

Ultra Unification

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Abstract

Strong, electromagnetic, and weak forces were unified in the Standard Model (SM) with spontaneous gauge *symmetry breaking*. These forces were further conjectured to be unified in a simple Lie group gauge interaction in the Grand Unification (GUT). In this work, we propose a theory beyond the SM and GUT by adding new gapped Topological Phase Sectors consistent with the nonperturbative global anomaly cancellation and cobordism constraints (especially from the baryon minus lepton number $\mathbf{B} - \mathbf{L}$, the electroweak hypercharge Y , and the mixed gauge-gravitational anomaly). Gapped Topological Phase Sectors are constructed via *symmetry extension*, whose low energy contains unitary Lorentz invariant topological quantum field theories (TQFTs): either 3+1d non-invertible TQFT (long-range entangled gapped phase), or 4+1d invertible or non-invertible TQFT (short-range or long-range entangled gapped phase). Alternatively, there could also be right-handed “sterile” neutrinos, gapless unparticle physics, more general interacting conformal field theories, or gravity with topological cobordism constraints, or their combinations to altogether cancel the mixed gauge-gravitational anomaly. We propose that a new high-energy physics frontier beyond the conventional 0d particle physics relies on the new Topological Force and Topological Matter including gapped extended objects (gapped 1d line and 2d surface operators or defects, etc., whose open ends carry deconfined fractionalized particle or anyonic string excitations) or gapless conformal matter. Physical characterizations of these gapped extended objects require the mathematical theories of cohomology, cobordism, or category. Although weaker than the weak force, Topological Force is infinite-range or long-range which does not decay in the distance, and mediates between the linked worldvolume trajectories via fractional or categorical statistical interactions.

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“Die einzig wahre Unsterblichkeit liegt in den eigenen Kindern.”

“The only true immortality resides in one’s children.”

“Meine Arbeit ist Ihnen gewidmet. Sie können dies sicherlich in Ihrem Herzen fühlen.”

“My work is dedicated to you. You can surely feel this in your heart.”

Intermezzo in A major. Andante teneramente, Six Pieces for Piano, Op. 118

Johannes Brahms in 1893

1 Introduction and Summary

Unification is a central theme in theoretical physics. In 1864-1865, Maxwell [1] unified the electricity and magnetism into the electrodynamics theory, where the derived electromagnetic wave manifests the light phenomena. In 1961-1967, Glashow-Salam-Weinberg (GSW) [2–5] made landmark contributions to the electroweak theory of the unified electromagnetic and weak forces between elementary particles, including the prediction of the weak neutral current. The GSW theory together with the strong force [6, 7] is now known as the Standard Model (SM), which is verified to be theoretically and experimentally essential to describe the subatomic high energy physics (HEP). In 1974, Georgi-Glashow hypothesized that at a higher energy, the three gauge interactions of the SM would be merged into a single electronuclear force under a simple Lie group gauge theory, known as the Grand Unification or Grand Unified Theory (GUT) [8, 9].

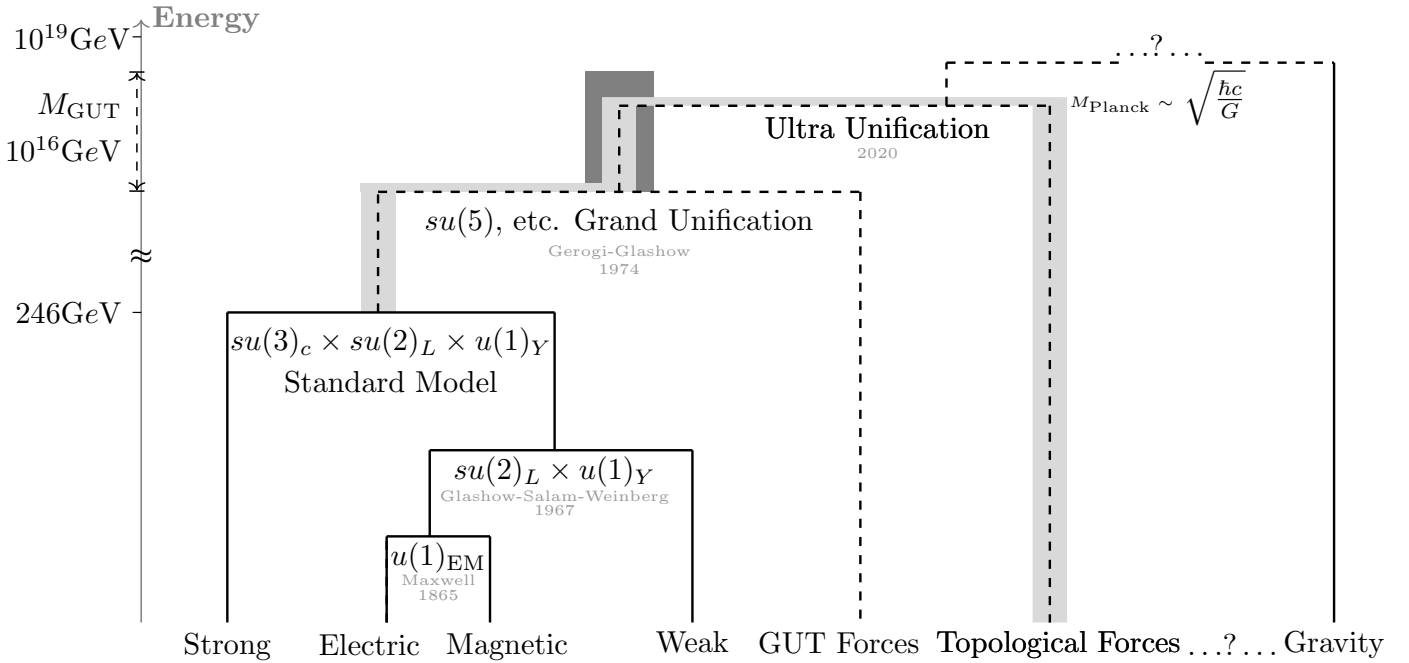


Figure 1: Unification of forces and interactions in fundamental physics. In fact, the proposed Ultra Unification can still play a role of unification even if Grand Unification (GUT) is not favored by Nature or verified by experiments. Namely, based on the anomaly and cobordism constraint, we can still propose the Standard Model (SM) + gapped Topological Phase Sector or gapless conformal sector + Topological Force in a consistent way without GUT. The *dark* gray area schematically shows the possible energy scales for various GUT scenarios, such as the $su(5), so(10), \dots, so(18)$ GUT around 10^{16}GeV . The *light* gray area schematically suggests that the possible energy gap Δ_{TQFT} for Topological Phase Sector can range from as low energy as the SM, to as high energy to somewhere within the GUT scales (e.g., below the $so(10)$ GUT scale). The dashed lines mean hypothetical unifications that have not yet been confirmed by experiments. Forces are arranged from the strongest to the weakest (horizontally from the left to the right) in the electroweak Higgs vacuum.

In this work, following our previous investigations based on nonperturbative global anomalies and cobordism constraints [10–12],¹ we propose an Ultra Unification that a new Topological Force comes into a theme of unification joining with three known fundamental forces and other hypothetical GUT forces. (See

¹Related global anomalies and cobordism constraints on SM, GUT, and BSM are explored also in [13–16]. Freed gave a generalized cohomology description of the $su(5)$ GUT anomalies [13]. Garcia-Etxebarria-Montero [14] and Davighi-Gripaios-Lohitsiri [16] used Atiyah-Hirzebruch spectral sequence (AHSS) to compute the classification of global anomalies. The author with Wen [15] and with Wan [10] used Adams spectral sequence (ASS) [17], Thom-Madsen-Tillmann spectra [18, 19], and

Figure 1.) More concretely, there is a new gapped Topological Phase Sector whose underlying dynamical gauge interactions are the Topological Forces. Alternatively, there could also be “right-handed sterile” neutrinos, gapless or more general interacting conformal field theories, or their combinations to altogether cancel the mixed gauge-gravitational anomaly (enumerated in Sec. 2.3). In a modern perspective, we should view the SM and GUT all as **effective field theories (EFT)** suitable below certain energy scales. Whenever “elementary particles” are mentioned, they only mean to be “elementary field quanta with respect to a given EFT.” Likewise, Ultra Unification should be viewed as an EFT which contains SM or GUT but also additional gapped Topological Phase Sectors with low energy Lorentz invariant unitary topological quantum field theories (TQFTs) of Schwarz type (which is the 4d analog of the 3d Chern-Simons-Witten theories [30–32]), or additional neutrinos, or additional gapless or conformal sectors. Topological Force and the gapped Topological Phase Sector here have specific physical and mathematical meanings, which we will clarify in Sec. 2.6. Before digging into Topological Phase Sector, we should state the assumptions and the logic that lead to the assertion of Ultra Unification, in Sec. 2.²

2 Logic to Ultra Unification

2.1 Assumptions

Our logic leading to Ultra Unification starts with the three **Assumptions** mostly given by Nature and broadly confirmed by experiments:

Freed-Hopkins theorem [20] to obtain the classification of all invertible quantum anomalies. In addition, there in Ref. [10], we can also fully characterize the dd ’t Hooft anomaly [21] of global symmetry G as $(d+1)d$ cobordism invariants, precisely as topological terms of cohomology classes and fermionic topological invariants. The cobordism invariants can be read from the Adams chart in Ref. [10]. Thus we focus on employing Ref. [10] result. By classifying all the invertible quantum anomalies, we must include

- all *local anomalies (perturbative anomalies)*: captured by perturbative Feynman diagram loop calculations, classified by the integer $\mathbb{Z}$ classes (the free classes). e.g., Adler-Bell-Jackiw (ABJ) anomalies [22, 23], perturbative local gravitational anomalies [24]. Typically the local anomalies are detectable via infinitesimal gauge or diffeomorphism transformations that can be continuously deformed from the identity.
- all *global anomalies (nonperturbative anomalies)*: classified by finite abelian groups as a product of $\mathbb{Z}_n$ (the torsion classes) for some positive integer n . e.g., Witten SU(2) [25] and the new SU(2) anomalies [26], global gravitational anomalies [27]. Typically the global anomalies are detectable only via large gauge or diffeomorphism transformations that *cannot* be continuously deformed from the identity.

More examples of dd anomalies characterized by $(d+1)d$ cobordism invariants can be found in [28, 29].

²**Conventions:** We follow the conventions of Ref. [10–12]. We denote nd for n -dimensional spacetime. We also follow the modern condensed matter or extreme quantum matter terminology on the interacting phases of quantum matter [33, 34]. For example,

- *Long-range entangled gapped* topological phases, whose low energy describes the noninvertible TQFTs, are known as intrinsic topological orders, which include examples of fractional quantum Hall states.
- *Short-range entangled gapped* topological phases protected by some global symmetry G , whose low energy describes the invertible TQFTs, are known as symmetry-protected topological states (SPTs) [35], which include examples of topological insulators and topological superconductors [36, 37].
- By a noninvertible TQFT, it means that the absolute value partition function $|\mathbf{Z}(M)| \neq 1$ on a generic spacetime manifold M with nontrivial topology (e.g., cycles or homology classes).
- By an invertible TQFT (iTQFT), it means that the absolute value partition function $|\mathbf{Z}(M)| = 1$ on any spacetime manifold M with any topology. Thus $\mathbf{Z}(M)$ implies the existence of an inverted phase $\mathbf{Z}'(M) \equiv \mathbf{Z}(M)^{-1}$ which defines another iTQFT $\mathbf{Z}'(M)$ that can cancel with the original iTQFT $\mathbf{Z}(M)$, as the stacking of two iTQFTs become a trivial vacuum $\mathbf{Z}(M) \cdot \mathbf{Z}'(M) = 1$ for any M .
- By a trivial gapped vacuum with no TQFT or trivial TQFT, it means that the partition function $\mathbf{Z}(M) = 1$ on any spacetime manifold M with any topology.

We absolutely should distinguish the above beyond-Ginzburg-Landau quantum phases (long-range entangled and short-range entangled states) from the within-Ginzburg-Landau symmetry-breaking phases (long-range and short-range orders and correlations). See more in Sec. 3.5.

1. **Standard Model gauge group G_{SM_q}** : The Standard Model gauge theory has a local Lie algebra $su(3) \times su(2) \times u(1)$, but the global structure of Lie group G_{SM_q} has four versions:³

$$G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_q}, \quad \text{with } q = 1, 2, 3, 6. \quad (2.1)$$

All the quantum numbers of quarks and leptons are compatible with the representations of any version of $q = 1, 2, 3, 6$. To confirm which version is used by Nature, it requires the experimental tests on extended objects such as 1d line or 2d surface operators (see recent expositions in [38–41]).

SU(5) and Spin(10) gauge group: Conventionally, people write the Georgi-Glashow model [8] as the $su(5)$ GUT and Fritzsch-Minkowski model [9] as the $so(10)$ GUT because they have the local Lie algebra $su(5)$ and $so(10)$, respectively. However, they have the precise global Lie group $\text{SU}(5)$ and $\text{Spin}(10)$, respectively. Only $q = 6$, we are allowed to have the embedding of SM gauge group $G_{\text{SM}_6} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6} \subset \text{SU}(5)$ into the $su(5)$ GUT (see more discussions later in (2.13)).

2. **Observed 15 Weyl fermions per generation by experiments**: So far the HEP experiments only confirmed the 15 Weyl fermions per generation of SM.⁴ The experimentalists have *not* yet confirmed the existence of 16th Weyl fermion for each generation. The 16th Weyl fermion (or possibly even more than 16 fermions per generation) is also known as the right-handed neutrino ν_R or the sterile neutrino. Namely, given the number of generation (or family) $N_{\text{gen}} = 3$, we have $15 \times N_{\text{gen}} = 45$ Weyl fermions confirmed in the SM. Similarly, we consider $15 \times N_{\text{gen}} = 45$ Weyl fermions applicable to the $su(5)$ GUT.⁵ (The quantum numbers and representations of the elementary particles in SM and GUT can be found in Table 1 and 2 of [11].) In terms of the quantum numbers of the 15 Weyl fermions per generation of $su(3) \times su(2) \times u(1)$ SM and that of $su(5)$ GUT as chiral gauge theories, in the left-handed (L) Weyl spinor basis, they are in the representations:

$$(\bar{\mathbf{3}}, \mathbf{1}, 1/3)_L \oplus (\mathbf{1}, \mathbf{2}, -1/2)_L \oplus (\mathbf{3}, \mathbf{2}, 1/6)_L \oplus (\bar{\mathbf{3}}, \mathbf{1}, -2/3)_L \oplus (\mathbf{1}, \mathbf{1}, 1)_L \text{ of } su(3) \times su(2) \times u(1) \\ \sim \bar{\mathbf{5}} \oplus \mathbf{10} \text{ of } su(5).$$

Adding the 16th Weyl fermion (the sterile neutrino) as $(\mathbf{1}, \mathbf{1}, 0)_L$ of $su(3) \times su(2) \times u(1)$ gives us $\bar{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{1}$ of $su(5)$ also $\mathbf{16}^+$ of $so(10)$ (precisely the 16-dimensional spinor representation of $\text{Spin}(10)$).

3. **A variant discrete Baryon minus Lepton number ($\mathbf{B} - \mathbf{L}$) is preserved at high energy**: We hypothesize a discrete X symmetry, which is a modified version of $(\mathbf{B} - \mathbf{L})$ number up to some electroweak hypercharge Y [42] is preserved (preserved at least at a higher energy):⁶

$$X \equiv 5(\mathbf{B} - \mathbf{L}) - 4Y. \quad (2.2)$$

The importance of this discrete symmetry $\mathbb{Z}_{4,X}$ (as a mod 4 symmetry of $\text{U}(1)_X$), in the context of global anomalies for SM and GUT is emphasized by Garcia-Etxebarria-Montero [14].

It is easy to check that (e.g., see the Table 1 and 2 of [11]):

- all the particles from $\bar{\mathbf{5}}$ of $su(5)$ GUT has a $\text{U}(1)_X$ charge -3 .
- all the particles from $\mathbf{10}$ of $su(5)$ GUT has a $\text{U}(1)_X$ charge $+1$.

³We denote the lower-case $su, so, \dots$ for the Lie algebra and the upper-case $\text{SU}, \text{SO}, \dots$ for the Lie group.

⁴Here Weyl fermions are spacetime Weyl spinors, which is $\mathbf{2}_L$ of $\text{Spin}(1, 3) = \text{SL}(2, \mathbb{C})$ with a complex representation in the Lorentz signature. On the other hand, the Weyl spinor is $\mathbf{2}_L$ of $\text{Spin}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$ with a pseudoreal representation in the Euclidean signature.

⁵However, the $so(10)$ GUT requires 16 Weyl fermions per generation due to the fermions sit at the $\mathbf{16}$ of the $\text{Spin}(10)$. This fact is used to argue the possibility of topological quantum phase transition between the energy scale of the 15n Weyl-fermion $su(5)$ GUT and 16n Weyl-fermion $so(10)$ GUT in Ref. [12].

⁶Follow [10–12], we choose the convention that the $\text{U}(1)_{\text{EM}}$ electromagnetic charge is $Q_{\text{EM}} = T_3 + Y$. The $\text{U}(1)_{\text{EM}}$ is the unbroken (not Higgsed) electromagnetic gauge symmetry and $T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a generator of $\text{SU}(2)_{\text{weak}}$. The electroweak Higgs field is in the representation $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ of $su(3) \times su(2) \times u(1)$.

- the singlet right-handed neutrino (if any) is in $\mathbf{1}$ of $su(5)$ GUT with a $U(1)_X$ charge $+5$.
- all the fermions of SM has a $\mathbb{Z}_{4,X}$ charge $+1$.
- the electroweak Higgs ϕ has a $U(1)_X$ charge -2 , thus a $\mathbb{Z}_{4,X}$ charge $+2$.

The $\mathbb{Z}_{4,X}$ also contains the fermion parity $\mathbb{Z}_2^F$ (whose operator $(-1)^F$ gives (-1) to all fermions) as a normal subgroup (so $\mathbb{Z}_{4,X}$ generator square $X^2 = (-1)^F$ as the fermion parity):

$$U(1)_X \supset \mathbb{Z}_{4,X} \supset \mathbb{Z}_2^F. \quad (2.3)$$

By looking at these consistent $\mathbb{Z}_{4,X}$ quantum number of SM particles, it is natural to hypothesize the discrete X symmetry plays an important role at a higher energy above the SM energy scale.

In Sec. 2.2, we review the anomaly and cobordism constraints given in [10–12]. Readers can freely skip the technical discussions on anomalies, and directly go to the final logic step lead to Ultra Unification in Sec. 2.3.

2.2 Anomaly and Cobordism Constraints

Based on the three mild and widely accepted assumptions listed in Sec. 2.1, we then impose the constraints from all invertible quantum anomalies via the cobordism calculation on SM and GUT models. The purpose is to check the consistency of the 15n Weyl fermion SM and GUT models:

Check: Perturbative local and nonperturbative global anomalies classified via cobordism.

The classification of dd 't Hooft anomalies of global symmetries G is equivalent to the classification of $(d+1)d$ invertible TQFTs with G -symmetry defined on a G -structure manifold,⁷ given by the cobordism group data $\Omega_G^d \equiv \text{TP}_d(G)$ defined in Freed-Hopkins [20].⁸ The symmetry

$$G \equiv \left(\frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}} \right) \equiv G_{\text{spacetime}} \ltimes_{N_{\text{shared}}} G_{\text{internal}} \quad (2.4)$$

contains the spacetime symmetry $G_{\text{spacetime}}$ and the internal symmetry G_{internal} .⁹ Our perspective is that:

- We can treat the spacetime-internal G as a global symmetry, and we view the anomaly associated with G as 't Hooft anomalies [21] of G symmetry.
- Then, we can ask all obstructions to dynamically gauging the G_{internal} as a gauge group, which give rise to all the dynamical gauge anomaly cancellation conditions that any consistent gauge theory must obey.

To proceed, we follow the results of [10, 11, 43], the relevant total spacetime-internal symmetry G for the SM_q is $G = \text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\text{SM}_q}$ with $q = 1, 2, 3, 6$, and for the $su(5)$ GUT is $G = \text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5)$. Only the $q = 6$ case of SM_6 can be embedded into the $su(5)$ GUT.

⁷For the QFT setup, we only require the category of smooth, differentiable, and triangulable manifolds.

⁸Let us compare the cobordism group $\Omega_G^d \equiv \text{TP}_d(G)$ defined in Freed-Hopkins [20] and the more familiar bordism group Ω_d^G . Here the cobordism group $\Omega_G^d \equiv \text{TP}_d(G)$ *not only* contains $\text{Hom}(\Omega_d^{G,\text{tors}}, U(1))$ (the Pontryagin dual of the torsion subgroup (= tors) of the bordism group Ω_d^G), *but also* contains the integer $\mathbb{Z}$ classes (the free part) descended from the free part of the bordism group $\Omega_{d+1}^{G,\text{free}}$ of one higher dimension. In other words,

- The classification of $(d-1)d$ *nonperturbative global anomalies* can be read from the torsion part (the finite subgroup part) of cobordism group $\Omega_G^{d,\text{tors}} \equiv \text{TP}_d^{\text{tors}}(G)$. It can also be read from the torsion part of the bordism group $\Omega_d^{G,\text{tors}}$ data.
- The classification of $(d-1)d$ *perturbative local anomalies* can be read from the free part (the $\mathbb{Z}$ classes) of cobordism group $\Omega_G^{d,\text{free}} \equiv \text{TP}_d^{\text{free}}(G)$, also from the free part of the bordism group $\Omega_{d+1}^{G,\text{free}}$ data.

In this work, we concern the most for $(d-1) = 4$ and $d = 5$.

⁹• The $G_{\text{spacetime}}$ is the spacetime symmetry, such as the spacetime rotational symmetry $\text{SO} \equiv \text{SO}(d)$ or the fermionic graded spacetime rotational Spin group symmetry $\text{Spin} \equiv \text{Spin}(d)$.

