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LINNIK'S PROBLEM IN FIBER BUNDLES OVER QUADRATIC HOMOGENEOUS VARIETIES

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Abstract. We compute the statistics of $\mathrm{SL}_d(\mathbb{Z})$ matrices lying on level sets of an integral polynomial defined on $\mathrm{SL}_d(\mathbb{R})$, a result that is a variant of the well known theorem proved by Linnik about the equidistribution of radially projected integral vectors from a large sphere into the unit sphere.

Using the above result we generalize the work of Aka, Einsiedler and Shapira in various directions. For example, we compute the joint distribution of the residue classes modulo q and the properly normalized orthogonal lattices of primitive integral vectors lying on the level set $-(x_1^2+x_2^2+x_3^2)+x_4^2=N$ as $N\to\infty$, where the normalized orthogonal lattices sit in a submanifold of the moduli space of rank-3 discrete subgroups of $\mathbb{R}^4$.

1. Introduction

1.1. **Linnik type problems.** To put our work in historical context, we will now recall a well known work of Linnik and its generalizations.

Consider for an integral homogeneous polynomial $P:\mathbb{R}^d\to\mathbb{R}$ and for $m\in\mathbb{Z}$ the level set

$$\mathcal{H}_m(P,\mathbb{R})\overset{\text{def}}{=}P^{-1}(\{m\})=\left\{\mathbf{v}\in\mathbb{R}^d\mid P(\mathbf{v})=m\right\},$$

and let

$$\mathcal{H}_{m,\text{prim}}(P,\mathbb{Z})\overset{\text{def}}{=}\mathcal{H}_m(f,\mathbb{R})\cap\mathbb{Z}^d_{\text{prim}}=\left\{\mathbf{v}\in\mathbb{Z}^d_{\text{prim}}\mid P(\mathbf{v})=m\right\},$$

where $\mathbb{Z}^d_{\text{prim}}$ denotes the set of primitive integral vectors in $\mathbb{R}^d$.

Assuming that the cardinalities of $\mathcal{H}_{m_i,\text{prim}}(P,\mathbb{Z})$ diverge to infinity along a sequence $\{m_i\}_{i=1}^{\infty}\subseteq\mathbb{N}$, it is natural to study the limiting statistics of $\mathcal{H}_{m_i}(P,\mathbb{Z})$ when projected radially into $\mathcal{H}_1(P,\mathbb{Z})$.

Linnik appears to be the first to consider the above problem in his seminal work (see [Lin68]) by computing the weak-* limits of the uniform probability measures μ_m on the unit sphere supported on $\frac{1}{\sqrt{m}}\mathcal{H}_{m,\text{prim}}(x^2+y^2+z^2,\mathbb{Z})$ as $m\to\infty$. Under suitable congruence conditions, Linnik was able to prove that μ_m converges towards the natural measure on $\mathbb{S}^2$ by developing a method known today as *Linnik's Ergodic method*, which has an arithmetic-dynamical nature.

Following Linnik's original work, the above problem was studied further by Linnik and his collaborators, see [Mal75] for a review, and more recently by a variety of other authors employing dynamical or harmonic analysis tools, see for example the definitely not exhaustive list [EMV10, GO03, MV06, EO06, BO12].

1.1.1. *Linnik type problem in SL_d.* The main results of our paper (see Theorems 3.7 and 3.8), concern a problem which falls into a broader category of Linnik type problems in an ambient manifold *that is not necessarily the Euclidean space.*

More explicitly, we will replace Euclidean space with $\mathrm{SL}_d(\mathbb{R})$ and primitive integral vectors with $\mathrm{SL}_d(\mathbb{Z})$. We will consider an integral polynomial $P:\mathrm{SL}_d(\mathbb{R})\to\mathbb{R}$ such that its level sets $\mathcal{Z}_T(\mathbb{R})=P^{-1}(\{T\})$ have a transitive action of a fixed group $G\le\mathrm{SL}_d(\mathbb{R})\times\mathrm{SL}_d(\mathbb{R})$ and such that there exists a G-equivariant projection $\pi_T:\mathcal{Z}_T(\mathbb{R})\to\mathcal{Z}_{T_0}(\mathbb{R})$, where $\mathcal{Z}_{T_0}(\mathbb{R})$ is a chosen reference level set. Then, similarly to the Linnik type problems above, we will consider (properly) normalized counting measure supported on $\mathcal{Z}_{T_0}(\mathbb{R})$ of the form $\mu_T\overset{\text{def}}{=}$

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$\frac{1}{c(T)}\sum_{x\in\mathcal{Z}_T(\mathbb{Z})}\delta_{\pi_T(x)}$, where $\mathcal{Z}_T(\mathbb{Z})\overset{\text{def}}{=}\mathcal{Z}_T(\mathbb{R})\cap\mathrm{SL}_d(\mathbb{Z})$, which are infinite, locally finite, atomic measures.

Our main result will state that, under certain conditions on the range of T,

$$\lim_{T\to\infty}\mu_T(f)=\mu_{\mathcal{Z}}(f),\ \forall f\in C_c(\mathcal{Z}_{T_0}(\mathbb{R})),$$

where $\mu_{\mathcal{Z}}$ will be a measure on $\mathcal{Z}_{T_0}(\mathbb{R})$ induced by the G-action.

1.2. **On the work of Aka, Einsiedler and Shapira.** Our original motivation for this paper comes from the work of Aka, Einsiedler and Shapira that can be found in [AES16b] and [AES16a]. We will extend [AES16a] in various directions using the limiting distribution of the measures μ_T discussed in Section 1.1.1 (see Theorems 4.8 and 4.9).

Remark. This paper relies on the method of the proof of [AES16a], and since analogue problems in dimension $d=3$ are treated by different set of tools (see e.g. [AES16b, Kha19]), the case of dimension $d=3$ is not treated in this paper.

We will now recall the main results of [AES16a]. Fix $d\geq 4$ and consider X_{d-1} the space of $(d-1)$-unimodular lattices in $\mathbb{R}^{d-1}$. The space of *shapes* of $(d-1)$-lattices is given by

$$\mathcal{S}_{d-1}\overset{\text{def}}{=}\mathrm{SO}_{d-1}(\mathbb{R})\backslash X_{d-1}\cong\mathrm{SO}_{d-1}(\mathbb{R})\backslash\mathrm{SL}_{d-1}(\mathbb{R})/\mathrm{SL}_{d-1}(\mathbb{Z})$$

which is simply the space of full rank lattices in $\mathbb{R}^{d-1}$ identified up-to a rotation.

For $\mathbf{v}\in\mathbb{R}^d$, we denote by $\mathbf{v}^\perp$ the orthogonal hyperplane to $\mathbf{v}$ with respect to the usual Euclidean inner product, and for $\mathbf{v}\in\mathbb{Z}^d_{\text{prim}}$ we define

$$\Lambda_{\mathbf{v}}\overset{\text{def}}{=}\mathbf{v}^\perp\cap\mathbb{Z}^d,$$

which is a rank $(d-1)$-discrete subgroup of $\mathbb{R}^d$.

We embed $\mathcal{S}_{d-1}$ into the space of rank $(d-1)$-discrete subgroups of $\mathbb{R}^d$ by identifying the horizontal plane $\mathbb{R}^{d-1}\times\{0\}\subseteq\mathbb{R}^d$ with $\mathbb{R}^{d-1}$. Then, by scaling the $\Lambda_{\mathbf{v}}$'s into unimodular lattices and by rotating them into $\mathbb{R}^{d-1}\times\{0\}$, we obtain their "shape" in $\mathcal{S}_{d-1}$. More explicitly, for a rank $(d-1)$-discrete subgroup $\Lambda\leq\mathbb{R}^d$, we denote by $\mathrm{covol}(\Lambda)$ the volume of a fundamental domain of Λ in the hyperplane containing Λ with respect the volume form obtained by the restriction of the Euclidean inner product to this hyperplane. An elementary argument (see e.g. [AES16a]) shows that

$$\mathrm{covol}(\Lambda_{\mathbf{v}})=\sqrt{\sum_{i=1}^d v_i^2}\overset{\text{def}}{=}\|\mathbf{v}\|\,,\ \forall\mathbf{v}\in\mathbb{Z}^d_{\text{prim}}.$$

By choosing $\rho_{\mathbf{v}}\in\mathrm{SO}_d(\mathbb{R})$ such that $\rho_{\mathbf{v}}\mathbf{v}=\mathbf{e}_d$, we get that $\rho_{\mathbf{v}}(\|\mathbf{v}\|^{-1/d-1}\Lambda_{\mathbf{v}})$ is a unimodular lattice in $\mathbb{R}^{d-1}\cong\mathbb{R}^{d-1}\times\{0\}$. We denote by $K\cong\mathrm{SO}_{d-1}(\mathbb{R})$ the subgroup of $\mathrm{SO}_d(\mathbb{R})$ stabilizing $\mathbf{e}_d$, and we define $\mathrm{shape}(\Lambda_{\mathbf{v}})\in\mathcal{S}_{d-1}$ by

$$\mathrm{shape}(\Lambda_{\mathbf{v}})\overset{\text{def}}{=}K\rho_{\mathbf{v}}(\|\mathbf{v}\|^{-1/d-1}\Lambda_{\mathbf{v}}),$$

which is well defined as a function of $\mathbf{v}\in\mathbb{Z}^d_{\text{prim}}$ (see (4.4) which extends the definition of "shape" function to the moduli space of $(d-1)-$discrete subgroups of $\mathbb{R}^d$).

