 below.

Theorem 5.1 (Bootstrap). Let $\varepsilon^{*}>0$ be sufficiently small, $0<\zeta<\frac{1}{12}$. For any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, let $T=T(M)$ be small enough, satisfying (3.5), and fix $T_{*} \in(0, T)$. Suppose that there exists $t^{*} \in\left(0, T_{*}\right)$ such that $u$ admits the unique geometrical decomposition (4.1) on $\left[t^{*}, T_{*}\right]$ and the following estimates hold for $\kappa:=\nu_{*}-3(\geq 2)$ :
(i) For the reminder term,

$$
\begin{equation*}
\|\nabla R(t)\|_{L^{2}} \leq(T-t)^{\kappa}, \quad\|R(t)\|_{L^{2}} \leq(T-t)^{\kappa+1} \tag{5.1}
\end{equation*}
$$

(ii) For the modulation parameters, $1 \leq j \leq K$,

$$
\begin{align*}
& \left|\lambda_{j}(t)-\omega_{j}(T-t)\right|+\left|\gamma_{j}(t)-\omega_{j}^{2}(T-t)\right| \leq(T-t)^{\kappa+1+\zeta},  \tag{5.2}\\
& \left|\alpha_{j}(t)-x_{j}\right|+\left|\beta_{j}(t)\right| \leq(T-t)^{\frac{\kappa}{2}+1+\zeta}  \tag{5.3}\\
& \left|\theta_{j}(t)-\left(\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| \leq(T-t)^{\kappa-1+\zeta} \tag{5.4}
\end{align*}
$$

Then, there exists $t_{*} \in\left[0, t^{*}\right)$ such that (4.1) holds on the larger interval $\left[t_{*}, T_{*}\right]$ and the coefficients in estimates (5.1)-(5.4) can be refined to $1 / 2$, i.e., for any $t \in\left[t_{*}, T_{*}\right]$, $1 \leq j \leq K$,

$$
\begin{align*}
& \|\nabla R(t)\|_{L^{2}} \leq \frac{1}{2}(T-t)^{\kappa}, \quad\|R(t)\|_{L^{2}} \leq \frac{1}{2}(T-t)^{\kappa+1}  \tag{5.5}\\
& \left|\lambda_{j}(t)-\omega_{j}(T-t)\right|+\left|\gamma_{j}(t)-\omega_{j}^{2}(T-t)\right| \leq \frac{1}{2}(T-t)^{\kappa+1+\zeta}  \tag{5.6}\\
& \left|\alpha_{j}(t)-x_{j}\right|+\left|\beta_{j}(t)\right| \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+1+\zeta}  \tag{5.7}\\
& \left|\theta_{j}(t)-\left(\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| \leq \frac{1}{2}(T-t)^{\kappa-1+\zeta} \tag{5.8}
\end{align*}
$$

In order to prove Theorem [5.1, we may take $t_{*} \in\left[0, t^{*}\right)$, sufficiently close to $t^{*}$, such that $u$ still has the geometrical decomposition (4.1) on the larger interval $\left[t_{*}, T_{*}\right]$ (this is possible because the Jacobian of transformation is continuous in time). Moreover, by
virtue of the local well-posedness theory and the $C^{1}$-regularity of modulation parameters, taking $t_{*}$ possibly closer to $t^{*}$, we have that for any $t \in\left[t_{*}, t_{n}\right]$,

$$
\begin{align*}
& \|\nabla R(t)\|_{L^{2}} \leq 2(T-t)^{\kappa}, \quad\|R(t)\|_{L^{2}} \leq 2(T-t)^{\kappa+1}  \tag{5.9}\\
& \left|\lambda_{j}(t)-\omega_{j}(T-t)\right|+\left|\gamma_{j}(t)-\omega_{j}^{2}(T-t)\right| \leq 2(T-t)^{\kappa+1+\zeta},  \tag{5.10}\\
& \left|\alpha_{j}(t)-x_{j}\right|+\left|\beta_{j}(t)\right| \leq 2(T-t)^{\frac{\kappa}{2}+1+\zeta},  \tag{5.11}\\
& \left|\theta_{j}(t)-\left(\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| \leq 2(T-t)^{\kappa-1+\zeta} . \tag{5.12}
\end{align*}
$$

By (5.10) and (5.12), we may also take $T$ sufficiently small such that (3.4) holds, and thus Lemma 3.1 is applicable below.

Remark 5.2. We infer from (5.10) that for $T$ small enough, $\lambda_{j}, \gamma_{j}, P$ are comparable with $T-t$, i.e.,

$$
\begin{equation*}
C_{1}(T-t) \leq \lambda_{j}, \gamma_{j}, P \leq C_{2}(T-t) \tag{5.13}
\end{equation*}
$$

where $C_{1}, C_{2}$ are positive constants independent of $\varepsilon$.
By virtue of Proposition 4.3 and Proposition 5.6 below, we have the refined estimate for the modulation parameters below.

Lemma 5.3. Assume estimates (5.1)-(5.4) to hold with $T$ sufficiently small. Then, there exists $C>0$ such that

$$
\begin{equation*}
\operatorname{Mod}(t) \leq C(T-t)^{\kappa+3}, \quad \forall t \in\left[t_{*}, T_{*}\right] \tag{5.14}
\end{equation*}
$$

Moreover, by equation (2.11), the remainder $R$ satisfies the equation

$$
\begin{equation*}
i \partial_{t} R+\Delta R+(f(u)-f(U))+b \cdot \nabla R+c R=-\eta \tag{5.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta=i \partial_{t} U+\Delta U+f(U)+b \cdot \nabla U+c U . \tag{5.16}
\end{equation*}
$$

The estimates of $U_{j}, R$ and $\eta$ are contained in Lemmas 5.4 and 5.5 below.
Lemma 5.4. Assume estimates (5.1)-(5.4) to hold and let $\varepsilon_{j}$ be defined in (4.10). Then, there exists $C>0$ such that for all $t \in\left[t_{*}, T_{*}\right], 1 \leq j \leq K$,

$$
\begin{aligned}
& \left\|\varepsilon_{j}(t)\right\|_{L^{2}}=\left\|R_{j}(t)\right\|_{L^{2}} \leq C(T-t)^{\kappa+1}, \quad \lambda_{j}^{-1}\left\|\nabla \varepsilon_{j}(t)\right\|_{L^{2}}=\left\|\nabla R_{j}(t)\right\|_{L^{2}} \leq C(T-t)^{\kappa}, \\
& \left\|U_{j}(t)\right\|_{L^{2}}=\|Q\|_{L^{2}}, \quad\left\|\nabla U_{j}(t)\right\|_{L^{2}}+\left\|\frac{x_{j}-\alpha_{j}(t)}{\lambda_{j}(t)} \cdot \nabla U_{j}(t)\right\|_{L^{2}} \leq C(T-t)^{-1}
\end{aligned}
$$

Lemma 5.5. Assume estimates (5.1)-(5.4) to hold with $T$ sufficiently small. Then, there exists a constant $C>0$ such that for $t \in\left[t_{*}, T_{*}\right]$ and multi-index $\nu$ with $|\nu| \leq 2$,

$$
\begin{equation*}
\left\|\partial^{\nu} \eta(t)\right\|_{L^{2}} \leq C(T-t)^{\kappa+1-|\nu|} . \tag{5.17}
\end{equation*}
$$

The proof is postponed to the Appendix for simplicity.
The remainder of Section 5 is devoted to the proof of Theorem 5.1. We first derive the estimates of the localized mass and energy in Subsection 5.1, and then in Subsection 5.2 we derive the key monotonicity property of the generalized energy, involving a Morawetz type term and localized funcitons, which actually constitutes the most technical part of this section. The detailed proof of Theorem 5.1 is then given in Subsection 5.3, We shall assume estimates (5.1)-(5.4) to hold on $\left[t_{*}, T_{*}\right] \subseteq[0, T)$ with $T$ small enough and satisfying (3.5) throughout Subsections 5.15.5.3.

### 5.1. Estimates of localized mass and energy.

Proposition 5.6 (Estimate of localized mass). There exists $C>0$ such that for any $t \in\left[t_{*}, T_{*}\right]$ and $1 \leq j \leq K$,

$$
\begin{equation*}
2 \operatorname{Re} \int \overline{U_{j}} R_{j} d x+\int|R(t)|^{2} \Phi_{j} d x=\mathcal{O}\left((T-t)^{2 \kappa+2}\right) \tag{5.18}
\end{equation*}
$$

where $R_{j}:=R \Phi_{j}$ with $\Phi_{j}$ the local functions defined in (3.2).
Remark 5.7. (i). The estimate (5.18) allows us to control the scalar product along the direction $Q$ when applying the localized coercivity in Corollary 3.4.
(ii). It should be mentioned that, the proof of (5.18) relies on the analysis of the localized mass $\int|u|^{2} \Phi_{j} d x$, instead of the usual whole mass $\|u\|_{L^{2}}^{2}$. This is quite different from the single bubble case in [55, 58].

Proof of Proposition 5.6. Using (4.1) and Lemma 3.1 we have that for some $\delta>0$,

$$
\begin{aligned}
\int|u|^{2} \Phi_{j} d x & =\int|U|^{2} \Phi_{j} d x+\int|R|^{2} \Phi_{j} d x+2 \operatorname{Re} \int \bar{U} R \Phi_{j} d x \\
& =\int|U|^{2} \Phi_{j} d x+\int|R|^{2} \Phi_{j} d x+2 \operatorname{Re} \int \overline{U_{j}} R_{j} d x+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right)
\end{aligned}
$$

which yields that
$\left.\left|2 \operatorname{Re} \int\left(\overline{U_{j}} R_{j}\right)(t) d x+\int\right| R(t)\right|^{2} \Phi_{j} d x\left|\leq\left|\int\right| u(t)\right|^{2} \Phi_{j} d x-\int\left|u\left(T_{*}\right)\right|^{2} \Phi_{j} d x \mid$

$$
\begin{equation*}
+\left.\left|\int\right| u\left(T_{*}\right)\right|^{2} \Phi_{j} d x-\int|U(t)|^{2} \Phi_{j} d x \left\lvert\,+C e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right. \tag{5.19}
\end{equation*}
$$

For the first term on the right-hand side of (5.19), we use equation (2.11) to get

$$
\begin{align*}
\frac{d}{d t} \int|u|^{2} \Phi_{j} d x & =\operatorname{Im} \int\left(2 \bar{u} \nabla u+b|u|^{2}\right) \cdot \nabla \Phi_{j} d x \\
& \leq \int_{\left|x-x_{l}\right| \geq 4 \sigma, 1 \leq l \leq K} 2|\bar{u} \nabla u|+|b||u|^{2} d x \tag{5.20}
\end{align*}
$$

where $\sigma$ is given by (3.1). By (5.11), we may take $t_{*}$ close to $T_{*}$ such that $\left|x_{l}(t)-\alpha_{l}(t)\right| \leq \sigma$ for any $t \in\left[t_{*}, T_{*}\right], 1 \leq l \leq K$. This along with (3.8) and (4.1) yields that

$$
\begin{aligned}
& \left.\left.\left|\frac{d}{d t} \int\right| u\right|^{2} \Phi_{j} d x \right\rvert\, \leq \leq C \int_{\left|x-\alpha_{l}\right| \geq 3 \sigma, 1 \leq l \leq K}|U+R||\nabla U+\nabla R|+|U+R|^{2} d x \\
& \leq C\left(\|R\|_{L^{2}}^{2}+\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}+\sum_{l=1}^{K}\left(\int_{|y| \geq \frac{3 \sigma}{\lambda_{l}}}|Q|^{2} d y\right)^{\frac{1}{2}}\|\nabla R\|_{L^{2}}\right. \\
&\left.\quad+\sum_{l=1}^{K} \int_{|y| \geq \frac{3 \sigma}{\lambda_{l}}}|Q|^{2}+\lambda_{l}^{-2}\left|\nabla Q_{l}\right|^{2} d y\right) \\
& \leq C\left(\|R\|_{L^{2}}^{2}+\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right)
\end{aligned}
$$

where $\delta>0$. Hence, we obtain that for some $\delta>0$,

$$
\begin{align*}
& \left.\left|\int\right| u(t)\right|^{2} \Phi_{j} d x-\int\left|u\left(T_{*}\right)\right|^{2} \Phi_{j} d x \mid \\
\leq & C \int_{t}^{T_{*}}\|R(s)\|_{L^{2}}^{2}+\|R(s)\|_{L^{2}}\|\nabla R(s)\|_{L^{2}} d s+C e^{-\frac{\delta}{T-t}} . \tag{5.21}
\end{align*}
$$

Regarding the second term on the right-hand side of (5.19), we apply Lemma 3.1 to extract the main blow-up profile $U_{j}$

$$
\int|U(t)|^{2} \Phi_{j} d x=\int\left|U_{j}(t)\right|^{2} \Phi_{j} d x+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right)
$$

Since

$$
\int\left|U_{j}(t)\right|^{2} \Phi_{j} d x=\int|Q|^{2} d y+\int|Q(y)|^{2}\left(\Phi_{j}\left(\lambda_{j}(t) y+\alpha_{j}(t)\right)-1\right) d y
$$

and for some $\delta>0$,

$$
\int|Q(y)|^{2}\left(1-\Phi_{j}\left(\lambda_{j}(t) y+\alpha_{j}(t)\right)\right) d y \leq \int_{|y| \geq \frac{3 \sigma}{\lambda_{j}(t)}} Q^{2}(y) d y \leq C e^{-\frac{\delta}{T-t}}
$$

we infer that

$$
\begin{equation*}
\int|U(t)|^{2} \Phi_{j} d x=\|Q\|_{L^{2}}^{2}+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) \tag{5.22}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\int\left|u\left(T_{*}\right)\right|^{2} \Phi_{j} d x=\int\left|\sum_{l=1}^{K} S_{l}\left(T_{*}\right)\right|^{2} \Phi_{j} d x=\|Q\|_{2}^{2}+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) \tag{5.23}
\end{equation*}
$$

We infer from (5.22) and (5.23) that

$$
\begin{equation*}
\left.\left|\int\right| u\left(T_{*}\right)\right|^{2} \Phi_{j} d x-\int|U(t)|^{2} \Phi_{j} d x \left\lvert\, \leq C e^{-\frac{\delta}{T-t}} .\right. \tag{5.24}
\end{equation*}
$$

Therefore, plugging (5.21) and (5.24) into (5.19) we obtain

$$
\left.\left|2 \operatorname{Re} \int\left(\overline{U_{j}} R_{j}\right)(t) d x+\int\right| R(t)\right|^{2} \Phi_{j} d x \left\lvert\, \leq C\left(\int_{t}^{T_{*}}\|R\|_{L^{2}}^{2}+\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} d s+e^{-\frac{\delta}{T-t}}\right)\right.
$$

which along with (5.9) yields (5.18) for $T$ small enough and finishes the proof.
Theorem 5.8 below contains the estimate of the variation of energy. Unlike the deterministic case, the energy (1.6) is no longer conserved and the corresponding variation plays an important role in the derivation of the refined estimate of the modulation parameter $\beta$ later (see Lemma 5.14 below).

Proposition 5.8 (Variation of the energy). There exists $C>0$ such that for any $t \in$ $\left[t_{*}, T_{*}\right]$,

$$
\begin{equation*}
\left|E(u(t))-E\left(u\left(T_{*}\right)\right)\right| \leq C(T-t)^{\kappa+1} \tag{5.25}
\end{equation*}
$$

Proof. The proof is quite similar to that of [58, Theorem 5.6], based on the GagliardoNirenberg inequality (3.28) and the estimate (4.23) of the spatial functions of noises under Assumption (A1). Actually, as in [58, (5.20)], we have

$$
\begin{align*}
\frac{d}{d t} E\left(u_{n}\right)= & -2 \sum_{k=1}^{N} B_{k} \operatorname{Re} \int \nabla^{2} \phi_{k}\left(\nabla u_{n}, \nabla \overline{u_{n}}\right) d x+\frac{1}{2} \sum_{k=1}^{N} B_{k} \int \Delta^{2} \phi_{k}\left|u_{n}\right|^{2} d x \\
& +\frac{2}{d+2} \sum_{k=1}^{N} B_{k} \int \Delta \phi_{k}\left|u_{n}\right|^{2+\frac{4}{d}} d x-\sum_{j=1}^{d} \operatorname{Im} \int \nabla\left(\sum_{k=1}^{N} \partial_{j} \phi_{k} B_{k}\right)^{2} \cdot \nabla u_{n} \overline{u_{n}} d x . \tag{5.26}
\end{align*}
$$

This yields that

$$
\begin{aligned}
\left|\frac{d}{d t} E\left(u_{n}\right)\right| \leq C\|R\|_{H^{1}}^{2}+C \sum_{k=1}^{K} \sum_{j=1}^{d} & \left(\int\left(\left|\nabla^{2} \phi_{k}\right|+\left|\Delta \phi_{k}\right|+\left|\partial_{j} \phi_{k} \nabla \partial_{j} \phi_{k}\right|\right)\left(|\nabla U|^{2}+|U|^{2+\frac{4}{d}}\right) d x\right. \\
+ & \left.\int\left(\left|\Delta^{2} \phi_{k}\right|+\left|\partial_{j} \phi_{k} \nabla \partial_{j} \phi_{k}\right|\right)|U|^{2} d x\right)
\end{aligned}
$$

which, via the change of variables, can be further bounded by, up to some constant,
$\|R\|_{H^{1}}^{2}+\sum_{k, l=1}^{K} \int(T-t)^{-2} \sum_{|\nu| \leq 2}\left|\partial^{\nu} \phi_{k}\right|\left(\lambda_{l} y+\alpha_{l}\right)\left(\left|\nabla Q_{l}\right|^{2}+\left|Q_{l}\right|^{2+\frac{4}{d}}\right)+\sum_{|\nu| \leq 4}\left|\partial^{\nu} \phi_{k}\right|\left(\lambda_{l} y+\alpha_{l}\right)|Q|^{2} d y$.
Thus, using (4.23) and (5.9) we obtain

$$
\left|\frac{d}{d t} E\left(u_{n}\right)\right| \leq C(T-t)^{\kappa},
$$

which immediately yields (5.25), thereby finishing the proof.
5.2. Monotonicity of generalized energy. This subsection is mainly devoted to the monotonicity property of a new generalized energy, which is the key ingredient in the proof of the bootstrap estimate (5.5) of the remainder.

It should be mentioned that, unlike the single blow-up point case in [55, 58], the new generalized energy (5.28) below includes also the localized functions in an appropriate way, such that the different profiles can be decoupled completely and the key monotonicity property is still preserved.

More precisely, let $\chi(x)=\psi(|x|)$ be a smooth radial function on $\mathbb{R}^{d}$, where $\psi$ satisfies $\psi^{\prime}(r)=r$ if $r \leq 1, \psi^{\prime}(r)=2-e^{-r}$ if $r \geq 2$, and

$$
\begin{equation*}
\left|\frac{\psi^{\prime \prime \prime}(r)}{\psi^{\prime \prime}(r)}\right| \leq C, \quad \frac{\psi^{\prime}(r)}{r}-\psi^{\prime \prime}(r) \geq 0 . \tag{5.27}
\end{equation*}
$$

Let $\chi_{A}(x):=A^{2} \chi\left(\frac{x}{A}\right), A>0, f(u):=|u|^{\frac{4}{d}} u$, and $F(u):=\frac{d}{2 d+4}|u|^{2+\frac{4}{d}}$. We shall also use the notations $f^{\prime}(U, R) \cdot R$ and $f^{\prime \prime}(U, R) \cdot R^{2}$ as in (3.18) and (3.19), respectively.

We define the generalized energy by

$$
\begin{aligned}
I(t):= & \frac{1}{2} \int|\nabla R|^{2} d x+\frac{1}{2} \sum_{j=1}^{K} \int \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j} d x-\operatorname{Re} \int F(u)-F(U)-f(U) \bar{R} d x \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R \bar{R} \Phi_{j} d x .
\end{aligned}
$$

The key monotonicity property of the generalized energy is formulated in Theorem 5.9 below.

Theorem 5.9. There exist $C_{1}, C_{2}(A), C_{3}>0$ such that for any $t \in\left[t_{*}, T_{*}\right]$

$$
\begin{equation*}
\frac{d I}{d t} \geq C_{1} \sum_{j=1}^{K} \frac{1}{\lambda_{j}} \int\left(\left|\nabla R_{j}\right|^{2}+\frac{1}{\lambda_{j}^{2}}\left|R_{j}\right|^{2}\right) e^{-\frac{\left|x-\alpha_{j}\right|}{A \lambda_{j}}} d x-C_{2}(A)(T-t)^{2 \kappa}-C_{3} \varepsilon^{*}(T-t)^{2 \kappa-1} . \tag{5.29}
\end{equation*}
$$

Remark 5.10. Theorem 5.9 yields that the derivative of the generalized energy is almost positive, up to some error terms, and thus the generalized energy is almost monotone. We also mention that, the error term of order $(T-t)^{2 \kappa-1}$ corresponds to the frequencies
$\left\{\omega_{j}\right\}_{j=1}^{K}$, and the small coefficient $\varepsilon^{*}$ is important later in the derivation of the bootstrap estimate (5.5) of the remainder $R$, and also in the iteration arguments in the proof of uniqueness.

In order to prove Theorem 5.9, we separate $I$ into two parts $I=I^{(1)}+I^{(2)}$, where

$$
\begin{align*}
I^{(1)} & :=\frac{1}{2} \int|\nabla R|^{2} d x+\frac{1}{2} \sum_{j=1}^{K} \int \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j} d x-\operatorname{Re} \int F(u)-F(U)-f(U) \bar{R} d x,  \tag{5.30}\\
I^{(2)} & :=\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R \bar{R} \Phi_{j} d x . \tag{5.31}
\end{align*}
$$

Below we treat $I^{(1)}$ and $I^{(2)}$ separately in Lemmas 5.11 and 5.12, Let us first show the estimate of $I^{(1)}$.

Lemma 5.11. Consider the situations as in Theorem 5.9. Then, for every $t \in\left[t_{*}, T_{*}\right]$ we have that for some $C_{1}, C_{2}>0$,

$$
\begin{aligned}
& \frac{d I^{(1)}}{d t} \geq \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}}\left\|R_{j}\right\|_{L^{2}}^{2}-\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{2}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}}\left|R_{j}\right|^{2}+\frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2}{\overline{U_{j}}}^{2} R_{j}^{2} d x \\
& \quad-\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}} \operatorname{Re} \int\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla \overline{U_{j}}\left\{\frac{2}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}\left|R_{j}\right|^{2}\right. \\
& \left.\quad \quad \frac{1}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} \overline{U_{j}} R_{j}^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)\left|U_{j}\right|^{\frac{4}{d}-4} U_{j}^{3}{\overline{R_{j}}}^{2}\right\} d x \\
& \quad-C_{1}(T-t)^{2 \kappa}-C_{2} \varepsilon^{*}(T-t)^{2 \kappa-1} .
\end{aligned}
$$

Proof. Using the identities

$$
\partial_{t} F(u)=\operatorname{Re}\left(f(u) \partial_{t} \bar{u}\right), \quad \partial_{t} f(U)=\partial_{z} f(U) \partial_{t} U+\partial_{\bar{z}} f(U) \partial_{t} \bar{U},
$$

and the expansion (3.21) we have

$$
\begin{aligned}
\frac{d I^{(1)}}{d t}= & \operatorname{Im}\left\langle\Delta R+f(u)-f(U), i \partial_{t} R\right\rangle-\lambda_{j}^{-2} \operatorname{Im}\left\langle R_{j}, i \partial_{t} R\right\rangle \\
& -\dot{\lambda}_{j} \lambda_{j}^{-3} \int|R|^{2} \Phi_{j} d x-\operatorname{Re}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, \partial_{t} U\right\rangle
\end{aligned}
$$

Then, in view of (5.15), we obtain

$$
\begin{aligned}
\frac{d I^{(1)}}{d t}= & -\sum_{j=1}^{K} \dot{\lambda}_{j} \lambda_{j}^{-3} \operatorname{Im} \int|R|^{2} \Phi_{j} d x-\sum_{j=1}^{K} \lambda_{j}^{-2} \operatorname{Im}\left\langle f^{\prime}(U) \cdot R, R_{j}\right\rangle \\
& -\operatorname{Re}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, \partial_{t} U\right\rangle-\sum_{j=1}^{K} \lambda_{j}^{-2} \operatorname{Im}\left\langle R \nabla \Phi_{j}, \nabla R\right\rangle \\
& -\sum_{j=1}^{K} \lambda_{j}^{-2} \operatorname{Im}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, R_{j}\right\rangle-\operatorname{Im}\left\langle\Delta R-\sum_{j=1}^{K} \lambda_{j}^{-2} R_{j}+f(u)-f(U), \eta\right\rangle \\
3) \quad & -\operatorname{Im}\left\langle\Delta R-\sum_{j=1}^{K} \lambda_{j}^{-2} R_{j}+f(u)-f(U), b \cdot \nabla R+c R\right\rangle=: \sum_{j=1}^{7} I_{t, j}^{(1)},
\end{aligned}
$$

where $\eta$ is given by (5.16).

