

# RESCALED-EXPANSIVE FLOWS: UNSTABLE SETS AND TOPOLOGICAL ENTROPY

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ABSTRACT. In this work we introduce and explore a rescaled-theory of local stable and unstable sets for rescaled-expansive flows and its applications to topological entropy. We introduce a rescaled version of the local unstable sets and the unstable points. We find conditions for points of the phase space to exhibit non-trivial connected pieces of such unstable sets. We apply these results to the problem of proving positive topological entropy for rescaled-expansive flows with non-singular Lyapunov stable sets.

## 1. INTRODUCTION

The property of expansiveness introduced by R. Uitz in 1950 is a landmark of the dynamical systems theory. Its great success is in part due to its proximity to the hyperbolic theory and its close relationship with many important topics of the dynamical systems theory, such as the stability theory and the entropy theory. Very soon expansiveness was perceived as a source of complex dynamical behavior. Indeed, many expansive systems exhibit chaotic features. We refer the reader to [1] for a detailed exposition of the dynamical properties of expansive homeomorphisms.

The concept of expansive flow was introduced in [8] by R. Bowen and P. Walters to describe the behavior of axiom A flows, but it is not appropriate to deal with flows exhibiting singularities accumulated by regular orbits, such as the Lorenz attractor. For these flows, M. Komuro introduced in [16] the concept of  $k^*$ -expansiveness. Later, other versions of expansiveness were introduced [4, 5, 7, 28]

Several authors have been interested on the chaotic behavior of expansive flows and topological entropy is among the most important ways of measuring the complexity of a dynamical system. Indeed, positive entropy is an indicative of chaotic behavior. In many contexts expansiveness is related to positive topological entropy. For instance, in [10] A. Fathi showed that any expansive homeomorphism has positive topological entropy, if the phase space has positive topological dimension. This was generalized by H. Kato in [15] for continuum-wise expansiveness (by applying techniques developed by R. Mañé in [20]) and for expansive flows in [3] by A. Arbieto, W. Cordeiro and M.J. Pacífico. These results are of topological character as they do not assume any differentiable structure on the phase space of the system.

To our best knowledge there is no result of topological nature regarding the positiveness of the entropy of a  $k^*$ -expansive flow. In this work we aim to explore implications of expansiveness to the topological entropy of expansive singular flows. The main reasons for this lack of results are the following:

- (1) The existence of many distinct versions of expansiveness for singular flows.

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- (2) These results are strongly dependent on the uniform expansiveness property, but expansive singular flows may not satisfy this property.
- (3) The non existence of cross-sections for singularities and the loss of control over the size and the time of the cross sections at regular points.

Actually, the techniques of [3,15] to prove positive entropy are strongly supported on the existence of non-trivial connected local stable or unstable sets. Unfortunately, the above listed facts may forbid the existence of such local stable sets. Some examples will be considered in Section 4.2.

In this article we deal with the rescaled-expansiveness property (R-expansiveness for short) introduced by L. Wen and X. Wen in [28]. Our goal is to study the topological entropy of R-expansive flows. To achieve it, we introduce a rescaled version of local stable and unstable sets for singular flows based on the dynamics of the holonomy maps along orbits of regular points. We obtain conditions to R-expansive flows admit non-trivial pieces of R-unstable sets with "hyperbolic behavior" and study their influence in the topological entropy of R-expansive flows. This choice of expansiveness is made due to its closeness with  $k^*$ -expansiveness, but also due its suitability to work with flow boxes, providing us a nice control over the holonomy maps between cross-sections which is essential to the study of stable and unstable sets.

This text is organized as follows: In section 2 we establish the basic notation and the basic setting used through this work. In section 3 we study R-expansiveness in more details and explore its role in the existence of local stable/unstable sets. Section 4 is devoted to study the entropy of R-expansive flows, providing to the reader some new examples of R-expansive flows and explaining how they are related to our results.

## 2. PRELIMINARIES

This section is devoted to establish the basic setting where we will work on. Throughout this paper  $M$  denotes a compact and boundary-less smooth Riemannian manifold. Let us denote  $g$  for the Riemannian metric of  $M$ . In addition, we denote  $d$  and  $\|\cdot\|$  for the distance induced on  $M$  and the norm induced on  $TM$  by the Riemannian metric  $g$ , respectively.

**Definition 2.1.** A  $C^r$ -flow  $\phi$  on  $M$  is a  $C^r$ -map  $\phi : \mathbb{R} \times M \rightarrow M$  satisfying the following conditions:

- (1)  $\phi(0, x) = x$ , for every  $x \in M$ .
- (2)  $\phi(t + s, x) = \phi(t, \phi(s, x))$ , for every  $t, s \in \mathbb{R}$  and every  $x \in M$ .

Throughout this work we will always assume  $r \geq 1$ . In this case,  $\phi$  generates a velocity vector field that will be denoted by  $X$ . Let us denote by  $\phi_t$  the map  $\phi(t, \cdot) : M \rightarrow M$ , when  $t$  is fixed. The *orbit* of a point  $x$  is the set

$$O(x) = \{\phi_t(x); t \in \mathbb{R}\}.$$

We say that  $x \in M$  is a *singularity* if  $\phi_t(x) = x$  for all  $t \in \mathbb{R}$ . A point  $x \in M$  is *periodic* if it is not a singularity and there exists  $t > 0$  such that  $\phi_t(x) = x$ . The sets of singularities and periodic points are denoted by  $\text{Sing}(\phi)$  and  $\text{Per}(\phi)$ , respectively. We say that a set  $\Lambda$  is *invariant* if  $\phi_t(\Lambda) = \Lambda$ , for every  $t \in \mathbb{R}$ .

**Definition 2.2.** Let  $\Lambda$  be a compact and invariant set. We say that  $\Lambda$  is *Lyapunov stable* if for any  $\varepsilon > 0$ , there is some  $\delta > 0$  such that if  $x \in B_\delta(\Lambda)$ , then  $\phi_t(x) \in B_\varepsilon(\Lambda)$ , for every  $t \geq 0$ . We say that  $\Lambda$  is an *attractor* if:

- (1)  $\phi|_\Lambda$  is *transitive*, i.e., there is  $x \in \Lambda$  such that  $\overline{O(x)} = \Lambda$ .
- (2) There is an open neighborhood  $U$  of  $\Lambda$  satisfying:
  - (a)  $\overline{\phi_t(U)} \subset U$  for any  $t > 0$  and
  - (b)  $\Lambda = \bigcap_{t \geq 0} \phi_t(U)$ .

Every attractor is Lyapunov stable, but the converse does not hold. A neighborhood of  $\Lambda$  as in the above definition is called *isolating neighborhood*. We say that an attractor  $\Lambda$  is *non-periodic* if it is not a periodic orbit. Let  $\Lambda \subset M$  be compact and invariant.

Now we recall the definition of topological entropy for flows. Fix  $\varepsilon > 0$  and  $t > 0$ . We say that a pair of points is *t- $\varepsilon$ -separated by  $\phi$*  if there is some  $0 \leq s \leq t$  such that  $d(\phi_s(x), \phi_s(y)) > \varepsilon$ . Also, a subset  $E \subset M$  is *t- $\varepsilon$ -separated* if any pair of distinct points of  $E$  is *t- $\varepsilon$ -separated by  $\phi$* . For  $\Lambda \subset M$ , let  $s_t(\varepsilon, \Lambda)$  denote the maximal cardinality of a *t- $\varepsilon$ -separated* subset of  $\Lambda$ . This number is finite due to the compactness of  $M$ . We define the *topological entropy* of  $\phi$  on  $\Lambda$  to be the number  $h(\phi, \Lambda)$  defined by

$$h(\phi, \Lambda) = \lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log s_t(\varepsilon, \Lambda).$$

**Definition 2.3.** The *topological entropy* of  $\phi$  is the number  $h(\phi) = h(\phi, M)$ .

The problem of finding positive topological entropy for expansive systems was first considered in the 80's and 90's by Fathi, Kato and Lewowicz independently, see [10, 13, 15]. Its version for expansive flows is proved by A. Arbieto, W. Cordeiro and M. J. Pacifico in [3].

