SOME VARIATIONAL PRINCIPLES FOR THE METRIC MEAN DIMENSION OF A SEMIGROUP ACTION

FAGNER B. RODRIGUES*, THOMAS JACOBUS, AND MARCUS V. SILVA

ABSTRACT. In this manuscript we show that the metric mean dimension of a free semigroup action satisfies three variational principles: (a) the first one is based on a definition of Shapira's entropy, introduced in [22] for a singles dynamics and extended for a semigroup action in this note; (b) the second one treats about a definition of Katok's entropy for a free semigroup action introduced in [8]; (c) lastly we consider the local entropy function for a free semigroup action and show that the metric mean dimension satisfies a variational principle in terms of such function. Our results are inspired in the ones obtained by [19], [28], [24] and [23].

1. Introduction

The aim of this note is to explore the notion of metric mean dimension for a free semigroup action. The notion of metric mean dimension for a dynamical system $f:(X,d)\to (X,d)$, denoted by $\mathrm{mdim}_M(X,\phi,d')$, was introduced in [18] and may be related to the problem of whether or not a given dynamical system can be embedded in the shift space $(([0,1]^{\mathbb{N}})^{\mathbb{Z}},\sigma)$. It refines the topological entropy for systems with infinite entropy, which, in the case of a manifold of dimension greater than one, form a residual subset of the set consisting of homeomorphisms defined on the manifold (see [29]). In fact, every system with finite topological entropy has metric mean dimension equals to zero. The metric mean dimension depends on the metric d, therefore it is not a topological invariant. However, for a metrizable topological space X, $\mathrm{mdim}_M(X,\phi) = \mathrm{inf}_{d'} \, \mathrm{mdim}_M(X,\phi,d')$ is invariant under topological conjugacy, where the infimum is taken over all the metrics on X which induce the topology on X. By the other hand, as showed in [19] and in [28], the metric mean dimension is strongly related with the ergodic behaviour of the system, since it satisfies a kind of variational principle.

In [8] the authors considered the compact metric space $(Y^{\mathbb{N}}, D)$ and (X, d), where (Y, d_Y) is a compact metric space and D is the product metric induced by d_Y . In this setting they introduced the notion of metric mean dimension for a free semigroup action and proved that for certain classes of random walks; the ones induced by homogeneous probability measures on Y, it is possible to obtain a kind of Bufetov's formula (see [4] for Bufetov's formula for the topological entropy of a free semigroup action).

Our main goal here is to consider a compactly generated free semigroup of continuous maps acting on a compact metric and prove that the metric mean dimension satisfies several variational principles: (a) the first one is based on a definition of Shapira's entropy, introduced in [22] for a singles dynamics and extended for a semigroup action in this note; (b) the second one treats about a definition of Katok's

Date: November 10, 2021.

²⁰¹⁰ Mathematics Subject Classification. Primary: 37A05, 37A35.

Key words and phrases. On variational principle for the metric mean dimension for free semi-group action

^{*}Corresponding author e-mail: fagnerbernardini@gmail.com.

entropy for a free semigroup action introduced in [8]; (c) lastly we consider the local entropy function for a free semigroup action and show that the metric mean dimension satisfies a variational principle in terms of such function. Our results are inspired in the ones obtained by [19], [28], [24] and [23]. As a second objective, we extend the definition of metric mean dimension when we have a compactly generated semigroup and the topological entropy is the one defined in [14]. In this context we obtain a partial variational principle for the metric mean dimension.

This paper is organized as follows. In Section 2 we present the main definitions and the main results. In Section 3 we recall some results and definitions about box dimension, homogeneous measures and G-homogeneous measures. In Section 4 we prove the main theorems.

2. Definitions and Main results

We start recalling the main concepts we use and describing the systems we will work with.

2.1. Metric mean dimension of a map. Let (X, d) be a compact metric space. Given a continuous map $f: X \to X$ and a non-negative integer n, define the dynamical metric $d_n: X \times X \to [0, \infty)$ by

$$d_n(x,z) = \max \{d(x,z), d(f(x),f(z)), \dots, d(f^n(x),f^n(z))\}$$

which generates the same topology as d. Having fixed $\varepsilon > 0$, we say that a set $E \subset X$ is (n,ε) -separated by f if $d_n(x,z) > \varepsilon$ for every $x,z \in E$. In the particular case of n=1, we will call such a set ε -separated. Denote by $s(f,n,\varepsilon)$ the maximal cardinality of all (n, ε) -separated subsets of X by f. Due to the compactness of X, the number $s(f, n, \varepsilon)$ is finite for every $n \in \mathbb{N}$ and $\varepsilon > 0$. We say that $R \subset X$ is a (n,ε) -spanning set if for any $x\in X$ there exists $z\in R$ such that $d_n(x,z)<\varepsilon$. When n=1, we say that the set is ε -spanning. Let $r(n,\varepsilon)$ be the minimum cardinality of the (n, ε) -spanning subsets of X.

Definition 2.1. The lower metric mean dimension of f with respect to the fixed metric d is given by

$$\underline{\mathrm{mdim}}_{M}\left(X, f, d\right) = \liminf_{\varepsilon \to 0^{+}} \frac{h(f, \varepsilon)}{|\log \varepsilon|}$$

where

$$h(f,\varepsilon)=\limsup_{n\to\infty}\,\frac{1}{n}\,\log s(f,n,\varepsilon).$$
 Similarly, the *upper metric mean dimension* of f with respect to d is the limit

$$\overline{\mathrm{mdim}}_{M}\left(X, f, d\right) = \limsup_{\varepsilon \to 0^{+}} \frac{h(f, \varepsilon)}{|\log \varepsilon|}.$$

Clearly, $\underline{\mathrm{mdim}}_{M}\left(X,f,d\right)=\overline{\mathrm{mdim}}_{M}\left(X,f,d\right)=0$ whenever the topological entropy of f, given by $h_{\text{top}}(f) = \lim_{\varepsilon \to 0^+} h(f, \varepsilon)$, is finite.

2.2. Compactly generated semigroup action of continuous maps. Let (X,d)and (Y, d_Y) be compact metric spaces and $(g_y)_{y \in Y}$ be a family of continuous maps $g_y \colon X \to X$. Denote by G the free semigroup having the set $G_1 = \{g_y \colon y \in Y\}$ as generator, where the semigroup operation \circ is the composition of maps. Let $\mathbb S$ be the induced free semigroup action

$$\mathbb{S} \colon \quad G \times X \quad \to \quad X \\ (g, x) \quad \mapsto \quad g(x)$$

which is said to be compactly generated by Y, and denote by T_G the associated skew product given by

$$T_G: Y^{\mathbb{N}} \times X \to Y^{\mathbb{N}} \times X$$

 $(\omega, x) \mapsto (\sigma(\omega), g_{\omega_1}(x)),$ (2.1)

where $\omega = (\omega_1, \omega_2, \dots)$ is an element of the full unilateral space of sequences $Y^{\mathbb{N}}$ and σ denotes the shift map acting on $Y^{\mathbb{N}}$. It will be a standing assumption that T_G is a continuous map. If for every $n \in \mathbb{N}$ and $\omega = (\omega_1, \omega_2, \dots) \in Y^{\mathbb{N}}$ we write

$$f_{\omega}^n = g_{\omega_n} \dots g_{\omega_1}$$

then

$$T_G^n(\omega, x) = \left(\sigma^n(\omega), f_\omega^n(x)\right).$$

Consider the set $G_1^* = G_1 \setminus \{id\}$ and, for each $n \in \mathbb{N}$, let G_n^* denote the space of concatenations of n elements in G_1^* . Similarly, define $G = \bigcup_{n \in \mathbb{N}_0} G_n$, where $G_0 = \{id\}$ and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{\omega_n} \dots g_{\omega_2} g_{\omega_1}$, with $g_{\omega_j} \in G_1$ (for notational simplicity's sake we will use $g_j g_i$ instead of the composition $g_j \circ g_i$). In what follows, we will assume that the generator set G_1 is minimal, meaning that no function $g_y \in G_1$, for $y \in Y$, can be expressed as a composition of the remaining generators. To summon an element \underline{g} of G_n^* , we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$. Each element \underline{g} of G_n may be seen as a word which originates from the concatenation of n elements in G_1 . Yet, different concatenations may generate the same element in G. Nevertheless, in the computations to be done, we shall consider different concatenations instead of the elements in G they create.

- 2.3. Random walks. A random walk \mathbb{P} on $Y^{\mathbb{N}}$ is a Borel probability measure in this space of sequences which is invariant by the shift map σ . For instance, we may consider a finite subset $F = \{p_1, \ldots, p_k\}$ of Y, a probability vector (a_1, \cdots, a_k) (that is, a selection of positive real numbers a_i such that $\sum_{i=1}^k a_i = 1$), the probability measure $\nu = \sum_{i=1}^k a_i \delta_{p_i}$ on F and the Borel product measure $\mathbb{P}_{\nu} = \nu^{\mathbb{N}}$ on $Y^{\mathbb{N}}$. Such a \mathbb{P}_{ν} will be called a Bernoulli measure, which is said to be symmetric if $a_i = \frac{1}{k}$ for every $i \in \{1, \cdots, k\}$, in which case we denote it by \mathbb{P}_k . If Y is a Lie group, a natural symmetric random walk is given by $\nu^{\mathbb{N}}$ where ν is the Haar measure. We denote by $\mathscr{P}(Y^{\mathbb{N}})$ the space of Borel probability measures on $Y^{\mathbb{N}}$ and by $\mathscr{P}_B(Y^{\mathbb{N}})$ its subset of Bernoulli elements. It will be clear later on that the role of each random walk is to point out a particular complex feature of the dynamics, here defined in terms of either the topological entropy (definition in Subsection 2.4) or the metric mean dimension (definition in Subsection 2.6).
- 2.4. Topological entropy of an action \mathbb{S} . Given $\varepsilon > 0$ and $\underline{g} := g_{\omega_n} \dots g_{\omega_2} g_{\omega_1} \in G_n$, the *n*th-dynamical ball $B_n(x, g, \varepsilon)$ is the set

$$B_n(x,\underline{g},\varepsilon) := \left\{ z \in X : d(\underline{g}_j(z),\underline{g}_j(x)) \leqslant \varepsilon, \ \forall \ 0 \leqslant j \leqslant n \right\}$$

where, for every $0 \le j \le n$, the notation \underline{g}_j stands for the concatenation $g_{\omega_j} \dots g_{\omega_2} g_{\omega_1}$ in G_j , and $\underline{g}_0 = id$. Observe that this is a classical ball with respect to the dynamical metric $d_{\underline{g}}$ defined by

$$d_{\underline{g}}(x,z) := \max_{0 \leq j \leq n} d(\underline{g}_{j}(x), \underline{g}_{j}(z)). \tag{2.2}$$

Notice also that both the dynamical ball and the dynamical metric depend on the underlying concatenation of generators $g_{\omega_n} \dots g_{\omega_1}$ and not on the semigroup element g, since the latter may have distinct representations.