As we shall see below, the main orders of $\frac{d I^{(1)}}{d t}$ are contributed by the first three terms $I_{t, 1}^{(1)}, I_{t, 2}^{(1)}$ and $I_{t, 3}^{(1)}$, the fourth term will contributes the error of order $(T-t)^{2 \kappa-1}$ for which we shall treat Case (I) and Case (II) separately, while the remaining three terms are of the negligible order $(T-t)^{2 \kappa}$.
(i) Estimate of $I_{t, 1}^{(1)}$. Since by (5.14), $\left|\frac{\lambda_{j} \dot{\lambda}_{j}+\gamma_{j}}{\lambda_{j}^{4}}\right| \leq C \frac{M o d}{\lambda_{j}^{4}} \leq C(T-t)^{\kappa-1}$, it follows from (5.1) that

$$
\begin{align*}
I_{t, 1}^{(1)} & =\sum_{j=1}^{K}\left(\frac{\gamma_{j}}{\lambda_{j}^{4}} \int|R|^{2} \Phi_{j} d x-\frac{\lambda_{j} \dot{\lambda}_{j}+\gamma_{j}}{\lambda_{j}^{4}} \int|R|^{2} \Phi_{j} d x\right) \\
& =\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int|R|^{2} \Phi_{j} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) \\
& \geq \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}}\left\|R_{j}\right\|_{L^{2}}^{2}-C(T-t)^{2 \kappa}, \tag{5.34}
\end{align*}
$$

where we also used the inequality $\Phi_{j} \geq \Phi_{j}^{2}$ in the last step.
(ii) Estimates of $I_{t, 2}^{(1)}$ and $I_{t, 3}^{(1)}$. We apply Lemma 3.1 to decouple different blow-up profiles to obtain
$I_{t, 2}^{(1)}+I_{t, 3}^{(1)}=-\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im} \int \frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}^{2}{\overline{R_{j}}}^{2} d x-\sum_{j=1}^{K} \operatorname{Re} \int f^{\prime \prime}\left(U_{j}, R_{j}\right) \cdot R_{j}^{2} \partial_{t} \overline{U_{j}} d x+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right)$.
Then, for each $1 \leq j \leq K$, straightforward computations show that (see also the proof of [58, (5.46), (5.49)])

$$
\begin{align*}
I_{t, 2}^{(1)}+I_{t, 3}^{(1)}= & -\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{2}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}}\left|R_{j}\right|^{2}+\frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2}{\overline{U_{j}}}^{2} R_{j}^{2} \\
- & \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}} \operatorname{Re} \int\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla \overline{U_{j}}\left\{\frac{2}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}\left|R_{j}\right|^{2}\right. \\
& \left.+\frac{1}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} \overline{U_{j}} R_{j}^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)\left|U_{j}\right|^{\frac{4}{d}-4} U_{j}^{3}{\overline{R_{j}}}^{2}\right\} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.35}
\end{align*}
$$

(iii) Estimate of $I_{t, 4}^{(1)}$. We consider Case (I) and Case (II) separately. First, in Case (I), since $\sum_{j=1}^{K} \nabla \Phi_{j}(x)=0$, we see that

$$
\begin{aligned}
\left|I_{t, 4}^{(1)}\right| & =\sum_{j=1}^{K}\left(\frac{1}{\lambda_{j}^{2}}-\frac{1}{\omega^{2}(T-t)^{2}}\right) \operatorname{Im}\left\langle R \nabla \Phi_{j}, \nabla R\right\rangle \\
& \leq \sum_{j=1}^{K} \frac{\left|\lambda_{j}-\omega(T-t)\right|\left|\lambda_{j}+\omega(T-t)\right|}{\lambda_{j}^{2} \omega^{2}(T-t)^{2}}\left\|R \nabla \Phi_{j}\right\|_{L^{2}}\|\nabla R\|_{L^{2}} \\
& \leq \sum_{j=1}^{K} \frac{\left|\lambda_{j}-\omega(T-t) \| \lambda_{j}+\omega(T-t)\right|}{\lambda_{j}^{2} \omega^{2}(T-t)}\left(\frac{\|R\|_{L^{2}}^{2}}{(T-t)^{2}}+\|\nabla R\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

Since $\left|\omega-\omega_{j}\right| \leq \varepsilon^{*}$ for any $1 \leq j \leq K$, using (5.10) and (5.13) we see that

$$
\begin{aligned}
& \frac{\left|\lambda_{j}-\omega(T-t)\right|\left|\lambda_{j}+\omega(T-t)\right|}{\lambda_{j}^{2} \omega^{2}(T-t)} \\
\leq & \frac{\left(\left|\lambda_{j}-\omega_{j}(T-t)\right|+\left|\left(\omega_{j}-\omega\right)(T-t)\right|\right)\left|\lambda_{j}+\omega(T-t)\right|}{\lambda_{j}^{2} \omega^{2}(T-t)} \\
\leq & C\left((T-t)^{\kappa-1+\zeta}+\varepsilon^{*}(T-t)^{-1}\right) .
\end{aligned}
$$

This along with (5.9) yields that

$$
\begin{equation*}
\left|I_{t, 4}^{(1)}\right| \leq C\left((T-t)^{3 \kappa-1+\zeta}+\varepsilon^{*}(T-t)^{2 \kappa-1}\right) \leq C\left((T-t)^{2 \kappa}+\varepsilon^{*}(T-t)^{2 \kappa-1}\right) . \tag{5.36}
\end{equation*}
$$

In Case (II), we see that

$$
\begin{equation*}
\left|I_{t, 4}^{(1)}\right| \leq \sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}}\left\|\nabla \Phi_{j}\right\|_{L^{\infty}}\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} \tag{5.37}
\end{equation*}
$$

Since $\left|\nabla \Phi_{j}\right| \leq C \sigma^{-1} \leq C \varepsilon^{*}$ in Case (II), using (5.9) and (5.13) we have

$$
\begin{equation*}
\left|I_{t, 4}^{(1)}\right| \leq C(T-t)^{-1}\left(\frac{\|R\|_{L^{2}}^{2}}{(T-t)^{2}}+\|\nabla R\|_{L^{2}}^{2}\right) \leq C \varepsilon^{*}(T-t)^{2 \kappa-1} . \tag{5.38}
\end{equation*}
$$

(iv) Estimate of $I_{t, 5}^{(1)}$. Since

$$
\begin{equation*}
|U(t)| \leq C(T-t)^{-\frac{d}{2}}, \tag{5.39}
\end{equation*}
$$

using (3.23), (3.28), (3.29) and (5.9) we get

$$
\begin{align*}
\left|I_{t, 5}^{(1)}\right| & \leq C(T-t)^{-2}\left(\int|U|^{\frac{4}{d}-1}|R|^{3} d x+\|R\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}\right) \\
& \leq C(T-t)^{-2}\left((T-t)^{-2+\frac{d}{2}}\|R\|_{L^{2}}^{3-\frac{d}{2}}\|\nabla R\|_{L^{2}}^{\frac{d}{2}}+\|R\|_{H^{1}}^{\frac{4}{d}+2}\right) \leq C(T-t)^{2 \kappa} . \tag{5.40}
\end{align*}
$$

(v) Estimate of $I_{t, 6}^{(1)}$. Regarding $I_{t, 6}^{(1)}$, since by (3.20) and (3.22),

$$
|f(u)-f(U)|=\left|f^{\prime}(U, R) \cdot R\right| \leq C\left(|U|^{\frac{4}{d}}+|R|^{\frac{4}{d}}\right)|R| \leq C\left((T-t)^{-2}+|R|^{\frac{4}{d}}\right)|R|
$$

using the integration by parts formula, (3.29), (5.9) and the estimate (5.17) of $\eta$, we obtain

$$
\begin{align*}
\left|I_{t, 6}^{(1)}\right| & \leq C\left(\|\nabla \eta\|_{L^{2}}\|\nabla R\|_{L^{2}}+(T-t)^{-2}\|R\|_{L^{2}}\|\eta\|_{L^{2}}+\|R\|_{H^{1}}^{\frac{4}{d}+1}\|\eta\|_{L^{2}}\right) \\
& \leq C(T-t)^{2 \kappa} . \tag{5.41}
\end{align*}
$$

(vi) Estimate of $I_{t, 7}^{(1)}$. The last term $I_{t, 7}^{(1)}$ can be estimated similarly as in the proof of [58, Lemma 5.10]. Precisely, using the explicit expressions (2.12) of $b, \sup _{0 \leq t \leq T}\left|B_{k}\right|<\infty$, a.s., $1 \leq k \leq N$, and integration by parts formula we have

$$
\begin{align*}
& \quad\left|\operatorname{Im}\left\langle\Delta R-\lambda_{j}^{-2} R_{j}+f(u)-f(U), b \cdot \nabla R\right\rangle\right| \\
& \leq C \sum_{k=1}^{N}\left(\left|\int \nabla^{2} \phi_{k}(\nabla \bar{R}, \nabla R) d x\right|+\left.\left|\int \Delta \phi_{k}\right| \nabla R\right|^{2} d x \mid+(T-t)^{-2}\|R\|_{L^{2}}^{2}\right. \\
& \left.\quad+\int|R|^{\frac{4}{d}+1}\left|\nabla \phi_{k} \cdot \nabla \bar{R}\right| d x+\left|\int\left(f(u)-f(U)-|R|^{\frac{4}{d}} R\right) \nabla \phi_{k} \cdot \nabla \bar{R} d x\right|\right) \\
& \leq C(T-t)^{2 \kappa}+C \sum_{k=1}^{N}\left|\int\left(f(u)-f(U)-|R|^{\frac{4}{d}} R\right)\left(\nabla \phi_{k} \cdot \nabla \bar{R}\right) d x\right| \tag{5.42}
\end{align*}
$$

where in the last step we also used Hölder's inequality and the inequality

$$
\begin{equation*}
\|R\|_{L^{p}}^{p} \leq C\|R\|_{L^{2}}^{p+d-\frac{1}{2} d p}\|\nabla R\|_{L^{2}}^{\frac{1}{2} d p-d} \leq C(T-t)^{\kappa p+p+d-\frac{1}{2} d p}, \quad \forall p \geq 2 . \tag{5.43}
\end{equation*}
$$

Moreover, by Lemma 3.1, the last term in (5.42) is bounded by

$$
\begin{aligned}
& C \sum_{k=1}^{4 / d} \sum_{j=1}^{K} \int\left|R^{k} U_{j}^{1+\frac{4}{d}-k} \nabla \phi_{k} \cdot \nabla \overline{R_{j}}\right| d x+C e^{-\frac{\delta}{T-t}} \\
\leq & C \sum_{j=1}^{4 / d} \sum_{k=1}^{K} \int \lambda_{j}^{d}\left|\left(R^{k} U_{j}^{1+\frac{4}{d}-k} \nabla \phi_{k} \cdot \nabla \overline{R_{j}}\right)\left(\lambda_{j} y+\alpha_{j}\right)\right| d y+C e^{-\frac{\delta}{T-t}} .
\end{aligned}
$$

which, via (4.23) and (5.43), can be further bounded by

$$
C \sum_{j=1}^{4 / d} \sum_{k=1}^{K}(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-k\right)+\nu_{*}}\|R\|_{H^{1}}\|R\|_{L^{2 k}}^{k}+C e^{-\frac{\delta}{T-t}} \leq C(T-t)^{2 \kappa} .
$$

Hence, we obtain

$$
\begin{equation*}
\left|\operatorname{Im}\left\langle\Delta R-\lambda_{j}^{-2} R_{j}+f(u)-f(U), b \cdot \nabla R\right\rangle\right| \leq C(T-t)^{2 \kappa} \tag{5.44}
\end{equation*}
$$

Similarly, since $|U(t)| \leq C(T-t)^{-\frac{d}{2}}$ and $\|c\|_{L^{\infty}\left(t_{*}, T_{*} ; L^{\infty}\right)}<\infty$, using (5.9) and (5.43) we get

$$
\begin{align*}
& \left|\operatorname{Im}\left\langle\Delta R-\lambda_{j}^{-2} R_{j}+f(u)-f(U), c R\right\rangle\right| \\
\leq & \|R\|_{H^{1}}^{2}+(T-t)^{-2}\|R\|_{L^{2}}^{2}+\sum_{k=1}^{1+\frac{4}{d}}(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-k\right)}\|R\|_{L^{k+1}}^{k+1} \\
\leq & C(T-t)^{2 \kappa} \tag{5.45}
\end{align*}
$$

Thus, we conclude from (5.44) and (5.45) that

$$
\begin{equation*}
\left|I_{t, 7}^{(1)}\right| \leq C(T-t)^{2 \kappa} . \tag{5.46}
\end{equation*}
$$

Therefore, plugging estimates (5.34), (5.35), (5.36), (5.38), (5.40), (5.41) and (5.46) into (5.33) we obtain (5.32) and finish the proof of Lemma 5.11.

Lemma 5.12. For all $t \in\left[t_{*}, T_{*}\right]$, we have that for some $C(A)>0$,

$$
\begin{aligned}
\frac{d I^{(2)}}{d t} \geq & -\sum_{j=1}^{K} \frac{\gamma_{j}}{4 \lambda_{j}^{4}} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|R_{j}\right|^{2} d x+\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{2}} \operatorname{Re} \int \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right) d x \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}} \operatorname{Re} \int \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla \overline{U_{j}}\left\{\frac{2}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}\left|R_{j}\right|^{2}\right. \\
(5.47) \quad & \left.\quad+\frac{1}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} \overline{U_{j}} R_{j}^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)\left|U_{j}\right|^{\frac{4}{d}-4} U_{j}^{3}{\overline{R_{j}}}^{2}\right\} d x-C(A)(T-t)^{2 \kappa} .
\end{aligned}
$$

Remark 5.13. The difficulty in the proof of Lemma 5.1 d lies in the analysis of the interactions between the remainders, of which the perturbation order is of only polynomial type. This is different from the situation in Lemma 5.11, where the interactions involving $U_{j}$ are very weak, because of the exponential decay of the ground state. The point here is to gain additional decays from the functions $\partial^{\nu} \chi_{A}$, where $|\nu| \geq 2$.

Proof of Lemma 5.12. Straightforward computations show that

$$
\begin{align*}
\frac{d I^{(2)}}{d t}= & -\sum_{j=1}^{K} \frac{\dot{\lambda}_{j} \gamma_{j}-\lambda_{j} \dot{\gamma}_{j}}{2 \lambda_{j}^{2}} \operatorname{Im}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R, R_{j}\right\rangle \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im}\left\langle\partial_{t}\left(\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\right) \cdot \nabla R, R_{j}\right\rangle+\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Im}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, \partial_{t} R\right\rangle \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right), \partial_{t} R\right\rangle \\
= & \sum_{j=1}^{K}\left(I_{t, j 1}^{(2)}+I_{t, j 2}^{(2)}+I_{t, j 3}^{(2)}+I_{t, j 4}^{(2)}\right) . \tag{5.48}
\end{align*}
$$

We shall estimate $I_{t, j k}^{(2)}, 1 \leq k \leq 4$, separately. The main contributions come from the last two terms $I_{t, j 3}^{(2)}$ and $I_{t, j 4}^{(2)}$, which requires a delicate analysis of the interactions between remainders.
(i) Estimate of $I_{t, j 1}^{(2)}$ and $I_{t, j 2}^{(2)}$. Since $\sup _{y}\left|\nabla^{2} \chi_{A}(y)(1+|y|)\right| \leq C(A)$, by Lemmas 5.3 and 5.4.

$$
\begin{align*}
\left|\partial_{t}\left(\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\right)\right|= & \left\lvert\, \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \frac{\dot{\lambda}_{j} \lambda_{j}+\gamma_{j}}{\lambda_{j}^{2}}\right.\right. \\
& \left.-\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \frac{\gamma_{j}}{\lambda_{j}^{2}}+\frac{\lambda_{j} \dot{\alpha}_{j}-2 \beta_{j}}{\lambda_{j}^{2}}+\frac{2 \beta_{j}}{\lambda_{j}^{2}}\right) \mid \\
\leq & C(A) \lambda_{j}^{-2}\left(\operatorname{Mod}_{j}+P_{j}\right) \leq C(A)(T-t)^{-1} . \tag{5.49}
\end{align*}
$$

Taking into account $\left|\frac{\dot{\lambda}_{j} \gamma_{j}-\lambda_{j} \dot{\gamma}_{j}}{2 \lambda_{j}^{2}}\right| \leq C \frac{\operatorname{Mod}_{j}(t)}{\lambda_{j}^{3}} \leq C(T-t)^{\kappa}$ and (5.9) we obtain

$$
\begin{align*}
\left|I_{t, j 1}^{(2)}+I_{t, j 2}^{(2)}\right| & \leq C(A)\left((T-t)^{\kappa}\|\nabla R\|_{L^{2}}\|R\|_{L^{2}}+(T-t)^{-1}\|\nabla R\|_{L^{2}}\|R\|_{L^{2}}\right) \\
& \leq C(A)(T-t)^{2 \kappa} \tag{5.50}
\end{align*}
$$

(ii) Estimate of $I_{t, j 3}^{(2)}$. We claim that

$$
\begin{align*}
I_{t, j 3}^{(2)}= & -\frac{\gamma_{j}}{4 \lambda_{j}^{4}} \operatorname{Re} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|R_{j}\right|^{2} d x+\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|\nabla R_{j}\right|^{2} d x \\
& -\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.51}
\end{align*}
$$

In order to prove (5.51), we infer from (3.21) and equation (5.15) that

$$
\begin{equation*}
I_{t, j 3}^{(2)}=-\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, \Delta R+f^{\prime}(U) \cdot R+f^{\prime \prime}(U, R) \cdot R^{2}+(b \cdot \nabla+c) R+\eta\right\rangle \tag{5.52}
\end{equation*}
$$

The main contributions come from the terms involving $\Delta R$ and $f^{\prime}(U) \cdot R$.
First, since for any $j \neq l,\left|x-\alpha_{j}\right| \geq 4 \sigma$ on the support of $R_{l}$, taking $t_{*}$ close to $T$ we may let $\left|x_{j}-\alpha_{j}\right| \leq \sigma$, and thus $\left|x-\alpha_{j}\right| \geq 3 \sigma$ on the support $R_{l}$. By the integration by
parts formula, for $1 \leq j \neq l \leq K$,

$$
\begin{align*}
& \left|\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j} \Delta \overline{R_{l}} d x\right| \\
\leq & \left|\frac{\gamma_{j}^{2}}{2 \lambda_{j}^{2}} \operatorname{Re} \int_{\left|x-\alpha_{j}\right| \geq 3 \sigma} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \nabla R_{j} \cdot \nabla \overline{R_{l}} d x\right| \\
& +\left|\frac{\gamma_{j}^{3}}{2 \lambda_{j}^{3}} \operatorname{Re} \int_{\left|x-\alpha_{j}\right| \geq 3 \sigma} \nabla \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla \overline{R_{l}} R_{j} d x\right| \\
= & : K_{1}+K_{2} . \tag{5.53}
\end{align*}
$$

The key observation here is that, because of the decay of $\partial^{\nu} \chi_{A},|\nu| \geq 2$, the different remainders $R_{j}$ and $R_{l}$ have weak interactions of order $(T-t)^{2 \kappa}$, which is important for the bootstrap estimate of the remainder.

To be precise, since

$$
\Delta \chi(y)=\psi^{\prime \prime}(|y|)+(d-1) \psi^{\prime}(|y|)|y|^{-1} \leq C|y|^{-1}, \quad \text { if }|y| \geq 2
$$

we have

$$
\begin{equation*}
K_{1} \leq C \frac{\gamma_{j}}{\lambda_{j}^{2}}\left(\frac{\lambda_{j} A}{3 \sigma}\right)\|R\|_{H^{1}}^{2} \leq C A(T-t)^{2 \kappa} \tag{5.54}
\end{equation*}
$$

Similarly, since

$$
\partial_{j} \Delta \chi(y)=\psi^{\prime \prime \prime}(|y|) \frac{y_{j}}{|y|}+(d-1)\left(\psi^{\prime \prime}(|y|) \frac{y_{j}}{|y|^{2}}-\psi^{\prime}(|y|) \frac{y_{j}}{|y|^{3}}\right) \leq C|y|^{-2}, \quad \text { if }|y| \geq 2
$$

we get

$$
\begin{equation*}
K_{2} \leq C \frac{\gamma_{j}}{\lambda_{j}^{3}}\left(\frac{\lambda_{j} A}{3 \sigma}\right)^{2}\|\nabla R\|_{L^{2}}\left\|R_{j}\right\|_{L^{2}} \leq C(T-t)^{2 \kappa+1} \tag{5.55}
\end{equation*}
$$

Hence, plugging (5.54) and (5.55) into (5.53) we conclude that the interactions between different remainders $R_{j}$ and $R_{l}$ have the negligible order $(T-t)^{2 \kappa}$, i.e.,

$$
\left|\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j} \Delta \overline{R_{l}} d x\right| \leq C(T-t)^{2 \kappa}
$$

This along with the integration by parts formula yields that

$$
\begin{align*}
-\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, \Delta R\right\rangle= & -\frac{\gamma_{j}}{4 \lambda_{j}^{4}} \operatorname{Re} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|R_{j}\right|^{2} d x \\
& +\frac{\gamma_{j}^{2}}{2 \lambda_{j}^{2}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|\nabla R_{j}\right|^{2} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.56}
\end{align*}
$$

We also apply Lemma 3.1 to obtain

$$
\begin{equation*}
\operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, f^{\prime}(U) \cdot R\right\rangle=\operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) \tag{5.57}
\end{equation*}
$$

Moreover, since by (3.23) and (5.39),

$$
\left|f^{\prime \prime}(U, R) \cdot R^{2}\right| \leq C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R|^{2} \leq C\left((T-t)^{-2+\frac{d}{2}}+|R|^{\frac{4}{d}-1}\right)|R|^{2},
$$

using (3.31), (5.9) and (5.43) we have

$$
\begin{align*}
& \left|\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}, f^{\prime \prime}(U, R) \cdot R^{2}\right\rangle\right| \\
\leq & C(A) \int(T-t)^{-1}\left((T-t)^{-2+\frac{d}{2}}+|R|^{\frac{4}{d}-1}\right)|R|^{3} d x \\
\leq & C(A)\left((T-t)^{-3+\frac{d}{2}}\|R\|_{L^{3}}^{3}+(T-t)^{-1}\|R\|_{H^{1}}^{\frac{4}{d}+2}\right) \leq C(A)(T-t)^{2 \kappa} . \tag{5.58}
\end{align*}
$$

Furthermore, using Hölder's inequality, (5.9) and (5.17) we have

$$
\begin{align*}
& \left|\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j},(b \cdot \nabla+c) R+\eta\right\rangle\right| \\
\leq & C(A)(T-t)^{-1}\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}+\|R\|_{L^{2}}^{2}\right)+C(A)(T-t)^{-1}\|\eta\|_{L^{2}}\|R\|_{L^{2}} \\
\leq & C(A)(T-t)^{2 \kappa} . \tag{5.59}
\end{align*}
$$

Hence, plugging (5.56), (5.57), (5.58) and (5.59) into (5.52) we obtain (5.51), as claimed.
(i) Estimate of $I_{t, j 4}^{(2)}$. We claim that

$$
\begin{align*}
I_{t, j 4}^{(2)}= & \frac{\gamma_{j}}{\lambda_{j}^{2}} \operatorname{Re} \int \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right) d x-\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|\nabla R_{j}\right|^{2} d x \\
0) & -\frac{\gamma_{j}}{\lambda_{j}}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R_{j}, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.60}
\end{align*}
$$

For this purpose, we infer from equation (5.15) and Lemma 3.1 that

$$
\begin{align*}
& I_{t, j 4}^{(2)}=-\frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right),\right. \\
& \left.61) \quad \Delta R+f^{\prime}\left(U_{j}\right) \cdot R_{j}+f^{\prime \prime}(U, R) \cdot R^{2}+(b \cdot \nabla+c) R+\eta\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) . \tag{5.61}
\end{align*}
$$

We first show that

$$
\begin{align*}
& -\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right), \Delta R\right\rangle  \tag{5.62}\\
= & \operatorname{Re} \int \frac{2}{\lambda_{j}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right)-\frac{1}{\lambda_{j}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|\nabla R_{j}\right|^{2} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) .
\end{align*}
$$

In order to prove (5.62), using integration by parts formula we see that

$$
\begin{align*}
& -\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R_{j}, \Delta R\right\rangle \\
= & \operatorname{Re} \int \frac{1}{\lambda_{j}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \bar{R}\right)-\frac{1}{\lambda_{j}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \nabla R_{j} \cdot \nabla \bar{R} d x \\
& -\sum_{1 \leq k, l \leq d} \operatorname{Re} \int \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{l} R_{j} \partial_{k l} \bar{R} d x . \tag{5.63}
\end{align*}
$$

Then, as in the proof of (5.56), using the decay of $\partial_{\nu} \chi_{A}$ with $|\nu|=2$ we obtain that the interactions between different remainders have negligible contributions and thus

$$
\begin{align*}
& -\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R_{j}, \Delta R\right\rangle \\
= & \operatorname{Re} \int \frac{1}{\lambda_{j}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right)-\frac{1}{\lambda_{j}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|\nabla R_{j}\right|^{2} d x \\
& -\sum_{1 \leq k, l \leq d} \operatorname{Re} \int \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{l} R_{j} \partial_{k l} \bar{R} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.64}
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
& -\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R \Phi_{j}, \Delta R\right\rangle \\
= & \operatorname{Re} \int \frac{1}{\lambda_{j}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)(\nabla R, \nabla \bar{R}) \Phi_{j}+\sum_{1 \leq k, l \leq d} \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{k l} R \partial_{l} \bar{R} \Phi_{j} d x \\
& +\operatorname{Re} \int\left(\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R\right)\left(\nabla \bar{R} \cdot \nabla \Phi_{j}\right) d x \\
= & \operatorname{Re} \int \frac{1}{\lambda_{j}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right)+\sum_{1 \leq k, l \leq d} \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{k l} \bar{R} \partial_{l} R \Phi_{j} d x \\
& +\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.65}
\end{align*}
$$

Moreover, for the two terms involving $\partial_{k l} \bar{R}$ in (5.64) and (5.65), we see that the cancellation appears and the integration by parts formula and (5.1) give

$$
\begin{align*}
& \operatorname{Re} \int \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{k l} \bar{R} \partial_{l} R \Phi_{j}-\partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{l} R_{j} \partial_{k l} \bar{R} d x \\
= & -\operatorname{Re} \int \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R \partial_{l} \Phi_{j} \partial_{k l} \bar{R} d x \\
= & \operatorname{Re} \int \frac{1}{\lambda_{j}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{l} R \bar{R} \partial_{l} \Phi_{j}+\partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{l} \Phi_{j} \partial_{l} R \partial_{k} \bar{R} d x \\
& +\operatorname{Re} \int \partial_{k} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \partial_{k l} \Phi_{j} \partial_{l} R \bar{R} d x=\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.66}
\end{align*}
$$

Thus, plugging (5.64)-(5.66) into (5.63) we obtain (5.62), as claimed.
We also apply Lemma 3.1 to decouple different profiles between $\left\{U_{j}\right\}$ and $\left\{R_{j}\right\}$ to obtain that, similarly to (5.57),

$$
\begin{aligned}
& \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right), f^{\prime}(U) \cdot R\right\rangle \\
= & \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R_{j} \Phi_{j}\right), f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) .
\end{aligned}
$$

Since $\left|x-\alpha_{j}\right| \geq 3 \sigma$ on the support of $1-\Phi_{j}$, we have the exponential decay of $U_{j}$ that $\left|U_{j}(x)\right| \leq C \lambda_{j}^{-\frac{d}{2}} e^{-\delta \frac{3 \sigma}{\lambda_{j}}}$, and thus
$\left.\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R \Phi_{j}\right), f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle=\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right)$.

Hence, we obtain

$$
\begin{align*}
& \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right), f^{\prime}(U) \cdot R\right\rangle \\
= & \frac{\gamma_{j}}{\lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R_{j}, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) . \tag{5.67}
\end{align*}
$$

Moreover, using Hölder's inequality, (5.9) and (5.17) we easily get

$$
\begin{equation*}
\left|\frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla R_{j}+\nabla R \Phi_{j}\right), f^{\prime \prime}(U) \cdot R^{2}+(b \cdot \nabla+c) R+\eta\right\rangle\right| \leq C(T-t)^{2 \kappa} . \tag{5.68}
\end{equation*}
$$

Hence, we conclude from (5.62), (5.67) and (5.68) that (5.60) holds.
Now, putting the estimates (5.50), (5.51) and (5.60) altogether we obtain

$$
\begin{aligned}
\frac{d I^{(2)}}{d t}= & -\sum_{j=1}^{K} \frac{\gamma_{j}}{4 \lambda_{j}^{4}} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left|R_{j}\right|^{2} d x+\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{2}} \operatorname{Re} \int \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\left(\nabla R_{j}, \nabla \overline{R_{j}}\right) d x \\
& -\sum_{j=1}^{K} \operatorname{Re}\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) R_{j}+\frac{\gamma_{j}}{\lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla R_{j}, f^{\prime}\left(U_{j}\right) \cdot R_{j}\right\rangle+\mathcal{O}\left((T-t)^{2 \kappa}\right) .
\end{aligned}
$$

Because the profiles are decoupled completely, we can treat each profile individually by using similar computations as in the proof of [58, Lemma 5.11] and thus obtain that the third term on the right-hand-side above is equal to

$$
\begin{align*}
\sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}} \operatorname{Re} \int \nabla & \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla \overline{U_{j}}\left\{\frac{2}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}\left|R_{j}\right|^{2}\right. \\
& \left.+\frac{1}{d}\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}-2} \overline{U_{j}} R_{j}^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)\left|U_{j}\right|^{\frac{4}{d}-4} U_{j}^{3}{\overline{R_{j}}}^{2}\right\} d x \tag{5.69}
\end{align*}
$$

which immediately yields (5.47), thereby finishing the proof of Lemma 5.12.
We are now ready to prove Theorem 5.9 ,
Proof of Theorem 5.9. At this stage, the blow-up profiles are decoupled in (5.32) and (5.47), up to the acceptance order $\mathcal{O}\left((T-t)^{2 \kappa}\right)$, and thus we are able to treat each profile separately by using similar arguments as in the proof of [58, Theorem 5.8]. For the reader's convenience, we sketch the proof below.