The main result of [AES16b] and [AES16a] was the joint equidistribution of the normalized probability counting measures supported on

$$\left\{\left(\mathrm{shape}(\Lambda_{\mathbf{v}}),\frac{1}{\sqrt{T}}\mathbf{v}\right)\mid\mathbf{v}\in\mathcal{H}_{\text{prim},T}(\mathbb{Z})\right\}\subseteq\mathcal{S}_{d-1}\times\mathbb{S}^{d-1},$$

where $\mathbb{S}^{d-1}\subseteq\mathbb{R}^d$ denotes the unit sphere.

1.2.1. *Some historical context for* [AES16b] *and* [AES16a] *and subsequent works.* Statistics of shapes of subgroups of $\mathbb{Z}^d$ were studied by W. Roelcke in [Roe56], H. Maass in [Maa59], and much later by W. Schmidt in [Sch98, Sch15] who proved more general results using elementary counting techniques. Schmidt's theorem was given a dynamical approach in [Mar10], and T. Horesh and Y. Karasik recently in [HK20] extended Schmidt's results to "higher" moduli spaces using the technique of [GN12].

A considerably more refined problem concerning the shapes of subgroups of $\mathbb{Z}^d$ lying in sparse subsets was first studied in [EMSS16] and then in [AES16b] and [AES16a]. We note the recent works [ERW17, Kha19, AEW19, BB20, AMW21] which extend and refine [AES16b, AES16a, EMSS16] in a various directions.

In this paper we continue the preceding line of research and generalize the results of [AES16a]. In a rough description, we will consider tuples of the form $(\mathrm{shape}(\Lambda_{\mathbf{v}}), \mathbf{v}, \mathbf{v} \bmod q)$ for integral $\mathbf{v} \in \mathbb{Z}^d_{\mathrm{prim}} \cap Q^{-1}(\{T\})$ where Q is a non-singular integral quadratic form which can be either positive definite, or of signature $(1, d-1)$, and moreover, we will consider "higher" moduli spaces.

1.2.2. *AES type result in two sheeted hyperboloids.* We now give a special case of our results. We fix $d \geq 4$, we let $Q(\mathbf{x}) = -(\sum_{i=1}^{d-1} x_i^2) + x_d^2$ and we consider the group $\mathrm{SO}_Q(\mathbb{R}) \leq \mathrm{SL}_d(\mathbb{R})$ which preserves Q. For $T \in \mathbb{R}$, we denote

$$\mathcal{H}_T(\mathbb{R}) \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^d \mid Q(\mathbf{x}) = T\right\},$$

and we let

$$\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z}) \stackrel{\text{def}}{=} \mathcal{H}_T(\mathbb{R}) \cap \mathbb{Z}^d_{\mathrm{prim}}.$$

In this paper we will concentrate on $T > 0$ because the stabilizers in $\mathrm{SO}_Q(\mathbb{R})$ of vectors in $\mathcal{H}_T(\mathbb{R})$ are compact, which is important for the method that we use. We recall by Theorem 6.9 of [BHC62] that $\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})$ is finite, and for $N \in \mathbb{N}$ we consider the following measure on $\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R})$ defined by

$$\nu_N \stackrel{\text{def}}{=} \frac{1}{|\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{\mathbf{v} \in \mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})} \delta_{\left(\mathrm{shape}(\Lambda_{\mathbf{v}}), \frac{1}{\sqrt{N}}\mathbf{v}\right)}.$$

Note that $\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R})$ is a quotient of $\mathrm{SL}_{d-1}(\mathbb{R})/\mathrm{SL}_{d-1}(\mathbb{Z}) \times \mathrm{SO}_Q(\mathbb{R})$ by a compact group, and on the former space there is a choice of a natural measure (for more details, see Section 4.3.2), which gives, by taking the pushforward under the natural projection, a product measure on $\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R})$ which we denote by $\mu_{\mathcal{S}_{d-1}} \otimes \mu_{\mathcal{H}_1}$.

Theorem 1.1. *For all $f \in C_c(\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R}))$ it holds that*

$$\lim_{N \to \infty} \nu_N(f) = \mu_{\mathcal{S}_{d-1}} \otimes \mu_{\mathcal{H}_1}(f).$$

By adding congruence assumptions on $N \in \mathbb{N}$, we obtain the following joint distribution of the radial projection into $\mathcal{H}_1(\mathbb{R})$, the shapes of orthogonal lattices and the residue classes of the vectors in $\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})$ as $N \to \infty$.

We choose $q \in \mathbb{N}$ and we define for $a \in \mathbb{Z}/(q)$

$$\mathcal{H}_a(\mathbb{Z}/(q)) \stackrel{\text{def}}{=} \left\{\mathbf{x} \in (\mathbb{Z}/(q))^d \mid Q(\mathbf{x}) = a\right\}.$$

For $N \in \mathbb{N}$ and $q \in \mathbb{N}$ we consider the following measures on $\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R}) \times \mathcal{H}_{N (\mathrm{mod} q)}(\mathbb{Z}/(q))$ defined by

$$\nu_N^q \stackrel{\text{def}}{=} \frac{1}{|\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{\mathbf{v} \in \mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})} \delta_{\left(\mathrm{shape}(\Lambda_{\mathbf{v}}), \frac{1}{\sqrt{N}}\mathbf{v}, \mathbf{v} \ (\mathrm{mod}\ q)\right)}$$

Theorem 1.2. *Let $q \in 2\mathbb{N}+1$ and let $a \in (\mathbb{Z}/(q))^{\times}$ be an invertible residue mod q. Assume that $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ satisfy*

$$T_n \ (\mathit{mod}\ q) = a \in (\mathbb{Z}/(q))^{\times}, \ \forall n \in \mathbb{N}.$$

Then for all $f \in C_c(\mathcal{S}_{d-1} \times \mathcal{H}_1(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$ *it holds that*

$$\lim_{n\to\infty} \nu^q_{T_n}(f) = \mu_{\mathcal{S}_{d-1}} \otimes \mu_{\mathcal{H}_1} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q))}(f),$$

where $\mu_{\mathcal{H}_a(\mathbb{Z}/(q))}$ *is the uniform probability measure on* $\mathcal{H}_a(\mathbb{Z}/(q))$*.*

1.3. Structure of the paper.

- Section 2 discusses some conventions, standing assumptions and basic facts that will be used throughout the paper.
- Section 3 discusses the manifolds $\mathcal{Z}_T(\mathbb{R}) \subseteq \mathrm{SL}_d(\mathbb{R})$ and presents our main "Linnik type" results, see Theorems 3.7 and 3.8.
- Section 4 discusses moduli spaces of discrete subgroups of $\mathbb{Z}^d$ and states our results concerning them refining [AES16a], see Theorems 4.8 and 4.9. We note that the latter results may also be interpreted conceptually as a Linnik type result.
- Section 5 proves Theorems 4.8 and 4.9 of the moduli spaces using Theorems 3.7 and 3.8 of the $\mathrm{SL}_d(\mathbb{R})$-submanifolds.
- The rest of the paper is devoted to proving Theorems 3.7 and 3.8. The scheme is roughly as follows:
  - Section 7 generalizes the proof of [AES16a, Theorem 3.1] concerning the equidistribution of a sequence of compact orbits in an S-arithemetic space, which builds on the results of [GO11].
  - Sections 8-10 exploit the equidistribution of orbits proved in Section 7 to prove Theorems 3.7 and 3.8 by revisiting the method of [AES16a]. The preceding method is outlined in Section 8.

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2. Some conventions, standing assumptions and basic facts

We denote by R a unital commutative ring, and for $d \in \mathbb{N}$ we view R^d as column vectors. We will denote for $1 \le i \le d$ by $\mathbf{e}_i \in R^d$ the standard basis vectors, and for $\mathbf{x} \in R^d$ we denote by $x_1, ..., x_d \in R$ the components of $\mathbf{x}$, namely $\mathbf{x} = \sum_{i=1}^d x_i \mathbf{e}_i$, where $x_i \in R$.

When $\mathcal{V}(\mathbb{Z}) \subseteq \mathbb{Z}^d$ is defined by the solutions of a collection of polynomials with integer coefficients, we denote by $\mathcal{V}(R)$ its solutions in R^d. For $q \in \mathbb{N}$ we denote by $\vartheta_q : \mathbb{Z} \to \mathbb{Z}/(q)$ the reduction map modulo q, and we observe that it induces a map $\vartheta_q : \mathcal{V}(\mathbb{Z}) \to \mathcal{V}(\mathbb{Z}/(q))$.

Throughout the paper we will consider

$$\mathrm{SL}_d(R) \overset{\mathrm{def}}{=} \{g \in M_d(R) \mid \det(g) = 1\},$$

$$\mathrm{ASL}_{d-1}(R) \overset{\mathrm{def}}{=} \left\{ \begin{pmatrix} m & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} \mid m \in \mathrm{SL}_{d-1}(R),\ \mathbf{v} \in R^{d-1} \right\},$$

and for an integral symmetric matrix $M \in M_d(\mathbb{Z})$ we let

$$\mathrm{SO}_Q(R) \overset{\mathrm{def}}{=} \left\{ g \in \mathrm{SL}_d(R) \mid g^t M g = M \right\},$$

where Q is the quadratic form whose companion symmetric matrix is M. We make the convention that a quadratic form $Q : \mathbb{Z}^d \to \mathbb{Z}$ is integral if Q has an *integral* companion matrix M, and we say that Q is non-degenerate if $\mathrm{disc}(Q) \overset{\mathrm{def}}{=} \det(M) \neq 0$.