Given $\underline{g} = g_{\omega_n} \dots g_{\omega_1} \in G_n$, we say that a set $K \subset X$ is $(\underline{g}, n, \varepsilon)$ -separated if $d_{\underline{g}}(x, z) > \varepsilon$ for any two distinct elements $x, z \in K$. The largest cardinality of any $(\underline{g}, n, \varepsilon)$ -separated subset on X is denoted by $s(\underline{g}, n, \varepsilon)$ (or, equivalently, $s(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon)$). A set $K \subset X$ is said to be $(\underline{g}, n, \varepsilon)$ -spanning if for every $x \in X$ there is $k \in K$ such that $d_{\underline{g}}(x, k) \leqslant \varepsilon$. The smallest cardinality of any $(\underline{g}, n, \varepsilon)$ -spanning subset on X is denoted by $b(\underline{g}, n, \varepsilon)$ (or $b(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon)$).

Definition 2.2. The topological entropy of the semigroup action \mathbb{S} with respect to a fixed set of generators G_1 and a random walk \mathbb{P} in $Y^{\mathbb{N}}$ is given by

$$h_{\mathrm{top}}(\mathbb{S}, \mathbb{P}) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^{\mathbb{N}}} s(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon) d \, \mathbb{P}(\omega)$$

where $\omega = \omega_1 \omega_2 \cdots \omega_n \cdots$. The topological entropy of the semigroup action \mathbb{S} is then defined by

$$h_{\text{top}}(\mathbb{S}) = \sup_{\mathbb{P}} h_{\text{top}}(\mathbb{S}, \mathbb{P}).$$

We observe that the semigroup may have multiple generating sets, and the dynamical or ergodic properties (as the topological entropy) depend on the chosen generator set. More information regarding these concepts in the case of finitely generated free semigroup actions may be read in [5, 6, 7].

2.5. Entropy function. Let (X, d) be a compact metric space. For each $\varepsilon > 0$ and $x \in X$, define

$$h_d(x,\varepsilon) = \inf\{B(K,\mathbb{S},\varepsilon) : K \text{ is compact neighbourhood of } x\},\$$

where

$$B(K, \mathbb{S}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \left(\int_{\Sigma_p^+} b(K, g_{\omega_n} \dots g_{\omega_1}, \varepsilon) \ d\mathbb{P}(\omega) \right),$$

and $b(K, g_{\omega_n} \dots g_{\omega_1}, \varepsilon)$ denotes the minimum cardinality of a $(g_{\omega_n} \dots g_{\omega_1}, \varepsilon)$ -spanning set. As $h_d(x, \varepsilon)$ increases as ε decreases to zero, it is well defined the following

$$h_d(x) = \lim_{\varepsilon \to 0^+} h_d(x, \varepsilon) \tag{2.3}$$

and it is less or equal to $h_{top}(X, \mathbb{S})$. In fact, it depends only on the topology of X and we can denote by $h_{top}(x)$.

Definition 2.3. Let $\mathbb{S}: G \times X \to X$ be a continuous finitely generated free semigroup action. The function $h_{top}: X \to [0, h_{top}(X, \mathbb{S})], x \mapsto h_{top}(x)$ is called the entropy function of \mathbb{S} .

Since $B(K, \mathbb{S}, \varepsilon) \leq S(K, \mathbb{S}, \varepsilon) \leq B(K, \mathbb{S}, \varepsilon/2)$, we have

$$h_{top}(x) = \lim_{\varepsilon \to 0} \inf \{ S_d(K, \mathbb{S}, \varepsilon) : K \text{ is compact neighbourhood of } x \}.$$

By [21, Theorem C] we have that

$$\sup_{x \in X} \lim_{\varepsilon \to 0} h_d(x, \varepsilon) = h_{top}(\mathbb{S}, \mathbb{P}).$$

2.6. Metric mean dimension of a semigroup action. Let (X,d) be a compact metric space and \mathbb{S} be the free semigroup action induced on (X,d) by a family of continuous maps $(g_y \colon X \to X)_{y \in Y}$. The following definition for the semigroup setting was introduced in [9].

Definition 2.4. The upper and lower metric mean dimension of the free semigroup action \mathbb{S} on (X, d) with respect to a fixed set of generators G_1 and a random walk \mathbb{P} in $Y^{\mathbb{N}}$ are given respectively by

$$\overline{\mathrm{mdim}}_{M} \left(X, \mathbb{S}, d, \mathbb{P} \right) = \lim_{\varepsilon \to 0^{+}} \frac{h(X, \mathbb{S}, \mathbb{P}, \varepsilon)}{-\log \varepsilon} \\
\underline{\mathrm{mdim}}_{M} \left(X, \mathbb{S}, d, \mathbb{P} \right) = \lim_{\varepsilon \to 0^{+}} \frac{h(X, \mathbb{S}, \mathbb{P}, \varepsilon)}{-\log \varepsilon}$$

where

$$h(X, \mathbb{S}, \mathbb{P}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^{\mathbb{N}}} s(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon) \ d\mathbb{P}(\omega). \tag{2.4}$$

Our first result shows that the metric mean dimension of a semigroup action may be computed in terms of the entropy function.

Theorem A. Let (X,d) be a compact metric space and S be the free semigroup action induced on (X,d) by a family of continuous maps $(g_u: X \to X)_{u \in Y}$. Then

$$\overline{{\it mdim}}_{M}\left(X,\mathbb{S},d,\mathbb{P}\right) = \limsup_{\varepsilon \,\to\, 0^{+}} \frac{\sup_{x \in X} h_{d}(x,\varepsilon)}{-\log \varepsilon},$$

for every $\mathbb{P} \in \mathcal{M}(Y^{\mathbb{N}})$.

2.7. **Katok's entropy.** In [7] the authors considered an extension of the Katok's entropy when the dynamical systems under consideration is a free semigroup action.

Definition 2.5. Given probability measure \mathbb{P} on $Y^{\mathbb{N}}$ and a Borel probability measure ν on X, $\delta \in (0,1)$ and $\varepsilon > 0$, define

$$h_{\nu}^{K}(\mathbb{S}, \varepsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_{p}^{+}} s_{\nu}(g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) d\mathbb{P}(\omega)$$
 (2.5)

where $\omega = \omega_1 \, \omega_2 \cdots \omega_n \cdots$,

$$s_{\nu}(g_{\omega_n}\dots g_{\omega_1}, n, \varepsilon, \delta) = \inf_{\{E\subseteq X: \ \nu(E)>1-\delta\}} \ s(g_{\omega_n}\dots g_{\omega_1}, n, \varepsilon, E)$$

and $s(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, E)$ denotes the maximal cardinality of the $(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon)$ -separated subsets of E.

The entropy of the semigroup action $\mathbb S$ with respect to ν and $\mathbb P$ is defined by

$$h_{\nu}^{K}(\mathbb{S}, \mathbb{P}) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_{p}^{+}} s_{\nu}(g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) d\mathbb{P}(\omega)$$
 (2.6)

Observe that the previous limit is well defined due to the monotonicity of the function

$$(\varepsilon, \delta) \mapsto \frac{1}{n} \log \int_{\Sigma_p^+} s_{\nu}(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta) \ d\mathbb{P}(\omega)$$

on the unknowns ε and δ . Moreover, if the set of generators is $G_1 = \{Id, f\}$, we recover the notion proposed by Katok for a single dynamics f.