Combining (5.32) and (5.47) altogether and then using the renormalized variable $\varepsilon_{j}$ in (4.10) we obtain that for all $t \in\left[t_{*}, T_{*}\right]$,

$$
\begin{align*}
\frac{d I}{d t} \geq & \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}}\left(\int \nabla^{2} \chi\left(\frac{y}{A}\right)\left(\nabla \varepsilon_{j}, \nabla \overline{\varepsilon_{j}}\right) d y+\int\left|\varepsilon_{j}\right|^{2} d y-\int\left(1+\frac{4}{d}\right) Q^{\frac{4}{d} \varepsilon_{j, 1}^{2}}+Q^{\frac{4}{d}} \varepsilon_{j, 2}^{2} d y\right. \\
& \left.\quad-\frac{1}{4 A^{2}} \int \Delta^{2} \chi\left(\frac{y}{A}\right)\left|\varepsilon_{j}\right|^{2} d y\right) \\
& +\sum_{j=1}^{K} \frac{2}{d} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int\left(A \nabla \chi\left(\frac{y}{A}\right)-y\right) \cdot \nabla Q Q^{\frac{4}{d}-1}\left(\left(1+\frac{4}{d}\right) \varepsilon_{j, 1}^{2}+\varepsilon_{j, 2}^{2}\right) d y \\
0) \quad & \quad C \varepsilon^{*}(T-t)^{2 \kappa-1}-C(A)(T-t)^{2 \kappa}, \tag{5.70}
\end{align*}
$$

where $\varepsilon_{j, 1}$ and $\varepsilon_{j, 2}$ denote the real and imaginary parts of $\varepsilon_{j}$, respectively.

Then, since

$$
\int \nabla^{2} \chi\left(\frac{y}{A}\right)\left(\nabla \varepsilon_{j}, \nabla \overline{\varepsilon_{j}}\right) d y \geq \int \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)\left|\nabla \varepsilon_{j}\right|^{2} d y
$$

applying Corollary 3.4 with $\phi(x):=\psi^{\prime \prime}(|x|)$, Lemma 4.4 and Proposition 5.6, and using the estimate $\operatorname{Scal}\left(\varepsilon_{j}\right) \leq C(T-t)^{2 \kappa+4}$ we obtain for some $\widetilde{C}>0$

$$
\begin{align*}
\frac{d I}{d t} \geq & \widetilde{C} \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)\left(\left|\varepsilon_{j}\right|^{2}+\left|\nabla \varepsilon_{j}\right|^{2}\right) d y-\sum_{j=1}^{K} \frac{1}{4 A^{2}} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int \Delta^{2} \chi\left(\frac{y}{A}\right)\left|\varepsilon_{j}\right|^{2} d y \\
& +\sum_{j=1}^{K} \frac{2}{d} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int\left(A \nabla \chi\left(\frac{y}{A}\right)-y\right) \cdot \nabla Q Q^{\frac{4}{d}-1}\left(\left(1+\frac{4}{d}\right) \varepsilon_{j, 1}^{2}+\varepsilon_{j, 2}^{2}\right) d y \\
& -C \varepsilon^{*}(T-t)^{2 \kappa-1}-C(A)(T-t)^{2 \kappa} . \tag{5.71}
\end{align*}
$$

Taking into account that for $A$ large enough,

$$
\frac{1}{4 A^{2}}\left|\Delta^{2} \chi\left(\frac{y}{A}\right)\right| \leq \frac{1}{4} \widetilde{C} \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right),
$$

and

$$
\frac{2}{d}\left(2+\frac{4}{d}\right)\left|A \nabla \chi\left(\frac{y}{A}\right)-y\right|\left|\nabla Q Q^{\frac{4}{d}-1}\right| \leq \frac{1}{4} \widetilde{C} \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)
$$

we arrive at

$$
\frac{d I}{d t} \geq \frac{1}{2} \widetilde{C} \sum_{j=1}^{K} \frac{\gamma_{j}}{\lambda_{j}^{4}} \int \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)\left(\left|\varepsilon_{j}\right|^{2}+\left|\nabla \varepsilon_{j}\right|^{2}\right) d y-C \varepsilon^{*}(T-t)^{2 \kappa-1}-C(A)(T-t)^{2 \kappa}
$$

Therefore, as $\psi^{\prime \prime}(r) \geq \delta e^{-r}$ for some $\delta>0$, we obtain (5.29) and finish the proof.
5.3. Proof of bootstrap estimates. In this subsection we prove the crucial bootstrap estimates in Theorem 5.1. To begin with, we first obtain the refined estimate for the modulation parameter $\beta$.

Lemma 5.14 (Refined estimate for $\beta$ ). There exists $C>0$ such that for all $t \in\left[t_{*}, T_{*}\right]$,

$$
\begin{equation*}
\sum_{j=1}^{K}\left|\beta_{j}(t)\right|^{2} \leq C \sum_{j=1}^{K}\left|\omega_{j}^{2} \lambda_{j}^{2}(t)-\gamma_{j}^{2}(t)\right|+C(T-t)^{\kappa+3} \tag{5.72}
\end{equation*}
$$

Remark 5.15. Unlike single bubble case in [58], the proof of Lemma 5.14 requires the localized mass in Proposition 5.6 and also a delicate treatment of the localized function $\Phi_{j}$ and the radial function $\phi_{A}$ in Corollary 3.4. in order to derive the coercivity of energy.

Proof of Lemma 5.14. Using the expansion (3.27) of $F(u)=\frac{d}{2 d+4}|u|^{2+\frac{4}{d}}$ we have

$$
\begin{align*}
E(u)= & \frac{1}{2} \int|\nabla U|^{2} d x-\frac{d}{2 d+4} \int|U|^{2+\frac{4}{d}} d x-\operatorname{Re} \int\left(\Delta U+|U|^{\frac{4}{d}} U\right) \bar{R} d x \\
& +\frac{1}{2} \operatorname{Re} \int|\nabla R|^{2}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x+o\left(\|R\|_{H^{1}}^{2}\right) . \tag{5.73}
\end{align*}
$$

Note that, by Lemma 3.1 and the explicit expression (4.3) of $U_{j}$,

$$
\begin{align*}
& \frac{1}{2} \int|\nabla U|^{2} d x-\frac{d}{2 d+4} \int|U|^{2+\frac{4}{d}} d x \\
= & \sum_{j=1}^{K}\left(\frac{1}{2} \int\left|\nabla U_{j}\right|^{2} d x-\frac{d}{2 d+4} \int\left|U_{j}\right|^{2+\frac{4}{d}} d x\right)+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right) \\
= & \sum_{j=1}^{K}\left(\frac{\left|\beta_{j}\right|^{2}}{2 \lambda_{j}^{2}}\|Q\|_{L^{2}}^{2}+\frac{\gamma_{j}^{2}}{8 \lambda_{j}^{2}}\|y Q\|_{L^{2}}^{2}\right)+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right), \tag{5.74}
\end{align*}
$$

and

$$
\begin{equation*}
\operatorname{Re} \int\left(\Delta U+|U|^{\frac{4}{d}} U\right) \bar{R} d x=\sum_{j=1}^{K} \operatorname{Re} \int\left(\Delta U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}\right) \overline{R_{j}} d x+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right) \tag{5.75}
\end{equation*}
$$

Taking into account Proposition 5.6 and rearranging the terms according to the orders of $R$ we obtain that for $T$ small enough,

$$
\begin{align*}
E(u)= & E(u)+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Re} \int \overline{U_{j}} R_{j}+\frac{1}{2}|R|^{2} \Phi_{j} d x+\mathcal{O}\left((T-t)^{2 \kappa}\right) \\
= & \sum_{j=1}^{K}\left(\frac{\left|\beta_{j}\right|^{2}}{2 \lambda_{j}^{2}}\|Q\|_{L^{2}}^{2}+\frac{\gamma_{j}^{2}}{8 \lambda_{j}^{2}}\|y Q\|_{L^{2}}^{2}\right)-\sum_{j=1}^{K} \operatorname{Re} \int\left(\Delta U_{j}-\frac{1}{\lambda_{j}^{2}} U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}\right) \overline{R_{j}} d x \\
& +\frac{1}{2} \operatorname{Re} \int|\nabla R|^{2}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x \\
& +\mathcal{O}\left((T-t)^{2 \kappa}\right) . \tag{5.76}
\end{align*}
$$

On one hand, using the identity (4.28) and the change of variables we get

$$
\begin{align*}
& \operatorname{Re} \int\left(\Delta U_{j}-\frac{1}{\lambda_{j}^{2}} U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}\right) \overline{R_{j}} d x \\
= & \frac{1}{\lambda_{j}^{2}} \operatorname{Im} \int\left(\gamma_{j} \Lambda Q_{j}-2 \beta_{j} \cdot \nabla Q_{j}\right) \overline{\varepsilon_{j}} d x+\frac{1}{\lambda_{j}^{2}} \operatorname{Re} \int\left|\beta_{j}-\frac{\gamma_{j}}{2} y\right|^{2} Q_{j} \overline{\varepsilon_{j}} d x, \tag{5.77}
\end{align*}
$$

which along with the almost orthogonality in Lemma 4.4 and (5.11) yields that

$$
\begin{equation*}
\left|\sum_{j=1}^{K} \operatorname{Re} \int\left(\Delta U_{j}-\frac{1}{\lambda_{j}^{2}} U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}\right) \overline{R_{j}} d x\right| \leq C\|R\|_{L^{2}} \leq C(T-t)^{2 \kappa+1} \tag{5.78}
\end{equation*}
$$

On the other hand, we claim that there exist $\widetilde{c}, C>0$ such that for the quadratic terms of $R$ on the right-hand side of (5.76),

$$
\begin{align*}
E_{2}(u): & =\operatorname{Re} \int|\nabla R|^{2}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x \\
\geq & \geq \widetilde{c}\left(\int|\nabla R|^{2}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j} d x\right) \\
& -C\left((T-t)^{-1}\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\|R\|_{H^{1}}^{2}+(T-t)^{2 \kappa+2}\right) . \tag{5.79}
\end{align*}
$$

(Note that, this does not follow directly from Corollary 3.4, because the localized function $\Phi_{j}$ does not satisfy the conditions there.)

For this purpose, using the partition of unity and Lemma 3.1 we have

$$
\begin{aligned}
E_{2}(u)= & \sum_{j=1}^{K} \operatorname{Re} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right) \Phi_{j}-\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}^{2} \bar{R}^{2} d x \\
& +\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right)
\end{aligned}
$$

In order to obtain the coercivity of the energy, we use $\phi_{A, j}(x):=\phi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)$ with $\phi_{A}(x)$ as in Corollary 3.4 and the renormalized variable $\widetilde{\varepsilon}_{j}$ defined by

$$
R(t, x)=\lambda_{j}^{-\frac{d}{2}} \widetilde{\varepsilon}_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}}
$$

to reformulate $E_{2}(u)$ as follows

$$
\begin{align*}
E_{2}(u)= & \sum_{j=1}^{K} \operatorname{Re} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right) \phi_{A, j}-\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}\left|U_{j}\right|^{\frac{4}{d}-2} U_{j}^{2} \bar{R}^{2} d x \\
& +\sum_{j=1}^{K} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right)\left(\Phi_{j}-\phi_{A, j}\right) d x+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) \\
= & \sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Re} \int\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right) \phi_{A}-\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}}\left|\widetilde{\varepsilon}_{j}\right|^{2}-\frac{2}{d} Q^{\frac{4}{d}-2} \bar{Q}_{j}^{2} \widetilde{\varepsilon}_{j}^{2} d y \\
& +\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \int\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) \\
0) & : \sum_{j=1}^{K} E_{21, j}+\sum_{j=1}^{K} E_{22, j}+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) . \tag{5.80}
\end{align*}
$$

Since $Q_{j}=Q+\mathcal{O}\left(P\langle y\rangle^{2} Q\right)$, the localized coercivity in Corollary 3.4, the estimate $\operatorname{Scal}\left(\widetilde{\varepsilon}_{j}\right) \leq C(T-t)^{2 \kappa+4}$ and (5.1) yield immediately that there exists $\widetilde{c}_{j}>0$ such that

$$
\begin{align*}
E_{21, j} & \geq \frac{1}{\lambda_{j}^{2}} \operatorname{Re} \int\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right) \phi_{A}-\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}}\left|\widetilde{\varepsilon}_{j}\right|^{2}-\frac{2}{d} Q^{\frac{4}{d} \widetilde{\varepsilon}_{j}^{2} d y-C(T-t)^{-1}\|R\|_{L^{2}}^{2}} \\
& \geq \widetilde{c}_{j} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right) \phi_{A, j} d x-C \frac{1}{\lambda_{j}^{2}} \operatorname{Scal}\left(\widetilde{\varepsilon}_{j}\right)-C(T-t)^{-1}\|R\|_{L^{2}}^{2} \\
.81) & \geq \widetilde{c}_{j} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right) \phi_{A, j} d x-C(T-t)^{2 \kappa+2}-C(T-t)^{-1}\|R\|_{L^{2}}^{2} . \tag{5.81}
\end{align*}
$$

Moreover, set $\widetilde{c}:=\min \left\{\frac{1}{2}, \widetilde{c}_{j}, 1 \leq j \leq K\right\}>0$. Since $\left.\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) \geq 0$ if $\left|y \cdot \mathbf{v}_{\mathbf{1}}\right| \leq \frac{3 \sigma}{\lambda_{j}}$, we get

$$
\begin{aligned}
E_{22, j} \geq & \frac{\widetilde{c}}{\lambda_{j}^{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \leq \frac{3 \sigma}{\lambda_{j}}}\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y \\
& +\frac{1}{\lambda_{j}^{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{3 \sigma}{\lambda_{j}}}\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y .
\end{aligned}
$$

By the positivity of $\Phi_{j}$ and the exponential decay of $\phi_{A}$, the second term on the right-hand side above is bounded from below by

$$
\begin{aligned}
& \frac{\widetilde{c}}{\lambda_{j}^{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{3 \sigma}{\lambda_{j}}}\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y-\frac{1-\widetilde{c}}{\lambda_{j}^{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{3 \sigma}{\lambda_{j}}}\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right) \phi_{A} d y \\
\geq & \frac{\widetilde{c}}{\lambda_{j}^{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{3 \sigma}{\lambda_{j}}}\left(\left|\nabla \widetilde{\varepsilon_{j}}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y-\frac{1-\widetilde{c}}{\lambda_{j}^{2}} e^{-\frac{3 \sigma}{A \lambda_{j}}\left\|\widetilde{\widetilde{j}}_{j}\right\|_{H^{1}} .}
\end{aligned}
$$

This yields that for $t$ close to $T$,

$$
\begin{align*}
E_{22, j} & \geq \frac{\widetilde{c}}{\lambda_{j}^{2}} \int\left(\left|\nabla \widetilde{\varepsilon}_{j}\right|^{2}+\left|\widetilde{\varepsilon}_{j}\right|^{2}\right)\left(\Phi_{j}\left(\lambda_{j} y+\alpha_{j}\right)-\phi_{A}(y)\right) d y-\frac{1-\widetilde{c}}{\lambda_{j}^{2}} e^{-\frac{3 \sigma}{A \lambda_{j}}\left\|\widetilde{\varepsilon}_{j}\right\|_{H^{1}}} \\
& \geq \widetilde{c} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{j}^{2}}|R|^{2}\right)\left(\Phi_{j}-\phi_{A, j}\right) d x-C e^{-\frac{\delta}{T-t}}\|R\|_{H^{1}}^{2} . \tag{5.82}
\end{align*}
$$

Then, plugging (5.81) and (5.82) into (5.80) we obtain (5.79), as claimed.
Therefore, taking into account $u\left(T_{*}\right)=S_{T}\left(T_{*}\right)$ we have

$$
\begin{aligned}
E\left(u\left(T_{*}\right)\right) & =\frac{1}{2} \int\left|\sum_{j=1}^{K} \nabla S_{j}\left(T_{*}\right)\right|^{2} d x-\frac{d}{2 d+4} \int\left|\sum_{j=1}^{K} S_{j}\left(T_{*}\right)\right|^{2+\frac{4}{d}} d x \\
& =\sum_{j=1}^{K} \frac{\omega_{j}^{2}}{8}\|y Q\|_{L^{2}}^{2}+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\right)
\end{aligned}
$$

and then plugging (5.78) and (5.79) into (5.76) we arrive at

$$
\begin{aligned}
\sum_{j=1}^{K} \frac{1}{2 \lambda_{j}^{2}}\left|\beta_{j}(t)\right|^{2}\|Q\|_{L^{2}}^{2} \leq & \sum_{j=1}^{K} \frac{1}{8 \lambda_{j}^{2}}\|y Q\|_{L^{2}}^{2}\left|\omega_{j}^{2} \lambda_{j}^{2}(t)-\gamma_{j}^{2}(t)\right| \\
& +\left|E(u)(t)-E(u)\left(T_{*}\right)\right|+\mathcal{O}\left((T-t)^{2 \kappa}\right),
\end{aligned}
$$

which, via Theorem 5.8 and (5.13), yields (5.72) and finishes the proof.
We are now ready to prove the bootstrap estimates in Theorem 5.1.
Proof of Theorem 5.1. (i) Estimate of R. On one hand, by (3.27),

$$
\begin{aligned}
I= & \frac{1}{2} \int|\nabla R|^{2}+\frac{1}{2} \sum_{j=1}^{K} \operatorname{Re} \int \frac{1}{\lambda_{j}^{2}}|R|^{2} \Phi_{j}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x \\
& +\mathcal{O}\left((T-t)^{-2+\frac{d}{2}} \sum_{k=3}^{2+\frac{4}{d}}\|R\|_{H^{1}}^{k}\right)+\mathcal{O}\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right),
\end{aligned}
$$

which, via (5.13) and (5.79), yields that for some $0<\widetilde{c}<1$,

$$
\begin{aligned}
I \geq & \widetilde{c}\left(|\nabla R|_{L^{2}}^{2}+\frac{1}{(T-t)^{2}}\|R\|_{L^{2}}^{2}\right) \\
& -C\left(e^{-\frac{\delta}{T-t}}\|R\|_{H^{1}}^{2}+(T-t)^{-1}\|R\|_{L^{2}}^{2}+(T-t)^{\frac{d}{2}}\|R\|_{H^{1}}^{2}+\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) .
\end{aligned}
$$

Then, taking $T$ small enough we get that

$$
\begin{equation*}
I(t) \geq \frac{1}{2} \widetilde{c}\left(\|\nabla R\|_{L^{2}}^{2}+\frac{1}{(T-t)^{2}}\|R\|_{L^{2}}^{2}\right) . \tag{5.83}
\end{equation*}
$$

On the other hand, Theorem 5.9 yields that for any $t \in\left[t_{*}, T_{*}\right]$,

$$
\begin{equation*}
\frac{d I}{d t} \geq-C(A)(T-t)^{2 \kappa}-C \varepsilon^{*}(T-t)^{2 \kappa-1} \tag{5.84}
\end{equation*}
$$

Thus, combining (5.83) and (5.84) and using the fundamental theorem of calculus we obtain that for any $t \in\left[t_{*}, T_{*}\right]$,

$$
\frac{1}{2} \widetilde{c}\left(\|\nabla R(t)\|_{L^{2}}^{2}+\frac{1}{(T-t)^{2}}\|R(t)\|_{L^{2}}^{2}\right) \leq I\left(T_{*}\right)+\int_{t}^{T_{*}} C \varepsilon^{*}(T-r)^{2 \kappa-1}+C(A)(T-r)^{2 \kappa} d r
$$

Taking into account $I\left(T_{*}\right)=0$ we obtain

$$
\begin{equation*}
\|\nabla R(t)\|_{L^{2}}^{2}+\frac{1}{(T-t)^{2}}\|R(t)\|_{L^{2}}^{2} \leq \frac{C \varepsilon^{*}}{\kappa \widetilde{c}}(T-t)^{2 \kappa}+\frac{2 C(A)}{(2 \kappa+1) \widetilde{c}}(T-t)^{2 \kappa+1} \tag{5.85}
\end{equation*}
$$

which yields (5.5) immediately, as long as $\varepsilon^{*}$ and $T$ are sufficiently small such that

$$
\frac{C}{\kappa \widetilde{c}} \varepsilon^{*}+\frac{2 C(A)}{(2 \kappa+1) \widetilde{c}} T \leq \frac{1}{8} .
$$

(ii) Estimates of $\lambda_{j}$ and $\gamma_{j}$. Since $\left(\frac{\gamma_{j}}{\lambda_{j}}\right)\left(T_{*}\right)=\omega_{j}$ and by (5.14),

$$
\begin{equation*}
\left|\frac{d}{d t}\left(\frac{\gamma_{j}}{\lambda_{j}}\right)\right|=\frac{\left|\lambda_{j}^{2} \dot{\gamma}_{j}-\lambda_{j} \dot{\lambda}_{j} \gamma_{j}\right|}{\lambda_{j}^{3}} \leq 2 \frac{M o d}{\lambda_{j}^{3}} \leq C(T-t)^{\kappa} \tag{5.86}
\end{equation*}
$$

we infer that for $T$ small enough such that $C T^{4 \zeta} \leq \frac{1}{2}$,

$$
\begin{equation*}
\left|\left(\frac{\gamma_{j}}{\lambda_{j}}\right)(t)-\omega_{j}(t)\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\frac{\gamma_{j}}{\lambda_{j}}\right)\right| d r \leq C(T-t)^{\kappa+1} \leq \frac{1}{2}(T-t)^{\kappa+6 \zeta} \tag{5.87}
\end{equation*}
$$

This along with (5.14) yields that

$$
\left|\frac{d}{d t}\left(\lambda_{j}-\omega_{j}(T-t)\right)\right|=\left|\dot{\lambda}_{j}+\frac{\gamma_{j}}{\lambda_{j}}+\omega_{j}-\frac{\gamma_{j}}{\lambda_{j}}\right| \leq \frac{M o d}{\lambda_{j}}+\frac{1}{2}(T-t)^{\kappa+6 \zeta} \leq C(T-t)^{\kappa+6 \zeta},
$$

which implies that for $T$ possibly smaller such that $C T^{\zeta} \leq \frac{1}{2}$,

$$
\begin{equation*}
\left|\lambda_{j}-\omega_{j}(T-t)\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\lambda_{j}-\omega_{j}(T-r)\right)\right| d r \leq \frac{1}{2}(T-t)^{\kappa+1+5 \zeta} \tag{5.88}
\end{equation*}
$$

thereby yielding the estimate of $\lambda_{j}$ in (5.6).
Similarly, by (5.14) and (5.87),

$$
\left|\frac{d}{d t}\left(\gamma_{j}-\omega_{j}^{2}(T-t)\right)\right|=\left|\dot{\gamma}_{j}+\frac{\gamma_{j}^{2}}{\lambda_{j}^{2}}+\omega_{j}^{2}-\frac{\gamma_{j}^{2}}{\lambda_{j}^{2}}\right| \leq \frac{M o d}{\lambda_{j}^{2}}+C\left|\omega_{j}-\frac{\gamma_{j}}{\lambda_{j}}\right| \leq C(T-t)^{\kappa+6 \zeta}
$$

which along with $\gamma_{j}\left(T_{*}\right)=\omega_{j}^{2}\left(T-T_{*}\right)$ yields that

$$
\left|\gamma_{j}(t)-\omega_{j}^{2}(T-t)\right| \leq \int_{t}^{T_{*}}\left|\frac{d}{d r}\left(\gamma_{j}(r)-\omega_{j}^{2}(T-r)\right)\right| d r \leq C(T-t)^{\kappa+1+6 \zeta}
$$

Hence, for $T$ very small such that $C T^{\zeta} \leq \frac{1}{2}$ we obtain

$$
\begin{equation*}
\left|\gamma_{j}-\omega_{j}^{2}(T-t)\right| \leq \frac{1}{2}(T-t)^{\kappa+1+5 \zeta} \tag{5.89}
\end{equation*}
$$

which yields the estimate of $\gamma_{j}$ in (5.6).
(iii) Estimates of $\beta_{j}$ and $\alpha_{j}$. We use the refined estimate of $\beta_{j}$ in Lemma 5.14 to get

$$
\begin{equation*}
\left|\beta_{j}\right|^{2} \leq C \sum_{j=1}^{K}\left|\omega_{j}^{2} \lambda_{j}^{2}-\gamma_{j}^{2}\right|+C(T-t)^{\kappa+3} \leq C \sum_{j=1}^{K}\left(\lambda_{j}^{2}\left|\omega_{j}-\frac{\gamma_{j}}{\lambda_{j}}\right|+\lambda_{j}^{\kappa+3}\right) \tag{5.90}
\end{equation*}
$$

which along with (5.13) and (5.87) yields that for $T$ small enough,

$$
\begin{equation*}
\left|\beta_{j}\right| \leq C \sum_{j=1}^{K}\left(\lambda_{j}\left|\omega_{j}-\frac{\gamma_{j}}{\lambda_{j}}\right|^{\frac{1}{2}}+\lambda_{j}^{\frac{\kappa}{2}+\frac{3}{2}}\right) \leq C(T-t)^{\frac{\kappa}{2}+1+3 \zeta} \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+1+2 \zeta} \tag{5.91}
\end{equation*}
$$

Moreover, since $\alpha_{j}\left(T_{*}\right)=x_{j}$ and by (5.14) and (5.91),

$$
\begin{equation*}
\left|\dot{\alpha}_{j}\right|=\left|\frac{\lambda_{j} \dot{\alpha}_{j}-2 \beta_{j}}{\lambda_{j}}+\frac{2 \beta_{j}}{\lambda_{j}}\right| \leq \frac{M o d}{\lambda_{j}}+\frac{2\left|\beta_{j}\right|}{\lambda_{j}} \leq C(T-t)^{\frac{\kappa}{2}+2 \zeta}, \tag{5.92}
\end{equation*}
$$

we infer that for sufficiently small $T$,

$$
\begin{equation*}
\left|\alpha_{j}(t)-x_{j}\right| \leq \int_{t}^{T_{*}}\left|\dot{\alpha}_{j}(r)\right| d r \leq C|T-t|^{\frac{\kappa}{2}+1+2 \zeta} \leq \frac{1}{2}(T-t)^{\frac{\kappa}{2}+1+\zeta} \tag{5.93}
\end{equation*}
$$

which yields (5.7).
(iv) Estimate of $\theta_{j}$. By (5.13), (5.88) and (5.91),

$$
\begin{align*}
\left|\frac{d}{d t}\left(\theta_{j}-\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| & =\left|\frac{\lambda_{j}^{2} \dot{\theta}_{j}-1-\left|\beta_{j}\right|^{2}}{\lambda_{j}^{2}}+\frac{\left|\beta_{j}\right|^{2}}{\lambda_{j}^{2}}+\frac{1}{\lambda_{j}^{2}}-\frac{1}{\omega_{j}^{2}(T-t)^{2}}\right| \\
& \leq \frac{M o d}{\lambda_{j}^{2}}+\frac{\left|\beta_{j}\right|^{2}}{\lambda_{j}^{2}}+\frac{\left|\lambda_{j}-\omega_{j}(T-t)\right|\left|\lambda_{j}+\omega_{j}(T-t)\right|}{\omega_{j}^{2} \lambda_{j}^{2}(T-t)^{2}} \\
& \leq C(T-t)^{\kappa-2+5 \zeta}, \tag{5.94}
\end{align*}
$$

which yields that for $t$ sufficiently small,

$$
\begin{align*}
\left|\theta_{j}-\left(\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| & \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\theta-\omega_{j}^{-2}(T-r)^{-1}+\vartheta_{j}\right)\right| d r \\
& \leq C(T-t)^{\kappa-1+5 \zeta} \leq \frac{1}{2}(T-t)^{\kappa-1+4 \zeta} \tag{5.95}
\end{align*}
$$

thereby yielding (5.8). The proof of Theorem 5.1 is complete.