**2.1. Expansiveness.** We start by giving the definition of expansiveness which was introduced by R. Bowen and P. Walters in [8].

**Definition 2.4.** A flow  $\phi$  is *expansive* if for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that the following holds: If  $x, y \in M$ ,  $\rho: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying  $\rho(0) = 0$  and

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta$$

for every  $t \in \mathbb{R}$ , then  $y \in \phi_{[-\varepsilon, \varepsilon]}(x)$ . We say that a compact and invariant set  $\Lambda \subset M$  is *expansive* if the flow restricted to  $\Lambda$  is expansive.

**Theorem 2.5** ([3]). *Let  $\phi$  be a continuous flow and suppose  $\dim(M) > 1$ . If  $\phi$  is expansive then  $h(\phi) > 0$ .*

**Remark 2.6.** The previous result was actually proved in the context of *CW*-expansive flows which, on its turn, contains the expansive flows. In this work we will not go into details about *CW*-expansive flows, since this concept will not be addressed here.

The property of expansiveness was designed to be the model for the expansive behavior displayed by axiom A and Anosov flows. Unfortunately, this concept does not capture the singular behavior of flows such as the Lorenz Attractor. To cover these flows, M. Komuro introduced in [16] the following concept:

**Definition 2.7.** A flow  $\phi$  is  $k^*$ -*expansive* if for every  $\varepsilon > 0$ , there is some  $\delta > 0$  such that the following holds: if  $x, y \in M$ ,  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing homomorphism and

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta$$

for every  $t \in \mathbb{R}$ , then there are  $t_0, s \in \mathbb{R}$  such that  $|s| \leq \varepsilon$  and  $y = \phi_{t_0+s}(x)$ . We say that a compact and invariant set  $\Lambda \subset M$  is  $k^*$ -expansive if the flow restricted to  $\Lambda$  is  $k^*$ -expansive.

**Remark 2.8.** Expansive flows do not exist in surfaces (see [8] for details), while in [4] surface  $k^*$ -expansive flows are classified. On the other hand, it was proved in [29] that every surface flow has zero topological entropy. Thus, there is no result similar to Theorem 2.5 for  $k^*$ -expansiveness for  $\dim(M) = 2$ .

### 3. RESCALED EXPANSIVENESS

In this section we give the definition of the main concept used in this work: The rescaled-expansiveness and study its influence on the existence of non-trivial local R-stable and R-unstable sets.

**3.1. Cross sections and flow boxes.** The techniques used in [3] to obtain positive entropy for expansive flows are strongly supported in the fact that if  $\phi$  is non-singular, then one can always choose cross-sections for every point with uniform size (see [8]). This allows us to use flow-boxes with uniform size to study the dynamics. Unfortunately, it does not hold for singular flows. To see this, let  $x \in M$  be a regular point for  $\phi$ . The normal space of  $x$  in  $T_x M$  is the set

$$\mathcal{N}(x) = \{v \in T_x M; v \perp X(x)\}.$$

Let us denote  $\mathcal{N}_r(x) = \mathcal{N}(x) \cap \mathcal{B}_r(0)$ , where  $\mathcal{B}_r(0)$  is the ball in  $T_x M$  of radius  $r$  and centered at 0. The tubular flow theorem for smooth flows asserts that for any regular point  $x$  there are  $\eta_x > 0$  and  $r_x > 0$  such that the set

$$N_{r_x}(x) = \exp_x(\mathcal{N}_{r_x}(x))$$

is a cross section of time  $\eta_x$  through  $x$ , *i.e.*, for any  $y \in N_{r_x}(x)$  we have that

$$\phi_{[-\eta_x, \eta_x]}(y) \cap N_{r_x}(x) = \{y\}.$$

Furthermore, any  $y \in N_{r_x}(x)$  is regular. Another important consequence of the tubular flow theorem is that it allows us to work with *holonomy maps* generated by flows. To make the previous assertion precise, let  $x \in M$  be a regular point, fix some  $t \in \mathbb{R}$  and suppose that  $N_\varepsilon(\phi_t(x))$  is a cross section. Then by the tubular flow theorem, there are  $r_x > 0$  and a continuous function  $\tau : N_{r_x}(x) \rightarrow \mathbb{R}$  such that  $\phi_{\tau(y)}(y) \in N_{r_x}(\phi_t(x))$  for all  $y \in N_{r_x}(x)$  and  $\tau(x) = t$ . In this way, we define the holonomy map

$$P_{x,t} : N_{r_x}(x) \rightarrow N_\varepsilon(\phi_t(x))$$

by setting  $P_{x,t}(y) = \phi_{\tau(y)}(y)$ .

One of the main difficulties in the use of cross-sections and holonomy maps for singular flows is that the radius  $r_x$  may go to zero when  $x$  approaches some singularity. The next result allow us to have a better control on these cross-sections. Indeed, it gives us an explicit relation between the constants  $\varepsilon$  and  $r_x$  used above. Before stating the result, let us fix the following notation that will be used throughout this paper

$$N_\varepsilon^r(x) = N_{\varepsilon\|X(x)\|}(x).$$

**Theorem 3.1** ([28]). *Suppose that  $X$  is a  $C^1$ -vector field and let  $\phi$  be the flow induced by  $X$ . Then there exist  $L > 0$  and  $\beta_0 > 0$  such that for any  $0 < \beta < \beta_0$ ,  $t > 0$  and  $x \in M \setminus \text{Sing}(\phi)$  we have:*

- (1) *The set  $\phi|_{[-\beta, \beta]}(N_\beta^r(x))$  is a flow box, in particular it does not contain singularities.*
- (2) *The ball  $B_{\frac{1}{3}\beta\|X(x)\|}(x)$  is contained on  $\phi|_{[-\beta, \beta]}(N_\beta^r(x))$*
- (3) *The holonomy map*

$$P_{x,t} : N_{\frac{\beta}{L^t}}^r(X) \rightarrow N_\beta^r(\phi_t(x))$$

*is well defined and injective. Moreover, for any  $y \in N_{\frac{\beta}{L^t}}^r(X)$  we have*

$$d(\phi_s(x), \phi_s(y)) \leq \beta\|X(\phi_s(x))\|$$

*for any  $0 \leq s \leq t$ . The same statement is valid for  $t < 0$ .*

**3.2. R-expansiveness.** Based on the above ideas, L. Wen and X. Wen in [28] introduced a *rescaled* version of expansiveness by considering that the distance of separation of the orbits is rescaled by the size of the velocity vector field.

**Definition 3.2.** A  $C^r$ -flow  $\phi$  on  $M$  is *R-expansive* (or *rescaled expansive*) if for every  $\varepsilon > 0$  there is some  $\delta > 0$  such that: if  $x, y \in M$ ,  $\rho : \mathbb{R} \rightarrow \mathbb{R}$  is a increasing continuous function and

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \delta\|X(\phi_t(x))\|$$

for every  $t \in \mathbb{R}$ , then  $\phi_{\rho(t)}(y) \in \phi|_{[t-\varepsilon, t+\varepsilon]}(x)$  for any  $t \in \mathbb{R}$ . We say that a compact and invariant set  $\Lambda \subset M$  is R-expansive if the flow restricted to  $\Lambda$  is R-expansive.

