

About the value of the two dimensional Lévy's constant

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Abstract

We give a numerical approximation of the Lévy constant on the growth of the denominators of the best Diophantine approximations in dimension 2 with respect to the euclidean norm. This constant is expressed as an integral on a surface of dimension 7. We reduce the computation of this integral to a triple integral, whose numerical evaluation was carried out in [3].

1 Introduction

In 1936, Aleksandr Khintchin showed that there exists a constant K such that the denominators $(q_n)_{n \geq 0}$ of the convergents of the continued fraction expansions of almost all real numbers θ satisfy

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n = K$$

Soon afterward, Paul Lévy gave the explicit value of the constant,

$$K = \frac{\pi^2}{12 \ln 2}.$$

In [2], this result is extended to the denominators of best Diophantine approximations to vectors in \mathbb{R}^d and even to matrices in $M_{d,c}(\mathbb{R})$. The value of the limit is given by an integral $\int_S d\mu_S$ over a codimension one submanifold S in the space of lattices $\mathrm{SL}(d+1, \mathbb{R})/\mathrm{SL}(d+1, \mathbb{Z})$ (see Section 2.2 below). However, apart from the case of $d = 1$, this integral is very difficult to calculate. The aim of this document is to give a numerical approximation of the integral associated with best Diophantine approximations to vectors in \mathbb{R}^2 .

This document is organized as follows. We first give the definition of the submanifold S together with a parametrization of S . Then we give an explicit formula for the measure μ_S induced by the flow. The two difficult parts of the work are the explicit description of the domain of integration, i.e., the subset of parameters corresponding to S , and the calculation of the integral $\int_S d\mu_S$. This is done in the two last sections. For more details on best Diophantine approximation we refer the reader to [2].

2 Definitions of S and its parametrization

2.1 Definition

The surface S is the set of unimodular lattices Λ in \mathbb{R}^3 such that there exist two independent vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in Λ such that:

- $|u_3|$ and $\sqrt{v_1^2 + v_2^2}$ are $< r = |v_3| = \sqrt{u_1^2 + u_2^2}$,
- the only nonzero points of Λ in the cylinder $C(r) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \max(\sqrt{x_1^2 + x_2^2}, |x_3|) \leq r\}$ are $\pm u$ and $\pm v$.

Let μ_G be the Haar measure in $G = \mathrm{SL}(3, \mathbb{R})$. Let μ be the measure in the space of unimodular lattices $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ invariant by the left action of G , induced by μ_G . In turn, let μ_S be the measure induced by μ and the diagonal flow

$$g_t = \mathrm{diag}(e^t, e^t, e^{-2t}), t \in \mathbb{R}$$

on S .

2.2 The main formula

In [2], it is proved that for Lebesgue-almost all $\theta \in \mathbb{R}^2$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\theta) = \frac{2\mu(\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z}))}{\mu_S(S)}$$

where $(q_n(\theta))_n$ is the sequence of best approximation denominators of θ associated with the standard Euclidean norm in \mathbb{R}^2 .

2.3 Parametrization of S

Let Λ be a lattice in S and let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be the two vectors associated with Λ by the definition of S . By Lemma 10 of [2], there exists a vector $w \in \Lambda$ such that u, v, w is a basis of Λ . We can suppose u_3 and $v_3 \geq 0$ w.l.g.. There is a rotation k_θ in $\mathrm{SO}(3, \mathbb{R})$ that fixes e_3 and such that $rk_\theta e_1 = (u_1, u_2, 0)$ where $r = \sqrt{u_1^2 + u_2^2}$. If M is the 3×3 matrix whose columns are u, v and w , we have

$$\Lambda = M\mathbb{Z}^3,$$

$$M = rk_\theta \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & a_2 & c_2 \\ b & 1 & c_3 \end{pmatrix}$$

where $b \in [0, 1[$, $a_1^2 + a_2^2 < 1$ and

$$\det M = r^3((1 - a_1b)(-c_2) - a_2(bc_1 - c_3)) = 1.$$

Therefore, we can parametrize S with the seven parameters

$$\theta, b, a_1, a_2, c_1, c_2, c_3.$$

The problem is now to find a subset of parameters Ω_7 such that

- $\pm u$ and $\pm v$ are the only nonzero vector of Λ in the cylinder $C(r)$,
- for every $\Lambda \in S$ there exists exactly one 7-tuple $(\theta, b, a_1, a_2, c_1, c_2, c_3)$ such that $\Lambda = M\mathbb{Z}^3$,

see the section about the Domain of integration.