• The G_{internal} is the internal symmetry, such as G_{internal} in the SM as $G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_q}$ with $q = 1, 2, 3, 6$. We also have $G_{\text{internal}} = \text{SU}(5)$ in the $su(5)$ GUT and $G_{\text{internal}} = \text{Spin}(10)$ in the $so(10)$ or $\text{Spin}(10)$ GUT. The N_{shared} is the shared common normal subgroup symmetry between $G_{\text{spacetime}}$ and G_{internal} . The “semi-direct product $\ltimes$ ” extension is due to a group extension from G_{internal} by $G_{\text{spacetime}}$. For a trivial extension, the semi-direct “ $\ltimes$ ” becomes a direct product “ $\times$ ”.

2.2.1 Standard Models

Ref. [10] considers the classification of G -anomalies in 4d given by the $d=5$ cobordism group $\Omega_G^d \equiv \text{TP}_d(G)$ for all **Standard Models** of SM_q with an extra discrete $\mathbb{Z}_{4,X}$ symmetry:

$$\text{TP}_{d=5}(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\text{SM}_q}) = \begin{cases} \mathbb{Z}^5 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{16}, & q = 1, 3. \\ \mathbb{Z}^5 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}, & q = 2, 6. \end{cases} \quad (2.5)$$

Here we summarize the anomaly classification for the Standard Model $G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y}{\mathbb{Z}_q}$ obtained in [10, 11, 43]. Below we write the 4d anomalies in terms of the 5d cobordism invariants or invertible topological quantum field theories (iTQFTs). For perturbative local 4d anomalies, we can also write them customarily as the 6d anomaly polynomials, and their cubic terms of gauge or gravitational couplings in the one-loop triangle Feynman diagram. Here is the list of classifications of anomalies:¹⁰

1. $\text{U}(1)_Y^3$: 4d $\mathbb{Z}$ class local pure gauge anomaly from 5d $\text{CS}_1^{\text{U}(1)} c_1(\text{U}(1))^2$ and 6d $c_1(\text{U}(1))^3$.
2. $\text{U}(1)_Y\text{-SU}(2)^2$: 4d $\mathbb{Z}$ class local pure gauge anomaly from 5d $\text{CS}_1^{\text{U}(1)} c_2(\text{SU}(2))$ and 6d $c_1(\text{U}(1))c_2(\text{SU}(2))$.

¹⁰Here we follow the conventions of [10–12, 28]:

- We can characterize anomalies via (perturbative) *local* anomalies or (nonperturbative) *global* anomalies.
- We can also characterize anomalies via their induced fields: *pure gauge* anomalies, *mixed gauge-gravity* anomalies, or *gravitational* anomalies (those violate the general covariance under coordinate reparametrization; i.e. diffeomorphism).
- The $c_j(G)$ is the j th Chern class of the associated vector bundle of the principal G -bundle.
- We will use CS_{2n-1}^V to denote the Chern-Simons $(2n-1)$ -form for the Chern class (if V is a complex vector bundle) or the Pontryagin class (if V is a real vector bundle). The relation between the Chern-Simons form and the Chern class is $c_n(V) = d\text{CS}_{2n-1}^V$ where the d is the exterior differential and the $c_n(V)$ is regarded as a closed differential form in de Rham cohomology. The $w_j(TM)$ is the j -th Stiefel-Whitney class of spacetime tangent bundle TM of the base manifold M .
- The PD is defined as the Poincaré dual. We define the product notation $c\eta$ between a cohomology class c and a fermionic invariant η via the Poincaré dual PD of cohomology class, thus $c\eta \equiv \eta(\text{PD}(c))$. We use the $\smile$ notation for the cup product between cohomology classes. We often make the cup product $\smile$ and the Poincaré dual PD implicit.
- The μ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = (\frac{\sigma - F \cdot F}{8})(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(1))))$ is related to $\frac{1}{8}(\sigma - F \cdot F)(\text{PD}(c_1(\text{U}(1))))$. Here $\cdot$ is the intersection form of M^4 . The F is the characteristic 2-surface in a 4-manifold M^4 , it obeys the condition $F \cdot x = x \cdot x \pmod 2$ for all $x \in H_2(M^4, \mathbb{Z})$. By the Freedman-Kirby theorem: $(\frac{\sigma - F \cdot F}{8})(M^4) = \text{Arf}(M^4, F) \pmod 2$, here $\text{Arf}(M^4, F)$ is defined to be $\text{Arf}(q)$ where $q: H_1(F, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is a quadratic form associated with the characteristic surface $F \subset M^4$ [44].
- The $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator as a cobordism invariant of the bordism group $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$.
- The η' is a mod 4 index of 1d Dirac operator as a cobordism invariant of the bordism group $\Omega_1^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_4$.
- The Arf invariant [45] is a 2d cobordism invariant of $\Omega_2^{\text{Spin}} = \mathbb{Z}_2$. The Arf appears to be the low energy iTQFT of a 1+1d Kitaev fermionic chain [46], whose boundary hosts a single 0+1d real Majorana zero mode on each of open ends.
- We use the notation “ $\sim$ ” to indicate the two sides are equal in that dimension up to a total derivative term.
- Because of the $\mathbb{Z}_{4,X} \supset \mathbb{Z}_2^F$, we have a short exact sequence $0 \rightarrow \mathbb{Z}_2^F \rightarrow \mathbb{Z}_{4,X} \rightarrow \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F} \rightarrow 0$. Together with the SO structure,

$$1 \rightarrow \mathbb{Z}_2^F \rightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \rightarrow \text{SO} \times \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F} \rightarrow 1. \quad (2.6)$$

Below we use the standard convention, all cohomology classes are pulled back to the manifold M along the maps, thus the gauge field in $H^1(\text{BA}, A)$ can be pulled back to the gauge field in $H^1(M, A)$ for some abelian group A . Eqn. (2.6) is similar to $1 \rightarrow \mathbb{Z}_2^F \rightarrow \text{Spin}^c \equiv \text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1) \rightarrow \text{SO} \times \frac{\text{U}(1)}{\mathbb{Z}_2^F} \rightarrow 1$, so below we compare Spin^c with $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge fields.

- The Spin^c gauge field A is *not* an ordinary abelian gauge field, due to $\frac{1}{2}w_2(TM) = \frac{dA}{2\pi} \pmod \mathbb{Z}$ thus $w_2(TM) = c_1 = \frac{d(2A)}{2\pi} \pmod 2$ [47]; but the $2A$ is an ordinary abelian gauge field in $H^1(M, \frac{\text{U}(1)}{\mathbb{Z}_2^F})$.
- The $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge field $\mathcal{A}_{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}}$ denoted as $\mathcal{A}_{\mathbb{Z}_4}$ in brief is *not* an ordinary abelian gauge field. We define the cohomology classes of background gauge field $\mathcal{A}_{\mathbb{Z}_2} \in H^1(M, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$, which on a $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -manifold M obeys a constraint: $w_2(TM) = \mathcal{A}_{\mathbb{Z}_2}^2 = \text{Sq}^1(\mathcal{A}_{\mathbb{Z}_2}) = \frac{1}{2}\delta\mathcal{A}_{\mathbb{Z}_2}$ with a coboundary operator δ , where $\text{Sq}^1(\mathcal{A}_{\mathbb{Z}_2})$ is defined as the Steenrod square map $H^1(M, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}) \xrightarrow{\text{Sq}^1} H^2(M, \mathbb{Z}_2^F)$. If we can lift the $\mathcal{A}_{\mathbb{Z}_2}$ gauge field to $p^*(\mathcal{A}_{\mathbb{Z}_2})$ via a pullback $\text{Spin} \times \mathbb{Z}_{4,X} \xrightarrow{p} \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$, this requires $w_2(TM) = \mathcal{A}_{\mathbb{Z}_2}^2 = 0$. *Only if* this pullback exists, then $\mathcal{A}_{\mathbb{Z}_4} = p^*(\mathcal{A}_{\mathbb{Z}_2})$ becomes an ordinary abelian $\mathbb{Z}_4$ -gauge field such that $\mathcal{A}_{\mathbb{Z}_2} = \mathcal{A}_{\mathbb{Z}_4} \pmod 2$, so $\mathcal{A}_{\mathbb{Z}_2}^2 = (\mathcal{A}_{\mathbb{Z}_4} \pmod 2)^2 = 0$ because $\frac{1}{2}\delta\mathcal{A}_{\mathbb{Z}_2} = \frac{1}{2}\delta(\mathcal{A}_{\mathbb{Z}_4} \pmod 2) = 2(\frac{1}{4}\delta\mathcal{A}_{\mathbb{Z}_4}) = 0 \pmod 2$.

3. $U(1)_Y$ - $SU(3)^2$: 4d $\mathbb{Z}$ class local pure gauge anomaly from 5d $CS_1^{U(1)} c_2(SU(3))$ and 6d $c_1(U(1))c_2(SU(3))$.
4. $U(1)_Y$ -(gravity) 2 : 4d $\mathbb{Z}$ class local mixed gauge-gravity anomaly from 5d $\mu(PD(c_1(U(1))))$ and 6d $\frac{c_1(U(1))(\sigma-F \cdot F)}{8} \equiv \frac{1}{8}(\sigma - F \cdot F)(PD(c_1(U(1))))$.
5. $SU(3)^3$: 4d $\mathbb{Z}$ class local pure gauge anomaly from 5d $\frac{1}{2}CS_5^{SU(3)}$ and 6d $\frac{1}{2}c_3(SU(3))$.
6. Witten $SU(2)$ anomaly: 4d $\mathbb{Z}_2$ class global mixed gauge-gravity anomaly from 5d $\tilde{\eta} c_2(SU(2)) \equiv \tilde{\eta}(PD(c_2(SU(2))))$ and 6d $\text{Arf } c_2(SU(2)) \equiv \text{Arf}(PD(c_2(SU(2))))$. However, the original Witten $\mathbb{Z}_2$ global anomaly becomes *mutated*:
 - When $q = 1$ or 3 , Witten anomaly *mutated* to a 4d $\mathbb{Z}_4$ class global mixed gauge-gravity anomaly, given by a 5d cobordism invariant $\eta' c_2(SU(2)) \equiv \eta'(PD(c_2(SU(2))))$. There is a short exact sequence $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$ where the $(\mathcal{A}_{\mathbb{Z}_2})c_2(SU(2))$ sits at the $\mathbb{Z}_2$ normal subgroup and the original Witten anomaly $\tilde{\eta} c_2(SU(2))$ sits at the $\mathbb{Z}_2$ quotient group, while the mutated $\eta' c_2(SU(2))$ sits at the $\mathbb{Z}_4$ total group. If we write the group element $k = 2k_N + k_Q \in \mathbb{Z}_4$ via two data (k_N, k_Q) from the normal and quotient groups, then $k\eta' = (2k_N + k_Q)\eta' = k_N \mathcal{A}_{\mathbb{Z}_2} + k_Q \tilde{\eta}$, also $k\eta' c_2(SU(2)) = (2k_N + k_Q)\eta' c_2(SU(2)) = k_N \mathcal{A}_{\mathbb{Z}_2} c_2(SU(2)) + k_Q \tilde{\eta} c_2(SU(2))$.
 - When $q = 2$ or 6 , Witten anomaly *mutated* to become part of 4d $\mathbb{Z}$ class local mixed gauge-gravity anomaly (firstly explained in [48]), given by a 5d cobordism invariant $\frac{1}{2}CS_1^{U(2)} c_2(U(2)) \sim \frac{1}{2}c_1(U(2))CS_3^{U(2)}$. There is a short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$ where the $CS_1^{U(1)} c_2(SU(2)) \sim c_1(U(1))CS_3^{SU(2)}$ sits at the $\mathbb{Z}$ normal subgroup and the original Witten anomaly $\tilde{\eta} c_2(SU(2))$ sits at the $\mathbb{Z}_2$ quotient group, while the mutated $\frac{1}{2}CS_1^{U(2)} c_2(U(2)) \sim \frac{1}{2}c_1(U(2))CS_3^{U(2)}$ sits at the $\mathbb{Z}$ total group. If we write the group element $k = 2\frac{k_{\text{even}}}{2} + (k \bmod 2) \in \mathbb{Z}$ via two data $(\frac{k_{\text{even}}}{2}, k \bmod 2)$ from the normal and quotient groups, then $k\frac{1}{2}CS_1^{U(2)} c_2(U(2)) = \frac{k_{\text{even}}}{2}CS_1^{U(1)} c_2(SU(2)) + (k \bmod 2)\tilde{\eta} c_2(SU(2))$.
7. $(\mathcal{A}_{\mathbb{Z}_2})c_2(SU(2))$: 4d $\mathbb{Z}_2$ global gauge anomaly given by a 5d cobordism invariant $\mathcal{A}_{\mathbb{Z}_2} \smile c_2(SU(2))$.
 - When $q = 1$ or 3 , as explained earlier, it fuses with Witten anomaly to become a $\mathbb{Z}_4$ global anomaly.
 - When $q = 2$ or 6 , this $\mathbb{Z}_2$ global anomaly occurs.
8. $(\mathcal{A}_{\mathbb{Z}_2})c_2(SU(3))$: 4d $\mathbb{Z}_2$ global gauge anomaly.
 - When $q = 1$ or 2 , this is given by a 5d cobordism invariant $\mathcal{A}_{\mathbb{Z}_2} \smile c_2(SU(3))$.
 - When $q = 3$ or 6 , this is given by a 5d cobordism invariant $\mathcal{A}_{\mathbb{Z}_2} \smile c_2(U(3))$.
9. $\eta' c_1(U(1))^2$: 4d $\mathbb{Z}_4$ global mixed gauge-gravity anomaly.
 - When $q = 1$, this is given by a 5d cobordism invariant $\eta' c_1(U(1))^2 \equiv \eta'(PD(c_1(U(1))^2))$.
 - When $q = 2$, this is given by a 5d cobordism invariant $\eta' c_1(U(2))^2 \equiv \eta'(PD(c_1(U(2))^2))$.
 - When $q = 3$, this is given by a 5d cobordism invariant $\eta' c_1(U(3))^2 \equiv \eta'(PD(c_1(U(3))^2))$.
 - When $q = 6$, this is given by a 5d cobordism invariant $\eta' c_1(U(2))^2 \sim \eta' c_1(U(3))^2$.
10. $\eta(PD(\mathcal{A}_{\mathbb{Z}_2}))$: 4d $\mathbb{Z}_{16}$ global mixed gauge-gravity anomaly is given by a 5d cobordism invariant $\eta(PD(\mathcal{A}_{\mathbb{Z}_2}))$. The $\eta(PD(\mathcal{A}_{\mathbb{Z}_2}))$ is the value of Atiyah-Patodi-Singer (APS [49]) eta invariant $\eta \in \mathbb{Z}_{16}$ on the Poincaré dual (PD) submanifold of $\mathcal{A}_{\mathbb{Z}_2}$. It descends from the bordism group calculation of $\Omega_5^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$. Many previous works had also explored these 5d $\mathbb{Z}_{16}$ fermionic invariants [50–52]. The relation between 5d $\eta(PD(\mathcal{A}_{\mathbb{Z}_2}))$ and 4d η is given by the Smith homomorphism [53, 54].

2.2.2 Georgi-Glashow $su(5)$ Grand Unification

Ref. [10, 43] consider the classification of G -anomalies in 4d given by the $d=5$ cobordism group $\Omega_G^d \equiv \text{TP}_d(G)$ for the $su(5)$ GUT with an extra discrete $\mathbb{Z}_{4,X}$ symmetry:

$$\text{TP}_{d=5}(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5)) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_{16}. \quad (2.7)$$

Below we also write down the 4d anomalies in terms of the 5d cobordism invariants or iTQFTs, or the 6d anomaly polynomials:

1. $SU(5)^3$: 4d $\mathbb{Z}$ class local gauge anomaly. It is given by a 5d cobordism invariant $\frac{1}{2}CS_5^{SU(5)}$, or more precisely $\frac{1}{2}((\mathcal{A}_{\mathbb{Z}_2})^2 CS_3^{SU(5)} + CS_5^{SU(5)})$ including the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ contribution.
2. $(\mathcal{A}_{\mathbb{Z}_2})c_2(SU(5))$: 4d $\mathbb{Z}_2$ class global gauge anomaly. It is given by a 5d cobordism invariant $(\mathcal{A}_{\mathbb{Z}_2})c_2(SU(5))$ which detects a mixed anomaly between the $\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}$ gauge field and $SU(5)$ gauge field.
3. $\eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))$: 4d $\mathbb{Z}_{16}$ global mixed gauge-gravity anomaly is given by a 5d cobordism invariant $\eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))$ which detects a mixed anomaly between the $\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}$ gauge field and the gravity. This is again the same $\mathbb{Z}_{16}$ global anomaly from $\Omega_5^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$.

2.3 Consequences lead to Ultra Unification

Ref. [11, 12] checked all the above local and global anomalies enlisted in Sec. 2.2 vanished for the SM_q with $q = 1, 2, 3, 6$ and for the $su(5)$ GUT with 15 Weyl fermions per generation, except the 4d $\mathbb{Z}_{16}$ class global anomaly (the mixed gauge-gravitational anomaly probed by the discrete $\mathbb{Z}_{4,X}$ symmetry and fermionic spacetime rotational symmetry $\text{Spin}(d)$ background fields) may *not* be completely canceled.¹¹ The cobordism invariant for any $\mathbf{v} \in \mathbb{Z}_{16}$ corresponds to a 5d iTQFT partition function

$$\mathbf{Z}_{5\text{d-iTQFT}}^{(\mathbf{v})} \equiv \exp\left(\frac{2\pi i}{16} \cdot \mathbf{v} \cdot \eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))\right) \Big|_{M^5}, \quad \text{with } \eta \equiv \eta_{\text{Pin}^+} \in \mathbb{Z}_{16}, \quad \mathbf{v} \in \mathbb{Z}_{16}. \quad (2.8)$$

Given a $G \supseteq \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure, the cohomology class $\mathcal{A}_{\mathbb{Z}_2} \in H^1(M, \mathbb{Z}_2)$ is the generator from $H^1(B(\mathbb{Z}_{4,X}/\mathbb{Z}_2^F), \mathbb{Z}_2)$. To compute $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \dots} = \mathbb{Z}_{16} \times \dots$, we use the Madsen-Tillmann (MT) spectra [19] to obtain $MT(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}) = M\text{Spin} \wedge \Sigma^{-2}M(\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$ [10], where $\wedge$ is the smash product and the Σ denotes a suspension.¹² Although the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge field $\mathcal{A}_{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}}$ denoted as $\mathcal{A}_{\mathbb{Z}_4}$ in brief is *not* an ordinary abelian gauge field (see footnote 10), the Thom spectra $M(\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$ suggests that the cobordism invariant depends on the $\mathcal{A}_{\mathbb{Z}_2}$ gauge field (instead of the $\mathcal{A}_{\mathbb{Z}_4}$ gauge field) in the cohomology class $H^1(M, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$. The $\eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))$ is the value of APS eta invariant $\eta \in \mathbb{Z}_{16}$ on the Poincaré dual (PD) submanifold of the cohomology class $\mathcal{A}_{\mathbb{Z}_2}$. This PD takes $\cap \mathcal{A}_{\mathbb{Z}_2}$ (of the cohomology H^1) from 5d to 4d

¹¹We should briefly compare the perspectives of Garcia-Etxebarria-Montero [14], Davighi-Gripaios-Lohitsiri [16], and our previous result [11, 12], and those with Wen [15] or with Wan [10]. In terms of the relevancy to the 4d anomalies of SM and $su(5)$ GUT given the spacetime-internal symmetry G ,

- Garcia-Etxebarria-Montero [14] checked: $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$, $\text{Spin} \times SU(n)$, $\text{Spin} \times \text{Spin}(n)$.
- Wang-Wen [15] checked: $G = \frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^F}$, $\frac{\text{Spin} \times \text{Spin}(10)}{\mathbb{Z}_2^F}$, $\text{Spin} \times SU(5)$.
- Davighi-Gripaios-Lohitsiri [16] checked: $G = \text{Spin} \times G_{SM_q}$, $\text{Spin} \times \text{Spin}(n)$, and other GUTs.
- Wan-Wang [10, 43] checked: $G = \text{Spin} \times G_{SM_q}$, $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{SM_q}$, $\text{Spin} \times SU(5)$, $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(5)$, $\text{Spin} \times \text{Spin}(n)$, $\frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^F}$ e.g., $n = 10, 18$, and other GUTs.

Thus, given by the starting assumption in Sec. 2.1, only Wan-Wang [10, 43] contains the complete anomaly classification data for $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{SM_q}$ and $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(5)$ that we need for completing the argument. So we must have to employ the results of Ref. [10]. Although Ref. [14, 16] checked several global anomalies, but they do not exhaust checking all anomalies that we need for $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{SM_q}$ and $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(5)$.

Indeed, specifically for $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times G_{SM_q}$ and $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(5)$, only Ref. [10, 43] exhausted the cobordism classifications for all their possible anomalies, and completed the anomaly cancellation checks on these groups.

¹²For a pointed topological space $\mathcal{X}$, the Σ denotes a suspension $\Sigma \mathcal{X} = S^1 \wedge \mathcal{X} = (S^1 \times \mathcal{X}) / (S^1 \vee \mathcal{X})$ where $\wedge$ and $\vee$ are smash product and wedge sum (a one point union) of pointed topological spaces respectively.

(of the homology H_4). The APS eta invariant $\eta \equiv \eta_{\text{Pin}^+} \in \mathbb{Z}_{16}$ is the cobordism invariant of the bordism group $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$. The notation “ $|_{M^5}$ ” means the evaluation on this invariant on a 5-manifold M^5 . Note that $1 \rightarrow \mathbb{Z}_2^F \rightarrow \text{Pin}^+ \rightarrow \text{O} \rightarrow 1$ and $1 \rightarrow \text{Spin} \rightarrow \text{Pin}^+ \rightarrow \mathbb{Z}_2^T \rightarrow 1$ with the orthogonal group O and time-reversal symmetry $\mathbb{Z}_2^T$, the Madsen-Tillmann (MT) spectra says $M\text{TPin}^+ = M\text{Spin} \wedge \Sigma^1 MT\mathbb{Z}_2^T = M\text{Spin} \wedge S^1 \wedge MT\mathbb{Z}_2^T$. We summarize the consequences and implications of this $\mathbb{Z}_{16}$ anomaly non-vanishing for 15n Weyl fermions in Sec. 2.3.

