Wormholes and out-of-time ordered correlators in
gauge/gravity duality

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ABSTRACT: We calculate the four-wave scattering amplitude in the background of an AdS traversable wormhole in 2+1 dimensions created by a nonlocal coupling of AdS boundaries in the BTZ black hole background. The holographic dual of this setup is a pair of CFTs coupled via a double-trace deformation, the scattering amplitude giving the out-of-time ordered correlation function (OTOC) in CFT. A short-living wormhole exhibits fast chaos (fast scrambling), with the Lyapunov exponent growing linearly with the temperature $T$, and in some cases even reaching the conjectured maximum value $2\pi T$ found in thermal black hole backgrounds. Drastic slowdown of scrambling is obtained for long-living wormholes, where OTOC grows exponentially slowly, eventually dropping to zero when the double-trace coupling strength is large enough. Our findings have parallels in strongly coupled disordered field theory models, and may indicate certain limitations of wormhole teleportation protocols previously studied in the literature.
1 Introduction

Until recently, wormhole geometries have held the status of an intriguing curiosity but little more than that. They have a number of exciting properties: they are topologically nontrivial, they might allow closed timelike curves, i.e., time travel, and if traversable they can transfer a finite amount of matter to the "other side" without breaking apart. Quite early on, it was shown [1, 2] that a wormhole requires the violation of even the weakest energy condition, the null energy condition (NEC). To do that, one needs either nonclassical matter or exotic, non-minimally coupled classical matter such as conformally coupled scalars or nonlocal interactions. Recently, physically motivated examples of such interactions have been found to give traversable wormholes in asymptotically flat space, with help of Landau-quantized fermions in magnetic field [3] or cosmic strings [4], and in AdS space, through nonlocal coupling via a double-trace deformation [5–9] or via a quotient by a discrete isommetry [10, 11], or again through cosmic strings [12]. The 2+1-dimensional wormhole obtained by Gao, Jafferis and Wall (GJW) from a double-trace deformation of the BTZ black hole [5] is a particularly natural setup for our interests: being asymptotically
AdS, it has a field theory dual, and living in $2+1$ dimensions the dual is a proper $(1+1$-dimensional) quantum field theory rather than quantum mechanics (i.e., $0+1$-dimensional theory). Higher-dimensional analogue of the GJW protocol is also possible and has been constructed in [7]. The only wormhole with a rigorously known field theory dual is the eternal AdS$_2$ wormhole of [8], shown in that paper to describe two coupled Sachdev-Ye-Kitaev (SYK) models. In higher dimensions, things are less clear, but for sure we have two strongly interacting field theories coupled via a double-trace coupling; an explicit example is given in [9].

Our goal is to examine what the nonlocal coupling and traversability from boundary to boundary means for dynamics, chaos and information transfer in dual field theory. Intimate relation of wormholes to quantum information and even the black hole entropy problem was found in [13], in the framework of the celebrated ER=EPR proposal [14, 15] and the concepts of scrambling [16, 17] and firewalls [18]. The "diffusion of information" on the black hole horizon turns out to be related to the decay rate of weakly non-equilibrium (linear response) correlation functions known as out-of-time ordered correlators (OTOC) [19–21]. Essentially, OTOC diagnoses chaos during the equilibration of the quantum field theory/many body system after a perturbation.¹ Another way of saying it is the scrambling concept of [16]: the OTOC growth rate determines the timescale over which a small package of information distributes itself over macroscopic distances (over the whole black hole horizon in this case); supposing pure exponential growth $e^{\lambda t}$, the exponent $\lambda$ is usually dubbed Lyapunov exponent, in (weak) analogy to classical Lyapunov exponents. If black holes are indeed the fastest scramblers in nature as proposed by Susskind and others, then the entropy problem is particularly sharp, as any piece of matter falling into a black hole will likely quickly get out as Hawking radiation, being thus entangled to both the black hole interior and the radiation outside the black hole.

Where do the wormholes arise in the above story? The insight of [13] is that a particle passing from left to right infinity through a traversable AdS wormhole describes the teleportation from left to right subsystem of the dual field theory (actually, pair of field theories). The module of the commutator of two observables, one from the left and the other from the right subsystem, is clearly related to OTOC (for details see [13] and the dicussion in the last section of this paper). One would therefore think that the OTOC behavior, i.e., the speed of the scrambling/the strength of chaos, will know about opening the wormhole. Coming to the GJW wormhole, we know that OTOC in the BTZ background at finite temperature exhibits maximum Lyapunov exponent $\lambda = 2\pi T$; what happens when the double-trace coupling is turned on and a wormhole opens? Naively, we expect much slower chaos, i.e., much smaller $\lambda$: first, a wave packet on a black hole horizon interacts with the huge number of degrees of freedom inside and that is why it scrambles so quickly – this is not the case for a wormhole, which has no horizon and its throat carries no internal degrees of freedom; second, a particle can go through the wormhole on the other side, and this means it does not stay forever in the interior, where the redshift is high and the scattering amplitude can

¹This kind of chaos is essentially quasiclassical, and is only weakly related to the chaos in the deep quantum regime, described by the random matrix theory [22].
grow very large.

Some work was done on OTOC calculations in various non-black hole systems. On the gravity side, [23] consider fuzzball and other microstate configurations alternative to black holes and find that indeed the scrambling slows down considerably in absence of a horizon. In [24] the authors find increased chaos on the boundary of an AdS$_3$-asymptotic geometry; [25] studies classical chaos in fuzzball backgrounds; anisotropic scrambling and space-dependent Lyapunov exponents are discussed in [26]. In wormhole backgrounds, to the best of our knowledge, mainly the field theory side was studied. Most relevant for us is the recent study [27] which considers two coupled SYK models (i.e., eternal AdS$_2$ wormhole) and finds exponentially small Lyapunov exponent at temperature $T$. Despite the mismatch in spacetime dimensions (our model being dual to a 1+1-dimensional theory), we will find something similar in one corner of the phase diagram. Since the SYK model is quite amenable to calculations, several interesting works have described chaos, OTOC and phase transitions in this and related models [28–35]. In [28] true quantum chaos, i.e., eigenvalue distribution is studied, as opposed to OTOC, and it is found that opening up a wormhole (coupling the two models) in general suppresses chaos. Our work confirms that line of reasoning, but only to some extent: running a bit ahead, we can say that only very long-living wormholes drastically reduce chaos, whereas a short-lasting double-trace coupling only has a small effect: the existence of a horizon, even in the past, influences the overall scattering amplitude.

The wormhole teleportation concept has also attracted attention. We are mainly inspired by the protocol of [36] where again the SYK model/eternal AdS$_2$ wormhole is exploited to teleport qubits in the basis of Dirac spinors, i.e., pairs of Majorana spinors in the SYK Hamiltonian. Alternative protocols are discussed, e.g., in [37–39], again from the SYK viewpoint. In this work we do not consider teleportation in any detail, we merely want to see how the wormhole Lyapunov exponents influence the teleportation in principle. The preliminary finding is a curious tradeoff: the bigger the wormhole the easier it is, but takes longer time, because the scrambling is slow. This line of reasoning clearly needs further work, which we hope to do in near future.

Operationally, we follow the tried recipe of OTOC calculation, explicated most saliently in [21]: we consider a bulk scalar field inserted at spatial infinity at time $t = 0$ and another bulk scalar inserted at time $t > 0$, scattering in the interior and reaching the other boundary. For a wormhole background however several complications arise. Not only is the geometry more complicated, but also time-dependent as the throat opens at some point in time; on top of that, the absence of infinite boost (present at the black hole horizon) invalidates some simple approximations which can be used for black holes. A very general method for calculating OTOC in coordinate representation, suitable also for time-dependent backgrounds, was found in [40] (see also [41] for related work). However, for our wormhole this method turned out too complicated. We thus perform a perturbative calculation for a weak double-trace coupling, when the time-dependent nature of the geometry can be tackled perturbatively. Our approach is thus doubly perturbative: the double-trace cou-

\[\text{One could, of course, consider some other field; we stick to the Klein-Gordon field for simplicity.}\]
pling/wormhole throat size $\gamma$ is assumed to be small compared to the initial black hole mass $M$, and the infalling waves are assumed to have small energy $\sim p$ compared to both; we thus have $p \ll \gamma \ll M$. This is somewhat limiting of course, but we will show that it covers a big chunk of the phase diagram and is good enough to gain some insight.

In the next section we set the stage: first we quickly recapitulate the setup of opening a traversable wormhole via a double trace deformation, then we introduce two approximations that simplify the subsequent calculations and calculate the wormhole metric in the whole spacetime (in [5] the metric is not given explicitly), and finally derive the bulk-to-boundary scalar propagators in this geometry. In section 3 we write the scattering amplitude for OTOC, discuss the complications arising from time-dependent metric and absence of the horizon, and finally calculate the OTOC; to that end we also describe the geodesics in the wormhole background. Section 4 describes the behavior of the Lyapunov exponents as a function of wormhole size and time duration; we also try to understand the result physically, in particular on field theory side. Section 5 concludes the paper with a discussion of our findings in the light of wormhole teleportation protocols.

2 Setup: metric and Klein-Gordon equation in traversable AdS wormholes

2.1 Simplified GJW wormholes and their metrics

Let us first briefly recapitulate from [5] how the wormhole temporarily opens up and becomes traversable by a double-trace deformation. We start from the maximally extended BTZ black hole, containing two boundaries dual to two initially decoupled CFTs, thus describing a pure state through the thermofield dynamics (TFD) double $\sum_n e^{-\beta E_n} |n\rangle |n\rangle$, with $\beta = 1/T$ the inverse temperature of the black hole and $|n\rangle$ being the CFT states with energy $E_n$ [15] (living in 1+1-dimensional spacetime with coordinates $(t, \phi)$). Now, [5] couple the CFTs via the interaction term $\delta H$ in the Hamiltonian:

$$H \equiv H_L - H_R \mapsto H + \delta H$$

$$\delta H = -\gamma \int dt \int d^D x \chi(t, \phi) O_L(t, \phi) O_R(-t, \phi),$$

(2.1)

where $\gamma = \text{const.}$ is the coupling strength, $\chi(t, \phi)$ encapsualtes the spacetime dependence of the coupling and $O_{L,R}$ are some operators in the left and right CFT respectively. The first line defines the unperturbed Hamiltonian which contains two identical CFTs on the left- and right-hand side respectively. The holographic dictionary on double-trace deformations [42, 43] allows one to calculate the correction to the two-point correlation functions. For a massive scalar of mass $m$ and conformal dimension $\Delta = 1 + \sqrt{1 + m^2}$, the field that we consider, this was done in [5] for both the bulk-to-bulk ($G$) and bulk-to-boundary ($K$) propagator. The former allows one to express the wormhole-generating stress-energy tensor. To that end, we introduce the usual Kruszkal coordinates:

$$U/V = -e^{2\pi t}, \quad (1 - UV)/(1 + UV) = r/r_h.$$

(2.2)
Now the stress-energy tensor of the bulk scalar of conformal dimension $\Delta$ is obtained by definition as

$$T_{UU}(U) = \lim_{U \to U'} \partial_U \partial_{U'} G(U, U')$$

$$G(t, t') = \gamma \sin \pi \Delta \int dt_1 \chi(t_1) K(t' + t_1 - i\beta/2) K(t - t_1) + (t' \mapsto t). \tag{2.3}$$