In [28] the authors proved that for a compact metric space (X,d) and a continuous map $f:X\to X$ holds the following variational principle for the metric mean dimension

$$\overline{\mathrm{mdim}}_{M}\left(X,f,d,\mathbb{P}\right) = \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} h_{\nu}^{K}(X,f,\varepsilon,\delta)}{-\log \varepsilon},$$

which, in the case where the dynamical systems is given by a free semigroup action, may be extend as:

Theorem B. Let (X,d) be a compact metric space and S be the free semigroup action induced on (X,d) by a family of continuous maps $(g_y: X \to X)_{y \in Y}$. Then

$$\overline{mdim}_{M}\left(X,\mathbb{S},d,\mathbb{P}\right) \geq \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} h_{\nu}^{K}(\mathbb{S},\mathbb{P},\varepsilon,\delta)}{-\log \varepsilon},$$

for every $\mathbb{P} \in \mathcal{M}(Y^{\mathbb{N}})$. If $\mathbb{P} = \gamma^{\mathbb{N}}$, with γ an homogeneous probability measure on Y, then

$$\overline{mdim}_{M}\left(X,\mathbb{S},d,\mathbb{P}\right) = \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} h_{\nu}^{K}(\mathbb{S},\mathbb{P},\varepsilon,\delta)}{-\log \varepsilon}.$$

2.8. Entropy of an open cover for a free semigroup action. Consider $\mathbb{P} \in \mathcal{M}(Y^{\mathbb{N}})$. Let $\mathcal{U} = \{U_1, \dots, U_k\}$ be a finite open cover of X. For each $\omega \in Y^{\mathbb{N}}$ and $n \in \mathbb{N}$ define

$$\mathcal{U}(\omega, n) = \left\{ U_{i_0} \cap (f_{\omega}^1)^{-1}(U_{i_1}) \cap \dots \cap (f_{\omega}^{n-1})^{-1}(U_{i_{n-1}}) : U_{i_i} \in \mathcal{U} \right\}.$$

Let $N_{\nu}(\mathcal{U}, w, n)$ is the minimal cardinal of a subcover of $\mathcal{U}(w, n)$. Finally, define

$$h_{top}(\mathcal{U}, \mathbb{S}, \mathbb{P}) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^{\mathbb{N}}} N(\mathcal{U}, \omega, n) \ d\mathbb{P}(\omega).$$

As a consequence of [27, Theorem 2.4] we have that

$$h_{top}(Y^{\mathbb{N}} \times X, \mathbb{S}, \mathbb{P}) = \sup_{\mathcal{U}} h_{top}(\mathcal{U}, \mathbb{S}, \mathbb{P}),$$

where the open covers under consideration in the above supremum are those which are finite and with finite topological entropy.

2.9. Shapira's entropy of a semigroup action. For $\nu \in \mathcal{M}(X)$, for $\delta \in (0,1)$ let $N_{\nu}(\mathcal{U}, w, n, \delta)$ the minimal cardinal of a subcover of $\mathcal{U}(w, n)$, up to a set of ν -measure less than $\delta > 0$. Define

$$h_{\nu}^{\mathcal{S}}(\mathcal{U}, \mathbb{S}, \mathbb{P}) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^{\mathbb{N}}} N_{\nu}(\mathcal{U}, \omega, n, \delta) \ d\mathbb{P}(\omega). \tag{2.7}$$

We call $h_{\nu}(\mathcal{U}, \mathbb{S}, \mathbb{P})$ the metric entropy of the cover \mathcal{U} with respect to ν . As

$$N_{\nu}(\mathcal{U}, \omega, n, \delta) \leq N(\mathcal{U}, \omega, n)$$
 for every $\delta \in (0, 1)$,

we have that $h_{\nu}^{\mathcal{S}}(\mathcal{U}, \mathbb{S}, \mathbb{P}) \leq h_{top}(\mathcal{U}, \mathbb{S}, \mathbb{P})$. It is important to mention that when $G_1 = \{f\}$, our definition coincides with the classical one given in [22].

Before we state our theorem we need to introduce some notation. Associated to an open cover \mathcal{U} of X, let $\tilde{\mathcal{U}} = \{[i] \times V : i = 1, \dots, p \text{ and } V \in \mathcal{U}\}$ and, for $\nu \in \mathcal{M}(X)$, denote $\Pi(\sigma, \nu)_{erg}$ the set of T_G -invariant measures so that the marginal in Σ_p^+ is σ -invariant and ν is the marginal in X.

Theorem C. Let (X,d) be a compact metric space and S be the free semigroup action induced on (X,d) by a finite family of continuous maps $(g_i: X \to X)_{i=1}^p$. Under the above conditions we have that

(a)
$$h_{top}(\mathcal{U}, \mathbb{S}, \eta_p) = h_{top}(\tilde{\mathcal{U}}, T_G) - \log p;$$

(b)
$$h_{top}(\mathcal{U}, \mathbb{S}, \overline{\eta_p}) = \sup \left\{ h_{\nu}^S(\mathcal{U}, \mathbb{S}, \eta_p) : \nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{erg} \neq \emptyset \right\},$$

where $\eta_{\underline{p}} = \left(\frac{1}{p}, \dots, \frac{1}{p}\right)^{\mathbb{N}}.$

As a direct consequence of Theorem C and [7] we have the following.

Corollary 1. Let (X,d) be a compact metric space and S be the free semigroup action induced on (X,d) by a finite family of continuous maps $(g_i \colon X \to X)_{i=1}^p$. Then

$$h_{top}(\mathbb{S}, \eta_{\underline{p}}) = \sup_{\mathcal{U}} \sup_{\{\nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} h_{\nu}^{S}(\mathcal{U}, \mathbb{S}, \eta_{\underline{p}}))$$
$$= h_{top}(F_{G}) - \log p,$$

where $\eta_{\underline{p}} = \left(\frac{1}{p}, \dots, \frac{1}{p}\right)^{\mathbb{N}}$.

In [23] it was proved that, for a compact metric space (X, d) and a continuous map $f: X \to X$,

$$\overline{\mathrm{mdim}}_{M}\left(X,f,d\right) = \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} \inf_{\mathrm{diam}(\mathcal{U}) \leq \varepsilon} h_{\nu}^{S}(\mathcal{U},f)}{-\log \varepsilon}.$$

In the next theorem we extend such result to the semigroup setting.

Theorem D. Let (X,d) be a compact metric space and \mathbb{S} be the free semigroup action induced on (X,d) by a family of continuous maps $(g_y\colon X\to X)_{y\in Y}$. If $\mathbb{P}=\gamma^{\mathbb{N}}$ and $\gamma\in\mathcal{M}(Y)$ is homogeneous, then

$$\overline{mdim}_{M}\left(X,\mathbb{S},d,\mathbb{P}\right) = \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X): \Pi(\sigma,\nu)_{erg} \neq \emptyset\}} \inf_{diam(\mathcal{U}) \leq \varepsilon} h_{\nu}^{S}(\mathbb{S},\mathcal{U})}{-\log \varepsilon}.$$

2.10. **Ghys-Langevan-Walczack entropy.** Ghys, Langevin and Walczak proposed in [14] the following definition of topological entropy of a semigroup action given by a finitely generated . A subset E of a compact metric space (X, d_X) is (n, ε) -separated points by elements of $\mathbb S$ if for any $x \neq y$ in E there exists $0 \leq j \leq n$ and $g \in G_j$ such that $d(g(x), g(y)) > \varepsilon$. The topological entropy of the semigroup action $\mathbb S$, induced by a semigroup G generated by a finite set G_1 of continuous maps, is given by

$$h_{GLW}(\mathbb{S}) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon)$$
 (2.8)

where $s(n,\varepsilon)$ is the largest cardinality of (n,ε) -separated points by elements of \mathbb{S} . Observe that, since X is compact, $s(n,\varepsilon)$ is finite for every $n\in\mathbb{N}$ and $\varepsilon>0$. Moreover, the map

$$\varepsilon > 0 \quad \mapsto \quad h_{GLW}(\mathbb{S}, \varepsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon)$$

is monotonic, so $h_{GLW}(\mathbb{S})$ is well defined (though it depends on the set G_1 of generators). This is a purely topological notion, independent of any previously fixed random walk on the semigroup. Observe also that

$$\sup_{g \in G_1} h_{\text{top}}(g) \leqslant h_{GLW}(\mathbb{S})$$

but this inequality may be strict (cf. [14]).

2.10.1. Metric mean dimension in the GLW setting. As a natural extension of the metric mean dimension for a single dynamics we can consider the upper GLW-metric mean dimension as

$$\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(\mathbb{S},\varepsilon)}{-\log \varepsilon}.$$
 (2.9)

As a direct consequence of the above definition we have that for any $\mathbb{P} \in \mathcal{M}(Y)$,

$$\overline{\mathrm{mdim}}_{M}\left(X,\mathbb{S},\mathbb{P},d\right) \leq \overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right),$$

and in the case where the generating setting consists of a single dynamics the two definitions coincide with the classical one.

2.10.2. Local measure entropy and measure metric mean dimension. For $n \in \mathbb{N}$ let

$$B_n^G(x,\varepsilon) = \{ y \in X : d(g(x),g(y)) < \varepsilon \text{ for all } g \in G_j, \ 0 \le j \le n \}$$

the dynamical ball of center x, radius ε and depth n. For any $\nu \in \mathcal{M}(X)$ the quantity

$$h_{\nu}^{G}(x) = \lim_{\varepsilon \to 0} h_{\nu}^{G}(x, \varepsilon)$$

where

$$h_{\nu}^G(x,\varepsilon) = \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(x,\varepsilon))$$

is called the local upper ν -measure entropy at the point x. If one takes \liminf with respect to n in the above definition we the local lower ν -measure entropy at the point x, denoted by $h_{\nu,G}(x)$. These quantities were defined and explored in [3], where the author proved that in the case of ν being a G-homogeneous measure $h_{\nu}^{G}(x) = h_{GLW}(\mathbb{S})$, for all $x \in X$ (see Section 3 for the definition of G-homogeneous measure).

In order to have a concept related to the metric mean dimension we define the local upper measure metric mean dimension as

$$\overline{\mathrm{mdim}}_{\nu}\left(X,\mathbb{S},d\right) = \limsup_{\varepsilon \to 0} \frac{h_{\nu}^{G}(x,\varepsilon)}{-\log \varepsilon}.$$
(2.10)

If one takes \liminf in ε we have the lower local upper measure metric mean dimension, denoted by $\underline{\mathrm{mdim}}_{\nu}\left(X,\mathbb{S},d\right)$.

If, instead of $h_{\nu}^{G}(x,\varepsilon)$ we consider $h_{\nu,G}(x,\varepsilon)$ we have the upper local lower measure metric mean dimension and lower local lower measure metric mean dimension, denoted by $\overline{\operatorname{mdim}}_{\nu}'\left(X,\mathbb{S},d\right)$ and $\underline{\operatorname{mdim}}_{\nu}'\left(X,\mathbb{S},d\right)$, respectively.