- Ref. [14] used the existence of $\mathbb{Z}_{16}$ global anomaly and its anomaly cancellation to verify the conventional lore: the 16 Weyl fermions per generation scenario, by introducing a right-handed neutrino per generation.
- Ref. [11, 12] take the $\mathbb{Z}_{16}$ anomaly as a secret entrance to find hidden new sectors beyond the Standard Model, given the fact the HEP experiments only have detected 15 Weyl fermions per generation thus far.

Consequences Follow Sec. 2.1 and 2.2, the $\mathbb{Z}_{16}$ anomaly index for both the SM_q ($q = 1, 2, 3, 6$) and the $su(5)$ GUT is the $(15 = -1 \bmod 16)$ per generation, and ($N_{\text{gen}} = 3$). Given the right-handed neutrino number $n_{\nu_{j,R}}$ (each $\nu_{j,R}$ has $\mathbb{Z}_{4,X}$ charge 1) for the j -th generation, and the new hidden sectors’ anomaly index $\mathbf{v}_{\text{new sectors}}$, we have the following anomaly cancellation condition to cancel the $\mathbb{Z}_{16}$ anomaly:

$$\boxed{-(N_{\text{gen}} = 3) + \left(\sum_{j=e,\mu,\tau,\dots} n_{\nu_{j,R}} \right) + \mathbf{v}_{\text{new sectors}} = 0 \bmod 16.} \quad (2.9)$$

The question is: How to cancel the $\mathbb{Z}_{16}$ anomaly? We enlist as many Scenarios as possible below.

1. **Standard Lore:** We can introduce the right-handed neutrino (the 16th Weyl fermion) number $n_{\nu_{j,R}} = 1$ for each generation (so $n_{\nu_{e,R}} = n_{\nu_{\mu,R}} = n_{\nu_{\tau,R}} = 1$ for electron, muon, and tau neutrinos). In this case, there is no new hidden sector.
 - (1a). **Massless:** $\mathbb{Z}_{4,X}$ can be preserved if fermions are gapless.
 - (1b). **Dirac mass:** $\mathbb{Z}_{4,X}$ is also preserved by the Yukawa-Higgs-Dirac Lagrangian, but $\mathbb{Z}_{4,X}$ is spontaneously broken by the Higgs condensate to give a **Dirac** mass gap.
 - (1c). **Majorana mass:** $\mathbb{Z}_{4,X}$ is broken explicitly by **Majorana** mass term.
2. **Proposals in Ref. [11, 12]:** Ref. [11, 12] proposed other novel ways to cancel the $\mathbb{Z}_{16}$ anomaly. Consequently, we can introduce new hidden sectors beyond the SM and the $su(5)$ GUT:
 - (2a). $\mathbb{Z}_{4,X}$ -symmetry-preserving anomalous gapped 4d topological quantum field theory (TQFT).¹³ We also call the finite *energy gap*

$$\Delta_{\text{TQFT}} \equiv E_{\text{excited}} - E_{\text{ground states}} \quad (2.10)$$

for the first excitation(s) above the ground state sectors of this 4d TQFT as **Topological mass** gap. The underlying quantum system has a 4d intrinsic topological order. We name the anomaly index $\mathbf{v}_{4d}$ for this anomalous 4d TQFT.

- (2b). $\mathbb{Z}_{4,X}$ -symmetry-preserving 5d invertible topological quantum field theory (iTQFT) given by the 5d cobordism invariant in (2.8). The underlying quantum system has a 5d symmetry-protected topological state (SPTs) with an *extra bulk 5th dimension* whose 4d boundary can live the 4d Standard Model world. We name the anomaly index $\mathbf{v}_{5d}$ to specify the boundary 4d anomaly of this 5d iTQFT.

¹³In the context of 3d boundary and 4d bulk, a novel surface topological order was firstly pointed out by Vishwanath-Senthil in an insightful work [55]. Later on many people follow up on developing the surface topological order constructions (see overviews in [33, 34] and [56]). Ref. [11] generalizes this condensed matter idea to find an anomalous symmetric 4d TQFT living on the boundary of 5d fermionic SPTs (2.8). Beware that although many 4d TQFTs listed below have naïve abelian gauge groups (like $[\mathbb{Z}_2]$ or $[\mathbb{Z}_4]$), but due to their fermionic nature and $\mathbb{Z}_2^F$ -symmetry, these 4d TQFT can exhibit nonabelian fusion rules or nonabelian braiding statistics, see Sec. 3.2 for discussions.

- (2c). $\mathbb{Z}_2^F$ -symmetry-preserving 5d bulk $[\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}]$ -gauged TQFT and 4d boundary $[\mathbb{Z}_{4,X}]$ -gauged TQFT. Overall the spacetime rotational symmetry is the fermionic Spin group $\text{Spin}(d)$ graded the bosonic rotation special orthogonal group $\text{SO}(d)$ by the fermion parity $\mathbb{Z}_2^F$.
- (2d). 5d bulk $[\mathbb{Z}_2^F \times \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}]$ -gauged TQFT and 4d boundary $[\mathbb{Z}_2^F \times \mathbb{Z}_{4,X}]$ -gauged TQFT. Overall the spacetime rotational symmetry is the bosonic special orthogonal group $\text{SO}(d)$. If the diffeomorphism symmetry of $\text{SO}(d)$ or $\text{Spin}(d)$ is further dynamically gauged, the outcome new sector may be a gravity theory or a topological gravity theory.
- (2e). $\mathbb{Z}_{4,X}$ -symmetry-breaking gapped phase (e.g. Ginzburg-Landau paradigm phase or 4d TQFT).
- (2f). $\mathbb{Z}_{4,X}$ -symmetry-preserving or $\mathbb{Z}_{4,X}$ -symmetry-breaking gapless phase, e.g., extra massless theories, free or interacting conformal field theories (CFTs). The interacting CFT with scale invariant gapless energy spectrum is also related to **unparticle physics** [57] in the high-energy phenomenology community.

Scenarios (2a) and (2b), and their linear combinations, are the *root phases* for new hidden gapped Topological Phase Sector. For example, *breaking* part of the global symmetries in the linear combined (2a) and (2b) would give rise to Scenario (2e). For another example, *gauging* part of the global symmetries in the linear combined (2a) and (2b) would give rise to other Scenarios:

- Gauging $[\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}]$ in 5d and gauging $[\mathbb{Z}_{4,X}]$ in 4d gives rise to Scenario (2c).
- Gauging $[\mathbb{Z}_2^F \times \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}]$ in 5d and gauging $[\mathbb{Z}_2^F \times \mathbb{Z}_{4,X}]$ in 4d gives rise to Scenario (2d).

The underlying quantum systems in Scenario (2a) and (2c) have the symmetry-enriched topologically ordered state (SETs) in a condensed matter terminology.

We name the combination of the above anomaly cancellation scenarios, including the standard lore (right-handed neutrinos in Scenario 1) and the new proposals (all enlisted in Scenario 2) beyond the SM and the GUT, as the **Ultra Unification**. Although introducing additional CFTs (Scenario (2f)) to cancel the anomaly is equally fascinating, we instead mostly focus on introducing the gapped Topological Phase Sector due to high-energy physics phenomenology (HEP-PH) constraints (see a summary in [11]). A central theme of Ultra Unification suggesting a new HEP frontier is that

$$\begin{aligned} &\textbf{Ultra Unification: HEP-PH provides Gapped Extended Objects} \\ &\text{or Gapless Conformal Objects beyond Particle Physics.} \end{aligned} \tag{2.11}$$

Some more comments:

- These gapped extended objects (of 1d, 2d, 3d, ...) are formulated mathematically in terms of *gapped* TQFT extended operators (1d line, 2d surface, 3d brane, etc., topological operators), beyond the 0d particles.
- These extended operators are heavy in the sense that they sit at the energy scale above the TQFT energy gap Δ_{TQFT} (so at or above the scale of E_{excited} in (2.10)).
- These extended operators are heavy in the sense that they have Topological mass and they can interact with dynamical gravity. So these gapped extended objects may be the Dark Matter candidate. If the Δ_{TQFT} is large, then the gapped extended objects are **heavy Dark Matter** candidates, in contrast the gapless conformal objects are **light Dark Matter** candidates.
- There are fractionalized anyonic excitations at the open ends of topological operators (1d line, 2d surface, 3d brane, etc.). In other words, the particle 1d worldline is the 1d line topological operator. The anyonic string 2d worldsheet is the 2d surface topological operator.¹⁴

¹⁴Of course we know that in above 3d spacetime (such as 4d and 5d that we concern), a 0d particle by itself can only have bosonic or fermionic statistics. A 0d particle does not have fractional or anyonic statistics [58] in above 3d spacetime. However, 1d worldline and 2d worldsheet can be linked in 4d. Triple and quadruple 2d worldsheets can be linked in 4d, etc (See [59–61] and [62–65]). These give rise to “anyonic statistics” to multi excitations of gapped particles or gapped strings in a 4d spacetime and above.

In summary, based on the anomaly cancellation and cobordism constraints, we propose that the SM and Georgi-Glashow $su(5)$ GUT (with 15 Weyl fermions per generation, and with a discrete baryon minus lepton number $\mathbb{Z}_{4,X}$ preserved) contains a new hidden sector that can be a linear combination of above Scenarios [11, 12]. In particular, we can focus on the new hidden sectors given by a 4d TQFT (with the anomaly index ν_{4d}) and a 5d iTQFT (with the 4d boundary's anomaly index ν_{5d}), so (2.9) becomes

$$\boxed{-(N_{\text{gen}} = 3) + \left(\sum_{j=e,\mu,\tau,\dots} n_{\nu_{j,R}} \right) + \nu_{4d} - \nu_{5d} = 0 \pmod{16}.} \quad (2.12)$$

2.4 Symmetry Breaking vs Symmetry Extension: Dirac or Majorana masses vs Topological mass

The distinctions between **Dirac mass**, **Majorana mass**, and **Topological mass** are already explored in Ref. [11]. They represent the Scenarios (1b), (1c), and (2a) respectively in Sec. 2.3. Here we summarize their essences that:

- **Symmetry breaking:** Dirac mass and Majorana mass are induced by *symmetry breaking* — either global symmetry breaking or gauge symmetry breaking, for example via the Anderson-Higgs mechanism or through Yukawa-Higgs term. More precisely, we start from a symmetry group (specifically here an internal symmetry, global or gauged) G , and we break G down to an appropriate subgroup $G_{\text{sub}} \subseteq G$ to induce quadratic mass term for matter fields. Mathematically we write an *injective* homomorphism ι :

$$G_{\text{sub}} \xrightarrow{\iota} G.$$

For example, in Anderson-Higgs mechanism, for a Bardeen-Cooper-Schrieffer type $\mathbb{Z}_2$ -gauged superconductor, we have $G_{\text{sub}} = \mathbb{Z}_2$ and $G = \text{U}(1)$ electromagnetic gauge group. For the SM electroweak Higgs mechanism, we have $G_{\text{sub}} = \frac{\text{SU}(3) \times \text{U}(1)_Y}{\mathbb{Z}_{\text{gcd}(q,3)}}$ and $G = G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_{\text{EM}}}{\mathbb{Z}_q}$ with $q = 1, 2, 3, 6$ and the appropriate greatest common divisor (gcd).

- **Symmetry extension:** Topological mass as the energy gap above a TQFT with 't Hooft anomaly (of a spacetime-internal symmetry G) can be induced by *symmetry extension* [56]. The symmetry extension mechanism extended the original Hilbert space (with nonperturbative global anomalies) to an enlarged Hilbert space by adding extra degrees of freedom to the original quantum system (see [66, 67] for explicit quantum Hamiltonian lattice constructions). The enlarged Hilbert space is meant to trivialize the 't Hooft-anomaly in G in an extended $\tilde{G}$. The pullback r^* of the map

$$\tilde{G} \xrightarrow{r} G$$

can be understood as part of the group extension in an exact sequence [56], while in general this can be generalized as the fibrations of their classifying spaces and higher classifying spaces [68–70]. In many simplified cases, we have a *surjective* homomorphism r in a short-exact sequence of group extension:

$$1 \longrightarrow N_{\text{normal}} \longrightarrow \tilde{G} \xrightarrow{r} G \longrightarrow 1,$$

where $G = \tilde{G} / N_{\text{normal}}$ becomes a quotient group of $\tilde{G}$ whose normal subgroup is N_{normal} . The G -anomaly becoming anomaly-free in the extended $\tilde{G}$ requires the essential use of algebraic topology criteria, such as the Lydon-Hochschild-Serre spectral sequence method [56]. We will explain further details in Sec. 3.2.1.

2.5 Gauging a discrete Baryon B, Lepton L, and Electroweak Hypercharge Y

We provide some more logical motivations why we should preserve $\mathbb{Z}_{4,X}$ and dynamically gauge $\mathbb{Z}_{4,X}$ with $X \equiv 5(\mathbf{B} - \mathbf{L}) - 4Y$, at a higher energy (Scenario (2c) and (2d) in Sec. 2.3):

1. First, as stated before, the $\mathbb{Z}_{4,X}$ is a good global symmetry read from the quantum numbers of SM particles and SM path integral kinematically. It is a global symmetry that has *not yet* been dynamically gauged in the G_{SM_q} nor in the $\text{SU}(5)$ of the $su(5)$ GUT.
2. The $\mathbb{Z}_{4,X} = Z(\text{Spin}(10))$ sits at the center subgroup $\mathbb{Z}_4$ of the $\text{Spin}(10)$ for the $so(10)$ GUT [14]. Thus the $\mathbb{Z}_{4,X}$ must be dynamically gauged, if the $so(10)$ GUT is a correct path to unification *at a higher energy*. It is natural to consider the following group embedding from the GUT to the SM [10–12]:

$$\frac{\text{Spin}(d) \times \text{Spin}(10)}{\mathbb{Z}_2^F} \supset \text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5) \supset \text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}. \quad (2.13)$$

- It is worthwhile mentioning that the $\mathbb{Z}_{16}$ global anomaly (occurred in the cobordism group for the $G = \text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5)$ and $\text{Spin}(d) \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\text{SM}_q}$) disappears in the case of $G = \frac{\text{Spin}(d) \times \text{Spin}(10)}{\mathbb{Z}_2^F}$.

So the $\mathbb{Z}_{4,X}$ can be an anomalous symmetry in the SM_q and the $su(5)$ GUT, but the $\mathbb{Z}_{4,X}$ is an anomaly-free symmetry in the $so(10)$ GUT.¹⁵

- The 15n Weyl fermion SM_q or the 15n Weyl fermion $su(5)$ GUT alone may have a $\mathbb{Z}_{16}$ global anomaly, while they can become anomaly-free at a higher-energy 16n Weyl fermion $so(10)$ GUT [15]. This fact motivates Ref. [12] to propose an analogous concept of topological quantum phase transition happens between two energy scales: (1) above the energy scale of the SM_q or the $su(5)$ GUT, (2) below the energy scale of the $so(10)$ GUT. (See the alternative interpretations of topological quantum phase transition by deforming the SM in Sec. 3.5.)

3. Global symmetry must be gauged or broken in quantum gravity. If for the above reasons, we ask the $\mathbb{Z}_{4,X}$ -symmetry to be preserved, then the $\mathbb{Z}_{4,X}$ must be dynamically gauged at a higher energy for the sake of quantum gravity.¹⁶

¹⁵The anomalous symmetry means a non-onsite symmetry in the condensed matter terminology, that cannot be realized acting only locally on a 0-simplex (a point). The anomaly-free symmetry means an onsite symmetry in the condensed matter terminology, that acts only locally on a 0-simplex (a 0d point), which can be easily gauged by coupling to dynamical variables living on 1-simplices (1d line segments).

¹⁶String theory landscape and swampland program develops the similar concepts of the use of cobordism for quantum gravity, see [71] and References therein. This can be understood as the deformation classes of quantum gravity. The deformation classes of quantum field theory is also proposed by Seiberg in [72]. In our context, we propose that the whole quantum system including the low energy SM plus additional hidden sectors, must correspond to the trivial group element 0 class in $\text{TP}_{d=5}(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\text{SM}_q})$. Similarly, the whole quantum system including the $su(5)$ GUT plus additional hidden sectors, must correspond to the trivial group element 0 class in $\text{TP}_{d=5}(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5))$.

- *Anomaly matching*: To take a step back, for a usual quantum field theory, one can try to match the index ν of 't Hooft anomaly of ultraviolet high energy (UV) with infrared low energy (IR). The index ν is a renormalization group (RG) flow invariant but possibly can be nonzero. This is the *anomaly matching* of the index ν between UV and IR theories.

- *Anomaly cancellation*: Here in contrast, in our case, we consider the whole quantum system (low energy and high energy) into account, due to our assumption that the $\mathbb{Z}_{4,X}$ is preserved thus gauged at the quantum gravity scale, we must have the system *anomaly matched* to a trivial group element with the total index $\nu = 0$ in the cobordism class (similar to [71]). We may also quote this cancellation as the anomaly matching to zero.

If we ignore the dynamical gravity, we can make some comments about the UV completion of Ultra Unification with a local-tensor product Hilbert space (namely, as a regularized quantum lattice model):

- If the $\mathbb{Z}_{4,X} \supseteq \mathbb{Z}_2^F$ is treated as a global internal symmetry (thus not dynamically gauged), the Ultra Unification requires a fermionic Hilbert space with local gauge-invariant fermionic operators such that the $\mathbb{Z}_{4,X} \supseteq \mathbb{Z}_2^F$ acts onsite in the 4d-5d coupled system. Namely, the system can be defined on a manifold with a fermionic $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -structure with its $\mathbb{Z}_{4,X}$ generator square $X^2 = (-1)^F$ as the fermion parity.

- If the $\mathbb{Z}_{4,X} \supseteq \mathbb{Z}_2^F$ is dynamically gauged at higher energy (as it should be), the Ultra Unification requires a bosonic Hilbert space with only local gauge-invariant bosonic operators in the 4d-5d coupled system, without any global symmetry. Namely, the system can be defined on a manifold with a bosonic SO -structure.

Overall, to have the Ultra Unification applied to our Universe's quantum vacuum, it is convenient to regard the $\mathbb{Z}_{4,X}$ looks like a global symmetry (thus a fermionic Hilbert space) at a lower energy around SM scales, but the $\mathbb{Z}_{4,X}$ is eventually dynamically gauged at higher energy (thus eventually a bosonic Hilbert space at the deep UV completion). Moreover, since the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge field is not an ordinary abelian discrete gauge field but with the extra constraint $w_2(TM) = \mathcal{A}_{\mathbb{Z}_2}^2$, we may require to sum over the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -gauge bundle and $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -gauge field $\mathcal{A}_{\mathbb{Z}_4}$ altogether properly.

2.6 Topological Phase Sector and Topological Force

Topological Force and the gapped Topological Phase Sector have specific physical and mathematical meanings in our context. We should clarify what they are, and then what they are not in the next:

1. **Topological Phase Sector:** In Sec. 2.3 and (2.12), we propose a Topological Phase Sector beyond the Standard Model (BSM) includes an appropriate linear combination of the following theories (selecting the best scenario to fit into the HEP phenomenology):
 - (a). **4d long-range entangled gapped topological phase** with an energy gap (named the gap Δ_{TQFT}) whose low energy physics is characterized by a **4d noninvertible topological quantum field theories**. This 4d TQFT is a Schwarz type unitary TQFT (which is the 4d analog of the 3d Chern-Simons-Witten theories [31, 32]).
 - This 4d TQFT has a 't Hooft anomaly [21] of a global symmetry G . We named the anomaly index ν_{4d} for this 4d TQFT.
 - Proper mathematical tools to study this 4d TQFT requires the category or higher category theories.
 - (b). **5d short-range entangled gapped topological phase** with an energy gap whose low energy physics is characterized by a **5d invertible topological quantum field theory (iTQFT)**. This 5d iTQFT is also a unitary TQFT. But the iTQFT is nontrivial distinct from a trivial gapped vacuum only in the presence of a global symmetry G , see Footnote 2. A G -symmetric iTQFT is mathematically given by a G -cobordism invariant, classified by an appropriate cobordism group $\Omega_G^d \equiv \text{TP}_d(G)$, defined in the Freed-Hopkins classification of invertible topological phases (TP) [20].
 - The boundary of this 5d iTQFT has a 4d 't Hooft anomaly of a global symmetry G . We named the anomaly index ν_{5d} for this 5d iTQFT.
 - Proper mathematical tools to study this 4d TQFT requires characteristic classes, cohomology, and cobordism theories.
 - (c). **5d long-range entangled gapped topological phase** with an energy gap whose low energy physics is characterized by a **5d TQFT**. This is the case when the discrete X symmetry is dynamically gauged, stated in Scenario (2c) and (2d) in Sec. 2.3.
2. **Topological Force:** In the context of Sec. 2.3, Topological Force is a discrete gauge force mediated between the linked worldvolume trajectories (1d worldlines, 2d worldsheets from gapped extended operators) via fractional or categorical statistical interactions (See Sec. 5 and 6 of [11]).
 - **Bosonic finite group gauge theory:** The conventional discrete gauge theories are **bosonic** types of finite group gauge theories [73, 74]. Bosonic types mean that their ultraviolet (UV) completion only requires a local tensor product Hilbert space of local (gauge-invariant) bosonic operators; the UV completion does not require local (gauge-invariant) fermionic operators. The underlying TQFT does not require the spin structures and can be defined on non-Spin manifolds (such as the oriented SO structures). The TQFTs are known as bosonic or non-Spin TQFTs.
 - **Fermionic finite group gauge theory:** The discrete gauge theories for our gapped Topological Phase Sectors for the beyond SM hidden sector are **fermionic** types of finite group gauge theories [52, 69]. Fermionic types mean that their UV completion must require local (gauge-invariant) fermionic operators. The underlying TQFT requires the additional spin structures defined on Spin manifolds. The TQFTs are known as Spin TQFTs. In fact the 4d TQFT in Sec. 2.3 requires the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure and can be defined on the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ manifolds (including both Spin manifolds and some non-Spin manifolds).