Here, $G$ is the first-order (in $\gamma$) corrected bulk propagator and $K$ is the zeroth-order (pure BTZ) bulk-to-boundary propagator. For the full derivation of the above result we refer the reader to [5, 6]. The symmetry $T_{UU}(U) = T_{VV}(V)$ is exact at the horizon, where $V = 0$ or $U = 0$ holds, so there is only one remaining varying coordinate, and the two wormhole mouths are symmetric with respect to $U \leftrightarrow V$. The resulting metric $g_{\mu\nu}$ reads:

$$ds^2 = -\frac{4L^2}{(1 + UV)^2} dU dV + \frac{r^2}{(1 + UV)^2} d\phi^2 + \frac{\gamma h(U, V) dU^2 + \gamma h(V, U) dV^2}{1 + UV}, \tag{2.4}$$

where $L$ is the AdS radius that we may put to unity and do so in the rest of the paper, and the symmetry of the stress-energy tensor implies the symmetry in the wormhole geometry, so that $h_{UU}(U, V) = h_{VV}(V, U) \equiv h(U, V) \equiv \tilde{h}(V, U)$. The function $h$ is determined from the Einstein equations. From now on we assume full isotropy in the angle $\phi$; this means we do not consider diffusion and spatial dependence of OTOC, but only time dependence, in a circular system (in other words, we consider spherical wormhole perturbations as in [19]). Now from (2.3-2.4) we can write down the one independent Einstein equation and its solution:

$$V \partial_V h - U \partial_U h - 2h + 2\gamma \frac{1 - UV}{1 + UV} T_{UU}(U) = 0 \tag{2.5}$$

$$h(U, V) = \frac{2}{U^2} \frac{1 - UV}{1 + UV} \int_{U_0}^U dU' T_{UU}(U'). \tag{2.6}$$

In order to explicitly calculate $h$ from (2.6), we need to insert a specific form of $\chi$ into (2.3). We consider two opposite regimes.

**Fast wormhole.** We consider first the idea given in [6], where the double-trace coupling is made instantaneous. In other words, we turn the coupling on at $t = t_0$ and turn it off at $t = t_f$, and then take the limit $t_f \to t_0$. This drastically simplifies the expressions for the average null energy, as found in [6], and also for the metric as we will now see. We can first expand the general expression (2.3) in small $U - U_0$ (of course, $t_0$ determines $U_0$) and integrate to find the stress tensor:

$$T_{UU}(U) = 2\gamma \Delta \frac{\Gamma(1/2 - \Delta)}{\Gamma(1 - \Delta)} B \left( -\frac{1}{U_0^2}, \frac{1}{2} + \Delta, -2\Delta \right) (U - U_0)^{-2\Delta - 1} \Theta(U - U_0), \tag{2.7}$$

where $B$ is the Euler beta function. For this $T_{UU}$, the Einstein equations at leading order in $\gamma$ (2.5-2.6) yield:

$$h(U, V) = 2\Delta \sqrt{\pi} \frac{\Gamma(1 - \Delta)}{\Gamma(3/2 - \Delta)} B \left( -\frac{1}{U_0^2}, \frac{1}{2} + \Delta, -2\Delta \right) (U - U_0)^{-2\Delta - 1} \frac{1 - UV}{1 + UV} \Theta(U - U_0). \tag{2.8}$$
Here we see a possible issue, hinted at in [6] for essentially the same model. The above expression may diverge, if \( T_{UU}(U \to 0) \sim U^\alpha \) with \( \alpha \leq -2 \). For (2.8), this means \( \Delta \geq 1/2 \). There is, however, a simple way to tackle the regime \( \Delta \geq 1/2 \). Observe first that for the marginal point \( \Delta = 1/2 \), when (2.8) is only logarithmically divergent; regularizing as \( U - U_0 \mapsto U - U_0 + \epsilon \) and taking the \( \epsilon \to 0 \) limit after the integration yields the same expression for \( h \) as we would obtain if the stress-energy tensor itself (rather than the coupling \( \chi \)) were proportional to \( \delta(U - U_0) \). This motivates us to assume (there is no controlled way to formulate this, as the singularity in (2.8) is simply nonintegrable for \( \Delta \geq 1/2 \)) that for \( \Delta \geq 1/2 \) the meaningful fast wormhole limit has the stress tensor

\[
T_{UU}(U) = -\gamma \delta(U - U_0)/U_0. \tag{2.9}
\]

Now the metric is

\[
h(U, V) = -\frac{2U_0}{U^2} \frac{1 - UV}{1 + UV} \Theta(U - U_0). \tag{2.10}
\]

This result is still perfectly physical; the Dirac delta in the stress tensor poses no problems, it simply means we formally glue together two solutions.\(^3\) We encompass both the \( \Delta \geq 1/2 \) case and the \( \Delta < 1/2 \) case under the name of fast wormhole.

**Slow wormhole.** The other important regime is the opposite case, when the perturbation is turned on for a very long time after \( U_0 \); we call this regime the slow wormhole. It will turn out that the time during which the wormhole is open is more important than its size, so it will be very instructive to compare the fast and slow wormholes. The stress tensor for the slow wormhole was obtained already in [5] by expanding around \( U \to \infty \). The stress-energy tensor and the metric correction are

\[
T_{UU}(U) = 4\gamma \Delta^2 U^{-2\Delta-2} \log U \log \frac{U_f}{U_0}, \tag{2.11}
\]

\[
h(U, V) = \frac{8\Delta^2}{(1 - 2\Delta)^2} \log \frac{U_f}{U_0} \frac{1 - UV}{1 + UV} \frac{1 + (1 - 2\Delta) \log V}{V^{2\Delta+1}}, \tag{2.12}
\]

where \( U_0 \) and \( U_f \) are the moments when the wormhole is turned on and off. Note that the strict \( U_0 \to 0, U_f \to \infty \) limit is not consistent: we cannot reach the eternal wormhole solution by starting from the finite-time double-trace deformation but we can study the long-time regime which is enough for our purposes. Therefore, we adopt \( U_0 = 1/U_f \to 0 \) and expand in \( U_0 \) small.\(^4\)

### 2.1.1 Wormhole metric in radial coordinates

For the solution of the Klein-Gordon equation and some other applications, it is convenient to have the wormhole metric also in \((t, r, \phi)\) coordinates. This is a harder nut to crack, and we could not obtain a closed-form expression. Instead, we match the solution in the throat region (small \( r \)) and the solution in the outer region (large \( r \)). The throat region develops

\(^3\)In other words, here the wormhole-opening perturbation is itself a shock wave; this is distinct from the fact that the OTOC perturbation always contains a shock wave component, and in BH backgrounds, as we know [19], OTOC is made solely from shock waves at leading order.

\(^4\)Remember that \( U_0 \to 0 \) means \( t_0 \to -\infty \) and \( U_0 \to \infty \) corresponds to \( t_0 \to \infty \).
near the black hole horizon and therefore is likely close to an AdS metric (indeed, the asymptotically flat 3+1-dimensional wormhole studied in [3] has AdS throat geometry). We introduce the new radial coordinate as
\[ \frac{r - r_h}{r_h} = \gamma \rho, \]  
and consider the limit \( \gamma \to 0, \rho \to \infty \) with \( \gamma \rho \to \text{const.} < 1 \); in other words, the mouth lives at large \( \rho \) but the deviation of the throat metric from the BH metric is still finite and small because the wormhole opening scale \( \gamma \) is assumed to be small. Since the metric is time-dependent, we need to consider different epochs in time separately. Remember that \( g_{UU} = h \) and \( g_{VV} = \tilde{h} \) are only significant in magnitude for certain times (or \( U, V \) coordinates). We thus introduce the regimes (0) where \( h, \tilde{h} \) are both negligible (I) only \( h \) is significant (II) both \( h \) and \( \tilde{h} \) are significant and (III) only \( \tilde{h} \) is significantly nonzero. Of course, the regime (0) is at leading order the same as the black hole metric. This picture becomes particularly simple for the Dirac delta model (2.10) where the regimes are sharply delineated. In the regime (I) we have \( U > U_0, V < U_0 \), regime (II) is \( U, V > U_0 \) and the regime (III) has \( U < U_0, V > U_0 \). Actually, we can write \( h \) explicitly as
\[ h(t, r) = -2U_0 e^{-2\gamma t} \frac{r + r_h}{r - r_h}. \]  
From this we see that the regime (I) implies \( t < t_0 \), (II) means \( t \approx t_0 \) and in (III) we have \( t > t_0 \); for \( t \ll t_0 \) and \( t \gg t_0 \) we are in the BH regime (0). In each region we simply start from the solutions (2.8) or (2.10) for \( h \), expanding in a series around \( r = r_h \) for the throat region or around \( r = \infty \) for the outer region.

Let us start from the throat region. In the regime (I) we get
\[ ds^2_I = (1 + \gamma^2 r_h^2 \rho^2) (-dt^2 + d\phi^2) + \frac{d\rho^2}{1 + \gamma^2 r_h^2 \rho^2} + \frac{2\gamma U_0 r_h^2}{1 + \gamma^2 r_h^2 \rho^2} dt d\rho, \]  
a nonstationary metric as it feels the onset of the perturbation. The regime (II) is stationary at leading order:
\[ ds^2_{II} = (1 + \gamma^2 r_h^2 \rho^2) (-dt^2 + d\phi^2) + \frac{d\rho^2}{1 + \gamma^2 r_h^2 \rho^2}. \]  
Notice that this AdS factor is not merely a remainder of the original near-horizon AdS throat, as the latter is AdS \( \times \mathbb{R} \) for a BTZ black hole, and also our rescaled coordinate \( \rho \) actually blows up for \( \gamma = 0 \). As could be expected, the third region has the same form as (2.15) with inverted time \( t \to -t \).

In the outer region, the solution must be close to the BTZ black hole. Considering again three regimes as before, in the regime (I) we get
\[ ds^2_I = -(r^2 - \tilde{r}_h^2) dt^2 + \frac{dr^2}{r^2 - \tilde{r}_h^2} + r^2 d\phi^2 + \frac{4\gamma r_h^2 U_0}{r^2 - \tilde{r}_h^2} dt dr, \]  
where \( \tilde{r}_h \equiv r_h(1 - \gamma U_0) \). The regime (III) has identical metric except that \( t \to -t \). In the regime (II) the metric is
\[ ds^2_{II} = -(r^2 - \tilde{r}_h^2) dt^2 + \frac{dr^2}{r^2 - \tilde{r}_h^2} + r^2 d\phi^2. \]
In principle, the next step would be to match the metric solutions both "vertically", along the radial coordinate, for $\rho \to \infty$, and "horizontally", along the time axis. The outcome is a series expansion in $r - r_h$ or equivalently $1/\rho$, that we do not write here as it is a very cumbersome expression, which will not appear explicitly in later analysis (and anyway we have the full explicit solution in Kruszkal coordinates). Instead we proceed to the solution of the Klein-Gordon equation. We do that in each of the two regions (outer and inner) separately and then find the bulk-to-boundary propagator; in this way we need nothing beyond the asymptotic metrics already found.

2.2 Klein-Gordon equation in the wormhole background

The solution of the equation of motion for a scalar in wormhole background is an all-around useful goody to have for many purposes also outside the scope of this paper (stability analysis, equilibrium correlation functions and spectral functions, etc). We first solve the Klein-Gordon equation in each region (outer, throat) and in each regime (I,II,III) separately. The "horizontal" matching over time can then be easily done directly, and matching along $r$ requires a series expansion. The case $\Delta \geq 1/2$ results in relatively simple expressions so let us show this case in full detail.