**Theorem 3.3** ([7]). *Let  $\phi$  be a  $k^*$ -expansive flow. If  $\text{Sing}(\phi)$  is a hyperbolic set, then  $\phi$  is R-expansive.*

**3.3. Stable and Unstable sets.** Here we give a new definition of local stable and unstable sets for regular points of a singular flow, based on holonomy maps and rescaled distances. Fix some regular point  $x \in M$ . By Theorem 3.1 for any  $0 < \beta \leq \beta_0$  the set  $N_\beta^r(x)$  is a cross section of radius  $\beta\|X(x)\|$  for the flow. Moreover, for any  $t > 0$  the holonomy map  $P_{x,t}$  is well defined on  $N_{\frac{\beta}{L^t}}^r(x)$  and if  $y \in N_{\frac{\beta}{L^t}}^r(x)$ , the orbit segment between  $y$  and  $P_{x,t}(y)$  belongs to the  $\beta$ -rescaled tubular neighborhood of  $O(x)$ . Let us fix  $x \in M \setminus \text{Sing}(\phi)$ ,  $t > 0$  and  $\beta > 0$ .

**Definition 3.4.** The  $\beta$ - $t$ - $R$ -stable and  $\beta$ - $t$ - $R$ -unstable local sets of  $x$  are respectively

$$S_\beta(t, x) = \left\{ y \in N_{\frac{\beta}{L^t}}^r(x); d(P_{x,nt}(x), P_{x,nt}(y)) \leq \frac{\beta}{L^t}\|X(P_{x,nt}(x))\|, \forall n \in \mathbb{N} \right\}$$

$$U_\beta(t, x) = \left\{ y \in N_{\frac{\beta}{L^t}}^r(x); d(P_{x,-nt}(x), P_{x,-nt}(y)) \leq \frac{\beta}{L^t}\|X(P_{x,-nt}(x))\|, \forall n \in \mathbb{N} \right\}$$

Let us present a useful characterization of R-expansiveness in terms of R-stable and R-unstable sets.

**Proposition 3.5.** *The flow  $\phi$  is R-expansive if, and only if, there exists  $\delta > 0$  such that for any regular point  $x \in M$  and any  $t > 0$ , one has  $S_\delta(t, x) \cap U_\delta(t, x) = \{x\}$ .*

*Proof.* Fix a regular point  $x$ ,  $\beta > 0$  small enough and let  $0 < \varepsilon < \beta$ . Let  $0 < \delta < \varepsilon$  be given by the R-expansiveness of  $\phi$  related to  $\varepsilon$ . Now suppose that

$$y \in S_\delta(t, x) \cap U_\delta(t, x).$$

Since  $y \in S_\delta(t, x)$  we have

$$d(P_{x,nt}(x), P_{x,nt}(y)) < \frac{\delta}{L^t} \|X(P_{x,nt}(x))\|$$

for any non-negative integer  $n$ . By the Theorem 3.1 we obtain a reparametrization  $\rho$  such that  $d(\phi_s(x), \phi_{\rho(s)}(y)) \leq \delta \|X(\phi_t(x))\|$  for every  $s \geq 0$ . On the other hand, since  $y \in U_\delta(t, x)$ , a similar argument allows us to modify the reparametrization  $\rho$ , in order to obtain  $d(\phi_s(x), \phi_{\rho(s)}(y)) \leq \delta \|X(\phi_t(x))\|$  for every  $s \leq 0$ . Now R-expansiveness implies that  $y \in \phi_{[-\varepsilon, \varepsilon]}(x)$ , but since  $x, y \in N_{\frac{\delta}{L^t}}^r(x)$ , we have that  $y = x$ .

Conversely, denote  $B = \sup_{x \in M} \{\|X(x)\|\}$ , fix  $0 < \varepsilon < \beta$  and let  $0 < \delta < \varepsilon$  be such that  $S_{\frac{\delta}{B}}(t, x) \cap U_{\frac{\delta}{B}}(t, x) = \{x\}$  for any regular point  $x$  and any  $t > 0$ . Fix  $t > 0$  such that  $L^t > 1$  and suppose there exist a reparametrization  $h$  and two points  $x, y$  satisfying  $d(\phi_s(x), \phi_{h(s)}(y)) \leq \frac{\delta}{BL^t} \|X(\phi_s(x))\|$  for  $s \in \mathbb{R}$ . This implies in particular that

$$d(x, y) < \frac{\delta}{BL^t} \|X(x)\|.$$

Since  $\delta < \varepsilon < \beta$ , Theorem 3.1 implies that there exists some

$$|s_0| < \frac{\delta}{BL^t} \|X(x)\| \leq \delta \leq \varepsilon$$

such that  $y_0 = \phi_{s_0}(y) \in N_\delta^r(x)$ . More generally, since

$$d(\phi_{nt}(x), \phi_{h(nt)}(y)) < \frac{\delta}{BL^t} \|X(\phi_{nt}(x))\|,$$

for any  $n \in \mathbb{Z}$ , there exists  $|s_n| < \varepsilon$  such that  $y_n = \phi_{h(nt)+s_n}(y) \in N_\delta^r(\phi_{nt}(x))$ . But last fact implies that the set  $\{y_n\}$  is the orbit of  $y_0$  under the holonomy maps  $\{P_{x,nt}\}$ . In addition, one has that  $y_0 \in S_\delta(t, x) \cap U_\delta(t, x)$  and therefore we must have  $y_0 = x$ . Then  $\phi_{s_0}(y) = x$  and the flow  $\phi$  is R-expansive.  $\square$

An interesting fact about the above characterization is that we do not need to concern about reparametrizations, since we are only working with the holonomy maps generated by  $\phi$ .

**3.4. Uniformity.** For the remaining of this section, we are assuming that the flows in consideration are R-expansive and the constant  $\delta$  given by Proposition 3.5 will be called a constant of R-expansiveness of  $\phi$ .

**Remark 3.6.** Notice that if  $\delta$  is a R-expansiveness constant for  $\phi$ , then any  $0 < \delta' < \delta$  is also an R-expansiveness constant for  $\phi$ .

Next we work in order to obtain versions of some well know results about non-singular expansive flows to the R-expansive case. For  $\Lambda \subset M$ , we denote

$$A_\Lambda = \inf_{x \in \Lambda} \{\|X(x)\|\}.$$

**Theorem 3.7** (Uniform R-expansiveness). *Suppose  $\phi$  is R-expansive with constant of R-expansiveness  $\delta$  and let  $\Lambda \subset M$  be a non-singular, compact and invariant set. Then for any  $0 < \eta \leq \delta A_\Lambda$  and  $t > 0$ , there exists  $J > 0$  such that if  $x \in \Lambda$  and  $y \in N_\delta^r(x)$  with  $d(x, y) > \eta$ , then there is  $-J \leq i \leq J$  such that*

$$d(P_{x,it}(x), P_{x,it}(y)) \geq \delta \|X(P_{x,it}(x))\|.$$

*Proof.* Suppose the result is false. Thus there are  $\eta > 0$ ,  $t > 0$ , sequences  $x_n \in \Lambda$ ,  $y_n \in N_\delta^r(x_n)$ ,  $m_n \rightarrow \infty$ , such that  $d(x_n, y_n) > \eta$  and

$$d(P_{x_n,it}(x_n), P_{x_n,it}(y_n)) \leq \delta \|X(P_{x_n,it}(x_n))\|$$

for  $-m_n \leq i \leq m_n$ . By compactness of  $\Lambda$  we can suppose that  $x_n \rightarrow x \in \Lambda$ ,  $y_n \rightarrow y \in M$ . Then we have  $\text{diam}(N_\delta^r(x_n)) > \eta > 0$  for any  $x_n$ . Since  $X$  is a  $C^1$ -vector field, the normal direction of  $X$  varies continuously with  $x$ , so we have that  $y \in N_\delta^r(x)$ . But now, the continuity of the holonomy maps implies that

$$d(P_{x,it}(x), P_{x,it}(y)) \leq \delta \|X(P_{x,it}(x))\|,$$

for every  $i \in \mathbb{Z}$  and then  $x = y$ , a contradiction, since  $d(x, y) > \eta$ .  $\square$