We consider the right $\mathrm{SO}_Q(\mathbb{R})$ linear action on R^d given by

$$\mathbf{v} \cdot \rho \overset{\mathrm{def}}{=} \rho^{-1}\mathbf{v},\ \ \rho \in \mathrm{SO}_Q(R),\ \ \mathbf{v} \in R^d, \tag{2.1}$$

and for $\mathbf{v} \in R^d$ we let

$$\mathbf{H}_{\mathbf{v}}(R) \overset{\mathrm{def}}{=} \{g \in \mathrm{SO}_Q(R) \mid g\mathbf{v} = \mathbf{v}\}.$$

Standing Assumption. *Throughout the paper Q denotes an integral, non-degenerate quadratic form in $d \geq 4$ variables such that $Q(\mathbf{e}_d) > 0$ and $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ is compact.*

Definition 2.1. For $q \in \mathbb{N}$ we will say that Q is non-singular modulo q if $\mathrm{disc}(Q)(\mathrm{mod}\ q) \in (\mathbb{Z}/(q))^{\times}$.

2.0.1. *Linear action of SL_d by the Cartan involution.* Let $\theta : \mathrm{SL}_d(R) \to \mathrm{SL}_d(R)$ be the involutive automorphism given by

$$\theta(g) \overset{\mathrm{def}}{=} \left(g^t\right)^{-1}.$$

In the paper we will consider the left action of $\mathrm{SL}_d(R)$ on R^d given by

$$g \cdot \mathbf{v} \overset{\mathrm{def}}{=} \theta(g)\mathbf{v},\ \ g \in \mathrm{SL}_d(R) \tag{2.2}$$

(where the right hand-side denotes matrix multiplication of $\mathbf{v}$ by $\theta(g)$) and we denote the translation map of $\mathbf{e}_d$ by

$$\tau(g) \overset{\mathrm{def}}{=} \theta(g)\mathbf{e}_d,\ \ g \in \mathrm{SL}_d(R). \tag{2.3}$$

The main motivation that led us to consider the action above (and not the usual left SL_d linear action) is that the vector $\tau(g) \in R^d$ is orthogonal to the first $d-1$ columns of g with respect to the Euclidean inner product, as we now explain.

For $\mathbf{x}, \mathbf{y} \in R^d$ we define the Euclidean bi-linear form $\langle \mathbf{x}, \mathbf{y} \rangle \overset{\mathrm{def}}{=} \sum_{i=1}^d x_i y_i$. An important property of θ is the invariance

$$\langle \theta(g) \cdot \mathbf{x}, g \cdot \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle\,,\ \forall g \in \mathrm{SL}_d(R), \tag{2.4}$$

which in particular implies

$$\langle \tau(g), g\mathbf{e}_j \rangle = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{i,j}. \tag{2.5}$$

2.0.2. *Concerning the covolume and the left action of $SL_d(R)$ on R^d.* For a discrete subgroup $\Lambda \leq \mathbb{R}^d$ of rank $d-1$, we define $\mathrm{covol}(\Lambda)$ to be the volume of a fundamental domain of Λ in the hyperplane containing Λ, with respect to the volume form obtained by the restriction of the Euclidean inner product to this hyperplane.

For $g \in \mathrm{SL}_d(\mathbb{R})$ we denote by $\hat{g} \in M_{d\times d-1}(\mathbb{R})$ the matrix formed by the first $d-1$ columns of g, and we note for $\Lambda = \hat{g}\mathbb{Z}^{d-1}$ (the discrete subgroup of rank $d-1$ having the columns of $\hat{g}$ as a $\mathbb{Z}$-basis) the formula

$$\mathrm{covol}(\Lambda)^2 = \det\left(\hat{g}^t \hat{g}\right).$$

Lemma 2.2. *We have for $g \in \mathrm{SL}_d(\mathbb{R})$ that*

$$\mathrm{covol}(\hat{g}\mathbb{Z}^{d-1}) = \|\tau(g)\|\,,$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

Proof. We first note that

$$\left(g^t g\right)^{-1} = \mathrm{adj}(g^t g),$$

where $\mathrm{adj}\,(\cdot)$ denotes the matrix adjugate, and we observe that the d,d entry of the matrix $\mathrm{adj}(g^t g)$ is $\det\left(\hat{g}^t\hat{g}\right) = \mathrm{covol}(\Lambda)^2$. In particular, the d,d entry of the matrix $\mathrm{adj}(g^t g)$ can be expressed by $\left\langle \mathbf{e}_d, \left(g^t g\right)^{-1}\mathbf{e}_d \right\rangle$, hence

$$\begin{aligned}\mathrm{covol}(\hat{g}\mathbb{Z}^{d-1})^2 &= \left\langle \mathbf{e}_d, \left(g^t g\right)^{-1}\mathbf{e}_d \right\rangle = \left\langle \mathbf{e}_d, g^{-1}\theta(g)\mathbf{e}_d \right\rangle \\ &= \langle \theta(g)\mathbf{e}_d, \theta(g)\mathbf{e}_d \rangle = \|\theta(g)\mathbf{e}_d\|^2 \\ &= \|\tau(g)\|^2 .\end{aligned}$$

□

3. Linnik type problem in $\mathrm{SL}_d(\mathbb{R})$

The structure of this section is as follows:

- Section 3.1 introduces the one-parameter family of subvarieties $\mathcal{Z}_T \subseteq \mathrm{SL}_d$ mentioned in Section 1.1.1, and discusses basic facts concerning them.
- Section 3.2 defines a natural homeomorphism between a subvariety $\mathcal{Z}_T(\mathbb{R})$ to a reference subvariety $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$.
- Section 3.3 presents our main results (Theorems 3.7 and 3.8).

3.1. **Subvarieties of $\mathbf{SL}_d$.** Let Q be a quadratic form as in our Standing Assumption and recall that R denotes a unital commutative ring. For $T \in R$, we let

$$\mathcal{H}_T(R) \stackrel{\text{def}}{=} \{\mathbf{v} \in R^d \mid Q(\mathbf{v}) = T\},$$

and we consider

$$\mathcal{Z}_T(R) \stackrel{\text{def}}{=} \tau^{-1}(\mathcal{H}_T(R)), \tag{3.1}$$

(see (2.3) to recall τ) namely

$$\mathcal{Z}_T(R) \stackrel{\text{def}}{=} \{g \in \mathrm{SL}_d(R) \mid (Q \circ \tau)(g) = Q\left((g^t)^{-1}\mathbf{e}_d\right) = T\}.$$

Note that $Q \circ \tau : \mathrm{SL}_d(\mathbb{Z}) \to \mathbb{Z}$ is an integral polynomial.

3.1.1. *Concerning the* $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})$ *action.* We recall the $\mathrm{SL}_d(R)$ action given in (2.2), and we observe that the stabilizer subgroup of $\mathrm{SL}_d(R)$ stabilizing $\mathbf{e}_d$ is $\mathrm{ASL}_{d-1}(R)$, which allows us to conclude

$$\mathrm{SL}_d(R)/\mathrm{ASL}_{d-1}(R) \cong \tau\left(\mathrm{SL}_d(R)\right). \tag{3.2}$$

In light of (3.2), $\mathrm{SL}_d(R)$ can be thought of as a union of fibers $\tau^{-1}(\mathbf{v}),\ \mathbf{v} \in R^d$, where each fiber is an $\mathrm{ASL}_{d-1}(R)$-right coset, and according to (3.1), $\mathcal{Z}_T(R)$ is the union of those fibers of vectors in $\tau\left(\mathrm{SL}_d(R)\right) \cap \mathcal{H}_T(R)$, which leads to the identification

$$\mathcal{Z}_T(R)/\mathrm{ASL}_{d-1}(R) \cong \tau\left(\mathrm{SL}_d(R)\right) \cap \mathcal{H}_T(R). \tag{3.3}$$

We consider the following right action of $\mathrm{SO}_Q(R)$ on $\mathrm{SL}_d(R)/\mathrm{ASL}_{d-1}(R)$ defined by

$$(g\mathrm{ASL}_{d-1}(R)) \cdot \rho \stackrel{\text{def}}{=} \theta\left(\rho\right)^{-1} g\mathrm{ASL}_{d-1}(R), \tag{3.4}$$

and we observe that the above action is equivalent to the right $\mathrm{SO}_Q(R)$ action (2.1) on the orbit $\tau\left(\mathrm{SL}_d(R)\right) \subseteq R^d$, namely

$$\tau(g\mathrm{ASL}_{d-1}(R) \cdot \rho) = \tau(g) \cdot \rho. \tag{3.5}$$

In view of (3.5), it is natural to consider the $\left(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1}\right)(R)$ action on $\mathcal{Z}_T(R)$ from the right by

$$g \cdot (\rho, \eta) \stackrel{\text{def}}{=} \theta(\rho)^{-1} g\eta,\ \ g \in \mathcal{Z}_T(R),\ \ (\eta, \rho) \in (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R), \tag{3.6}$$

and continuing with our description of $\mathcal{Z}_T(R)$ as a union of fibers of vectors in $\tau(\mathrm{SL}_d(R)) \cap \mathcal{H}_T(R)$, we interpret $g\eta$ as a "move" in the fiber of $\mathbf{v} \stackrel{\text{def}}{=} \tau(g)$ and by $\theta(\rho)^{-1}g\eta$ as a "transition" of $g\eta$ into the fiber of $\rho^{-1}\mathbf{v}$ (using (3.5)), which allows us to conclude (more formally, by (3.3) and (3.5)) that

$$\mathcal{Z}_T(R)/\left(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1}\right)(R) \cong \left(\tau\left(\mathrm{SL}_d(R)\right) \cap \mathcal{H}_T(R)\right)/\mathrm{SO}_Q(R). \tag{3.7}$$

We have the following corollary from (3.7).

