

## NONCOMMUTATIVE GEOMETRY OF ELLIPTIC SURFACES

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ABSTRACT. We recast elliptic surfaces over the projective line in terms of the non-commutative tori and one-parameter families of the periodic continued fractions. The correspondence is used to study the Picard numbers, the ranks and the minimal models of such surfaces. As an example, we calculate the Picard numbers of elliptic surfaces with fibers having complex multiplication.

## 1. INTRODUCTION

Our paper deals with two types of objects: the well known – elliptic curves and surfaces, and the less known – non-commutative tori and certain families of periodic continued fractions. By magic and glory of mathematics, they happen to be the same, the second (less known) exposing an intrinsic structure, while the first (well known) following a historic pattern. We hope that the reader will find the second type more natural and accurate description of the elliptic curves and surfaces.

Recall that elliptic surfaces are bundles over a base curve of genus  $g \geq 0$  with fibers of genus one. We assume further that  $g = 0$  and denote by  $\mathbf{Q}(t)$  the field of rational functions on the base  $\mathbf{Q}P^1$ . The elliptic curve over the field  $\mathbf{Q}(t)$  can be identified with an elliptic surface [Schütt & Shioda 2019] [5, Chapter 5] which we write  $\mathcal{E}_{\mathbf{Q}(t)}$  by an abuse of notation. The generic fiber of  $\mathcal{E}_{\mathbf{Q}(t)}$  will be denoted by  $\mathcal{E}_t$ . Silverman's specialization theorem says that the homomorphism  $\mathcal{E}_{\mathbf{Q}(t)} \rightarrow \mathcal{E}_t$  is injective for all but finitely many rational values of  $t$ . In particular, the rank of  $\mathcal{E}_t$  is related to the rank of  $\mathcal{E}_{\mathbf{Q}(t)}$ .

The less known (but equivalent) definition of the fiber  $\mathcal{E}_t$  is as follows. The Sklyanin algebra  $S(\alpha, \beta, \gamma)$  is a free  $\mathbf{C}$ -algebra on four generators  $\{x_1, \dots, x_4\}$  satisfying six quadratic relations  $x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3)$ ,  $x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3$ ,  $x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4)$ ,  $x_1x_3 + x_3x_1 = x_4x_2 - x_2x_4$ ,  $x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2)$  and  $x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2$ , where  $\alpha, \beta, \gamma \in \mathbf{C}$  and  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ . The algebra  $S(\alpha, \beta, \gamma)$  is twisted homogeneous coordinate ring of an elliptic curve in the Jacobi form  $\mathcal{E} \subset \mathbf{C}P^3$ , i.e. as an intersection of the quadrics  $u^2 + v^2 + w^2 + z^2 = \frac{1-\alpha}{1+\beta}v^2 + \frac{1+\alpha}{1-\gamma}w^2 + z^2 = 0$  [Stafford & van den Bergh 2001] [7, Example 8.5]. The norm-closure of a self-adjoint representation of the algebra  $S(\alpha, \beta, \gamma)$  by bounded linear operators on a Hilbert space  $\mathcal{H}$  is a non-commutative torus  $\mathcal{A}_\theta$ , i.e. a  $C^*$ -algebra generated by a pair of unitary operators  $u$  and  $v$  satisfying the commutation relation  $vu = e^{2\pi i\theta}uv$  for a real constant  $\theta$ . The  $\mathcal{A}_\theta$  is said to have real multiplication (RM), if  $\theta$  is a quadratic irrationality given by  $k$ -periodic continued fraction  $[b_1, \dots, b_N; \overline{a_1, \dots, a_k}]$ . The map  $F : \mathcal{E}_t \mapsto \mathcal{A}_\theta$  is a functor, such that if  $t \in \mathbf{Q}$ , then  $F(\mathcal{E}_t)$  are non-commutative tori with RM [4, Section 1.3].

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**Example 1.1.** ([3]) Let  $\mathcal{E}_{\mathbf{Q}(t)}$  be a surface given by the affine equation:

$$y^2 = x(x-1) \left( x - \frac{t-2}{t+2} \right), \quad t \in \{3, 4, 5, \dots\}. \quad (1.1)$$

Then  $F$  acts by the formula:

$$\mathcal{E}_{\mathbf{Q}(t)} \mapsto \mathcal{A}_{[t-1; \overline{1, t-2}]}. \quad (1.2)$$

The aim of our note is a generalization of (1.2) to the rest of elliptic surfaces  $\mathcal{E}_{\mathbf{Q}(t)}$  in terms of the continued fractions  $[b_1, \dots, b_N; \overline{a_1, \dots, a_k}]$ , see Theorem 1.4. The result is used to calculate the Picard number, the rank and the minimal model for the  $\mathcal{E}_{\mathbf{Q}(t)}$ , see Theorem 1.5. The following notation will be used.

**Definition 1.2.** (Brock, Elkies & Jordan [1]) We denote by  $V_{N,k}(\mathbf{C})$  an affine variety defined by the polynomials in variables  $a_{1 \leq i \leq k}$  and  $b_{1 \leq j \leq N}$  satisfying the following obvious equality:

$$[b_1, \dots, b_N, a_1, \dots, a_k; \overline{a_1, \dots, a_k}] = [b_1, \dots, b_N; \overline{a_1, \dots, a_k}]. \quad (1.3)$$

*Remark 1.3.* The equations of the  $V_{N,k}(\mathbf{C})$  date back to [Euler 1765] [2]. The corresponding projective variety was studied in [4, Section 6.2.1]. Our notation  $V_{N,k}(\mathbf{C})$  corresponds to the variety  $V(\mathcal{B})_{N,k}$  of [Brock, Elkies & Jordan 2021] [1].