**Remark 3.8.** In [17] it is proved that any expansive flow is uniformly expansive. So in the previous result we have  $\phi|_\Lambda$  is expansive and therefore it is uniformly expansive, by the non-singularity of  $\Lambda$ . But with this restriction we only obtain a uniform time of separation between the orbits in  $\Lambda$ . On the other hand, our result gives a uniform time in which  $\Lambda$  expels any point  $x$  in a neighborhood of  $\Lambda$ , if  $x$  is not too close to  $\Lambda$ .

Next we use the uniform R-expansiveness to obtain some type of "hyperbolic behaviour" for the R-stable (R-unstable) sets of points away from singularities. i.e. these sets need to contract uniformly in the future (in the past).

**Proposition 3.9** (Uniform contraction). *For any  $0 < \eta < \delta A_\Lambda$  and any  $t > 0$ , there is  $J > 0$  such that*

$$P_{x,nt}(S_\delta(t, x)) \subset S_\eta(t, P_{x,nt}(x)) \text{ and } P_{x,-nt}(U_\delta(t, x)) \subset U_\eta(t, P_{x,nt}(x))$$

for every  $n \geq J$  and every  $x \in \Lambda$ .

*Proof.* Let us fix  $0 < \eta < \delta A_\Lambda$  and  $t > 0$ . Let  $J$  be given by the previous theorem. suppose there exists  $x \in \Lambda$  such that

$$P_{x,nt}(S_\delta(t, x)) \not\subset S_\eta(t, P_{x,nt}(x)).$$

Then there is some  $y \in S_\delta(t, x)$  and  $n > N$  satisfying  $d(P_{x,nt}(x), P_{x,nt}(y)) > \eta$ . By the choice of  $J$  we must have

$$d(P_{x,(n+i)t}(x), P_{x,(n+i)t}(y)) > \delta \|X(P_{x,(n+i)t}(x))\|$$

for some  $-J \leq i \leq J$ , but this is impossible, since  $n > J$ .  $\square$

The following result is an easy corollary of Proposition 3.9.

**Proposition 3.10.** *If for some  $t > 0$ , we have  $y \in S_\varepsilon(t, x)$ , then  $\omega(x) = \omega(y)$ .*

Before to prove the proposition, let us make some remarks that will be used on next results. By Proposition 3.9, if  $y \in S_\varepsilon(t, x)$ , then  $d(P_{x,nt}(x), P_{x,nt}(y)) \rightarrow 0$  as  $n \rightarrow \infty$ . In addition, Theorem 3.1 implies that  $d(\phi_t(x), \phi_t(y)) \rightarrow 0$  as  $t \rightarrow \infty$ .

*Proof.* Let  $z \in \omega(x)$  and suppose that  $y \in S_\varepsilon(t, x)$ . If  $t_k \rightarrow \infty$  is such that  $\phi_{t_k}(x) \rightarrow z$ , then the previous remarks implies that  $\phi_{t_k}(y) \rightarrow z$  and therefore  $z \in \omega(y)$ . The contrary inclusion is analogous.  $\square$

**Corollary 3.11.** *Let  $x$  be a periodic point with period  $\pi(x) = t$ . For every  $0 < \eta \leq A_{O(x)}$  there exists  $J$  such that:*

$$P_{x,nt}(S_\delta(t, x)) \subset S_\eta(t, P_{x,nt}(x)) \text{ and } P_{x,-nt}(U_\delta(t, x)) \subset U_\eta(t, P_{x,-nt}(x)),$$

for every  $n \geq J$ .

**3.5. Rescaled-stable points.** Let us now introduce the concept of R-stable and R-unstable points. We would like to mention that the results here are inspired on the techniques developed in [13]. Define the positive and negative  $n$ - $\varepsilon$ -R-dynamical ball centered at  $x$ , respectively by:

$$N_t^r(x, n, \varepsilon) = \{y \in N_\varepsilon^r(x); d(P_{x,it}(x), P_{x,it}(y)) \leq \varepsilon \|X(P_{x,it}(x))\|, 0 \leq i \leq n\}.$$

and

$$N_{-t}^r(x, n, \varepsilon) = \{y \in N_\varepsilon^r(x); d(P_{x,it}(x), P_{x,it}(y)) \leq \varepsilon \|X(P_{x,it}(x))\|, -n \leq i \leq 0\}.$$

**Definition 3.12.** We say that  $x \in M \setminus \text{Sing}(\phi)$  is an R-stable (R-unstable) point of  $\phi$  if for every  $t > 0$ , the set  $\{S_\varepsilon(t, x)\}_{\varepsilon > 0}$  ( $\{U_\varepsilon(t, x)\}_{\varepsilon > 0}$ ) is a neighborhood basis for  $x$  on  $N_\delta^r(x)$ . In other words, if for every  $\varepsilon > 0$ , there is some  $\eta > 0$  such that if  $y \in N_\eta^r(x)$  and  $d(x, y) \leq \frac{\eta}{L^t} \|X(x)\|$ , then

$$d(P_{x,nt}(x), P_{x,nt}(y)) \leq \varepsilon \|X(P_{x,nt}(x))\|$$

for every  $n \geq 0$  ( $n \leq 0$ ).

Next theorem is a trivial consequence of the definitions and then we shall omit its proof.

**Theorem 3.13.** *If  $\overline{O(x)} \cap \text{Sing}(\phi) = \emptyset$ , then are equivalent:*

- (1)  $x$  is a R-stable point.
- (2)  $S_\delta(t, x)(x)$  is a neighborhood of  $x$  on  $N_\delta^r(x)$ .
- (3) There is some  $0 < \varepsilon_0 < \delta$  such that for any  $0 < \varepsilon < \varepsilon_0$  and  $t > 0$  we have

$$S_\varepsilon(t, x) = N_t^r(x, T, \varepsilon),$$

where  $T$  is given by Theorem 3.9 with respect to  $\varepsilon$ .

**Remark 3.14.** An equivalent result clearly holds for R-unstable points.

Hereafter we will always suppose  $x \in \Lambda$ , where  $\Lambda$  is a compact invariant set without singularities. Before to state our next result, we would like to state a result from [14] which will be used in our next proof. Let us denote  $C_\phi(M)$  for the set of non-negative functions  $f : M \rightarrow [0, \infty)$  such that  $f(x) = 0$  if, and only if  $x \in \text{Sing}(\phi)$ . Note that for any  $\delta > 0$ , the functions  $\delta \|X(x)\|$  belongs to  $C_\phi(M)$ . Next result will help us to find continuity properties for the R-holonomy maps.