3 Induced measure on S

We use Siegel normalization of the Haar measure on $G = \mathrm{SL}(3, \mathbb{R})$, see [1] Lecture XV. For a Borel set $B \subset \mathrm{SL}(3, \mathbb{R})$,

$$\mu_G(B) = \mathrm{Lebesgue}_{\mathbb{R}^9}(B')$$

where $B' = \{tM : M \in B, t \in [0, 1]\}$. Consider the parametrization of $\mathrm{GL}_+(3, \mathbb{R})$ given by

$$M = g_t r k_\theta \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & a_2 & c_2 \\ b & 1 & c_3 \end{pmatrix}.$$

In these coordinates the standard volume form in \mathbb{R}^9 is¹

$$3r^8 d(r, t, \theta, a_1, a_2, b, c_1, c_2, c_3).$$

We wish to replace r with the homogeneous coordinate $\rho := \Delta^{1/3}$ where

$$\Delta = \det M = r^3 \eta, \quad \text{and} \quad \eta := (1 - a_1 b)(-c_2) - a_2(bc_1 - c_3) > 0.$$

Rewrite Lebesgue volume form in the new coordinates as

$$\frac{\Delta^2 d(\Delta, t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3} = \frac{3\rho^8 d(\rho, t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3}.$$

The local form of Haar measure with Siegel's normalization is

$$d\mu_G = \int_0^1 \frac{3\rho^8 d(t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{\eta^3} d\rho = \frac{d(t, \theta, a_1, a_2, b, c_1, c_2, c_3)}{3((1 - a_1 b)(-c_2) - a_2(bc_1 - c_3))^3}.$$

Using this normalization, we can quote Siegel's formula and bypass the computation of the first return time:

$$\mathrm{vol}(\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})) = \frac{\zeta(2)\zeta(3)}{3}.$$

¹This was obtained by computing the determinant of a 9-by-9 matrix. The factor r^8 is expected as r is homogeneous of degree one. For $g_t = \exp(\mathrm{diag}(\lambda_1 t, \lambda_2 t, \lambda_3 t))$ the leading coefficient generalizes to $\lambda_1 - \lambda_3$.

The local form for the induced measure on the transversal S is now

$$d\mu_S = \frac{d(\theta, a_1, a_2, b, c_1, c_2, c_3)}{3((1 - a_1b)(-c_2) - a_2(bc_1 - c_3))^3}.$$

Levy's constant is $d = 2$ times the average return time (see the main formula), or

$$K = L_{2,1} = \frac{2\zeta(2)\zeta(3)}{3\mu_S(\Omega_7)}$$

where Ω_7 is any domain of integration parametrizing the transversal S .

4 Domain of integration

The suspension of the transversal gives a fundamental domain for the action of $\mathrm{SL}(3, \mathbb{Z})$ that is invariant under $\mathrm{SO}(2)$ rather than $\mathrm{SO}(3)$. The quotient of the transversal by the circle action is a 6 dimensional space that we shall realize as a fiber bundle over the 3 dimensional base $\Omega_2 \times [0, 1[$ where

$$\Omega_2 = \{a = a_1 + ia_2 \in \mathbb{C} : |a| < 1, |a - 1| \geq 1\}$$

parametrizes the possible configurations of $v = (a_1, a_2, 1)$ and $u = (1, 0, b)$ on the boundary of the standard unit cylinder $C_0 = C(1)$. Given a lattice in our transversal S we rescale and rotate so that the systole cylinder is given by C_0 with the vectors u and v on its ∂_+ - and ∂_- -faces. Ambiguity in the normal form can be ignored since it occurs on a set of positive codimension. The lattice is determined by specifying the third vector $w = (c_1, c_2, c_3)$ that forms an integral basis for the lattice, positively oriented by requiring $c_2 < 0$. We have some freedom for the choice of w , which we will describe next. Once this choice is made, the projection from the 6 dimensional total space to the base will have been specified.