The integer points of  $V_{N,k}(\mathbf{C})$  parametrize the  $k$ -periodic continued fractions  $[b_1, \dots, b_N; \overline{a_1, \dots, a_k}]$  and  $\dim V_{N,k}(\mathbf{C}) = N + k - 2$ . The  $V_{N,k}(\mathbf{C})$  is a fiber bundle over the Fermat-Pell conic  $\mathcal{P} : Cx^2 - Bxy + Ay^2 = (-1)^k A$  with the fiber map  $\pi : V_{N,k}(\mathbf{C}) \rightarrow \mathcal{P}$ , see [Brock, Elkies & Jordan 2021] [1] or (2.8) for the details. By  $t \in \mathbf{C}P^1$  we denote a rational parametrization of  $\mathcal{P}$ . In what follows, we consider only regular maps, i.e. given by polynomials over the base field. Our main result can be formulated as follows.

**Theorem 1.4.** For each elliptic surface  $\mathcal{E}_{\mathbf{Q}(t)}$  there exists a section  $\mathcal{P} \rightarrow U_{b_1, \dots, b_N; a_1, \dots, a_k}$  of a sub-bundle  $(U_{b_1, \dots, b_N; a_1, \dots, a_k}, \mathcal{P}, \pi')$  of the fiber bundle  $(V_{N,k}(\mathbf{C}), \mathcal{P}, \pi)$ , such that:

$$F(\mathcal{E}_{\mathbf{Q}(t)}) = \left\{ \mathcal{A}_{[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]} \mid a_i(t), b_j(t) \in \mathbf{N}, t \in \mathcal{P} \right\}. \quad (1.4)$$

Denote by  $\mathcal{F} = (\pi')^{-1}(t)$  a fiber of the bundle  $(U_{a_1, \dots, a_N; b_1, \dots, b_k}, \mathcal{P}, \pi')$  over  $t \in \mathcal{P}$ . For  $p(t) \in \mathbf{Z}[t]$  let  $\mathcal{E}_{p(t)}^{CM}$  be an elliptic surface whose fibers over  $t \in \mathbf{Q}$  have complex multiplication (CM) by the ring of integers of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-1-p^2(t)})$ , see Section 4 for an example. Denote by  $\mathcal{E}_{\mathbf{Q}(t)}^{\min}$  the minimal model of the surface  $\mathcal{E}_{\mathbf{Q}(t)}$ . An application of theorem 1.4 is as follows.

**Theorem 1.5.** Let  $\mathcal{E}_{\mathbf{Q}(t)}$  be a surface satisfying equation (1.4) and  $\rho(\mathcal{E}_{\mathbf{Q}(t)})$  its Picard number over  $\mathbf{Q}$ . The following is true:

- (i) the Picard number  $\rho(\mathcal{E}_{\mathbf{Q}(t)}) = N + k$ ;
- (ii) the rank  $r(\mathcal{E}_{\mathbf{Q}(t)}) = \dim \mathcal{F}$ ;
- (iii) the  $\mathcal{E}_{\mathbf{Q}(t)}^{\min}$  is  $\mathbf{C}$ -isomorphic to the  $\mathcal{E}_{p(t)}^{CM}$  for some  $p(t) > 0$ .

The article is organized as follows. Preliminary facts can be found in Section 2. The proofs of theorems 1.4 and 1.5 are given in Section 3. The Picard numbers of the elliptic surfaces with generic fiber having complex multiplication are calculated in Section 4.

## 2. PRELIMINARIES

We briefly review continued fractions, elliptic surfaces and non-commutative tori. For a detailed exposition we refer the reader to [Brock, Elkies & Jordan 2021] [1], [Schütt & Shioda 2019] [5, Chapter 5] and [4, Chapter 1], respectively.

**2.1. Continued fractions.** By an infinite continued fraction one understands an expression of the form:

$$[c_1, c_2, c_3, \dots] := c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \dots}}, \quad (2.1)$$

where  $c_1$  is an integer and  $c_2, c_3, \dots$  are positive integers. The continued fraction (2.1) converges to an irrational number and each irrational number has a unique representation by (2.1). The expression (2.1) is called  $k$ -periodic, if  $c_{i+k} = c_i$  for all  $i \geq N$  and a minimal index  $k \geq 1$ . We shall denote the  $k$ -periodic continued fraction by

$$[b_1, \dots, b_N, \overline{a_1, \dots, a_k}], \quad (2.2)$$

where  $(a_1, \dots, a_k)$  is the minimal period of (2.1). The continued fraction (2.2) converges to one of the irrational root of a quadratic polynomial

$$Ax^2 + Bx + C \in \mathbf{Z}[x]. \quad (2.3)$$

Conversely, the irrational root of any quadratic polynomial (2.3) has a representation by the continued fraction (2.2). Notice that the following two continued fractions define the same irrational number:

$$[b_1, \dots, b_N, \overline{a_1, \dots, a_k}] = [b_1, \dots, b_N, a_1, \dots, a_k, \overline{a_1, \dots, a_k}]. \quad (2.4)$$

But it is well known, that two infinite continued fraction with at most finite number of distinct entries must be related by the linear fractional transformation given by a matrix  $\mathcal{E} \in GL_2(\mathbf{Z})$ . Therefore equation (2.4) can be written in the form

$$x = \frac{E_{11}x + E_{12}}{E_{21}x + E_{22}}, \quad (2.5)$$

where  $\mathcal{E} = (E_{ij}) \in GL_2(\mathbf{Z})$  and  $x = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$ .