Corollary 3.1. *The following hold:*

(1) $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ *acts transitively on* $\mathcal{Z}_T(\mathbb{R})$ *for all* $T > 0$.
(2) Let $q \in 2\mathbb{N}+1$*, and assume that* Q *is non-singular modulo* q *(Definition 2.1). Then* $\left(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1}\right)(\mathbb{Z}/(q))$ *acts transitively on* $\mathcal{Z}_a(\mathbb{Z}/(q))$ *for all* $a \in (\mathbb{Z}/(q))^\times$.

(3) *There are finitely many* $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{Z})$ *orbits in* $\mathcal{Z}_N(\mathbb{Z})$ *for all* $N \in \mathbb{N}$*, and moreover*

$$|\mathcal{Z}_N(\mathbb{Z})/(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{Z})| = |\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|$$

Proof. To show (1) and (2), we observe that using (3.7), it is sufficient to prove that $\mathrm{SO}_Q(R)$ acts transitively on $\mathcal{H}_T(R)$ when $R \in \{\mathbb{R}, \mathbb{Z}/(q)\}$ and $T \in R$ are as specified in (1) and (2)

The claim for $R = \mathbb{R}$ follows from Witt's Theorem.

We now proceed to prove (2) by going along the lines of the proof of [Cas78, Chapter 8, Lemma 3.3]. Let p be an odd prime and let $k \in \mathbb{N}$. We may consider the following involution (a generalized reflection)

$$\tau_{\mathbf{v}} : \left(\mathbb{Z}/(p^k)\right)^d \to \left(\mathbb{Z}/(p^k)\right)^d,$$

defined for $\mathbf{v} \in \left(\mathbb{Z}/(p^k)\right)^d$ such that $Q(\mathbf{v}) \in \left(\mathbb{Z}/(p^k)\right)^\times$, by

$$\tau_{\mathbf{v}}(\mathbf{x}) \overset{\mathrm{def}}{=} \mathbf{x} - \frac{2Q(\mathbf{x},\mathbf{v})}{Q(\mathbf{v})}\mathbf{v},$$

where $Q(\mathbf{x},\mathbf{y}) \overset{\mathrm{def}}{=} \frac{1}{4}\left(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}-\mathbf{y})\right)$ for $\mathbf{x},\mathbf{y} \in (\mathbb{Z}/(q))^d$ is the associated bi-linear form of Q. By observing that $Q(\tau_{\mathbf{v}}(\mathbf{x})) = Q(\mathbf{x})$ and $\det(\tau_{\mathbf{v}}) = -1$, we deduce that $\tau_{\mathbf{u}_1} \circ \tau_{\mathbf{u}_2} \in \mathrm{SO}_Q(\mathbb{Z}/(p^k))$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \left(\mathbb{Z}/(p^k)\right)^d$ such that $Q(\mathbf{u}_1), Q(\mathbf{u}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$.

We now show that for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}_a(\mathbb{Z}/(p^k))$ with $a \in \left(\mathbb{Z}/(p^k)\right)^\times$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in \left(\mathbb{Z}/(p^k)\right)^d$ such that $Q(\mathbf{u}_1), Q(\mathbf{u}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$ and $\tau_{\mathbf{u}_1} \circ \tau_{\mathbf{u}_2}(\mathbf{v}_1) = \mathbf{v}_2$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}_a(\mathbb{Z}/(p^k))$ with $a \in \left(\mathbb{Z}/(p^k)\right)^\times$. We observe that $Q(\mathbf{v}_1+\mathbf{v}_2)+Q(\mathbf{v}_1-\mathbf{v}_2) = 4a \in \left(\mathbb{Z}/(p^k)\right)^\times$, which implies that either $Q(\mathbf{v}_1+\mathbf{v}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$ or $Q(\mathbf{v}_1-\mathbf{v}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$. Assuming that $Q(\mathbf{v}_1-\mathbf{v}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$, we may consider $\tau_{\mathbf{v}_1-\mathbf{v}_2}$ and we observe that $\tau_{\mathbf{v}_1-\mathbf{v}_2}\mathbf{v}_1 = \mathbf{v}_2$. Assuming the existence of $\mathbf{u} \in \left(\mathbb{Z}/(p^k)\right)^d$ such that $Q(\mathbf{u}) \in \left(\mathbb{Z}/(p^k)\right)^\times$ and $Q(\mathbf{u},\mathbf{v}_1) = 0$, we note that $\tau_{\mathbf{u}}(\mathbf{v}_1) = \mathbf{v}_1$, which implies in turn that $\tau_{\mathbf{v}_1-\mathbf{v}_2} \circ \tau_{\mathbf{u}}(\mathbf{v}_1) = \mathbf{v}_2$. To prove the existence of the above $\mathbf{u}$, we note that $Q(\mathbf{v}_1) \bmod p$ is non-zero, which implies that the restriction of the form $Q \pmod p$ to the vector space

$$V = \left\{\mathbf{x} \in (\mathbb{Z}/(p))^d \mid Q(\mathbf{x},\mathbf{v}_1) = 0 \bmod p\right\}$$

gives a non-singular form, proving in turn that there exists $\tilde{\mathbf{u}} \in V$ such that $Q(\tilde{\mathbf{u}})$ is non-zero mod p. Using [Ser73, Section 2, Theorem 1] (Hensel's Lemma for several variables) for the polynomial $f(\mathbf{x}) = Q(\mathbf{x},\mathbf{v}_1)$ (by lifting $\mathbf{v}_1$ to a $\mathbb{Z}_p^d$ vector) we deduce that there exists $\mathbf{u} \in \left(\mathbb{Z}/(p^k)\right)^d$ such that $\mathbf{u} = \tilde{\mathbf{u}} \bmod p$ and $Q(\mathbf{u},\mathbf{v}_1) = 0$, and in particular, since $\mathbf{u} = \tilde{\mathbf{u}} \bmod p$, we get $Q(\mathbf{u}) \in \left(\mathbb{Z}/(p^k)\right)^\times$. If on the other-hand it holds that $Q(\mathbf{v}_1+\mathbf{v}_2) \in \left(\mathbb{Z}/(p^k)\right)^\times$, then we have

$$\tau_{\mathbf{v}_2} \circ \tau_{\mathbf{v}_1+\mathbf{v}_2}(\mathbf{v}_1) = \mathbf{v}_2.$$

With this we have proved (2) for q being a power of an odd prime, and the result for a general $q \in 2\mathbb{N}+1$ follows by the Chinese remainder theorem.

Finally, to validate (3), note that for $T > 0$

$$\left(\tau\left(\mathrm{SL}_d(\mathbb{Z})\right) \cap \mathcal{H}_T(\mathbb{Z})\right)/\mathrm{SO}_Q(\mathbb{Z}) = \left(\mathbb{Z}^d_{\mathrm{prim}} \cap \mathcal{H}_T(\mathbb{Z})\right)/\mathrm{SO}_Q(\mathbb{Z}) = \mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z}).$$

□

3.1.2. *Stabilizers subgroups of* $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R)$. We now discuss some facts concerning the stabilizer subgroup of $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R)$ stabilizing $g \in \mathrm{SL}_d(R)$ by the right action (3.6). For the following recall that $\mathbf{H}_{\tau(g)}(R) \leq \mathrm{SO}_Q(R)$ denotes the stabilizer of $\tau(g) \in R^d$ by the $\mathrm{SO}_Q(\mathbb{R})$ action on R^d (to recall, see (2.1)).

Lemma 3.2. *Let* $g \in \mathrm{SL}_d(R)$ *and consider the group*

$$\mathbf{L}_g(R) \overset{def}{=} \left\{\left(w, g^{-1}\theta(w)g\right) \mid w \in \mathbf{H}_{\tau(g)}(R)\right\}. \tag{3.8}$$

Then $\mathbf{L}_g(R) \leq (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R)$ *is the stabilizer subgroup of* g *by the action* (3.6).

Proof. To show that $\mathbf{L}_g(R) \leq (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R)$ we observe that for all $w \in \mathbf{H}_{\tau(g)}(R)$ it holds that

$$\tau\left(g^{-1}\theta(w)g\right) = \theta(g^{-1})w\tau(g) = \theta(g^{-1})\tau(g) = e_d,$$

which implies that $g^{-1}\theta(w)g \in \mathrm{ASL}_{d-1}(R)$.

Next, as the reader should easily verify, all elements of $\mathbf{L}_g(R)$ stabilize g. For the other inclusion, let $(\rho,\eta) \in (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(R)$ be such that

$$g = g\cdot(\rho,\eta) = \theta(\rho^{-1})g\eta. \tag{3.9}$$

By rewriting (3.9), we get

$$g^{-1}\theta(\rho)g = \eta, \tag{3.10}$$

and we observe that to finish the proof, we need show that $\rho \in \mathbf{H}_{\tau(g)}(R)$. Indeed, we have

$$\rho^{-1}\tau(g) = \tau(\theta(\rho^{-1})g) \underbrace{=}_{\mathrm{ASL}_{d-1}(R)\text{ invariance}} \tau(\theta(\rho^{-1})g\eta) \underbrace{=}_{(3.9)} \tau(g).$$

□

3.1.3. *The form Q^*.* We will now go over some technical facts that we need about the groups $\theta(\mathrm{SO}_Q(\mathbb{R}))$ and $\theta(\mathbf{H}_{\mathbf{v}}(\mathbb{R}))$ for $\mathbf{v} \in R^d$ (which appears in the second factor of $\mathbf{L}_g(\mathbb{R})$). In a summary, we will show that $\theta(\mathrm{SO}_Q(\mathbb{R}))$ is identified with $\mathrm{SO}_{Q^*}(\mathbb{R})$ for a (rational) quadratic form Q^* defined below, and the subgroup $\theta(\mathbf{H}_{\mathbf{v}}(\mathbb{R}))$ is identified with the subgroup of $\mathrm{SO}_{Q^*}(\mathbb{R})$ that preserves the orthogonal hyperplane to $\mathbf{v}$ with respect to the Euclidean inner product.