In the throat region, the AdS-like geometry leads to Bessel functions in the solutions as could be expected. Expanding over energy eigenvalues $\omega$ and angular momentum eigenvalues $\ell$, the equation of motion for the wavefunction $\Phi(t,\rho,\phi) = \exp(-\omega t + i\ell \phi)\phi(\rho)$ reads:

$$\phi''(\rho) + \left(\frac{3}{\rho} - \frac{4\sigma r U_0^2 \rho^2}{\rho^4}\right)\phi'(\rho) + \frac{\omega^2 - \ell^2 - m^2}{\rho^2} \phi(\rho) = 0,$$

where $\sigma = -1, 0, 1$ for regions (I,II,III), respectively. In the regime (II) we easily get

$$\Phi_{\text{throat}}^{\text{II}}(t,\rho,\phi) = \frac{1}{\rho} e^{-\omega t + i\ell \phi} K_{\sqrt{1 + m^2}} \left(\sqrt{\ell^2 - \omega^2 \rho}\right), \quad \omega < \ell,$n

$$\Phi_{\text{throat}}^{\text{II}}(t,\rho,\phi) = \frac{1}{\rho} e^{-\omega t + i\ell \phi} J_{\sqrt{1 + m^2}} \left(\sqrt{\omega^2 - \ell^2 \rho}\right), \quad \omega > \ell.$$

(2.20)

In the other two regimes we can eliminate the extra term (proportional to $\sigma$ in (2.19)) by transforming:

$$\Phi_{I,III}^{\text{throat}} = c_{I,III} \Phi_{\text{throat}}^{\text{II}} \exp \left(\frac{\pm 2i \gamma U_0 \omega r_h^2}{3 \rho^3}\right).$$

(2.21)

We also need to match the constant in front in order to have a smooth solution. In this way we get

$$\Phi_{I,III}^{\text{throat}}(t,\rho,\phi) = \left(1 + 2 \gamma U_0 \sqrt{1 + m^2} \frac{\omega^2}{\ell^2 - \omega^2}\right) \frac{1}{\rho} e^{-\omega t + i\ell \phi} \frac{2i \gamma U_0 \omega r_h^2}{3 \rho^3} K_{\sqrt{1 + m^2}} \left(\sqrt{\ell^2 - \omega^2 \rho}\right),$$

(2.22)

for $\omega < \ell$ and analogously, with $K \rightarrow J$, for $\omega > \ell$. This solution matches the form (2.21), and for small $\omega$ it yields (2.20) at leading order. We have chosen the boundary conditions so that the solution is well-behaving in the interior of the throat ($\rho \to 0$). This is the expected behavior for the bulk-to-boundary propagator: it should be well-behaving in IR (whereas in
the UV it reflects the presence of a Dirac delta source, i.e., has a non-normalizable mode. Now consider the outer region. The equation reads

$$\phi''(r) + \frac{3r^2 - \tilde{r}_h^2}{r(r^2 - \tilde{r}_h^2)} \phi'(r) + \frac{r^2 (\omega^2 - m^2 (r^2 - \tilde{r}_h^2)) - 2\sigma r \gamma U_0 \omega \tilde{r}_h^2 \omega^2 - \tilde{r}_h^2}{r^2 (r^2 - \tilde{r}_h^2)} \phi(r) = 0,$$  

(2.23)

where $\sigma$ has the same meaning as before, and $\tilde{r}_h$ as in (2.17). In the regime (II) the solution is the hypergeometric function, as already found, e.g., in [44, 45]:

$$\Phi_{II}^{out}(t, r, \phi) = C_{II} e^{-\omega t + i\ell \phi} r^{\omega - \Delta}(r^2 - \tilde{r}_h^2)^{-\omega/2} F_1 \left( a, b, \Delta; \frac{r^2}{\tilde{r}_h^2} \right)$$

$$a = \frac{\tilde{\ell}}{2} - \frac{\tilde{\omega}}{2} + \frac{\Delta}{2}, \quad b = -\frac{\tilde{\ell}}{2} - \frac{\tilde{\omega}}{2} + \frac{\Delta}{2}. \quad (2.24)$$

We have introduced the notation $\tilde{\ell} \equiv \ell/\tilde{r}_h$, $\tilde{\omega} \equiv \omega/\tilde{r}_h$, and picked the branch that remains smooth in the interior (for $r \to r_h$), which is appropriate for the Feynmann propagator. As expected, this solution diverges on the AdS boundary, behaving in general as $r^{-\Delta}$, which corresponds to a source. The other branch of the solution behaves in the interior as $(r - r_h)^{\pm \tilde{\omega}}$ and this solution would be chosen for the advanced/retarded propagator. In the regimes (I,III) we can again reduce the equation to the $\sigma = 0$ case by introducing

$$\Phi_{I,III}^{out} = \Phi_{II}^{out} \exp \left[ \mp \gamma U_0 \left( \frac{\omega r - \tilde{r}_h^2}{r^2 - \tilde{r}_h^2} - \omega \arctan \frac{r}{r_h} \right) \right]. \quad (2.25)$$

Finally, the matching gives

$$\Phi^{out}(t, r, \phi) = \left( 1 + 2i \gamma U_0 \frac{\omega}{\sqrt{\omega^2 - \tilde{r}_h^2 r^2}} + \ldots \right) \Phi_{II}^{out}(t, \rho, \phi) \exp \left[ i \gamma |\omega| U_0 \left( \frac{r}{r^2 - \tilde{r}_h^2} - \arctan \frac{r}{r_h} \right) \right], \quad (2.26)$$

where we have shown only the leading-order term in the matching expansion (in this case we have found no closed-form solution akin to (2.22)). This concludes the solution of the Klein-Gordon equation. Now we will feed these results into the bulk-to-boundary propagator.

### 2.2.1 Bulk-to-boundary propagator

To remind, the bulk-to-boundary propagator satisfies the homogenous equation of motion in the bulk and behaves as the Dirac delta at the boundary when properly rescaled: $\lim_{r \to \infty} r^{D-\Delta} K(r; x, x') = \delta(x - x')$. Analytical expressions both for bulk-to-bulk and bulk-to-boundary propagators in the BTZ black hole background are known [45]. For the latter, it reads

$$K_{BTZ}(r; t, t' = 0; \phi, \phi' = 0) = \frac{r_h^{2\Delta}}{2\Delta + 1 \pi} \left( r \cosh r_h \phi - (r^2 - r_h^2)^{1/2} \cosh r_h t \right)^{-\Delta}. \quad (2.27)$$

Translation invariance allows putting $t'$ and $\phi'$ to zero. The result for $K$ is usually obtained from the bulk-to-bulk propagator, which is in turn obtained through the method of images.
from pure AdS$_3$ spacetime. Since the wormhole is not simply related to pure AdS$_3$ anymore, it is awkward to use the method of images. We instead construct $G$ by definition, as the sum of eigenmodes, and then find $K$ at leading order by expanding around $r' \to \infty$. Conveniently, we need only the throat and outer region solutions (2.22,2.26) for that (not the matched solution in the whole spacetime). Performing the sum

$$K(r; t, t'; \phi, \phi') = \lim_{r' \to \infty} \sum_{\ell} \int d\omega e^{i(\omega - \omega')\phi} \Phi^{out}(t', r', \phi') \Phi^{throat}(t, \rho(r), \phi)$$

(2.28)

leads us to the following result at leading order in $\gamma$:

$$K(r; t, t'; \phi, \phi') = \left[ 1 - \frac{2\gamma UV}{1 + UV} \left( \frac{2UV}{1+UV} \right)^2 [e^{-\tilde{r}_h U} - e^{\tilde{r}_h V} + \cosh (\tilde{r}_h (\phi - \phi'))]^2 \right]^{\Delta} \times e^{\frac{-\gamma t_0}{4}} \left[ e^{-\tilde{r}_h U} - e^{\tilde{r}_h V} + \cosh (\tilde{r}_h (\phi - \phi')) \right]^{-\Delta}.$$  

(2.29)

This is the final step of this rather cumbersome calculation. The cases $\Delta < 1/2$ and the slow wormhole are even worse; they are still straightforward but the algebra is very tedious. We have performed all the series expansions in the Mathematica package and have not bothered to simplify the expressions into a humanly readable form. We will give only the leading order results for the scattering amplitude, i.e., OTOC itself, where the expressions are a bit simpler. In the next section we will often need also the propagators in mixed variables $K(U, V; t', \phi, \phi')$, i.e., in terms of $U = U(t, r)$ and $V = V(t, r)$. It is easy to change variables in (2.29), so we see no reason to write out explicitly this form of the propagator. Finally, we remind the reader that we will mostly ignore the diffusion in $\phi$, so we could put $\phi = \phi' = 0$ from the beginning but prefer to have the most general form of the propagator for possible later use.

### 3 The scattering amplitude and OTOC

#### 3.1 Setting the stage

Let us first remind ourselves of the definition of OTOC and its connection to the bulk scattering amplitude. We may motivate the out-of-time ordered correlation function by noticing that the module of the commutator of some operator $A$ at time $t$ and $B$ at time $t = 0$ contains both time-ordered and time-disordered quantities:

$$\langle [A(t), B(0)]^2 \rangle = 2\langle [A(t)B(0)]^2 \rangle = 2\langle [A(t)B(0)]^2 \rangle + 2\langle A(t)B(0)^2 \rangle + 2\langle A(t)B(0)B^2(0) \rangle,$$

taking into account the invariance of the expectation value to cyclic permutations. While the second, time-ordered term presumably factorizes at long times, the first, the OTOC term, does not. The two-time commutator itself can be interpreted as the response of the system at time $t$ to the insertion of some perturbation at time 0, in analogy to the Loschmidt echo. For a more detailed physical discussion we refer the reader, e.g., to [21, 32]. Let us focus on the combination $A(t)A(t)B(0)B^2(0)$. From now on we take $A$ and $B$ to be operators dual to the bulk scalar (Klein-Gordon)
fields with different conformal dimensions $\Delta_1$ and $\Delta_2$. The wormhole-opening perturbation is generated by the field with dimensions $\Delta_1$, i.e., the conformal dimension $\Delta$ from the previous section is $\Delta_1$ in the OTOC setup. As explicated in [19–21], the fundamental holographic relation connecting the OTOC to a bulk scattering amplitude can be schematically represented as

$$D(t, 0) \equiv \langle A(t)B(0)A(t)B(0) \rangle = \int dP \langle \mathrm{IN}(P) | \Psi_1^\dagger(t, x_1) \Psi_2^\dagger(0, x_2) \Psi_3^\dagger(t, x_3) \Psi_2(0, x_4) | \mathrm{OUT}(P) \rangle \langle \mathrm{OUT}(P) \rangle = |\mathrm{IN}(P)\rangle e^{iS_c},$$

(3.1)

where $P$ is any set of variables that characterizes the IN and OUT states, and the eikonal approximation implies that the phase shift equals the classical action $S_c$. The bulk fields $\Psi_{1,2}$ with masses $m_{1,2}$ are dual to the field theory operators $A, B$ with conformal dimensions $\Delta_{1,2}$. Here we have schematically denoted the spatial coordinates on the boundary by $x$ and from now on we specialize to the $2+1$ dimensions with $x = \phi$ (although in the end we will ignore the angular dynamics anyway).