Remark 2.6. All the above definitions could be made in terms of dynamical balls.

In the case the where the ambient space X is an oriented manifold it admits a volume form dV which induces a natural volume measure ν_v on the Borel sets defined as

$$\nu_v(A) = \int_A dV.$$

The next gives a kind of partial variational principle for the metric mean dimension of the group action in terms of the volume measure.

Theorem E. Let (G, G_1) be a finitely generated group of homeomorphisms of a compact closed and oriented manifold (M, d). Let $s \in (0, \infty)$ and ν_v the natural volume on M. If

$$\overline{mdim}_{\nu_{v}}\left(x,d\right)\leq s \text{ for all } x\in M \text{ then } \overline{mdim}_{M}^{GLW}\left(X,\mathbb{S},d\right)\leq s.$$

Our last theorem shows that, in the case where the group action admits a strongly G-homogeneous measure ν we have an equality between the local measure metric measure mean dimension of ν and the metric mean dimension of the group action (see Section 3 for the definition of strongly G-homogeneous measure).

Theorem F. Let (X,d) be a compact metric space and \mathbb{S} be the semigroup action induced on (X,d) by a finite family of continuous maps $(g_i: X \to X)_{i=1}^p$.

(a) If $\nu \in \mathcal{M}(X)$ is strongly G-homogeneous then

$$\overline{{\operatorname{mdim}}}_{M}^{GLW}\left(X,\mathbb{S},d\right) = \limsup_{\varepsilon \to 0} \frac{h_{\nu}^{G}(x,\varepsilon)}{-\log \varepsilon}.$$

(b) Let ν be a Borel measure on X and $s \in (0, \infty)$. If

$$\inf_{x\in X}\overline{{mdim}}_{\nu}^{\prime}\left(x,d\right)\geq s\quad then\quad \overline{{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right)\geq s.$$

3. Some facts about homogeneous measures and G-homogeneous measures

In order to obtain a text as self-contained as possible, in this section we recall the definitions of upper box dimension, homogeneous measure and G-homogeneous measure.

3.1. Upper box dimension. Let (Y, d_Y) be a compact metric space.

Definition 3.1. The upper box dimension of (Y, d_Y) is given by

$$\overline{\dim}_B Y = \limsup_{\varepsilon \to 0^+} \frac{\log N(\varepsilon)}{|\log \varepsilon|},\tag{3.1}$$

where $N(\varepsilon)$ stands for the maximal cardinality of an ε -separated set in (Y, d_Y) .

Consider now a Borel probability measure ν on Y.

Definition 3.2. The upper box dimension of ν is given by

$$\overline{\dim}_B \nu = \lim_{\delta \to 0^+} \inf \Big\{ \overline{\dim}_B Z \colon \ Z \subset Y \quad \text{and} \quad \nu(Z) \geqslant 1 - \delta \Big\}.$$

It is worth mentioning that, although the upper box dimension of a set Z coincides with the upper box dimension of its closure, the upper box dimension of a probability measure is intended to estimate the size of subsets rather than the entire support of the measure (that is, the smallest closed subset with full measure). Indeed, it may happen that $\overline{\dim}_B \nu < \overline{\dim}_B (\operatorname{supp} \nu)$ (cf. Example 7.1 in [26]). We refer the reader to [11, 26] for excellent accounts on dimension theory.

3.2. Homogeneous measures. Let ν be a Borel probability measure on the compact metric space (Y, d_Y) . A balanced measure should give the same probability to any two balls with the same radius, but this is in general a too strong demanding. Instead, we weaken the request in the following way.

Definition 3.3. We say that ν is homogeneous if there exists L>0 such that

$$\nu(B(y_1, 2\varepsilon)) \leqslant L \nu(B(y_2, \varepsilon)) \qquad \forall y_1, y_2 \in \text{supp } \nu \quad \forall \varepsilon > 0.$$
 (3.2)

For instance, the Lebesgue measure on [0,1], atomic measures and probability measures absolutely continuous with respect to the latter ones, with densities bounded away from zero and infinity, are examples of homogeneous probability measures. We denote by \mathcal{H}_Y the set of such homogeneous Borel probability measures on Y.

By definition, every homogeneous measure satisfies

$$\nu(B(y, 2\varepsilon)) \leqslant L\nu(B(y, \varepsilon)) \qquad \forall y \in \text{supp } \nu \quad \forall \varepsilon > 0$$
 (3.3)

and, as $\nu(B(y_1,\varepsilon)) \leq \nu(B(y_1,2\varepsilon))$,

$$\nu(B(y_1,\varepsilon)) \leqslant L\nu(B(y_2,\varepsilon)) \qquad \forall y_1, y_2 \in \text{supp } \nu \quad \forall \varepsilon > 0.$$
 (3.4)

A measure ν satisfying (3.3) is said to be a doubling measure. Although the two concepts (3.3) and (3.4) are unrelated in general, if Y is a subset of an Euclidean space \mathbb{R}^k then any probability ν satisfying (3.4) is a doubling measure. Indeed, as there is a constant C_k such that $Leb(B(y,r)) = C_k r^k$ for every $y \in Y$ and every r > 0, any ball $B(y, 2\varepsilon)$ can be covered by at most 2^k balls of radius ε ; we now apply (3.2). For a discussion on conditions on Y which ensure the existence of homogeneous measures and further relations between homogeneity and the doubling property we refer the reader to [3, Section 4] and references therein.

- 3.3. G-homogeneous measures. For a compactly generated semigroup by a continuous family $(g_y: X \to X)_{y \in Y}$ acting on a metric space, we say that a Borel measure $\nu \in \mathcal{M}(X)$ is G-homogeneous if
 - (a) $\nu(K) < \infty$, for any compact set $K \subset X$;
 - (b) there exists $K_0 \subset X$ such that $\nu(K_0) > 0$;
 - (c) for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and c > 0 such that

$$\nu(B_n^G(x,\delta(\varepsilon))) \le c \cdot \nu(B_n^G(y,\varepsilon))$$

holds for any $n \in \mathbb{N}$ and all $x, y \in X$. In the case where $\delta(\varepsilon) = O(\varepsilon)$ we say that ν is strongly G-homogeneous.

As examples of spaces which admit a strongly G-homogeneous measure we have the following:

- 1. The canonical volume form dV on a closed, compact and oriented Riemannian manifold X determines a strongly G-homogeneous measure ν if G is a finitely generated group of isometries.
- 2. If X is a locally compact topological group, μ is a right invariant measure and G is a finitely generated group by $G_1 = \{id_X, T_1, T_1^{-1}, T_2, T_2^{-1}, \dots, T_p, T_p^{-1}\}$, a finite and symmetric set of homeomorphisms, then μ is strongly G-homogeneous (see [3, Proposition 4.6]).

4. Proofs

In this section we prove our main results.

4.1. **Proof of Theorem A.** It is clear from the definition of the entropy function that $h_d(x,\varepsilon) \leq h(X,\mathbb{S},\mathbb{P},\varepsilon)$, for all $x \in X$, and it implies that

$$\overline{\mathrm{mdim}}_{M}\left(X,\mathbb{S},d,\mathbb{P}\right) \geq \limsup_{\varepsilon \to 0} \frac{\sup_{x \in X} h_{d}(x,\varepsilon)}{-\log \varepsilon}.$$

To prove the converse inequality we start noticing that, for a fixed $\varepsilon>0$, if $X=\bigcup_{i=1}^k F_i$, finite union of closed sets, then $B(X,\mathbb{S},\varepsilon,\mathbb{P})\leq \max_i B(F_i,\mathbb{S},\varepsilon,\mathbb{P})$. Then cover X by closed balls of radius 1, say $\mathcal{B}_1=\{B_1^1,\ldots,B_{\ell_1}^1\}$ such cover. Let $B_{j_1}^1$ be the closed ball in the given cover where the maximum occurs. Now cover $B_{j_1}^1$ by a finite family of closed balls of radius at most $\frac{1}{2}$ denoted by $\{B_1^2,\ldots,B_{\ell_2}^2\}$. Again there exists $B_{j_2}^2\in\mathcal{B}_2$ for which $B(X,\mathbb{S},\varepsilon,\mathbb{P})\leq B(B_{j_2}^2,\mathbb{S},\varepsilon,\mathbb{P})$. Following by induction, for each $k\in\mathbb{N}$, there exists a closed ball of radius at most $\frac{1}{k}$ so that $B(X,\mathbb{S},\varepsilon,\mathbb{P})\leq B(B_{j_k}^k,\mathbb{S},\varepsilon,\mathbb{P})$. Moreover, by the previous construction we have a sequence of nested closed balls $\{B_{j_k}^k\}_{k\in\mathbb{N}}$ whose diameter goes to zero. So, there exists $\bar{x}=\cap_{k\in\mathbb{N}}B_{j_k}^k$ and for any closed neighbourhood F of \bar{x} we have $B_{j_k}^x\subset F$, for $k\in\mathbb{N}$ large enough. It gives

$$h_d(\bar{x}, \varepsilon) \geq B(F, \mathbb{S}, \varepsilon, \mathbb{P}) \geq B(B_{i_k}^k, \mathbb{S}, \varepsilon, \mathbb{P}) \geq B(X, \mathbb{S}, \varepsilon, \mathbb{P}).$$

Hence

$$\limsup_{\varepsilon \to 0^+} \frac{\sup_{x \in X} h_d(x,\varepsilon)}{-\log \varepsilon} \ge \overline{\mathrm{mdim}}_M \left(X,\mathbb{S},d,\mathbb{P}\right)$$

and it finishes the proof.