**Lemma 3.15** ([14]). *Let  $\phi$  be a continuous flow on  $M$ .*

- (1) For any  $e \in C_\phi(M)$  and  $T > 0$  we can find  $r \in C_\phi(M)$  such that: if  $d(x, y) \leq r(x)$ , then

$$d(\phi_t(x), \phi_t(y)) \leq e(\phi_t(x)),$$

for every  $t \in [-T, T]$

(2) For any  $e \in C_\phi(M)$  there is some  $r \in C_\phi(M)$  such that

$$r(x) \leq \max\{e(y); y \in B_{r(x)}(x)\}.$$

**Proposition 3.16.** *Every R-stable point in  $\Lambda$  which is recurrent is periodic.*

*Proof.* Suppose that  $x \in \Lambda$  is recurrent and R-stable. Fix  $\eta > 0$  such that  $N_\eta^r(x) \subset S_\varepsilon(t, x)$ . Since  $x$  is recurrent we can find a sequence  $t_k \rightarrow \infty$  such that  $\phi_{t_k}(x) \rightarrow x$ . We assume that  $\phi_{t_k}(x) \in B_\delta^r(x)$  for all  $k \geq 1$ .

Since Theorem 3.1 implies that  $B_\delta^r(x)$  is contained on the R-flow box of  $N_\delta^r(x)$ , we find a sequence of times  $n_k t \rightarrow \infty$  such that  $P_{x, n_k t}(x) \rightarrow x$ .

Let  $r \in C_\phi(M)$  be a function given by item 2 of Lemma 3.15 such that

$$0 < r(x) \leq \frac{\eta}{4} \|X(x)\|.$$

Let  $T$  be given by Theorem 3.9, with respect to  $r(x)$  and fix a sequence  $n_k$  with  $n_k > T$  such that  $P_{x, n_k t}(x) \in N_{r(x)}(x)$ . Then by Proposition 3.9 we have that

$$P_{x, n_k t}(N_\eta^r(x)) \cap N_{r(x)}(x) \subset S_{\frac{\eta}{4}}(t, P_{x, n_k t}(x)).$$

Lemma 3.15 implies

$$P_{x, n_k t}(N_\eta^r(x) \cap N_{r(x)}(x)) \subset N_{\frac{\eta}{2}}^r(x).$$

Now, if we apply again  $P_{x, n_k t}$  to  $P_{x, n_k t}(N_\eta^r(x) \cap N_{r(x)}(x))$ , we obtain that

$$P_{x, 2n_k t}(P_{x, n_k t}(N_\eta^r(x) \cap N_{r(x)}(x))) \subset N_{\frac{\eta}{2}}^r(x).$$

Finally, Proposition 3.9 implies that

$$\bigcap_{j=1}^{\infty} P_{x, j n_k t}(\overline{N_\eta^r(x)}) = \{z\}$$

and by construction we have that  $z$  is periodic for  $\{P_{x, nt}\}$ . This implies that  $z$  is periodic for  $\phi$  and by the previous proposition, we have that  $x \in \omega(x) = \omega(z) = O(z)$ . This finishes the proof.  $\square$

In the next results we will see that the R-stable points of a non-singular subset  $\Lambda$  of a R-expansive flows are formed by periodic orbits which are isolated from  $\Lambda$ .

**Proposition 3.17.** *Suppose  $\phi$  is R-expansive and let  $x \in M$  be such that  $\overline{O(x)} \cap \text{Sing}(\phi) = \emptyset$ . If  $x$  is a R-stable point, then there is a neighborhood of  $x$  on  $N_\delta^r(x)$  formed by R-stable points.*

*Proof.* Suppose that  $x$  is a R-stable point and fix  $0 < 4\varepsilon < \delta$  such that

$$\left( \bigcup_{t \geq 0} \overline{N_\varepsilon^r(\phi_t(x))} \right) \cap \text{Sing}(\phi) = \emptyset$$

Since  $x$  is R-stable, then there is some  $0 < \eta < \varepsilon$  such that  $N_\eta^r(x) \subset S_\varepsilon(t, x)$ . This implies that there is  $A > 0$  such that if  $y \in N_\eta^r(x)$ , then  $\inf_{t \geq 0} \{\|X(\phi_t(y))\|\} > A > 0$ .

Now fix  $\nu > 0$  and set  $0 < \gamma \leq \nu A$ . Fix  $y \in N_\eta^r(x)$ . Proposition 3.9 combined with Theorem 3.1 implies that we can find  $T > 0$  such that

$$d(P_{y, nt}(y), P_{y, nt}(z)) \leq \frac{\gamma}{Lt}$$

for any  $z \in B_\eta(x)$  and for any  $n \geq N$ . Finally, the continuity of the holonomy maps allows us to find  $\mu > 0$  (Lemma 3.15) such that if  $z \in N_\eta^r(x)$  and  $d(z, y) < \mu$ , then

$$d(P_{y,nt}(y), P_{y,nt}(z)) \leq \frac{\gamma}{L^t}$$

for  $0 \leq n \leq N_\eta$ . But this implies  $N_\mu^r(y) \subset S_\nu(y, t)$  and therefore,  $y$  is R-stable.  $\square$

**Lemma 3.18.** *Let  $\Lambda$  be a non-singular set and fix  $t > 0$ . Suppose  $x \in \Lambda$  is an R-stable point. There exist  $\rho > 0$  and  $T > 0$  such that*

$$N_\rho^r(P_{x,-nt}(x)) \subset S_{\frac{\delta}{3}}(t, P_{x,-nt}(x)),$$

for every  $n \geq N$ .

*Proof.* Let  $\varepsilon > 0$  be such that  $N_\varepsilon^r(x) \subset S_{\frac{\delta}{3}}(t, x)$  and let  $T_\varepsilon$  be given by Theorem 3.7. By continuity, we can find  $\rho > 0$  such that  $N_\rho^r(y) \in N_{-t}^r(y, T_\varepsilon, \varepsilon)$ , for any  $y \in \Lambda$ . We claim that the Lemma holds for  $T = 2T_\varepsilon$ . Indeed, first notice that

$$N_t^r(P_{x,-nt}(x), n, \varepsilon) \subset S_{\frac{\delta}{3}}(t, P_{x,-nt}(x)),$$

for every  $n \geq 0$ . If the lemma does not hold, there should be  $n \geq T$  and

$$y \in N_\rho^r(P_{x,-nt}(x)) \setminus N_t^r(P_{x,-nt}(x), n, \varepsilon).$$

In particular we can find

$$z \in N_\rho^r(P_{x,-nt}(x)) \cap \partial N_t^r(P_{x,-nt}(x), n, \varepsilon)$$

Therefore there is  $k > T_\varepsilon$  such that

$$d(P_{x,-(n+k)t}(x), P_{x,-(n+k)t}(z)) \geq \varepsilon A_\Lambda.$$

But now, Theorem 3.7 implies

$$\max_{|j| \leq T_\varepsilon} \{d(P_{x,-(n+k+j)t}(x), P_{x,-(n+k+j)t}(z))\} > \frac{\delta}{3} A_\Lambda$$

contradicting the choice of  $z$ .  $\square$

**Theorem 3.19.** *Let  $\phi$  be a R-expansive flow and  $\Lambda \subset M$  be a compact invariant set without singularities. If  $x \in \Lambda$  is a R-stable or R-unstable point, then  $x$  is periodic.*

*Proof.* We will only present the proof for R-stable points, since the case of R-unstable points is totally analogous. Suppose  $x \in \Lambda$  is a R-stable point, let  $\delta > 0$  be a R-expansiveness constant of  $\phi$  and fix  $t > 0$ . By the compactness of  $\Lambda$ , we can find a sequence  $-n_k \rightarrow \infty$  such that  $P_{x,n_k t} \rightarrow z \in \Lambda$ . In particular,  $z$  is a regular point. Let  $0 < \gamma < \delta$  be such that  $N_\gamma^r(y)$  is well defined and it is a cross-section of time  $t$ , for every  $y \in \Lambda$ . Let  $\rho > 0$  and  $T > 0$  be given by Lemma 3.18.