Our strategy will be to treat the wormhole opening $\gamma$ perturbatively, so the whole OTOC calculation that we perform essentially builds up on the calculation in the BH background. Let us thus first remind the reader how it works for a BH, emphasizing those points that will change when the wormhole opens. The bulk fields $\Psi_{1,2}$ are sourced from the boundary and thus generated by the field with dimensions $\Delta_1$ in the OTOC setup. Let us thus first remind the reader how it works for a BH, emphasizing those points that will change when the wormhole opens. The bulk fields $\Psi_{1,2}$ are sourced from the boundary and thus generated by the field with dimensions $\Delta_1$ in the OTOC setup.

where $p, q$ are the momenta. The overlap $\langle \Psi_1 \Psi_2 | \Psi_3 \Psi_4 \rangle$ in (3.1) for Klein-Gordon wave-functions is given by $\oint \sqrt{-g} \Psi_1^\dagger \Psi_2^\dagger \partial_\Sigma \Psi_3 \Psi_4 |\Sigma\rangle$, where the integral is performed along any bulk slice $\Sigma$. Following [21], we can evaluate it at a distance $\epsilon \to 0$ from the horizon $U = 0$. We thus need to find the wavefunctions (3.2) at this slice in the perturbed background.

Here comes the hallmark of the BH calculation: the perturbed background is a shock-wave geometry, where the metric changes discontinuously and can be obtained by gluing two BH solutions, with masses $M$ and $M + p$, where $p$ is the momentum of the incoming wave. This is because the stress-energy tensor in the eikonal approximation is of the form $p\delta(X_\mu(\tau) - x_\mu)$, i.e., it has the form of the stress-energy tensor for a point particle with the geodesic $X_\mu(\tau)$, and the infinite redshift at the horizon implies that to a very good approximation the backreaction only happens at $U = 0$ and $V = 0$. This means that the metric perturbations are of the form $\delta g_{UU} \propto p V \Theta(U)$ and analogously for $\delta g_{VV}$. In this background, the incoming functions $\Psi_{1,2}$ are just those from (3.2), the function $\Psi_3$ is transformed as

$$\Psi_3^\dagger(t, \phi) = \int dV \int d\phi' e^{iq\sqrt{\bar{v}}} \langle K(U, \bar{V} = V - pU \Theta(U); t, \phi, \phi') A(t, \phi') | U = 0 \rangle = e^{iq\nu \epsilon^\nu \epsilon^\mu \epsilon^\nu \epsilon^\mu \Theta(U)} \Psi_3,$$

(3.3)

In principle, the wormhole-generating perturbation could be due to an altogether different field, with dimension distinct from both $\Delta_1$ and $\Delta_2$, but apparently we do not lose in generality by taking it equal to $\Delta_1$. 


and analogously for $\Psi_4$. We have exploited the fact that the momenta are the generators of translations and we have written $p_U$, $q_V$ in terms of $p^U$, $q^V$ because the latter will appear in the stress-energy tensor of the perturbation. Now we can write out the amplitude in (3.1) in full detail. Rewriting the derivatives in terms of the momenta we get

\[ D_{BH}(t,0) = \int d\phi' \int d\phi'' \int dp_U \int dp_V \Psi_1^\dagger \Psi_2^\dagger p_U p_V e^{i(g_{\mu U} p^\mu + g_{\nu V} p^\nu)} \Psi_3 \Psi_4 e^{i S_c} = \]

\[ = \int d\phi' \int d\phi'' \int dp^\kappa g_{\kappa U} g_{\Lambda V} \Psi_1^\dagger \Psi_2^\dagger g_{\mu U} p^\mu g_{\nu V} p^\nu \Psi_3 \Psi_4 e^{i(g_{\mu U} p^\mu + g_{\nu V} p^\nu)} e^{i S_c} = \]

\[ = K^\dagger(0,V;V) K^\dagger(U,0;0) K(0,V;U) K(U,0;0) g_{\kappa U} g_{\Lambda V} g_{\mu U} g_{\nu V} p^\mu p^\nu e^{i(g_{\mu U} p^\mu + g_{\nu V} p^\nu + S_c)}. \]

(3.4)

We have adopted a shorthand notation in several respects in the above equation: it is understood that the wavefunctions $\Psi_{1,2,3,4}$ depend on $t$ and $\phi$, a spherical shock is assumed so that in fact the $\phi$-dependence drops out and the angular integrals are trivial, thus in the last row the dependence of the propagators on $\phi$ is not written explicitly and also the integrals are written in a shorthand notation (they are the same integrals from the previous line). Note that the phase factors cancel out, except for the extra phase obtained from the shock wave scattering. In order to complete the calculation, we need to supply the classical, i.e., on-shell action, which is obtained at leading order from the linear approximation as $S_c = \int \delta g_{\mu \nu} t^\mu \nu$, and $t^\mu \nu$ is the stress-energy tensor for a point particle.

Let us now sit back and think what will change when we try to follow the same path for a wormhole. The general formula (3.1) remains. Different geometry however means different solutions to the Klein-Gordon equations, different propagators, different bulk wavefunctions, different geodesics and different backreaction. In detail, this means the following:

1. The geodesic $X_\mu(\tau)$ that enters the stress-energy tensor will critically differ from the BH geodesic for small $U$ or small $V$ – this is where falling into the horizon is replaced by the tunnelling through the wormhole throat. This effect can be studied perturbatively in $\gamma$. Locally, it will merely decrease the Lyapunov exponent by a factor linear in $\gamma$; but the different global shape of the trajectory, i.e., the fact that it can reach the other side or get trapped in the throat, may have dramatic consequences, leading to recurrences in the OTOC and eventually zero Lyapunov exponent for large-mouth long-living wormholes.

2. The WH geometry is explicitly time-dependent, thus the momenta are not conserved anymore. However, the momentum nonconservation can also be treated perturbatively in $\gamma$ so we can write the momenta as $p = p_\infty + \gamma(\ldots)$ – the sum of the asymptotic momentum and the wormhole-induced correction. This will influence the stress-energy tensor of the perturbation $t_{\mu \nu}$ as well as the amplitude calculation (3.4) and can have drastic consequences: the extra terms in $S_c$ stemming from the change in momentum can make the Lyapunov exponent exponentially small. Therefore, even though the initial effect is perturbative, its integral (the amplitude that determines OTOC) can change nonperturbatively.
3. The backreaction will be more complicated than just a shock wave. Even though the stress-energy tensor is still proportional to a Dirac delta in the eikonal approximation, the perturbation $\delta g_{\mu\nu}$ will be of the form shock wave plus smooth corrections. There is always a shock wave component whenever eikonal approximation is used because point particles and rings always source a shock-wave metric [46–48], however for a general background metric the shock wave alone does not in general solve the Einstein equations. But the final influence of this effect is not very dramatic: it changes the numbers, i.e., the value of the Lyapunov exponents and the scrambling time but not the qualitative properties of OTOC.

4. Different propagators and bulk wavefunctions will likewise only introduce quantitative, not qualitative corrections.

The exciting things happen as a consequence of (1) and (2) in the above list, and now we describe how this happens. We first write the geodesic equations and solve them for $UV$ small, and then we consider the large-scale geometry of geodesics; these results allow us to write the stress-energy tensor in the eikonal approximation. The second step will be the calculation of the backreaction, resulting in the metric correction $\delta g_{\mu\nu}$. Then it is easy to compute the on-shell action, and the final step is the calculation of the scattering amplitude from the ingredients previously obtained.

3.2 The calculation

3.2.1 The geodesic equation

Near-mouth behavior. In BH background, $U = 0$ is a geodesic, and in that case $p^V$ is the only nonzero component of the momentum. Clearly, this is not the case for a wormhole. Therefore, $p^U$ and $p^V$ are both nonzero, and the metric receives corrections in $UU$, $VV$, $UV$ and $\phi\phi$ components both from incoming and outgoing waves (the remaining components are zero by symmetry). To see that, start from the equation for the radial geodesic:

$$V'' + g^{UV} \partial_V g_{UV} (V')^2 + \gamma h g^{UV} U'' + \frac{1}{2} \gamma g^{UV} \left( \partial_U h (U')^2 - \partial_U \tilde{h} (V')^2 \right) + \gamma g^{UV} \partial_V h U' V' = 0,$$

where $U = U(\tau)$, $V = V(\tau)$ and also $h = h(U(\tau), V(\tau))$ and $\tilde{h} = h(V(\tau), U(\tau))$; by $g_{UV}$ we denote the component of the unperturbed wormhole metric (2.4). The second equation is equivalent to the above, with $U \leftrightarrow V$.\footnote{This also means $h \leftrightarrow \tilde{h}$.} For $U \ll 1$ (and for $V$ small analogously), we can expand the equation (3.5) about the BH geodesic $U = 0$, $V = \tau$ quadratically in $\gamma$. After some algebra, the solution reads:

$$V'(\tau) = \int_0^\tau \frac{d\tau'}{1 + \frac{\gamma}{4} \int_0^{\tau'} \partial_V h (V(\tau''), U(\tau''))} + O(\gamma^3) = \tau + O(\gamma).$$

$$U'(\tau) = \frac{\gamma}{4} \int_0^\tau d\tau' \int_0^{\tau'} \partial_U h (V(\tau''), U(\tau'')) + O(\gamma^3)$$

\footnote{Of course, the form of $V(\tau)$ depends on the gauge choice but we can always pick the gauge where $V = \tau$.}
In the second line we have emphasized that the solution is an order-$\gamma$ correction of the BH solution. Inserting the solution (2.6) for $h$, we further get

$$U(\tau) = \gamma T_{UU}(0) - \gamma \int d\tau' \tau'^2 h(\tau', 0) + O(\gamma^3)$$

$$V(\tau) = \tau + \gamma \int d\tau' \tau'^2 h(\tau', 0) + O(\gamma^3).$$

(3.8)

(3.9)

Now inserting a specific wormhole model, in our case (2.8), (2.10) or (2.12), we obtain the equation for the geodesic (the trajectory equation). From (3.8-3.9), it reads simply:

$$U - \gamma f(V) = 0, \quad f(V) = \int_0^V d\tau h(\tau, 0) \tau^2 + O(\gamma^3),$$

(3.10)

so it is easy to write down explicitly given the function $h$.

The expansion (3.8-3.9), valid in the region of $UV$ small, precisely where the redshift is the highest (of order $O(1/\gamma)$), captures the leading local backreaction effect, and we will shortly calculate the leading contribution to the stress-energy tensor. The global difference from the BH case – the fact that the orbit continues through the throat – does not matter at leading order as the backreaction deep in the throat and further is subleading.