We should emphasize that our Topological Phase Sector and Topological Force are *not* the kinds of Chern class topological terms which are already summed over in the continuous Lie group gauge theory. Namely, our Topological Phase Sector and Topological Force are **not** the followings:

— **The θ -term with or without a dynamical θ -axion** [75, 76], well-known as $\theta F \wedge F$ or $\theta F \tilde{F}$ in the particle physics, is in fact related to the **second Chern class** $c_2(V_G)$ and the square of the **first Chern class** $c_1(V_G)$ of the associated vector bundle of the gauge group G :

$$\frac{\theta}{8\pi^2} \text{Tr}(F \wedge F) = \frac{\theta}{2} c_1(V_G)^2 - \theta c_2(V_G). \quad (2.14)$$

In particular, here we consider G as the $U(N)$ or $SU(N)$ gauge group, so we can define the Chern characteristic classes associated with complex vector bundles. The V_G is the associated vector bundle of the principal G bundle. This θ -term is a topological term, but it is summed over as a weighted factor to define a Yang-Mills gauge theory partition function [38, 77, 78]. This θ -term does **not** define a quantum system or a quantum phase of matter by itself, distinct from our 4d TQFT (with intrinsic topological order) and 5d iTQFT (with SPTs) as certain unitary quantum phases of matter by themselves.

— **The instantons** [79, 80] or **the sphalerons** [81], are also not the Topological Phase Sector and Topological Force in our context. Instantons and sphalerons are again the objects with nontrivial Chern class integrated over the spacetime manifold. Those objects are already defined as part of the SM and GUT continuous group gauge theories.

As we will mention in Sec. 3.1, we can also include (1) the θ -term with or without a dynamical θ -axion, (2) instantons, and (3) sphalerons, into the Standard Model and the $su(5)$ GUT path integral. These objects are already in the old paradigm of the SM and GUT models. These objects do not affect the anomaly cancellation and cobordism constraints (especially the $\mathbb{Z}_{16}$ global anomaly) discussed in Sec. 2.2.¹⁷ These objects belong to the Standard Model and the $su(5)$ GUT path integral (Sec. 3.1), not to the Topological Phase Sector (TQFT) path integral (Sec. 3.2), but they all can be included as part of the Ultra Unification path integral (Sec. 3).

3 Ultra Unification Path Integral

In this section, we provide the functional path integral (i.e., partition function) $\mathbf{Z}_{\text{UU}}$ of Ultra Unification, which includes the standard paradigm of the Standard Model path integral $\mathbf{Z}_{\text{SM}}$ or the Georgi-Glashow $su(5)$ GUT path integral $\mathbf{Z}_{\text{GUT}}$ in Sec. 3.1. Then we provide the Topological Phase Sector TQFT path integral $\mathbf{Z}_{\text{TQFT}} \equiv \mathbf{Z}_{\text{5d-iTQFT}} \cdot \mathbf{Z}_{\text{4d-TQFT}}$ in Sec. 3.2

3.1 Standard Model and the $su(5)$ GUT Path Integral coupled to $X \equiv 5(\mathbf{B} - \mathbf{L}) - 4Y$

3.1.1 Standard Model Path Integral coupled to X

Now we describe the Standard Model (SM) path integral in the Minkowski (or Lorentz) signature:

$$\mathbf{Z}_{\text{SM}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}][\mathcal{D}A][\mathcal{D}\phi] \dots \exp(i S_{\text{SM}}[\psi, \bar{\psi}, A, \phi, \dots, \mathcal{A}_{\mathbb{Z}_4}]|_{M^4}). \quad (3.1)$$

¹⁷However, those θ terms in Yang-Mills gauge theory may affect the higher anomalies involving higher generalized global symmetries, see for example [77, 78].

The ... depends on the details of which variant versions of SM that we look at (e.g., adding axions or not). In the schematic way, we have the action S :

$$S_{\text{SM}} = \int_{M^4} \left(\text{Tr}(F_I \wedge \star F_I) - \frac{\theta_I}{8\pi^2} g_I^2 \text{Tr}(F_I \wedge F_I) \right) + \int_{M^4} \left(\bar{\psi} (i \not{D}_{A, \mathcal{A}_{Z_4}}) \psi + |D_{\mu, A, \mathcal{A}_{Z_4}} \phi|^2 - U(\phi) - (\psi_L^\dagger \phi \psi_R + \text{h.c.}) \right) d^4x. \quad (3.2)$$

But more precisely we really need more details in the Lagrangian $\mathcal{L}$ with Weyl fermions, with $S \equiv \int \mathcal{L} d^4x$:

$$\begin{aligned} \mathcal{L}_{\text{SM}} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\theta\text{-Chern}} + \mathcal{L}_{\text{Weyl}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa-Higgs}} \\ &= \sum_{I=1,2,3} -\frac{1}{4} F_{I,\mu\nu}^a F_I^{a\mu\nu} - \frac{\theta_I}{64\pi^2} g_I^2 \epsilon^{\mu\nu\mu'\nu'} F_{I,\mu\nu}^a F_{I,\mu'\nu'}^a + \psi_L^\dagger (i \bar{\sigma}^\mu D_{\mu, A, \mathcal{A}_{Z_4}}) \psi_L + \psi_R^\dagger (i \sigma^\mu D_{\mu, A, \mathcal{A}_{Z_4}}) \psi_R \\ &\quad + |D_{\mu, A, \mathcal{A}_{Z_4}} \phi|^2 - U(\phi) - (\psi_L^\dagger \phi \psi_R + \text{h.c.}). \end{aligned} \quad (3.3)$$

Here come some Remarks:

1. Yang-Mills gauge theory [82] has the action $S_{\text{YM}} = \int \text{Tr}(F \wedge \star F)$ and Lagrangian $\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$. The F is the Lie algebra valued field strength curvature 2-form $F = dA - igA \wedge A$, with its Hodge dual $\star F$, all written in differential forms. In the trace “Tr” we pick up a Lie algebra representation $\mathbf{R}$ whose Lie algebra generators T^a labeled by “a.” We have also the subindex $I = 1, 2, 3$ to specify the SM Lie algebra sectors $u(1)$, $su(2)$, or $su(3)$.

More precisely $F = \frac{1}{2} F_{\mu\nu} (dx^\mu \wedge dx^\nu) = \frac{1}{2} F_{\mu\nu}^a T^a (dx^\mu \wedge dx^\nu)$, and we define the commutator $[T^b, T^c] = i f^{bcd} T^d$ with a structure constant f^{bcd} , then $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{bca} A_\mu^b A_\nu^c$. Note that $\text{Tr}(T^a T^b) = C(\mathbf{R}) \delta^{ab}$ for some constant of representation $\mathbf{R}$. Here for the fundamental representation $\mathbf{R}$, we take $\text{Tr}(T^a T^a) = \frac{1}{2}$. Then we have $\text{Tr}(F \wedge \star F) = (-1)^s \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) d^4x = (-1)^s (\frac{1}{4}) F_{\mu\nu}^a (F^a)^{\mu\nu} d^4x$ with the $(-1)^s$ as the sign of the determinant of the spacetime metric. Here $(-1)^s = -1$ in the Minkowski signature. Notice that here we normalize the $u(1)$ Yang-Mills in (3.2) and (3.3) slightly differently from the conventional $u(1)$ Maxwell theory by scaling a factor $\text{Tr}(1)$ of the $u(1)$ by a $\frac{1}{2}$.

The θ -term and dynamical θ -axion: We can also introduce the Chern class topological θ -term $S_{\theta\text{-Chern}} = -\int \frac{\theta}{8\pi^2} g^2 \text{Tr}(F \wedge F)$ and $\mathcal{L}_{\theta\text{-Chern}} = -\frac{\theta}{64\pi^2} g^2 \epsilon^{\mu\nu\mu'\nu'} F_{\mu\nu}^a F_{\mu'\nu'}^a$. Given a $U(N)$ or $SU(N)$ bundle V_G and its field strength $\hat{F}$, the first and second Chern classes are given by $c_1(V_G) = \frac{\text{Tr} F}{2\pi}$ and $c_2(V_G) = -\frac{1}{8\pi^2} \text{Tr}(F \wedge F) + \frac{1}{8\pi^2} (\text{Tr} F) \wedge (\text{Tr} F)$, so that $\frac{\theta}{8\pi^2} \text{Tr}(F \wedge F) = \frac{\theta}{2} c_1(V_G)^2 - \theta c_2(V_G)$. If a dynamical θ -axion [75, 76] is introduced, it requires a summation of the compact θ in the path integral measure $\int [\mathcal{D}\theta]$.

There is an overall constant that can be absorbed into the field A and coupling g redefinition.¹⁸

The path integral $\int [\mathcal{D}A]$ for continuous Lie group gauge field theory (here $U(1)$, $SU(N)$, $U(N)$ for the SM and $su(5)$ GUT), really means (1) the summation of all inequivalent principal gauge bundles P_A , and then (2) the summation of all inequivalent gauge connections $\tilde{A}$ (under a given specific principal gauge bundles P_A), where $\tilde{A}$ is a (hopefully globally defined physically) 1-form gauge connection. So we physically define:

$$\int [\mathcal{D}A] \dots \equiv \sum_{\text{gauge bundle } P_A} \int [\mathcal{D}\tilde{A}] \dots$$

¹⁸For example, by redefining $A \rightarrow A' = \frac{1}{g} A$ and $F \rightarrow F' = \frac{1}{g} F$, then $F' = dA' - iA' \wedge A'$ and $F_{\mu\nu}'^a = \partial_\mu A_\nu'^a - \partial_\nu A_\mu'^a + f^{bca} A_\mu'^b A_\nu'^c$. Then we can also write

$$S_{\text{YM}} + S_{\theta\text{-Chern}} = \frac{1}{g^2} \int \text{Tr}(F' \wedge \star F') - \int \frac{\theta}{8\pi^2} \text{Tr}(F' \wedge F'), \text{ and } \mathcal{L}_{\text{YM}} + \mathcal{L}_{\theta\text{-Chern}} = -\frac{1}{4g^2} F_{\mu\nu}'^a F'^{a\mu\nu} - \frac{\theta}{64\pi^2} \epsilon^{\mu\nu\mu'\nu'} F_{\mu\nu}'^a F_{\mu'\nu'}^a.$$

2. Dirac fermion theory has $S_{\text{Dirac}} = \int \bar{\psi}(\mathbf{i}\not{D}_A)\psi \, d^4x$ and $\mathcal{L}_{\text{Dirac}} = \bar{\psi}(\mathbf{i}\not{D})\psi$ with the Dirac spinor ψ defined as a section of the spinor bundles. The $\bar{\psi}(\mathbf{i}\not{D})\psi$ is an inner product in the complex vector space with the Dirac operator $\not{D}_A$ as a natural linear operator in the vector space. The path integral $\int[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]$ is (1) the summation of all inequivalent spinor bundles, and then (2) the summation of all inequivalent sections (as spinors) of spinor bundles (under a given specific spinor bundle). We require the spin geometry and spin manifold, in particular we require the $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure.

In fact preferably we present not in the Dirac spinor basis, but we present all of (3.3) in the Weyl spinor basis (below).

3. Weyl fermion theory has $S_{\text{Weyl}} = \int \mathcal{L}_{\text{Weyl}} \, d^4x = \int \psi_L^\dagger(\mathbf{i}\bar{\sigma}^\mu D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}})\psi_L \, d^4x$. Weyl spinor bundle splits the representation of the Dirac spinor bundle. Weyl spinor again is defined as the section of Weyl spinor bundle. The Weyl spacetime spinor is in $\mathbf{2}_L$ of $\text{Spin}(1,3) = \text{SL}(2,\mathbb{C})$ with a complex representation in the Lorentz signature, or $\mathbf{2}_L$ of $\text{Spin}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$ with a pseudoreal representation in the Euclidean signature. We also write the analogous right-handed Weyl fermion theory. The σ^μ and $\bar{\sigma}^\mu$ are the standard spacetime spinor rotational $su(2)$ Lie algebra generators. We will emphasize and illuminate the meanings of covariant derivative $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ altogether in Remark 5.
4. Higgs theory has $S_{\text{Higgs}} = \int \mathcal{L}_{\text{Higgs}} \, d^4x = \int (|D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}\phi|^2 - U(\phi)) \, d^4x$ with gauged kinetic and potential terms. The Higgs field bundle is typically a trivial complex line bundle.¹⁹ The Higgs scalar field is the section of a field bundle. The electroweak Higgs is in complex value $\mathbb{C}$ and also in $\mathbf{2}$ of $\text{SU}(2)$ gauge field. Again by doing summation $\int[\mathcal{D}\phi]$ we (1) sum over the field bundles, and (2) sum over the section of each field bundle. We illuminate the meanings of covariant derivative $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ altogether in Remark 5.
5. Covariant derivative operator $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ in (3.3) is defined as:

$$D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}} \equiv \nabla_\mu - \mathbf{i}g \, q_{\mathbf{R}} A_\mu - \mathbf{i}q_X \mathcal{A}_{\mathbb{Z}_4,\mu}. \quad (3.4)$$

Placed on a curved spacetime (with a non-dynamical metric, only with background gravity) requires a covariant derivative ∇_μ , and a spin connection for the spinors.

Comments about the term $g \, q_{\mathbf{R}} A$ in a differential form (e.g., quantum numbers read from Table 1 in [11]):

$$g \, q_{\mathbf{R}} A \equiv (q_e A_{u(1),\mu} + g_{su(2)} \sum_{a=1}^3 \frac{\zeta^a}{2} A_{su(2),\mu}^a + g_{su(3)} \sum_{a=1}^8 \frac{\tau^a}{2} A_{su(3),\mu}^a) dx^\mu. \quad (3.5)$$

The ζ^a and τ^a are the rank-2 and rank-3 Lie algebra generator matrix representations for $su(2)$ and $su(3)$ respectively. The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acting on ψ_L contains the $su(2)$ gauge field. The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acting on ψ_R does not contain the $su(2)$ gauge field, because the $su(2)$ weak interaction is a maximally parity violating chiral gauge theory. The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acting only on quarks (both ψ_L and ψ_R) contain the $su(3)$ gauge field. The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acting on ϕ contains the $su(2) \times u(1)$ gauge field.

Comments about the term $q_X \mathcal{A}_{\mathbb{Z}_4}$ (e.g., quantum numbers read from Table 1 and 2 in [11]):

- The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acts on all left-handed SM Weyl fermion ψ_L via its $\mathcal{A}_{\mathbb{Z}_4}$ charge $q_X = 1$.
- The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acts on all right-handed SM Weyl fermion ψ_R via its $\mathcal{A}_{\mathbb{Z}_4}$ charge $q_X = -1$.
- The $D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}$ acts on the electroweak Higgs ϕ via its $\mathcal{A}_{\mathbb{Z}_4}$ charge $q_X = 2$.

The subtle part is that $\mathcal{A}_{\mathbb{Z}_4}$ should be treated as a cohomology class, such as a cohomology or cochain gauge field. The $\mathcal{A}_{\mathbb{Z}_4} \in H^1(M, \mathbb{Z}_4)$ is the generator from $H^1(B\mathbb{Z}_{4,X}, \mathbb{Z}_4)$. In physics, for the continuum QFT theorists who prefer to think $\mathbb{Z}_{4,X} \subset \text{U}(1)_X$ as a continuum gauge field breaking down to a discrete $\mathbb{Z}_{4,X}$, we can introduce an extra $\mathbb{Z}_4$ charge new Higgs field φ and its potential $V(\varphi)$:

$$\psi_L^\dagger(\mathbf{i}\bar{\sigma}^\mu D_{\mu,\mathcal{A}_{\mathbb{Z}_4}})\psi_L + \psi_R^\dagger(\mathbf{i}\sigma^\mu D_{\mu,\mathcal{A}_{\mathbb{Z}_4}})\psi_R + |(\partial_\mu - \mathbf{i}2\mathcal{A}_{\mathbb{Z}_4,\mu})\phi|^2 + |(\partial_\mu - \mathbf{i}4\mathcal{A}_{\mathbb{Z}_4,\mu})\varphi|^2 + V(\varphi) + \dots \quad (3.6)$$

¹⁹In Higgs theory, people in general do not consider nontrivial complex line bundles for Higgs field. But it may be amusing to consider the alternative.

There are extra superconductivity-like term $\phi^2\varphi^\dagger + (\phi^\dagger)^2\varphi$ does not break the $\mathbb{Z}_4$. Their $\mathbb{Z}_4$ or $U(1)$ transformations are:

$$\begin{aligned}\psi_L &\rightarrow \psi_L e^{i\frac{2\pi}{4}}, \quad \psi_R \rightarrow \psi_R e^{-i\frac{2\pi}{4}}, \quad \phi \rightarrow \phi e^{i2\frac{2\pi}{4}}, \quad \varphi \rightarrow \varphi. \\ \psi_L &\rightarrow \psi_L e^{i\Theta}, \quad \psi_R \rightarrow \psi_R e^{-i\Theta}, \quad \phi \rightarrow \phi e^{i2\Theta}, \quad \varphi \rightarrow \varphi e^{i4\Theta}.\end{aligned}$$

In the $\langle\varphi\rangle \neq 0$ condensed Higgs phase as a discrete $\mathbb{Z}_4$ gauge theory, which we can dualize the theory as a level-4 BF theory [83]. The formulation starts from adding $\int[D\varphi]$ in the path integral measure, and it ends with a 2-form $\mathcal{B}$ and 1-form gauge field $\mathcal{A}_{\mathbb{Z}_4}$

$$\int [D\mathcal{B}][D\mathcal{A}_{\mathbb{Z}_4}] \exp(i\frac{4}{2\pi} \int_{M^4} \mathcal{B} \wedge d\mathcal{A}_{\mathbb{Z}_4} + \dots). \quad (3.7)$$

But more precisely, we really should formulate in terms of a cohomology/cochain TQFT and taking care of the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure, which we will do in Sec. 3.2 (also in Sec. 5 of [11]).

6. Yukawa-Higgs-Dirac term has $S_{\text{Yukawa-Higgs-Dirac}} = \int \mathcal{L}_{\text{Yukawa-Higgs-Dirac}} d^4x = \int (\psi_L^\dagger \phi \psi_R + \text{h.c.}) d^4x$. In this case, we pair the $\psi_L^\dagger$'s $\bar{\mathbf{2}}$ of $SU(2)$ with the ϕ 's $\mathbf{2}$ of $SU(2)$, and vice versa pair ψ_L with $\phi^\dagger$ to get an $SU(2)$ singlet. The right-handed ψ_R here is (meant to be) an $SU(2)$ singlet. This Yukawa-Higgs-Dirac term at the kinetic level also preserves the $\mathbb{Z}_{4,X}$, although the Higgs vacuum expectation value (vev) breaks the $\mathbb{Z}_{4,X}$ dynamically.

7. Yukawa-Higgs-Majorana term with Weyl fermion:

We can add Yukawa-Higgs-Majorana term for Weyl fermions. For example, for the left-handed ψ_L , we can add a dimension-5 operator:

$$S_{\text{Yukawa-Higgs-Majorana}} = \int -(\psi_L^\dagger \phi (\phi i\sigma^2 \psi_L^*) + \text{h.c.}) d^4x = \int -(\psi_L^\dagger \phi (\phi i\sigma^2 \psi_L^*) + (-\psi_L^T i\sigma^2 \phi^*) \phi^\dagger \psi_L) d^4x.$$

Again the σ^2 is from the σ^μ of the spacetime spinor rotational $su(2)$ Lie algebra generators. Renormalizability is not an issue because we are concerned the effective field theory. For the right-handed ψ_R , we can add a dimension-3 operator for some Majorana mass coupling M :

$$S_{\text{Yukawa-Higgs-Majorana}} = \int -M(\psi_R^T (i\sigma^2) \psi_R + \text{h.c.}) d^4x = \int -M(\psi_R^T (i\sigma^2) \psi_R + \psi_R^\dagger (-i\sigma^2) \psi_R^*) d^4x,$$

which breaks the lepton number conservation. However, in either cases, both Yukawa-Higgs-Majorana terms above break the $\mathbb{Z}_{4,X}$ explicitly. So they are not encouragingly favored if we pursue the $\mathbb{Z}_{4,X}$ -preserving theory at least at higher energy.

We have presented above the Standard Model coupled to a discrete X gauge field in the path integral (3.1), the action (3.2), and the Lagrangian (3.3). Below we can quickly modify a few terms to obtain the $su(5)$ Grand Unification coupled to a discrete X gauge field.

3.1.2 The $su(5)$ Grand Unification Path Integral coupled to X

We have the $su(5)$ GUT path integral coupled to X :

$$\mathcal{Z}_{\text{GUT}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \int [D\psi][D\bar{\psi}][D\mathcal{A}][D\phi] \dots \exp(i S_{\text{GUT}}[\psi, \bar{\psi}, \mathcal{A}, \phi, \dots, \mathcal{A}_{\mathbb{Z}_4}]|_{M^4}). \quad (3.8)$$

We should write all $\bar{\mathbf{5}}$ and $\mathbf{10}$ of the $SU(5)$ as the left-handed Weyl fermions ψ_L , so there are 15 Weyl fermions ψ_L per generation. In Sec. 2.3, we may or may not introduce the right-handed neutrinos here denoted as χ_R . In the schematic way, we have the action:

$$\begin{aligned}S_{\text{GUT}} = \int_{M^4} &\left(\text{Tr}(F \wedge \star F) - \frac{\theta}{8\pi^2} g^2 \text{Tr}(F \wedge F) \right) + \int_{M^4} \left(\psi_L^\dagger (i\bar{\sigma}^\mu D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}) \psi_L + \chi_R^\dagger (i\sigma^\mu D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}}) \chi_R \right. \\ &\left. + |D_{\mu,A,\mathcal{A}_{\mathbb{Z}_4}} \phi|^2 - U(\phi) - (\psi_L^\dagger \phi (i\sigma^2 \psi_L'^*) + \text{h.c.}) + \dots \right) d^4x. \quad (3.9)\end{aligned}$$

We are left now only with an $SU(5)$ gauge field whose 1-form connection written as:

$$g \, q_{\mathbf{R}} \, A = (g_{su(5)} \sum_{a=1}^{24} T^a A_{su(5),\mu}^a) dx^\mu. \quad (3.10)$$

We require the T^a as the rank-5 and rank-10 Lie algebra generator matrix representations for $su(5)$ to couple to $\mathbf{5}$ and $\mathbf{10}$ of $SU(5)$ respectively. Yukawa-Higgs pairs the appropriate ψ_L and ψ'_L Weyl fermions. We may or may not introduce the Majorana mass terms to χ_R in the $\dots$, while the consequences are already discussed (which break the $\mathbb{Z}_{4,X}$ explicitly) in the Remark 7. The discussions about this path integral (3.8) directly follow the above Remarks 1-7, so we should not repeat.