Let $M \in M_d(\mathbb{Z})$ be the companion matrix of the form Q, namely

$$Q(\mathbf{x}) = \mathbf{x}^t M\mathbf{x}.$$

We recall that Q is a non-degenerate integral form, which implies that $M \in \mathrm{GL}_d(\mathbb{Q})$, and we define the rational form Q^* by

$$Q^*(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}^t M^{-1}\mathbf{x}. \tag{3.11}$$

Remark. The form Q^* can be defined more intrinsically as follows. Let $Q(\cdot,\cdot)$ the bi-linear form associated to Q. Since Q is non-degenerate, the map

$$l^Q : \mathbb{R}^d \to \left(\mathbb{R}^d\right)^*$$

where $\left(\mathbb{R}^d\right)^*$ denotes the dual space, defined by $l^Q(\mathbf{x}) \stackrel{\text{def}}{=} Q(\cdot,\mathbf{x})$ is a linear isomorphism. The form Q^* can be identified as the form on $\left(\mathbb{R}^d\right)^*$ which is makes the map l^Q an isometry.

Lemma 3.3. *We have that $\theta(\mathrm{SO}_Q(\mathbb{R})) = \mathrm{SO}_{Q^*}(\mathbb{R})$. Moreover, let $g \in \mathrm{SL}_d(\mathbb{Z})$ such that $Q(\tau(g)) \neq 0$, then:*

(1) We have that $\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right) = \{\rho \in \mathrm{SO}_{Q^*}(\mathbb{R}) \mid \rho(M\tau(g)) = M\tau(g)\}$.

(2) It holds that $(M\tau(g))^{\perp(Q^*)} = Span_{\mathbb{R}}\{\mathbf{g}_1,...,\mathbf{g}_{d-1}\}$, where $(M\tau(g))^{\perp(Q^*)}$ denotes the orthogonal hyperplane to $M\tau(g)$ with respect to Q^* , and $\mathbf{g}_i$ is the i'th column of g. Moreover

$$\mathbb{R}^d = (M\tau(g))^{\perp(Q^*)} \oplus Span_{\mathbb{R}}\{M\tau(g)\}.$$

Proof. To show that $\theta(\mathrm{SO}_Q(\mathbb{R}))$ is the group preserving the form Q^*, we observe that

$$\rho^t M\rho = M \iff$$

$$\theta\left(\rho^t M\rho\right) = \theta(M) \iff$$

$$\theta(\rho)^t M^{-1}\theta(\rho) = M^{-1}. \tag{3.12}$$

Next, to prove that the subgroup $\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right) \leq \mathrm{SO}_{Q^*}(\mathbb{R})$ is the stabilizer of $M\tau(g)$, we observe by (3.12) that

$$\theta(\rho)\left(M\tau(g)\right) = M\rho\tau(g),$$

and since M is invertible, we deduce that

$$M\rho\tau(g) = M\tau(g) \iff \rho\tau(g) = \tau(g),$$

namely $\theta(\rho)$ stabilizes $M\tau(g)$ if and only if ρ stabilizes $\tau(g)$.

Next, to show (2), we note that

$$Q^*(M\tau(g)) = Q(\tau(g)) \neq 0,$$

which by [Cas78, Lemma 1.3] shows that

$$\mathbb{R}^d = (M\tau(g))^{\perp(Q^*)} \oplus \mathrm{Span}_{\mathbb{R}}\left\{M\tau(g)\right\}.$$

Note that the bi-linear form B_{Q^*} determined by Q^* is given by

$$B_{Q^*}(\mathbf{u}_1, \mathbf{u}_2) = \left\langle \mathbf{u}_1, M^{-1}\mathbf{u}_2 \right\rangle.$$

Let $\mathbf{g}_i$ be the i'th column of g, then

$$B_{Q_*}(\mathbf{g}_i, M\tau(g)) = \left\langle \mathbf{g}_i, M^{-1}M\tau(g) \right\rangle = \left\langle g\mathbf{e}_i, \left(g^t\right)^{-1}\mathbf{e}_d \right\rangle = \delta_{i,d},$$

which proves that $(M\tau(g))^{\perp(Q^*)} = \mathrm{Span}_{\mathbb{R}}\left\{\mathbf{g}_1, ..., \mathbf{g}_{d-1}\right\}$. □

3.2. **The equivariant isomorphism.** Our goal now is to describe a one-parameter group $\{a_T\}_{T>0} \le \mathrm{SL}_d(\mathbb{R})$ such that $a_T \in \mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$ for all $T > 0$, and such that the stabilizer group $\mathbf{L}_{a_T}(\mathbb{R}) \le \mathrm{SO}_Q \times \mathrm{ASL}_{d-1}(\mathbb{R})$ of a_T is independent of T. This will allow us to define a $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ equivariant map $\mathcal{Z}_{T_1}(\mathbb{R}) \to \mathcal{Z}_{T_2}(\mathbb{R})$, for $T_i > 0$.

We note that $Q(\tau(I_d)) = Q(\mathbf{e}_d) \neq 0$, and by Lemma 3.3,(2) we obtain

$$\mathbb{R}^d = \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\} \oplus \mathrm{Span}_{\mathbb{R}}\left\{M\mathbf{e}_d\right\}$$

where $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}$ and $\mathrm{Span}_{\mathbb{R}}\left\{M\mathbf{e}_d\right\}$ are invariant spaces under the ordinary left $\theta\left(\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\right)$-linear action.

Definition 3.4. For $T > 0$ we define $a_T \in \mathrm{SL}_d(\mathbb{R})$ to be the unique matrix which acts on $P_0 \overset{\mathrm{def}}{=} \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}$ by scalar multiplication of a factor of $T^{\frac{1}{2(d-1)}}$ and on $P_0^{\perp(Q^*)} \overset{\mathrm{def}}{=} \mathrm{Span}_{\mathbb{R}}\{M\mathbf{e}_d\}$ by scalar multiplication of a factor of $T^{-1/2}$.

Corollary 3.5. *It holds that* $a_T \in \mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$, $\forall T > 0$, *and* $\mathbf{L}_{a_T}(\mathbb{R}) = \mathbf{L}_{I_d}(\mathbb{R})$.

Proof. In order to validate that $a_T \in \mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$, we show below that

$$\tau(a_T) = \sqrt{T}\mathbf{e}_d. \tag{3.13}$$

We have

$$\left\langle \mathbf{e}_i, \tau(a_T) \right\rangle = \left\langle \mathbf{e}_i, \left(a_T^t\right)^{-1}\mathbf{e}_d \right\rangle =$$

$$\left\langle a_T^{-1}\mathbf{e}_i, \mathbf{e}_d \right\rangle \underbrace{=}_{\text{Definition 3.4}} T^{\frac{1}{2}}\delta_{i,d},$$

which implies (3.13). Next, since P_0 and $P_0^{\perp(Q^*)}$ are invariant spaces under the left linear $\theta\left(\mathbf{H}_{\mathbf{e}_d}(\mathbb{Q})\right)$ action, and since a_T acts by scalar multiplication on each of these spaces, it follows that a_T is in the center of $\theta\left(\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\right)$. Therefore

$$\begin{aligned}\mathbf{L}_{a_T}(\mathbb{R}) &= \left\{\left(w, (a_T)^{-1}\theta(w)a_T\right) \mid w \in \mathbf{H}_{\tau(a_T)}(\mathbb{R})\right\} \\ &= \left\{(w, \theta(w)) \mid w \in \mathbf{H}_{\tau(a_T)}(\mathbb{R})\right\}.\end{aligned}$$

Now we have by (3.13) that $\mathbf{H}_{\tau(a_T)}(\mathbb{R}) = \mathbf{H}_{\sqrt{T}\mathbf{e}_d}(\mathbb{R}) = \mathbf{H}_{\mathbf{e}_d}(\mathbb{R}) = \mathbf{H}_{\tau(I_d)}(\mathbb{R})$, which in turn implies that $\mathbf{L}_{a_T}(\mathbb{R}) = \mathbf{L}_{I_d}(\mathbb{R})$. □

For the rest of the paper we denote

$$H \overset{\text{def}}{=} L_{I_d}(\mathbb{R}) = \{(w, \theta(w)) \mid w \in \mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\}. \tag{3.14}$$

By Corollary 3.5 we have for all $T > 0$ the identification

$$\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \cong H\backslash(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R}) \tag{3.15}$$

by the orbit map

$$H\,(\rho, \eta) \mapsto \theta(\rho^{-1})a_T\eta.$$

We define

$$\pi_{\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}),$$

by

$$\pi_{\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}}\left(\theta(\rho^{-1})a_T\eta\right) \overset{\text{def}}{=} \theta(\rho^{-1})I_d\eta = \theta(\rho^{-1})\eta, \tag{3.16}$$

which is clearly equivariant with respect to the action of $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ on $\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$ and $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ (since a_T has the same stabilizer $\forall T > 0$).

3.2.1. *The natural measure on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$.* We now define a $(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ invariant measure on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ using the identification (3.15). We choose Haar measures $m_{\mathrm{SO}_Q(\mathbb{R})}$, $m_{\mathrm{ASL}_{d-1}(\mathbb{R})}$ on $\mathrm{SO}_Q(\mathbb{R})$ and $\mathrm{ASL}_{d-1}(\mathbb{R})$ respectively with a normalization we discuss in Section 4.3.2, and we observe that H is compact (by (3.14), we have $H \cong \mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$, and recall that $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ is compact under our Standing Assumption). Then on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ we can define the following measure

$$\mu_{\mathcal{Z}} \overset{\text{def}}{=} (\pi_H)_* m_{\mathrm{SO}_Q(\mathbb{R})} \otimes m_{\mathrm{ASL}_{d-1}(\mathbb{R})}, \tag{3.17}$$

where $\pi_H : (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R}) \to H\backslash(\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ is the natural quotient map.