**Global structure.** However, one effect that can be important already at leading order in $\gamma$ is the possibility of an orbit which goes back and forth multiple times, contributing to the backreaction whenever it passes near $UV = 0$. This is a difficult topic. Geodesics in the wormhole geometry are nonintegrable, since the only conserved quantity is the angular momentum (which is trivial anyway for a spherical wave); energy is not conserved because the geometry changes explicitly at $U_0$. We will give a few numerical examples to demonstrate that multiple windings are an exception in fast wormholes but almost a rule in slow wormholes. This is logical: a fast wormhole only has the left-right coupling for an instant, thus most orbits do not have time to go back and forth; for a slow wormhole there is ample time. A convenient way to see this is to start from the Lagrangian in the Schwarzschild coordinates and formulate the effective potential $V_{\text{eff}}$:

$$L = \frac{\dot{R}^2}{r_h^2 - R^2} + \frac{\dot{\tilde{R}}^2 - \tau_h^2 + R^2}{r_h^2 - R^2} \left( e^{2r_h \tau_h} - e^{-2r_h \tau_h} \right)$$

$$V_{\text{eff}} = - \left( r_h^2 - R^2 \right)^2 + \gamma r_h^2 \frac{\left( e^{2r_h \tau_h} - e^{-2r_h \tau_h} \right)}{(r_h + R)^2}.$$ 

(3.11)

(3.12)

It is understood that $R = R(\tau)$, and the angular momentum $L_z$ is put to zero. Now we can plot the effective potential as a function of the wormhole coupling $\gamma$ and the wormhole creation time $U_0$ (inserting of course the expressions for $h, \tilde{h}$ for a chosen wormhole model). Fig. 1 shows the effective potential for a fast wormhole with $\Delta = 3/4$, a fast wormhole with $\Delta = 1/4$ and a slow wormhole with $\Delta = 1/4$, for a range of $\gamma$ and $U_0$ values. Two effects are obvious (1) the effective potential, flat at the horizon for a BH ($\gamma = 0$), develops a well whose depth grows for large left-right coupling $\gamma$ and for early switching of the wormhole (2) while fast wormholes never develop a very deep well, the depths grows drastically in the
Figure 1. The effective potential for a spherical shell (effectively a particle with \( L_z = 0 \)) in the background of a fast wormhole with \( \Delta = 3/4 \) (solid lines), fast wormhole with \( \Delta = 1/4 \) (dashed lines) and slow wormhole with \( \Delta = 1/4 \) (dotted lines), each for \( \gamma = 0, 0.1, 0.2, 0.3 \) (black, blue, magenta red); fast wormholes have \( U_0 = 0.5 \) and the slow wormhole has \( U_0 = 0.1 \). For \( \gamma = 0 \) all three geometries reduce to a BTZ black hole at temperature \( T = 1/4\pi \), with flat potential on the horizon. For nonzero coupling, a well develops for small \( R \), but the well is shallow for fast wormholes and quite deep for slow wormholes, implying many bound states which however are not infinitely stable – their lifetime grows with the height of the maxima of \( V_{\text{eff}} \) at intermediate \( r \) values. The panels (A, B, C) zoom in successively in the small \( r \) region, to show clearly both the shallow wells and the deep wells. For very large \( r \) (not in the range of the plots) all potential wells acquire the universal form imposed by the AdS asymptotics.

slow wormhole approximation. We could try to study the high-lying bound states in deep wells within WKB formalism and count them analytically, but we postpone such detailed investigations of the throat dynamics for later work. For now we are content to conclude that slow wormholes have many bound states\(^9\) which become denser and denser as they approach the top of the well. Therefore, orbits can likely spend a long time inside the throat. In OTOC calculations, we will find that precisely for large-mouth slow wormholes, when the potential well is deepest, the chaos slowdown is infinite, i.e., the Lyapunov exponent is zero.

Let us also look at the orbits. In Fig. 2 we show a geodesic in a fast wormhole geometry and for reference also for a BTZ black hole. Of course, a wormhole replaces the horizon by a throat so instead of falling into the singularity the particle goes to the other AdS boundary; this may repeat several times if there are windings. But for small \( UV \), the two orbits are remarkably close, as we see also by looking at the coordinates in proper time \( U(\tau), V(\tau) \): they only diverge from the BTZ BH in a perturbative way. This suggests a practical method for computing the backreaction: we focus on the region where the boost is maximized, i.e., the minimal \( UV \) part of the orbit, where its stress tensor can be obtained as a series expansion in \( \gamma \) around the stress tensor for a particle/spherical wave in the BH geometry. The redshift is now finite everywhere, but it has a sharp maximum in the far IR region which is thus still the crucial one for scattering.

Stress-energy tensor. Now we can calculate the stress-energy tensor of the infalling wave by definition. The total backreaction is due to two incoming and two outgoing waves; we can write the equations for one incoming wave and in the end add the contributions from the other waves obtained by symmetry \( U \leftrightarrow V \). Starting from the single-particle action

\(^9\)These bound states are not infinitely long-living because the well in the center is of finite depth, meaning that the particle will eventually escape.
Figure 2. A typical orbit (red) in a fast wormhole background for $\Delta = 1/100, \gamma = 0.1$ (A) and for $\Delta = 1/4, \gamma = 0.5$ (B), in Kruszkal coordinates $(U, V)$. The $U$- and $V$-axis form the horizon for $\gamma = 0$ (black hole), the singularity is reached for $UV = 1$ (dotted black lines) and the AdS boundaries are at $UV = -1$ (dashed black lines); the original bifurcation surface is at $B_1$ and the point of left/right future horizon crossing is $B_2$. In (C) we show the coordinates $-U(\tau)$ (blue) and $V(\tau)$ (red) for $\Delta = 1/2$, and $\gamma = 0$ (BH, dashed lines) or $\gamma = 0.1$ (wormhole, solid lines). The divergence from the BH geodesic $U = 0, V = \tau$ is relatively small, justifying a perturbative treatment.

$$S_p = -\int d\tau \sqrt{-g} \dot{X}^\mu X^\nu, \text{ with } X_\mu = (U, V),$$ 

$$t_{\mu\nu} = 2 \sqrt{-g} \delta g_{\mu\nu} = \frac{1}{\sqrt{-g}} \delta(U - U(\tau)) g_{\mu\alpha} g_{\nu\beta} \dot{X}^\alpha \dot{X}^\beta \frac{U(\tau)}{U(\tau)} = \begin{pmatrix} t_{UU} & t_{UV} & 0 \\ t_{UV} & t_{VV} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.13)$$

The components $t_{UU}, t_{VV}$ are due to the wormhole opening and are proportional to the wormhole size $\gamma$. The (non-conserved) momenta $p^U(\tau), p^V(\tau)$ read

$$p^U(\tau) = -\gamma p^V(\tau = 0) g_{UU} h(V(t), U(t)) + O(\gamma^3), \quad p^V(\tau) = p^V(\tau = 0) + O(\gamma^3), \quad (3.14)$$

so we are in fact somewhat lucky: even to second order the $p^V$ momentum is still approximately constant. From now on we denote the asymptotic momentum by $p \equiv p^V(\tau = 0)$ (and likewise $q \equiv q^U(\tau = 0)$ for the other wave) to label the orbit. The first order correction is always zero, so we state the leading, second-order results:

$$t_{UU}(U, V) = \frac{\dot{\gamma} g_{UV}^2}{\sqrt{-g}} \left( 1 + \gamma^2 (g^{UV})^2 \frac{\dot{h}}{h} \right), \quad (3.15)$$

$$t_{VV}(U, V) = \frac{\dot{\gamma}^2 \dot{h}^2}{4 \sqrt{-g}}, \quad (3.16)$$

$$t_{UV}(U, V) = \frac{\dot{\gamma} g_{UV} \dot{h}}{2 \sqrt{-g}} \left( 1 - \frac{\gamma^2 (g^{UV})^2 \dot{h}}{4} \right), \quad (3.17)$$

with $\dot{\delta} \equiv \delta(U - \gamma f(V))$, the trajectory equation. In line with our perturbative treatment, we expand (with help of (3.8)):

$$\dot{\delta} = \delta(U) - \gamma \int_0^\tau d\tau' \tau^2 h(\tau', 0) + O(\gamma^3), \quad (3.18)$$

which suffices to obtain the backreaction to second order.
3.2.2 Backreaction: shock wave and beyond

The next step is the backreaction of the stress tensor (3.15-3.17). As we already mentioned, the presence of the Dirac delta in the stress-energy tensor, inherent to the eikonal approximation, means that the metric change $\delta g_{\mu\nu}$ will contain a shock wave: two vacuum solutions\textsuperscript{10} glued together along the surface normal to the trajectory given by $U - \gamma f(V) = 0$. Such solutions were first constructed for a BH in [46, 47] and studied in detail in [48]. This latter paper constructs the shock wave solution for a number of rather general metrics, however our wormhole does not fit into any of the classes considered there; it is therefore no surprise that a pure shock-wave solution does not exist in our case. We thus look for a solution which is a smooth correction of a shock wave. Following [48], the shock wave can be formulated (equivalently to the gluing picture) as a discontinuous coordinate change with an (as yet undetermined) discontinuity $c(U,V)$:

$$
(U,V) \mapsto (\tilde{U},\tilde{V}) = (U, V - c(U,V) \Theta(U - \gamma f(V)))
$$

$$
(dU,dV) \mapsto (d\tilde{U},d\tilde{V}) = (dU,dV - c(U,V) \delta dU + \gamma c(U,V) \delta f'(V)dV,dV)
$$

$$
\tilde{\delta} \equiv \delta(U - \gamma f(V)). \tag{3.19}
$$

The last line is the equation of trajectory, already mentioned in relation to the geodesic equations. The above coordinate change influences all the tensors, in particular the metric $g_{\mu\nu}$ and the background stress-energy tensor $T_{\mu\nu}$. The metric correction from the background (2.4) now totals the shock-wave contribution $\Delta g_{\mu\nu}$ plus the smooth contribution $\delta g_{\mu\nu}$:

$$
g_{\mu\nu} \mapsto g_{\mu\nu} + c(U,V) \Delta g_{\mu\nu} + \delta g_{\mu\nu}
$$

$$
\Delta g_{\mu\nu} = \begin{pmatrix}
-2g_{UV} & -\gamma \tilde{h} + 2\gamma g_{UV} f'(V) + \gamma^2 \tilde{h} & 0 \\
-\gamma \tilde{h} + 2\gamma g_{UV} f'(V) + \gamma^2 \tilde{h} & 2\gamma^2 \tilde{h} f'(V) & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{3.20}
$$

For symmetry reasons, the nonzero components of the smooth part are $UU$, $VV$, $UV$ and $\phi\phi$. Similar reasoning holds for the stress-energy tensor: being a second-rank tensor, $T_{\mu\nu}$ transforms the same way as the metric. Adding up $\delta T_{\mu\nu}$ and the direct contribution $t_{\mu\nu}$ from (3.15,3.17), the total stress-energy tensor is now

$$
T_{UU} \mapsto T_{UU} + t_{UU} - 2c(U,V) \delta T_{UV}
$$

$$
T_{UV} \mapsto T_{UV} + t_{UV} - c(U,V) \delta T_{VV} - c(U,V) \delta f' f'(V) + c(U,V) \delta^2 T_{VV} + c(U,V) \delta \gamma T_{UV} f'(V) + c(U,V) \delta \gamma^2 T_{VV}
$$

$$
T_{VV} \mapsto T_{VV} + t_{VV} + 2c(U,V) \delta \gamma T_{UV} f'(V). \tag{3.21}
$$

Now, given a WH model, i.e., the function $T_{UU}(U)$, we can in principle write down and solve the Einstein equations. The unknown jump $c$ is found by matching the metric perturbation to its stress tensor and integrating across $U = f(V)$ to find the coefficient in front of the Dirac delta. In this whole discussion, we have disregarded the terms with the square of the Dirac delta. Terms of the form $\delta^2(\ldots)$ regularly appear in shock-wave calculations but

\textsuperscript{10}Vacuum in the sense that $t_{\mu\nu} = 0$; of course the background is still the wormhole, with nonzero $T_{\mu\nu}$.
what they mean is that the coefficients in front of $\delta^2(U)$ should vanish, as explained already in [48]. This is indeed the case upon inserting the final solution into these terms, thus here we find it justified to simply leave out such terms from the very beginning, so as not to clutter the notation too much.