## 6. Existence of multi-bubble solutions

In this section, we shall fix $\varepsilon^{*}>0$ to be sufficiently small, and for any $0<\varepsilon \leq \varepsilon^{*}$, take $\tau^{*}$ to be very small such that for a large universal constant $C$,

$$
C\left(1+\max _{1 \leq j \leq K}\left|x_{j}\right|\right) \tau_{*}^{\frac{1}{4}} \leq \frac{1}{2}
$$

For any $T \in\left(0, \tau^{*}\right)$, take any increasing sequence $\left\{t_{n}\right\}$ converging to $T$ and consider the approximating solutions $u_{n}$ satisfying the equation

$$
\left\{\begin{array}{l}
i \partial_{t} u_{n}+\Delta u_{n}+\left|u_{n}\right|^{\frac{4}{d}} u_{n}+(b \cdot \nabla+c) u_{n}=0,  \tag{6.1}\\
u_{n}\left(t_{n}\right)=\sum_{j=1}^{K} S_{j}\left(t_{n}\right)
\end{array}\right.
$$

where the coefficients $b, c$ are given by (2.12) and (2.13) respectively, and $S_{j}$ are the pseudo-conformal blow-up solutions defined in (2.5), $1 \leq j \leq K$. We also note that for each $n \geq 1, t_{n}$ plays the same role as $T_{*}$ in the previous sections.

We first have the uniform estimates of approximating solutions in Theorem 6.1 below.
Theorem 6.1 (Uniform estimates). For $n$ large enough, $u_{n}$ admits the unique geometrical decomposition $u_{n}=\omega_{n}+R_{n}$ on $\left[0, t_{n}\right]$ as in (4.1), and estimates (5.1) -(5.4) hold on $\left[0, t_{n}\right]$. Moreover, we have

$$
\begin{equation*}
\sup _{t \in\left[0, t_{n}\right]}\left\|x u_{n}(t)\right\|_{L^{2}} \leq C, \tag{6.2}
\end{equation*}
$$

and for any $t \in\left[0, t_{n}\right]$,

$$
\begin{equation*}
\left\|R_{n}(t)\right\|_{\Sigma} \leq C(T-t)^{\kappa} \tag{6.3}
\end{equation*}
$$

Proof. The proof of (5.1)-(5.4) is quite similar to that of [58, Theorem 5.1], mainly based on the bootstrap estimate in Theorem 5.1 and the abstract bootstrap principle (see, e.g., [60, Proposition 1.21]). For simplicity, the details are omitted here.

Below we prove estimates (6.2) and (6.3). Let $\varphi(x) \in C^{1}\left(\mathbb{R}^{d}, \mathbb{R}\right)$ be a radial cutoff function such that $\varphi(x)=0$ for $|x| \leq r$, and $\varphi(x)=(|x|-r)^{2}$ for $|x|>r$, where $r=2 \max _{1 \leq j \leq K}\left\{\left|x_{j}\right|, 1\right\}$. Note that, $|\nabla \varphi| \leq C \varphi^{\frac{1}{2}}$.

Using integration by parts formula we have for some constant $C>0$ independent of $n$,

$$
\begin{align*}
& \left.\left|\frac{d}{d t} \int\right| u_{n}\right|^{2} \varphi d x\left|=\left|\operatorname{Im} \int\left(2 \overline{u_{n}} \nabla u_{n}+b\left|u_{n}\right|^{2}\right) \cdot \nabla \varphi d x\right|\right. \\
\leq & C \int_{\left|x-x_{j}\right| \geq 1,1 \leq j \leq K}\left(\left|u_{n}\right|\left|\nabla u_{n}\right|+\left|u_{n}\right|^{2}\right) \varphi^{\frac{1}{2}} d x \\
\leq & C\left(\left(\int_{\left|x-x_{j}\right| \geq 1,1 \leq j \leq K}\left|\nabla u_{n}\right|^{2} d x\right)^{\frac{1}{2}}+\left(\int_{\left|x-x_{j}\right| \geq 1,1 \leq j \leq K}\left|u_{n}\right|^{2} d x\right)^{\frac{1}{2}}\right)\left(\int\left|u_{n}\right|^{2} \varphi d x\right)^{\frac{1}{2}} \tag{6.4}
\end{align*}
$$

By (4.1) and the exponential decay of the ground state,

$$
\begin{equation*}
\left.\left|\int_{\left|x-x_{j}\right| \geq 1,1 \leq j \leq K}\right| u_{n}(t)\right|^{2}+\left|\nabla u_{n}(t)\right|^{2} d x \left\lvert\, \leq C\left(\left\|R_{n}(t)\right\|_{H^{1}}^{2}+e^{-\frac{\delta}{T-t}}\right) .\right. \tag{6.5}
\end{equation*}
$$

Thus, taking into account the uniform estimate (5.1) we get

$$
\begin{align*}
\left.\left.\left|\frac{d}{d t} \int\right| u_{n}(t)\right|^{2} \varphi d x \right\rvert\, & \leq C\left(\left\|R_{n}(t)\right\|_{H^{1}}+e^{-\frac{\delta}{2(T-t)}}\right)\left(\int\left|u_{n}(t)\right|^{2} \varphi d x\right)^{\frac{1}{2}} \\
& \leq C(T-t)^{\kappa}\left(\int\left|u_{n}(t)\right|^{2} \varphi d x\right)^{\frac{1}{2}} . \tag{6.6}
\end{align*}
$$

Moreover, using the boundary condition $u_{n}\left(t_{n}\right)=\sum_{j=1}^{K} S_{j}\left(t_{n}\right)$ and $R_{n}\left(t_{n}\right)=0$ we have

$$
\begin{equation*}
\left.\left|\int\right| u_{n}\left(t_{n}\right)\right|^{2} \varphi d x \left\lvert\, \leq C e^{-\frac{\delta}{T-t_{n}}}\right. \tag{6.7}
\end{equation*}
$$

Thus, integrating from $t$ to $t_{n}$ we get for $t \in\left[0, t_{n}\right]$,

$$
\begin{equation*}
\int\left|u_{n}(t)\right|^{2} \varphi d x \leq C(T-t)^{2 \kappa+2}+C e^{-\frac{\delta}{T-t}} \leq C(T-t)^{2 \kappa+2} \tag{6.8}
\end{equation*}
$$

which yields that

$$
\begin{align*}
& \int\left|R_{n}(t)\right|^{2} \varphi d x \leq C\left(\int\left|U_{n}(t)\right|^{2} \varphi d x+\int\left|u_{n}(t)\right|^{2} \varphi d x\right) \\
& \leq C(T-t)^{2 \kappa+2}+C e^{-\frac{\delta}{T-t}} \leq C(T-t)^{2 \kappa+2}  \tag{6.9}\\
& 42
\end{align*}
$$

But $\varphi(x) \geq \frac{1}{2}|x|^{2}$ for $|x|$ large enough. Hence, we infer that for $t \in\left[0, t_{n}\right]$

$$
\begin{equation*}
\int\left|x u_{n}(t)\right|^{2} d x \leq C\left(\int\left|u_{n}(t)\right|^{2} \varphi d x+\int\left|u_{n}(t)\right|^{2} d x\right) \leq C \tag{6.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\int\left|x R_{n}(t)\right|^{2} d x \leq C\left(\int\left|R_{n}(t)\right|^{2} \varphi d x+\int\left|R_{n}(t)\right|^{2} d x\right) \leq C(T-t)^{2 \kappa+2} \tag{6.11}
\end{equation*}
$$

Therefore, taking into account (5.1) we obtain (6.2) and (6.3) and finish the proof.
Proof of Theorem [2.14. Let $\tau^{*}, \varepsilon^{*}$ be as in Theorem6.1] and let $T \in\left(0, \tau^{*}\right], \varepsilon \in\left(0, \varepsilon^{*}\right]$ be fixed below. By virtue of Theorem 6.1, we have the geometrical decomposition

$$
\begin{equation*}
u_{n}=U_{n}+R_{n}, \quad \forall t \in\left[0, t_{n}\right], \tag{6.12}
\end{equation*}
$$

with

$$
\begin{equation*}
u_{n}\left(t_{n}\right)=S_{T}\left(t_{n}\right):=\sum_{j=1}^{K} S_{j}\left(t_{n}\right), \quad R_{n}\left(t_{n}\right)=0 \tag{6.13}
\end{equation*}
$$

where $U_{n}=\sum_{j=1}^{K} U_{n, j}$ is as in (4.2) with the modulation parameters $\mathcal{P}_{n, j}$ satisfying

$$
\begin{align*}
\mathcal{P}_{n, j}\left(t_{n}\right): & =\left(\lambda_{n, j}\left(t_{n}\right), \alpha_{n, j}\left(t_{n}\right), \beta_{n, j}\left(t_{n}\right), \gamma_{n, j}\left(t_{n}\right), \theta_{n, j}\left(t_{n}\right)\right)  \tag{6.14}\\
& =\left(\omega_{j}\left(T-t_{n}\right), x_{j}, 0, \omega_{j}^{2}\left(T-t_{n}\right), \omega_{j}^{-2}\left(T-t_{n}\right)^{-1}+\vartheta_{j}\right) .
\end{align*}
$$

and $S_{j}$ are the pseudo-conformal blow-up solutions given by (2.5), $1 \leq j \leq K$. Moreover, the uniform estimates (5.1)-(5.4) hold on $\left[0, t_{n}\right]$.

In particular, $\left\{u_{n}(0)\right\}$ are uniformly bounded in $\Sigma$, and thus $u_{n}(0)$ converges weakly to some $u_{0} \in \Sigma$.

We claim that $u_{n}(0)$ indeed converges strongly in $L^{2}$, i.e.,

$$
\begin{equation*}
u_{n}(0) \rightarrow u_{0}, \quad \text { in } L^{2}, \text { as } n \rightarrow \infty \tag{6.15}
\end{equation*}
$$

This follows immediately from the uniform integrability of $\left\{u_{n}(0)\right\}$, that is, by (6.2),

$$
\begin{equation*}
\sup _{n \geq 1}\left\|u_{n}(0)\right\|_{L^{2}(|x|>A)} \leq \frac{1}{A} \sup _{n \geq 1}\left\|x u_{n}(0)\right\|_{L^{2}(|x|>A)} \leq \frac{C}{A} \rightarrow 0, \text { as } A \rightarrow \infty \tag{6.16}
\end{equation*}
$$

Thus, by virtue of (6.15) and the $L^{2}$ local well-posedness theory (see, e.g. (4) we obtain a unique $L^{2}$-solution $u$ to (6.1) on $[0, T)$ satisfying that $u(0)=u_{0}$, and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n}-u\right\|_{C\left([0, t] ; L^{2}\right)}=0, \quad t \in[0, T) . \tag{6.17}
\end{equation*}
$$

Moreover, since $u_{0} \in H^{1}$, using the $H^{1}$ local well-posedness result (see, e.g., [5]) we also have $u \in C\left([0, t] ; H^{1}\right)$ for any $0<t<T$. Such solution is indeed the desirable blow-up solution that explodes at the given $K$ points $\left\{x_{j}\right\}_{j=1}^{K}$.

As a matter of fact, let

$$
\begin{equation*}
\left(\lambda_{0, j}, \alpha_{0, j}, \beta_{0, j}, \gamma_{0, j}, \theta_{0, j}\right):=\left(\omega_{j}(T-t), x_{j}, 0, \omega_{j}^{2}(T-t), \omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right), \tag{6.18}
\end{equation*}
$$

and $Q_{0, j}(t, y):=Q(y) e^{i\left(\beta_{0, j}(t) \cdot y-\frac{\gamma_{0, j}(t)}{4}|y|^{2}\right)}$. Since

$$
\begin{align*}
U_{n, j}-S_{j}= & \left(\lambda_{n, j}^{-\frac{d}{2}}-\lambda_{0, j}^{-\frac{d}{2}}\right) Q_{n, j}\left(t, \frac{x-\alpha_{n, j}}{\lambda_{n, j}}\right) e^{i \theta_{n, j}} \\
& +\lambda_{0, j}^{-\frac{d}{2}}\left(Q_{n, j}\left(t, \frac{x-\alpha_{n, j}}{\lambda_{n, j}}\right)-Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right) e^{i \theta_{n, j}}\right. \\
& +\lambda_{0, j}^{-\frac{d}{2}}\left(Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)-Q_{0, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\right) e^{i \theta_{n, j}} \\
& +\lambda_{0, j}^{-\frac{d}{2}} Q_{0, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\left(e^{i \theta_{n, j}}-e^{i \theta_{0, j}}\right), \tag{6.19}
\end{align*}
$$

using the change of variables and (5.13) we infer that, if $M:=1+\max _{1 \leq j \leq K}\left|x_{j}\right|$,

$$
\begin{aligned}
\left\|x\left(U_{n}-S_{T}\right)\right\|_{L^{2}} \leq & \lambda_{0, j}^{-\frac{d}{2}} \||x|\left(Q_{n, j}\left(t, \frac{x-\alpha_{n, j}}{\lambda_{n, j}}\right)-Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right) \|_{L^{2}}\right. \\
& +\lambda_{0, j}^{-\frac{d}{2}}\left\||x|\left(Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)-Q_{0, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\right)\right\|_{L^{2}} \\
& +C M\left(\left|\frac{\lambda_{n, j}^{\frac{d}{2}}-\lambda_{0, j}^{\frac{d}{2}}}{\lambda_{0, j}^{\frac{d}{2}}}\right|+\left|\theta_{n, j}-\theta_{0, j}\right|\right)\|\langle y\rangle Q\|_{L^{2} .}
\end{aligned}
$$

Then, taking into account the uniform estimates (5.2)-(5.4) of the modulation parameters and the well localized property of $Q$ we get that for $T$ small enough such that $M T \leq 1$,

$$
\begin{aligned}
&\left\|x\left(U_{n}-S_{T}\right)\right\|_{L^{2}} \leq C M \sum_{j=1}^{K}\left(\left|\frac{\lambda_{0, j}}{\lambda_{n, j}}-1\right|+\left|\frac{\alpha_{n, j}-\alpha_{0, j}}{\lambda_{n, j}}\right|+\left|\beta_{n, j}-\beta_{0, j}\right|\right. \\
&\left.+\left|\gamma_{n, j}-\gamma_{0, j}\right|+\left|\frac{\lambda_{n, j}^{\frac{d}{2}}-\lambda_{0, j}^{\frac{d}{2}}}{\lambda_{0, j}^{\frac{d}{2}}}\right|+\left|\theta_{n, j}-\theta_{0, j}\right|\right) \\
& \leq C M(T-t)^{\frac{\kappa}{2}+\zeta} \leq C(T-t)^{\frac{\kappa}{2}-1+\zeta},
\end{aligned}
$$

where $C$ is a universal constant, independent of $n, t, M, T$.
Similarly, we have (see also the proof of [58, Theorem 2.12])

$$
\begin{gathered}
\left\|U_{n}-S_{T}\right\|_{L^{2}} \leq C \sum_{j=1}^{K}\left(\left|\frac{\lambda_{0, j}}{\lambda_{n, j}}-1\right|+\left|\frac{\alpha_{n, j}-\alpha_{0, j}}{\lambda_{n, j}}\right|+\left|\beta_{n, j}-\beta_{0, j}\right|++\left|\gamma_{n, j}-\gamma_{0, j}\right|\right. \\
\left.+\left|\frac{\lambda_{n, j}^{\frac{d}{2}}-\lambda_{0, j}^{\frac{d}{2}}}{\lambda_{0, j}^{\frac{d}{2}}}\right|+\left|\theta_{n, j}-\theta_{0, j}\right|\right) \leq C(T-t)^{\frac{\kappa}{2}+\zeta},
\end{gathered}
$$

and

$$
\begin{gathered}
\left\|\nabla U_{n}-\nabla S_{T}\right\|_{L^{2}} \leq C \sum_{j=1}^{K}\left(\frac{1}{\lambda_{0, j}}\left|\frac{\lambda_{0, j}}{\lambda_{n, j}}-1\right|+\left|\frac{\alpha_{n, j}-\alpha_{0, j}}{\lambda_{0, j} \lambda_{n, j}}\right|+\left|\frac{\beta_{n, j}-\beta_{0, j}}{\lambda_{0, j}}\right|+\left|\frac{\gamma_{n, j}-\gamma_{0, j}}{\lambda_{0, j}}\right|\right. \\
\left.+\left|\frac{\lambda_{n, j}^{1+\frac{d}{2}}-\lambda_{0, j}^{1+\frac{d}{2}}}{\lambda_{n, j}^{1+\frac{d}{2}}}\right|+\left|\frac{\theta_{n, j}-\theta_{0, j}}{\lambda_{0, j}}\right|\right) \leq C(T-t)^{\frac{\kappa}{2}-1+\zeta .} .
\end{gathered}
$$

Hence, we conclude that

$$
\begin{equation*}
\left\|U_{n}(t)-S_{T}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{\kappa}{2}-1+\zeta} \tag{6.20}
\end{equation*}
$$

This along with (5.1) and (6.12) yields that

$$
\begin{equation*}
\left\|u_{n}(t)-S_{T}(t)\right\|_{\Sigma} \leq\left\|U_{n}(t)-S_{T}(t)\right\|_{\Sigma}+\left\|R_{n}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{\kappa}{2}-1+\zeta} \tag{6.21}
\end{equation*}
$$

where $C$ is independent of $n$.
Hence, in view of (6.17), we infer that for some subsequence (still denoted by $\{n\}$ ),

$$
u_{n}(t)-S_{T}(t) \rightharpoonup u(t)-S_{T}(t), \text { weakly in } \Sigma, \text { as } n \rightarrow \infty,
$$

which yields that

$$
\left\|u(t)-S_{T}(t)\right\|_{\Sigma} \leq \liminf _{n \rightarrow \infty}\left\|u_{n}(t)-S_{T}(t)\right\|_{\Sigma} \leq C(T-t)^{\frac{\kappa}{2}-1+\zeta} .
$$

Therefore, the proof of Theorem 2.14 is complete.

## 7. Uniqueness of multi-bubble solutions

7.1. Geometrical decomposition. In this subsection we obtain the geometrical decomposition and uniform estimates for the blow-up solution constructed in the proof of Theorem 2.14 which, actually, are inherited from those of the approximating solutions.

Let us start with the boundedness of the remainders in the more regular space $H^{\frac{3}{2}}$, which will be used in Theorem 7.5 later to derive the key monotonicity formula of the generalized functional defined on the difference.

Below we use the same notations $u, u_{n}, U_{n}, R_{n}$ and $t_{n}$ as in Section 6. We also set $M:=\max _{1 \leq j \leq K}\left|x_{j}\right|+1$ and keep using the notation $\kappa:=\nu_{*}-3$.
Proposition 7.1. Assume (A0) and (A1) with $\nu_{*} \geq 5$. Then, for $T$ small enough such that

$$
\begin{equation*}
C\left(1+\max _{1 \leq j \leq K}\left|x_{j}\right|\right) T^{\frac{1}{4}} \leq \frac{1}{2} \tag{7.1}
\end{equation*}
$$

where $C$ is a large universal constant, independent of $\varepsilon, T, n$, we have

$$
\begin{equation*}
\left\|R_{n}(t)\right\|_{H^{\frac{3}{2}}} \leq(T-t)^{\kappa-2}, \quad t \in\left[0, t_{n}\right) \tag{7.2}
\end{equation*}
$$

Proof. We rewrite the equation (5.15) of $R_{n}$ as follows

$$
\begin{equation*}
i \partial_{t} R_{n}+\Delta R_{n}+b \cdot \nabla R_{n}+c R_{n}=-\eta_{n}-f\left(R_{n}\right)-\left(f\left(u_{n}\right)-f\left(U_{n}\right)-f\left(R_{n}\right)\right) \tag{7.3}
\end{equation*}
$$

with $R_{n}\left(t_{n}\right)=0$ and $\eta_{n}$ as in (5.16), where $U$ is replaced by $U_{n}$ given by (6.12). Then, applying the operator $\langle\nabla\rangle^{\frac{3}{2}}$ to both sides of (7.3) we obtain

$$
\begin{align*}
& i \partial_{t}\left(\langle\nabla\rangle^{\frac{3}{2}} R_{n}\right)+\Delta\left(\langle\nabla\rangle^{\frac{3}{2}} R_{n}\right)+(b \cdot \nabla+c)\left(\langle\nabla\rangle^{\frac{3}{2}} R_{n}\right) \\
= & {\left[b \cdot \nabla+c,\langle\nabla\rangle^{\frac{3}{2}}\right] R_{n}-\langle\nabla\rangle^{\frac{3}{2}} \eta_{n}-\langle\nabla\rangle^{\frac{3}{2}} f\left(R_{n}\right)-\langle\nabla\rangle^{\frac{3}{2}}\left(f\left(u_{n}\right)-f\left(U_{n}\right)-f\left(R_{n}\right)\right), } \tag{7.4}
\end{align*}
$$

where $\left[b \cdot \nabla+c,\langle\nabla\rangle^{\frac{3}{2}}\right]$ is the commutator $(b \cdot \nabla+c)\langle\nabla\rangle^{\frac{3}{2}}-\langle\nabla\rangle^{\frac{3}{2}}(b \cdot \nabla+c)$. We regard (7.4) as the equation for the unknown $\langle\nabla\rangle^{\frac{3}{2}} R_{n}$ and apply the Strichartz estimates and local smoothing estimates (see [64, Theorem 2.13]) to get

$$
\begin{align*}
\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)} \leq & C\left(\left\|\left[b \cdot \nabla+c,\langle\nabla\rangle^{\frac{3}{2}}\right] R_{n}\right\|_{L^{2}\left(t, t_{n} ; H_{1}^{-\frac{1}{2}}\right)}+\left\|\langle\nabla\rangle^{\frac{3}{2}} \eta_{n}\right\|_{L^{\frac{4+2 d}{4+d}}\left(t, t_{n} ; L^{\frac{4+2 d}{4+d}}\right)}\right. \\
& +\left\|\langle\nabla\rangle^{\frac{3}{2}}\left(f\left(R_{n}\right)\right)\right\|_{L^{\frac{4+2 d}{4+d}}\left(t, t_{n} ; L^{\frac{4+2 d}{4+d}}\right)} \\
& \left.+\left\|\langle\nabla\rangle^{\frac{3}{2}}\left(f\left(u_{n}\right)-f\left(U_{n}\right)-f\left(R_{n}\right)\right)\right\|_{L^{2}\left(t, t_{n} ; H_{1}^{-\frac{1}{2}}\right)}\right)=: \sum_{j=1}^{4} R_{j} . \tag{7.5}
\end{align*}
$$

Below we estimate each term $R_{j}, 1 \leq j \leq 4$, separately.
(i) Estimate of $R_{1}$. Since by Assumption ( $A 0$ ), $\left|\partial^{\nu} b\right|+\left|\partial^{\nu} c\right| \leq C\langle x\rangle^{-2}$ for any multiindex $\nu$, using the calculus of pseudo-differential operators (see, e.g., [64) and (5.1) we get

$$
\begin{equation*}
R_{1} \leq C\left\|R_{n}\right\|_{L^{2}\left(t, t_{n} ; H_{-1}^{1}\right)} \leq C(T-t)^{\frac{1}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{1}\right)} \leq C(T-t)^{\kappa+\frac{1}{2}} . \tag{7.6}
\end{equation*}
$$

(ii) Estimate of $R_{2}$. Using (4.14) and (5.16) we have the pointwise estimate of $\eta_{n}$ that for any multi-index $\nu$ with $|\nu| \leq 2$ and for $y:=\frac{x-\alpha_{n, j}}{\lambda_{n, j}}$,

$$
\begin{aligned}
\left|\partial_{x}^{\nu} \eta_{n}(x)\right| & \leq \sum_{j=1}^{K} \lambda_{n, j}^{-4-\frac{d}{2}}\left|\partial_{y}^{\nu}\left(\left(\lambda_{n, j} \widetilde{b} \cdot \nabla+\lambda_{n, j}^{2} \widetilde{c}\right) Q_{n, j}\right)\right|+\sum_{j=1}^{K} \lambda_{n, j}^{-4-\frac{d}{2}} \operatorname{Mod}_{n}\langle y\rangle^{2}\left|\sum_{|\widetilde{\nu}| \leq|\nu|+1} \partial_{y}^{\widetilde{\nu}} Q_{n, j}\right| \\
& =: \sum_{j=1}^{K} \eta_{1, j}+\eta_{2, j},
\end{aligned}
$$

where $\widetilde{b}$ and $\widetilde{c}$ are as in (4.22). Note that, for $p:=\frac{4+2 d}{4+d}$, by (4.23),

$$
\begin{aligned}
\left\|\eta_{1, j}\right\|_{L^{p}} & \leq \sum_{j=1}^{K} \lambda_{n, j}^{-\frac{d}{2}-4} \lambda_{n, j}^{\frac{d}{p}}\left\|\partial_{y}^{\nu}\left(\left(\lambda_{n, j} \widetilde{b} \cdot \nabla+\lambda_{n, j}^{2} \widetilde{c}\right) Q_{n, j}\right)\right\|_{L^{p}} \\
& \leq \sum_{j=1}^{K} \lambda_{n, j}^{d\left(\frac{1}{p}-\frac{1}{2}\right)-4} \lambda_{n, j}^{\nu_{*}+1} \leq(T-t)^{\kappa+\frac{d}{2+d}} .
\end{aligned}
$$

Moreover, by (5.14),

$$
\left\|\eta_{2, j}\right\|_{L^{p}} \leq C(T-t)^{d\left(\frac{1}{p}-\frac{1}{2}\right)-4} \operatorname{Mod}_{n} \leq C(T-t)^{\kappa-\frac{2}{2+d}} .
$$

Hence, we obtain that for any multi-index $\nu$ with $|\nu| \leq 2$,

$$
\left\|\partial_{x}^{\nu} \eta_{n}\right\|_{L^{p}} \leq C(T-t)^{\kappa-\frac{2}{2+d}} .
$$

This yields that

$$
\begin{equation*}
R_{2} \leq\left\|\eta_{n}\right\|_{L^{\frac{4+2 d}{4+d}}\left[\left[t, t_{n}\right] ; W^{\left.2, \frac{4+2 d}{4+d}\right)}\right.} \leq C(T-t)^{\frac{4+d}{4+2 d}+\kappa-\frac{2}{2+d}} \leq C(T-t)^{\kappa} . \tag{7.7}
\end{equation*}
$$