3.2 Topological Phase Sector and TQFT Path Integral coupled to $X \equiv 5(\mathbf{B} - \mathbf{L}) - 4Y$

The 15n Weyl fermion SM and $su(5)$ GUT path integrals, (3.1) and (3.8), are *not* gauge invariant under the $\mathbb{Z}_{4,X}$ gauge transformation only when the gravitational background is turned on to probe the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -structure via the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge field $\mathcal{A}_{\mathbb{Z}_4}$.

- If the $\mathbb{Z}_{4,X}$ is only coupled to a background gauge field, this only means the system has 't Hooft anomaly under the $\mathbb{Z}_{4,X}$ anomalous symmetry and the spacetime (Spin group) coordinate reparametrization transformations (e.g., the Euclidean rotation or Lorentz boost part of diffeomorphism).
- If the $\mathbb{Z}_{4,X}$ is dynamically gauged and preserved at high energy, then we must append a new sector to make the whole theory well-defined.

In any case, follow one of the scenario in Sec. 2.3, we now provide a path integral including Topological Quantum Field Theories (TQFTs) to make the whole theory free from the $\mathbb{Z}_{16}$ global anomaly (2.12):

$$(-(N_{\text{gen}} = 3) + n_{\nu_{e,R}} + n_{\nu_{\mu,R}} + n_{\nu_{\tau,R}} + \mathfrak{v}_{4d} - \mathfrak{v}_{5d}) = 0 \pmod{16}.$$

1. The 5d iTQFT partition function is given by (2.8):

$$\mathbf{Z}_{5d\text{-iTQFT}}^{(\mathfrak{v}_{5d})}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \exp\left(\frac{2\pi i}{16} \cdot \mathfrak{v}_{5d} \cdot \eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))\right)\Big|_{M^5}, \quad \text{with } \mathfrak{v}_{5d} \in \mathbb{Z}_{16}, \quad \mathcal{A}_{\mathbb{Z}_2} \in H^1(M, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}). \quad (3.11)$$

2. We propose the full gauge-invariant path integral, invariant under the mixed gauge-gravity transformation (i.e., gauge-diffeomorphism) of $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure and free from its $\mathbb{Z}_{16}$ global anomaly as follows. The SM version employs (3.1) into:

$$\boxed{\mathbf{Z}_{\text{UU}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}/4d\text{-SM+TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}}^{(-\mathfrak{v}_{5d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4d\text{-TQFT}}^{(\mathfrak{v}_{4d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{\text{SM}}^{(n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}})}[\mathcal{A}_{\mathbb{Z}_4}].} \quad (3.12)$$

The GUT version employs (3.8) into:

$$\boxed{\mathbf{Z}_{\text{UU}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}/4d\text{-GUT+TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}}^{(-\mathfrak{v}_{5d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4d\text{-TQFT}}^{(\mathfrak{v}_{4d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{\text{GUT}}^{(n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}})}[\mathcal{A}_{\mathbb{Z}_4}].} \quad (3.13)$$

3.2.1 Symmetry extension $[\mathbb{Z}_2] \rightarrow \text{Spin} \times \mathbb{Z}_{4,X} \rightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ and a 4d fermionic discrete gauge theory

Below we ask whether we can construct a fully gauge-diffeomorphism invariant 5d-4d coupled partition function preserving the $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ structure:

$$\mathbf{Z}_{5d\text{-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4d\text{-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}]. \quad (3.14)$$

Preserving the $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ structure means that under the spacetime coordinate background transformation (i.e., diffeomorphism) and the $\mathcal{A}_{\mathbb{Z}_4}$ background gauge transformation, the 5d-4d coupled partition function is still fully gauge-diffeomorphism invariant.

First, we can rewrite the 5d iTQFT partition function (3.11) on a 5d manifold M^5 into

$$\begin{aligned} \mathbf{Z}_{5\text{d-iTQFT}}^{(\mathbf{v})}[\mathcal{A}_{\mathbb{Z}_4}] &= \exp\left(\frac{2\pi i}{16} \cdot \mathbf{v} \cdot \eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))\right)\Bigg|_{M^5} \\ &= \exp\left(\frac{2\pi i}{16} \cdot \mathbf{v} \cdot \left(8 \cdot \frac{p_1(TM)}{48}(\text{PD}(\mathcal{A}_{\mathbb{Z}_2})) + 4 \cdot \text{Arf}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^3)) + 2 \cdot \tilde{\eta}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^4)) + (\mathcal{A}_{\mathbb{Z}_2})^5\right)\right)\Bigg|_{M^5} \\ &= \exp\left(\frac{2\pi i}{16} \cdot \mathbf{v} \cdot \left(8 \cdot \frac{\sigma}{16}(\text{PD}(\mathcal{A}_{\mathbb{Z}_2})) + 4 \cdot \text{Arf}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^3)) + 2 \cdot \tilde{\eta}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^4)) + (\mathcal{A}_{\mathbb{Z}_2})^5\right)\right)\Bigg|_{M^5}, \quad (3.15) \end{aligned}$$

for a generic $\mathbf{v} = -N_{\text{generation}} \in \mathbb{Z}_{16}$.

- The $p_1(TM)$ is the first Pontryagin class of spacetime tangent bundle TM of the manifold M . Via the Hirzebruch signature theorem, we have $\frac{1}{3} \int_{\Sigma^4} p_1(TM) = \sigma(\Sigma^4) = \sigma = \frac{1}{8\pi^2} \int \text{Tr}(R(\omega) \wedge R(\omega))$ on a 4-manifold Σ^4 , where σ is the signature of Σ^4 while ω is the 1-connection of tangent bundle and $R(\omega)$ is the Riemann curvature 2-form of ω .

So in (3.15), we evaluate the $\frac{1}{3} \int_{\Sigma^4} p_1(TM) = \sigma$ on the Poincaré dual (PD) of Σ^4 manifold of the $(\mathcal{A}_{\mathbb{Z}_2})$ cohomology class within the M^5 .²⁰

- The $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator as a cobordism invariant of $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$. The 1d manifold generator of $\tilde{\eta}$ is a circle S^1 with a periodic boundary condition (i.e., Ramond) for the fermion.
- The Arf invariant [45] is a mod 2 cobordism invariant of $\Omega_2^{\text{Spin}} = \mathbb{Z}_2$, whose quantum matter realization is the 1+1d Kitaev fermionic chain [46] whose each open end hosts a 0+1d Majorana zero mode.
- The $(\mathcal{A}_{\mathbb{Z}_2})^5$ is a mod 2 class purely bosonic topological invariant, which corresponds to a 5d bosonic SPT phase given by the group cohomology class data $H^5(B\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, which is also one of the $\mathbb{Z}_2$ generators in $\Omega_5^{\text{SO}}(B\mathbb{Z}_2)$.

1. When $\mathbf{v}$ is odd, such as $\mathbf{v} = 1, 3, 5, 7, \dots \in \mathbb{Z}_{16}$,

Ref. [51] suggested that the symmetry-extension method [56] cannot construct a symmetric gapped TQFT. Furthermore, Cordova-Ohmori [88] proves that a symmetry-preserving gapped TQFT phase is impossible for this odd $\mathbf{v} \in \mathbb{Z}_{16}$ anomaly from $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_{16}$. The general statement in [88] is that given an anomaly index $\mathbf{v} \in \Omega_5^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_{16}$, we can at most construct a fully symmetric gapped TQFT *if and only if* $4 \mid 2\mathbf{v}$. Namely, 4 has to be a divisor of $2\mathbf{v}$. Apparently, the $4 \mid 2\mathbf{v}$ is true only when $\mathbf{v}$ is even.

Since $\mathbf{v} = -N_{\text{generation}}$, the case of $\mathbf{v} = 1$ (for a single generation) and $\mathbf{v} = 3$ (for three generations) are particularly important for the high energy physics phenomenology. This means that we are not able to directly construct any 4d symmetric gapped TQFT that explicitly matches the same $\mathbb{Z}_{16}$ anomaly for one right-handed neutrino ($\mathbf{v} = 1$) or three right-handed neutrinos ($\mathbf{v} = 3$) alone.

2. When $\mathbf{v}$ is even, such as $\mathbf{v} = 2, 4, 6, 8, \dots \in \mathbb{Z}_{16}$, Ref. [41, 51, 67, 87] suggested that the symmetry-extension method [56] can trivialize the 't Hooft anomaly. Furthermore, Cordova-Ohmori [88] shows that there is no obstruction to construct a symmetry-preserving gapped TQFT phase for any even $\mathbf{v}_{\text{even}} \in \mathbb{Z}_{16}$. We can verify the claim by rewriting (3.15) in terms of the $(\frac{\mathbf{v}_{\text{even}}}{2}) \in \mathbb{Z}_8$ index:

$$\begin{aligned} \mathbf{Z}_{5\text{d-iTQFT}}^{(\mathbf{v}_{\text{even}}=2)} &= \exp\left(\frac{2\pi i}{16} \cdot \mathbf{v}_{\text{even}} \cdot \eta(\text{PD}(\mathcal{A}_{\mathbb{Z}_2}))\right)\Bigg|_{M^5} = \exp\left(\frac{2\pi i}{8} \cdot \left(\frac{\mathbf{v}_{\text{even}}}{2}\right) \cdot \left(\text{ABK}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^3))\right)\right)\Bigg|_{M^5} \\ &= \exp\left(\frac{2\pi i}{8} \cdot \left(\frac{\mathbf{v}_{\text{even}}}{2}\right) \cdot \left(4 \cdot \text{Arf}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^3)) + 2 \cdot \tilde{\eta}(\text{PD}((\mathcal{A}_{\mathbb{Z}_2})^4)) + (\mathcal{A}_{\mathbb{Z}_2})^5\right)\right)\Bigg|_{M^5}, \quad (3.16) \end{aligned}$$

²⁰Here is a caveat: We know that $\frac{p_1(TM)}{48} = \frac{\sigma}{16} \in \mathbb{Z}$ for 4d spin manifolds which makes (3.15) computable. But we leave the precise analogous revised expression of $\frac{p_1(TM)}{48} = \frac{\sigma}{16}$ on unoriented manifolds (such as Pin^+ manifolds) as a torsion class [84–86] in the parallel work [87].

with a 2d Arf-Brown-Kervaire (ABK) invariant which is also known as the Pin^- -structure $\mathbb{Z}_8$ -class of iTQFT of the 1+1d Fidkowski-Kitaev fermionic chain [89, 90] with a time reversal $T^2 = +1$ symmetry. Notice that (3.16) can become trivialized if we can trivialize the $(\mathcal{A}_{\mathbb{Z}_2})^3$ factor. In fact, the $(\mathcal{A}_{\mathbb{Z}_2})^3$ can be trivialized by the symmetry extension [56], written in terms of the group extension of a short exact sequence:

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_{4,X} \rightarrow \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F} \rightarrow 0.$$

Namely, the 2-cocycle topological term $(\mathcal{A}_{\mathbb{Z}_2})^3$ in $H^3(B(\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}), U(1))$ becomes a coboundary once we lifting the $\frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F}$ -gauge field $\mathcal{A}_{\mathbb{Z}_2}$ to a $\mathbb{Z}_{4,X}$ gauge field in $H^3(B\mathbb{Z}_{4,X}, U(1))$. So this suggests that the following symmetry extension for the spacetime-internal symmetry, written in terms of the group extension of a short exact sequence:²¹

$$1 \rightarrow [\mathbb{Z}_2] \rightarrow \text{Spin} \times \mathbb{Z}_{4,X} \rightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \rightarrow 1. \quad (3.17)$$

can fully trivialize any even $\mathbf{v}_{\text{even}} \in \mathbb{Z}_{16}$ cobordism invariant given in (3.16). The $[\mathbb{Z}_2]$ means that we can gauge the *anomaly-free* normal subgroup $[\mathbb{Z}_2]$ in the total group $\text{Spin} \times \mathbb{Z}_{4,X}$. This symmetry extension (3.17) also means that a 4d $[\mathbb{Z}_2]$ gauge theory preserves the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ symmetry while also saturates the even $\mathbf{v}_{\text{even}} \in \mathbb{Z}_{16}$ anomaly. This 4d $[\mathbb{Z}_2]$ gauge theory is the *anomalous* symmetric gapped non-invertible TQFT (with 't Hooft anomaly of $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ -symmetry) desired in the Scenario (2a).

Since a symmetric anomalous 4d TQFT only exists with even $\mathbf{v}_{\text{even}} \in \mathbb{Z}_{16}$, below we formulate the path integral $\mathbf{Z}_{4\text{d-TQFT}}^{(\mathbf{v}_{4\text{d}}=2)}[\mathcal{A}_{\mathbb{Z}_4}]$ of the root phase $\mathbf{v}_{4\text{d}} = 2$. We generalize the boundary TQFT construction in the Section 8 of [52]. With $\mathbf{v}_{\text{even}} = 2 \in \mathbb{Z}_{16}$, we have (3.14) with the input of 5d bulk iTQFT (3.16), then we can explicitly construct the partition function on a 5d manifold M^5 with a 4d boundary $M^4 \equiv \partial M^5$ as,²²

$$\begin{aligned} \mathbf{Z}_{5\text{d-iTQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4\text{d-TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] &= \sum_{c \in \partial'^{-1}(\partial[\text{PD}(\mathcal{A}^3)])} e^{\frac{2\pi i}{8} \text{ABK}(c \cup \text{PD}(\mathcal{A}^3))} \\ &\cdot \frac{1}{2^{|\pi_0(M^4)|}} \sum_{\substack{a \in C^1(M^4, \mathbb{Z}_2), \\ b \in C^2(M^4, \mathbb{Z}_2)}} (-1)^{\int_{M^4} a(\delta b + \mathcal{A}^3)} \cdot e^{\frac{2\pi i}{8} \text{ABK}(c \cup \text{PD}'(b))}. \end{aligned} \quad (3.18)$$

We write the mod 2 cohomology class $\mathbb{Z}_2$ gauge field as $\mathcal{A} \equiv \mathcal{A}_{\mathbb{Z}_2} \in H^1(M^5, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$, while the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ gauge field $\mathcal{A}_{\mathbb{Z}_4}$ is not an ordinary abelian discrete gauge field but with the extra constraint $w_2(TM) = \mathcal{A}_{\mathbb{Z}_2}^2 = \mathcal{A}^2$. Here come some remarks on this 5d bulk iTQFT-4d boundary TQFT partition function (3.18):

- 1). The $(\text{PD}(\mathcal{A}^3))$ is a 2d manifold taking the Poincaré dual (PD) of 3-cocycle $\mathcal{A}^3$ in the M^5 ; but the 2d manifold $(\text{PD}(\mathcal{A}^3))$ may touch the 4d boundary $\partial M^5 = M^4$. The 1d boundary $\partial(\text{PD}(\mathcal{A}^3))$ can be regarded as the 1d intersection between the 2d $(\text{PD}(\mathcal{A}^3))$ and the M^4 .
- 2). More precisely, for a $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ manifold M^5 with a boundary, we have used the Poincaré-Lefschetz duality for a manifold with boundaries:

$$\mathcal{A}^3 \in H^3(M^5, \mathbb{Z}_2) \xrightarrow{\cong} H_2(M^5, M^4, \mathbb{Z}_2) \ni \text{PD}(\mathcal{A}^3). \quad (3.19)$$

²¹See more discussions in Sec. 5 of [11], and in [87].

²²We use the $\smile$ notation for the cup product between cohomology classes, or between a cohomology class and a fermionic topological invariant (paired via a Poincaré dual PD). We often make the cup product $\smile$ and the Poincaré dual PD implicit. We use the $\cup$ notation for the surgery gluing the boundaries of two manifolds within relative homology classes. So the $M_1 \cup M_2$ means gluing the boundary $\partial M_1 = \bar{\partial} M_2$ such that the common orientation of $\bar{\partial} M_2$ is the reverse of ∂M_2 .

- 3). For any pair (S, S') , where S' is a subspace of S , the short exact sequence of chain complexes

$$0 \rightarrow C_*(S') \rightarrow C_*(S) \rightarrow C_*(S, S') \rightarrow 0, \quad (3.20)$$

with $C_n(S, S') \equiv C_n(S)/C_n(S')$, induces a long exact sequence of homology groups

$$\cdots \rightarrow H_n(S') \rightarrow H_n(S) \rightarrow H_n(S, S') \xrightarrow{\partial} H_{n-1}(S') \rightarrow \cdots. \quad (3.21)$$

Here $H_n(S, S')$ is the relative homology group, and ∂ is the boundary map.

- Take $(S, S') = (\text{PD}(\mathcal{A}^3), \partial\text{PD}(\mathcal{A}^3))$, we denote the boundary map by ∂ :

$$H_2(\text{PD}(\mathcal{A}^3), \partial\text{PD}(\mathcal{A}^3)) \xrightarrow{\partial} H_1(\partial\text{PD}(\mathcal{A}^3)). \quad (3.22)$$

Here $\text{PD}(\mathcal{A}^3)$ is not a closed 2-manifold, but which has a boundary closed 1-manifold $\partial\text{PD}(\mathcal{A}^3)$.

- Take $(S, S') = (M^4 = \partial M^5, \partial\text{PD}(\mathcal{A}^3))$, we denote another boundary map by ∂_1 :

$$H_2(M^4, \partial\text{PD}(\mathcal{A}^3)) \xrightarrow{\partial_1} H_1(\partial\text{PD}(\mathcal{A}^3)). \quad (3.23)$$

Both $M^4 = \partial M^5$ and $\partial\text{PD}(\mathcal{A}^3)$ are closed manifolds, of 4d and 1d, respectively.

- 4). Now, the c is defined as a 2d surface living on the boundary M^4 . The $\partial_1 c$ uses the boundary map (3.23)'s ∂_1 of c on the M^4 . We can compensate the 2-surface $\text{PD}(\mathcal{A}^3)$ potentially with a 1-boundary, by gluing it with c to make a closed 2-surface. To do so, we require both $\text{PD}(\mathcal{A}^3)$ and c share the same 1d boundary.
- 5). The $c \in \partial_1^{-1}(\partial[\text{PD}(\mathcal{A}^3)])$ also means

$$\partial_1 c = \partial[\text{PD}(\mathcal{A}^3)] = [\partial\text{PD}(\mathcal{A}^3)]. \quad (3.24)$$

- The $[\text{PD}(\mathcal{A}^3)]$ means the fundamental class and the relative homology class of the 2d manifold $\text{PD}(\mathcal{A}^3)$.
 - The $\partial[\text{PD}(\mathcal{A}^3)] = [\partial\text{PD}(\mathcal{A}^3)]$ means the boundary of the fundamental class (via the boundary ∂ map in (3.22)) is equivalent to the fundamental class of the boundary ∂ of $\text{PD}(\mathcal{A}^3)$. Beware that the two ∂ operations in $\partial[\text{PD}(\mathcal{A}^3)] = [\partial\text{PD}(\mathcal{A}^3)]$ have different meanings.
 - Eqn. (3.24) exactly matches the requirement that the bulk 2-surface $\text{PD}(\mathcal{A}^3)$ (living in M^5) and the boundary 2-surface c (living on M^4) share the same 1d boundary.
- 6). Note that when M^5 is a closed manifold with no boundary $M^4 = \partial M^5 = \emptyset$ thus $c = \emptyset$, then the term

$$\sum_{c \in \partial_1^{-1}(\partial[\text{PD}(\mathcal{A}^3)])} e^{\frac{2\pi i}{8} \text{ABK}(c \cup \text{PD}(\mathcal{A}^3))} \text{ is equivalently reduced to } e^{\frac{2\pi i}{8} \text{ABK}(\text{PD}(\mathcal{A}^3))} = \mathbf{Z}_{5\text{d-iTQFT}}^{(\nu_{\text{even}}=2)}.$$

This is a satisfactory consistent check, consistent with the 5d bulk-only iTQFT at $\nu_{\text{even}} = 2$ in (3.16).

- 7). The $a \in C^1(M^4, \mathbb{Z}_2)$ means that a is a 1-cochain, and the $b \in C^2(M^4, \mathbb{Z}_2)$ means that b is a 2-cochain. The factor $(-1)^{\int_{M^4} a(\delta b + \mathcal{A}^2)} = \exp(i\pi \int_{M^4} a(\delta b + \mathcal{A}^2))$ gives the weight of the 4d $\mathbb{Z}_2$ gauge theory. The $a\delta b$ term is the level-2 BF theory written in the mod 2 class.²³ The δb is a coboundary operator δ acting on b . The path integral sums over these distinct cochain classes.

²³The continuum QFT version of this $\mathbb{Z}_2$ gauge theory is $\exp(i \int_{M^4} \frac{2}{2\pi} a db + \frac{1}{\pi^2} a \mathcal{A}^2)$, where the b integration over a closed 2-cycle, $\oint b$, can be $n\pi$ with some integer $n \in \mathbb{Z}$. The a and $\mathcal{A}$ integration over a closed 1-cycle, $\oint a$ and $\oint \mathcal{A}$, can be $n\pi$ with some integer $n \in \mathbb{Z}$.

Another alternative possibility of 4d TQFT of (3.18) can be $\frac{1}{2|\pi_0(M^4)|} \sum_{\substack{a \in C^1(M^4, \mathbb{Z}_2) \\ b \in C^2(M^4, \mathbb{Z}_2)}} (-1)^{\int_{M^4} b(\delta a + \mathcal{A}^2)} \cdot e^{\frac{2\pi i}{8} \text{ABK}(c \cup \text{PD}'(\mathcal{A}a))}.$

In contrast to (3.25), this 4d TQFT is constructed out of a different trivialization $w_2(TM) = \mathcal{A}^2 = \delta a = 0$ under the pullback (3.17) to a $\text{Spin} \times \mathbb{Z}_{4,X}$ -structure. The continuum version of this $\mathbb{Z}_2$ gauge theory has a different expression as $\exp(i \int_{M^4} \frac{2}{2\pi} b da + \frac{1}{\pi^2} b \mathcal{A}^2)$. Either 4d $\mathbb{Z}_2$ gauge theory sits at the normal subgroup $[\mathbb{Z}_2]$ of the group extension (3.17). Although the $[\mathbb{Z}_2]$ is abelian, this 4d TQFT actually exhibits *non-abelian topological order* due to the fermionic nature of $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ and the fermionic invariant ABK.