The fast and slow wormhole shock waves. Let us now do this explicitly for the fast wormhole. Consider first the regime $\Delta \geq 1/2$ with the Dirac delta metric (2.10), which is easier. The solution reads:

$$c(U,V) = 2p - 2\gamma^2 pU_0 \Theta(V - U_0) \frac{2V - 1}{V}$$

$$\delta g_{\phi\phi} = \frac{2p^2}{1 - p^2} g_{\phi\phi}$$

$$\delta g_{UV} = 2\gamma p \frac{\text{arctanh}(UV)}{(1 + UV)^2}, \quad \delta g_{UU} = \delta g_{VV} = 0.$$  

Notice that (3.23) in fact has nothing to do with the wormhole (it is $\gamma$-independent), it is simply the higher-order correction to the linear shock wave. Also, this whole story rests on the small $\gamma$ expansion and thus is only valid when $UV \ll \gamma < 1$, i.e., one should not worry that the hyperbolic arctangent grows exponentially for large $UV$. For $\Delta < 1/2$ the algebra is more tedious; when the dust settles, $\delta g_{\phi\phi}$ is expectedly the same as before while $c(U,V)$ and $\delta g_{UV}$ differ:

$$c(U,V) = 2p - 2\gamma^2 p \left( \frac{2\delta}{1 - 2\delta} \right)^2 (1 + (1 - 2\delta) \log (U - U_0)) U^{1-2\delta} \Theta(V - U_0)$$

$$\delta g_{UV} = \gamma p \exp \left( \frac{3 - 2\Delta}{2 - 4\Delta} \right) U^{3/2 + \Delta} (1 + (1 - 2\Delta) \log U).$$

For the slow wormhole, again only these two quantities change:

$$c(U,V) = 2p - 2\gamma^2 pU_0^2 (eV)^{-\Delta-1} (\log eV)$$

$$\delta g_{UV} = \frac{\gamma p}{(1 + UV)^2}.$$  

This completes the solution for the wormhole geometry perturbed by the eikonal scalar waves. The primary qualitative feature is the deformation of the shock wave, i.e., the wavefront has the form determined by the trajectory equation, and the amplitude is likewise spacetime-dependent. This effectively introduces long-time and nonlocal correlations that can kill the fast scrambling.

Now we can put the pieces together to express the on-shell action. We start from the textbook linearized gravity-matter action, transformed as in [21]:

$$S_c = \frac{1}{4} \int d^3x \sqrt{-g} \left( c g_{\mu\nu} + \delta g_{\mu\nu} \right) t^{\mu\nu}. \quad (3.29)$$

Note that the matter-independent kinetic term (of the form $\delta g_{\mu\nu} \partial^2 \delta g^{\mu\nu}$) is in fact included in (3.29) as it equals minus one half of the metric-matter terms of the form $\delta g_{\mu\nu} t^{\mu\nu}/2$. The Dirac delta terms, coming from the shock wave, will only contribute along the line
\( U = \gamma f(V) \), but the smooth terms will contribute to the integral in (3.29) in the whole space. Now, on top of the waves \( \Psi_{1,3} \) we add up also the waves \( \Psi_{2,4} \) with asymptotic momentum \( q \), which are easily obtained by symmetry from the solutions already found.

Again, the simplest case is the Dirac delta regime of the fast wormhole:

\[
S_{\gamma}^{(\Delta \geq 1/2)} = pq(1 - 4\gamma U_0) - \frac{3}{8} \gamma^2 U_0^3 (p^2 + q^2 \Theta (t - t_0)) .
\] (3.30)

This can be compared to the black hole result \( S_c = pq \):\(^{11}\) the main effect is the appearance of terms \( p^2 \) and \( q^2 \). In the other regime we get:

\[
S_{\gamma}^{(\Delta < 1/2)} = pq(1 - 4\gamma U_0) - \left( \frac{\Delta}{1 - 2\Delta} \right)^2 \exp \left( \frac{3 - 2\Delta}{2 - 4\Delta} \right) (p^2 - q^2 \Theta (t - t_0)) - \\
-14\gamma^2 \left( \frac{\Delta}{1 - 2\Delta} \right)^4 (p^2 + q^2 \Theta (t - t_0)),
\] (3.31)

the main difference being that now there is already a first-order effect in \( \gamma \).

In the long-time (slow wormhole) limit we expand in small \( U_0 \) and take the leading term. We get:

\[
S_{\gamma}^{(\text{slow})} = pq - 2\gamma U_0^{-2-3\Delta} \left( 4pqU_0 \log U_0 + (p^2 + q^2) \log U_0^2 + O(U_0^2) \right) .
\] (3.32)

For \( U_0 \to 0 \), the BH contribution \( pq \) is suppressed by a factor of \( U_0^{3\Delta+2} \) (ignoring the subleading logarithms), and the other \( pq \)-proportional term is suppressed by a factor of \( U_0 \).

Therefore, the dominant contribution is just

\[
S_{\gamma}^{(\text{slow})} = -4\gamma U_0^{-2-3\Delta} \log U_0 (p^2 + q^2) .
\] (3.33)

Here the qualitative difference from BH scrambling is obvious: fast scrambling is the consequence of the on-shell action being equal to the center-of-mass momentum squared [17], which equals \( pq \). Now this term is suppressed by \( U_0 \) small, and the scrambling is determined mainly by the asymmetric contributions \( p^2 \) and \( q^2 \): we can already guess something will change dramatically compared to the black hole scrambling.

### 3.2.3 The scattering amplitude

Now we get to the nastiest part of the business, calculating the scattering amplitude in the time-dependent, horizonless wormhole background. Such situations are considered in the seminal work [40] where the authors rewrite the amplitude in the coordinate representation in time-ordered form with the help of the Green identity. This is a more elegant and physically transparent way than what we do here, however in the wormhole geometry we have found it difficult to find the boundary surfaces along which to integrate the Green identity, so we have not succeeded in the time-ordered method here. Instead, we will rewrite (3.4) in terms of coordinates and nonconserved momenta with explicit proper-time dependence, and then we will again make a perturbative expansion of the momenta around their asymptotic values: according to (3.14), one momentum changes at the \( O(\gamma) \) level.

\(^{11}\)We pick the units so that \( 4\pi G = 1 \).
and the other remains constant at order $O(\gamma^2)$, thus perturbative treatment makes this approach quite doable.

Let us thus rewrite the scattering amplitude (3.4) for a wormhole. Since the momenta are not constant anymore, the coordinate shifts are not generated by $p,q$ and we have to write them explicitly, as derivatives along coordinates. Therefore, the wavefunction (3.3), and in general shifted wavefunctions of the form $\Psi(U, V - F(U, V))$, are now written as

$$\exp[-iF(U, V)\partial_{nv}]\Psi(U, V).$$

The derivative $\partial_{nv}$ that generates the translation is orthogonal to the wavefront, meaning it is not parallel to $U$ or $V$ axis since the BH geodesics $U=0$ are now replaced by the trajectory equation $U - \gamma f(V) = 0$. The same considerations hold mutatis mutandis for the trajectory $V - \gamma f(U) = 0$ and the operator $\partial_{nv}$.\footnote{One might wonder why we label the directions by $nv$, $n_V$ instead, since the whole point is precisely that these are not in the direction of the $U/V$ axis anymore. The reason is that, since our approach is perturbative, we can continuously deform our shockwaves to the BH shock waves, thus in the BH limit $\gamma \to 0$ indeed $nv$ becomes the direction of the $U$ axis and likewise for $V$.}

Let us first write the derivative $\partial_{nv}$ for a general trajectory given by $F(U, V) = 0$. Introducing $\Delta_{U,V} \equiv \partial_{U,V} F(U, V)$, we get:

$$\partial_{nv} = n_U \partial_U + n_V \partial_V = \frac{\gamma \Delta_V}{\sqrt{(1-\gamma \Delta_V)^2 + (\gamma \Delta_U)^2}} \partial_U + \frac{1-\gamma \Delta_U}{\sqrt{(1-\gamma \Delta_V)^2 + (\gamma \Delta_U)^2}} \partial_V =

(\gamma \Delta_V + \gamma^2 \Delta_U \Delta_V) \partial_U + \left(1 - \frac{\gamma^2 \Delta_U^2}{2}\right) \partial_V + O(\gamma^2),$$

and analogously for $\partial_{nv}$, replacing everywhere $\partial_U \leftrightarrow \partial_V$. For the trajectory we have derived with help of (3.8-3.9), written in the form $U - \gamma f(V) = 0$, we get $\Delta_{U,V} = 1$, $\Delta_{V} = -\gamma f'(V)$.

Finally, the integral over the phase space in (3.4) will now be the integral over the configuration space, where we integrate over the coordinates $U, V, \phi$ and over the asymptotic momenta $p, q$, as the latter determine the initial conditions of the trajectory, which are necessary to fully determine the configuration.\footnote{In a homogenous stationary spacetime it is enough to know the conserved momenta, as the orbit can always be translated in spacetime but in our case translations in time are not symmetries, so we need to label each configuration by the coordinates and initial conditions, which can be mapped to initial momenta.}

The expression for the amplitude is now (with the notation $\int \equiv \int dU \int dV \int d\phi \int d\phi' \int dp \int dq$ and leaving out the angular dependence everywhere):

$$\langle \text{IN} | \text{OUT} \rangle = \int \Psi_4^\dagger(U, V; t) \Psi_2^\dagger(U, V; 0) \partial_{nv} \Psi_3(U, V - \gamma f(V); t) \partial_{nv} \Psi_4(U - \gamma f(U), V; 0) e^{iS_c} =

= \int \Psi_4^\dagger(U, V; t) \Psi_2^\dagger(U, V; 0) \partial_{nv} \left(e^{-i\gamma f(V)\partial_V} \Psi_3(U, V; t) \right) \partial_{nv} \left(e^{-i\gamma f(U)\partial_U} \Psi_4(U, V; 0) \right) e^{iS_c} =

= \int K^\dagger \tilde{K}^\dagger \partial_{nv} \left(e^{-i\gamma f(V)\partial_V} K \right) \partial_{nv} \left(e^{-i\gamma f(U)\partial_U} \tilde{K} \right) e^{iS_c},$$

(3.35)

where in the last row we have written $K \equiv K (\gamma f(V), V; t)$, $\tilde{K} \equiv K (U, \gamma f(U); 0)$. All the terms in this expression are now known: the expressions for the derivatives $\partial_{nv}, \partial_{nv}$ are to be inserted from (3.34), the propagators were found in earlier sections, and the equation (3.10) determines the function $f$. However, (3.35) is quite messy. We really need to expand
it perturbatively in $\gamma$:

$$
\langle \text{IN}|\text{OUT} \rangle \equiv \int \mathcal{A} = \int (A_0 + \gamma A_1 + \gamma^2 A_2 + \ldots) .
$$

(3.36)

The terms up to the second order are

$$
A_0 = k\tilde{k}^\dagger \partial_U k\partial_V \tilde{k}, \quad A_1 = -2\gamma k\tilde{k}^\dagger \partial_U k\partial_V \tilde{k}
$$

$$
A_2 = -(f'_U + f'_V) k\tilde{k}^\dagger \partial_U k\partial_V \tilde{k} - f'_U k\tilde{k}^\dagger \partial_U k\partial_U \tilde{k} - f'_V k\tilde{k}^\dagger \partial_U k\partial_V \tilde{k},
$$

(3.37)

(3.38)

where for brevity we write $f_U \equiv f(U)$, $f_V \equiv f(V)$, and $k \equiv K(0;V,t)$, $\tilde{k} \equiv K(U,0;0)$. Equations (3.36-3.38) are the final outcome of our formalism for OTOC calculation. The rest is just algebra (actually, elementary integrals, saddle-point integration and transformations with hypergeometric functions), but the outcome of this algebra is the core of the paper – the behavior of OTOC and the Lyapunov exponents. We devote the next section to a detailed discussion of these matters.