(iii) Estimate of $R_{3}$. Using the product rule in Lemma 3.6, (3.29) and (5.1) we get

$$
\begin{align*}
\left\|\langle\nabla\rangle^{\frac{3}{2}}\left(f\left(R_{n}\right)\right)\right\|_{L^{\frac{4+2 d}{4+d}}} & \leq C\left\|\langle\nabla\rangle^{\frac{3}{2}} R_{n}\right\|_{L^{2}}\left\|R_{n}\right\|_{L^{\frac{4+4 d}{d}}}^{\frac{4}{d}} \\
& \leq C\left\|R_{n}\right\|_{H^{\frac{3}{2}}}\left\|R_{n}\right\|_{H^{1}}^{\frac{4}{d}} \leq C(T-t)^{\frac{4}{d} \kappa}\left\|R_{n}\right\|_{H^{\frac{3}{2}}}, \tag{7.8}
\end{align*}
$$

which yields that

$$
\begin{equation*}
R_{3} \leq C(T-t)^{\frac{4}{d} \kappa+\frac{4+d}{4+2 d}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)} . \tag{7.9}
\end{equation*}
$$

(iv) Estimate of $R_{4}$. We first see that

$$
\begin{aligned}
R_{4} \leq & C\left\|\mid\langle x\rangle\left(f\left(u_{n}\right)-f\left(U_{n}\right)-f\left(R_{n}\right)\right)\right\|_{L^{2}\left(t, t_{n} ; H^{1}\right)} \\
\leq & C \sum_{j=1}^{4 / d}\left(\left\|\langle x\rangle\left|U_{n}\right|^{1+\frac{4}{d}-j}\left|R_{n}\right|^{j}\right\|_{L^{2}\left(t, t_{n} ; L^{2}\right)}+\left\|\langle x\rangle\left|\nabla U_{n}\right|\left|U_{n}\right|^{\frac{4}{d}-j}\left|R_{n}\right|^{j}\right\|_{L^{2}\left(t, t_{n} ; L^{2}\right)}\right. \\
& \quad+\left\|\langle x\rangle\left|U_{n}\right|^{1+\frac{4}{d}-j}\left|\nabla R_{n}\left\|\left|R_{n}\right|^{j-1}\right\|_{L^{2}\left(t, t_{n} ; L^{2}\right)}\right)\right. \\
= & \sum_{j=1}^{4 / d}\left(R_{4, j 1}+R_{4, j 2}+R_{4, j 3}\right) .
\end{aligned}
$$

Since $\left|\alpha_{j}\right| \leq\left|x_{j}-\alpha_{j}\right|+\left|x_{j}\right| \leq M, 1 \leq j \leq K$, and $\sup _{y \in \mathbb{R}^{d}}\langle y\rangle Q(y)<\infty$, we infer that

$$
\begin{equation*}
\left\|\langle x\rangle U_{n}(t)\right\|_{L^{\infty}} \leq \sum_{j=1}^{K} \sup _{y \in \mathbb{R}^{d}}\left\langle\lambda_{j} y+\alpha_{j}\right\rangle \lambda_{j}^{-\frac{d}{2}} Q(y) \leq C M(T-t)^{-\frac{d}{2}}, \tag{7.11}
\end{equation*}
$$

which along with (5.43) yields that

$$
\begin{equation*}
R_{4, j 1} \leq C M(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-j\right)+\frac{1}{2}}\left\|R_{n}\right\|_{\left.C\left(t, t t_{n}\right] ; L^{2 j}\right)}^{j} \leq C M(T-t)^{\kappa-\frac{1}{2}} \tag{7.12}
\end{equation*}
$$

Moreover, since for any $1<p<\infty$,

$$
\begin{equation*}
\left\|\langle x \rangle \left|\nabla U_{n}\left\|\left.U_{n}\right|^{\frac{4}{d}-j}\right\|_{L^{p}} \leq C M(T-t)^{\frac{d}{2}(j-1)-3+\frac{d}{p}},\right.\right. \tag{7.13}
\end{equation*}
$$

using Hölder's inequality and (5.43) we get that

$$
\begin{align*}
R_{4, j 2} & \leq C\left\|\langle x \rangle \left|\nabla U_{n}\left\|\left.U_{n}\right|^{\frac{4}{d}-j}\right\|_{L^{2}\left(t, t_{n} ; L^{p}\right)}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; L^{q j}\right)}^{j}\right.\right. \\
& \leq C M(T-t)^{\frac{d}{2}(j-1)-\frac{5}{2}+\frac{d}{p}}(T-t)^{j\left(\kappa+1-\frac{d}{2}\right)+\frac{d}{q}} \leq C M(T-t)^{\kappa-\frac{3}{2}}, \tag{7.14}
\end{align*}
$$

where $p, q$ are any positive numbers such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.
It remains to treat the last term $R_{4, j 3}$. In the case where $j=1$ we have

$$
R_{4, j 3} \leq C M(T-t)^{-\frac{3}{2}}\left\|\nabla R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; L^{2}\right)} \leq C M(T-t)^{\kappa-\frac{3}{2}} .
$$

In the case where $d=1$ and $2 \leq j \leq \frac{4}{d}$, using Sobolev's embedding $H^{1}(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ we get

$$
R_{4, j 3} \leq C M(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-j\right)+\frac{1}{2}}\left\|R_{n}\right\|_{\left.C\left(t, t t_{n}\right] ; H^{1}\right)}^{j} \leq C M(T-t)^{2 \kappa-1}
$$

Furthermore, in the case where $d=2$ and $2 \leq j \leq \frac{4}{d}$, using the Sobolev embedding $H^{\frac{1}{2}, 2}\left(\mathbb{R}^{2}\right) \hookrightarrow L^{\frac{2 d}{d-1}}\left(\mathbb{R}^{2}\right)$ instead we get

$$
\begin{aligned}
R_{4, j 3} & \leq C M(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-j\right)+\frac{1}{2}}\left\|R_{n}^{j-1}\right\|_{C\left(\left[t, t_{n}\right] ; L^{2 d}\right)}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; W^{1, \frac{2 d}{d-1}}\right)} \\
& \leq C M(T-t)^{-\frac{d}{2}\left(1+\frac{4}{d}-j\right)+\frac{1}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{1}\right)}^{j-1}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)} \\
& \leq C M(T-t)^{\kappa-\frac{3}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)} .
\end{aligned}
$$

Hence we conclude that

$$
\begin{equation*}
R_{4, j 3} \leq C M\left((T-t)^{\kappa-\frac{3}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)}+(T-t)^{\kappa-\frac{3}{2}}\right) \tag{7.15}
\end{equation*}
$$

Thus, plugging (7.12)-(7.15) into (7.10) we obtain

$$
\begin{equation*}
R_{4} \leq C M\left((T-t)^{\kappa-\frac{3}{2}}+(T-t)^{\kappa-\frac{3}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)}\right) . \tag{7.16}
\end{equation*}
$$

Therefore, plugging the estimates (7.6), (7.7), (7.9) and (7.16) into (7.5) we obtain

$$
\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)} \leq C M\left((T-t)^{\kappa-\frac{3}{2}}+(T-t)^{\kappa-\frac{3}{2}}\left\|R_{n}\right\|_{C\left(\left[t, t_{n}\right] ; H^{\frac{3}{2}}\right)}\right) .
$$

Hence, taking $T$ very small such that (7.1) holds, we obtain (7.2) and finish the proof.
Proposition 7.2 below shows that, in the case where $\kappa \geq 3$ (i.e., $\nu_{*} \geq 6$ ), one may enhance the approximation (2.15) of $u$ and $u_{n}$ in the space $\dot{H}^{\frac{3}{2}}$.
Proposition 7.2. Consider the situation as in Proposition 7.1 but with $\nu_{*} \geq 6$. Then, we have

$$
\begin{equation*}
\left\|u_{n}(t)-S_{T}(t)\right\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\frac{1}{2}(\kappa-3)+\zeta} \tag{7.17}
\end{equation*}
$$

where $\zeta \in\left(0, \frac{1}{12}\right)$. In particular, for the blow-up solution $u$ constructed in Theorem 2.14.

$$
\begin{equation*}
\left\|u(t)-S_{T}(t)\right\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\frac{1}{2}(\kappa-3)+\zeta}, \tag{7.18}
\end{equation*}
$$

and we also have the strong $H^{1}$ convergence that for any $t \in(0, T)$,

$$
\begin{equation*}
\left\|u_{n}-u\right\|_{C\left([0, t] ; H^{1}\right)} \rightarrow 0, \text { as } n \rightarrow \infty \tag{7.19}
\end{equation*}
$$

Proof. Since

$$
\left\|u_{n}(t)-\sum_{j=1}^{K} S_{j}(t)\right\|_{\dot{H}^{\frac{3}{2}}} \leq\left\|R_{n}(t)\right\|_{\dot{H}^{\frac{3}{2}}}+\sum_{j=1}^{K}\left\|U_{n, j}(t)-S_{j}(t)\right\|_{\dot{H}^{\frac{3}{2}}},
$$

in view of Proposition [7.1, we only need to prove that for each $1 \leq j \leq K$,

$$
\begin{equation*}
\left\|U_{n, j}(t)-S_{j}(t)\right\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\frac{1}{2}(\kappa-3)+\zeta} \tag{7.20}
\end{equation*}
$$

For this purpose, we let $\left(\lambda_{0, j}, \alpha_{0, j}, \beta_{0, j}, \gamma_{0, j}, \theta_{0, j}\right)$ and $Q_{0, j}(t, y)$ be as in the proof of Theorem 2.14. Then, as in (6.19), we decompose

$$
\begin{aligned}
U_{n, j}-S_{j}= & \left(\lambda_{n, j}^{-\frac{d}{2}}-\lambda_{0, j}^{-\frac{d}{2}}\right) Q_{n, j}\left(t, \frac{x-\alpha_{n, j}}{\lambda_{n, j}}\right) e^{i \theta_{n, j}}+\lambda_{0, j}^{-\frac{d}{2}}\left(Q_{n, j}\left(t, \frac{x-\alpha_{n, j}}{\lambda_{n, j}}\right)-Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\right) e^{i \theta_{n, j}} \\
& +\lambda_{0, j}^{-\frac{d}{2}}\left(Q_{n, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)-Q_{0, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\right) e^{i \theta_{n, j}}+\lambda_{0, j}^{-\frac{d}{2}} Q_{0, j}\left(t, \frac{x-\alpha_{0, j}}{\lambda_{0, j}}\right)\left(e^{i \theta_{n, j}}-e^{i \theta_{0, j}}\right) \\
= & T_{1}+T_{2}+T_{3}+T_{4}
\end{aligned}
$$

Note that

$$
\left\|T_{1}\right\|_{\dot{H}^{\frac{3}{2}}}=\left(\lambda_{n, j}^{-\frac{d}{2}}-\lambda_{0, j}^{-\frac{d}{2}}\right) \lambda_{n, j}^{\frac{d}{2}-\frac{3}{2}}\|Q\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\kappa-\frac{3}{2}+\zeta} .
$$

Moreover, for $T_{2}$, we have
$\left\|T_{2}\right\|_{\dot{H}^{\frac{3}{2}}}=\lambda_{0, j}^{-\frac{d}{2}}\left\|\xi^{\frac{3}{2}}\left(\lambda_{n, j}^{d} e^{-i \alpha_{n, j} \xi} \widehat{Q_{n, j}}\left(\lambda_{n, j} \xi\right)-\lambda_{0, j}^{d} e^{-i \alpha_{0, j} \xi} \widehat{Q_{n, j}}\left(\lambda_{0, j} \xi\right)\right)\right\|_{L^{2}} \leq C(T-t)^{\frac{1}{2}(\kappa-3)+\zeta}$.
Regarding $T_{3}$ and $T_{4}$ we have the bounds

$$
\left\|T_{3}\right\|_{\dot{H}^{\frac{3}{2}}} \leq C \lambda_{0, j}^{-\frac{3}{2}}\left(\left|\beta_{n, j}-\beta_{0, j}\right|+\left|\gamma_{n, j}-\gamma_{0, j}\right|\right) \leq C(T-t)^{\frac{1}{2}(\kappa-1)+\zeta},
$$

and

$$
\left\|T_{4}\right\|_{\dot{H}^{\frac{3}{2}}}=\lambda_{0, j}^{-\frac{3}{2}}\left|\theta_{n, j}-\theta_{0, j}\right|\|Q\|_{\dot{H}^{\frac{3}{2}}} \leq C(T-t)^{\kappa-\frac{5}{2}+\zeta} .
$$

Thus, putting the estimates above altogether we obtain (7.20), and thus (7.17) follows.
In view of (6.17) and (7.17), we also infer that

$$
\begin{equation*}
u_{n}(t)-S_{T}(t) \rightharpoonup u(t)-S_{T}(t), \text { weakly in } \dot{H}^{\frac{3}{2}}, \text { as } n \rightarrow \infty \tag{7.21}
\end{equation*}
$$

which yields (7.18) immediately. Moreover, the strong convergence (7.19) in $H^{1}$ follows from the strong $L^{2}$ convergence (6.17), the uniform $H^{\frac{3}{2}}$ boundedness (7.2) and standard interpolation arguments. Therefore, the proof is complete.

Below we show that the constructed blow-up solution $u$ indeed admits the geometrical decomposition as in Proposition 4.1 on the maximal existing time interval $[0, T)$.

For each $0<t<T$ fixed, Lemma 5.3 yields that the derivatives of modulation parameters $\dot{\mathcal{P}}_{n}$ are uniformly bounded on $[0, t]$, and thus $\mathcal{P}_{n}$ are equicontinuous on $[0, t], n \geq 1$. Then, by the Arzelà-Ascoli Theorem, $\mathcal{P}_{n}$ converges uniformly on $[0, t]$ up to some subsequence (which may depend on $t$ ). But, using the diagonal arguments one may extract a universal subsequence (still denoted by $\{n\})$ such that for some $\mathcal{P}:=\left(\mathcal{P}_{1}, \cdots, \mathcal{P}_{K}\right)$,
where $\mathcal{P}_{j}:=\left(\lambda_{j}, \alpha_{j}, \beta_{j}, \gamma_{j}, \theta_{j}\right) \in C\left([0, t] ; R^{2 d+3}\right), 1 \leq j \leq K$, and for every $t \in(0, T)$, one has

$$
\begin{equation*}
\mathcal{P}_{n} \rightarrow \mathcal{P} \text { in } C\left([0, t] ; R^{(2 d+3) K}\right) . \tag{7.22}
\end{equation*}
$$

Then, taking into account the uniform estimates (5.1)-(5.4) we obtain that for each $t \in[0, T)$ and for $1 \leq j \leq K$,

$$
\begin{align*}
& \left|\lambda_{j}(t)-\omega_{j}(T-t)\right|+\left|\gamma_{j}(t)-\omega_{j}^{2}(T-t)\right| \leq(T-t)^{\kappa+1+\zeta}  \tag{7.23}\\
& \left|\alpha_{j}(t)-x_{j}\right|+\left|\beta_{j}(t)\right| \leq(T-t)^{\frac{\kappa}{2}+1+\zeta}  \tag{7.24}\\
& \left|\theta_{j}(t)-\left(\omega_{j}^{-2}(T-t)^{-1}+\vartheta_{j}\right)\right| \leq(T-t)^{\kappa-1+\zeta} \tag{7.25}
\end{align*}
$$

In particular, as in Remark 5.2, $\lambda_{j}, \gamma_{j}, P$ are comparable to $T-t$, i.e., there exist $C_{1}, C_{2}>$ 0 , independent of $\varepsilon, T$, such that

$$
\begin{equation*}
C_{1}(T-t) \leq \lambda_{j}, \gamma_{j}, P \leq C_{2}(T-t), \quad \forall 0 \leq t<T \tag{7.26}
\end{equation*}
$$

Let

$$
\begin{equation*}
U(t, x):=\sum_{j=1}^{K} \lambda_{j}^{-\frac{d}{2}} Q_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right) e^{i \theta_{j}}\left(:=\sum_{j=1}^{K} U_{j}(t, x)\right) \tag{7.27}
\end{equation*}
$$

with

$$
\begin{equation*}
Q_{j}(t, y):=Q(y) e^{i\left(\beta_{j}(t) \cdot y-\frac{1}{4} \gamma_{j}(t)|y|^{2}\right)} \tag{7.28}
\end{equation*}
$$

define $\varrho_{j}$ as in (4.6), and define the remainder $R$ by

$$
\begin{equation*}
R(t, x):=u(t, x)-U(t, x), \quad 0 \leq t<T, x \in \mathbb{R}^{d} . \tag{7.29}
\end{equation*}
$$

Then, for each $0<t<T$, using the explicit expression (7.27) of $U$ and the convergence (7.22) of mudulation parameters, we infer that $\langle x\rangle^{2} U_{n} \rightarrow\langle x\rangle^{2} U$ in $C\left([0, t] ; H^{1}\right), U_{n} \rightarrow U$ in $C\left([0, t] ; H^{\frac{3}{2}}\right), \Lambda U_{n} \rightarrow \Lambda U$ in $C\left([0, t] ; L^{2}\right)$, and $\varrho_{n, j} \rightarrow \varrho_{j}$ in $C\left([0, t] ; L^{2}\right)$. Then, in view of (6.17) and (7.21), we obtain that $R_{n} \rightarrow R$ in $C\left([0, t] ; L^{2}\right)$ and $R_{n}(t) \rightharpoonup R(t)$ weakly in $H^{\frac{3}{2}}$, which by interpolation yields that $R_{n}(t) \rightarrow R(t)$ in $H^{1}$.

Hence, in view of (5.1) and Proposition 7.1, we get that for $T=T(M)$ small enough satisfying (7.1) and for any $t \in[0, T)$,

$$
\begin{equation*}
\|R(t)\|_{L^{2}} \leq(T-t)^{\kappa+1}, \quad\|R(t)\|_{H^{1}} \leq(T-t)^{\kappa}, \quad\|R(t)\|_{H^{\frac{3}{2}}} \leq(T-t)^{\kappa-2} \tag{7.30}
\end{equation*}
$$

Furthermore, the following orthogonality conditions hold on $[0, T)$ for each $1 \leq j \leq K$ :

$$
\begin{align*}
& \operatorname{Re} \int\left(x-\alpha_{j}\right) U_{j} \bar{R} d x=0, \quad \operatorname{Re} \int\left|x-\alpha_{j}\right|^{2} U_{j} \bar{R} d x=0 \\
& \operatorname{Im} \int \nabla U_{j} \bar{R} d x=0, \quad \operatorname{Im} \int \Lambda U_{j} \bar{R} d x=0, \quad \operatorname{Im} \int \varrho_{j} \bar{R} d x=0 . \tag{7.31}
\end{align*}
$$

We also see that the modulation parameters in $\mathcal{P}$ are $C^{1}$ functions. Actually, similar arguments as in the proof of Proposition 4.3 show that the modulation equations $\operatorname{Mod}_{n}$ can be expressed in terms of the inner products of polynomials of $\partial^{\nu} U_{n}$ and $R_{n}$, where $|\nu| \leq 2$. Hence, the convergence of $\partial^{\nu} U_{n}$ and $R_{n}$ also yields that of $\mathcal{P}_{n}$, which in turn yields the desirable $C^{1}$-regularity of the modulation parameter $\mathcal{P}$.

Thus, similarly to Lemma 5.3, we have
Lemma 7.3. There exists $C>0$ such that

$$
\begin{equation*}
\operatorname{Mod}(t) \leq C \underset{49}{C(T-t)^{\kappa+3}}, \quad t \in[0, T) \tag{7.32}
\end{equation*}
$$

7.2. Energy estimate of the difference. Below we assume Assumption (A1) with $\nu_{*} \geq 11$. Let $v$ be any blow-up solution to (2.11) satisfying (2.16), i.e.,

$$
\begin{equation*}
\left\|v(t)-\sum_{j=1}^{K} S_{j}(t)\right\|_{H^{1}} \leq C(T-t)^{3+\zeta}, \quad 0<t<T \tag{7.33}
\end{equation*}
$$

Set

$$
\begin{equation*}
w:=v-u=\sum_{j=1}^{K} w_{j}, \text { with } w_{j}:=w \Phi_{j}, \tag{7.34}
\end{equation*}
$$

where the localization functions $\Phi_{j}$ are given by (3.2), $1 \leq j \leq K$. Since $\kappa:=\nu_{*}-3 \geq 8$, it follows from (2.16) and (7.33) that

$$
\begin{equation*}
\|w(t)\|_{H^{1}} \leq C(T-t)^{3+\zeta} \tag{7.35}
\end{equation*}
$$

For $1 \leq j \leq K$, define the renormalized variable $\epsilon_{j}$ by

$$
\begin{equation*}
w_{j}(t, x):=\lambda_{j}(t)^{-\frac{d}{2}} \epsilon_{j}\left(t, \frac{x-\alpha_{j}(t)}{\lambda_{j}(t)}\right) e^{i \theta_{j}(t)} \tag{7.36}
\end{equation*}
$$

Set

$$
\begin{equation*}
D(t):=\|\nabla w(t)\|_{L^{2}}^{2}+\sum_{j=1}^{K} \frac{\left\|w_{j}(t)\right\|_{L^{2}}^{2}}{\lambda_{j}^{2}(t)} . \tag{7.37}
\end{equation*}
$$

We note that

$$
\begin{equation*}
\left\|w_{j}(t)\right\|_{L^{2}} \leq C(T-t) \sqrt{D(t)}, \quad\left\|w_{j}(t)\right\|_{H^{1}} \leq C \sqrt{D(t)}, \quad 1 \leq j \leq K \tag{7.38}
\end{equation*}
$$

Moreover, as in (3.15), we set $\operatorname{Scal}_{j}(t):=\operatorname{Scal}\left(\epsilon_{j}\right)$, i.e., for $\epsilon_{j, 1}:=\operatorname{Re} \epsilon_{j}, \epsilon_{j, 2}:=\operatorname{Im} \epsilon_{j}$,

$$
\begin{equation*}
\left.\operatorname{Scal}_{j}(t):=\left\langle\epsilon_{j, 1}, Q\right\rangle^{2}+\left\langle\epsilon_{j, 1}, y Q\right\rangle^{2}+\left.\left\langle\epsilon_{j, 1},\right| y\right|^{2} Q\right\rangle^{2}+\left\langle\epsilon_{j, 2}, \nabla Q\right\rangle^{2}+\left\langle\epsilon_{j, 2}, \Lambda Q\right\rangle^{2}+\left\langle\epsilon_{j, 2}, \rho\right\rangle^{2}, \tag{7.39}
\end{equation*}
$$

which actually measures the deviations of the remainder term $\epsilon_{j}$ with respect to the six instable directions of the linearized operator $L$.

By equation (2.11), $w$ satisfies the equation

$$
\begin{equation*}
i \partial_{t} w+\Delta w+f(u+w)-f(u)+b \cdot \nabla w+c w=0, \quad t \in(0, T) \tag{7.40}
\end{equation*}
$$

and $\lim _{t \rightarrow T}\|w(t)\|_{H^{1}}=0$, due to (7.35).
The strategy to prove that $w \equiv 0$ is to show that $D \equiv 0$. As in the proof of the bootstrap estimate of remainder in Section 5 above, such result will be derived from the estimate of a generalized energy. This leads to the definition of $\widetilde{I}$ below

$$
\begin{align*}
\widetilde{I}:= & \frac{1}{2} \int|\nabla w|^{2} d x+\frac{1}{2} \sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \int|w|^{2} \Phi_{j} d x-\operatorname{Re} \int F(u+w)-F(u)-f(u) \bar{w} d x \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla w \bar{w} \Phi_{j} d x . \tag{7.41}
\end{align*}
$$

Note that, $u$ and $w$ play similar roles as $U$ and $R$ in (5.28), respectively.
Lemma 7.4 relates the generalized energy $\underset{50}{ } \widetilde{I}(t)$ and the two quantities $D(t)$ and $S c a l_{j}(t)$.

Lemma 7.4. For $t \in[0, T)$, there exist $C_{1}, C_{2}, C_{3}>0$ such that

$$
\begin{equation*}
C_{1} D(t)-C_{2} \sum_{j=1}^{K} \frac{S c a l_{j}(t)}{\lambda_{j}^{2}} \leq \widetilde{I}(t) \leq C_{3} D(t) \tag{7.42}
\end{equation*}
$$

Proof. We first show that, in the formulation of the generalized energy $\widetilde{I}$, one may replace the blow-up solution $u$ with the main blow-up profile $U$ given by (7.27), at the cost of the error $\mathcal{O}((T-t) D(t))$, i.e.,

$$
\begin{align*}
& \operatorname{Re} \int F(u+w)-F(u)-f(u) \bar{w} d x \\
= & \operatorname{Re} \int F(U+w)-F(U)-f(U) \bar{w} d x+\mathcal{O}((T-t) D(t)) . \tag{7.43}
\end{align*}
$$

In order to prove (7.43), using Taylor's expansion (3.21) we see that

$$
\begin{align*}
& \operatorname{Re} \int F(u+w)-F(u)-f(u) \bar{w} d x  \tag{7.44}\\
= & \operatorname{Re} \int F(U+w)-F(U)-f(U) \bar{w} d x+\mathcal{O}\left(\int\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R||w|^{2} d x\right) .
\end{align*}
$$

By (3.29), (5.39) and (7.30),

$$
\begin{equation*}
\int|U|^{\frac{4}{d}-1}\left|R\left\|\left.w\right|^{2} d x \leq C(T-t)^{-\frac{d}{2}\left(\frac{4}{d}-1\right)}\right\| R\left\|_{L^{2}}\right\| w\left\|_{H^{1}}^{2} \leq C(T-t)\right\| w \|_{H^{1}}^{2}\right. \tag{7.45}
\end{equation*}
$$

Moreover, by (3.29), (7.30) and (7.35),

$$
\begin{equation*}
\int\left(|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)\left|R\left\|\left.w\right|^{2} d x \leq C\left(\|R\|_{H^{1}}\|w\|_{H^{1}}^{\frac{4}{d}+1}+\|R\|_{H^{1}}^{\frac{4}{d}}\|w\|_{H^{1}}^{2}\right) \leq C(T-t)\right\| w \|_{H^{1}}^{2} .\right. \tag{7.46}
\end{equation*}
$$

Hence, plugging (7.45) and (7.46) into (7.44) and using (7.38) we obtain (7.43), as claimed.
Next we analyze the right-hand-side of (7.43). Note that, by (3.21),

$$
\begin{align*}
\operatorname{Re}(F(U+w)-F(U)-f(U) \bar{w})= & \frac{1}{2}\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|w|^{2}+\frac{1}{d}|U|^{\frac{4}{d}-2} \operatorname{Re}\left(U^{2} \bar{w}^{2}\right) \\
& +\mathcal{O}\left(\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^{3}\right) . \tag{7.47}
\end{align*}
$$

Note that

$$
\begin{equation*}
\int\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^{3} d x \leq C(T-t)^{-2}\left(\|w\|_{H^{1}}^{3}+\|w\|_{H^{1}}^{2+\frac{4}{d}}\right) \leq C(T-t) D(t) \tag{7.48}
\end{equation*}
$$

Moreover, a direct application of Hölder's inequality also shows that the last term on the right-hand-side of (7.41) is bounded by

$$
\begin{equation*}
C\|w\|_{L^{2}}\|\nabla w\|_{L^{2}} \leq C(T-t) D(t) . \tag{7.49}
\end{equation*}
$$

Thus, we conclude from (7.43), (7.47), (7.48) and (7.49) that

$$
\begin{align*}
\widetilde{I}= & \frac{1}{2} \operatorname{Re} \int|\nabla w|^{2}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}}|w|^{2} \Phi_{j}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|w|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{w}^{2} d x \\
& +\mathcal{O}((T-t) D(t)) . \tag{7.50}
\end{align*}
$$

Arguing as in the proof of (5.79) we have that for some $C_{1}, C_{2}>0$,

$$
\begin{aligned}
\widetilde{I} \geq & C_{1}\left(\int|\nabla w|^{2}+\sum_{j=1}^{K} \lambda_{j}^{-2}|w|^{2} \Phi_{j} d x\right) \\
& -C_{2}\left((T-t) D+(T-t)^{-1}\|w\|_{L^{2}}^{2}+\sum_{j=1}^{K} \lambda_{j}^{-2} S c a l_{j}+e^{-\frac{\delta}{T-t}}\|w\|_{H^{1}}^{2}\right) .
\end{aligned}
$$

Then, taking $T$ small enough such that $C_{2}\left(2(T-t)+e^{-\frac{\delta}{T-t}}\right) \leq \frac{1}{2} C_{1}, \forall t \in[0, T]$, and taking into account (7.37) and $\Phi_{j} \geq \Phi_{j}^{2}$, we get

$$
\begin{aligned}
\widetilde{I} & \geq C_{1}\left(\int|\nabla w|^{2}+\sum_{j=1}^{K} \lambda_{j}^{-2}|w|^{2} \Phi_{j} d x\right)-\frac{1}{2} C_{1} D(t)-C_{2} \sum_{j=1}^{K} \lambda_{j}^{-2}(t) S c a l_{j}(t) \\
& \geq \frac{1}{2} C_{1} D(t)-C_{2} \sum_{j=1}^{K} \lambda_{j}^{-2}(t) S c a l_{j}(t) .
\end{aligned}
$$

But, since $|U(t)| \leq C(T-t)^{-\frac{d}{2}}$, using Hölder's inequality, (7.37) and (7.50) we also have

$$
\widetilde{I}(t) \leq C D(t)
$$

Therefore, combining two estimates together we obtain (7.42) and finish the proof.
Similarly to Theorem 5.9, we have the monotonicity property of $\frac{d \widetilde{I}}{d t}$.