The non-abelian nature of this 4d TQFT is similar to the non-abelian nature of 3d $\mathbb{Z}_2$ gauge theory obtained from gauging the $\mathbb{Z}_2$ -onsite symmetry of the odd class of 2+1d fermionic topological superconductor from the $\Omega_3^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_8$ classification (e.g., Section 8 of [63], and [52]). The braiding and fusions statistics of vortices of these TQFTs are *non-abelian*. Moreover, we may require *additional symmetry extension* beyond (3.17) to construct generic symmetric 4d TQFTs.

The variation of a gives the equation of motion $(\delta b + \mathcal{A}^3) = 0 \pmod{2}$. In the path integral, we can integrate out a to give the same constraint $(\delta b + \mathcal{A}^3) = 0 \pmod{2}$. This is precisely the trivialization of the second cohomology class,

$$\mathcal{A}^3 = (\mathcal{A}_{\mathbb{Z}_2})^3 = \delta b \pmod{2} \quad (3.25)$$

so the 3-cocycle becomes a 3-coboundary which splits to a 2-cochain b . This exactly matches the condition imposed by the symmetry extension (3.17): $1 \rightarrow [\mathbb{Z}_2] \rightarrow \text{Spin} \times \mathbb{Z}_{4,X} \rightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \rightarrow 1$, where the $\mathcal{A}^3$ term in $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ becomes trivialized as a coboundary $\mathcal{A}^3 = \delta b$ (so $\mathcal{A}^3 = 0$ in terms of a cohomology or cocycle class) in $\text{Spin} \times \mathbb{Z}_{4,X}$.

- 8). The $\frac{1}{2^{|\pi_0(M^4)|}}$ factor mod out the gauge redundancy for the boundary 4d $\mathbb{Z}_2$ gauge theory. The $\pi_0(M^4)$ is the zeroth homotopy group of M^4 , namely the set of all path components of M^4 . Thus, we only sum over the gauge equivalent classes in the path integral.
- 9). The $\text{ABK}(c \cup \text{PD}(\mathcal{A}^3))$ is defined on a 2d manifold with Pin^- structure. Recall the M^5 has the $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ structure. If M^5 is closed, then there is a natural Smith map to induce the 2d Pin^- structure on the closed surface via $\text{PD}(\mathcal{A}^3)$. However, the M^5 has a boundary M^4 , so $\text{PD}(\mathcal{A}^3)$ may not be closed — the previously constructed closed 2-surface $(c \cup \text{PD}(\mathcal{A}^3))$ is meant to induce a 2d Pin^- structure.²⁴ Then we compute the ABK on this closed 2-surface $(c \cup \text{PD}(\mathcal{A}^3))$.
- 10). Let us explain the other term $\text{ABK}(c \cup \text{PD}'(\mathcal{A}a))$ of the 4d boundary TQFT in (3.18). The PD' is the Poincaré dual on $M^4 = \partial M^5$. Since the fundamental classes of M^5 and $M^4 = \partial M^5$ are related by

$$[M^5] \in H_5(M^5, M^4, \mathbb{Z}_2) \xrightarrow{\partial} H_4(M^4, \mathbb{Z}_2) \ni [M^4] \equiv [\partial M^5]. \quad (3.26)$$

Here we have the following relations:²⁵

$$\begin{aligned} \text{PD} &= [M^5] \cap, \\ \text{PD}' &= [M^4] \cap = [\partial M^5] \cap = \partial[M^5] \cap, \end{aligned} \quad (3.27)$$

$$\partial' \text{PD}'(b) = \text{PD}'(\delta b) = \text{PD}'(\mathcal{A}^3) = \partial[\text{PD}(\mathcal{A}^3)] = \partial_1 c. \quad (3.28)$$

- The cap product $\cap$ here is to define PD homology class, such that $\text{PD}(\mathcal{A}) = [M^5] \cap \mathcal{A}$.
- Here we use $[M^4] = [\partial M^5] = \partial[M^5]$: the fundamental class of boundary of M^5 gives the boundary of fundamental class.
- The (3.27)'s first equality $\partial' \text{PD}'(b) = \text{PD}'(\delta b)$ uses the coboundary operator δ on the cohomology class b .
- The (3.27)'s second equality $\text{PD}'(\delta b) = \text{PD}'(\mathcal{A}^3)$ uses the condition $\delta \mathcal{A} = 0$ and the trivialization condition (3.25): $\delta b = \mathcal{A}^3$.
- The (3.27)'s third equality $\text{PD}'(\mathcal{A}^3) = \partial[\text{PD}(\mathcal{A}^3)]$, we use “the naturality of the cap product”. There are natural pushforward and pullback maps on homology and cohomology, related by the projection formula, also known as “the naturality of the cap product.”
- The (3.27)'s last equality $\partial[\text{PD}(\mathcal{A}^3)] = \partial_1 c$ is based on (3.24). Importantly, as a satisfactory consistency check, this also shows that the 2-surface c obeys:

$$\text{the } c \text{ in } \text{ABK}(c \cup \text{PD}(\mathcal{A}^3)) \text{ is the same } c \text{ in } \text{ABK}(c \cup \text{PD}'(b)). \quad (3.29)$$

²⁴We do not yet know whether it is always possible to induce a unique 2d Pin^- structure on $(c \cup \text{PD}(\mathcal{A}^3))$ for any possible pair of data $(M^5, M^4 = \partial M^5)$ given any M^5 with $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ structure. However, we claim that it is possible to find some suitable M^4 so that the 2d $(c \cup \text{PD}(\mathcal{A}^3))$ has Pin^- induced, thus in this sense the $\text{ABK}(c \cup \text{PD}(\mathcal{A}^3))$ is defined. For physics purposes, it is enough that we can firstly focus on studying the theory on these types of $(M^5, M^4 = \partial M^5)$.

²⁵Let us clarify the notations: ∂ , ∂' , and ∂_1 . The boundary notation ∂ may mean as (1) taking the boundary, or (2) in the boundary map of relative homology class in (3.22). It should be also clear to the readers that

- the ∂ is associated with the operations on objects living in the bulk M^5 or ending on the boundary M^4 ,
- while the ∂' is associated with the operations on objects living on the boundary M^4 alone.
- The ∂_1 is defined as another boundary map in (3.23).

- As before, the union $(c \cup \text{PD}'(b))$ is a closed 2-surface and we induce a Pin^- structure on this 2-surface. So we can compute the ABK on this closed 2-surface $(c \cup \text{PD}'(b))$ on the M^4 .²⁶

In summary, we have constructed the 4d fermionic discrete gauge theory in (3.18) preserving the $(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X})$ -structure, namely it is a $(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X})$ -symmetric TQFT but with $\mathbf{v}_{\text{even}} = 2 \in \mathbb{Z}_{16}$ anomaly. This can be used to compensate the anomaly $\mathbf{v} = -N_{\text{generation}} \bmod 16$ with $N_{\text{generation}} = 2$, two generations of missing right-handed neutrinos. We could not however directly construct the symmetric gapped TQFT for $\mathbf{v}$ is odd (thus symmetric TQFTs not possible for $N_{\text{generation}} = 1$ or 3), due to the obstruction found in [51, 88].

3.3 General Principle

The discussion in Sec. 3.2.1 says that only for the even integer $\mathbf{v}_{4d,\text{even}} \in \mathbb{Z}_{16}$ does the $(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X})$ -symmetry-preserving TQFT exist. This prompts us to improve the SM version (3.12) into:

$$\mathbf{Z}_{\text{UU}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}/4d\text{-SM+TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}}^{(-\mathbf{v}_{5d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4d\text{-TQFT}}^{(\mathbf{v}_{4d,\text{even}})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{\text{SM}}^{(n_{\nu_e,R}, n_{\nu_\mu,R}, n_{\nu_\tau,R})}[\mathcal{A}_{\mathbb{Z}_4}]. \quad (3.30)$$

The GUT version (3.13) should be adjusted into:

$$\mathbf{Z}_{\text{UU}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}/4d\text{-GUT+TQFT}}[\mathcal{A}_{\mathbb{Z}_4}] \equiv \mathbf{Z}_{5d\text{-iTQFT}}^{(-\mathbf{v}_{5d})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{4d\text{-TQFT}}^{(\mathbf{v}_{4d,\text{even}})}[\mathcal{A}_{\mathbb{Z}_4}] \cdot \mathbf{Z}_{\text{GUT}}^{(n_{\nu_e,R}, n_{\nu_\mu,R}, n_{\nu_\tau,R})}[\mathcal{A}_{\mathbb{Z}_4}]. \quad (3.31)$$

Also the anomaly constraint (2.12) becomes:

$$(-(N_{\text{gen}} = 3) + n_{\nu_e,R} + n_{\nu_\mu,R} + n_{\nu_\tau,R} + \mathbf{v}_{4d,\text{even}} - \mathbf{v}_{5d}) = 0 \bmod 16. \quad (3.32)$$

This implies that the existence of symmetry-preserving 4d TQFT sector requires the following:²⁷

$$(n_{\nu_e,R} + n_{\nu_\mu,R} + n_{\nu_\tau,R} - \mathbf{v}_{5d}) \text{ must be an odd integer.}$$

Now we have derived an Ultra Unification path integral in (3.30) and (3.31), including 4d SM (3.1), 4d GUT (3.8), 5d iTQFT (3.15) and 4d TQFT (3.18), comprising many Scenarios and their linear combinations enlisted in Sec. 2.3: (1a), (1b), (1c), (2a), and (2b). Then we can dynamically gauge the appropriate bulk-boundary global symmetries, promoting the theory to a bulk gauge theory in Scenarios (2c) and (2d), or break some of the (global or gauge) symmetries to (2e).

²⁶Similar to Footnote 24, we can find some suitable M^4 so that the 2d Pin^- is induced, thus in this sense the ABK($c \cup \text{PD}'(b)$) is defined.

²⁷The nonperturbative global anomaly cancellation constraint $(-(N_{\text{gen}} = 3) + n_{\nu_e,R} + n_{\nu_\mu,R} + n_{\nu_\tau,R} + \mathbf{v}_{4d,\text{even}} - \mathbf{v}_{5d}) = 0 \bmod 16$ provides the capacity for many kinds of the Ultra Unification model building. For example, the 4d $\mathbb{Z}_{4,X}$ -symmetry preserving TQFT sector can take the index $\mathbf{v}_{4d,\text{even}} = 0, 2, 4, \dots$ for any even integer. There are many (perhaps countably infinite) types of TQFTs for each index $\mathbf{v}_{4d,\text{even}}$. But to be more economic, we can ask for the minimum degrees of freedom required by a TQFT for any given index $\mathbf{v}_{4d,\text{even}}$. Also for HEP phenomenological purposes, by taking account of the experimentally observed neutrino mass eigenstates splitting:

- if one use the *conventional quadratic mass mechanism* to generate the observed neutrino masses, one may propose to have at least two generations of right-handed neutrinos, which means that a possible phenomenological input $n_{\nu_e,R} + n_{\nu_\mu,R} + n_{\nu_\tau,R} \geq 2$. A viable Ultra Unification candidate can be, for example, $n_{\nu_e,R} + n_{\nu_\mu,R} + n_{\nu_\tau,R} = 2$, $\mathbf{v}_{4d,\text{even}} = 2$, and $\mathbf{v}_{5d} = 1$, which saturates the anomaly cancellation and some phenomenological constraints.

- if we use the interacting topological mass Δ_{TQFT} and its topological defect energy subgap Δ_{sub} to account for the observed neutrino masses (see Sec. 3.5 and Fig. 9 for illustrations), then we may have different choices of the number of right-handed neutrinos, written as $(\sum_{j=e,\mu,\tau,\dots} n_{\nu_j,R})$ in (2.9).

So far, we mainly use the cobordism theory to study the invertible anomalies and invertible topological field theories, and we also use the cohomology data to construct non-invertible topological quantum field theories. However, once the discrete symmetries (such as $\mathbb{Z}_{4,X}$) are dynamically gauged, it is more natural to use the mathematical category or higher category theories to characterize the Topological Phase Sectors.

General Principle

In summary, we propose a general principle behind the Ultra Unification:

1. We start with a QFT in general as an effective field theory (EFT) given some full spacetime-internal symmetry G , say in a D dimensional (Dd) spacetime.
2. We check the anomaly and cobordism constraint given by G via computing $\Omega_G^{D+1} \equiv \text{TP}_{D+1}(G)$.
3. We check the anomaly index of the Dd QFT/EFT constrained by cobordism $\Omega_G^{D+1} \equiv \text{TP}_{D+1}(G)$.
4. If all anomalies are canceled, then we do not require any new hidden sector to define Dd QFT/EFT.
5. If some anomalies are not canceled, we have two perspectives, *either* regarding the G -symmetry as a global symmetry with 't Hooft anomaly at a lower energy; *or* regarding the G -symmetry and its anomaly still persist at a higher energy (thus we can further assume the G symmetry is dynamically gauged at a higher energy due to “no global symmetry in quantum gravity reasonings”), then either
 - (1) we need to break some symmetry out of G to eliminate the anomaly, or
 - (2) we extend the symmetry G to an appropriate $\tilde{G}$ to trivialize nonperturbative global anomalies in $\Omega_{\tilde{G}}^{D+1} \equiv \text{TP}_{D+1}(\tilde{G})$, or
 - (3) we propose new hidden sectors appending to Dd QFT/EFT, with a schematic path integral (say if we add Dd-TQFT or CFT and (D+1)d-iTQFT onto the original theory):

$$\boxed{\mathbf{Z}_{\substack{(D+1)\text{d-iTQFT/} \\ \text{Dd-QFT/EFT+TQFT or CFT}}}[\mathcal{A}] \equiv \mathbf{Z}_{(D+1)\text{d-iTQFT}}[\mathcal{A}] \cdot \mathbf{Z}_{\text{Dd-TQFT or CFT}}[\mathcal{A}] \cdot \mathbf{Z}_{\text{Dd-QFT/EFT}}[\mathcal{A}].} \quad (3.33)$$

Here $\mathcal{A}$ is a generic G -symmetry background field, that is also to be dynamically gauged at a higher energy.

3.4 Detect Topological Phase Sectors and the Essence of Ultra Unification

Detect Topological Forces

By looking at Fig. 1, in the Higgs vacuum where our Standard Model effective field theory (SM EFT) resides in, we have only detected the Strong, Electromagnetic, and Weak in the subatomic physics. The GUT forces are weaker than the Weak force, and the Topological Force is further weaker than the GUT and Weak forces. So how could we experimentally detect Topological Forces?

Notice that the gravity is further weaker than all other forces. (So how could we experimentally detect gravity?) But the gravity has accumulative effects that only have the gravitational attractions. Without doubt, the gravity has been detected by everyone and by all astrophysics and cosmology observations. The gravity had been detected first in the human history among all the forces!

Similarly, although Topological Force is also weak (but stronger than the gravity), Topological Force is infinite-range or long-range which does *not decay in the long distance*, and mediates between the linked worldline/worldsheet/worldvolume trajectories of the charged (point-like or extended) objects via fractional or categorical anyonic statistical interactions. So in principle, *we may have already experienced Topological Force in our daily life, in a previously scientifically unnoticed way.*²⁸

²⁸For example, if the $\mathbb{Z}_{4,X}$ is dynamically gauged, there is a dynamical discrete gauge Wilson line connecting all SM fermions living in 4d SM or GUT (e.g., quarks and electrons in our body). Namely, the SM fermions can live at the open ends of the $\mathbb{Z}_{4,X}$ gauged Wilson line. Thus there could be long-distance topological interactions and communications between the $\mathbb{Z}_{4,X}$ -gauge charged objects.

Moreover, the $\mathbb{Z}_{4,X}$ gauged Wilson line as a $\mathcal{A}_{\mathbb{Z}_4}$ gauge field on the 4d theory can be leaked into the 5d bulk theory as a $\mathcal{A}_{\mathbb{Z}_2}$ gauge field. In the 5d bulk, a nontrivial link configuration can be charged under the other end of $\mathcal{A}_{\mathbb{Z}_2}$ [11].

It is worthwhile to emphasize that the new sectors that we propose (the 4d gapped phase with TQFT, the 4d gapless phase with CFT, and the 5d gapped phase with iTQFT or TQFT) are *invisible* to SM gauge forces ($su(3) \times su(2) \times u(1)$) also *invisible* to the $su(5)$ GUT forces. However, these new sectors are *detectable* via Topological Forces (i.e., statistical interaction via discrete gauge forces). These new sectors are also *detectable* via the gravitations.

Neutrino Oscillations and Dark Matter

In fact, Ref. [11] had proposed that the Topological Force may cause (thus be detected by) the phenomena of neutrino oscillations. We can consider the Majorana zero modes of the vortices in the 4d TQFT defects. The left-handed neutrinos (confirmed by the experiments) are nearly gapless/massless. When the left-handed neutrinos traveling through the 4d TQFT defects, we may observe nearly gapless neutrino flavor oscillations interfering with the Majorana zero modes trapped by the vortices/vortex strings/monopoles in the 4d TQFT defects (see Sec. 3.5 and Fig. 9 for illustrations). On the other hand, the gapped heavy excitations (point or extended objects) of Topological Phase Sector may be a significant contribution to Dark Matter [91].

If the TQFT energy gap Δ_{TQFT} is large, the heavy excitations above the Δ_{TQFT} may contribute the *heavy Dark Matter*. In contrast, if the Δ_{TQFT} is small compared to the SM's particle masses, or if the new hidden sector contains CFT, then the new sector contributes the *light Dark Matter*.

The Essence of Ultra Unification

Finally, we come to the essence of **Ultra Unification**. What is unified after all? We have united the **Strong, Electromagnetic, Weak, GUT forces, and Topological forces into the same theory** in the Ultra Unification QFT/TQFT path integral (that this theory can also be coupled to the curved spacetime geometry and gravity, at least well-defined in a background non-dynamical way). See Fig. 2, Fig. 3, Fig. 4, and Fig. 5 for illustrations.

However, the Grand Unification [8, 9] united the three gauge interactions of the SM into a single electronuclear force under a simple Lie group gauge theory. Do we have any equivalent statement to also unite **Strong, Electromagnetic, Weak, GUT forces and Topological forces into a single force** at a high enough energy? We believe that the definite answer relies on studying the details of analogous topological quantum phase transitions [12, 91] and the parent effective field theory that describes the phase transition and neighborhood phases, such as those explored in 4d [70, 92–95]. The underlying mathematical structure suggests a 4d version of particle-vortex duality, S-duality, T-duality, or mirror symmetry. See Fig. 6, Fig. 7, and Fig. 8 for illustrations.

3.5 Summary of Ultra Unification and Quantum Matter in Drawings

Let us summarize what we have done in this work in drawings.

- Fig. 2: We have started from the Nature given Standard Model (SM) quarks and leptons, and their quantum numbers (see Tables in [11]), in three generations.
- Fig. 3: We have included gauge forces and various Higgs for SM and Grand Unification (GUT).
- Fig. 4: After the essential check of the anomaly and cobordism constraints, the detection of the $\mathbb{Z}_{16}$ global anomaly for 15n Weyl fermion SM and GUT implies that we can choose (as one of many options) to realize our 4d world living on an extra-dimensional 5d invertible TQFT (a 5d topological superconductor, mathematically a 5d cobordism invariant). Here Fig. 4 may be understood as a one-brane 4d world with an extra-large fifth dimension.

In a metaphysics sense, perhaps the long-distance Topological Force might be related to some of the unexplained mysterious phenomena. In any case, every phenomenon and every law of Nature should be explained by mathematics and physics principles.

- Fig. 5: **Ultra Unification** (UU) incorporates the SM, GUT, and Topological forces into the same theory (that this theory can also be coupled to curved spacetime geometry and gravity in a background non-dynamical way). (1) The upper left has the 4d SM and GUT. (2) The upper right has the 4d non-invertible TQFT. (3) The bottom has the 5d invertible TQFT (alternatively 5d non-invertible TQFT if we dynamically gauge the full $\mathbb{Z}_{4,X}$). The three sectors can communicate with each other mediated via the dynamical $\mathbb{Z}_{4,X}$ gauge forces. Here Fig. 5 may be understood as a multi-branes or two-brane 4d world with an extra fifth dimension. The issues of mirror fermion doubling [96] on the mirror world, depending on the precise anomaly index on the mirror sector, may be fully trivially gapped (if anomaly-free), may contain a mirror chiral gauge theory or unparticle conformal field theory, or may be topological order gapped with a low energy TQFT. The issues of gapping mirror fermions are tackled in the past starting from Eichten-Preskill [97] and by many recent works [15, 98–108].

- Fig. 6: In Quantum Matter terminology, we show that SM and GUT belong to a framework of a continuous gauge field theory, Anderson-Higgs (global or gauge) symmetry-breaking mass, and Ginzburg-Landau paradigm. In contrast, the new sectors that we introduce are beyond Ginzburg-Landau paradigm. The new sectors include a fermionic discrete gauge theory, symmetry-extension topological mass, and modern issues on symmetry, topology, nonperturbative interactions, and short/long-range entanglements.

- Fig. 7 and Fig. 8: In general, our SM vacuum (with possible BSM corrections that we denote SM*) may live in a landscape of quantum vacua (e.g., see recent works [109, 110]). Possible quantum vacua tuning parameters may be the GUT-Higgs potential or other QFT parameters that can induce quantum phase transitions from SM to neighbor GUT vacua in an SM deformation class [72, 111]. UU may not only be just a higher-energy effective field theory of SM*, but also provide a parent effective field theory to go between SM, SM*, or different GUT vacua, from the left-hand sided 15n-Weyl-fermion model plus a 4d TQFT, a 4d CFT, or a 5d iTQFT, to the right-hand sided 16n-Weyl-fermion model in 4d, via a topological quantum phase transition through an energy-gap-closing gapless quantum critical region.