The space of possibilities for w is a connected component of the complement of the union of C_0 with all its translates under the action of the discrete subgroup $\mathbb{Z}u + \mathbb{Z}v$. The topological boundary is tessellated by the domain $G(a, b)$ where this surface meets ∂C_0 . The orthogonal projection of $G(a, b)$ onto the (c_1, c_3) -plane is a rectilinear domain that we will denote by

$$F(a, b).$$

The choice of w is such that its orthogonal projection lies in $F(a, b)$; that is,

$$(c_1, c_3) \in F(a, b) \quad \text{and} \quad -c_2 \geq \sqrt{1 - c_1^2}.$$

$F(a, b)$ is rectilinear since curved arcs on $\partial G(a, b)$ map to horizontal segments.

4.1 Intersection patterns

For the precise description of $F(a, b)$ we need to know which cylinders appear on $\partial G(a, b)$. By symmetry, it suffices to consider the case $a_2 > 0$.

Lemma 1. $C_{\pm u}, C_{\pm v}, C_{\pm(u+v)}$ and $C_{\pm(v-u)}$ are the only translates whose intersection with C_0 has nonempty interior, unless $|2a - 1| < 2$, when there is an additional pair $C_{\pm(2v-u)}$.

Proof. The hypothesis on C_{mu+nv} is satisfied by (m, n) in the intersection of the infinite strip $|bm + n| < 2$ and an ellipse that is contained in the vertical strip $|m| < 2|a_1|/|a|$. The pair $(2, -1)$ lies in the ellipse if and only if $|2a - 1| < 2$. \square

Lemma 2. $C_{\pm(2v-u)}$ is disjoint from $\partial C_0 \setminus U$ where U is the union of the cylinders in Lemma 1.

Proof. By symmetry, it suffices to show $C_{2v-u} \cap \partial C_0 \subset \text{Int } C_{v-u} \cup C_v$, which in terms of euclidean disks is the condition $D_0 \cap D_{2a-1} \subset \text{Int } D_a \cup D_{a-1}$. For this, it is enough to verify that $D_0 \cap D_{2a-1}$ is contained in the open disk of radius $1/2$ centered at $a - 1/2$, i.e.

$$\left| \chi_-(2a - 1) - a + \frac{1}{2} \right| < \frac{1}{2}$$

which is equivalent to $|2a - 1| < 2$. Here, the notation

$$\chi_{\pm}(a) := \frac{a}{2} \pm \frac{ia}{|a|} \sqrt{1 - \frac{|a|^2}{4}}$$

denotes the two points where the circles $|z| = 1$ and $|z - a| = 1$ meet. \square

To show that a cylinder meets $\partial G(a, b)$ in an arc involves checking that the arc is disjoint from the other 7 cylinders. There does not seem to be an easier way to carry out this tedious task, e.g. to verify that the cylinders $C_{\pm u}$ and $C_{\pm v}$ appear on $\partial G(a, b)$ in every possible scenario.² We shall skip the verification of these intuitive claims and simply identify the criteria for the appearance of the other 4 cylinders. It is perhaps surprising that the answer is always determined by the sign of $|a - \xi| - 1$ where ξ denotes the sixth root of unity in the first quadrant.

Lemma 3. $C_{\pm(u-v)}$ appear iff $|a - \xi| < 1$ while $C_{\pm(u+v)}$ appear iff $|a - \xi| > 1$.

Proof. Considerations of elementary nature lead to the following characterizations: C_{u-v} appears iff $d(1 - a, \bar{\xi}) < 1$ while C_{v-u} appears iff $d(a - 1, -\xi) < 1$; C_{u+v} appears iff $d(a + \bar{\xi}, 1) > 1$ while C_{-u-v} appears iff $d(-a - \bar{\xi}, -1) > 1$. In each case the sign of $|a - \xi| - 1$ is nonzero. \square

The shape of $F(a, b)$ also depends on how the cylinders intersect with C_0 .

Lemma 4. $G(a, b)$ has overlap with the bottom of C_0 iff $|a + \xi| < 1$.

Proof. Note that the point where the curved faces of C_{-v} and C_{-u-v} meet the plane $z = -1$ is described by the complex number $-a - \xi$. \square

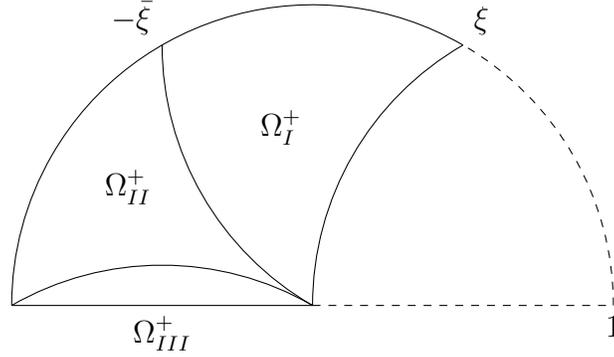


Figure 1: Three subregions of Ω_2^+ .