Theorem 7.5. There exist $C_{1}, C_{2}, C_{3}>0$ such that for any $t \in[0, T)$,

$$
\begin{align*}
\frac{d \widetilde{I}}{d t} \geq & \sum_{j=1}^{K} \frac{C_{1}}{\lambda_{j}} \int\left(\left|\nabla w_{j}\right|^{2}+\frac{1}{\lambda_{j}^{2}}\left|w_{j}\right|^{2}\right) e^{-\frac{\left|x-\alpha_{j}\right|}{A \lambda_{j}}} d x \\
& -C\left(D(t)+\sum_{j=1}^{K} \frac{S c a l_{j}(t)}{\lambda_{j}^{3}(t)}\right)-C_{3} \varepsilon^{*} \frac{D(t)}{T-t} \tag{7.51}
\end{align*}
$$

Proof. Similarly to (5.33) and (5.48), using equation (7.40) we compute

$$
\begin{align*}
\frac{d \widetilde{I}}{d t}= & -\sum_{j=1}^{K} \frac{\dot{\lambda}_{j}}{\lambda_{j}^{3}} \operatorname{Im} \int|w|^{2} \Phi_{j} d x-\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime}(u) \cdot w, w_{j}\right\rangle-\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} u\right\rangle \\
& -\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle w \nabla \Phi_{j}, \nabla w\right\rangle-\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, w_{j}\right\rangle \\
& -\sum_{j=1}^{K} \operatorname{Im}\left\langle\Delta w-\frac{1}{\lambda_{j}^{2}} w_{j}+f(u+w)-f(u), b \cdot \nabla w+c w\right\rangle \\
& -\sum_{j=1}^{K} \frac{\dot{\lambda}_{j} \gamma_{j}-\lambda_{j} \dot{\gamma}_{j}}{2 \lambda_{j}^{2}} \operatorname{Im}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot \nabla w, w_{j}\right\rangle \\
& +\sum_{j=1}^{K} \frac{\gamma_{j}}{2 \lambda_{j}} \operatorname{Im}\left\langle\partial_{t}\left(\nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\right) \cdot \nabla w, w_{j}\right\rangle \\
& +\sum_{j=1}^{K} \operatorname{Im}\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) w_{j}+\frac{\gamma_{j}}{2 \lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla w_{j}+\nabla w \Phi_{j}\right), \partial_{t} w\right\rangle \\
= & \sum_{j=1}^{9} \widetilde{I}_{t, j} . \tag{7.52}
\end{align*}
$$

Below we replace the each appearance of $u$ by the blow-up profile $U$ in the terms $\widetilde{I}_{t, 2}$, $\widetilde{I}_{t, 3}, \widetilde{I}_{t, 5}, \widetilde{I}_{t, 6}$ and $\widetilde{I}_{t, 9}$.
(i) Estimate of $\widetilde{I}_{t, 2}$. Since by (3.18),

$$
\begin{equation*}
\left|f^{\prime}(u) \cdot w-f^{\prime}(U) \cdot w\right| \leq C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|R||w|, \tag{7.53}
\end{equation*}
$$

using Lemma 3.1 we get

$$
\left|\widetilde{I}_{t, 2}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime}(U) \cdot w, w_{j}\right\rangle\right| \leq C \sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \int\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|R \| w|\left|w_{j}\right| d x
$$

By Hölder's inequality, (3.28), (5.39), (5.43) and (7.30), we have that for $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, $\frac{1}{\lambda_{j}^{2}} \int|U|^{\frac{4}{d}-1}\left|R\left\|w\left|\left\|w_{j} \left\lvert\, d x \leq(T-t)^{-4+\frac{d}{2}}\right.\right\| R\left\|_{L^{p}}\right\| w\left\|_{L^{q}}\right\| w_{j}\left\|_{L^{2}} \leq(T-t)^{\kappa-3+\frac{d}{p}}\right\| w\left\|_{L^{2}}^{2-\frac{d}{p}}\right\| \nabla w \|_{L^{2}}^{\frac{d}{p}}\right.\right.\right.$. Then, by Young's inequality $a b \leq \frac{a^{\tilde{p}}}{\widetilde{p}}+\frac{b \tilde{p}^{{ }^{\prime}}}{\widetilde{p}^{\prime}}$ with $\widetilde{p}=\frac{2 p}{d}, \frac{1}{\widetilde{p}}+\frac{1}{\widetilde{p}^{\prime}}=1$ and $0<\frac{d}{p}<1$,

$$
\frac{1}{\lambda_{j}^{2}} \int|U|^{\frac{4}{d}-1}\left|R\|w\| w_{j}\right| d x \leq(T-t)^{(\kappa-3) \widetilde{p}+d}\left\|w_{j}\right\|_{L^{2}}^{2}+\left\|\nabla w_{j}\right\|_{L^{2}}^{2} \leq C D(t)
$$

Moreover, by (3.29) and (7.30),

$$
\frac{1}{\lambda_{j}^{2}} \int|R|^{\frac{4}{d}}\left|w_{j}\left\|w \mid d x \leq C(T-t)^{-2}\right\| R\left\|_{H^{1}}^{\frac{4}{d}}\right\| w\left\|_{H^{1}}^{2} \leq C\right\| w \|_{H^{1}}^{2} \leq C D(t),\right.
$$

and

$$
\frac{1}{\lambda_{j}^{2}} \int|w|^{\frac{4}{d}}|R|\left|w_{j}\right| d x \leq C(T-t)^{-2}\|R\|_{L^{2}}\|w\|_{H^{1}}^{1+\frac{4}{d}} \leq C\|w\|_{H^{1}}^{2} \leq C D(t)
$$

Hence, we conclude that

$$
\begin{equation*}
\widetilde{I}_{t, 2}=-\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime}(U) \cdot w, w_{j}\right\rangle+\mathcal{O}(D(t)) \tag{7.54}
\end{equation*}
$$

(ii) Estimate of $\widetilde{I}_{t, 3}$. Since by (7.29),

$$
\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} u\right\rangle=\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} U\right\rangle+\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} R\right\rangle
$$

we shall treat the two terms on the right-hand side separately below.
First, using (3.19) we see that

$$
\begin{equation*}
\left|f^{\prime \prime}(u, w) \cdot w^{2}-f^{\prime \prime}(U, w) \cdot w^{2}\right| \leq C\left(|U|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R||w|^{2}, \tag{7.55}
\end{equation*}
$$

which yields that

$$
\begin{align*}
& \left|\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} U\right\rangle-\operatorname{Re}\left\langle f^{\prime \prime}(U, w) \cdot w^{2}, \partial_{t} U\right\rangle\right| \\
\leq & C \sum_{j=1}^{K}\left\|\partial_{t} U_{j}\right\|_{L^{\infty}} \int\left(|U|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R||w|^{2} d x \tag{7.56}
\end{align*}
$$

Note that, by (4.14), $\left\|\partial_{t} U_{j}\right\|_{L^{\infty}} \leq C(T-t)^{-2-\frac{d}{2}}$, and by Gagliardo-Nirenberg's inequality (3.28),
$\left\|\partial_{t} U_{j}\right\|_{L^{\infty}} \int|U|^{\frac{4}{d}-2}\left|R\left\|\left.w\right|^{2} d x \leq C(T-t)^{-4+\frac{d}{2}}\right\| R\left\|_{L^{2}}\right\| w\left\|_{L^{4}}^{2} \leq C(T-t)^{\kappa+\frac{d}{2}-3}\right\| w\left\|_{L^{2}}^{2-\frac{d}{2}}\right\| \nabla w \|_{L^{2}}^{\frac{d}{2}}\right.$,
which, via Young's inequality $a b \leq \frac{a^{p}}{p}+\frac{b^{q}}{q}$ with $p=\frac{4}{4-d}$ and $q=\frac{4}{d}$, can be bounded by

$$
C(T-t)^{\left(\kappa+\frac{d}{2}-3\right) \frac{4}{d-2}}\|w\|_{L^{2}}^{2}+\|\nabla w\|_{L^{2}}^{2} \leq C(T-t)^{-2}\|w\|_{L^{2}}^{2}+\|\nabla w\|_{L^{2}}^{2} \leq C D(t)
$$

Moreover, by Hölder's inequality, (3.31), (5.43), (7.30) and (7.35),

$$
\begin{aligned}
\left\|\partial_{t} U_{j}\right\|_{L^{\infty}} \int\left(|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R \| w|^{2} d x & \leq C(T-t)^{-2-\frac{d}{2}}\left(\|R\|_{L^{\frac{8}{d}-2}}^{\frac{4}{d}-1}\|w\|_{H^{1}}^{2}+\|R\|_{L^{2}}\|w\|_{H^{1}}^{\frac{4}{d}}\right) \\
& \leq C\|w\|_{H^{1}}^{2} \leq C D(t)
\end{aligned}
$$

Hence, we obtain

$$
\begin{equation*}
\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} U\right\rangle=\operatorname{Re}\left\langle f^{\prime \prime}(U, w) \cdot w^{2}, \partial_{t} U\right\rangle+\mathcal{O}(D(t)) \tag{7.57}
\end{equation*}
$$

Next we show that

$$
\begin{equation*}
\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} R\right\rangle=\mathcal{O}(D(t)) \tag{7.58}
\end{equation*}
$$

For this purpose, using equation (5.15) we get

$$
\left|\operatorname{Re}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \partial_{t} R\right\rangle\right|=\left|\operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \Delta R+f(u)-f(U)+b \cdot \nabla R+c R+\eta\right\rangle\right| .
$$

Note that, by (7.30),

$$
\begin{align*}
\left|\operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \Delta R\right\rangle\right| & \leq C\|R\|_{\dot{H}^{\frac{3}{2}}}\left\|f^{\prime \prime}(u, w) \cdot w^{2}\right\|_{\dot{H}^{\frac{1}{2}}} \\
& \leq C(T-t)^{\kappa-2}\left\|f^{\prime \prime}(u, w) \cdot w^{2}\right\|_{\dot{H}^{\frac{1}{2}}} . \tag{7.59}
\end{align*}
$$

Using (3.32), (7.30) and $\|U(t)\|_{H^{1}} \leq C(T-t)^{-1}$ we get

$$
\begin{align*}
& \left\|f^{\prime \prime}(u, w) \cdot w^{2}\right\|_{H^{\frac{1}{2}}} \leq C \sum_{j=2}^{1+\frac{4}{d}}\|u\|_{H^{1}}^{1+\frac{4}{d}-j}\|w\|_{H^{1}}^{j} \leq C \sum_{j=2}^{1+\frac{4}{d}}\left(\|U\|_{H^{1}}^{1+\frac{4}{d}-j}+\|R\|_{H^{1}}^{1+\frac{4}{d}-j}\right)\|w\|_{H^{1}}^{j} \\
& \quad \leq C \sum_{k=2}^{1+\frac{4}{d}}\left((T-t)^{-\left(1+\frac{4}{d}-j\right)}+(T-t)^{\kappa\left(1+\frac{4}{d}-j\right)}\right)(T-t)^{(3+\zeta)(j-2)}\|w\|_{H^{1}}^{2} \\
& \quad \leq C(T-t)^{1-\frac{4}{d}}\|w\|_{H^{1}}^{2}, \tag{7.60}
\end{align*}
$$

$$
\left|\operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \Delta R\right\rangle\right| \leq C\|w\|_{H^{1}}^{2} \leq C D(t) .
$$

Moreover, since

$$
\begin{equation*}
|f(u)-f(U)| \leq C\left(|U|^{\frac{4}{d}}+|R|^{\frac{4}{d}}\right)|R|, \tag{7.62}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|f^{\prime \prime}(u, w) \cdot w^{2}\right| \leq C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^{2}, \tag{7.63}
\end{equation*}
$$

by (3.31) and (7.30) we get

$$
\begin{equation*}
\left|\operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, f(u)-f(U)+b \cdot \nabla R+c R\right\rangle\right| \leq C\|w\|_{H^{1}}^{2} \leq C D(t) . \tag{7.64}
\end{equation*}
$$

Furthermore, using (3.31), (5.17) and (7.63) again we also have

$$
\begin{align*}
\left|\operatorname{Im}\left\langle f^{\prime \prime}(u, w) \cdot w^{2}, \eta\right\rangle\right| & \leq C\left((T-t)^{-2+\frac{d}{2}}+\|R\|_{H^{1}}^{\frac{4}{d}-1}+\|w\|_{H^{1}}^{\frac{4}{d}-1}\right)\|\eta\|_{L^{2}}\|w\|_{H^{1}}^{2} \\
& \leq C\|w\|_{H^{1}} \leq C D(t) . \tag{7.65}
\end{align*}
$$

Thus, combining estimates (7.61), (7.64) and (7.65) we obtain (7.58), as claimed.
Therefore, we infer from (7.57) and (7.65) that

$$
\begin{equation*}
\widetilde{I}_{t, 3}=\operatorname{Re}\left\langle f^{\prime \prime}(U, w) \cdot w^{2}, \partial_{t} U\right\rangle+\mathcal{O}(D(t)) . \tag{7.66}
\end{equation*}
$$

(iii) Estimate of $\widetilde{I}_{t, 5}$. Using (7.55) we have

$$
\left|\widetilde{I}_{t, 5}+\sum_{j=1}^{K} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime \prime}(U, w) \cdot w^{2}, w_{j}\right\rangle\right| \leq C(T-t)^{-2} \int\left(|U|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R||w|^{2}\left|w_{j}\right| d x .
$$

By Hölder's inequality, (3.29), (5.39), (7.30) and (7.35),

$$
\begin{aligned}
(T-t)^{-2} \int|U|^{\frac{4}{d}-2}|R||w|^{2}\left|w_{j}\right| d x & \leq(T-t)^{-2-\frac{d}{2}\left(\frac{4}{d}-2\right)}\|R\|_{L^{2}}\|w\|_{H^{1}}^{3} \\
& \leq C(T-t)^{d+\kappa+\zeta}\|w\|_{H^{1}}^{2} \leq C D(t) .
\end{aligned}
$$

Similarly, using (3.29), (7.30) and (7.35) we also have

$$
\begin{aligned}
(T-t)^{-2} \int\left(|R|^{\frac{4}{d}-2}+|w|^{\frac{4}{d}-2}\right)|R \| w|^{2}\left|w_{j}\right| d x & \leq C(T-t)^{-2}\left(\|R\|_{H^{1}}^{\frac{4}{d}-1}\|w\|_{H^{1}}^{3}+\|R\|_{L^{2}}\|w\|_{H^{1}}^{\frac{4}{d}+1}\right) \\
& \leq C D(t) .
\end{aligned}
$$

Hence, we obtain

$$
\begin{equation*}
\widetilde{I}_{t, 5}=-\sum_{j=1}^{N} \frac{1}{\lambda_{j}^{2}} \operatorname{Im}\left\langle f^{\prime \prime}(U, w) \cdot w^{2}, w_{j}\right\rangle+\mathcal{O}(D(t)) . \tag{7.67}
\end{equation*}
$$

(iv) Estimate of $\widetilde{I}_{t, 6}$. Since by (3.20),

$$
\begin{equation*}
|f(u+w)-f(u)-(f(U+w)-f(U))| \leq C\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R||w|, \tag{7.68}
\end{equation*}
$$ taking into account (3.29) we infer that

$$
\begin{aligned}
& \operatorname{Im}\langle f(u+w)-f(u)-(f(U+w)-f(U)),(b \cdot \nabla+c) w\rangle \\
\leq & C \int\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R\|w\|(b \cdot \nabla+c) w| d x \\
\leq & C\left[(T-t)^{-\frac{d}{2}\left(\frac{4}{d}-1\right)}+\|w\|_{H^{1}}^{\frac{4}{d}-1}+\|R\|_{H^{1}}^{\frac{4}{d}-1}\right]\|R\|_{H^{1}}\|w\|_{H^{1}}\|(b \cdot \nabla+c) w\|_{L^{2}} \\
\leq & C\|w\|_{H^{1}}^{2} \leq C D(t) .
\end{aligned}
$$

This yields that

$$
\begin{equation*}
\widetilde{I}_{t, 6}=-\sum_{j=1}^{K} \operatorname{Im}\left\langle\Delta w-\frac{1}{\lambda_{j}^{2}} w_{j}+f(U+w)-f(U), b \cdot \nabla w+c w\right\rangle+\mathcal{O}(D(t)) . \tag{7.69}
\end{equation*}
$$

(v) Estimate of $\widetilde{I}_{t, 9}$. The arguments are similar to those in the previous cases $(i)$ and (iv). Actually, using equation (7.40) we infer that

$$
\begin{gathered}
\widetilde{I}_{t, 9}=\operatorname{Im}\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) w_{j}+\frac{\gamma_{j}}{2 \lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla w_{j}+\nabla w \Phi_{j}\right),\right. \\
i \Delta w+i(f(u+w)-f(u))+i(b \cdot \nabla+c) w\rangle .
\end{gathered}
$$

Then, in view of (7.68) we see that

$$
\begin{aligned}
& \left\lvert\,\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) w_{j}+\frac{\gamma_{j}}{2 \lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla w_{j}+\nabla w \Phi_{j}\right), i(f(u+w)-f(u))\right\rangle\right. \\
& \left.\quad-\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) w_{j}+\frac{\gamma_{j}}{2 \lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla w_{j}+\nabla w \Phi_{j}\right), i(f(U+w)-f(U))\right\rangle \right\rvert\, \\
& \leq C(A) \int(T-t)^{-1}\left|w_{j}\right|\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R||w| d x \\
& \quad+C(A) \int\left|\nabla w_{j}+\nabla w \Phi_{j}\right|\left(|U|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R||w| d x .
\end{aligned}
$$

Arguing as in the proof of (7.54) and (7.69), respectively, we can also bound the two integrations on the right-hand side above by $C D(t)$. This yields that

$$
\begin{gather*}
\widetilde{I}_{t, 9}=\operatorname{Im}\left\langle\frac{\gamma_{j}}{2 \lambda_{j}^{2}} \Delta \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) w_{j}+\frac{\gamma_{j}}{2 \lambda_{j}} \nabla \chi_{A}\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right) \cdot\left(\nabla w_{j}+\nabla w \Phi_{j}\right),\right. \\
i \Delta w+i(f(U+w)-f(U))+i(b \cdot \nabla+c) w\rangle+\mathcal{O}(D(t)) . \tag{7.70}
\end{gather*}
$$

At this stage, we have replaced $u$ by the blow-up profile $U$ in (7.52). Then, arguing as in the proof of Theorem 5.9) with $R$ replaced by $\omega$ and using (7.35) we obtain (7.51) and thus finish the proof.

As a consequence of Lemma 7.4 and Theorem 7.5 we have
Theorem 7.6. For $t \in[0, T)$, we have that

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq C\left(\sum_{j=1}^{K} \sup _{t \leq s<T} \frac{S c a l_{j}(s)}{\lambda_{j}^{2}(s)}+\int_{t}^{T} \sum_{j=1}^{K} \frac{S c a l_{j}(s)}{\lambda_{j}^{3}(s)}+\varepsilon^{*} \frac{D(s)}{T-s} d s\right) . \tag{7.71}
\end{equation*}
$$

Proof. By Lemma 7.4 and Theorem 7.5, for $t<\tilde{t}<T$,

$$
\begin{aligned}
C_{1} D(t) & \leq \widetilde{I}(t)+C_{2} \sum_{j=1}^{K} \frac{S c a l_{j}(t)}{\lambda_{j}^{2}(t)}=\widetilde{I}(\widetilde{t})+C_{2} \sum_{j=1}^{K} \frac{S c a l_{j}(t)}{\lambda_{j}^{2}(t)}-\int_{t}^{\tilde{t}} \frac{d \widetilde{I}}{d s}(s) d s \\
& \leq C\left(D(\widetilde{t})+\sum_{j=1}^{K} \frac{S c a l_{j}(t)}{\lambda_{j}^{2}(t)}+\int_{t}^{\tilde{t}} D(s)+\sum_{j=1}^{K} \frac{S c a l_{j}(s)}{\lambda_{j}^{3}(s)}+\varepsilon^{*} \frac{D(s)}{T-s} d s\right),
\end{aligned}
$$

which yields that
$\sup _{t \leq s \leq \tilde{t}} D(s) \leq C\left(D(\widetilde{t})+\sum_{j=1}^{K} \sup _{t \leq s \leq \tilde{t}} \frac{S c a l_{j}(s)}{\lambda_{j}^{2}(s)}+(\widetilde{t}-t) \sup _{t \leq s \leq \tilde{t}} D(s)+\int_{t}^{\tilde{t}} \sum_{j=1}^{K} \frac{S c a l_{j}(s)}{\lambda_{j}^{3}(s)}+\varepsilon^{*} \frac{D(s)}{T-s} d s\right)$.
By (7.35), $D(\widetilde{t}) \rightarrow 0$ as $\widetilde{t} \rightarrow T$. Hence, letting $\tilde{t} \rightarrow T$ and taking $T$ sufficiently small we obtain (7.71) and finish the proof.
7.3. Control of the null space. In view of Theorem 7.6, the last step is to control the scalar products in $S c a l j$, that is, the growth along six unstable directions in the null space. The key result is formulated in Theorem 7.7 below. Then, at the end of this subsection, we finish the proof of the main uniqueness result.

Theorem 7.7. For $T$ small enough and for $1 \leq j \leq K$, there exist $C>0, \zeta \in(0,1)$ such that

$$
\begin{equation*}
\operatorname{Scal}_{j}(t) \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} D(s) \tag{7.72}
\end{equation*}
$$

To begin with, we first treat the estimate of the scalar product $\operatorname{Re}\left\langle U_{j}, w\right\rangle$.
Proposition 7.8. For $1 \leq j \leq K$, we have

$$
\begin{equation*}
\left|\operatorname{Re} \int U_{j}(t) \bar{w}(t) d x\right| \leq C(T-t)^{4+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.73}
\end{equation*}
$$

Remark 7.9. One may also use equation (7.88) below to obtain the bound

$$
\begin{equation*}
\left|\operatorname{Re} \int U_{j}(t) \bar{w}(t) d x\right|=\left|\operatorname{Re} \int Q e_{j}(t) d y\right| \leq C(T-t)^{3+\zeta} \sup _{t \leq s<T} \sqrt{D(s)}, \tag{7.74}
\end{equation*}
$$

where $e_{j}$ is defined in (7.82) below. Estimate (17.73) improves (7.74) by a factor $(T-t)$, by exploring the conservation law of mass.

Proof of Proposition 7.8. We first note from (7.34), $v=u+w$, that

$$
\int|v(t)|^{2} \Phi_{j} d x=\int|u(t)|^{2} \Phi_{j} d x+\int|w(t)|^{2} \Phi_{j} d x+2 \operatorname{Re}\left\langle w_{j}, u\right\rangle .
$$

Taking into account $u=U+R$, Lemma (3.1) (7.30) and (7.35) we get, for $T$ small enough,

$$
\begin{aligned}
\operatorname{Re}\left\langle w_{j}, u\right\rangle & =\operatorname{Re}\left\langle w, U_{j}\right\rangle+\operatorname{Re}\left\langle w_{j}, R\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}}\right) \\
& =\operatorname{Re}\left\langle U_{j}, w\right\rangle+\mathcal{O}\left((T-t)^{\kappa+1}\|w\|_{L^{2}}\right) .
\end{aligned}
$$

Hence, taking into account (7.35) and $\kappa \geq 3$ we obtain

$$
\begin{align*}
\operatorname{Re}\left\langle U_{j}(t), w(t)\right\rangle= & \frac{1}{2}\left(\int|v(t)|^{2} \Phi_{j} d x-\int|u(t)|^{2} \Phi_{j} d x\right) \\
& -\frac{1}{2} \int|w(t)|^{2} \Phi_{j} d x+\mathcal{O}\left((T-t)^{\kappa+1}\|w(t)\|_{L^{2}}\right) \\
= & \frac{1}{2}\left(\int|v(t)|^{2} \Phi_{j} d x-\int|u(t)|^{2} \Phi_{j} d x\right)+\mathcal{O}\left((T-t)^{3+\zeta}\|w(t)\|_{L^{2}}\right) \tag{7.75}
\end{align*}
$$

In order to estimate the first term on the right-hand side above, we note that for $\tilde{t} \in(t, T)$,

$$
\begin{align*}
\int|v(t)|^{2} \Phi_{j} d x-\int|u(t)|^{2} \Phi_{j} d x= & \int_{\tilde{t}}^{t}\left(\frac{d}{d s} \int|v|^{2} \Phi_{j} d x-\frac{d}{d s} \int|u|^{2} \Phi_{j} d x\right) d s \\
& +\left(\int|v(\tilde{t})|^{2} \Phi_{j} d x-\int|u(\tilde{t})|^{2} \Phi_{j} d x\right) \tag{7.76}
\end{align*}
$$

Similarly to (5.20), we have

$$
\frac{d}{d t} \int|v|^{2} \Phi_{j} d x=\operatorname{Im} \int\left(2 \bar{v} \nabla v+b|v|^{2}\right) \cdot \nabla \Phi_{j} d x
$$

Similar equation also holds for $u$. Then, taking into account $v=u+w$ and $u=U+R$ we get

$$
\begin{align*}
& \frac{d}{d t} \int|v|^{2} \Phi_{j} d x-\frac{d}{d t} \int|u|^{2} \Phi_{j} d x \\
= & \operatorname{Im} \int\left(2(\bar{w} \nabla u+\bar{u} \nabla w)+2 \bar{w} \nabla w+b\left(\bar{w} u+\bar{u} w+|w|^{2}\right)\right) \cdot \nabla \Phi_{j} d x \\
= & \operatorname{Im} \int_{\left|x-x_{l}\right| \geq 4 \sigma, 1 \leq l \leq K}(2(\bar{w} \nabla U+\bar{U} \nabla w)+b(\bar{w} U+\bar{U} w)) \cdot \nabla \Phi_{j} d x \\
& +\operatorname{Im} \int\left(2(\bar{w} \nabla R+\bar{R} \nabla w)+2 \bar{w} \nabla w+b\left(\bar{w} R+\bar{R} w+|w|^{2}\right)\right) \cdot \nabla \Phi_{j} d x, \tag{7.77}
\end{align*}
$$

which along with Lemma 3.1, integration by parts formula, Hölder's inequality and (7.30) yields that

$$
\begin{align*}
\left.\left.\left|\frac{d}{d t} \int\right| v\right|^{2} \Phi_{j} d x-\frac{d}{d t} \int|u|^{2} \Phi_{j} d x \right\rvert\, & \leq C\left(\|R\|_{H^{1}}+\|w\|_{H^{1}}+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}} \\
& \leq C(T-t)^{3+\zeta}\|w\|_{L^{2}}, \tag{7.78}
\end{align*}
$$

and thus

$$
\begin{equation*}
\left|\int_{t}^{\tilde{t}}\left(\frac{d}{d s} \int|v|^{2} \Phi_{j} d x-\frac{d}{d s} \int|u|^{2} \Phi_{j} d x\right) d s\right| \leq C(T-t)^{4+\zeta} \sup _{t \leq s \leq \tilde{t}}\|w(s)\|_{L^{2}} . \tag{7.79}
\end{equation*}
$$

Moreover, taking into account (7.33) we have

$$
\begin{equation*}
\left.\lim _{\tilde{t} \rightarrow T}\left|\int\right| v(\widetilde{t})\right|^{2} \Phi_{j} d x-\int|u(\widetilde{t})|^{2} \Phi_{j} d x \mid=0 \tag{7.80}
\end{equation*}
$$

Therefore, plugging (7.79) and (7.80) into (7.76) and passing to the limit $\tilde{t} \rightarrow T$ we arrive at

$$
\begin{equation*}
\left.\left|\int\right| v(t)\right|^{2} \Phi_{j} d x-\int|u(t)|^{2} \Phi_{j} d x \mid \leq C(T-t)^{4+\zeta} \sup _{t \leq s<T}\|w(s)\|_{L^{2}} \tag{7.81}
\end{equation*}
$$

which along with (7.38) and (7.75) yields $\left.\frac{(7.73)}{58}\right)$ thereby finishing the proof.