It is a classical fact from the theory of flows that there exists  $0 < \xi < t$  such that if  $y \in \phi_{[-\xi, \xi]}(z)$ , then

$$d(\phi_s(z), \phi_s(y)) \leq \frac{\delta}{3},$$

for every  $s \in \mathbb{R}$  (cf [8]). For every  $n_k \geq T$ , denote

$$V_k = \phi_{[-\xi, \xi]}(N_\rho^r(P_{x,n_k t}(x))).$$

Notice that  $z \in \text{int}(V_k)$ , if  $k$  is big enough. We claim that  $z$  is a R-stable point of  $\Lambda$ . Indeed, fix  $k > 0$  such that  $z \in \text{int}(V_k)$  and let  $V = N_\gamma^r(z) \cap V_k$ . Fix  $y \in V$ , and let  $-\xi < s_y, s_z < \xi$  be such that  $\phi_{s_y}(y), \phi_{s_z}(z) \in N_\rho^r(P_{x,-n_k t}(x))$ . For fixed  $n > 0$  we then have

$$\begin{aligned}
 & d(P_{z,nt}(z), P_{z,nt}(y)) \leq \\
 & d(P_{z,nt}(z), P_{x,nt}(\phi_{s_z}(z))) + d(P_{x,nt}(\phi_{s_z}(z)), P_{x,nt}(\phi_{s_y}(y))) + d(P_{x,nt}(\phi_{s_y}(y)), P_{z,nt}(y)) \\
 & \leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3},
 \end{aligned}$$

by Lemma 3.18 and the choice the choice of  $\xi$ . Therefore  $V \subset S_\delta(t, z)$ .

Now we conclude the proof by showing that  $x$  is a recurrent point. Fix  $\beta > 0$  such that  $B_\beta(x) \cap S_\delta(t, x)$  is an open set of  $N_\delta^r(x)$ . Define  $V_\xi = \phi_{[-\xi, \xi]}(V)$ . By Theorem 3.21, we can find  $k_0 > 0$  such that  $P_{x, -n_k} t(x) \in V_\xi$  and  $\text{diam}(P_{z, n_k} t(V)) \leq \frac{\beta}{2}$ , for every  $k \geq k_0$ . Let  $x_{k_0} = \phi_{s_0}(P_{x, -n_{k_0}} t(x)) \in V$ , where  $s \in [-\xi, \xi]$ . Thus

$$P_{z, n_{k_0}} t(x_{k_0}) \in B_\beta(x) \text{ and } x_{k_0} \in \phi_{[-\xi, \xi]}(x).$$

Finally, we can find  $k_1 > k_0$  such that if  $x_{k_1} = \phi_{s_1}(P_{x, -n_{k_1}} t(x)) \in V$ , with  $s_1 \in [-\xi, \xi]$ , then

$$P_{z, n_{k_0}}(x_{k_1}) \in B_\beta(x).$$

Therefore,  $x$  is recurrent and by Theorem 3.16, we obtain that  $x$  is periodic.  $\square$

**Corollary 3.20.** *If  $x \in \Lambda$  is a R-stable or R-unstable point, then  $O(x)$  is isolated from  $\Lambda$ . In particular,  $\Lambda$  contains at most a finite number of orbits of R-stable or R-unstable points.*

*Proof.* Let  $x \in \Lambda$  be a R-stable point. By Theorem 3.19  $O(x)$  is a compact set. Suppose  $O(x)$  is not isolated from  $\Lambda$ , then there is a sequence of points  $x_n \in \Lambda \setminus O(x)$  such that  $x_n \rightarrow x$ . By combining Theorems 3.17 and 3.19 we obtain that  $x_n$  is periodic for  $n$  sufficiently large, contradicting the expansiveness of  $\phi_t|_\Lambda$ .  $\square$

**3.6. Existence of non-trivial stable sets.** If  $A \subset X$  and  $x \in A$ , we denote by  $C(A, x)$  the connected component of  $A$  containing  $x$ . For any  $x \in M$ ,  $t > 0$  and  $\varepsilon > 0$ , we denote

$$CS_\varepsilon(t, x) = C(S_\varepsilon(t, x), x) \text{ and } CU_\varepsilon(t, x) = C(U_\varepsilon(t, x), x).$$

Our first result deals with the existence of connected pieces of local R-stable and R-unstable sets with large diameter. This problem was first solved for expansive homeomorphisms of surfaces by J. Lewowicz and K. Hiraide, who independently classified such systems (they are conjugate to pseudo-Anosov diffeomorphisms). This was later generalized for Bowen-Walters expansive flows in [17]. Let us denote  $\Gamma_\gamma^r(p)$  for the sphere of radius  $\gamma \|X(p)\|$  centered at  $p$ .

**Theorem 3.21.** *If  $\phi$  is an R-expansive flow,  $\Lambda \subset M$  is a compact invariant set without singular points and  $\delta > 0$  then there is  $\eta > 0$  such that*

$$CS_\delta(t, p) \cap \Gamma_\eta^r(p) \neq \emptyset \text{ and } CU_\delta(t, p) \cap \Gamma_\eta^r(p) \neq \emptyset.$$

for all  $t > 0$  and any point  $p \in \Lambda$  which is not R-stable or R-unstable.

*Proof.* The proof is based on the following claiming:

*Claim:* For every  $0 < \varepsilon < \delta$ , and  $\eta > 0$ , there is some  $K = K_{\varepsilon, \eta}$  such that

$$N_\eta^r(x) \not\subset N_t^r(x, K, \varepsilon) \text{ and } N_\eta^r(x) \not\subset N_{-t}^r(x, K, \varepsilon)$$

for every  $x \in \Lambda$ .

If the claim is false, we can find  $\varepsilon > 0$  and  $\eta > 0$  and a sequence of points  $x_k \in \Lambda$  such that  $N_\eta^r(x_k) \subset N_t^r(x_k, k, \varepsilon)$  for any  $k > 0$ . Now, if we suppose that  $x_k \rightarrow x$ , then  $x$  must be an R-stable point of  $\Lambda$  and this is a contradiction. The case of R-unstable points is analogous and the claim is proved.

Now fix  $x \in \Lambda$ ,  $0 < \varepsilon < \delta$  and let  $T > 0$  be given by Theorem 3.7. Let  $\eta > 0$  be such that if  $d(x, y) \leq \eta$ , then  $d(P_{x,nt}(x), P_{x,nt}(y)) \leq \varepsilon$  for  $|n| = 0, \dots, T$ . Fix some  $n \geq \max\{T, K_{\varepsilon, \eta}\}$ . By the claim, we have that

$$P_{x, -nt}(N_\eta^r(P_{x, nt}(x))) \not\subset C(N_t^r(x, n, \varepsilon), x).$$

Thus there is some

$$y_0 \in P_{x, -nt}(N_\eta^r(P_{x, nt}(x)) \cap \partial C(N_t^r(x, n, \varepsilon), x)).$$

In particular, this implies that for some  $0 \leq k \leq n$ , we have that

$$d(P_{x, kt}(x), P_{x, kt}(y_0)) = \varepsilon.$$

But now,  $k \notin [n - T, n - 1]$ , by the choice of  $\eta$ . Also  $k \notin [T, n - T]$ , otherwise there should exist some  $0 \leq j \leq n$  such that  $d(P_{x, jt}(x), P_{x, jt}(y_0)) > \delta$  contradicting  $y_0 \in N_t^r(x, n, \varepsilon)$ . Thus,  $k \in [0, T]$  and therefore  $d(x, y_0) > \eta$ , by the choice of  $\eta$ .