3.3. **Statistics of $\mathcal{Z}_N(\mathbb{Z})$ as $N \to \infty$.** We are now ready to discuss our main results. Let $N \in \mathbb{N}$ and consider the following atomic measure on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$

$$\nu_N^{\mathcal{Z}} = \frac{1}{|\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{x \in \mathcal{Z}_N(\mathbb{Z})} \delta_{\pi_{\mathcal{Z}_N}(x)}. \tag{3.18}$$

The following definition amounts to a congruence condition of the range of $N \in \mathbb{N}$ for which we are able to obtain the asymptotics of the measures ν_N.

Definition 3.6. Given a prime p and a rational quadratic form Q, we say that $\mathbf{v} \in \mathbb{Q}^d$ is (Q, p) co-isotropic if $\mathbf{H}_{\mathbf{v}}(\mathbb{Q}_p)$ (the stabilizer of $\mathbf{v}$ in the group $\mathrm{SO}_Q(\mathbb{Q}_p)$) is non-compact. We say that $N \in \mathbb{N}$ has the (Q, p) co-isotropic property if there exists $\mathbf{v} \in \mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})$ which is (Q, p) co-isotropic.

Remark. For $\mathbf{v} \in \mathbb{Q}^d$ we have $\mathbf{H}_{\mathbf{v}}(\mathbb{Q}_p)$ is non-compact if and only if $\exists \mathbf{u} \in \mathbb{Q}_p^d \otimes \mathbf{v}^{\perp(Q)}$ such that $Q(\mathbf{u}) = 0$, where $\mathbf{v}^{\perp(Q)}$ is the orthogonal hyperplane *with respect to Q*. We note that if Q is a rational quadratic form in $d \geq 6$ variables, then the form induced on $\mathbb{Q}_p^d \otimes \mathbf{v}^{\perp(Q)}$ is in $d \geq 5$ variables and by [Cas78] (see [Cas78, Lemma 1.7]), we obtain that any $\mathbf{v} \in \mathbb{Q}^d$ is (Q, p) co-isotropic, for any prime p.

Our main results are as follows.

Theorem 3.7. *Assume that $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ is a sequence of integers satisfying the (Q, p_0) co-isotropic property for some fixed odd prime p_0, and $T_n \to \infty$. Then for all $f \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}))$ we have that*

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Z}}(f) = \mu_{\mathcal{Z}}(f).$$

Next, for $N \in \mathbb{N}$ and $q \in \mathbb{N}$ we consider the following measure on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_{\vartheta_q(T)}(\mathbb{Z}/(q))$ given by

$$\nu_N^{\mathcal{Z},q} = \frac{1}{|\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{x \in \mathcal{Z}_N(\mathbb{Z})} \delta_{(\pi_{\mathcal{Z}_N}(x), \vartheta_q(x))}. \tag{3.19}$$

Theorem 3.8. *Let $q \in 2\mathbb{N}+1$. In addition to our Standing Assumption on the form Q, assume that Q is non-singular modulo q (see Definition 2.1). Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence of integers satisfying the (Q, p_0) co-isotropic property for some odd prime p_0, and assume that there is a fixed $a \in (\mathbb{Z}/(q))^{\times}$ such that for all $n \in \mathbb{N}$ it holds $\vartheta_q(T_n) = a$. Then, for all $f \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_a(\mathbb{Z}/(q)))$ we have that*

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Z},q}(f) = \mu_{\mathcal{Z}} \otimes \mu_{\mathcal{Z}_a(\mathbb{Z}/(q))}(f),$$

where $\mu_{\mathcal{Z}_a(\mathbb{Z}/(q))}$ is the uniform probability measure on $\mathcal{Z}_a(\mathbb{Z}/(q))$.

4. Moduli spaces - refinements of [AES16a]

This section discusses our results which generalize [AES16a]. We note that these results are also conceptually similar to the Linnik type results that we discussed in Section 1.1, and are roughly described as follows. We will introduce moduli spaces $\mathcal{Y}(\mathbb{R})$ and $\mathcal{X}(\mathbb{R})$ which are fiber bundles over $\mathbb{R}^d \smallsetminus \mathbf{0}$ with fibers that are isomorphic to $Y_{d-1} = \mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})$ and $X_{d-1} = \mathrm{SL}_{d-1}(\mathbb{R})/\mathrm{SL}_{d-1}(\mathbb{Z})$. Taking the preimage of a quadratic variety $\mathcal{H}_T(\mathbb{R}) \subseteq \mathbb{R}^d \smallsetminus \mathbf{0}$ by the projection map to $\mathbb{R}^d \smallsetminus \mathbf{0}$, we obtain for $\mathcal{M} \in \{\mathcal{Y}, \mathcal{X}\}$ a one parameter family of subbundles $\mathcal{M}_T(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$ over $\mathcal{H}_T(\mathbb{R})$, which are all isomorphic. We will define a geometrically motivated homeomorphism $\pi_{\mathcal{M}_T} : \mathcal{M}_T(\mathbb{R}) \to \mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R})$, and our main results, Theorems 4.8-4.9, will be about the distribution of $\pi_{\mathcal{M}_T}(\mathcal{M}_T(\mathbb{Z}))$ in $\mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R})$, where $\mathcal{M}_T(\mathbb{Z}) = \mathcal{M}(\mathbb{Z}) \cap \mathcal{M}_T(\mathbb{R})$.

The structure of this section is as follows:

- Sections 4.1-4.2 discuss $\mathcal{X}(\mathbb{R})$ and $\mathcal{Y}(\mathbb{R})$.
- Section 4.3 discusses the subbundles $\mathcal{M}_T(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$, $T > 0$, the homeomorphisms $\pi_{\mathcal{M}_T}$, and some natural measures on these subbundles.
- Section 4.4 states Theorems 4.8-4.9.
- Section 4.5 relying on Theorems 4.8-4.9 proves Theorems 1.1 and 1.2 from the introduction.

4.1. The moduli space of oriented rank $d-1$ discrete subgroups of $\mathbb{R}^d$.

Instead of considering the shapes of orthogonal lattices to integral vectors (which we introduced in Section 1.2), we may consider the orthogonal lattices "as is" by

$$\mathcal{X}(\mathbb{Z}) \overset{\text{def}}{=} \left\{ (\Lambda_{\mathbf{v}}, \mathbf{v}) \mid \mathbf{v} \in \mathbb{Z}^d_{\text{prim}},\ \Lambda_{\mathbf{v}} = \mathbb{Z}^d \cap \mathbf{v}^{\perp} \right\}.$$

We will now describe a homogeneous space $\mathcal{X}(\mathbb{R})$, which can be thought of as a natural ambient space that contains $\mathcal{X}(\mathbb{Z})$.

We let $X_{d-1,d}$ be the space of rank $(d-1)$-discrete subgroups of $\mathbb{R}^d$, and we define $\mathcal{X}(\mathbb{R}) \subseteq X_{d-1,d} \times \mathbb{R}^d \smallsetminus \mathbf{0}$ by

$$\mathcal{X}(\mathbb{R}) \overset{\text{def}}{=} \left\{ (\Lambda, \mathbf{v}) \in X_{d-1,d} \times \mathbb{R}^d \smallsetminus \mathbf{0} \mid \mathbf{v} \perp \Lambda,\ \mathrm{covol}(\Lambda) = \|\mathbf{v}\| \right\},$$

and as we now show, $\mathcal{X}(\mathbb{R})$ is a homogeneous space. We consider the left action of $\mathrm{SL}_d(\mathbb{R})$ on $X_{d-1,d} \times \mathbb{R}^d$ given by

$$g \cdot (\Lambda, \mathbf{v}) \overset{\text{def}}{=} (g\Lambda, \theta(g)\mathbf{v}),\ g \in \mathrm{SL}_d(\mathbb{R}). \tag{4.1}$$

Lemma 4.1. *It holds that*

$$\mathcal{X}(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R}) \cdot (\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d).$$

Proof. It is straightforward to verify that $\mathrm{SL}_d(\mathbb{R})$ acts transitively on $X_{d-1,d}$. The rest follows by (2.5) and Lemma 2.2. □

By noting that the stabilizer of $(\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d)$ is the subgroup $\mathrm{ASL}_{d-1}(\mathbb{Z})U \cong \mathrm{SL}_{d-1}(\mathbb{Z}) \ltimes \mathbb{R}^{d-1}$, where

$$U = \left\{ \begin{pmatrix} I_{d-1} & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} \mid \mathbf{v} \in \mathbb{R}^{d-1} \right\},$$

we deduce the identification

$$\mathcal{X}(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R})/\left(\mathrm{ASL}_{d-1}(\mathbb{Z})U\right).$$

By restricting the above $\mathrm{SL}_d(\mathbb{R})$ action on $\mathcal{X}(\mathbb{R})$ to $\mathrm{SL}_d(\mathbb{Z})$, we obtain the following observation.