- Fig. 9: UU provides alternative new ways to explain the mass of the neutrinos. The traditional seesaw mechanism argues that the right-handed sterile neutrinos have a *Majorana mass* m_M , while the left-handed paired with right-handed neutrinos get a *Dirac mass* m_D ; the mass eigenstate allows a small mass $\frac{m_D^2}{m_M} \ll m_D$; these quadratic masses break the $\mathbb{Z}_{4,X}$ symmetry. In contrast, UU replaces some of sterile neutrinos with a TQFT sector within the same anomaly index. TQFT energy gap Δ_{TQFT} is the *topological mass* gap. The Δ_{TQFT} may sit at any energy scale below GUT or Planck scale $M_{\text{GUT,Pl}}$. The symmetric TQFT can preserve $\mathbb{Z}_{4,X}$ but still give a topological mass; the TQFT may also have $\mathbb{Z}_{4,X}$ -topological defects which trap the zero modes. (Proliferating the topological defects restores the $\mathbb{Z}_{4,X}$ -symmetry driving to the symmetric TQFT phase.) The nearly gapless *left-handed neutrinos* ($\nu_{e,L}, \nu_{\mu,L}, \nu_{\tau,L}$) travel in waves and interact with TQFT defect's *zero modes*. Quantum interference between left-handed neutrinos and zero modes possibly causes *neutrino oscillations*. The energy spectrum near the defect (e.g., vortex points/strings) has some energy subgap $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,Pl}}}$ above the zero energy modes.²⁹ This interacting mechanism $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,Pl}}} \ll m_D$ (different from the traditional seesaw mechanism $\frac{m_D^2}{m_M} \ll m_D$) may also give the left-handed neutrinos small masses.

Of course, there could also be mixed scenarios (enumerated in Sec. 2.3) such that we have some right-handed neutrino sector and some TQFT sector, so both sectors can contribute the small $\frac{m_D^2}{m_M}$ and small $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,Pl}}}$ respectively to the mass origin of the left-handed neutrinos.

²⁹This dimensional analysis $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,Pl}}}$ is in analogy with the vortex energy subgap $\Delta_{\text{sub}} \simeq \frac{\Delta_{\text{SC}}^2}{E_F}$ of superconductor gap Δ_{SC} and Fermi energy E_F in the finite-density fermionic superconductor system. This idea is reported by the author in the Harvard String Lunch seminar (December 4, 2020).

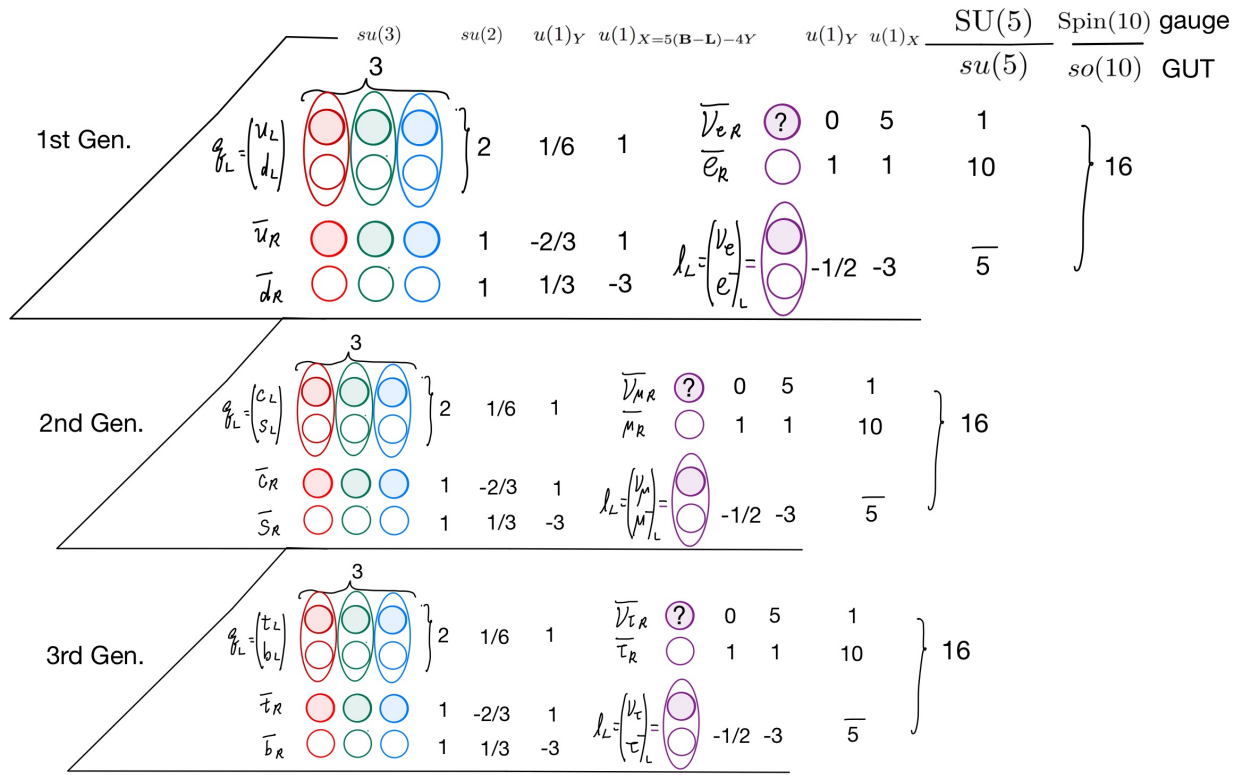


Figure 2: Standard Model, the $su(5)$ GUT and the $so(10)$ GUT quarks and leptons. Each Weyl fermion, represented by a color circle, is a Lorentz spinor $\mathbf{2}_L$ under the spacetime symmetry group $\text{Spin}(1,3)$.

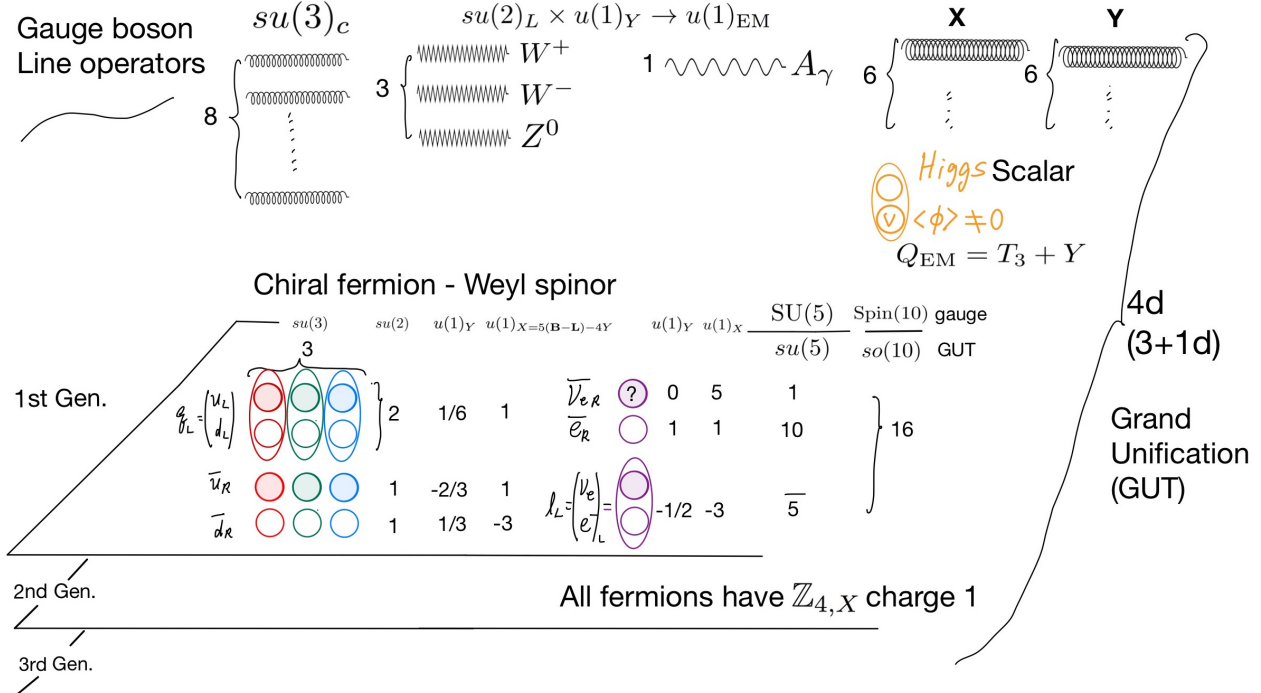
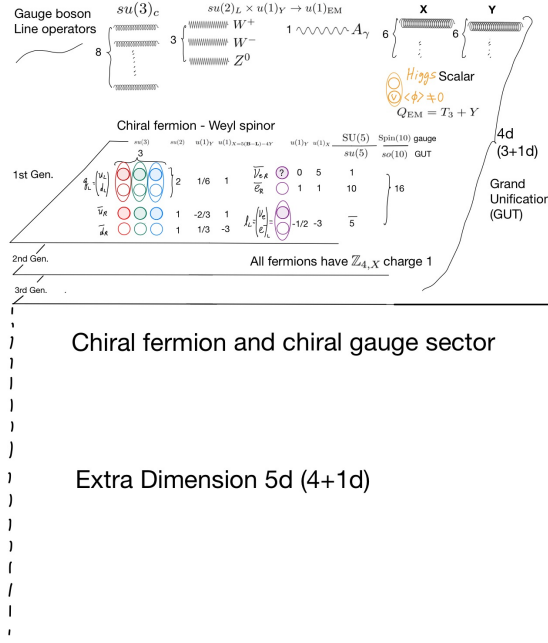


Figure 3: Include gauge forces and (various) Higgs to the Standard Model and GUT. Other heavier excitations such as a magnetic monopole would occur as the open end of 't Hooft (world-)line operator.



Chiral fermion and chiral gauge sector

Extra Dimension 5d (4+1d)

Figure 4: Detection of the $\mathbb{Z}_{16}$ global anomaly for 15n Weyl fermion SM and GUT. It can be realized as adding an extra-dimensional 5d invertible TQFT (i.e., a 5d cobordism invariant), where the 4d world lives on its boundary.

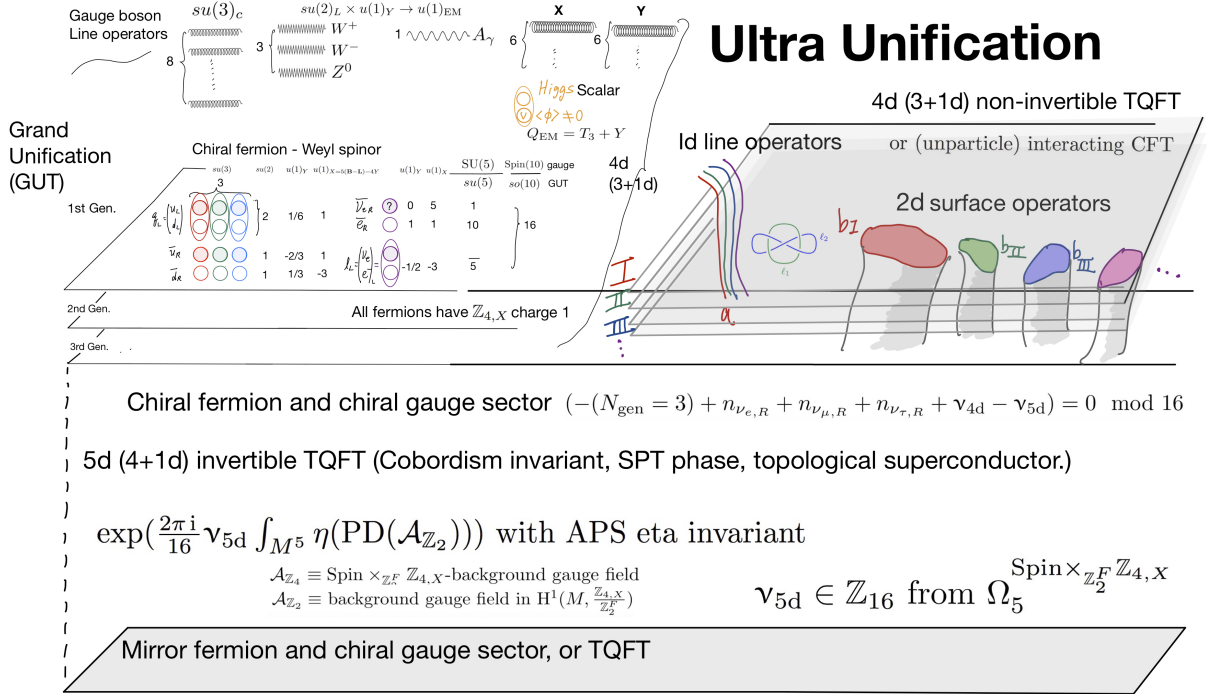


Figure 5: **Ultra Unification** incorporates the **Strong, Electromagnetic, Weak, GUT, and Topological Forces** into the same theory in the QFT/TQFT path integral (that this theory can also be coupled to curved spacetime geometry and gravity in a background non-dynamical way). (1) The upper left has the 4d SM and GUT. (2) The upper right has the 4d non-invertible TQFT. (3) The bottom has the 5d invertible TQFT (alternatively 5d non-invertible TQFT if we dynamically gauge the $\mathbb{Z}_{4,X}$ or $\mathbb{Z}_{4,X}/\mathbb{Z}_2^F$). The three sectors can communicate with each other mediated via the dynamical discrete gauge forces.

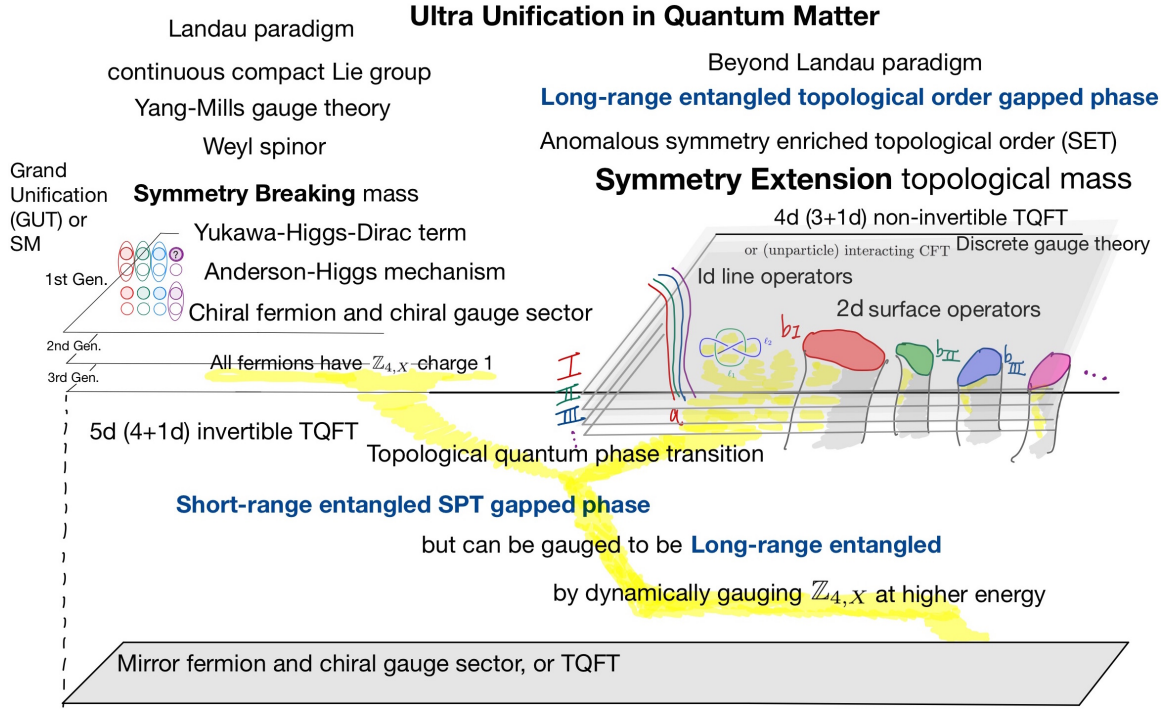


Figure 6: **Ultra Unification** summarized in terms of Quantum Matter terminology. We can trade some even number of right-handed neutrinos (the 16th Weyl fermions, possibly with *symmetry-breaking* masses) for the *symmetry-extension* topological order sector in 3+1d, via a topological quantum phase transition. Once the $\mathbb{Z}_{4,X}$ is dynamically gauged (e.g., at higher energy), the 3+1d theory and 4+1d bulk all are coupled and correlated with each other via the topological $\mathbb{Z}_{4,X}$ -gauge force. In a colloquial sense, our Standard Model world may live with the neighbors of 3+1d *intrinsic topological order* or 3+1d *unparticle CFTs*, and also live on the boundary of some medium of 4+1d *topological quantum computer*.

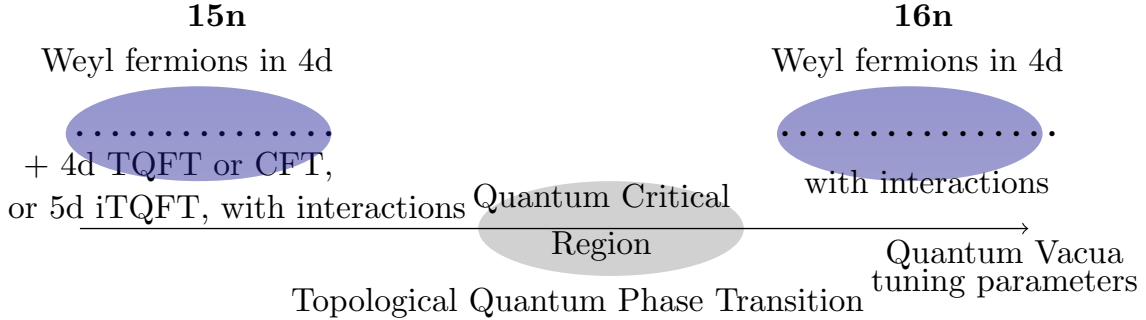


Figure 7: Ultra Unification contains all pertinent low-energy scenarios (enumerated in Sec. 2.3), such that we can have a topological quantum phase transition from the left-hand sided 15n-Weyl-fermion model plus 4d TQFT/CFT or 5d iTQFT, to the right-hand sided 16n-Weyl-fermion model in 4d (shown in two shaded blue regions as two different phases of quantum vacua with different low-energy and different ground state sectors), tuning through a *gapless quantum critical region* (shown as the schematic gray region). The 15 dots and 16 dots on the left and right regions represent the number of Weyl fermions. The horizontal axis indicates the possible quantum vacua tuning parameter(s). Quantum phase transition and quantum criticality may also exhibit supersymmetry phenomena.

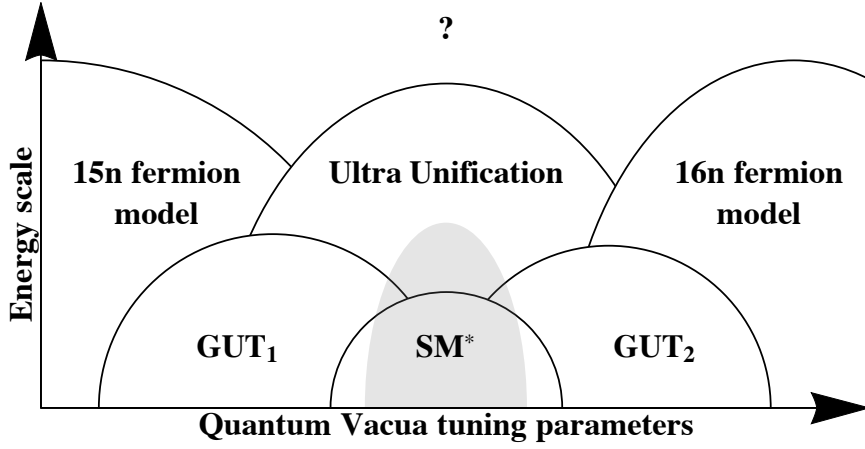


Figure 8: SM vacuum (with possible BSM corrections denote as SM*) may live in a landscape of quantum vacua in a quantum phase diagram (e.g., see recent works [109, 110]). Possible quantum vacua tuning parameters may be the GUT-Higgs potential or other QFT deformation parameters that can induce quantum phase transition from SM to neighbor GUT vacua in an SM deformation class [72, 111]. In Ref. [109, 110]’s viewpoint, Ultra Unification may not only be just a higher-energy effective field theory of SM*, but also provide a parent effective field theory to go between SM, SM*, or different GUT vacua, from the left-hand sided 15n-Weyl-fermion model plus 4d TQFT/CFT or 5d iTQFT, to the right-hand sided 16n-Weyl-fermion model in 4d, tuning through a *gapless topological quantum critical region* (the schematic gray region).

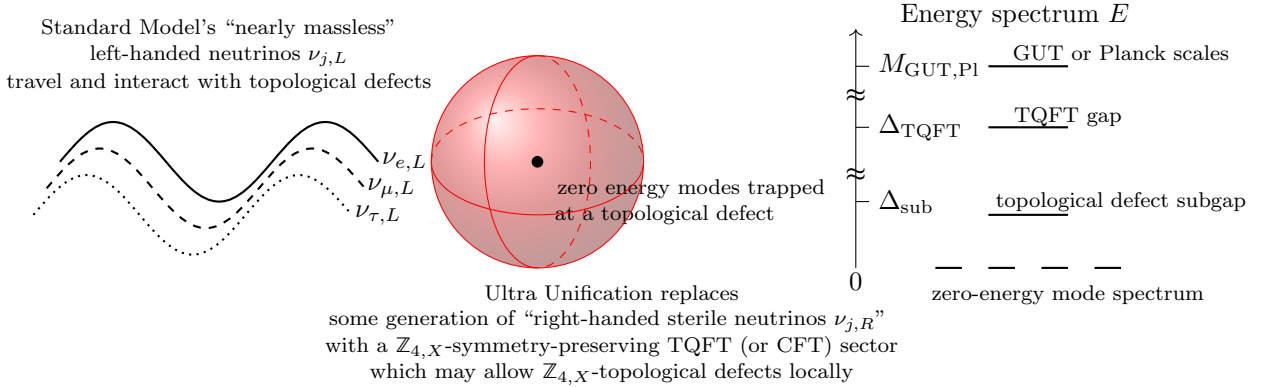


Figure 9: Ultra Unification (UU) also provides alternative new possible ways to explain the mass of the neutrinos. The traditional seesaw mechanism argues that the right-handed sterile neutrinos have a *Majorana mass* m_M , while the left-handed paired with right-handed neutrinos get a *Dirac mass* m_D ; the mass eigenstate allows a small mass $\frac{m_D^2}{m_M} \ll m_D$; these quadratic masses break the $\mathbb{Z}_{4,X}$ symmetry. In contrast, UU replaces some of sterile neutrinos to a TQFT sector with the same anomaly index, which can have a *topological mass* of TQFT energy gap Δ_{TQFT} . The Δ_{TQFT} may sit at any energy scale below GUT or Planck scale $M_{\text{GUT,PI}}$. The symmetric TQFT can preserve $\mathbb{Z}_{4,X}$ but still give a topological mass; the TQFT may also have $\mathbb{Z}_{4,X}$ -topological defects which trap the zero modes. (Proliferating the topological defects restores the $\mathbb{Z}_{4,X}$ -symmetry driving to the symmetric TQFT phase.) The nearly gapless *left-handed neutrinos* ($\nu_{e,L}, \nu_{\mu,L}, \nu_{\tau,L}$) travel in waves and interact with TQFT defect’s *zero modes*, which quantum interference possibly causes *neutrino oscillations*. The energy spectrum near the defect (such as vortex points/strings) has energy subgap $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,PI}}}$, in analogy with the vortex subgap $\Delta_{\text{sub}} \simeq \frac{\Delta_{\text{SC}}^2}{E_F}$ of superconductor gap Δ_{SC} and Fermi energy E_F . In contrast to the traditional seesaw mechanism $\frac{m_D^2}{m_M} \ll m_D$, this interacting mechanism $\Delta_{\text{sub}} \lesssim \frac{\Delta_{\text{TQFT}}^2}{M_{\text{GUT,PI}}} \ll m_D$ may also give the left-handed neutrinos small masses.