*Remark 2.1.* It is easy to see, that  $x$  in (2.5) is the root of quadratic polynomial (2.3) with  $A = E_{21}$ ,  $B = E_{22} - E_{11}$  and  $C = -E_{12}$ .

**Definition 2.2.** The Brock-Elkies-Jordan variety  $V_{N,k}(\mathbf{C}) \subset \mathbb{A}^{N+k}$  is an affine variety over  $\mathbf{Z}$  defined by the three equations:

$$\begin{cases} A[E_{22} - E_{11}](y_1, \dots, y_N, x_1, \dots, x_k) = BE_{21}(y_1, \dots, y_N, x_1, \dots, x_k) \\ -AE_{12}(y_1, \dots, y_N, x_1, \dots, x_k) = CE_{21}(y_1, \dots, y_N, x_1, \dots, x_k) \\ -BE_{12}(y_1, \dots, y_N, x_1, \dots, x_k) = C[E_{22} - E_{11}](y_1, \dots, y_N, x_1, \dots, x_k). \end{cases}$$

It is verified directly from remark 2.1 and the equality  $E_{11}E_{22} - E_{12}E_{21} = (-1)^k$ , that

$$CE_{21}^2 - BE_{21}E_{22} + AE_{22}^2 = (-1)^k A. \quad (2.6)$$

**Definition 2.3.** By the Fermat-Pell conic  $\mathcal{P}$  one understands the plane curve:

$$Cu^2 - Buv + Av^2 = (-1)^k A. \quad (2.7)$$

**Theorem 2.4. (Brock, Elkies & Jordan [1])** *The affine variety  $V_{N,k}(\mathbf{C})$  fibers over the Fermat-Pell conic  $\mathcal{P}$ , i.e. there exists a map  $\pi : V_{N,k}(\mathbf{C}) \rightarrow \mathcal{P}$ , such that*

$$\pi(y_1, \dots, y_N, x_1, \dots, x_k) = (E_{21}, E_{22}). \quad (2.8)$$

## 2.2. Elliptic surfaces.

**2.2.1. Surfaces.** An algebraic surface  $S$  is a variety of dimension two. An elliptic surface  $S$  over a curve  $C$  is a smooth projective surface with an elliptic fibration over  $C$ , i.e. a surjective morphism  $f : S \rightarrow C$  such that almost all fibers are smooth elliptic curves.

**2.2.2. Blow-ups.** The map  $\phi : S \dashrightarrow S'$  is called rational, if it is given by a rational function defined everywhere except for the poles of  $\phi$ . The map  $\phi$  is birational, if the inverse  $\phi^{-1}$  is a rational map.

A birational map  $\epsilon : S \dashrightarrow S'$  is called a blow-up, if it is defined everywhere except for a point  $p \in S$  and a rational curve  $C \subset S'$ , such that  $\epsilon^{-1}(C) = p$ . Every birational map  $\phi : S \dashrightarrow S'$  is composition of a finite number of the blow-ups, i.e.  $\phi = \epsilon_1 \circ \dots \circ \epsilon_k$ .

**2.2.3. Minimal models.** The surface  $S$  is called a minimal model, if any birational map  $S \dashrightarrow S'$  is an isomorphism. The minimal models exist and are unique unless  $S$  is a ruled surface. By the Castelnuovo Theorem, the surface  $S$  is a minimal model if and only if  $S$  does not contain rational curves  $C$  with the self-intersection index  $-1$ .

**2.3. Non-commutative tori.** The  $C^*$ -algebra is an algebra  $\mathcal{A}$  over  $\mathbf{C}$  with a norm  $a \mapsto \|a\|$  and an involution  $\{a \mapsto a^* \mid a \in \mathcal{A}\}$  such that  $\mathcal{A}$  is complete with respect to the norm, and such that  $\|ab\| \leq \|a\| \|b\|$  and  $\|a^*a\| = \|a\|^2$  for every  $a, b \in \mathcal{A}$ . Each commutative  $C^*$ -algebra is isomorphic to the algebra  $C_0(X)$  of continuous complex-valued functions on some locally compact Hausdorff space  $X$ . Any other algebra  $\mathcal{A}$  can be thought of as a noncommutative topological space.

The non-commutative torus  $\mathcal{A}_\theta$  is defined as a  $C^*$ -algebra generated by the unitary operators  $u$  and  $v$  satisfying the relation  $vu = e^{2\pi i \theta} uv$ . The  $\mathcal{A}_\theta$  is said to have real multiplication (RM), if  $\theta$  is a quadratic irrationality represented by the  $k$ -periodic continued fraction  $[b_1, \dots, b_N; \overline{a_1, \dots, a_k}]$ .