Lemma 4.2. *It holds that*

$$\mathcal{X}(\mathbb{Z}) = \mathrm{SL}_d(\mathbb{Z}) \cdot \left(\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d\right).$$

Proof. Since the columns of $g \in \mathrm{SL}_d(\mathbb{Z})$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$, and since $\tau(g)$ is orthogonal to the first $d-1$ columns of g (see (2.5)), we have

$$\begin{aligned}\Lambda_{\tau(g)} =& \mathrm{Span}_{\mathbb{Z}}\{g\mathbf{e}_1, .., g\mathbf{e}_{d-1}\}\\ =& g \cdot \mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}.\end{aligned}$$

Finally, we note that $\tau\left(\mathrm{SL}_d(\mathbb{Z})\right) = \mathbb{Z}^d_{\mathrm{prim}}$ (to recall τ, see (2.3)) □

We now observe that the map $\pi^{\mathcal{X}}_{vec} : \mathcal{X}(\mathbb{R}) \to \mathbb{R}^d \smallsetminus \mathbf{0}$ defined by

$$\pi^{\mathcal{X}}_{vec}\left((\Lambda, \mathbf{v})\right) \overset{\text{def}}{=} \mathbf{v}, \tag{4.2}$$

gives $\mathcal{X}(\mathbb{R})$ the structure of a fiber bundle with fibers isomorphic to X_{d-1}. Indeed

$$\begin{aligned}(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v}_0) =& \{(\Lambda, \mathbf{v}_0) \in X_{d-1,d} \times \{\mathbf{v}_0\} \mid \Lambda \perp \mathbf{v}_0,\ \mathrm{covol}(\Lambda) = \|\mathbf{v}_0\|\}\\ \cong& \{\Lambda \in X_{d-1,d} \mid \mathbf{v}_0 \perp \Lambda,\ \mathrm{covol}(\Lambda) = \|\mathbf{v}_0\|\}\\ \cong& X_{d-1}.\end{aligned}$$

4.1.1. *The extension of the "shape" map to $\mathcal{X}(\mathbb{R})$.* We now reconsider the map shape : $\mathbb{Z}^d_{\mathrm{prim}} \to \mathcal{S}_{d-1}$ from Section 1.2 and extend it to $\mathcal{X}(\mathbb{R})$.

We note that $\mathrm{SO}_d(\mathbb{R})$ acts on $\mathcal{X}(\mathbb{R})$ by

$$\rho \cdot (\Lambda, \mathbf{v}) \overset{\text{def}}{=} (\rho\Lambda, \rho\mathbf{v}),\quad \rho \in \mathrm{SO}_d(\mathbb{R}),\ (\Lambda, \mathbf{v}) \in \mathcal{X}(\mathbb{R}),$$

which is the restriction of (4.1) to $\mathrm{SO}_d(\mathbb{R})$, and we let $K \overset{\text{def}}{=} \mathrm{SO}_d(\mathbb{R}) \cap \mathrm{ASL}_{d-1}(\mathbb{R})$ be the stabilizer of $\mathbf{e}_d$ by the ordinary $\mathrm{SO}_d(\mathbb{R})$ left linear action on $\mathbb{R}^d$. Since $(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d)$ is identified with the space of of full rank lattices in $\mathbb{R}^{d-1}$, and since K acts on $(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d)$ by Euclidean rotations in the plane $\mathbf{e}_d^{\perp}$, we obtain that $\mathcal{S}_{d-1}$ identifies naturally with the space of K-orbits in $(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d)$. Since $(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d)$ is the $\mathrm{ASL}_{d-1}(\mathbb{R})$ orbit passing through $(\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d)$, we get that $(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d) \cong \mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U$, and we conclude that

$$\mathcal{S}_{d-1} \cong K\backslash\mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U. \tag{4.3}$$

Next, for $\mathbf{v} \in \mathbb{R}^d \smallsetminus \mathbf{0}$ we choose a $\rho_{\mathbf{v}} \in \mathrm{SO}_d(\mathbb{R})$ such that $\rho_{\mathbf{v}}\mathbf{v} = \|\mathbf{v}\|\,\mathbf{e}_d$, and for $t > 0$ we define $d_t \in \mathrm{SL}_d(\mathbb{R})$ by

$$d_t \overset{\text{def}}{=} \begin{pmatrix} t^{-1/(d-1)}I_{d-1} & \\ & t\end{pmatrix}.$$

Then

$$d_{\|\mathbf{v}\|}\rho_{\mathbf{v}} \cdot (\Lambda, \mathbf{v}) = \left((d_{\|\mathbf{v}\|}\rho_{\mathbf{v}})\Lambda, \mathbf{e}_d\right) \in (\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d),$$

and we note that $(d_{\|\mathbf{v}\|}\rho_{\mathbf{v}})\Lambda = \rho_{\mathbf{v}}(\|\mathbf{v}\|^{\,-1/(d-1)}\Lambda)$. We observe that the K orbit $K(d_{\|\mathbf{v}\|}\rho_{\mathbf{v}})\Lambda \subseteq (\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{e}_d)$ is independent of the choice of $\rho_{\mathbf{v}}$, and we define shape : $\mathcal{X}(\mathbb{R}) \to \mathcal{S}_{d-1}$ by

$$\mathrm{shape}(\Lambda, \mathbf{v}) \overset{\text{def}}{=} K(d_{\|\mathbf{v}\|}\rho_{\mathbf{v}})\Lambda,\ (\Lambda, \mathbf{v}) \in \mathcal{X}(\mathbb{R}). \tag{4.4}$$

4.2. **The space of unimodular lattices with a marked rational hyperplane.** As in [AES16a], we describe an object that extracts more information from a primitive vector $\mathbf{v}$ than we get from $\Lambda_{\mathbf{v}}$ by telling us how $\Lambda_{\mathbf{v}}$ is completed to $\mathbb{Z}^d$. Namely, for $\mathbf{v} \in \mathbb{Z}^d_{\text{prim}}$, we let $\mathbf{w} \in \mathbb{Z}^d_{\text{prim}}$ such that

$$\Lambda_{\mathbf{v}} \oplus \mathbf{w}\mathbb{Z} = \mathbb{Z}^d.$$

We say that $\mathbf{w}$ completes $\Lambda_{\mathbf{v}}$ in a positive direction if $\langle \mathbf{v}, \mathbf{w} \rangle > 0$. This data is concisely recorded by the triple $(\mathbb{Z}^d, \mathbf{v}^{\perp}, \mathbf{v})$ in a natural way, and motivates us to consider

$$\mathcal{Y}(\mathbb{Z}) \overset{\text{def}}{=} \left\{ (\mathbb{Z}^d, \mathbf{v}^{\perp}, \mathbf{v}) \mid \mathbf{v} \in \mathbb{Z}^d_{\text{prim}} \right\}.$$

As for $\mathcal{X}(\mathbb{R})$ and $\mathcal{X}(\mathbb{Z})$, we will now describe $\mathcal{Y}(\mathbb{R})$ as a homogeneous space that can be thought of as a natural ambient space containing $\mathcal{Y}(\mathbb{Z})$.

We let X_d be the space of unimodular lattices in $\mathbb{R}^d$ and we denote by $\mathrm{Gr}(d-1, d)$ the space of hyperplanes in $\mathbb{R}^d$. For $L \in X_d$ we define $\mathrm{Gr}(d-1,d)_L$ to be the space of L-rational hyperplanes, namely

$$\mathrm{Gr}(d-1,d)_L \overset{\text{def}}{=} \left\{ P \in \mathrm{Gr}(d-1,d) \mid P \cap L \text{ is a rank } (d-1)\text{-discrete group of } \mathbb{R}^d \right\},$$

and we define

$$\mathcal{Y}(\mathbb{R}) \overset{\text{def}}{=} \left\{ (L, P, \mathbf{v}) \in X_d \times \mathrm{Gr}(d-1,d) \times \mathbb{R}^d \mid P \in \mathrm{Gr}(d-1,d)_L,\ P \perp \mathbf{v},\ \|\mathbf{v}\| = \mathrm{covol}(L \cap P) \right\}. \tag{4.5}$$

We define a left action of $\mathrm{SL}_d(\mathbb{R})$ on $X_d \times \mathrm{Gr}(d-1,d) \times \mathbb{R}^d$ by

$$g \cdot (L, P, \mathbf{v}) \overset{\text{def}}{=} (gL, gP, \theta(g)\mathbf{v}),\ g \in \mathrm{SL}_d(\mathbb{R}). \tag{4.6}$$

Lemma 4.3. *It holds that* $\mathcal{Y}(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R}) \cdot \left(\mathbb{Z}^d, \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d\right)$.

Proof. It is well known that $\mathrm{SL}_d(\mathbb{R})$ acts transitively on X_d and that the stabilizer in $\mathrm{SL}_d(\mathbb{R})$ of a lattice L acts transitively on $\mathrm{Gr}(d-1,d)_L$. The rest follows by (2.5) and Lemma 2.2. $\square$

We observe that the stabilizer of $\left(\mathbb{Z}^d, \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d\right)$ is $\mathrm{ASL}_{d-1}(\mathbb{Z})$, hence

$$\mathcal{Y}(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}).$$

By restricting the action of $\mathrm{SL}_d(\mathbb{R})$ to $\mathrm{SL}_d(\mathbb{Z})$, we obtain the following observation which we leave the reader to verify.

Lemma 4.4. *We have* $\mathcal{Y}(\mathbb{Z}) = \mathrm{SL}_d(\mathbb{Z}) \cdot \left(\mathbb{Z}^d, \mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, .., \mathbf{e}_{d-1}\}, \mathbf{e}_d\right)$.