4 Lyapunov spectra and the phase diagram

Now we will describe the behavior of OTOC in various parameter regimes. The relevant variables are the conformal dimensions (bulk masses) $\Delta_{1,2}$, the wormhole coupling $\gamma$, wormhole dynamics (fast/slow) and notably the time $t$ itself – this is what we call the Lyapunov spectrum, the collection of exponents $\lambda$ which characterize the correlation decay in various epochs. We will solve approximately the OTOC integral (3.36) first for fast wormholes, then for slow wormholes. Then we will plot the spectrum of Lyapunov exponents for various cases and discuss the physical consequences.

4.1 Lyapunov spectra for fast wormholes

Before we take off, two technical remarks are in order. First, all results for OTOC are of course time-dependent functions multiplied by time-independent constants depending on $\Delta_{1,2}$ and $r_h$. These constant terms are not important for us as we are mainly interested in the correlation decay in time, not the absolute magnitude of the function $D(t,0)$. For this reason we always just leave out such constant terms. Second, we will emphasize the essence over the calculational details; therefore we sometimes leave out the full integral as calculated in the saddle-point approximation (when the expression is unpractically long) and give the asymptotic long-time dynamics in terms of exponentials or power laws.

Consider first the simplest case: the Dirac delta model of the fast wormhole. The propagators (2.29) are first be evaluated at $(U,0)$ or $(0,V)$, the function $f$ is easily fed from (3.10), and the integral $\int dU \int dV \int d\phi \int d\phi'$ is performed exactly, yielding

$$
D(\Delta \geq 1/2)(t,0) = \int d\tilde{p} \int d\tilde{q} q^{2\Delta_1 - 1} \tilde{q}^{2\Delta_2 - 1} \times
$$

$$
\times \exp \left[ 2\gamma (\tilde{p} - e^{\gamma t} \tilde{q}) + (1 - 4\gamma U_0) \tilde{p} \tilde{q} - \frac{3\gamma^2 U_0^2}{8} (\tilde{p}^2 + \tilde{q}^2 \Theta(t-t_0)) \right].
$$

(4.1)
The time $t_0$ is the time when the wormhole is turned on for a boundary observer: $t_0 = \log U_0/r_h < 0$ ($U_0$ is in the first quadrant of the $U-V$ plane, and $U_0 = 0$ corresponds to $t_0 = -\infty$, a wormhole turned on for all times). The momenta are rescaled as

$$\tilde{p} = p r_h^{-\gamma U_0/2} e^{\gamma t_0/4} \log 4^{-1}, \quad \tilde{q} = q r_h^{-\gamma U_0/2} e^{-\gamma t_0/4} \log 4^{-1}. \tag{4.2}$$

The integral over momenta in (4.1) is doable in the saddle-point approximation, as in [21]. The saddle-point approximation works best when one conformal dimension is much larger than the other. We must therefore consider the cases $\Delta_2 > \Delta_1$ and $\Delta_1 > \Delta_2$. The two are not symmetric because the operators with $\Delta_1$ are inserted at time $t$ and those with $\Delta_2$ at time $0$; however the qualitative behavior is the same in both cases so let us assume for concreteness that $\Delta_2$ is larger. In the regime $t < t_0$ we get

$$D^{(\Delta \geq 1/2)}(t, 0) = F_1 \left( \Delta_1, 1/2, \frac{2}{3\gamma^2 U_0^2} \left( (2\Delta_2 - 1) e^{r_h t} + 1 \right)^2 \right) +$$

$$+ \frac{\Gamma(\Delta_2 + 1/2)}{\Gamma(\Delta_2)} F_1 \left( \Delta_1 + 1/2, 3/2, \frac{2}{3\gamma^2 U_0^2} \left( (2\Delta_2 - 1) e^{r_h t} + 1 \right)^2 \right) \tag{4.3}$$

which asymptotically reads

$$D^{(\Delta \geq 1/2)}(t, 0) \sim (1 - (2\Delta_2 - 1) e^{2\pi T t})^{-2\Delta_1} \Rightarrow \lambda = 2\pi T, \tag{4.4}$$

thus the BH chaos limit is still saturated. The regime $t > t_0$ makes the integral much more complex, thus we just give the final outcome:

$$D(t, 0)^{(\Delta \geq 1/2)} \sim \left[ 1 - (2\Delta_2 - 1) e^{2\pi T t} \right]^{-2\Delta_1} \left[ 1 - (2\Delta_1 - 1) e^{2\pi t (\Delta_2 + \Delta_1)} \right]^{-2\Delta_2} \Rightarrow$$

$$\Rightarrow \tilde{\lambda} = 2\pi T \left( 1, 1 - \frac{\Delta_2 - \Delta_1}{2\Delta_2} \right). \tag{4.5}$$

Here we have introduced the vector of Lyapunov exponents $\tilde{\lambda}$ which in this case has two components, the first saturating the chaos bound but the second being lower. At first glance, it is only the larger exponent that makes sense, i.e., the growth is as fast as its fastest mode, but this is made more complicated by the overall dimensional scaling of OTOC (with the exponents $2\Delta_1$ and $2\Delta_2$) and the constants in front of the exponents. In this case the slower mode (with the smaller Lyapunov exponent) has the larger dimensional exponent $2\Delta_2$ but is in turn multiplied by the small parameter $\gamma$. We simply do not know which Lyapunov mode is dominant in a certain time window without plotting the OTOC for specific parameter values.

For the case $\Delta_1 < 1/2$, the integration over coordinates results in the following expression:

$$D^{(\Delta < 1/2)}(t, 0) = \int d\tilde{p} \int d\tilde{q} \tilde{p}^{\Delta_1 + 2(\Delta_2 - 1)} \tilde{q}^{\Delta_1 + 2(\Delta_2 - 1)} \times$$

$$\times \exp \left[ 2\gamma \left( \tilde{p} - e^{r_h t} \tilde{q} \right) + (1 - 4\gamma U_0) \tilde{p}^2 - \frac{3\gamma^2 U_0^2}{8} \left( \tilde{p}^2 + \tilde{q}^2 \Theta (t - t_0) \right) \right]. \tag{4.6}$$
Crucially, the factor $\tilde{t}$ is time-dependent:

$$\tilde{t} = 1 + (1/2 - \Delta_1)\tilde{r}_h \gamma t.$$  \hspace{1cm} (4.7)

with $\tilde{r}_h = r_h(1 - \gamma U_0)$ as in (2.17). The saddle-point integration for $\Delta_2 > \Delta_1$ gives

$$D(\Delta<1/2)(t,0) \sim \left[1 - (2 \Delta_2 - 1)e^{2\pi T t}\right]^{-2\Delta_1} \left[1 - (2\Delta_1 - 1)\gamma \frac{\Delta_1}{1 - 2\Delta_1} e^{2\pi T t(1 - (1-2\Delta_1)/(1-\Delta_1) r_h \gamma)}\right]^{-2\Delta_2}$$

$$\Rightarrow \tilde{\lambda} = 2\pi T \left(1, 1 - \frac{(1 - 2\Delta_1)(1 - \Delta_1)}{2\Delta_1}\right).$$  \hspace{1cm} (4.8)

There is again a pair of exponents, the second being below the chaos bound.

4.2 Lyapunov spectra for slow wormholes

In a slow wormhole the chaos becomes exponentially weak or nonexistent. This is the first sign of novel scrambling mechanisms in wormholes: so far we have only found a perturbative reduction of the Lyapunov exponent. We have seen in (3.32-3.33) that at leading order in $U_0$ (which is roughly the expansion in large $-t_0$, with $t_0 < 0$) the eikonal phase factorizes into $\sim \exp(-p^2 - q^2)$. If the propagator didn’t couple $p$ and $q$, OTOC would behave similar to TOC at long times, i.e., as a simple product of expectation values. However, the propagator couples $p$ and $q$. After integrating out the coordinate dependence, we arrive at:

$$D^{(\text{slow})}(t,0) = \int d\bar{p} \int d\bar{q} \bar{p}^{2\Delta_1 - 1} \bar{q}^{2\Delta_2 - 1} \left(\bar{p} + \bar{q}\right)^{\Delta_1 + \Delta_2 - 2} \times \exp \left[\left(1 - 2(2 + 3\Delta_1)^{-1} \gamma U_0^{-1 - 3\Delta_1}\right) \left(\bar{p}^2 + \bar{q}^2\right)\right].$$  \hspace{1cm} (4.9)

This integral is harder to crack than the previous ones, but we can still repeat the same strategy: first do the saddle-point integration over $q$ (obtaining generalized hypergeometric functions as a result), and then the $p$ integral is doable through a series expansion. When everything is over, we get:

$$D^{(\text{slow})}(t,0) \sim \left[1 - \exp \left(\left(2 + 3\Delta_1 - 2\gamma U_0^{-1 - 3\Delta_1}\right)^2 e^{-(2 + 3\Delta_1)/2\pi T t}\right)\right]^{-2\Delta_2}$$

$$\Rightarrow \lambda = \left(2 + 3\Delta_1 - 2\gamma U_0^{-1 - 3\Delta_1}\right)^2 e^{-(2 + 3\Delta_1)/2\pi T} \sim e^{-(2 + 3\Delta_1)/2\pi T}.$$  \hspace{1cm} (4.10)

Therefore, even though the growth exponent is positive, it is extremely small: not only is it exponentially small in small $T$, but also quadratically small in both small $\gamma$ and small $U_0$. In a sense, this is logical: when the horizon is removed in distant past, most orbits almost don’t see the horizon at all and do not get the exponential boost factor $e^{\gamma t}$ (which produces the growth rate $\propto T$). For the critical value $\gamma_c(\Delta_1) = (2 + 3\Delta_1) U_0^{-1 + 3\Delta_1}/2$, the exponent in (4.10) formally becomes negative. In fact, the leading term in (4.9) simply goes to zero, and the expression (4.10) is replaced by subleading terms (that we have neglected in Eq. (4.10)). We do not have a good analytical understanding of this regime, but the numerics and the analysis of bound states in the effective potential in Section 3.2.1 both suggest that a dense spectrum of short-living recurrences forms:

$$D^{(\text{slow})}(t,0) \sim \left[1 - \sum_n \exp(\lambda_n t)\right]^{-2\Delta_2} \Rightarrow \tilde{\lambda} = (\lambda_1, \lambda_2, \ldots).$$  \hspace{1cm} (4.11)
where \( \lambda_n \) can only be determined numerically, by directly integrating the amplitude equation (we hope to gain more analytical understanding in further work). The outcome is thus a large spectrum of exponents, the OTOC showing many recurrences as we will see in the plot of the function \( D(t,0) \) in the next section. At least in the cases we have studied, the peaks have a power-law envelope. We conjecture (but cannot prove) that in the strict eternal wormhole limit the spectrum becomes continuous. But already for the slow wormhole with \( \gamma > \gamma_c \) the power-law envelope implies zero Lyapunov exponent at long timescales: \( \lambda = 0 \).