The non-commutative tori provide a bridge between the elliptic surfaces and continued fractions. Namely, there exists a covariant functor  $F$  mapping the fibers  $\mathcal{E}_t$  of an elliptic surface  $S$  into the non-commutative tori  $\mathcal{A}_\theta$ , see [4, Section 1.3] for the details. The fibers  $\mathcal{E}_t$  defined over  $\mathbf{Q}$  or a finite extension of  $\mathbf{Q}$  map to the non-commutative tori  $F(\mathcal{E}_t)$  with RM.

## 3. PROOFS

**3.1. Proof of theorem 1.4.** For the sake of clarity, let us outline the main ideas. Let  $(V_{N,k}, \mathcal{P}, \pi)$  be a fiber bundle defined by the map (2.8). Using an exclusion process described by formulas (3.3) -(3.5) in below, we construct a sub-bundle:

$$(U_{b_1, \dots, b_N; a_1, \dots, a_k}, \mathcal{P}, \pi') \subset (V_{N,k}, \mathcal{P}, \pi) \quad (3.1)$$

depending on the point  $(b_1, \dots, b_N; a_1, \dots, a_k) \in V_{N,k}$  and whose fibers are  $r$ -dimensional. Each section  $\sigma$  of the bundle defines a family of the non-commutative tori  $\mathcal{A}_{[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]}$ , where  $t \in \mathcal{P}$  so that  $a_i(t_0) = a_i$  and  $b_j(t_0) = b_j$ . By the construction,

$$F(\mathcal{E}_{\mathbf{Q}(t)}) = \mathcal{A}_{[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]}. \quad (3.2)$$

Let us pass to a detailed argument.

*Proof.* (i) Let  $(b_1, \dots, b_N; a_1, \dots, a_k)$  be an integer point of the variety  $V_{N,k}$ . To construct a subvariety  $U_{b_1, \dots, b_N; a_1, \dots, a_k}$ , we denote by  $(u_1, \dots, u_m)$  the variables and by  $(c_1, \dots, c_m)$  the constants. Unless stated otherwise, it is assumed that  $c_1 = b_1, \dots, c_m = a_k$ , where  $m = N + k$ .

The equations (2.4) defining the variety  $V_{N,k}$  allow to exclude two variables, say,  $u_{m-1}$  and  $u_m$ , i.e. they become algebraically dependent on the variables  $\{u_i \mid 1 \leq i \leq m-2\}$ . Namely, one can write:

$$\begin{cases} u_{m-1} &= \frac{P_{m-1}(u_1, \dots, u_{m-2})}{Q_{m-1}(u_1, \dots, u_{m-2})} \\ u_m &= \frac{P_m(u_1, \dots, u_{m-2})}{Q_m(u_1, \dots, u_{m-2})} \end{cases} \quad (3.3)$$

for some polynomials  $P_{m-1}, Q_{m-1}, P_m, Q_m \in \mathbf{Z}[u_1, \dots, u_{m-2}]$ .

Since  $(c_1, \dots, c_m) \in V_{N,k}$ , one obtains a sub-variety of the  $V_{N,k}$  consisting of the points  $(u_1, \dots, u_{m-2}, c_{m-1}, c_m)$ . In view of the (3.3), such a sub-variety is given by the system of equations:

$$\begin{cases} P_m(u_1, \dots, u_{m-2}) &= c_m Q_m(u_1, \dots, u_{m-2}) \\ P_{m-1}(u_1, \dots, u_{m-2}) &= c_{m-1} Q_{m-1}(u_1, \dots, u_{m-2}). \end{cases} \quad (3.4)$$

It follows from (3.4), that again two variables, say,  $u_{m-3}$  and  $u_{m-2}$  become algebraically dependent on the variables  $\{u_i \mid 1 \leq i \leq m-4\}$ . We repeat the argument obtaining a sub-variety made of the points  $(u_1, \dots, u_{m-4}, c_{m-3}, c_{m-2}, c_{m-1}, c_m)$ .

It is clear, that the algorithm will stop when the following system of the polynomial equations in the variables  $u_i$  is satisfied:

$$\begin{cases} P_m(u_1, \dots, u_{m-2}) &= c_m Q_m(u_1, \dots, u_{m-2}) \\ P_{m-1}(u_1, \dots, u_{m-2}) &= c_{m-1} Q_{m-1}(u_1, \dots, u_{m-2}) \\ &\vdots \\ P_2(u_1, u_2) &= c_2 Q_2(u_1, u_2) \\ P_1(u_1, u_2) &= c_1 Q_1(u_1, u_2). \end{cases} \quad (3.5)$$

*Remark 3.1.* Notice that system (3.5) always has a solution, e.g. the trivial solution  $(c_1, \dots, c_m)$ . Example 1.1 shows that in fact such solutions can be a variety of dimension 1. Below we consider a general case in terms of the Krull dimension of a polynomial ring.

(ii) Let  $\mathcal{I}_{c_1, \dots, c_m}$  be an ideal generated by equations (3.5) in the polynomial ring  $\mathbf{C}[u_1, \dots, u_m]$ . Consider the ring  $\mathbf{C}[u_1, \dots, u_m]/\mathcal{I}_{c_1, \dots, c_m}$ . The Krull dimension of such a ring will be denoted by  $r + 1$ , see remark 3.2 for the notation.