Let us divide $\Omega_2^+ := \{a \in \Omega_2 : a_2 \geq 0\}$ into the following subregions as depicted in Figure 1:

$$\begin{aligned}\Omega_I^+ &= \{a \in \Omega_2 : |a - \xi| < 1\} \\ \Omega_{II}^+ &= \{a \in \Omega_2 : |a - \xi| \geq 1, |a + \xi| \geq 1\} \\ \Omega_{III}^+ &= \{a \in \Omega_2 : |a + \xi| < 1, \text{Im } a \geq 0\}\end{aligned}$$

Next, we describe $G(a, b)$ in each of the 3 main cases.

CASE $a \in \Omega_I^+$: $G(a, b)$ is bounded by the cylinders $C_{\pm u}$, $C_{\pm v}$ and $C_{\pm(v-u)}$ and consists of two subregions $G_+ \sqcup G_0$ joined along an arc, labeled \oplus , along the top rim of C_0 . The arcs along the boundary of each subregion in counter-clockwise order are labeled

$$\begin{aligned}G_+ &: \oplus, u, v, v - u \\ G_0 &: \oplus, v - u, -u, -u, -v, u - v, u - v, u\end{aligned}$$

CASE $a \in \text{Int } \Omega_{II}^+$: $G(a, b)$ is bounded by the cylinders $C_{\pm u}$, $C_{\pm v}$ and $C_{\pm(v+u)}$ and consists of two subregions as in the previous case, with arcs along the boundary of the subregions labeled

$$\begin{aligned}G_+ &: \oplus, u, v + u, v \\ G_0 &: \oplus, v, -u, -u, -v - u, -v, -v, u\end{aligned}$$

CASE $a \in \Omega_{III}^+$: $G(a, b)$ is bounded by the same cylinders in the previous case but now consists of three subregions $G_+ \sqcup G_0 \sqcup G_-$ bounded by the following arcs

$$\begin{aligned}G_+ &: \oplus, v + u, v \\ G_0 &: \oplus, v, v, -u, -v - u, -v - u, \ominus, -v, -v, u, v + u, v + u \\ G_- &: \ominus, -v - u, -v\end{aligned}$$

²Later, we shall show that the projection $F(a, b)$ forms a fundamental domain for the action of $\mathbb{Z}(1, b) + \mathbb{Z}(-a_1, 1)$, which indirectly shows that none of the arcs on $\partial F(a, b)$ can be blocked by the other cylinders.

where \ominus is an arc along the bottom rim of C_0 joining G_0 and G_- .

In the next section, we use the above description of $G(a, b)$ to arrive at an explicitly described region $\tilde{F}(a, b)$ that is *a priori* only known to contain $F(a, b)$. We will verify that $\tilde{F}(a, b)$ is a fundamental domain for the action of $\mathbb{Z}(1, b) + \mathbb{Z}(-a_1, 1)$, from which it follows that $\tilde{F}(a, b)$ provides an equivalent definition of $F(a, b)$. In anticipation of this conclusion and since there is no further need to distinguish between two sets, we shall drop the overscript when referring to $\tilde{F}(a, b)$ in the following section.

4.2 Explicit description of $F(a, b)$

Each vertical arc on $\partial F(a, b)$ is the projection (in the y -direction) of a linear segment on $\partial G(a, b)$ along which one of the cylinders in Lemma 1 is transverse to C_0 . Using complex notation for the projection (in the z -direction) of the linear segment, we arrive at the following table

vertical arc	z_0	$\kappa(z_0)$
C_u	ξ	$1/2$
C_{-u}	$-\xi$	$-1/2$
C_v	$\chi_+(a)$	$\kappa(a)$
C_{-v}	$\chi_-(-a)$	$\kappa(-\bar{a})$
C_{u-v}	$\chi_-(1-a)$	$\kappa(1-\bar{a})$
C_{v-u}	$\chi_+(a-1)$	$\kappa(a-1)$
C_{u+v}	$\chi_-(a+1)$	$\kappa(\bar{a}+1)$
C_{-u-v}	$\chi_+(-a-1)$	$\kappa(-a-1)$

where vertical arcs are labelled by the corresponding transverse cylinder and κ is given by

$$\kappa(a) := \operatorname{Re} \chi_+(a) = \frac{a_1}{2} - \frac{a_2}{|a|} \sqrt{1 - \frac{|a|^2}{4}}.$$