4.3 Phase diagram

Let us collect and systematize the findings obtained through the integrations in the previous subsections. This essentially amounts to a phase diagram of wormholes in terms of scalar perturbations:

- **Maximum chaos.** Fast wormholes always have a maximally chaotic mode, which retains the fast black hole scrambling mechanism with the exponent \( 2\pi T \). This is the consequence of the exponential boost factor \( e^{r_h t} \) at the horizon. Even though a wormhole opens at time \( t_0 \), there will always be orbits launched early enough from the AdS boundary that do not see the wormhole throat.

- **Fast chaos.** In the regime \( \Delta_1 < 1/2 \) always, and in the other regime for later times (i.e., late enough that the operator inserted at \( t = 0 \) has had time to reach the wormhole mouth), the fast wormhole Lyapunov exponent falls below the maximum, but is still linear in temperature, i.e., the BH-like boost is still present.

- **Slow chaos.** Slow wormholes are created in distant past, so that almost all orbits see the wormhole mouth. No wonder this regime has no linear scaling of \( \lambda \) with temperature: there is no exponential boost factor characteristic for the black hole.

- **No chaos.** For large enough coupling \( \gamma \), there will be many episodes of OTOC growth and decay, i.e., recurrences in field theory. In this case the Lyapunov spectrum is near-continuous, and the sum of a many exponential functions yields a power-law envelope. Therefore, the OTOC grows as a power law, with zero exponent. This is however only a phenomenological claim, as we do not have a good analytical control over this case.

We illustrate this picture with density plots of the Lyapunov exponent magnitude as a function of conformal dimensions. We first consider a fixed coupling \( \gamma \), for a fast wormhole and a slow wormhole, corresponding to the cases (4.5) and (4.8) above (Fig. 3). Plotting the largest Lyapunov exponent as a function of the conformal dimensions, it is immediately obvious that a slow wormhole exhibits a slow chaos phase. For a fast wormhole, it is not easy to differentiate between the maximum and fast chaos; the latter only happens in a small range of \((\Delta_1, \Delta_2)\) values (Fig. 3). Comparing the \( \gamma \)-dependence of the chaos exponent for the fast and slow wormhole in Fig. 4 makes the point: while in the fast case (A) \( \lambda \) reduces in accordance with (4.8), in the slow case (B) there is a line \( \gamma_c(\Delta_1) \) when the...
Figure 3. Dependence of the maximum Lyapunov exponent on the conformal dimensions of operators $A (\Delta_1)$ and $B (\Delta_2)$ for a wormhole at temperature $T = 0.05$ and coupling $\gamma = 0.3$, in the fast wormhole phase (A) and slow wormhole phase (B). The white lines denote the regimes of validity of the approximations in the calculation of the scattering amplitude ($\Delta_1 < \Delta_2$ or $\Delta_1 > \Delta_2$) and the wormhole metric ($\Delta_1 < 1/2$ or $\Delta_1 > 1/2$); the fact that the color map is smooth across these lines corroborates that the approximations work well in their respective domains. The panel (A2) is a zoom-in of (A1), to show the region where the Lyapunov exponent drops from its maximum value. The color map spans from red ($\lambda = 2\pi T = \text{max}$.) down to violet (lowest $\lambda$, exponentially small in $T$).

Lyapunov exponent drops to zero, through the mechanism outlined in the previous section – an infinite series of exponentials merges into a power law.

In order to better understand the dynamics of OTOC we plot also the time dependence $D(t, 0)$ for representative cases. In Fig. 5 we show $D(t, 0)$ for the Dirac delta fast wormhole (A), for the $\Delta < 1/2$ fast wormhole (B), for the slow wormhole in the slow-chaos phase (C) and slow wormhole in the no-chaos phase (D). For the fast wormhole there is a single peak, which rises and falls off exponentially. In the slow wormhole background, numerical
Figure 5. The OTOC function $D(t,0)$ for representative examples: fast wormhole with $\Delta_1 = 3/4$ (A) and $\Delta_1 = 1/4$ (B), both for $\Delta_2 = 2$ and $\gamma = 0.1$; slow wormhole with $\Delta_1 = 1/4$, $\Delta_2 = 3$ and $\gamma = 0.1$ (C) or $\gamma = 0.7$ (D). All plots are for the temperature $T = 0.31$. The colors code for different times $t_0(U_0)$ when the wormhole coupling is turned on: $U_0 = 1/2, 1, 2, 3, 4$ (fast wormhole) and $U_0 = (1/2, 1, 2, 3, 4) \times 10^{-2}$ (slow wormhole), in black, blue, magenta, red, orange, respectively. While the fast wormhole typically has a single peak (no recurrences), the slow wormhole generically has multiple peaks, which for supercritical $\gamma$ may merge into a near-continuum with a power-law envelope. This last point is asserted purely phenomenologically from numerics and we cannot prove it. In (D) the time axis is cut short in the negative $t$ direction in order to show more clearly the dense part of the spectrum.

integration yields multiple peaks. Eventually, for $\gamma > \gamma_c$, we find numerically a dense forest of peaks whose envelope is consistent with a power law, stemming from the dense energy levels near the top of the potential well in the effective potential $V_{\text{eff}} (3.12)$. Unfortunately, so far we are not able to find an analytical argument that the envelope is always a power law. At least for the specific case studied in 5(D) this is apparently true. This look like the most salient feature of the wormhole OTOC: nonlocal correlations introduced by the double-trace coupling prevent any finite rate (Lyapunov exponent) of the correlation decay – a power-law decay means zero exponent. In the final section we will try to explain the physical significance of this finding.

5 Discussion, teleportation and conclusions

The basic findings, summed up in the previous section, are in a sense expected: we know from [16, 17] that the black holes are the fastest scramblers,\textsuperscript{14} thus no wonder that removing the black hole horizon slows down the scrambling. Since fast wormholes still have the

\textsuperscript{14}From a rigorous viewpoint this is actually still a conjecture, but physical arguments accumulated in the meantime support it strongly.
horizon some of the time, it is also logical that their Lyapunov exponent is nonzero (albeit smaller than the maximal value). But even in the slow wormhole limit the Lyapunov exponent is nonzero, only very small, as long as we are below some critical coupling \( \gamma \) (and even then we actually only have strong indication but no proof that it drops exactly to zero).

In principle, this might be an artifact of the slow wormhole approximation, i.e., the fact that the wormhole is long-living but still not eternal. However, a recent work on the exact eternal SYK wormhole from field theory side [27] has found nonzero Lyapunov exponent with the same type of scaling \( \exp(-E_g/2T) \) where \( E_g \) is the energy gap in the wormhole phase. Is it more than a coincidence? Maybe yes because (1) it might be a universal effect independent of the details of the model (2) slow wormholes should have a smooth limit to eternal ones. Maybe no because (1) the eternal SYK wormhole is a 0 + 1-dimensional field theory, dual to 1 + 1-dimensional Jackiw-Teitelboim (JT) gravity, very different from general relativity in 2 + 1 dimension that we consider (2) it is unlikely that the energy gap can be matched to the constant of proportionality that we find \( \frac{2 + 3\Delta_1}{\pi} \). We hope to learn more on this question in the future.

One caveat is in order about our perturbative treatment. We expand in small wormhole coupling \( \gamma \). Therefore, one might worry that the critical \( \gamma \) above which the Lyapunov exponent drops to zero is beyond the scope of the perturbative treatment. However, the formal requirement for the series expansion is \( \gamma < 1 \) which is satisfied. Besides, the metric stays close to a black hole in most of the space as we can see, e.g., from the geodesic plots in Fig. 2. This suggests – but by no means proves – that our perturbative findings are sound.

An interesting work for us is [35]: it considers a wormhole, again from field theory side, as two coupled SYK models dual to the eternal JT wormhole. The authors find recurrences (multiple peaks) in transmission probability for the wormhole, qualitatively very similar to our plots; quite likely the OTOC sees these same recurrent states. Several authors [28, 29, 32] have found a slowdown of chaos even in a single SYK model when additional couplings are added, either to a slow-chaos ("Fermi-liquid-like") phase with \( \lambda \propto T^2 \). In [30] a quadratic perturbation leads to a zero-\( \lambda \) phase; the significance of these findings for us is unclear (because the SYK generalizations considered in these works differ from the wormhole coupling, and again because our system is non-stationary).

In the future we plan to deal specifically with one aspect of our findings, related to the wormhole teleportation protocols; out of several proposed scenarios, we are mainly inspired by [36]. That paper develops in detail the idea from [13] where the left and right AdS boundary, i.e., field theory serve as entanglement resources, and we want to teleport the state of the qubit Q, inserted in the left CFT, to the qubit T inserted in the right CFT, with reference R which starts out maximally entangled with Q. Translated into correlation functions, the correlation between Q and T is just the expectation value of the anticommutator (because the operators are fermionic) of the two inserted observables (on the left and on the right) in the presence of double-trace coupling, which can be expressed in terms of TOC (which quickly factorizes and reaches a constant value) and OTOC. The calculation in [36] finds that the mutual information between R and T is maximized when OTOC reaches its maximum absolute value. In their setup (SYK/eternal wormhole), this maximum scales as \( \gamma^{\Delta_1} \) in our notation and the timescale of transport as \( \log\gamma \). In our
higher-dimensional nonstationary setup, the timescale of transport is $1/\lambda$ which slows down drastically with increasing $\gamma$, but the overall magnitude of OTOC (although we have not studied it in detail) clearly grows at least as $\gamma^2$ (because the correction to the classical action is either of order $\gamma^2$ as in Eqs. (3.30,3.31) or of order $\gamma$ as in (3.33), and the additional factor of $\gamma$ comes from the correction of the propagator). Here we get a catch-22 if the protocol of [36] is applied to higher-dimensional non-stationary wormholes: high-fidelity of the teleportation requires strong wormhole coupling but in this case the whole procedure takes a long time.

The above discussion on teleportation also sensitively depends on dimensionality as well as on the timespan of the wormhole.\footnote{We thank to J. Pedraza for a discussion on this matter.} Poincare-symmetric eternal wormholes in AdS$_{D+1}$ for $D > 1$ are shown in [49] to have Planckian curvature (i.e., do not exist in the classical gravity regime). If either symmetry is reduced or the wormhole is not eternal there is no problem, as shown by the constructions in [9] and [5, 7], respectively. Therefore, we do not expect that any teleportation is possible (or at least not describable by classical gravity) in such cases. But for our non-eternal (even if long-living, i.e., slow) BTZ wormhole, in 2+1 dimensions, and all the more for the eternal wormhole in JT gravity [27], the teleportation should be possible. Thus, one might wonder why both [27] and our paper predict it to be exponentially slow. But (1) possibility of teleportation, even high-fidelity teleportation does not necessarily imply fast teleportation – these two notions are distinct: fidelity is about the faithfullness of the quantum state and speed is determined by the step in which the measurement outcome is transmitted classically (2) the simplistic assumption "fast OTOC=fast teleportation" does not hold in general; indeed, fast OTOC growth is a property of BH horizons, as we already discussed. The relation between OTOC and teleportation is subtler. In further work we will address this problem in detail.

Finally, another important goal for future work is to consider the quantum chaos and teleportation in more realistic strongly correlated systems, in particular the holographic non-Fermi liquids and strange metals. Once the appropriate background is constructed, our perturbative formalism for the scattering amplitude can be readily applied. This might provide experimentally relevant predictions for the equilibration of the system.

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