Below we estimate the growth in the remaining five unstable directions associated to the null space of the operator $L$.

We define the renormalized variables $\widetilde{e}_{j}$ and $e_{j}$ by

$$
\begin{equation*}
w(t, x)=\lambda_{j}(t)^{-\frac{d}{2}} \widetilde{e}_{j}\left(t, \frac{x-\alpha_{j}(t)}{\lambda_{j}(t)}\right) e^{i \theta_{j}(t)}, \text { with } \widetilde{e}_{j}(t, y)=e_{j}(t, y) e^{i\left(\beta_{j}(t) \cdot y-\frac{1}{4} \gamma_{j}(t)|y|^{2}\right)} \tag{7.82}
\end{equation*}
$$

and let $e_{j, 1}:=\operatorname{Re} e_{j}$ and $e_{j, 2}:=\operatorname{Im} e_{j}$, where $1 \leq j \leq K$. Note that, the renormalized variable $e_{j}$ is different from the previous one $\epsilon_{j}$ in (7.36). The advantage to introduce $e_{j}$ and $\widetilde{e}_{j}$ can be seen in Proposition 7.12 below, where the estimates of the unstable directions can be diagonalized in some sense.

Using the Taylor expansions (3.20), (3.21) we have

$$
\begin{equation*}
f(u+w)-f(u)=f^{\prime}(U) \cdot w+G_{1}, \tag{7.83}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{1}:=\left(\partial_{z} f\right)^{\prime}(U, R) \cdot R w+\left(\partial_{\bar{z}} f\right)^{\prime}(U, R) \cdot R \bar{w}+f^{\prime \prime}(u, w) \cdot w^{2} . \tag{7.84}
\end{equation*}
$$

We further split $f^{\prime}(U ; w)$ into three parts below

$$
\begin{align*}
f^{\prime}(U) \cdot w & =f^{\prime}\left(U_{j}\right) \cdot w+\sum_{l \neq j} f^{\prime}\left(U_{l}\right) \cdot w+\left[f^{\prime}(U) \cdot w-\sum_{l=1}^{K} f^{\prime}\left(U_{l}\right) \cdot w\right] \\
& =: f^{\prime}\left(U_{j}\right) \cdot w+G_{2}+G_{3} . \tag{7.85}
\end{align*}
$$

Note that, $G_{2}$ contains the blow-up profiles different from $U_{j}$, and $G_{3}$ contains the interactions between different blow-up profiles.

Moreover, let $G_{4}$ denote the lower order perturbations

$$
\begin{equation*}
G_{4}:=b \cdot \nabla w+c w, \tag{7.86}
\end{equation*}
$$

where $b, c$ are given by (2.12), (2.13), respectively.
Then, plugging (7.83), (7.85) and (7.86) into equation (7.40) we reformulate the equation of $w$ as follows

$$
\begin{equation*}
i \partial_{t} w+\Delta w+f^{\prime}\left(U_{j}\right) \cdot w=-\sum_{l=1}^{4} G_{l} \tag{7.87}
\end{equation*}
$$

The equation of renomalized variable $e_{j}$ is contained in Lemma 7.10 below, which is, actually, a consequence of several algebraic cancellations.

Lemma 7.10. For every $1 \leq j \leq K$, $e_{j}$ satisfies the equation

$$
\begin{equation*}
i \lambda_{j}^{2} \partial_{t} e_{j}+\Delta e_{j}-e_{j}+\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}} e_{j}+\frac{2}{d} Q^{\frac{4}{d}} \bar{e}_{j}=-\sum_{l=1}^{4} H_{l}+\mathcal{O}\left(\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) M o d_{j}\right), \tag{7.88}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{l}(t, y)=\lambda_{j}^{2+\frac{d}{2}} e^{-i \theta_{j}} e^{-i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)} G_{l}\left(t, \lambda_{j} y+\alpha_{j}\right), \quad 1 \leq l \leq 4 \tag{7.89}
\end{equation*}
$$

Proof. Using the identity

$$
\partial_{t} \widetilde{e}_{j}=\partial_{t} e_{j} e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}+i\left(\dot{\beta}_{j} \cdot y-\frac{1}{4} \dot{\gamma}_{j}|y|^{2}\right) \widetilde{e}_{j},
$$

we infer from (7.82) that, if $y:=\frac{x-\alpha_{j}}{\lambda_{j}}$,

$$
\begin{aligned}
\partial_{t} w=\lambda_{j}^{-2-\frac{d}{2}} e^{i \theta_{j}}( & -\frac{d}{2} \lambda_{j} \dot{\lambda}_{j} \widetilde{e}_{j}+\lambda_{j}^{2} \partial_{t} e_{j} e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}+i \lambda_{j}^{2} \dot{\beta}_{j} \cdot y \widetilde{e}_{j}-\frac{1}{4} i \lambda_{j}^{2} \dot{\gamma}_{j}|y|^{2} \widetilde{e}_{j} \\
& \left.-\lambda_{j} \dot{\alpha}_{j} \cdot \nabla \widetilde{e}_{j}-\lambda_{j} \dot{\lambda}_{j} y \cdot \nabla \widetilde{e}_{j}+i \lambda_{j}^{2} \dot{\theta}_{j} \widetilde{e}_{j}\right),
\end{aligned}
$$

which along with (4.7) yields that

$$
\begin{aligned}
i \partial_{t} w=\lambda_{j}^{-2-\frac{d}{2}} e^{i \theta_{j}}\{ & \left\{\gamma_{j} \Lambda \widetilde{e}_{j}+i \lambda_{j}^{2} \partial_{t} e_{j} e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}+\gamma_{j} \beta_{j} \cdot y \widetilde{e}_{j}-\frac{1}{4} \gamma_{j}^{2}|y|^{2} \widetilde{e}_{j}\right. \\
& \left.-2 i \beta_{j} \cdot \nabla \widetilde{e}_{j}-\widetilde{e}_{j}-\left|\beta_{j}\right|^{2} \widetilde{e}_{j}+\mathcal{O}\left(\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) M o d_{j}\right)\right\} .
\end{aligned}
$$

Then, taking into account the identities, similarly to (4.29) and (4.30),

$$
\begin{align*}
& \Lambda \widetilde{e}_{j}=\left(\Lambda e_{j}+i\left(\beta_{j} \cdot y-\frac{1}{2} \gamma_{j}|y|^{2}\right) e_{j}\right) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}  \tag{7.90}\\
& \nabla \widetilde{e}_{j}=\left(\nabla e_{j}+i\left(\beta_{j}-\frac{1}{2} \gamma_{j} y\right) e_{j}\right) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)} \tag{7.91}
\end{align*}
$$

we come to

$$
\begin{align*}
& i \partial_{t} w=\lambda_{j}^{-2-\frac{d}{2}} e^{i \theta_{j}} e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}\left\{i \lambda_{j}^{2} \partial_{t} e_{j}+i \gamma_{j} \Lambda e_{j}+\left|\beta_{j}-\frac{1}{2} \gamma_{j} y\right|^{2} e_{j}-2 i \beta_{j} \cdot \nabla e_{j}-e_{j}\right. \\
&  \tag{7.92}\\
& \left.+\mathcal{O}\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) \operatorname{Mod}_{j}\right\} .
\end{align*}
$$

Moreover, by (7.82), direct computations show that

$$
\begin{equation*}
\Delta \widetilde{e}_{j}=\left(\Delta e_{j}-i \gamma_{j} \Lambda e_{j}-\left|\beta_{j}-\frac{1}{2} \gamma_{j} y\right|^{2} e_{j}+2 i \beta_{j} \cdot \nabla e_{j}\right) e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)} \tag{7.93}
\end{equation*}
$$

Thus, combing (7.92) and (7.93) altogether we obtain that, after algebraic cancellations, $i \partial_{t} w+\Delta w=\lambda_{j}^{-2-\frac{d}{2}} e^{i \theta_{j}} e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}\left\{i \lambda_{j}^{2} \partial_{t} e_{j}-e_{j}+\Delta e_{j}+\mathcal{O}\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) M o d_{j}\right\}$, which along with (7.87) yields (7.88) and thus finishes the proof.

The contributions of the error terms $H_{l}, 1 \leq l \leq 4$, are actually negligible, which is the content of Lemma 7.11 below.

Lemma 7.11. Let $E$ belong to the generalized kernels of the linearized operator L, i.e., $E \in\left\{Q, y Q,|y|^{2} Q, \nabla Q, \Lambda Q, \rho\right\}$. Then, there exists $C>0, \zeta, \delta \in(0,1)$ such that

$$
\begin{align*}
& \int\left|H_{1}(t, y)\left\|E(y) \mid d y \leq C(T-t)^{3+\zeta}\right\| w \|_{L^{2}}\right.  \tag{7.94}\\
& \int\left(\left|H_{2}(t, y)\right|+\left|H_{3}(t, y)\right|\right)|E(y)| d y \leq C e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}},  \tag{7.95}\\
& \left|\int H_{4}(t, y) E(y) d y\right| \leq C(T-t)^{\nu_{*}+1}\|w\|_{L^{2}}, \tag{7.96}
\end{align*}
$$

where $\nu_{*}$ is the flatness index of the spatial functions of noise in Assumption (A1).
Proof. We first see that, by (7.89),

$$
\begin{equation*}
\int\left|H_{1}(t, y)\right||E(y)| d y \leq C(T-t)^{2-\frac{d}{2}} \int\left|G_{1}(t, x) E\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)\right| d x . \tag{7.97}
\end{equation*}
$$

Since by (3.18), (3.19) and (7.84),

$$
\left|G_{1}\right| \leq C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R||w|+C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}+|w|^{\frac{4}{d}-1}\right)|w|^{2},
$$

taking into account $\|E\|_{L^{\infty}}<\infty$ and using Hölder's inequality and (3.29) we obtain

$$
\begin{aligned}
& \int\left|H_{1}(t, y)\left\|E(y) \left\lvert\, d y \leq C(T-t)^{2-\frac{d}{2}}\right.\right\| w \|_{L^{2}}\left((T-t)^{-2+\frac{d}{2}}\|R\|_{L^{2}}+\|R\|_{H^{1}}^{\frac{4}{d}}\right.\right. \\
&\left.+(T-t)^{-2+\frac{d}{2}}\|w\|_{L^{2}}+\|R\|_{H^{1}}^{\frac{4}{d}-1}\|w\|_{H^{1}}+\|w\|_{H^{1}}^{\frac{4}{d}}\right)
\end{aligned}
$$

which along with (7.30) and (7.35) yields (7.94).
Moreover, since by (3.25), (7.82) and (7.85),

$$
\left|G_{2}(t, x)\right| \leq \sum_{l \neq j}\left|f^{\prime}\left(U_{l}\right) \cdot w\right| \leq(T-t)^{-2-\frac{d}{2}} Q^{\frac{4}{d}}\left(\frac{x-\alpha_{l}}{\lambda_{l}}\right)\left|e_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right)\right|
$$

we see from (7.89) that

$$
\begin{equation*}
\left|H_{2}(t, y)\right| \leq C \sum_{l \neq j} Q^{\frac{4}{d}}\left(\frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right)\left|e_{j}(t, y)\right| . \tag{7.98}
\end{equation*}
$$

This yields that

$$
\begin{align*}
\int\left|H_{2}(t, y) \| E(y)\right| d y & \leq C \sum_{l \neq j} \int Q^{\frac{4}{d}}\left(\frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right)\left|e_{j}(y) \| E(y)\right| d y \\
& \leq C \sum_{l \neq j} e^{-\frac{\delta}{T-t}}\left\|e_{j}\right\|_{L^{2}} \leq C e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}} \tag{7.99}
\end{align*}
$$

where $\delta \in(0,1)$, and the second inequality is due to the exponential decay of the ground state $Q$ and $E$.

Similarly, by the definition of $G_{3}$ and (5.39), $1 \leq l \leq K$,

$$
\left|G_{3}(t, x)\right| \leq C(T-t)^{-2-\frac{d}{2}} \sum_{l \neq h} Q\left(\frac{x-\alpha_{l}}{\lambda_{l}}\right) Q\left(\frac{x-\alpha_{h}}{\lambda_{h}}\right)\left|e_{j}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right)\right|,
$$

which along with (7.89) yields that

$$
\left|H_{3}(t, y)\right| \leq C \sum_{l \neq j} Q\left(\frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right)\left|e_{j}(t, y)\right| .
$$

Hence, similarly to (7.99), we get that for some $\delta \in(0,1)$,

$$
\int\left|H_{3}(t, y)\left\|E(y) \left\lvert\, d y \leq C e^{-\frac{\delta}{T-t}}\right.\right\| w \|_{L^{2}}\right.
$$

and thus (7.95) follows.
Regarding $H_{4}$, in view of (7.86), (7.89) and (7.91), we obtain

$$
H_{4}(t, y)=\lambda_{j} \widetilde{b} \cdot\left(\nabla e_{j}+i\left(\beta_{j}-\frac{1}{2} \gamma_{j} y\right) e_{j}\right)+\lambda_{j}^{2} \widetilde{c} e_{j}
$$

where $\widetilde{b}(t, y)=b\left(t, \lambda_{j} y+\alpha_{j}\right), \widetilde{c}(t, y)=c\left(t, \lambda_{j} y+\alpha_{j}\right)$. Using integration by parts formula we have

$$
\int \widetilde{b} \cdot \nabla e_{j} E d y=-\int \operatorname{div} \widetilde{b} E e_{j} d y-\int \widetilde{b} \cdot \nabla E e_{j} d y
$$

then using Hölder's inequality, (2.12), (2.13) and (4.23) we obtain (7.96) and finish the proof.

By virtue of the identities in (3.11) and Lemmas 7.10 and 7.11 we are now able to control the growth of $e_{j}$ along the remaining five unstable directions as stated in Proposition 7.12 below.

Proposition 7.12. Let $e_{j}$ be as in (17.82) and $e_{j, 1}:=\operatorname{Re}_{j}, e_{j, 2}:=\operatorname{Im} e_{j}$, Then, for $1 \leq j \leq K$ and for $T$ small enough we have

$$
\begin{align*}
& \frac{d}{d t}\left\langle e_{j, 2}, \Lambda Q\right\rangle=2 \lambda_{j}^{-2}\left\langle e_{j, 1}, Q\right\rangle+\mathcal{O}\left((T-t)^{2+\zeta} \sqrt{D(t)}\right)  \tag{7.100}\\
& \left.\left.\frac{d}{d t}\left\langle e_{j, 1},\right| y\right|^{2} Q\right\rangle=-4 \lambda_{j}^{-2}\left\langle e_{j, 2}, \Lambda Q\right\rangle+\mathcal{O}\left((T-t)^{2+\zeta} \sqrt{D(t)}\right)  \tag{7.101}\\
& \left.\frac{d}{d t}\left\langle e_{j, 2}, \rho\right\rangle=\left.\lambda_{j}^{-2}\left\langle e_{j, 1},\right| y\right|^{2} Q\right\rangle+\mathcal{O}\left((T-t)^{2+\zeta} \sqrt{D(t)}\right)  \tag{7.102}\\
& \frac{d}{d t}\left\langle e_{j, 2}, \nabla Q\right\rangle=\mathcal{O}\left((T-t)^{2+\zeta} \sqrt{D(t)}\right)  \tag{7.103}\\
& \frac{d}{d t}\left\langle e_{j, 1}, y Q\right\rangle=-2 \lambda_{j}^{-2}\left\langle e_{j, 2}, \nabla Q\right\rangle+\mathcal{O}\left((T-t)^{2+\zeta} \sqrt{D(t)}\right), \tag{7.104}
\end{align*}
$$

Proof. Let us take $\frac{d}{d t}\left\langle e_{j, 2}, \Lambda Q\right\rangle$ as an example to illustrate the arguments. Using equation (7.88) we have

$$
\frac{d}{d t}\left\langle e_{j, 2}, \Lambda Q\right\rangle=\lambda_{j}^{-2} \operatorname{Re} \int \Lambda Q\left(\Delta e_{j}-e_{j}+\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}} e_{j}+\frac{2}{d} Q^{\frac{4}{d}} \overline{e_{j}}\right) d y
$$

$$
\begin{equation*}
+\lambda_{j}^{-2} \sum_{l=1}^{4} \mathcal{O}\left(\int \Lambda Q H_{l} d y\right)+\lambda_{j}^{-2} \operatorname{Mod}_{j} \mathcal{O}\left(\int \Lambda Q\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) d y\right) . \tag{7.105}
\end{equation*}
$$

Note that, by the definition (3.9) of the operator $L_{+}$, the integration by parts formula and the identity $L_{+} \Lambda Q=-2 Q$ in (3.11),

$$
\begin{equation*}
\operatorname{Re} \int \Lambda Q\left(\Delta e_{j}-e_{j}+\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}} e_{j}+\frac{2}{d} Q^{\frac{4}{d}} \overline{e_{j}}\right) d y=-\int \Lambda Q L_{+} e_{j, 1} d y=2 \int Q e_{j, 1} d y \tag{7.106}
\end{equation*}
$$

Moreover, using Lemma 7.11 and (7.38) we infer that for each $1 \leq l \leq 4$,

$$
\begin{equation*}
\lambda_{j}^{-2}\left|\int \Lambda Q H_{l} d y\right| \leq C(T-t)^{1+\zeta}\|w\|_{L^{2}} \leq C(T-t)^{2+\zeta} \sqrt{D(t)} \tag{7.107}
\end{equation*}
$$

We also note from Lemma 7.3 that $\operatorname{Mod}(t) \leq C(T-t)^{\kappa+3} \leq C(T-t)^{3+\zeta}$. Then, using Hölder's inequality, the boundedness of $\left\|\langle y\rangle^{2} \Lambda Q\right\|_{L^{2}}$ and

$$
\left\|\widetilde{e}_{j}\right\|_{L^{2}}+\left\|\nabla \widetilde{e}_{j}\right\|_{L^{2}} \leq C\left(\|w\|_{L^{2}}+\lambda_{j}\|\nabla w\|_{L^{2}}\right) \leq C(T-t) \sqrt{D(t)},
$$

we get

$$
\begin{align*}
\lambda_{j}^{-2} \operatorname{Mod}_{j}\left|\int \Lambda Q\left(\langle y\rangle^{2}\left|\widetilde{e}_{j}\right|+\langle y\rangle\left|\nabla \widetilde{e}_{j}\right|\right) d y\right| & \leq C(T-t)^{1+\zeta}\left\|\langle y\rangle^{2} \Lambda Q\right\|_{L^{2}}\left(\left\|\widetilde{e}_{j}\right\|_{L^{2}}+\left\|\nabla \widetilde{e}_{j}\right\|_{L^{2}}\right) \\
& \leq C(T-t)^{2+\zeta} \sqrt{D(t)} . \tag{7.108}
\end{align*}
$$

Thus, plugging (7.106)-(7.108) into (7.105) yields immediately (7.102).
Similar arguments apply also to the remaining four estimates in Proposition 7.12, For simplicity, the details are omitted here.

We are now ready to prove the key estimate (7.72) in Theorem 7.7 ,
Proof of Theorem 7.7. Similarly to (7.39), we define the growth quantity associated to $e_{j}$ by

$$
\begin{equation*}
\left.\widetilde{S_{S c a l}^{j}}{ }_{j}(t):=\left\langle e_{j, 1}, Q\right\rangle^{2}+\left\langle e_{j, 1}, y Q\right\rangle^{2}+\left.\left\langle e_{j, 1},\right| y\right|^{2} Q\right\rangle^{2}+\left\langle e_{j, 2}, \nabla Q\right\rangle^{2}+\left\langle e_{j, 2}, \Lambda Q\right\rangle^{2}+\left\langle e_{j, 2}, \rho\right\rangle^{2} . \tag{7.109}
\end{equation*}
$$

As we shall see below, the two renomalized variables $\epsilon_{j}$ and $e_{j}$ defined in (7.36) and (7.82) respectively are almost the same up to the negligible error of order $\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)$, and thus the two quantities $S c a l_{j}$ and $\widetilde{S c a l}_{j}$ should be close to each other.

We first claim that for some $\zeta>0$,

$$
\begin{equation*}
\widetilde{S c a l}_{j}(t) \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} D(s) \tag{7.110}
\end{equation*}
$$

To this end, we use Proposition 7.8 and the change of variables to get

$$
\begin{equation*}
\left|\left\langle e_{j, 1}, Q\right\rangle\right| \leq C(T-t)^{4+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.111}
\end{equation*}
$$

which along with (7.100) yields that

$$
\left|\frac{d}{d t}\left\langle e_{j, 2}, \Lambda Q\right\rangle\right| \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} \sqrt{D(s)}
$$

Since by (7.35), $\lim _{t \rightarrow T}\|w(t)\|_{H^{1}}=0$, we infer that

$$
\begin{equation*}
\lim _{t \rightarrow T}\left\langle e_{j, 1}(t), \Lambda Q\right\rangle=0 \tag{7.112}
\end{equation*}
$$

This yields that

$$
\begin{equation*}
\left|\left\langle e_{j, 2}, \Lambda Q\right\rangle\right| \leq \int_{t}^{T}\left|\frac{d}{d s}\left\langle e_{j, 2}, \Lambda Q\right\rangle\right| d s \leq C(T-t)^{3+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.113}
\end{equation*}
$$

Then, plugging (7.113) into (7.101) yields

$$
\begin{equation*}
\left.\left|\left\langle e_{j, 1},\right| y\right|^{2} Q\right\rangle \mid \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.114}
\end{equation*}
$$

which along with (7.102) yields

$$
\begin{equation*}
\left|\left\langle e_{j, 2}, \rho\right\rangle\right| \leq C(T-t)^{1+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.115}
\end{equation*}
$$

We also see from (7.103) that

$$
\begin{equation*}
\left|\left\langle e_{j, 2}, \nabla Q\right\rangle\right| \leq C(T-t)^{3+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.116}
\end{equation*}
$$

which along with (7.104) yields that

$$
\begin{equation*}
\left|\left\langle e_{j, 1}, y Q\right\rangle\right| \leq C(T-t)^{2+\zeta} \sup _{t \leq s<T} \sqrt{D(s)} \tag{7.117}
\end{equation*}
$$

Thus, combining estimates (7.111)-(7.117) altogether we obtain (7.110), as claimed.
Next we show that there exist $C, \delta>0$ such that

$$
\begin{equation*}
\operatorname{Scal}_{j}(t)=\widetilde{S c a l}_{j}(t)+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w(t)\|_{L^{2}}^{2} \tag{7.118}
\end{equation*}
$$

Then, taking into account (7.26), (7.38) and (7.110) we obtain (7.72).
It remains to prove (7.118). For this purpose, by (7.27), (7.34), (7.82) and the change of variables,

$$
\operatorname{Re}\left\langle e_{j}, Q\right\rangle=\operatorname{Re}\left\langle w, U_{j}\right\rangle=\operatorname{Re}\left\langle w_{j}, U_{j}\right\rangle+\sum_{l \neq j} \operatorname{Re}\left\langle w_{l}, U_{j}\right\rangle
$$

Using Lemma 3.1 to decouple $w_{l}$ and $U_{j}, l \neq j$, and then using (7.36) we get

$$
\operatorname{Re}\left\langle e_{j}, Q\right\rangle=\operatorname{Re}\left\langle w_{j}, U_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}}\right)=\operatorname{Re}\left\langle\epsilon_{j}, Q_{j}\right\rangle+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}}\right)
$$

where $Q_{j}$ is given by (7.28). Then, using the fact

$$
\begin{equation*}
Q_{j}=Q+\mathcal{O}\left(P\langle y\rangle^{2} Q\right) \tag{7.119}
\end{equation*}
$$

we arrive at

$$
\begin{equation*}
\operatorname{Re}\left\langle e_{j}, Q\right\rangle=\operatorname{Re}\left\langle\epsilon_{j}, Q\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}} \tag{7.120}
\end{equation*}
$$

Similar arguments with slight modifications also yield that

$$
\begin{align*}
\operatorname{Re}\left\langle e_{j}, y Q\right\rangle & =\operatorname{Re}\left\langle\epsilon_{j}, y Q\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}},  \tag{7.121}\\
\left.\left.\operatorname{Re}\left\langle e_{j},\right| y\right|^{2} Q\right\rangle & \left.=\left.\operatorname{Re}\left\langle\epsilon_{j},\right| y\right|^{2} Q\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}}, \tag{7.122}
\end{align*}
$$

and taking into account $\rho_{j}=\rho+\mathcal{O}\left(P\langle y\rangle^{2} \rho\right)$ we also have

$$
\begin{equation*}
\operatorname{Re}\left\langle e_{j}, \rho\right\rangle=\operatorname{Re}\left\langle\epsilon_{j}, \rho\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}} \tag{7.123}
\end{equation*}
$$

Regarding $\operatorname{Im}\left\langle\Lambda Q, e_{j}\right\rangle$, using the change of variables and the identity (4.29) we have

$$
\begin{aligned}
\operatorname{Im}\left\langle e_{j}, \Lambda Q\right\rangle & =\operatorname{Im} \int \widetilde{e}_{j} \overline{\Lambda Q e^{i\left(\beta_{j} \cdot y-\frac{1}{4} \gamma_{j}|y|^{2}\right)}} d y \\
& =\operatorname{Im} \int w\left(\Lambda \overline{U_{j}}+i\left(\beta_{j} \cdot\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)-\frac{1}{2} \gamma_{j}\left|\frac{x-\alpha_{j}}{\lambda_{j}}\right|^{2}\right) \overline{U_{j}}\right) d x \\
& =\operatorname{Im} \int w_{j}\left(\Lambda \overline{U_{j}}+i\left(\beta_{j} \cdot\left(\frac{x-\alpha_{j}}{\lambda_{j}}\right)-\frac{1}{2} \gamma_{j}\left|\frac{x-\alpha_{j}}{\lambda_{j}}\right|^{2}\right) \overline{U_{j}}\right) d x+\mathcal{O}\left(e^{\left.-\frac{\delta}{T-t}\|w\|_{L^{2}}\right)}\right.
\end{aligned}
$$

where in the last step we also used Lemma 3.1 to decouple the different profiles $w_{l}$ and $U_{j}$, $l \neq j$, which merely contribute the exponentially small error. Then, using again (4.29), (7.36), the change of variables and (7.119) we obtain

$$
\begin{align*}
\operatorname{Im}\left\langle e_{j}, \Lambda Q\right\rangle & =\operatorname{Im} \int \epsilon_{j} \overline{\Lambda Q e^{i\left(\left.\beta_{j} \cdot y-\frac{1}{4} \gamma_{j} \right\rvert\, y^{2}\right)}} d y+\mathcal{O}\left(e^{-\frac{\delta}{T-t}}\|w\|_{L^{2}}\right) \\
& =\operatorname{Im}\left\langle\epsilon_{j}, \Lambda Q\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}} . \tag{7.124}
\end{align*}
$$

Using similar arguments, but with the identity (4.30) instead, we also have

$$
\begin{equation*}
\operatorname{Im}\left\langle e_{j}, \nabla Q\right\rangle=\operatorname{Im}\left\langle\epsilon_{j}, \nabla Q\right\rangle+\mathcal{O}\left(P+e^{-\frac{\delta}{T-t}}\right)\|w\|_{L^{2}} \tag{7.125}
\end{equation*}
$$

Therefore, combining estimates (7.120), (7.121), (7.122), (7.123), (7.124) and (7.125) altogether we obtain (7.118) and thus finish the proof of Theorem 7.7 .