(iii) Consider an  $(r + 1)$ -dimensional affine subvariety  $U_{c_1, \dots, c_m}$  of the  $V_{N,k}$  given by equations (3.5). As it was shown in item (ii), the points of the variety  $U_{c_1, \dots, c_m}$  can be written the form:

$$(R_1, \dots, R_{r+1}, c_{r+2}, \dots, c_m), \quad (3.6)$$

where  $R_i \in \mathbf{Z}[u_1, \dots, u_{r+1}]$ .

(iv) The Brock-Elkies-Jordan Theorem 2.4 says that the variety  $V_{N,k}$  fibers over the Fermat-Pell conic  $\mathcal{P}$ . We denote by  $\pi'$  a restriction the map  $\pi : V_{N,k} \rightarrow \mathcal{P}$  to the subvariety  $U_{c_1, \dots, c_m}$ . Assuming  $c_1 = b_1, \dots, c_m = a_k$ , one gets a fiber bundle

$$(U_{b_1, \dots, b_N; a_1, \dots, a_k}, \mathcal{P}, \pi') \quad (3.7)$$

consisting the  $r$ -dimensional fibers, the 1-dimensional base and the  $(r+1)$ -dimensional total space.

(v) Consider a global section  $\sigma : \mathcal{P} \rightarrow U_{b_1, \dots, b_N; a_1, \dots, a_k}$ . The  $\sigma$  is given by the polynomials:

$$b_1(t), b_2(t), \dots, b_N(t) \in \mathbf{Z}[t]; \quad a_1(t), a_2(t), \dots, a_k(t) \in \mathbf{Z}[t], \quad (3.8)$$

where all but  $r + 1$  polynomials are constants, compare with (3.6).

*Remark 3.2.* The number  $r$  is equal to the rank of the  $\mathcal{E}_{\mathbf{Q}(t)}$ , see item (ii) of theorem 1.5 to be proved in the next section.

(vi) It remains to show, that  $F(\mathcal{E}_{\mathbf{Q}(t)}) = \mathcal{A}_{[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]}$ , where  $a_i(t)$  and  $b_j(t)$  are given by formulas (3.8). We shall prove this fact adapting the argument of [3] to the case of the  $\mathcal{E}_{\mathbf{Q}(t)}$ . Namely, suppose that the  $\mathcal{E}_{\mathbf{Q}(t)}$  is given in the Legendre form:

$$y^2 = x(x - 1)(x - \alpha(t)), \quad \alpha(t) \in \mathbf{Q}(t). \quad (3.9)$$

(vii) Recall that if  $F(\mathcal{E}_{\mathbf{Q}(t)}) = \mathcal{A}_\theta$ , then

$$\begin{pmatrix} b-1 & 1 \\ b-2 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \theta \\ 1 \end{pmatrix}, \quad \text{where} \quad \frac{b-2}{b+2} = \alpha(t), \quad (3.10)$$

see [3, Theorem 1 & Corollary 1.2]. Since  $b = \frac{2(1+\alpha(t))}{1-\alpha(t)}$ , one can write (3.10) in the form:

$$\begin{pmatrix} \frac{3\alpha(t)+1}{1-\alpha(t)} & 1 \\ \frac{4\alpha(t)}{1-\alpha(t)} & 1 \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \theta \\ 1 \end{pmatrix}. \quad (3.11)$$

(viii) On the other hand, it follows from (2.5) that:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \theta \\ 1 \end{pmatrix}, \quad (3.12)$$

where  $\theta = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$ .

(ix) One can factorize matrix in (3.12) as follows:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \\ \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_N & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdots \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}, \quad (3.13)$$

see e.g. [Brock, Elkies & Jordan 2021] [1, Definition 2.4].

(x) It remains to compare (3.11) and (3.12), i.e.

$$E_{11} = \frac{3\alpha(t) + 1}{1 - \alpha(t)}, \quad E_{21} = \frac{4\alpha(t)}{1 - \alpha(t)}, \quad E_{12} = E_{22} = 1. \quad (3.14)$$

(xi) Since  $\alpha(t) \in \mathbf{Q}(t)$ , one gets  $E_{12}(t), E_{21}(t) \in \mathbf{Q}(t)$ . Moreover, clearing the denominators in (3.12) one can always assume  $E_{ij}(t) \in \mathbf{Z}[t]$ . One obtains from (3.13), that  $a_i(t), b_j(t) \in \mathbf{Z}[t]$ .

Theorem 1.4 is proved.  $\square$

### 3.2. Proof of theorem 1.5.

*Proof.* Let us prove item (i) of theorem 1.5. Recall that the Neron-Severi group  $NS(\mathcal{E}_{\mathbf{Q}(t)})$  is the abelian group of divisors on  $\mathcal{E}_{\mathbf{Q}(t)}$  modulo algebraic equivalence. The Picard number  $\rho(\mathcal{E}_{\mathbf{Q}(t)})$  is defined as the rank of the  $NS(\mathcal{E}_{\mathbf{Q}(t)})$ . Such a number is always finite.

The idea of the proof is based on an identification of the  $F(NS(\mathcal{E}_{\mathbf{Q}(t)}))$  with the convergents of the continued fraction  $[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]$ . By an elementary property of the continued fractions, all such convergents are rational functions of the first  $N + k$  convergents. Let us pass to a detailed argument.