Finally, we have that for any  $n \geq \max\{T, K_{\varepsilon, \eta}\}$  we have that  $C(N_t^r(x, n, \varepsilon), x)$  is a connected set with diameter greater than  $\eta$ . Thus by the compactness of the continuum hyperspace of  $M$  we have that the set

$$\bigcap_{n > 0} \overline{C(N_t^r(x, n, \varepsilon), x)}$$

is a connected set contained in  $S_\varepsilon(t, x)$  with diameter greater than  $\eta$ . Since the case for the R-unstable sets is analogous, the theorem is proved.  $\square$

#### 4. THE TOPOLOGICAL ENTROPY OF R-EXPANSIVE FLOWS AND SOME EXAMPLES

In this section we will explore the topological entropy of R-expansive flows. We will divide this section in two parts. The former deals with general R-expansive flows containing Lyapunov stable sets, while in the second part we will present some examples of R-expansive flows to which our results can be applied or not.

**4.1. R-expansive flows with non-singular Lyapunov stable sets.** In the next result, we derive some conditions to obtain positive topological entropy for R-expansive flows.

**Theorem 4.1.** *Let  $\phi$  be a R-expansive flow. If there exists a non-singular Lyapunov stable set  $\Lambda \subset M$ , such that  $\Lambda$  is not a finite union of compact orbits, then  $h(\phi) > 0$ .*

*Proof.* Let  $\Lambda$  be a non-singular set as in the hypothesis. Let  $\Lambda'$  be the set formed by the orbits of all R-stable points and the orbits of all R-unstable points of  $\Lambda$ . Since  $\Lambda$  is not a finite union of periodic orbits,  $\Lambda_0 = (\Lambda \setminus \Lambda') \neq \emptyset$ . Moreover, Corollary 3.20 implies  $\Lambda_0$  is compact, invariant, and without R-stable or R-unstable points. In addition, the Lyapunov stability of  $\Lambda$  implies that  $\Lambda_0$  is also Lyapunov stable. Now, Theorem 3.19 implies that there exists  $\eta > 0$  such that  $CU_\eta(t, x)$  is non-trivial for any  $x \in \Lambda_0$ . Let us now fix some constants:

- (1) Fix  $\delta > 0$  the constant of R-expansiveness of  $\phi$ .
- (2) Fix  $0 < \varepsilon \leq \delta$  such that  $\overline{B_\varepsilon(\Lambda_0)} \cap \text{Sing}(\phi) = \emptyset$ .
- (3) Let  $\gamma > 0$  be given by the Lyapunov stability of  $\Lambda_0$  with respect to  $\varepsilon$ .

(4) fix  $x \in \Lambda_0$  and let  $y \in CU_\eta(t, x)$  be such that  $y \neq x$ .

Proposition 3.9 implies that  $d(P_{x, -nt}(y), P_{x, -nt}(x)) \rightarrow 0$ , while Theorem 3.1 implies that in fact  $d(O(x), \phi_{-t}(y)) \rightarrow 0$ . On the other hand, the Lyapunov stability of  $\Lambda_0$  guarantees that  $\phi_t(y) \in B_\varepsilon(\Lambda_0)$  for any  $t \geq 0$  (see Figure 1).

Last facts imply that

$$\Lambda_1 = \overline{\bigcup_{x \in \Lambda_0} \bigcup_{t \in \mathbb{R}} \phi_t(CU_\eta(t, x))}$$

Is a compact and invariant set contained in  $\overline{B_\varepsilon(\Lambda_0)}$  and hence non-singular. In particular, it is expansive and has dimension greater than one, since it contains  $O(x)$  and  $U_\eta(t, x)$ . So, we conclude by Theorem 2.5 that  $h(\phi) > 0$ .

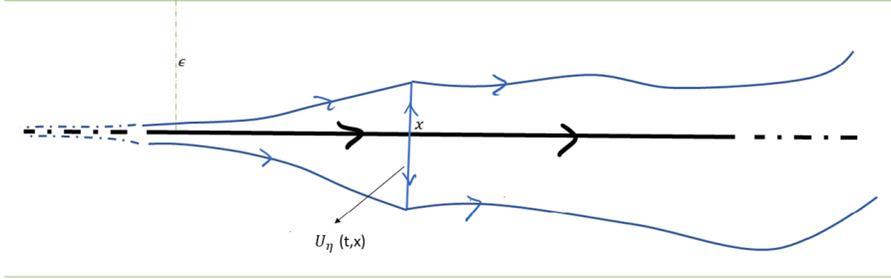


FIGURE 1. The idea behind the proof of Theorem 4.1

□

Notice that, in contrast with Theorem 2.5, in the previous result we are not assuming any dimensional hypothesis on  $\Lambda$ . Instead, our techniques allow us to construct a compact and invariant with enough dimension, even we are beginning with an one-dimensional set  $\Lambda$ . In addition, we obtain as an immediate consequence of Theorems 4.1 and 3.3 we obtain the following corollary:

**Corollary 4.2.** *Let  $\phi$  be a  $k^*$ -expansive flow such that  $\text{Sing}(\phi)$  is a hyperbolic set. Let  $\Lambda$  be a non-singular Lyapunov stable subset of  $M$ . If  $\Lambda$  is not a finite union of periodic orbits, then  $h(\phi) > 0$ .*

**4.2. Examples.** We end this work by presenting some new examples of R-expansive flows and showing their relation with the results we obtained. Our first example illustrates that, although R-expansiveness implies  $k^*$ -expansiveness under the hyperbolicity of  $\text{Sing}(X)$ , there are examples of non-trivial R-expansive flows that are far from being  $k^*$ -expansive. Furthermore, our results do not apply to this example, since it has zero topological entropy.

**Example 4.3.** Consider  $M = \mathbb{T}^3 = S^1 \times \mathbb{T}^2$ . We begin by defining a periodic flow  $\psi$  on  $M$  induced by a vector field with velocity constant and equal to one. Here we will see  $M$  as the product  $[-2, 2] \times \mathbb{T}^2$ , where the end points of  $[-2, 2]$  are identified. Let us consider on  $M$  the vector field  $X$  constant and equal to  $(1, 0, 0)$ . Thus  $X$  generates the flow  $\psi$  desired.

Now we modify this flow to obtain an R-expansive flow. First consider a smooth non-negative function  $\rho$  on  $M$  satisfying the following conditions:

- (1)  $\rho$  is constant along the fibers  $\{x\} \times \mathbb{T}^2$ .
- (2)  $\rho((x, y, z)) = 1$ , if  $(x, y, z) \in [-2, -1] \times \mathbb{T}^2$  or  $(x, y, z) \in [1, 2] \times \mathbb{T}^2$ .
- (3)  $\rho((x, y, z)) = 0$  if, and only if,  $(x, y, z) \in \{0\} \times \mathbb{T}^2$ .
- (4)  $\rho((x, y, z))$  decreases in  $[-1, 1] \times \mathbb{T}^2$ , as  $x \rightarrow 0$ .