In each case, $F(a, b)$ can be described as some larger rectangle $[a, b] \times [c, d]$ with two or more “corners” removed. These “corners” will be described by the notation:

$$\begin{aligned} NW(x, y) &:= [a, x[\times]y, d] & NE(x, y) &:=]x, b] \times]y, d] \\ SW(x, y) &:= [a, x[\times [c, y[& SE(x, y) &:=]x, b] \times [c, y[\end{aligned}$$

For $a \in \Omega_I^+$, $F(a, b)$ is given by

$$\left[\kappa(a-1), \frac{1}{2} \right] \times [0, 1] \quad \text{minus} \quad SW(-1/2, 1-b) \cup SE(\kappa(1-\bar{a}), b)$$

while for $a \in \operatorname{Int} \Omega_{II}^+$ it is given by

$$\left[\kappa(a), \frac{1}{2} \right] \times [-b, 1] \quad \text{minus} \quad SW(-1/2, 1-b) \cup SE(\kappa(-\bar{a}), 0)$$

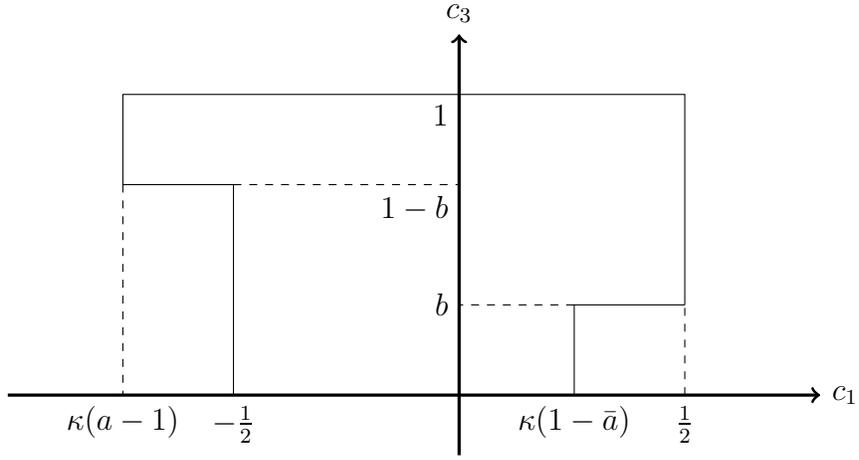


Figure 2: $F(a, b)$ in the case $a = \frac{3i}{10} \in \Omega_I^+$ and $b = .3$

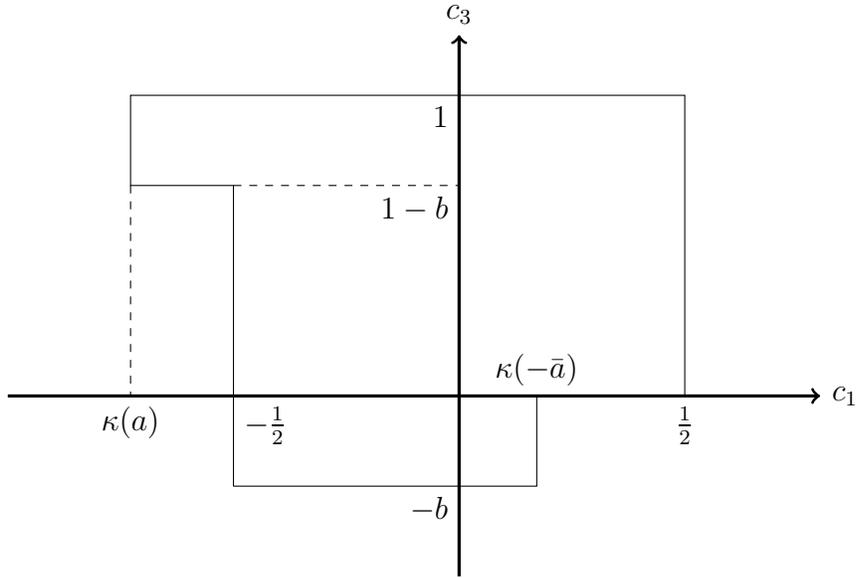


Figure 3: $F(a, b)$ in the case $a = \frac{-9+3i}{10} \in \Omega_{II}^+$ and $b = .3$