4.2. **Proof of Theorem B.** First we notice that for any $\nu \in \mathcal{M}(X)$ and $\delta > 0$, $\nu(X) > 1 - \delta$ and so, for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $\omega \in Y^{\mathbb{N}}$

$$s(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon) \ge s_{\nu}(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta).$$

It implies that, for any $\nu \in \mathcal{M}(X)$

$$h(X, \mathbb{S}, \mathbb{P}, \varepsilon) \ge h_{\nu}^{K}(X, \mathbb{S}, \mathbb{P}, \varepsilon, \delta).$$

Hence,

$$\overline{\mathrm{mdim}}_{M}\left(X, \mathbb{S}, d, \mathbb{P}\right) \ge \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} h_{\nu}^{K}(\mathbb{S}, \mathbb{P}, \varepsilon, \delta)}{-\log \varepsilon}.$$
(4.1)

If $\mathbb{P} = \gamma^{\mathbb{N}}$ with $\gamma \in \mathcal{H}_Y$, by (4.3) we know that for $\nu \in \mathcal{M}(X)$ and $\mu \in \Pi(\sigma, \nu) \neq \emptyset$,

$$h_{\nu}^{K}(X, \mathbb{S}, \mathbb{P}, \varepsilon, \delta) \ge \sup_{\mu \in \Pi(\sigma, \nu)} h_{\mu}^{K}(Y^{\mathbb{N}} \times X, T_{G}, \varepsilon, \delta) - \log N_{Z}(\varepsilon).$$

It follows that

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\nu \in \mathcal{M}(X)} h_{\nu}^{K}(\mathbb{S}, \mathbb{P}, \varepsilon, \delta)}{-\log \varepsilon} \ge \lim_{\delta \to 0} \limsup_{\varepsilon \to 0^{+}} \frac{\sup_{\mu \in \mathcal{M}_{T_{G}}(Y^{\mathbb{N}} \times X)} h_{\nu}^{K}(T_{G}, \varepsilon, \delta)}{-\log \varepsilon} - \overline{\dim}_{B}(\operatorname{supp}(\gamma))$$

$$= \overline{\operatorname{mdim}}_{M} \left(Y^{\mathbb{N}} \times X, T_{G}, D \times d \right) - \overline{\dim}_{B}(\operatorname{supp}(\gamma))$$

$$= \overline{\operatorname{mdim}}_{M} \left(X, \mathbb{S}, d, \mathbb{P} \right).$$

By (4.1) we have the desired equality and conclude the proof.

4.3. Proof of Theorem C. Take $i_0, \ldots, i_{n-1} \in \{1, \ldots, p\}, U_{j_0}, \ldots, U_{j_{n-1}} \in \mathcal{U}$ and consider

$$([i_0] \times U_{j_0}) \cap (T_G^{-1}([i_1] \times U_{j_1}) \cap \cdots \cap T_G^{-1}([i_{n-1}] \times U_{j_{n-1}})$$

= $[i_0 \dots i_{n-1}] \times (U_{j_0} \cap \cdots \cap (f_{\omega}^{n-1})^{-1}(U_{j_{n-1}})),$

where ω belongs to the cylinder set $[i_0 \dots i_{n-1}]$. If we denote by $\mathcal{U}(\omega, n) = \{V_{j_0} \cap \dots \cap (f_{\omega}^{n-1})^{-1}(V_{j_{n-1}}) : V_{j_{\ell}} \in \mathcal{U}\}$ the open cover of X induced by ω , we have that $N(\mathcal{U}, \omega, n)$ coincides with the minimum number of open sets of $\tilde{\mathcal{U}}^{(n)}$ necessary to cover $[i_0 \dots i_{n-1}] \times X$. So,

$$h_{top}(\mathcal{U}, \mathbb{S}, \eta_{\underline{p}}) + \log p = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{\underline{g} \in G_n} N(\mathcal{U}, \underline{g}, n) \right) + \log p$$
$$= \lim_{n \to \infty} \frac{1}{n} \log N(\tilde{\mathcal{U}}, T_G, n)$$
$$= h_{top}(\tilde{\mathcal{U}}, T_G).$$

It proves item (i).

For the second item take $\delta \in (0,1)$ and $\nu \in \mathcal{M}(X)$ so that $\Pi(\sigma,\nu)_{erg} \neq \emptyset$. For $\mu \in \Pi(\sigma,\nu)_{erg}$ we have that

$$\sum_{g \in G_n} N_{\nu}(\mathcal{U}, \underline{g}, n, \delta) = N_{\mu}(\mathcal{U}, T_G, n, \delta).$$

The equality comes from the fact that if

$$\sum_{\underline{g}\in G_n} N_{\nu}(\mathcal{U}, \underline{g}, n, \delta) > N_{\mu}(\mathcal{U}, T_G, n, \delta),$$

there exists a cylinder $[i_0 \dots i_{n-1}]$ so that $[i_0 \dots i_{n-1}] \times X$ is covered by at most $N_{\nu}(\mathcal{U}, \underline{g}, n, \delta) - 1$ open sets, where $w = i_0 \dots i_{n-1}$. As $(\pi_X)_*(\mu) = \nu$, it contradicts the minimality of $N_{\nu}(\mathcal{U}, g, n, \delta)$. So,

$$\sup_{\{\nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma,\nu)_{erg} \neq \emptyset\}} h_{\nu}^{S}(\mathcal{U},\mathbb{S})$$

$$= \sup_{\{\nu \in \mathcal{M}(X) \text{ and } \Pi(\sigma,\nu)_{erg} \neq \emptyset\}} \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{1}{p^{n}} \sum_{\underline{g} \in G_{n}} N_{\nu}(\mathcal{U},\underline{g},n) \right)$$

$$= \sup_{\mu \in \mathcal{E}_{T_{G}}(\Sigma_{p}^{+} \times X)} \lim_{n \to \infty} \frac{1}{n} \log N_{\mu}(\tilde{\mathcal{U}}, T_{G}, n) - \log p$$

$$= h_{top}(\tilde{\mathcal{U}}, T_{G}) - \log p$$

$$= h_{top}(\mathcal{U}, \mathbb{S}, \eta_{p}),$$

which concludes the proof of the second item.

4.4. **Proof of Theorem D.** Before we start the proof we observe that Definition 2.5 could be made in terms of spanning sets. More precisely, given $\varepsilon > 0$, a positive integer n and $\underline{g} = g_{\omega_n} \dots g_{\omega_1}$, we say that a subset A of $E \subset X$ is a $(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, E)$ —spanning set if for any $x \in E$ there exists $y \in A$ so that $D_{\underline{g}}(x,y) < \varepsilon$. By the compactness of X, given ε , n and \underline{g} as before, there exists a finite (g, n, ε, E) —spanning set.

We denote by $b(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, E)$ the minimum cardinality of a $(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, E)$ -spanning. For $\delta > 0$ we set

$$b_{\nu}(g_{\omega_n}\dots g_{\omega_1}, n, \varepsilon, \delta) = \inf_{\{E \subset X : \nu(E) > 1 - \delta\}} b(g_{\omega_n}\dots g_{\omega_1}, n, \varepsilon, E).$$

It is not difficult to see that

$$h_{\nu}^{K}(\mathbb{S}, \mathbb{P}) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_{p}^{+}} b_{\nu}(g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) d\mathbb{P}(\omega).$$

Let us proceed to the proof of the theorem. Fix $\varepsilon > 0$ and consider a positive integer $k = k(\varepsilon) \ge 1$ so that $\sum_{i \ge k} \frac{diam(Y)}{2^i} < \frac{\varepsilon}{2}$. For $\gamma \in \mathcal{H}_Y$, take $Z = \operatorname{supp}(\gamma)$ and choose a maximal $\frac{\varepsilon}{4}$ -separated set $E \subset Z$, whose cardinality is denoted by $N_Z(\varepsilon)$. By the definition of upper box dimension,

$$\limsup_{\varepsilon \to 0} \frac{N_Z(\varepsilon)}{-\log \varepsilon} = \overline{\dim}_B(Z).$$

For each $n \in \mathbb{N}$ and each point $(p_1, \dots, p_{n+k}) \in E^{n+k}$, consider the cylinder

$$C_{i_1...i_{n+k}} = \left\{ \omega \in Y^{\mathbb{N}} : \omega_i \in B\left(p_i, \frac{\varepsilon}{4}\right), \text{ for } i = 1, \dots, n+k \right\}.$$

Note that the collection of cylinders defined above covers $Z^{\mathbb{N}}$ and has diameter less than ε .