4 Acknowledgements and Bibliography

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References

- [1] J. C. Maxwell, *A dynamical theory of the electromagnetic field*, *Phil. Trans. Roy. Soc. Lond.* **155** 459–512 (1865).
- [2] S. L. Glashow, *Partial Symmetries of Weak Interactions*, *Nucl. Phys.* **22** 579–588 (1961).
- [3] A. Salam and J. C. Ward, *Electromagnetic and weak interactions*, *Phys. Lett.* **13** 168–171 (1964).
- [4] A. Salam, *Weak and Electromagnetic Interactions*, *Conf. Proc. C* **680519** 367–377 (1968).
- [5] S. Weinberg, *A Model of Leptons*, *Phys. Rev. Lett.* **19** 1264–1266 (1967).
- [6] D. J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys. Rev. Lett.* **30** 1343–1346 (1973).
- [7] H. D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, *Phys. Rev. Lett.* **30** 1346–1349 (1973).
- [8] H. Georgi and S. L. Glashow, *Unity of All Elementary Particle Forces*, *Phys. Rev. Lett.* **32** 438–441 (1974).
- [9] H. Fritzsch and P. Minkowski, *Unified Interactions of Leptons and Hadrons*, *Annals Phys.* **93** 193–266 (1975).
- [10] Z. Wan and J. Wang, *Beyond Standard Models and Grand Unifications: Anomalies, Topological Terms, and Dynamical Constraints via Cobordisms*, *JHEP* **07** 062 (2020), [[arXiv:1910.14668](#)].
- [11] J. Wang, *Anomaly and Cobordism Constraints Beyond the Standard Model: Topological Force*, [arXiv:2006.16996](#).
- [12] J. Wang, *Anomaly and Cobordism Constraints Beyond Grand Unification: Energy Hierarchy*, [arXiv:2008.06499](#).
- [13] D. S. Freed, *Pions and Generalized Cohomology*, *J. Diff. Geom.* **80** 45–77 (2008), [[arXiv:hep-th/0607134](#)].
- [14] I. Garcia-Etxebarria and M. Montero, *Dai-Freed anomalies in particle physics*, *JHEP* **08** 003 (2019), [[arXiv:1808.00009](#)].
- [15] J. Wang and X.-G. Wen, *A Non-Perturbative Definition of the Standard Models*, *Phys. Rev. Res.* **2** 023356 (2020), [[arXiv:1809.11171](#)].

³⁰Instead of writing or drawing an image of the author’s mental feelings, a piece of Johann Sebastian Bach’s music “The Goldberg Variations, BWV 988 (1861)” (e.g., Variatio 25. a 2 Clav.: Adagio) may illuminate this well.

- [16] J. Davighi, B. Gripaios and N. Lohitsiri, *Global anomalies in the Standard Model(s) and Beyond*, [arXiv:1910.11277](#).
- [17] J. F. Adams, *On the structure and applications of the Steenrod algebra*, *Comment. Math. Helv.* **32** 180–214 (1958).
- [18] R. Thom, *Quelques propriétés globales des variétés différentiables*, *Commentarii Mathematici Helvetici* **28** 17–86 (1954).
- [19] S. Galatius, I. Madsen, U. Tillmann and M. Weiss, *The homotopy type of the cobordism category*, *Acta Math.* **202** 195–239 (2009), [[arXiv:math/0605249](#)].
- [20] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, *ArXiv e-prints* (2016 Apr.), [[arXiv:1604.06527](#)].
- [21] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, *NATO Sci. Ser. B* **59** 135–157 (1980).
- [22] S. L. Adler, *Axial vector vertex in spinor electrodynamics*, *Phys. Rev.* **177** 2426–2438 (1969).
- [23] J. S. Bell and R. Jackiw, *A PCAC puzzle: $\pi^0 \rightarrow \gamma\gamma$ in the σ model*, *Nuovo Cim.* **A60** 47–61 (1969).
- [24] L. Alvarez-Gaume and E. Witten, *Gravitational Anomalies*, *Nucl. Phys.* **B234** 269 (1984).
- [25] E. Witten, *An $SU(2)$ Anomaly*, *Phys. Lett.* **117B** 324–328 (1982).
- [26] J. Wang, X.-G. Wen and E. Witten, *A New $SU(2)$ Anomaly*, *J. Math. Phys.* **60** 052301 (2019), [[arXiv:1810.00844](#)].
- [27] E. Witten, *Global Gravitational Anomalies*, *Commun. Math. Phys.* **100** 197 (1985).
- [28] Z. Wan and J. Wang, *Higher Anomalies, Higher Symmetries, and Cobordisms I: Classification of Higher-Symmetry-Protected Topological States and Their Boundary Fermionic/Bosonic Anomalies via a Generalized Cobordism Theory*, *Ann. Math. Sci. Appl.* **4** 107–311 (2019), [[arXiv:1812.11967](#)].
- [29] E. Witten and K. Yonekura, *Anomaly Inflow and the η -Invariant*, in *The Shoucheng Zhang Memorial Workshop Stanford, CA, USA, May 2-4, 2019*, 2019. [arXiv:1909.08775](#).
- [30] S.-S. Chern and J. Simons, *Characteristic forms and geometric invariants*, *Annals Math.* **99** 48–69 (1974).
- [31] A. S. Schwarz, *The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants*, *Lett. Math. Phys.* **2** 247–252 (1978).
- [32] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** 351–399 (1989).
- [33] X.-G. Wen, *Zoo of quantum-topological phases of matter*, *Rev. Mod. Phys.* **89** 041004 (2017), [[arXiv:1610.03911](#)].
- [34] T. Senthil, *Symmetry-Protected Topological Phases of Quantum Matter*, *Annual Review of Condensed Matter Physics* **6** 299–324 (2015 Mar.), [[arXiv:1405.4015](#)].
- [35] X. Chen, Z.-C. Gu, Z.-X. Liu and X.-G. Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, *Phys. Rev.* **B87** 155114 (2013), [[arXiv:1106.4772](#)].
- [36] M. Z. Hasan and C. L. Kane, *Colloquium: Topological insulators*, *Reviews of Modern Physics* **82** 3045–3067 (2010 Oct.), [[arXiv:1002.3895](#)].

- [37] X.-L. Qi and S.-C. Zhang, *Topological insulators and superconductors*, *Reviews of Modern Physics* **83** 1057–1110 (2011 Oct.), [[arXiv:1008.2026](#)].
- [38] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** 115 (2013), [[arXiv:1305.0318](#)].
- [39] D. Tong, *Line Operators in the Standard Model*, *JHEP* **07** 104 (2017), [[arXiv:1705.01853](#)].
- [40] J. P. Ang, K. Roumpedakis and S. Seifnashri, *Line Operators of Gauge Theories on Non-Spin Manifolds*, [arXiv:1911.00589](#).
- [41] Z. Wan, J. Wang and Y. Zheng, *Higher anomalies, higher symmetries, and cobordisms II: Lorentz symmetry extension and enriched bosonic / fermionic quantum gauge theory*, *Ann. Math. Sci. Appl.* **05** 171–257 (2020), [[arXiv:1912.13504](#)].
- [42] F. Wilczek and A. Zee, *Operator Analysis of Nucleon Decay*, *Phys. Rev. Lett.* **43** 1571–1573 (1979).
- [43] Z. Wan, J. Wang and et al, *Higher Anomalies, Higher Symmetries, and Cobordisms IV: Standard Models, Grand Unifications and Beyond (in preparation), to appear* (2019).
- [44] N. Saveliev, *Lectures on the topology of 3-manifolds*. De Gruyter Textbook. Walter de Gruyter and Co., Berlin, revised ed., 2012.
- [45] C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik 2, I*, *J. Reine Angew. Math.*, **183** 148–167 (1941).
- [46] A. Y. Kitaev, *Unpaired Majorana fermions in quantum wires*, *Phys. Usp.* **44** 131–136 (2001), [[arXiv:cond-mat/0010440](#)].
- [47] N. Seiberg and E. Witten, *Gapped Boundary Phases of Topological Insulators via Weak Coupling*, *PTEP* **2016** 12C101 (2016), [[arXiv:1602.04251](#)].
- [48] J. Davighi and N. Lohitsiri, *Anomaly interplay in $U(2)$ gauge theories*, *JHEP* **05** 098 (2020), [[arXiv:2001.07731](#)].
- [49] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry 1*, *Math. Proc. Cambridge Phil. Soc.* **77** 43 (1975).
- [50] Y. Tachikawa and K. Yonekura, *Why are fractional charges of orientifolds compatible with Dirac quantization?*, *arXiv e-prints* arXiv:1805.02772 (2018 May), [[arXiv:1805.02772](#)].
- [51] C.-T. Hsieh, *Discrete gauge anomalies revisited*, [arXiv:1808.02881](#).
- [52] M. Guo, K. Ohmori, P. Putrov, Z. Wan and J. Wang, *Fermionic Finite-Group Gauge Theories and Interacting Symmetric/Crystalline Orders via Cobordisms*, *Commun. Math. Phys.* **376** 1073 (2020), [[arXiv:1812.11959](#)].
- [53] A. Kapustin, R. Thorngren, A. Turzillo and Z. Wang, *Fermionic Symmetry Protected Topological Phases and Cobordisms*, *JHEP* **12** 052 (2015), [[arXiv:1406.7329](#)].
- [54] I. Hason, Z. Komargodski and R. Thorngren, *Anomaly Matching in the Symmetry Broken Phase: Domain Walls, CPT, and the Smith Isomorphism*, *SciPost Phys.* **8** 062 (2020), [[arXiv:1910.14039](#)].
- [55] A. Vishwanath and T. Senthil, *Physics of three dimensional bosonic topological insulators: Surface Deconfined Criticality and Quantized Magnetoelectric Effect*, *Phys. Rev. X* **3** 011016 (2013), [[arXiv:1209.3058](#)].

- [56] J. Wang, X.-G. Wen and E. Witten, *Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions*, *Phys. Rev. X* **8** 031048 (2018), [[arXiv:1705.06728](#)].
- [57] H. Georgi, *Unparticle physics*, *Phys. Rev. Lett.* **98** 221601 (2007), [[arXiv:hep-ph/0703260](#)].
- [58] F. Wilczek, ed., *Fractional statistics and anyon superconductivity*. 1990.
- [59] C. Wang and M. Levin, *Braiding statistics of loop excitations in three dimensions*, *Phys. Rev. Lett.* **113** 080403 (2014), [[arXiv:1403.7437](#)].
- [60] S. Jiang, A. Mesaros and Y. Ran, *Generalized Modular Transformations in (3+1)D Topologically Ordered Phases and Triple Linking Invariant of Loop Braiding*, *Phys. Rev. X* **4** 031048 (2014), [[arXiv:1404.1062](#)].
- [61] J. C. Wang and X.-G. Wen, *Non-Abelian string and particle braiding in topological order: Modular $SL(3, \mathbb{Z})$ representation and (3+1)-dimensional twisted gauge theory*, *Phys. Rev. B* **91** 035134 (2015 Jan.), [[arXiv:1404.7854](#)].
- [62] C. Wang and M. Levin, *Topological invariants for gauge theories and symmetry-protected topological phases*, *Phys. Rev. B* **91** 165119 (2015 Apr.), [[arXiv:1412.1781](#)].
- [63] P. Putrov, J. Wang and S.-T. Yau, *Braiding Statistics and Link Invariants of Bosonic/Fermionic Topological Quantum Matter in 2+1 and 3+1 dimensions*, *Annals Phys.* **384** 254–287 (2017), [[arXiv:1612.09298](#)].
- [64] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Surgery*, *Phys. Lett. B* **807** 135516 (2020), [[arXiv:1602.05951](#)].
- [65] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Topology: Quantum Surgery Formulas*, *Annals Phys.* **409** 167904 (2019), [[arXiv:1901.11537](#)].
- [66] A. Prakash, J. Wang and T.-C. Wei, *Unwinding Short-Range Entanglement*, *Phys. Rev. B* **98** 125108 (2018), [[arXiv:1804.11236](#)].
- [67] A. Prakash and J. Wang, *Unwinding Fermionic SPT Phases: Supersymmetry Extension*, *Phys. Rev. B* **103** 085130 (2021), [[arXiv:2011.13921](#)].
- [68] Y. Tachikawa, *On gauging finite subgroups*, *SciPost Phys.* **8** 015 (2020), [[arXiv:1712.09542](#)].
- [69] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo, H. Lin, P. Gao and S.-T. Yau, *Tunneling Topological Vacua via Extended Operators: (Spin-)TQFT Spectra and Boundary Deconfinement in Various Dimensions*, *PTEP* **2018** 053A01 (2018), [[arXiv:1801.05416](#)].
- [70] Z. Wan and J. Wang, *Adjoint QCD_4 , Deconfined Critical Phenomena, Symmetry-Enriched Topological Quantum Field Theory, and Higher Symmetry-Extension*, *Phys. Rev. D* **99** 065013 (2019), [[arXiv:1812.11955](#)].
- [71] J. McNamara and C. Vafa, *Cobordism Classes and the Swampland*, [arXiv:1909.10355](#).
- [72] N. Seiberg, *“Thoughts About Quantum Field Theory”*, <https://www.ias.edu/sites/default/files/sns/files/QFT.pdf>, (Talk at Strings 2019) (2019).
- [73] L. M. Krauss and F. Wilczek, *Discrete Gauge Symmetry in Continuum Theories*, *Phys. Rev. Lett.* **62** 1221 (1989).
- [74] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, *Commun.Math.Phys.* **129** 393 (1990).

- [75] S. Weinberg, *A New Light Boson?*, *Phys. Rev. Lett.* **40** 223–226 (1978).
- [76] F. Wilczek, *Problem of Strong P and T Invariance in the Presence of Instantons*, *Phys. Rev. Lett.* **40** 279–282 (1978).
- [77] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, Time Reversal, and Temperature*, *JHEP* **05** 091 (2017), [[arXiv:1703.00501](#)].
- [78] Z. Wan, J. Wang and Y. Zheng, *Quantum 4d Yang-Mills Theory and Time-Reversal Symmetric 5d Higher-Gauge Topological Field Theory*, *Phys. Rev.* **D100** 085012 (2019), [[arXiv:1904.00994](#)].
- [79] A. Belavin, A. M. Polyakov, A. Schwartz and Y. Tyupkin, *Pseudoparticle Solutions of the Yang-Mills Equations*, *Phys. Lett. B* **59** 85–87 (1975).
- [80] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, *Phys. Rev. D* **14** 3432–3450 (1976).
- [81] F. R. Klinkhamer and N. Manton, *A Saddle Point Solution in the Weinberg-Salam Theory*, *Phys. Rev. D* **30** 2212 (1984).
- [82] C. N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, *Phys. Rev.* **96** 191–195 (1954 Oct).
- [83] T. Banks and N. Seiberg, *Symmetries and Strings in Field Theory and Gravity*, *Phys. Rev.* **D83** 084019 (2011), [[arXiv:1011.5120](#)].
- [84] E. Witten, *Fermion Path Integrals And Topological Phases*, *Rev. Mod. Phys.* **88** 035001 (2016), [[arXiv:1508.04715](#)].
- [85] E. Witten, *The "Parity" Anomaly On An Unorientable Manifold*, *Phys. Rev.* **B94** 195150 (2016), [[arXiv:1605.02391](#)].
- [86] M. Guo, P. Putrov and J. Wang, *Time reversal, $SU(N)$ Yang-Mills and cobordisms: Interacting topological superconductors/insulators and quantum spin liquids in $3 + 1$ D*, *Annals of Physics* **394** 244–293 (2018 Jul), [[arXiv:1711.11587](#)].
- [87] P. Putrov, R. Thorngren, Z. Wan, J. Wang and et. al, (*in preparation*), *private conversations*.
- [88] C. Cordova and K. Ohmori, *Anomaly Constraints on Gapped Phases with Discrete Chiral Symmetry*, *Phys. Rev. D* **102** 025011 (2020), [[arXiv:1912.13069](#)].
- [89] L. Fidkowski and A. Kitaev, *Topological phases of fermions in one dimension*, *Phys. Rev. B* **83** 075103 (2011).
- [90] L. Fidkowski and A. Kitaev, *Effects of interactions on the topological classification of free fermion systems*, *Phys. Rev. B* **81** 134509 (2010).
- [91] (*in preparation*), *to appear*.
- [92] M. M. Anber and E. Poppitz, *Two-flavor adjoint QCD*, *Phys. Rev. D* **98** 034026 (2018), [[arXiv:1805.12290](#)].
- [93] C. Cordova and T. T. Dumitrescu, *Candidate Phases for $SU(2)$ Adjoint QCD_4 with Two Flavors from $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory*, [arXiv:1806.09592](#).
- [94] Z. Bi and T. Senthil, *Adventure in Topological Phase Transitions in $3+1$ -D: Non-Abelian Deconfined Quantum Criticalities and a Possible Duality*, *Phys. Rev. X* **9** 021034 (2019), [[arXiv:1808.07465](#)].

- [95] J. Wang, Y.-Z. You and Y. Zheng, *Gauge enhanced quantum criticality and time reversal deconfined domain wall: $SU(2)$ Yang-Mills dynamics with topological terms*, *Phys. Rev. Res.* **2** 013189 (2020), [[arXiv:1910.14664](#)].
- [96] H. B. Nielsen and M. Ninomiya, *A no-go theorem for regularizing chiral fermions*, *Phys. Lett. B* **105** 219 (1981).
- [97] E. Eichten and J. Preskill, *Chiral Gauge Theories on the Lattice*, *Nucl. Phys.* **B268** 179–208 (1986).
- [98] X.-G. Wen, *Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders*, *Phys. Rev.* **D88** 045013 (2013), [[arXiv:1303.1803](#)].
- [99] X.-G. Wen, *A lattice non-perturbative definition of an $SO(10)$ chiral gauge theory and its induced standard model*, *Chin. Phys. Lett.* **30** 111101 (2013), [[arXiv:1305.1045](#)].
- [100] J. Wang and X.-G. Wen, *Non-Perturbative Regularization of 1+1D Anomaly-Free Chiral Fermions and Bosons: On the equivalence of anomaly matching conditions and boundary gapping rules*, [arXiv:1307.7480](#).
- [101] Y. You, Y. BenTov and C. Xu, *Interacting Topological Superconductors and possible Origin of 16n Chiral Fermions in the Standard Model*, [arXiv:1402.4151](#).
- [102] Y.-Z. You and C. Xu, *Interacting topological insulator and emergent grand unified theory*, *Phys. Rev. B* **91** 125147 (2015 Mar.), [[arXiv:1412.4784](#)].
- [103] Y. BenTov and A. Zee, *Origin of families and $SO(18)$ grand unification*, *Phys. Rev.* **D93** 065036 (2016), [[arXiv:1505.04312](#)].
- [104] Y. Kikukawa, *Why is the mission impossible? – Decoupling the mirror Ginsparg-Wilson fermions in the lattice models for two-dimensional abelian chiral gauge theories*, *PTEP* **2019** 073B02 (2019), [[arXiv:1710.11101](#)].
- [105] Y. Kikukawa, *On the gauge invariant path-integral measure for the overlap Weyl fermions in $SO(10)$* , *PTEP* **2019** 113B03 (2019), [[arXiv:1710.11618](#)].
- [106] J. Wang and X.-G. Wen, *A Solution to the 1+1D Gauged Chiral Fermion Problem*, *Phys. Rev.* **D99** 111501 (2019), [[arXiv:1807.05998](#)].
- [107] S. S. Razamat and D. Tong, *Gapped Chiral Fermions*, *Phys. Rev. X* **11** 011063 (2021), [[arXiv:2009.05037](#)].
- [108] S. Catterall, *Chiral Lattice Theories From Staggered Fermions*, [arXiv:2010.02290](#).
- [109] J. Wang and Y.-Z. You, *Gauge Enhanced Quantum Criticality Beyond the Standard Model*, [arXiv:2106.16248](#).
- [110] J. Wang and Y.-Z. You, *Gauge Enhanced Quantum Criticality Between Grand Unifications: Categorical Higher Symmetry Retraction*, [arXiv:2111.10369](#).
- [111] J. Wang, Z. Wan and Y.-Z. You, *Cobordism and Deformation Class of the Standard Model*, [arXiv:2112.14765](#).
- [112] J. Wang, “*Ultra Unification*”, https://www.youtube.com/results?search_query=ultra+unification+juven+wang also harvard university particle physics lunch (november 30, 2020) and harvard university string lunch (december 4, 2020)., (Talk at Erwin Schrödinger International Institute for Mathematics and Physics (ESI). *Higher Structures and Field Theory. Week 1 - Topological Orders and Higher Structures on August 4th, 2020*) (2020).