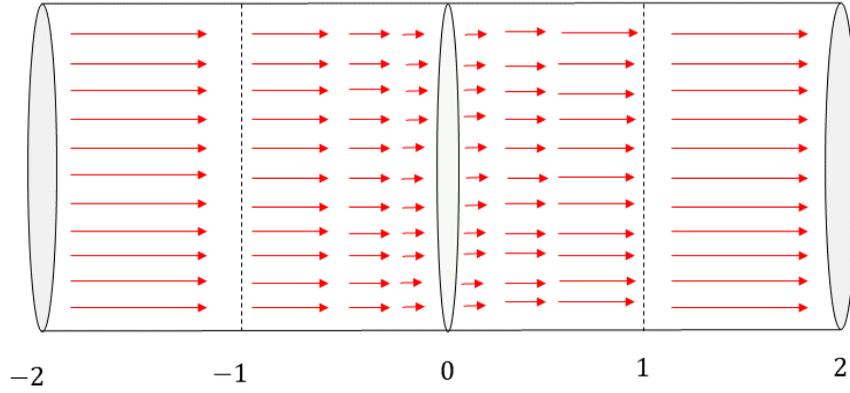


FIGURE 2. Chain-transitive R-expansive flow on  $\mathbb{T}^3$  with zero topological entropy.

Let  $\phi$  be the flow generated by the field  $\rho X$  (see Figure 2).

*Claim:  $\phi$  is R-expansive*

To prove the claim we proceed as follows. Fix some regular point  $p = (x, y, z) \in M$ . Notice that  $\{x\} \times \mathbb{T}^2$  is a cross-section through  $p$  for any time  $t > 0$ . So fix some  $t > 0$  and  $\delta > 0$ . The set  $S_\delta(t, x)(p)$  is formed by all the points  $q \in N_\delta^r(p)$  such that

$$d(P_{p,nt}(p), P_{p,nt}(q)) \leq \|X(P_{p,nt}(p))\|$$

for any  $n > 0$ . But by the choice of  $\rho$ , one  $\|X(P_{p,nt}(p))\| \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand,  $\phi$  acts isometrically on the fibers  $\{x\} \times \mathbb{T}^2$  and this implies that

$$d(P_{p,nt}(p), P_{p,nt}(q)) = d(p, q)$$

for any  $q \in N_\delta^r(p)$  and every  $n \in \mathbb{Z}$ . Thus we have that  $S_\delta(t, x)(p) = \{p\}$ . A similar argument shows that  $U_\delta(t, x)(p) = \{p\}$ . This proves that  $\phi$  is R-expansive. In addition,  $h(\phi) = 0$ , since the non-wandering set of  $\phi$  is formed by fixed points. Finally, notice that our results do not apply to this examples, since  $\overline{O(X)} \cap \text{Sing}(X) \neq \emptyset$ , for every  $x \in M$ .

**Remark 4.4.** In [22] it is proved that  $BW$ -expansive flows cannot exist on three-dimensional manifolds with fundamental group of sub-exponential growth. On the other hand, in [6] it is proved that the same does not hold for  $k^*$ -expansive flows, by exhibiting an example of such a flow on  $S^3$ . The previous example also shows that the same does not hold for  $R$ -expansive flows which are not  $k^*$ -expansive.

The next is an example to illustrate our main results.

**Example 4.5.** This example is a slight modification of the example of singular axiom A flow presented in [6], the major modification is that we replace the geometric Lorenz attractor used in that work by the Rovella attractor. Let  $A \subset \mathbb{R}^3$  be the geometric Rovella attractor defined as in [9]. For a detailed discussion on the construction and properties of the Rovella attractor we refer the reader to [26] and [21]. Here we need to keep in mind that the Rovella attractor is non-locally star and it is a homoclinic class. In particular,  $A$  has positive topological entropy.

By following techniques analogous to the techniques used in [6], one can obtain  $A$  by a flow  $X$  on  $\mathbb{R}^3$  with the following properties:

- (1) There exists a solid two-torus  $S \subset \mathbb{R}^3$  such that the vector field  $X$  is transversal to  $\partial S$  and points inwardly  $S$ .
- (2)  $A \subset \text{int}(S)$ .
- (3)  $A = \bigcap_{t \geq 0} \phi'_t(S)$ , where  $\phi'$  is the flow generated by  $X$ .

On the other hand, by considering  $-X$ , we also obtain a Rovella's repeller  $R$  delimited by a bi-torus  $S'$  and whose  $-X$  is transversal to  $S'$  and points outwardly  $S'$ . Let  $\phi : S \rightarrow S'$  be a diffeomorphism and define  $M = \frac{U \cup \overline{U'}}{x \sim \phi(x)}$ .

Notice that  $M$  is a closed manifold and the resulting vector field  $Y'$  satisfies  $\Omega(Y') = A \cup R$ . Moreover, the orbit of any wandering point goes to  $R$  in the past, to  $A$  in the future and crosses  $S$  in exactly one point. By Corollary 3.25 of [7] and Theorem 1 of [9] we have that  $Y'$  is  $R$ -expansive in  $A$  and  $R$ .

Notice that in order to apply our main results, we need to ensure that  $Y'$  is  $R$ -expansive. To obtain that we modify  $Y'$  based on the Example 4.3.

Remember that to obtain  $Y'$  we glued to surfaces  $S$  and  $S'$ . Let us denote here by  $S$  the glued resulting surface. Let  $V \subset M$  be a neighborhood of  $S$  such that  $U \cap (A \cup R) = \emptyset$  and let  $\rho : M \rightarrow R$  be a  $C^\infty$ -function such that:

- $\phi(x) \geq 0$ , for every  $x \in M$ .
- $\phi(x) = 1$ , for every  $x \in M \setminus V$ .
- $\phi(x) = 0$  if, and only if  $x \in S$ .

Next consider  $Y = \rho Y'$  and let  $\phi_t$  be the flow induced by  $Y$ . Notice that now we have  $\Omega(Y) = A \cup R \cup S$  and if  $x$  is a wandering point, then  $x$  satisfies only one of the two following behaviors.

- (1)  $\phi_t(x) \rightarrow A$  as  $t \rightarrow \infty$  and there is some  $p \in S$  such that  $\phi_t(x) \rightarrow p \in S$ , as  $t \rightarrow -\infty$ .
- (2) There is some  $p \in S$  such that  $\phi_t(x) \rightarrow p \in S$ , as  $t \rightarrow \infty$  and  $\phi_t(x) \rightarrow R$ , as  $t \rightarrow -\infty$ .

We claim that  $\phi_t$  is  $R$ -expansive. Indeed, again the  $R$ -expansiveness of  $\Omega(Y)$  is immediate. Now, if  $x$  is wandering and  $y \in \Omega(Y)$ , then  $x$  and  $y$  are separated by conditions (1) and (2) above. If  $x$  and  $y$  are wandering points, then if  $x$  satisfies condition (1) and  $y$  satisfies condition (2), then  $x$  and  $y$  are also trivially separated. Finally, if both  $x$  and  $y$  satisfy condition (1) then  $\phi_t(x) \rightarrow p_x$  and  $\phi_t(y) \rightarrow p_y$  as

$t \rightarrow -\infty$ . Since,  $Y(\phi_t(x)) \rightarrow 0$  as  $t \rightarrow -\infty$ , then

$$d(\phi_t(x), \phi_{h(t)}(y)) < \delta|Y(\phi_t(x))|,$$

for every  $t \in \mathbb{R}$ , if and only if,  $p_x = p_y$ . But this implies  $y \in O(x)$  and therefore  $Y$  is  $\mathbb{R}$ -expansive.

Finally, by the main result in [21]  $A$  and  $R$  are homoclinic classes. Therefore they contain horseshoes which are examples Lyapunov stable sets under the hypothesis of Theorem 4.1. Thus, we can obtain  $h(\phi) > 0$ .

## 5. DECLARATIONS

**Ethical Approval:** This declaration is not applicable.

**Competing Interests:** We hereby declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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