Now, for the fixed ε let \mathcal{U}_0 be an open cover of X with $\operatorname{diam}(\mathcal{U}_0) \leq \varepsilon$ and $\operatorname{Leb}(\mathcal{U}_0) \geq \varepsilon$. If \mathcal{U} is an open cover of X with diameter less or equal to $\frac{\varepsilon}{8}$, $\omega \in Y^{\mathbb{N}}$, as $\operatorname{Leb}(\mathcal{U}_0) \geq \operatorname{diam}(\mathcal{U})$, $\mathcal{U}(\omega, n)$ refines $\mathcal{U}_0(\omega, n)$. It implies that, for $\delta \in (0, 1)$, $N_{\nu}(\mathbb{S}, \mathcal{U}, g_{\omega_n} \dots g_{\omega_1}, n, \delta) \geq N_{\nu}(\mathbb{S}, \mathcal{U}_0, g_{\omega_n} \dots g_{\omega_1}, n, \delta)$. Thus, once

$$N_{\nu}(\mathbb{S}, \mathcal{U}, g_{\omega_n} \dots g_{\omega_1}, n, \delta) \ge s_{\nu}(\mathbb{S}, g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta) \ge b_{\nu}(\mathbb{S}, g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta),$$

for all $\omega \in Y^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have that

$$\int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}(\omega) \ge \int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}_{0}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}$$

$$\ge \int_{Y^{\mathbb{N}}} s_{\nu}(\mathbb{S}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) \ d\mathbb{P}$$

$$\ge \int_{Y^{\mathbb{N}}} b_{\nu}(\mathbb{S}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) \ d\mathbb{P}$$

$$\ge \sum_{\underline{i} = (i_{1} \dots i_{n+k})} \min_{\omega \in C_{\underline{i}} \cap \mathbb{Z}^{\mathbb{N}}} b_{\nu}(\mathbb{S}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) \times \min_{\underline{i}} \mathbb{P}(C_{\underline{i}} \cap \mathbb{Z}).$$
(4.2)

Now we notice that the image of $b_{\nu}(\mathbb{S}, \cdot, n, \varepsilon, \delta) : C_{\underline{i}} \to \mathbb{Z}_+$ has a minimum in \mathbb{Z}_+ and such minimum is attained by some $\omega^{(\underline{i})} \in C_{\underline{i}}$. So, This together with (4.2), the fact that \mathbb{P} is a product measure and the homogeneity assumption on γ imply that

$$\int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) d\mathbb{P}(\omega)
\geq \int_{Y^{\mathbb{N}}} b_{\nu}(g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) d\mathbb{P}(\omega)
\geq \left[\sum_{\underline{i} = (i_{1}, i_{2}, \dots, i_{n+K})} \min_{\omega \in C_{\underline{i}} \cap Z^{\mathbb{N}}} b_{\nu}(g_{\omega_{n}} \dots g_{\omega_{1}}, n, \varepsilon, \delta) \right] \times \min_{\underline{i}} \mathbb{P}(C_{\underline{i}} \cap Z^{\mathbb{N}})
\geq \sum_{\underline{i}} b_{\nu}(g_{\omega(\underline{i})}, n, \varepsilon, \delta) \times \min_{\underline{i}} \prod_{j=0}^{n+K-1} \gamma \left(B(p_{i_{j}}, \frac{\varepsilon}{4}) \cap Z \right)
\geq b_{\mu}(T_{G} \mid_{Z^{\mathbb{N}} \times X}, n, \varepsilon, \delta) \left(\frac{1}{L^{2}} \right)^{n+K} \left(\frac{1}{N_{Z}(\varepsilon)} \right)^{n+K}
\geq N_{\mu}(T_{G} \mid_{Z^{\mathbb{N}} \times X}, \mathcal{V}_{0}, n, \delta) \left(\frac{1}{L^{2}} \right)^{n+K} \left(\frac{1}{N_{Z}(\varepsilon)} \right)^{n+K}$$

where by $g_{\omega(\underline{i})}$ we mean $g_{\omega_n^{(\underline{i})}} \dots g_{\omega_1^{(\underline{i})}}$ if $\omega^{(\underline{i})}|_{[1,n]} = \omega_1^{(\underline{i})} \dots \omega_n^{(\underline{i})}$ and $\mu \in \Pi(\sigma,\nu)$, \mathcal{V}_0 is an open cover with $Leb(\mathcal{V}_0) \leq \varepsilon$ and L>0 is specified by the homogeneity of γ and does not depend on neither ε nor n. Notice that the inequality

$$\sum_{\underline{i}} b_{\nu}(g_{\omega(\underline{i})}, n, \varepsilon, \delta) \geqslant b_{\mu}(T_G \mid_{Z^{\mathbb{N}} \times X}, n, \varepsilon, \delta)$$

is a consequence of the fact that, if $\{x_1^{(i)}, \ldots, x_{b(g_{\omega^{(i)}}, n, \varepsilon)}\}$ is a $(g_{\omega^{(i)}}, n, \varepsilon)$ -spanning set for a subset $\overline{Z} \subset Z$, satisfying $\nu(\overline{Z}) \geq 1 - \delta$, with smallest cardinality, then

$$\bigcup_{i} \left\{ \left(\omega^{(\underline{i})}, x_{1}^{(\underline{i})}\right), \dots, \left(\omega^{(\underline{i})}, x_{b(g_{\omega^{(\underline{i})}}, n, \varepsilon)}^{(\underline{i})}\right) \right\}$$

is a (T_G, n, ε) -spanning set for $Y^{\mathbb{N}} \times \overline{Z}$ and $\mu(Y^{\mathbb{N}} \times \overline{Z}) = \nu(\overline{Z}) \geq 1 - \delta$. Besides, the inequality

$$\min_{\underline{i}} \prod_{i=0}^{n+K-1} \gamma \left(B(p_{i_j}, \frac{\varepsilon}{4}) \right) \geqslant \left(\frac{1}{L^2} \right)^{n+K} \left(\frac{1}{N_Z(\varepsilon)} \right)^{n+K}$$

is due to the homogeneity of γ , which implies that, for every $q \in \text{supp } \nu$, any p_{i_j} and all i,

$$\gamma \Big(B(p_{i_j}, \varepsilon) \Big) \geqslant \frac{1}{L} \gamma \Big(B(q, \varepsilon) \Big) \qquad \forall \varepsilon > 0$$

and the fact that, as $\bigcup_{e \in E} B(e, \frac{\varepsilon}{4}) = Z$,

$$1 = \gamma \left(\bigcup_{e \in E} B(e, \frac{\varepsilon}{4}) \right) \leqslant \sum_{e \in E} \gamma \left(B(e, \frac{\varepsilon}{4}) \right) \leqslant N_Z(\varepsilon) L \gamma \left(B(q, \frac{\varepsilon}{4}) \right)$$

thus

$$\gamma\left(B(q, \frac{\varepsilon}{4})\right) \geqslant \frac{1}{L} \frac{1}{N_Z(\varepsilon)}.$$

Then we notice that, by (4.3)

$$\sup_{\{\nu \in \mathcal{M}: \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} h_{\nu}^{S}(\mathbb{S}, \varepsilon, \mathbb{P}) \geq \sup_{\mu \in \mathcal{E}(T_{G})} h_{\mu}^{S}(T_{G}, \mathcal{V}_{0}) - \log N_{Z}(\varepsilon)$$

$$= h_{\text{top}}(T_{G}, \mathcal{V}_{0}) - \log N_{Z}(\varepsilon)$$

$$\geq h(T_{G}, 3\varepsilon) - \log N_{Z}(\varepsilon).$$

Therefore,

$$\limsup_{\varepsilon \to 0} \frac{h^{S}(\mathbb{S}, \varepsilon, \mathbb{P})}{-\log \varepsilon} \geqslant \overline{\mathrm{mdim}}_{M} \left(Z^{\mathbb{N}} \times X, T_{G}, D \times d \right) - \limsup_{\varepsilon \to 0^{+}} \frac{\log N_{Z}(\varepsilon)}{-\log \varepsilon}$$

$$= \overline{\mathrm{mdim}}_{M} \left(Z^{\mathbb{N}} \times X, T_{G}, D \times d \right) - \overline{\dim}_{B} Z$$

$$= \overline{\mathrm{mdim}}_{M} \left((\mathrm{supp} \ \nu)^{\mathbb{N}} \times X, T_{G}, D \times d \right) - \overline{\dim}_{B} \left(\mathrm{supp} \ \nu \right)$$

$$= \overline{\mathrm{mdim}}_{M} \left(X, \mathbb{S}, d, \mathbb{P} \right).$$

$$(4.4)$$

For the converse inequality we observe that

$$\int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}_{0}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}(\omega) \leq \int_{Y^{\mathbb{N}}} s_{\nu}(\mathbb{S}, Leb(\mathcal{U}_{0}), g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}(\omega)$$

$$\leq \sum_{\underline{i} = (i_{1} \dots i_{n+k})} \left[\max_{\omega \in C_{\underline{i}} \cap Z^{\mathbb{N}}} s_{\nu}(\mathbb{S}, Leb(\mathcal{U}_{0}), g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \times \mathbb{P}(C_{\underline{i}}) \right].$$

Now we notice that the image of $s_{\nu}(\mathbb{S}, \cdot, n, Leb(\mathcal{U}_0)) : C_{\underline{i}} \to \mathbb{Z}_+$ is contained in $[0, s(T_G, n, Leb(\mathcal{U}_0)))$. So, it has a maximum in \mathbb{Z}_+ and such maximum is attained by some $\omega^{(\underline{i})} \in C_i$. So, using the fact that

$$N_{\nu}(\mathbb{S}, \mathcal{U}_0, g_{\omega_n} \dots g_{\omega_1}, n, \delta) \leq s_{\nu}(\mathbb{S}, g_{\omega_n} \dots g_{\omega_1}, n, Leb(\mathcal{U}_0), \delta)$$

and

$$\sum_{\underline{i}=(i_1...i_{n+k})} s_{\nu}(\mathbb{S}, g_{\omega_n} \dots g_{\omega_1}, n, Leb(\mathcal{U}_0), \delta) \leq s_{\mu}(T_G, n, Leb(\mathcal{U}_0))$$

and that γ is homegeneous we obtain

$$\begin{split} &\int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}_{0}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}(\omega) \\ &\leq \int_{Y^{\mathbb{N}}} s_{\nu}(\mathbb{S}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, Leb(\mathcal{U}_{0}), \delta) \ d\mathbb{P} \\ &\leq \sum_{\underline{i} = (i_{1} \dots i_{n+k})} \left[s_{\nu}(\mathbb{S}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, Leb(\mathcal{U}_{0}), \delta) \times \max_{\omega \in C_{\underline{i}} \cap Z^{\mathbb{N}}} P(C_{\underline{i}}) \right] \\ &\leq s_{\mu}(T_{G}, n, Leb(\mathcal{U}_{0})) \left(\frac{1}{N_{Z}(Leb(\mathcal{U}_{0}))} \right)^{n+K} \\ &\leq N_{\mu}(T_{G}, \mathcal{V}_{0}, n) \left(\frac{1}{N_{Z}(Leb(\mathcal{U}_{0}))} \right)^{n+K}, \end{split}$$

where \mathcal{V}_0 is a finite collection of open sets which covers $Y^{\mathbb{N}} \times X$ up to a set of μ -measure less than δ , $\frac{\varepsilon}{4} \leq Leb(\mathcal{U}_0) = \operatorname{diam}(\mathcal{V}_0) \leq \varepsilon$ and $Leb(\mathcal{V}_0) \geq \frac{\varepsilon}{8}$.