(i) Consider a global section  $\sigma_i : \mathcal{P} \rightarrow \mathcal{E}_{\mathbf{Q}(t)}$  of the elliptic surface  $\mathcal{E}_{\mathbf{Q}(t)}$  with the base curve  $\mathcal{P}$ . The  $\sigma_i(\mathcal{P}) := \mathcal{P}_i$  is a genus zero curve on the surface  $\mathcal{E}_{\mathbf{Q}(t)}$ . Thus one can identify  $\mathcal{P}_i$  with a divisor of the  $\mathcal{E}_{\mathbf{Q}(t)}$ .

(ii) Denote by  $\{\mathcal{A}_{\frac{p}{q}} \mid p, q \in \mathbf{Z}\}$  the non-commutative tori with rational values of the parameter  $\theta = \frac{p}{q}$ . The  $\mathcal{A}_{\frac{p}{q}}$  correspond to the degenerate elliptic curves  $\mathcal{P}$ , i.e.  $F(\mathcal{P}) = \mathcal{A}_{\frac{p}{q}}$  [4, Section 1.3].

(iii) On the other hand, any non-commutative torus  $\mathcal{A}_\theta$  is the inductive limit of an ascending sequence of the  $\mathcal{A}_{\frac{p_i}{q_i}}$ , where  $\frac{p_i}{q_i}$  are convergents of the continued fraction of  $\theta$ , see e.g. [4, Section 3.5]. Consider a commutative diagram in Figure 1, where  $\mathcal{P}_i$  is the union of all divisors of the  $\mathcal{E}_{\mathbf{Q}(t)}$  obtained as a pull back of  $F$  as  $t$  runs through all admissible values.

(iv) The ascending sequence of the rational non-commutative tori:

$$\mathcal{A}_{\frac{p_1(t)}{q_1(t)}} \subset \mathcal{A}_{\frac{p_2(t)}{q_2(t)}} \subset \dots \quad (3.15)$$

gives rise to an infinite inclusion sequence of the divisors  $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$ . It is easy to see, that

$$NS(\mathcal{E}_{\mathbf{Q}(t)}) = \lim_{i \rightarrow \infty} \mathcal{P}_i. \quad (3.16)$$

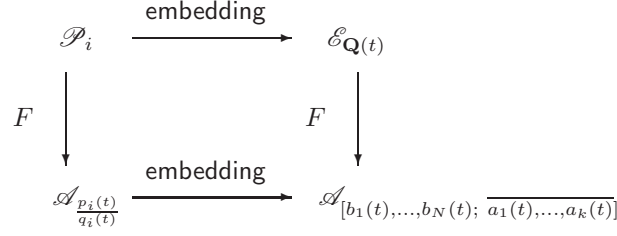


FIGURE 1.

(v) Let us evaluate the number of generators of the group  $NS(\mathcal{E}_{\mathbf{Q}(t)})$ . In view of (3.16), this question can be reduced to the number of generators of the sequence (3.15). Namely, given the periodic continued fraction  $[b_1(t), \dots, b_N(t); \overline{a_1(t), \dots, a_k(t)}]$ , how many algebraically independent convergents  $\frac{p_i(t)}{q_i(t)}$  are there?

(vi) It is easy to see, that the total number of the independent convergents is equal to  $N + k$ . The idea is simple: we recover the  $a_i(t)$  and  $b_j(t)$  from the first  $N + k$  convergents, and then express the remaining convergents  $\left\{ \frac{p_i(t)}{q_i(t)} \mid i > N + k \right\}$  as the rational functions of the  $a_i(t)$  and  $b_j(t)$ .

Namely, the first convergent  $\frac{p_1(t)}{q_1(t)}$  coincides with the  $b_1(t)$ . The second convergent  $\frac{p_2(t)}{q_2(t)} = \frac{p_1(t)}{q_1(t)} + \frac{1}{b_2(t)}$  and, therefore,  $b_2(t) = \left( \frac{p_2(t)}{q_2(t)} - \frac{p_1(t)}{q_1(t)} \right)^{-1}$ . Similarly, one gets  $b_3(t)$  as a rational function of the  $\frac{p_1(t)}{q_1(t)}$ ,  $\frac{p_2(t)}{q_2(t)}$  and  $\frac{p_3(t)}{q_3(t)}$ . Finally, the  $a_k(t)$  can be written as a rational function of the convergents  $\left\{ \frac{p_i(t)}{q_i(t)} \mid 1 \leq i \leq N + k \right\}$ .

Clearly, the remaining convergents  $\left\{ \frac{p_i(t)}{q_i(t)} \mid i > N + k \right\}$  depend algebraically on the  $a_i(t)$  and  $b_j(t)$  and, therefore, on the convergents  $\left\{ \frac{p_i(t)}{q_i(t)} \mid 1 \leq i \leq N + k \right\}$ .

(vii) We conclude from (i)-(vi), that the free abelian group  $NS(\mathcal{E}_{\mathbf{Q}(t)})$  has  $N + k$  generators. In particular, the Picard number of the surface  $\mathcal{E}_{\mathbf{Q}(t)}$  is given by the formula:

$$\rho(\mathcal{E}_{\mathbf{Q}(t)}) = N + k. \quad (3.17)$$

Item (i) of theorem 1.5 is proved.  $\square$

*Proof.* Let us prove item (ii) of theorem 1.5. Roughly speaking, to calculate the rank  $r(\mathcal{E}_{\mathbf{Q}(t)})$  we need to know the number of the convergents  $\left\{ \frac{p_i(t)}{q_i(t)} \mid 1 \leq i \leq N + k \right\}$  independent of the parameter  $t \in \mathcal{P}$ . Those correspond to the “horizontal” and “vertical” divisors  $\left\{ \mathcal{P}_i \mid F(\mathcal{P}_i) = \mathcal{A}_{\frac{p_i(t)}{q_i(t)}} \right\}$ , see [Schütt & Shioda 2019] [5, Section 6.1] for the terminology. We pass to a detailed argument.