Now, we are ready to prove the main uniqueness result, i.e., Theorem 2.15.
Proof of Theorem [2.7. First take any $\varepsilon \in\left(0, \varepsilon^{*}\right]$, where $\varepsilon^{*}$ is sufficiently small and is to be specified later. Then, let $T$ be sufficiently small and satisfy (7.1). We shall use iteration arguments to show that $D \equiv 0$.

By Theorem 7.6, we have for any $t \in[0, T)$,

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq C_{1} \sup _{t \leq s<T} \sum_{j=1}^{K} \frac{S c a l_{j}(s)}{\lambda_{j}^{2}(s)}+C_{1} \int_{t}^{T} \sum_{j=1}^{K} \frac{S c a l_{j}(s)}{\lambda_{j}^{3}(s)}+\varepsilon^{*} \frac{D(s)}{T-s} d s \tag{7.126}
\end{equation*}
$$

Then, in view of Theorem 7.7 and (7.35) we obtain that for some $\zeta>0$,

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq C_{2}(T-t)^{\zeta} \sup _{t \leq s<T} D(s)+C_{2} \varepsilon^{*} \int_{t}^{T} \frac{D(s)}{T-s} d s \tag{7.127}
\end{equation*}
$$

where $C_{2}$ is independent of $\varepsilon^{*}$. This yields that for $T$ even smaller such that $C_{2} T^{\zeta} \leq \frac{1}{2}$,

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq 2 C_{2} \varepsilon_{*} \int_{t}^{T} \frac{D(s)}{T-s} d s \tag{7.128}
\end{equation*}
$$

We also infer from (7.35) and (7.37) that

$$
D(s) \leq C_{3}(T-s)^{4+\zeta},
$$

where we may take $C_{3} \geq 1$, independent of $\varepsilon^{*}$.
Then, plugging this into (7.128) we get

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq 2 C_{2} \varepsilon^{*} \int_{t}^{T} C_{3}(T-s)^{3+\zeta} d s \leq\left(\frac{2 C_{2} C_{3} \varepsilon^{*}}{4+\zeta}\right)(T-t)^{4+\zeta} . \tag{7.129}
\end{equation*}
$$

But, plugging (7.129) into (7.128) again we can obtain the refined estimate

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq\left(\frac{2 C_{2} C_{3} \varepsilon^{*}}{4+\zeta}\right)^{2}(T-t)^{4+\zeta} . \tag{7.130}
\end{equation*}
$$

Thus, iterating similar arguments we infer that for any $k \geq 1$,

$$
\begin{equation*}
\sup _{t \leq s<T} D(s) \leq\left(\frac{2 C_{2} C_{3} \varepsilon^{*}}{4+\zeta}\right)^{k}(T-t)^{4+\zeta} \leq\left(\frac{2 C_{2} C_{3} \varepsilon^{*}}{4+\zeta}\right)^{k} \tag{7.131}
\end{equation*}
$$

where $C_{2}, C_{3}>0$ are independent of $\varepsilon^{*}$ and $k$.
Therefore, taking $\varepsilon^{*}$ small enough such that $\frac{2 C_{2} C_{3} \varepsilon^{*}}{4+\zeta}<1$ and using the arbitrariness of $k$ we infer that $D(t)=0$ for any $t \in[0, T)$, which yields immediately that $w \equiv 0$, and thus $v \equiv u$. The proof of Theorem 2.7 is complete.

## 8. Appendix

Proof of Lemma 3.1. First, using (3.4) and the separation of $\left\{x_{j}\right\}_{j=1}^{K}$ we have that for any $t^{*} \leq t<T_{*}$,

$$
\begin{equation*}
\left|\left(\alpha_{j}(t)-\alpha_{l}(t)\right) \cdot \mathbf{v}_{\mathbf{1}}\right| \geq 10 \sigma, \quad j \neq l, \tag{8.1}
\end{equation*}
$$

where $\sigma$ is given by (3.1). Note that

$$
\begin{equation*}
\max _{1 \leq j \neq l \leq K} \frac{\lambda_{j}(t)}{\lambda_{l}(t)} \leq c_{*}:=4 \max _{1 \leq j \neq l \leq K} \frac{\omega_{j}}{\omega_{l}}, \max _{1 \leq j \neq l \leq K}\left|\alpha_{j}-\alpha_{l}\right| \leq 2 M:=2\left(1+\max _{1 \leq j \leq K}\left|x_{j}\right|\right) \tag{8.2}
\end{equation*}
$$

In order to prove (3.6), we note that

$$
\begin{aligned}
& \left|\int_{\mathbb{R}^{d}}\right| x-\left.\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{j}\right|^{m}\left|G_{j}(t)\right| d x \\
\leq & C(T-t)^{-|\nu|+m+n-d} \int_{\mathbb{R}^{d}}\left|\frac{x-\alpha_{l}}{\lambda_{l}}\right|^{n}\left|\frac{x-\alpha_{j}}{\lambda_{j}}\right|^{m}\left|\partial^{\nu} g_{l}\left(t, \frac{x-\alpha_{l}}{\lambda_{l}}\right)\right|\left|g_{l}\left(t, \frac{x-\alpha_{j}}{\lambda_{j}}\right)\right| d x \\
\leq & C(T-t)^{-|\nu|+m+n} \int\left|\frac{\lambda_{j} y+\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right|^{n}|y|^{m}\left|\partial^{\nu} g_{l}\left(t, \frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right) \| g(y)\right| d y .
\end{aligned}
$$

Using (7.1) we have

$$
\left|\frac{\lambda_{j} y+\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right|^{n} \leq C\left(|y|^{n}+(T-t)^{-n} M^{n}\right) \leq C(T-t)^{-2 n}\langle y\rangle^{n} .
$$

It follows that

$$
\begin{aligned}
& \left|\int_{\mathbb{R}^{d}}\right| x-\left.\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{j}\right|^{m}\left|G_{j}(t)\right| d x \\
\leq & C(T-t)^{-|\nu|+m-n}\left(\int_{\left|y \cdot \mathbf{v}_{1}\right| \leq \frac{5 \sigma}{c_{*} \lambda_{j}}}+\int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{5 \sigma}{c_{*} \lambda_{j}}}\right)\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\left(t, \frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right) \| g(y)\right| d y .
\end{aligned}
$$

On one hand, in the region $\left\{y \in \mathbb{R}^{d}:\left|y \cdot \mathbf{v}_{\mathbf{1}}\right| \leq \frac{5 \sigma}{c_{*} \lambda_{j}}\right\}$, by (8.1) and (8.2),

$$
\left|\frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right| \geq\left|\frac{\left(\alpha_{j}-\alpha_{l}\right) \cdot \mathbf{v}_{\mathbf{1}}}{\lambda_{l}}\right|-\left|\frac{\lambda_{j}}{\lambda_{l}}\left(y \cdot \mathbf{v}_{\mathbf{1}}\right)\right| \geq \frac{5 \sigma}{\lambda_{l}} \rightarrow \infty, \text { as } T \rightarrow 0 .
$$

Then, in view of the exponential decay of $\partial_{\nu} g$, we obtain that

$$
\begin{aligned}
& \left.(T-t)^{-|\nu|+m-n} \int_{\left|y \cdot \mathbf{v}_{1}\right| \leq \frac{5 \sigma}{c_{*} \lambda_{l}}}\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\left(t, \frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right)\right| g(y) \right\rvert\, d y \\
\leq & C(T-t)^{-|\nu|+m-n} e^{-\frac{\delta_{1}}{T-t}}\left\|\langle y\rangle^{m+n} g\right\|_{L^{1}} \leq C e^{-\frac{\delta_{1}}{2(T-t)}},
\end{aligned}
$$

where $\delta_{1}>0$, and we also used $\sup _{r>0} r^{-|\nu|+m-n} e^{-\frac{\delta_{1}}{2 r}}<\infty$ in the last step. On the other hand, in the region $\left\{y:\left|y \cdot \mathbf{v}_{\mathbf{1}}\right| \geq \frac{5 \sigma}{c_{*} \lambda_{l}}\right\},|y| \geq\left|y \cdot \mathbf{v}_{\mathbf{1}}\right| \geq \frac{5 \sigma}{c_{*} \lambda_{l}} \rightarrow \infty$, as $t \rightarrow T$. Using the change of variables and the exponential decay of $g$ we get

$$
\begin{aligned}
& (T-t)^{-|\nu|+m-n} \int_{\left|y \cdot \mathbf{v}^{1}\right| \geq \frac{5 \sigma}{c_{*} \lambda_{l}}}\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\left(t,, \frac{\lambda_{j}}{\lambda_{l}} y+\frac{\alpha_{j}-\alpha_{l}}{\lambda_{l}}\right) \| g(y)\right| d y \\
\leq & C(T-t)^{-|\nu|+m-n} e^{-\frac{\delta_{2}}{T-t}}\left\|\langle y\rangle^{m+n} \left\lvert\, g(y)^{\frac{1}{2}}\right.\right\|_{L^{\infty}}\left\|\partial^{\nu} g_{l}\right\|_{L^{1}} \leq C e^{-\frac{\delta_{2}}{2(T-t)}}
\end{aligned}
$$

where $\delta_{2}>0$. Thus, combing two estimates above we obtain (3.6).
Regarding (3.7), since for $j \neq l,\left|\left(x-x_{l}\right) \cdot \mathbf{v}_{\mathbf{1}}\right| \geq 4 \sigma$ on the support of $\Phi_{j}$, we infer from (3.4) that $\left|\left(x-\alpha_{l}\right) \cdot \mathbf{v}_{\mathbf{1}}\right| \geq 3 \sigma$. Then, using the change of variables and the inequality

$$
\left|\frac{\lambda_{l} y+\alpha_{l}-\alpha_{j}}{\lambda_{j}}\right|^{m} \leq C M(T-t)^{-m}\langle y\rangle^{m} \leq C(T-t)^{-2 m}\langle y\rangle^{m}
$$

we get

$$
\begin{align*}
& \left|\int_{\mathbb{R}^{d}}\right| x-\left.\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{j}\right|^{m}|h| \Phi_{j} d x \mid \\
\leq & C(T-t)^{-|\nu|+m+n-\frac{d}{2}} \int_{\left|\left(x-x_{l}\right) \cdot \mathbf{v}_{1}\right| \geq 4 \sigma}\left|\frac{x-\alpha_{l}}{\lambda_{l}}\right|^{n}\left|\partial^{\nu} g_{l}\left(t, \frac{x-\alpha_{l}}{\lambda_{l}}\right)\right|\left|\frac{x-\alpha_{j}}{\lambda_{j}}\right|^{m}|h(x)| d x \\
\leq & C(T-t)^{-|\nu|-m+n+\frac{d}{2}} \int_{\left|y \cdot \mathbf{v}_{1}\right| \geq \frac{3 \sigma}{\lambda_{l}}}\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}(t, y)\right|\left|h\left(\lambda_{l} y+\alpha_{l}\right)\right| d y . \tag{8.3}
\end{align*}
$$

Then, using Cauchy's inequality and the exponential decay of $\partial^{\nu} g$ again we obtain that for $t$ close to $T$, the right-hand-side of (8.3) is bounded by

$$
\begin{aligned}
& C(T-t)^{-|\nu|-m+n+\frac{d}{2}} e^{-\frac{\delta_{3}}{T-t}}\left\|\langle y\rangle^{m+n} \left\lvert\, \partial^{\nu} g_{l} l^{\frac{1}{2}}\right.\right\|_{L^{2}}\left(\int\left|h\left(\lambda_{l} y+\alpha_{l}\right)\right|^{2} d y\right)^{\frac{1}{2}} \\
\leq & C(T-t)^{-|\nu|-m+n} e^{-\frac{\delta_{3}}{T-t}}\left\|\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\right|^{\frac{1}{2}}\right\|_{L^{2}}\|h\|_{L^{2}} \leq C e^{-\frac{\delta_{3}}{2(T-t)}}\|h\|_{L^{2}}
\end{aligned}
$$

where $\delta_{3}>0$ and in the last step we also used $\sup _{r>0} r^{-|\nu|-m+n} e^{-\frac{\delta_{3}}{2 r}}<\infty$. One can also bound the right-hand-side of (8.3) by

$$
\begin{aligned}
& C(T-t)^{-|\nu|+n+\frac{d}{2}} e^{-\frac{\delta_{3}}{T-t}}\left\|\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\right|^{\frac{1}{2}}\right\|_{L^{\infty}} \int\left|h\left(\lambda_{l} y+\alpha_{l}\right)\right| d y \\
\leq & C(T-t)^{-|\nu|-m+n-\frac{d}{2}} e^{-\frac{\delta_{3}}{T-t}}\left\|\langle y\rangle^{m+n}\left|\partial^{\nu} g_{l}\right|^{\frac{1}{2}}\right\|_{L^{\infty}}\|h\|_{L^{1}} \leq C e^{-\frac{\delta_{3}}{2(T-t)}}\|h\|_{L^{1}} .
\end{aligned}
$$

Hence, combining the two estimate together we obtain (3.7), thereby finishing the proof of Lemma 3.1

Proof of Corollary [3.4. Let $\tilde{f}:=f \phi_{A}^{\frac{1}{2}}$. Since $\nabla f \phi_{A}^{\frac{1}{2}}=\nabla \tilde{f}-\frac{\nabla \phi_{A}}{2 \phi_{A}} \tilde{f}$, we have

$$
\begin{aligned}
& \int\left(|\nabla f|^{2}+|f|^{2}\right) \phi_{A}-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} f_{1}^{2}-Q^{\frac{4}{d}} f_{2}^{2} d x \\
= & \int|\nabla \widetilde{f}|^{2}+|\widetilde{f}|^{2}-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} \widetilde{f}_{1}^{2}-Q^{\frac{4}{d}} \widetilde{f}_{2}^{2} d x \\
& -\int\left(1-\phi_{A}\right)\left(\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} f_{1}^{2}+Q^{\frac{4}{d}} f_{2}^{2}\right) d x \\
& +\frac{1}{4} \int\left|\frac{\nabla \phi_{A}}{\phi_{A}}\right|^{2}|\tilde{f}|^{2} d x-\operatorname{Re} \int \frac{\nabla \phi_{A}}{\phi_{A}} \cdot \nabla \widetilde{f} \widetilde{f} d x=: \sum_{i=1}^{4} K_{i} .
\end{aligned}
$$

Using Corollary 3.3 we have

$$
\begin{equation*}
K_{1} \geq C_{1}^{\prime}\|\widetilde{f}\|_{H^{1}}^{2}-C_{2}^{\prime} S \operatorname{cal}(\tilde{f}), \tag{8.5}
\end{equation*}
$$

where $C_{1}^{\prime}, C_{2}^{\prime}>0$. Moreover, we claim that there exist $C, \delta>0$ such that

$$
\begin{equation*}
S \operatorname{cal}(\widetilde{f}) \leq S \operatorname{cal}(f)+C e^{-\delta A}\|\widetilde{f}\|_{L^{2}}^{2} \tag{8.6}
\end{equation*}
$$

To this end, we see that

$$
\left\langle\widetilde{f}_{1}, Q\right\rangle=\left\langle f_{1}, Q\right\rangle+\left\langle\widetilde{f}_{1} \phi_{A}^{-\frac{1}{2}}\left(\phi_{A}^{\frac{1}{2}}-1\right), Q\right\rangle .
$$

Since on the support of $\phi_{A}^{\frac{1}{2}}-1,|x| \geq A$, by the exponential decay of $Q$ we have that for some $\delta^{\prime}>0$

$$
\left|\phi_{A}^{-\frac{1}{2}}(x) Q^{\frac{1}{2}}(x)\right| \leq C e^{-\frac{1}{2}|x|\left(\delta^{\prime}-A^{-1}\right)} \leq C e^{-\frac{1}{4} \delta^{\prime} A}, \quad|x| \geq A,
$$

where we also took $A$ large enough such that $\delta^{\prime}-A^{-1} \geq \frac{1}{2} \delta^{\prime}$. It follows that

$$
\begin{equation*}
\left|\left\langle\tilde{f}_{1} \phi_{A}^{-\frac{1}{2}}\left(\phi_{A}^{\frac{1}{2}}-1\right), Q\right\rangle\right| \leq C\|\widetilde{f}\|_{L^{2}}\left\|\phi_{A}^{-\frac{1}{2}} Q^{\frac{1}{2}}\right\|_{L^{\infty}(|x| \geq A)}\left\|Q^{\frac{1}{2}}\right\|_{L^{2}} \leq C e^{-\frac{1}{4} \delta^{\prime} A}\|\widetilde{f}\|_{L^{2}}, \tag{8.7}
\end{equation*}
$$

which yields that

$$
\left|\left\langle\widetilde{f}_{1}, Q\right\rangle-\left\langle f_{1}, Q\right\rangle\right| \leq C e^{-\frac{1}{4} \delta^{\prime} A}\|\widetilde{f}\|_{L^{2}} .
$$

Similar arguments apply also to the remaining five scalar products in $S c a l(f)$, and thus we obtain (8.6), as claimed.

Hence, we infer from (8.5) and (8.6) that for $C_{1}, C_{2}>0$,

$$
\begin{equation*}
K_{1} \geq C_{1}\|\tilde{f}\|_{H^{1}}^{2}-C_{2} S \operatorname{cal}(f) \tag{8.8}
\end{equation*}
$$

Regarding the second term $K_{2}$, we see that

$$
\begin{equation*}
K_{2}=\int\left(1-\phi_{A}\right) \phi_{A}^{-1}\left(\left(1+\frac{4}{d}\right) Q^{2} \widetilde{f}_{1}^{2}+Q^{2} \widetilde{\tilde{f}_{2}^{2}}\right) d x \leq C\left\|\phi_{A}^{-1} Q^{2}\right\|_{L^{\infty}(|x| \geq A)}\|\widetilde{f}\|_{L^{2}} \tag{8.9}
\end{equation*}
$$

Similar arguments as in the proof of (8.7) yield that for $A$ large enough $\left\|\phi_{A}^{-1} Q^{2}\right\|_{L^{\infty}(|x| \geq A)} \leq$ $C e^{-\delta A}$, where $C, \delta>0$. This implies that for $A$ large enough

$$
\begin{equation*}
K_{2} \leq C e^{-\delta A}\|\widetilde{f}\|_{L^{2}} \tag{8.10}
\end{equation*}
$$

We also note that, since $\left|\frac{\nabla \phi_{A}}{\phi_{A}}\right| \leq C A^{-1}$, by Hölder's inequality,

$$
\begin{equation*}
K_{3}+K_{4} \leq \frac{C}{A}\|\tilde{f}\|_{H^{1}}^{2} \tag{8.11}
\end{equation*}
$$

Thus, plugging (8.8), (8.10) and (8.11) into (8.4), we obtain that for $A$ possibly larger

$$
\begin{equation*}
\int\left(|\nabla f|^{2}+|f|^{2}\right) \phi_{A}-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} f_{1}^{2}-Q^{\frac{4}{d}} f_{2}^{2} d x \geq \frac{C_{1}}{2}\|\widetilde{f}\|_{H^{1}}^{2}-C_{2} S c a l(f) \tag{8.12}
\end{equation*}
$$

It remains to show that for $A$ large enough,

$$
\begin{equation*}
\frac{C_{1}}{2}\|\widetilde{f}\|_{H^{1}}^{2} \geq \frac{C_{1}}{8} \int\left(|\nabla f|^{2}+|f|^{2}\right) \phi_{A} d x \tag{8.13}
\end{equation*}
$$

To this end, since $\nabla \tilde{f}=\nabla f \phi_{A}^{\frac{1}{2}}+\frac{\nabla \phi_{A}}{2 \phi_{A}} \tilde{f}$ we have

$$
\begin{align*}
\frac{C_{1}}{2}\|\widetilde{f}\|_{H^{1}}^{2} & =\frac{C_{1}}{2} \int|f|^{2} \phi_{A} d x+\frac{C_{1}}{2} \int\left|\nabla f \phi_{A}^{\frac{1}{2}}+\frac{\nabla \phi_{A}}{2 \phi_{A}} \tilde{f}\right|^{2} d x \\
& =\frac{C_{1}}{2} \int\left(|f|^{2}+|\nabla f|^{2}\right) \phi_{A} d x+\frac{C_{1}}{2} \int \operatorname{Re}\left(\nabla f \phi_{A}^{\frac{1}{2}} \frac{\nabla \phi_{A}}{\phi_{A}} \widetilde{\tilde{f}}\right) d x+\frac{C_{1}}{8} \int\left|\frac{\nabla \phi_{A}}{\phi_{A}} \widetilde{f}\right|^{2} d x . \tag{8.14}
\end{align*}
$$

Note that, by Hölder's inequality and $\left|\frac{\nabla \phi_{A}}{\phi_{A}}\right| \leq C A^{-1}$,

$$
\begin{align*}
\left|\frac{C_{1}}{2} \int \operatorname{Re}\left(\nabla f \phi_{A}^{\frac{1}{2}} \frac{\nabla \phi_{A}}{\phi_{A}} \overline{\widetilde{f}}\right) d x\right| & \leq \frac{C_{1} C}{2 A}\left(\int|\nabla f|^{2} \phi_{A} d x\right)^{\frac{1}{2}}\left(\int|f|^{2} \phi_{A} d x\right)^{\frac{1}{2}} \\
& \leq \frac{C_{1}}{4} \int|\nabla f|^{2} \phi_{A} d x+\frac{4 C_{1} C^{2}}{A^{2}} \int|f|^{2} \phi_{A} d x . \tag{8.15}
\end{align*}
$$

We also have

$$
\begin{equation*}
\frac{C_{1}}{8} \int\left|\frac{\nabla \phi_{A}}{\phi_{A}} \widetilde{f}\right|^{2} d x \leq \frac{C_{1} C^{2}}{2 A^{2}} \int|f|^{2} \phi_{A} d x . \tag{8.16}
\end{equation*}
$$

Therefore, combining (8.12)- (8.16) and taking $A$ large enough such that $\frac{4 C^{2}}{A^{2}}+\frac{C^{2}}{2 A^{2}} \leq \frac{1}{8}$ we obtain (8.13), as claimed. This along with (8.12) yields (3.17).

Proof of Lemma 5.5. Using (4.14) and (5.16) we have

$$
\begin{equation*}
\left\|\partial^{\nu} \eta\right\|_{L^{2}} \leq C(T-t)^{-2-|\nu|} M o d+\sum_{j=1}^{K}\left\|\partial^{\nu}\left(b \cdot \nabla U_{j}+c U_{j}\right)\right\|_{L^{2}}+\left\|\partial^{\nu}\left(f(U)-\sum_{j=1}^{K} f\left(U_{j}\right)\right)\right\|_{L^{2}} \tag{8.17}
\end{equation*}
$$

Note that,

$$
\begin{aligned}
\left\|\partial^{\nu}\left(b \cdot \nabla U_{j}\right)\right\|_{L^{2}} & =\sum_{\nu_{1}+\nu_{2}}\left\|\partial^{\nu_{1}} b \cdot \partial^{\nu_{2}} \nabla U_{j}\right\|_{L^{2}} \\
& \leq C(T-t)^{-\left|\nu_{2}\right|-1} \sum_{j=1}^{K}\left(\int\left|\left(\partial^{\nu_{1}} b\right)\left(\lambda_{j} y+\alpha_{j}\right) \partial^{\nu_{2}} \nabla Q_{j}(y)\right|^{2} d y\right)^{\frac{1}{2}} .
\end{aligned}
$$

Since by (2.12) and (4.23),

$$
\left|\left(\partial^{\nu_{1}} b\right)\left(\lambda_{j} y+\alpha_{j}\right)\right| \leq C(T-t)^{\nu_{*}-\left|\nu_{1}\right|}\langle y\rangle^{\nu_{*}+1} .
$$

This yields that

$$
\begin{equation*}
\left\|\partial^{\nu}\left(b \cdot \nabla U_{j}\right)\right\|_{L^{2}} \leq C(T-t)^{\nu_{*}-|\nu|-1} \tag{8.18}
\end{equation*}
$$

Similarly, using (2.13) and (4.23) again we also have

$$
\begin{equation*}
\left\|\partial^{\nu}\left(c U_{j}\right)\right\|_{L^{2}} \leq C(T-t)^{\nu_{*}-|\nu|-1} \tag{8.19}
\end{equation*}
$$

Moreover, we have that for $T$ small enough,

$$
\begin{align*}
\left\|\partial^{\nu}\left(f(U)-\sum_{j=1}^{K} f\left(U_{j}\right)\right)\right\|_{L^{2}} & \leq C(T-t)^{-\frac{8}{d}\left(2+\frac{d}{2}\right)}\left(\sum_{k \neq l} \int\left|U_{k}\right|\left|U_{l}\right| d x\right)^{\frac{1}{2}} \\
& \leq C(T-t)^{-\frac{8}{d}\left(2+\frac{d}{2}\right)} e^{-\frac{\delta}{T-t}} \leq C(T-t)^{\nu_{*}-|\nu|-1} \tag{8.20}
\end{align*}
$$

Therefore, plugging (8.18), (8.19) and (8.20) into (8.17) we arrive at

$$
\left\|\partial^{\nu} \eta\right\|_{L^{2}} \leq C(T-t)^{-2-|\nu|} M o d+C(T-t)^{\nu_{*}-|\nu|-1}
$$

which, via (5.14) and $\nu_{*}=\kappa+3$, yields (5.17) immediately. The proof is complete.

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