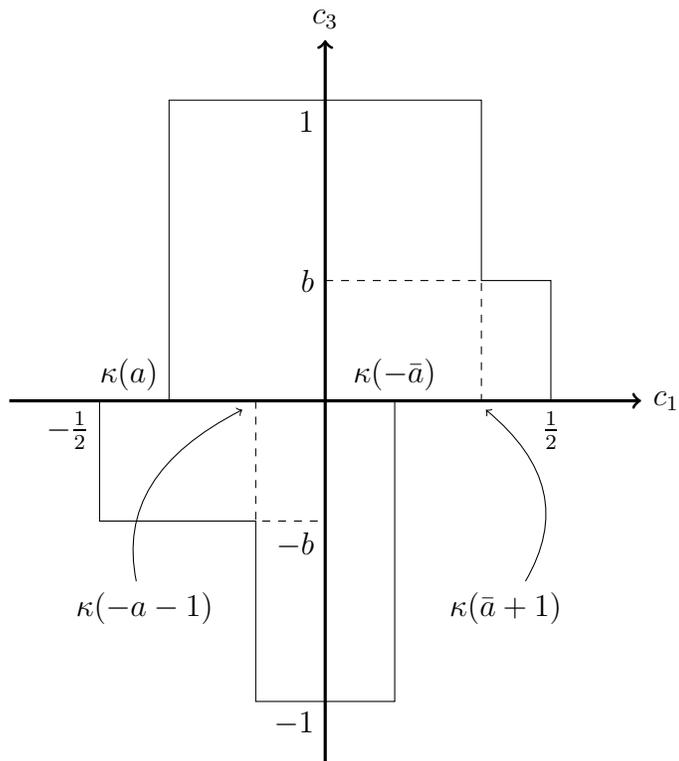


Figure 4: $F(a, b)$ in the case $a = \frac{-10+i}{20} \in \Omega_{III}^+$ and $b = .4$

and for $a \in \Omega_{III}^+$ by

$$\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1] \quad \text{minus} \quad \begin{array}{l} NW(\kappa(a), 0) \cup NE(\kappa(1+\bar{a}), b) \cup \\ SW(\kappa(-a-1), -b) \cup SE(\kappa(-\bar{a}), 0) \end{array}.$$

It is easy to verify directly from the explicit description that $F(a, b)$ is a fundamental domain for the action of $\mathbb{Z}(1, b) + \mathbb{Z}(a_1, 1)$. From this, it follows that $\text{area } F(a, b) = 1 - a_1 b$. This last claim can also be checked directly: Indeed, in the case $a \in \Omega_{II}^+$ this boils down to the identity $\kappa(-\bar{a}) - \kappa(a) = -a_1$, while for $a \in \Omega_I^+$ it is the same identity with $1 - \bar{a}$ instead of a . The case $a \in \Omega_{III}^+$ follows by observing that the total width of NE and NW corners is

$$\left(\kappa(a) + \frac{1}{2}\right) + \left(\frac{1}{2} - \kappa(\bar{a} + 1)\right) = \kappa(-\bar{a}) - \kappa(-a - 1).$$

5 Computation of $\mu_S(\Omega_7)$

We have now established the closed form expression

$$\begin{aligned} \mu_S(\Omega_7) &= \frac{4\pi}{3} \int_{\Omega_2^+} da_1 da_2 \int_0^1 db \int_{F(a,b)} dc_1 dc_3 \int_{\sqrt{1-c_1^2}}^{\infty} \frac{dc_2}{((1-a_1b)c_2 - a_2(bc_1 - c_3))^3} \\ &= \frac{2\pi}{3} \int_{\Omega_2^+} da_1 da_2 \int_0^1 db \left(\frac{1}{1-a_1b} \int_{F(a,b)} \frac{dc_1 dc_3}{\Xi^2} \right) \end{aligned}$$

where³

$$\Xi := (1 - a_1 b) \sqrt{1 - c_1^2} - a_2 (bc_1 - c_3).$$

Green's theorem applied to $\frac{1}{\Xi^2} = \frac{\partial Q}{\partial c_1} - \frac{\partial P}{\partial c_3}$ with $Q = 0$ and $P = \frac{1}{a_2 \Xi}$ implies

$$\mu_S(\Omega_7) = \int_{\Omega_2^+} da_1 da_2 \int_0^1 db \left(\frac{2\pi/3}{a_2(1-a_1b)} \int_{\partial F(a,b)} \frac{dc_1}{\Xi} \right).$$