(i) Let  $(U_{b_1, \dots, b_N; a_1, \dots, a_k}, \mathcal{P}, \pi')$  be the fiber bundle constructed in 1.4 and let  $\mathcal{F}$  be a fiber of the  $(U_{b_1, \dots, b_N; a_1, \dots, a_k}, \mathcal{P}, \pi')$  having dimension  $\dim \mathcal{F} = r$ . As shown



above, one can express  $a_i(t)$  and  $b_j(t)$  in terms of  $\frac{p_i(t)}{q_i(t)}$  and write points (3.6) of the variety  $U_{b_1, \dots, b_N; a_1, \dots, a_k}$  in the form:

$$(R_1, \dots, R_{r+1}, c_{r+2}, \dots, c_{N+k}), \quad (3.18)$$

where  $R_i \in \mathbf{Z} \left[ \frac{p_1(t)}{q_1(t)}, \dots, \frac{p_{r+1}(t)}{q_{r+1}(t)} \right]$ .

(ii) Recall the Tate-Shioda formula:

$$\rho(\mathcal{E}_{\mathbf{Q}(t)}) = r(\mathcal{E}_{\mathbf{Q}(t)}) + 2 + \sum_{v \in R} (m_v - 1), \quad (3.19)$$

where  $R$  is the finite set of singular fibers of the  $\mathcal{E}_{\mathbf{Q}(t)}$  and  $m_v$  is the number of components of the fiber  $v \in R$  [Shioda 1972] [6, Corollary 1.5]. In formula (3.19) the number of horizontal divisors of  $\mathcal{E}_{\mathbf{Q}(t)}$  is equal 2 and such of the vertical divisors is equal to  $\sum_{v \in R} (m_v - 1)$  [Schütt & Shioda 2019] [5, Section 6.1].

(iii) Since the horizontal and vertical divisors are not generic, they cannot depend on  $t \in \mathcal{P}$ . Thus the constants  $c_i$  in (3.18) represent the horizontal and vertical divisors. Comparing with the Tate-Shioda formula (3.19), one gets

$$r(\mathcal{E}_{\mathbf{Q}(t)}) = r = \dim \mathcal{F}. \quad (3.20)$$

Item (ii) of theorem 1.5 is proved.  $\square$

*Proof.* Let us prove item (iii) of theorem 1.5. Roughly speaking, the minimal model corresponds to the surface with the least Picard number among the surfaces in the birational equivalence class of the  $\mathcal{E}_{\mathbf{Q}(t)}$ , see e.g. [Schütt & Shioda 2019] [5, Section 4.5]. But from 1.5 (i) such a surface must minimize the sum  $N+k$ . We show that for the minimal model  $N = k = 1$ , so that  $F(\mathcal{E}_{p(t)}^{CM}) = \mathcal{A}_{[p(t); \overline{2p(t)}]}$  for a positive definite polynomial  $p(t) \in \mathbf{Z}[t]$ . The latter formula follows from a symmetry between the complex and real multiplication [4, Section 1.4.1]. We pass to a detailed argument.

(i) Let  $B$  be the birational equivalence class of the  $\mathcal{E}_{\mathbf{Q}(t)}$ . Recall that a birational map  $\{\mathcal{E} \dashrightarrow \mathcal{E}' \mid \mathcal{E}, \mathcal{E}' \in B\}$  is called dominant, if  $\rho(\mathcal{E}) > \rho(\mathcal{E}')$ . The surface  $\mathcal{E}^{\min} \in B$  is a minimal model if  $\mathcal{E}^{\min}$  is dominated by any other  $\mathcal{E} \in B$ .

(ii) One gets from 1.5 (i):

$$\rho(\mathcal{E}^{\min}) = \min_{\mathcal{E} \in B} (N + k), \quad (3.21)$$

where  $N \geq 0$  and  $k \geq 1$  satisfy (1.4). Let us find the minimal values of  $N$  and  $k$  separately.

(iii) The minimal value of  $k$  is equal to 1, since the period of continued fraction (1.4) cannot vanish. Thus we have  $k_{\min} = 1$ .

(iv) The minimal value of  $N$  is equal to 0, since continued fraction (1.4) can be purely periodic. Thus we have  $N_{\min} = 0$ .

(v) Using (iii) and (iv), one can write (1.4) in the form:

$$F(\mathcal{E}^{\min}) = \left\{ \mathcal{A}_{[a(t)]} \mid a(t) \in \mathbf{Z}[t] \right\}. \quad (3.22)$$

$D$	$\theta$	Picard number of surface $\mathcal{E}_{D(t)}$
2	$[1, 2]$	2
3	$[1, 1, 2]$	3
7	$[2, 1, 1, 1, 4]$	5
11	$[3, 3, 6]$	3
19	$[4, 2, 1, 3, 1, 2, 8]$	7
43	$[6, 1, 1, 3, 1, 5, 1, 3, 1, 1, 12]$	11
67	$[8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16]$	11
163	$[12, 1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1, 24]$	19

FIGURE 2.