$$\begin{split} h^{S}(\mathbb{S}, \varepsilon, \mathbb{P}) &= \sup_{\{\nu \in \mathcal{M}(X): \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} \inf_{\operatorname{diam}(\mathcal{U}) \leq \varepsilon} h^{S}_{\nu}(\mathbb{S}, \mathcal{U}, \mathbb{P}) \\ &\leq \sup_{\{\nu \in \mathcal{M}(X): \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} h^{S}_{\nu}(\mathbb{S}, \mathcal{U}_{0}, \mathbb{P}) \\ &= \sup_{\{\nu \in \mathcal{M}(X): \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} \limsup_{n \to \infty} \frac{1}{n} \log \int_{Y^{\mathbb{N}}} N_{\nu}(\mathbb{S}, \mathcal{U}_{0}, g_{\omega_{n}} \dots g_{\omega_{1}}, n, \delta) \ d\mathbb{P}(\omega) \\ &\leq \sup_{\{\nu \in \mathcal{M}(X): \Pi(\sigma, \nu)_{erg} \neq \emptyset\}} \sup_{\mu \in \Pi(\sigma, \nu)} \limsup_{n \to \infty} \frac{1}{n} \log N_{\mu}(T_{G}, \mathcal{V}_{0}, n, \delta) - \log N_{Z}(Leb(\mathcal{U}_{0})) \\ &= h_{top}(T_{G}, \mathcal{V}_{0}) - \log N_{Z}\left(\frac{\varepsilon}{4}\right) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log s\left(T_{G}, n, Leb(\mathcal{V}_{0})\right) - \log N_{Z}\left(\frac{\varepsilon}{4}\right) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log s\left(T_{G}, n, \frac{\varepsilon}{8}\right) - \log N_{Z}\left(\frac{\varepsilon}{4}\right). \end{split}$$

Hence,

$$\limsup_{\varepsilon \to 0} \frac{h^{S}(\mathbb{S}, \varepsilon, \mathbb{P})}{-\log \varepsilon} \le \limsup_{\varepsilon \to 0} \left[\frac{h\left(T_{G}, \frac{\varepsilon}{8}\right)}{-\log \varepsilon} - \frac{\log N_{Z}\left(\frac{\varepsilon}{4}\right)}{-\log \varepsilon} \right] \\
= \overline{\mathrm{mdim}}_{M} \left(Y^{\mathbb{Y}} \times X, T_{G}, D \times d \right) - \overline{\dim}_{B}(Z) \\
= \overline{\mathrm{mdim}}_{M} \left(X, \mathbb{S}, d, \mathbb{P} \right). \tag{4.5}$$

By (4.4) and (4.5) we obtain the result.

4.5. **Proof of Theorem E.** let ν_v be the natural volume measure on X and assume that $\underline{\mathrm{mdim}}_{\nu_v}(x,d) \geq s$, for all $x \in X$. Fix $\eta > 0$ and let

$$X_k = \left\{ x \in X : \frac{\limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(x,\varepsilon))}{-\log \varepsilon} > (s - \delta/2) \text{ for all } \varepsilon \in (0, \frac{1}{k}) \right\}.$$

By hypotheses, $X = \bigcup_{k \in \mathbb{N}} X_k$. For $\varepsilon \in (0, \frac{1}{5 \cdot k}]$ and $x \in X_k$, there exists $n(x) \in \mathbb{N}$ so that for any $N \ge n(x)$ we have

$$\nu_v(B_n^G(x,\varepsilon)) \ge e^{-(s+\delta)N \cdot -\log \varepsilon}$$

Since X is a compact Riemannian manifold it has bounded geometry (see [10] for more details on manifolds of bounded geometry). It implies that each function $f_m: X_k \to \mathbb{R}$ given by $f_m(x) := \nu_v(B_m^G(x, \varepsilon))$ is continuous and so

$$N_0 := \sup\{n(x) : x \in X_k\} < \infty.$$

By Vitali Covering Lemma, for any $N \geq N_0$ it is possible to choose from the cover $\mathcal{B}_N := \{\overline{B_N^G(x,\varepsilon)} : x \in X_K\}$ of $\overline{X_k}$ a subset $F_N \subset X_k$ and a family $\mathcal{D}_N := \{\overline{B_N^G(x,\varepsilon)} : x \in F_N\}$ of disjoint balls for which we have

$$X_k \subset \overline{X_k} \subset \bigcup_{x \in F_N} \overline{B_N^G(x, 5\varepsilon)} \subset \bigcup_{x \in F_N} B_N^G(x, 6\varepsilon)$$

and

$$\nu_v(B_N^G(x,\varepsilon)) \ge e^{-(s+\delta)N \cdot -\log \varepsilon}$$
 for all $x \in F_N$.

So, as the family \mathcal{D}_N is given by disjoint balls.

$$\sharp (F_N) \cdot e^{-(s+\delta)N \cdot -\log \varepsilon} = \sum_{x \in F_N} e^{-(s+\delta)N \cdot -\log \varepsilon} \le \sum_{x \in F_N} \nu_v(B_N^G(x,\varepsilon)) \le 1.$$

As

$$h_{GLW}(X_k, \mathbb{S}, 6\varepsilon) \le \limsup_{n \to \infty} \frac{1}{N} \log \sharp(F_N)$$

we have

$$\sup_{k \in \mathbb{N}} \overline{\mathrm{mdim}}_{M}^{GLW}\left(X_{k}, \mathbb{S}, d\right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(X_{k}, \mathbb{S}, 6\varepsilon)}{-\log \varepsilon} \leq s - \delta$$

since $X_k \subset X_{k+1}$ for all $k \in \mathbb{N}$ and $X = \bigcup_{k \in \mathbb{N}} X_k$ we have

$$\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right) \leq s - \delta.$$

As $\delta \geq 0$ may be considered arbitrary small we have

$$\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right)\leq s$$

and it finishes the proof.

4.6. **Proof of Theorem F.** The following lemma is an important tool in the proof.

Lemma 4.1. Let $\nu \in \mathcal{M}(X)$ be a G-homogeneous probability measure. Then

$$\overline{mdim}_{\nu}(x,d) = \overline{mdim}_{\nu}(y,d), \text{ for all } x,y \in X.$$

Proof. For $\varepsilon > 0$, by G-homogeneity, there exists $\delta(\varepsilon) > 0$ and c > 0 so that

$$\nu(B_n^G(x,\delta(\varepsilon))) \le c \cdot \nu(B_n^G(y,\varepsilon)),$$

and it implies

$$h_{\nu}^G(x,\delta(\varepsilon)) = \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(x,\delta(\varepsilon))) \leq \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(y,\varepsilon)) = h_{\nu}^G(y,\varepsilon),$$

and so

$$\limsup_{\varepsilon \to 0} \frac{h_{\nu}^G(x,\delta(\varepsilon))}{-\log \delta(\varepsilon)} \leq \limsup_{\varepsilon \to 0} \frac{h_{\nu}^G(y,\varepsilon)}{-\log \varepsilon}, \text{ for all } x,y \in X,$$

which gives $\overline{\mathrm{mdim}}_{\nu}(x,d) \leq \overline{\mathrm{mdim}}_{\nu}(y,d)$. By switching the roles of x and y in the previous computations one obtains the converse inequality and finishes the proof.

As a consequence of Lemma 4.1 we obtain that makes sense to define the measure metric mean dimension of a semigroup action with to respect of a G-homogeneous measure as the following:

$$\overline{\mathrm{mdim}}_{\nu}\left(\mathbb{S},d\right)=\limsup_{\varepsilon\to0}\frac{h_{\nu}^{G}(x,\varepsilon)}{-\log\varepsilon},\ \mathrm{for\ any}\ x\in X$$

since the limsup considered is constant in X.