The substitution

$$c_1 = \frac{2\tau}{1+\tau^2} \quad \text{and} \quad dc_1 = \frac{2(1-\tau^2)}{(1+\tau^2)^2} d\tau$$

and the observation

$$(1+\tau^2)\Xi = (1-a_1b)(1-\tau^2) - a_2(bc_1 - c_3)(1+\tau^2)$$

readily leads to

$$\begin{aligned} \frac{dc_1}{\Xi} &= \frac{2(1-\tau^2)d\tau}{(1+\tau^2)(1-a_1b+a_2c_3-2a_2b\tau-(1-a_1b-a_2c_3)\tau^2)} \\ &= \left(\frac{A+B\tau}{1+\tau^2} + \frac{C(a_2b+(1-a_1b-a_2c_3)\tau)+D}{1-a_1b+a_2c_3-2a_2b\tau-(1-a_1b-a_2c_3)\tau^2} \right) d\tau \end{aligned}$$

³Remark: it can be shown that $\Xi > 1/24$.

where

$$B = C = \frac{a_2 b A}{1 - a_1 b}, \quad D = -a_2 c_3 A, \quad \text{and} \quad A = \frac{2(1 - a_1 b)}{(1 - a_1 b)^2 + a_2^2 b^2}.$$

Since $A \tan^{-1} \tau + \frac{B}{2} \ln(1 + \tau^2)$ depends on c_1 but not on c_3 , the sum over the corners of $F(a, b)$ with alternating sign vanishes. The same reasoning applies to the first term of the decomposition

$$\frac{C}{2} \ln \{(1 - a_1 b + a_2 c_3) - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2\} = \frac{C}{2} \ln(1 + \tau^2) + \frac{C}{2} \ln \Xi.$$

The second term also vanishes when summed over the corners of $F(a, b)$ with alternating sign. To see this, we consider the pairing of the corners of $F(a, b)$ induced by the identification of vertical edges, noting that paired vertices are summed with opposite signs, and that Ξ is constant on each pair. It follows that

$$\mu_S(\Omega_\tau) = \int_{\Omega_2^+} da \int_0^1 db \frac{4\pi/3}{(1 - a_1 b)^2 + a_2^2 b^2} \int_{\partial F(a, b)} \frac{(-c_3) d\tau}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2}.$$

Since we can factor

$$(\phi_+ - 2a_2 b \tau - \phi_- \tau^2) \phi_- = (\sqrt{D} + a_2 b + \phi_- \tau)(\sqrt{D} - a_2 b - \phi_- \tau)$$

where $\phi_\pm = 1 - a_1 b \pm a_2 c_3$ and

$$D := (1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2) \geq |a|^2 b^2 - 2a_1 b + 1 - a_2^2 \geq a_2^2 \left(\frac{1}{|a|^2} - 1 \right) > 0$$

and noting that

$$\begin{aligned} \frac{d}{d\tau} \left(\ln \frac{\{\sqrt{(1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2)} + a_2 b + (1 - a_1 b - a_2 c_3) \tau\}^2}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2} \right) \\ = \frac{2\sqrt{(1 - a_1 b)^2 + a_2^2 (b^2 - c_3^2)}}{1 - a_1 b + a_2 c_3 - 2a_2 b \tau - (1 - a_1 b - a_2 c_3) \tau^2} \end{aligned}$$

the rational function of τ can readily be integrated to yield

$$\mu_S(\Omega_\tau) = \int_{\Omega_2^+} da \int_0^1 db \sum_{\nu} \frac{(-1)^\nu 2\pi c_3 / 3 \sqrt{D}}{(1 - a_1 b)^2 + a_2^2 b^2} \ln \left(\frac{\phi_+ - 2a_2 b \tau - \phi_- \tau^2}{(\sqrt{D} + a_2 b + \phi_- \tau)^2} \right)$$

where the sum is over the corners of $F(a, b)$ labelled in such a way that the sign of, say $(1/2, b)$ in the case $a \in \Omega_I$ and $(1/2, 0)$ in the other two cases, is given a plus sign.