(vi) For an explicit construction of the  $\mathcal{E}^{\min}$ , recall that one can add a finite tail  $\frac{1}{2}a(t)$  to the purely periodic fraction  $[\overline{a(t)}]$ . The obtained surface  $(\mathcal{E}^{\min})'$  will be isomorphic over  $\mathbf{C}$  to the original surface  $\mathcal{E}^{\min}$  [4, Theorem 1.3.1]. In other words, one gets from (3.22):

$$F((\mathcal{E}^{\min})') = \left\{ \mathcal{A}_{[p(t); \overline{2p(t)}]} \mid p(t) \in \mathbf{Z}[t] \right\}, \quad (3.23)$$

where  $p(t) := \frac{1}{2}a(t)$ .

(vii) Since  $[p(t); \overline{2p(t)}] = (1 + p^2(t))^{\frac{1}{2}}$  in (3.23), one can use an explicit formula for the functor  $F$  saying that fibers of the surface  $(\mathcal{E}^{\min})'$  must have complex multiplication by the ring of integers of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-1 - p^2(t)})$  [4, Theorem 1.4.1]. Thus  $(\mathcal{E}^{\min})' \cong \mathcal{E}_{p(t)}^{CM}$ .

This argument finishes the proof of item (iii) of theorem 1.5.  $\square$

#### 4. COMPLEX MULTIPLICATION

In this section we apply 1.5 (i) to calculate the Picard numbers of elliptic surfaces whose fibers of rational points (in  $\mathbf{Q}P^1$ ) have complex multiplication.

Let  $D > 1$  be a square-free integer. Denote by  $\mathcal{E}_{CM}^{(-D)}$  an elliptic curve with CM by the ring of integers of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-D})$ . It is known, that

$$F\left(\mathcal{E}_{CM}^{(-D)}\right) = \mathcal{A}_{RM}^{(D, f)}, \quad (4.1)$$

where  $\mathcal{A}_{RM}^{(D, f)}$  is a non-commutative torus with RM by the order of conductor  $f \geq 1$  in the ring of integers of the real quadratic field  $\mathbf{Q}(\sqrt{D})$  [4, Theorem 1.4.1]. In other words, one gets the  $\mathcal{A}_\theta$ , where

$$\theta = \begin{cases} \sqrt{f^2 D} = [b_1; \overline{a_1, a_2, \dots, a_2, a_1, 2b_1}], & \text{if } D \equiv 2, 3 \pmod{4} \\ \frac{1 + \sqrt{f^2 D}}{2} = [b_1; \overline{a_1, a_2, \dots, a_2, a_1, 2b_1 - 1}], & \text{if } D \equiv 1 \pmod{4}. \end{cases} \quad (4.2)$$

As usual, denote by  $(V_{1,k}(\mathbf{C}), \mathcal{P}, \pi)$  the fiber bundle corresponding to the continued fractions (4.2).

**Definition 4.1.** By  $\mathcal{E}_{D(t)}$  we understand an elliptic surface, such that for a section  $\sigma : \mathcal{P} \rightarrow U_{b_1; a_1, \dots, a_k}$  of the bundle  $(U_{b_1; a_1, \dots, a_k}, \mathcal{P}, \pi') \subset (V_{1,k}(\mathbf{C}), \mathcal{P}, \pi)$

$$F(\mathcal{E}_{D(t)}) = \begin{cases} \mathcal{A}_{[b_1(t); \overline{a_1(t), \dots, a_1(t), 2b_1(t)}]}, & \text{if } D \equiv 2, 3 \pmod{4} \\ \mathcal{A}_{[b_1(t); \overline{a_1(t), \dots, a_1(t), 2b_1(t)-1}]}, & \text{if } D \equiv 1 \pmod{4}. \end{cases} \quad (4.3)$$

**Corollary 4.2.** *The Picard number of the surface  $\mathcal{E}_{D(t)}$  is given by:*

$$\rho(\mathcal{E}_{D(t)}) = 1 + k, \quad (4.4)$$

where  $k$  is the length of period of the continued fraction (4.2).

**Example 4.3.** Let  $D = 2, 3, 7, 11, 19, 43, 67$  or  $163$ . The imaginary quadratic field  $\mathbf{Q}(\sqrt{-D})$  has class number one. Since such a number for the real quadratic field  $\mathbf{Q}(\sqrt{D})$  is also one, one gets  $f = 1$  from a symmetry equation, see [4, Theorem 1.4.1] for the details. In view of  $D \equiv 2, 3 \pmod{4}$ , we use the first line in the formulas (4.2). The Picard numbers of the surface  $\mathcal{E}_{D(t)}$  are shown in Figure 2. Whether such surfaces are minimal is an interesting open problem.

**Example 4.4.** Let the surface  $\mathcal{E}_{\mathbf{Q}(t)}$  be given by equation (1.1), see Example 1.1. Unlike 4.3, there is no complex multiplication on the fibers in this case. However, one gets from (1.2)  $N = 1$  and  $k = 2$ . Therefore the Picard number  $\rho(\mathcal{E}_{\mathbf{Q}(t)}) = 3$ .

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