Proposition 4.1. Let G be a compactly generated semigroup and ν be a strongly G-homogeneous probability measure on a compact metric space (X, d). Then

$$\overline{{mdim}}_{\nu}\left(X,\mathbb{S},d\right)=\overline{{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right).$$

Proof. Fix $\varepsilon > 0$ and take E a maximal (n, ε) -separated set in X. Then, by the maximality property of E, $B_n^G(x, \varepsilon/2) \cap B_n^G(y, \varepsilon/2)$ for any $x, y \in E$. In particular, for a fixed $x \in E$

$$\nu(X) \geq \sum_{y \in E} \nu\left(B_n^G(y, \varepsilon/2)\right) \geq s(n, \varepsilon) \cdot \nu(B_n^G(x, \varepsilon/2)).$$

By the G-homogeneity there exist $0 < \delta(\varepsilon) < \varepsilon$ and c > 0 so that $\nu(B_n^G(y, \delta(\varepsilon))) \le c \cdot \nu(B_n^G(x, \varepsilon/2))$, for all $x, y \in X$. It follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon)\leq \limsup_{n\to\infty}-\frac{1}{n}\log \nu(B_n^G(y,\delta(\varepsilon))).$$

Now, by the strongly G-homogeneity

$$\begin{split} \overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right) &= \limsup_{\varepsilon \to 0} \frac{h_{GLW}(\mathbb{S},\varepsilon)}{-\log \varepsilon} \\ &\leq \limsup_{\varepsilon \to 0} \frac{h_{\nu}^{G}(\delta(\varepsilon))}{-\log \delta(\varepsilon)} \frac{\log \delta(\varepsilon)}{\log \varepsilon} \\ &= \overline{\mathrm{mdim}}_{\nu}\left(X,\mathbb{S},d\right). \end{split}$$

and then $\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right)\leq\overline{\mathrm{mdim}}_{\nu}\left(\mathbb{S},d\right).$

For the opposite inequality, fix $\delta > 0$ and notice that that if F is a (n, ε) -spanning set of minimal cardinality $b(n, \varepsilon)$, then $X \subset \bigcup_{x \in F} B_n^G(x, 2\delta)$. Given $\varepsilon > 0$ there exist $0 < \delta(\varepsilon) < \varepsilon$ and c > 0 for which

$$\nu\left(B_n^G(x,2\delta(\varepsilon))\right) \le c \cdot \nu\left(B_n^G(y,\varepsilon)\right) \text{ for all } x,y \in X \text{ and } n \in \mathbb{N}.$$

It guarantees that

$$c \cdot b(n, \delta(\varepsilon)) \cdot \nu \left(B_n^G(y, \varepsilon) \right) \ge \nu(X) > 0$$

and so, by the strong G-homogeneity, we have

$$\overline{\operatorname{mdim}}_{M}^{GLW}(X, \mathbb{S}, d) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(\mathbb{S}, \delta(\varepsilon))}{-\log \delta(\varepsilon)}$$

$$\geq \limsup_{\varepsilon \to 0} \frac{h_{V}^{GLW}(\delta(\varepsilon))}{-\log \varepsilon} \frac{\log \varepsilon}{\log \delta(\varepsilon)}$$

$$= \overline{\operatorname{mdim}}_{\nu} (X, \mathbb{S}, d),$$

and it ends the proof.

Let us proceed to the proof of Theorem F. For the first we notice that it is a consequence of Proposition 4.1. For part (b) let ν be a Borel measure on X so that $\underline{\mathrm{mdim}}_{\nu}(x,d) \geq s$, for all $x \in X$. Fix $\eta > 0$ and let

$$X_k = \left\{ x \in X : \frac{\limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n^G(x,\varepsilon))}{-\log \varepsilon} > (s - \delta/2) \text{ for all } \varepsilon \in (0, \frac{1}{k}) \right\}.$$

By hypotheses, $X = \bigcup_{k \in \mathbb{N}} X_k$. It follows that $0 < \nu(X) \leq \sum_k \nu(X_k)$, which guarantees the existence of some $k_0 \in \mathbb{N}$ for which we have $\nu(X_{k_0}) > 0$. Again, we can wright $X_{k_0} = \bigcup_{N \in \mathbb{N}} X_{k_0,N}$ where

$$X_{k_0,N} = \left\{ x \in X_{k_0} : \frac{-\log \nu(B_n^G(x,\varepsilon))}{-n\log \varepsilon} > (s - \delta/2) \text{ for all } n \ge N \right\}.$$

In such case, there exists $N_0 \in \mathbb{N}$ for which $\nu(X_{k_0,N_0}) > 0$. In particular,

$$\nu(B_n^G(x,\varepsilon)) \leq e^{-n(s-\delta)\cdot(-\log\varepsilon)}, \text{ for all } x \in X_{k_0,N_0}, \varepsilon \in (0,\frac{1}{k}) \text{ and } n \geq N_0.$$

Now, for each integer $N \geq N_0$ consider the open cover of X_{k_0,N_0} given by $\mathcal{B}_N = \{B_N^G(x,\varepsilon) : x \in X_{k_0,N_0}\}$. In such case we have that for a subcover \mathcal{C} of \mathcal{B}_N

$$\inf_{\mathcal{C}} \cdot \sharp(\mathcal{C}) e^{-N(s-\delta)\cdot(-\log\varepsilon)} = \inf_{\mathcal{C}} \left\{ \sum_{B_N^G(x,\varepsilon)\in\mathcal{C}} e^{-N(s-\delta)\cdot(-\log\varepsilon)} \right\} \ge \nu(X_{k_0,N_0}).$$

As $cov(X, N, \varepsilon) \ge cov(X_{k_0, N_0}, N, \varepsilon)$, for all N and $\varepsilon > 0$, we have

$$\operatorname{cov}(X, N, \varepsilon)e^{-N(s-\delta)\cdot(-\log \varepsilon)} \ge \nu(X_{k_0, N_0}),$$

and it implies that

$$\limsup_{N \to \infty} \frac{1}{N} \log \operatorname{cov}(X, N, \varepsilon) e^{-N(s-\delta) \cdot (-\log \varepsilon)} \ge 0$$

and so,

$$h_{GLW}(X, \mathbb{S}, \varepsilon) \ge (s - \delta) \cdot (-\log \varepsilon).$$

Hence

$$\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(X,\mathbb{S},\varepsilon)}{-\log \varepsilon} \geq s - \delta.$$

As the inequality was obtained for an arbitrary δ we conclude that

$$\overline{\mathrm{mdim}}_{M}^{GLW}\left(X,\mathbb{S},d\right) = \limsup_{\varepsilon \to 0} \frac{h_{GLW}(X,\mathbb{S},\varepsilon)}{-\log \varepsilon} \geq s,$$

as part (b) states.

References

- L. Barreira and C. Wolf. Measures of maximal dimension for hyperbolic diffeomorphisms. Comm. Math. Phys. 239 (2003) 93-113.
- J. Bélair and S. Dubuc, Eds. Fractal Geometry and Analysis. NATO ASI Series C, vol. 346, 1989.
- [3] A. Biś. An analogue of the variational principle for group and pseudogroup actions. Ann. Inst. Fourier 63:3 (2013) 839–863.
- [4] A. Bufetov. Topological entropy of free semigroup actions and skew-product transformations.
 J. Dynam. Control Systems 5 (1999) 137–143.
- [5] M. Carvalho, F. Rodrigues, P. Varandas. Semigroups actions of expanding maps. J. Stat. Phys. 116:1 (2017) 114–136.
- [6] M. Carvalho, F. Rodrigues, P. Varandas. Quantitative recurrence for free semigroup actions. Nonlinearity 31:3 (2018) 864–886.
- [7] M. Carvalho, F. Rodrigues and P. Varandas. A variational principle for free semigroup actions. Adv. Math. 334 (2018) 450–487.
- [8] M. Carvalho, F. Rodrigues and P. Varandas. A variational principle for the metric mean dimension of free semigroup actions Ergod. Th. & Dynam. Sys.
- [9] M. Carvalho, F. Rodrigues and P. Varandas. Generic homeomorphisms have full metric mean dimension Ergod. Th. & Dynam. Sys.
- [10] Eldering J. Manifolds of Bounded Geometry. Normally Hyperbolic Invariant Manifolds. Atlantis Series in Dynamical Systems, vol 2. Atlantis Press, Paris.
- [11] K. Falconer. Fractal geometry: Mathematical foundations and applications. John Wiley & Sons, 1990.
- [12] M. Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps I. Math. Phys. Anal. Geom. 2:4 (1999) 323–415.
- [13] Y. Gutman, E. Lindenstrauss and M. Tsukamoto. Mean dimension of \mathbb{Z}^k -actions. Geom. Funct. Anal. Vol. 26 (2016) 778–817.
- [14] E. Ghys, R. Langevin and P. Walczak. Entropie géométrique des feuilletages. Acta Math. 160:1-2 (1988), 105–142.
- [15] B. Kloeckner. Optimal transport and dynamics of expanding circle maps acting on measures. Ergodic Theory & Dynam. Systems 33:2 (2013) 529–548.
- [16] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. J. Lond. Math. Soc. 16:3 (1977) 568–576.
- [17] E. Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. Publications Mathématiques de l'IHES 89:1 (1999) 227–262.
- [18] E. Lindenstrauss and B. Weiss. Mean topological dimension. Israel J. Math. 115 (2000) 1–24.
- [19] E. Lindenstrauss and M. Tsukamoto. Double variational principle for mean dimension. Geom. Funct. Anal. Vol. 29 (2019) 1048–1109.
- [20] E. Lindenstrauss. Mean dimension, small entropy factors and embedding maintheorem, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 1999, Vol. 89, Issue 1, 227–262.
- [21] F. Rodrigues, T. Jacobus and M. Silva. Entropy points and applications for free semigroup
- [22] Uri Shapira. Measure theoretical entropy of covers. Israel Journal of Mathematics, 158(1), 225-247, 2007.
- [23] Ruxi Shi. On variational principle for the metric mean dimension, arXiv:2101.02610v1 [math.DS](preprint).
- [24] Yonatan Gutman and Adam Spiewak. Around the variational principle for metric mean dimension. arXiv:2010.14772[math.DS](preprint).

- [25] Yonatan Gutman and Adam Spiewak. Metric mean dimension and analog compression. IEEE Transactions on Information Theory, 2020.
- [26] Ya. Pesin. Dimension Theory in Dynamical Systems: Contemporary Views and Applications. Lectures in Mathematics, Chicago Press, 1997.
- [27] Jingru Tang, Bing Li and Wen-Chiao Cheng Some properties on topological entropy of free semigroup action, Dynamical Systems, 33(1), 54-71, 2018.
- [28] Anibal Velozo and Renato Velozo. Rate distortion theory, metric mean dimension and measure theoretic entropy, arXiv:1707.05762 [math.DS](preprint).
- [29] Koichi Yano, A remark on the topological entropy of homeomorphisms. Inventiones mathematicae 59 (3) 215-220, 1980.

Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Brazil. $Email\ address:$ fagnerbernardini@gmail.com

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL. *Email address*: jacobus.math@gmail.com

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL. $Email\ address$: marcus4230gmail.com