Remark. $\sqrt{D} + a_2 b + \phi_- \tau > 0$. (*Proof.* First note that $\phi_+ - 2a_2 b \tau - \phi_- \tau^2 > 0$ because $\Xi > 0$. In the case $\phi_- > 0$ the factors $\sqrt{D} \pm (a_2 b + \phi_- \tau)$ have the same sign and cannot both be negative since their average is $\sqrt{D} > 0$. In the case $\phi_- \leq 0$, it suffices to show that $|\phi_-| \leq a_2 b$ (since $D > 0$ and $|\tau| \leq 1$) and this follows from $D = a_2^2 b^2 - |\phi_-| \phi_+ > 0$ and $|\phi_-| \leq \phi_+$.)

When evaluating the integrand, it is convenient to pair up terms with the same c_3 that are joined by a horizontal segment on $\partial F(a, b)$. If $\tau_- < \tau_+$ distinguishes the endpoints, then the integrand of the triple integral is the expression

$$\frac{2\pi c_3/\sqrt{D}}{(1-a_1b)^2 + a_2^2b^2} \ln \left\{ \left(\frac{\phi_+ - 2a_2b\tau_+ - \phi_- \tau_+^2}{\phi_+ - 2a_2b\tau_- - \phi_- \tau_-^2} \right) \left(\frac{\sqrt{D} + a_2b + \phi_- \tau_-}{\sqrt{D} + a_2b + \phi_- \tau_+} \right)^2 \right\}$$

summed over $c_3 \in \{b, 1-b, 1\}$ with signs $\{+, +, -\}$ for $a \in \Omega_I^+$; $c_3 \in \{-b, 1-b, 1\}$ with signs $\{+, +, -\}$ for $a \in \Omega_{II}^+$; and $c_3 \in \{-1, -b, b, 1\}$ with signs $\{+, +, -, -\}$ for $a \in \Omega_{III}^+$.

	c_3	sign	$c_1(\tau_-)$	$c_1(\tau_+)$
I	1	-	$\kappa(a-1)$.5
	$1-b$	+	$\kappa(a-1)$	-.5
	b	+	$\kappa(1-\bar{a})$.5
II	1	-	$\kappa(a)$.5
	$1-b$	+	$\kappa(a)$	-.5
	$-b$	+	-.5	$\kappa(-\bar{a})$
III	1	-	$\kappa(a)$	$\kappa(\bar{a}+1)$
	b	-	$\kappa(\bar{a}+1)$.5
	$-b$	+	-.5	$\kappa(-a-1)$
	-1	+	$\kappa(-a-1)$	$\kappa(-\bar{a})$

Note that τ as a function of c_1 is given by $\tau = \frac{1-\sqrt{1-c_1^2}}{c_1}$ apart from the removable singularity at $c_1 = 0$, so that

$$\tau(a) = \frac{2a_1 - \text{sgn}(a_1)|a|\sqrt{4-|a|^2}}{|a|^2 + \text{sgn}(a_1)2a_2}.$$

The expression that was fed into Octave is

$$\begin{aligned} 3\mu_S(\Omega_7) &= \int_{\Omega_2^+} da \int_0^1 db \sum_{c_3} \frac{(-1)^\nu 2\pi c_3}{((1-a_1b)^2 + a_2^2b^2)\sqrt{D}} \ln \frac{1-x}{1+x} \\ &= \int_{\Omega_2^+} da \int_0^1 db \sum_{c_3} \frac{(-1)^\nu 2\pi c_3 (\tau_+ - \tau_-)}{((1-a_1b)^2 + a_2^2b^2)(\phi_+ - \tau_+ \tau_- \phi_- - a_2b(\tau_+ + \tau_-))} \frac{1}{x} \ln \frac{1-x}{1+x} \end{aligned}$$

where

$$x = \frac{(\tau_+ - \tau_-)\sqrt{D}}{\phi_+ \tau_+ \tau_- \phi_- - a_2b(\tau_+ + \tau_-)}$$

and the value obtained by numerical integration is⁴

$$3\mu_S(\Omega_7) = 3.49277983865703\dots$$

which, using $\zeta(2) = 1.2020569031\dots$ and $\zeta(3) = 1.649340668\dots$, leads to

$$L_{2,1} = 1.13525697416719\dots$$

which is the value reported in [3].

⁴In [3], the factor of 3 is missing.

References

- [1] Siegel, Carl Ludwig. Lectures on the geometry of numbers. Springer-Verlag, Berlin, 1989.
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- [3] Seraphine Xieu, SFSU Applied Math Project, <http://math.sfsu.edu/cheung/xieu-amp.pdf>