4.2.1. *The projection to $\mathcal{X}(\mathbb{R})$.* A natural connection between $\mathcal{Y}(\mathbb{R})$ and $\mathcal{X}(\mathbb{R})$ is given by the projection $\pi_{\cap} : \mathcal{Y}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ defined by

$$\pi_{\cap}\left((L, P, \mathbf{w})\right) \overset{\text{def}}{=} (L \cap P, \mathbf{w}). \tag{4.7}$$

We observe that for $(\Lambda, \mathbf{v}) \in \mathcal{X}(\mathbb{R})$, the fiber $\pi_{\cap}^{-1}\left((\Lambda, \mathbf{v})\right)$ consists of the triples of the form

$$\left(\Lambda + \left(\mathbf{u} + \frac{1}{\mathrm{covol}(\Lambda)}\mathbf{v}\right)\mathbb{Z}, \Lambda \otimes \mathbb{R}, \mathbf{v}\right),$$

where $\mathbf{u} \in \Lambda \otimes \mathbb{R}$. Namely, the fiber $\pi_{\cap}^{-1}\left((\Lambda, \mathbf{v})\right)$ can be identified with $(\Lambda \otimes \mathbb{R})/\Lambda \cong \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$. In terms of coset spaces, we have

$$\pi_{\cap}\left(g\mathrm{ASL}_{d-1}(\mathbb{Z})\right) = g\left(\mathrm{ASL}_{d-1}(\mathbb{Z})U\right), \tag{4.8}$$

which implies that

$$\pi_{\cap}^{-1}\left(g(\mathrm{ASL}_{d-1}(\mathbb{Z})U)\right) = g\mathrm{ASL}_{d-1}(\mathbb{Z})U/\mathrm{ASL}_{d-1}(\mathbb{Z}) \cong \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}.$$

In particular, $\pi_{\cap}$ has compact fibers.

Remark. We note that the analogue space to $\mathcal{Y}(\mathbb{R})$ for dimensions $3 \leq k < d-1$ in d-space was recently considered in [AMW21] which studies a problem similar to the one addressed in the current paper.

4.2.2. $\mathcal{Y}(\mathbb{R})$ *as the space of oriented* $(d-1)$*-grids in* $\mathbb{R}^d$. We will now present another description of $\mathcal{Y}(\mathbb{R})$ that, in our opinion, is more geometrically transparent. This description more clearly connects $\mathcal{Y}(\mathbb{R})$ to the notion of grid shapes considered in [AES16a] (which actually motivated us to consider the space $\mathcal{Y}(\mathbb{R})$).

We recall the space of unimodular grids in $\mathbb{R}^{d-1}$ (translates of unimodular lattices)

$$Y_{d-1} \overset{\text{def}}{=} \left\{ \Lambda + \mathbf{u} \mid \Lambda \in X_{d-1},\ \mathbf{u} \in \mathbb{R}^{d-1} \right\}.$$

For a triple $(L, P, \mathbf{v}) \in \mathcal{Y}(\mathbb{R})$ we let $\mathbf{w} \in L$ such that $(L \cap P) \oplus \mathbf{w}\mathbb{Z} = L$ and $\langle \mathbf{w}, \mathbf{v} \rangle > 0$. We denote by $\pi_P^{\perp} : \mathbb{R}^d \to P$ the orthogonal projection. Then

$$\pi_P^{\perp}\left((L \cap P) + \mathbf{w}\right) = (L \cap P) + \pi_P^{\perp}(\mathbf{w}), \tag{4.9}$$

which can be viewed as a grid that sits in the hyperplane P, is independent of the choice of $\mathbf{w}$. By defining $f(L, P, \mathbf{v}) \overset{\text{def}}{=} ((L \cap P) + \pi_P^{\perp}(\mathbf{w}), \mathbf{v})$, obtain an identification of $\mathcal{Y}(\mathbb{R})$ with

$$\{(\Lambda + \mathbf{u}, \mathbf{v}) \mid \Lambda \in X_{d-1,d},\ \mathbf{u} \in \Lambda \otimes \mathbb{R},\ \mathbf{v} \perp \Lambda,\ \|\mathbf{v}\| = \operatorname{covol}(\Lambda)\}.$$

Using the above description of $\mathcal{Y}(\mathbb{R})$, we see that the projection $\pi_{vec}^{\mathcal{Y}} : \mathcal{Y}(\mathbb{R}) \to \mathbb{R}^d \smallsetminus \mathbf{0}$ defined by

$$\pi_{vec}^{\mathcal{Y}}\left((\Lambda + \mathbf{u}, \mathbf{v})\right) \overset{\text{def}}{=} \mathbf{v}, \tag{4.10}$$

endows $\mathcal{Y}(\mathbb{R})$ with the structure of a fiber bundle with fibers isomorphic to Y_{d-1}.

A quick summary - hierarchy of moduli spaces. We summarize the discussion concerning the moduli spaces by the following commuting diagram

$$\begin{tikzcd} \mathcal{Y}(\mathbb{R}) \arrow[r, "\pi_\cap"] \arrow[rr, bend right, "\pi_{vec}^{\mathcal{Y}}"'] & \mathcal{X}(\mathbb{R}) \arrow[r, "\pi_{vec}^{\mathcal{X}}"] & \mathbb{R}^d \smallsetminus \mathbf{0} \end{tikzcd} \tag{4.11}$$

and we note that in terms of coset spaces, the following diagram is equivalent to (4.11)

$$\begin{tikzcd} \mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}) \arrow[r] \arrow[rr, bend right] & \mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U \arrow[r] & \mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{R}) \end{tikzcd} \tag{4.12}$$

where all the maps are the natural projections.

4.3. **Moduli level sets, their measures and their isomorphisms.** Let Q be as in our Standing Assumption. For $T > 0$ we define

$$\mathcal{Y}_T(\mathbb{R}) \overset{\text{def}}{=} \left(\pi_{vec}^{\mathcal{Y}}\right)^{-1}(\mathcal{H}_T(\mathbb{R})), \quad \mathcal{X}_T(\mathbb{R}) \overset{\text{def}}{=} \left(\pi_{vec}^{\mathcal{X}}\right)^{-1}(\mathcal{H}_T(\mathbb{R})), \tag{4.13}$$

namely

$$\mathcal{X}_T(\mathbb{R}) \overset{\text{def}}{=} \{(\Lambda, \mathbf{v}) \in \mathcal{X}(\mathbb{R}) \mid Q(\mathbf{v}) = T\},$$

and

$$\mathcal{Y}_T(\mathbb{R}) \overset{\text{def}}{=} \{(L, P, \mathbf{v}) \in \mathcal{Y}(\mathbb{R}) \mid Q(\mathbf{v}) = T\}.$$

We note the following commuting diagram (which follows from (4.11)) that describes the hierarchy between the above moduli level sets

$$\begin{tikzcd} \mathcal{Y}_T(\mathbb{R}) \arrow[r, "\pi_\cap"] \arrow[rr, bend right, "\pi_{vec}^{\mathcal{Y}}"'] & \mathcal{X}_T(\mathbb{R}) \arrow[r, "\pi_{vec}^{\mathcal{X}}"] & \mathcal{H}_T(\mathbb{R}) \end{tikzcd} \tag{4.14}$$

Next, we define the integral points lying on the moduli level sets. We consider for $N \in \mathbb{N}$

$$\mathcal{H}_{N,\text{prim}}(\mathbb{Z}) \overset{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{Z}^d_{\text{prim}} \mid Q(\mathbf{x}) = N \right\},$$

and we define

$$\mathcal{X}_N(\mathbb{Z}) \overset{\text{def}}{=} \mathcal{X}(\mathbb{Z}) \cap \mathcal{X}_N(\mathbb{R}) = \{(\Lambda_{\mathbf{v}}, \mathbf{v}) \mid \mathbf{v} \in \mathcal{H}_{N,\text{prim}}(\mathbb{Z})\},$$

and

$$\mathcal{Y}_N(\mathbb{Z}) \stackrel{\text{def}}{=} \mathcal{Y}(\mathbb{Z}) \cap \mathcal{Y}_N(\mathbb{R}) = \left\{ (\mathbb{Z}^d, \mathbf{v}^\perp, \mathbf{v}) \mid \mathbf{v} \in \mathcal{H}_{N,\text{prim}}(\mathbb{Z}) \right\}.$$

We also note the following commuting diagram

$$\begin{tikzcd} \mathcal{Y}_N(\mathbb{Z}) \arrow[r, leftrightarrow, "\pi_\cap"] \arrow[rr, leftrightarrow, bend right, "\pi^{\mathcal{Y}}_{vec}"'] & \mathcal{X}_N(\mathbb{Z}) \arrow[r, leftrightarrow, "\pi^{\mathcal{X}}_{vec}"] & \mathcal{H}_{N,\text{prim}}(\mathbb{Z}) \end{tikzcd} \tag{4.15}$$

where $\longleftrightarrow$ denotes bijection.

4.3.1. *Maps between level sets.* We now define the homeomorphisms $\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and $\pi_{\mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$, by using a geometrically natural scaling transformation.

We define $\pi_{\mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\pi_{\mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}}(\Lambda, \mathbf{v}) \stackrel{\text{def}}{=} \left(\frac{1}{T^{1/2(d-1)}}\Lambda, \frac{1}{\sqrt{T}}\mathbf{v} \right), \ (\Lambda, \mathbf{v}) \in \mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}). \tag{4.16}$$

We now give an alternative description of (4.16) using the $\mathrm{SL}_d(\mathbb{R})$ action on $\mathcal{X}(\mathbb{R})$. For $\mathbf{v} \in \mathcal{H}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$ we define the unique matrix $S_{T,\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ that acts by scalar multiplication of a factor $T^{-\frac{1}{2(d-1)}}$ on $P = \mathbf{v}^\perp$ and that acts by scalar multiplication of a factor $T^{1/2}$ on the line $\mathbb{R}\mathbf{v}$. Then, it follows for $(\Lambda, \mathbf{v}) \in \mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$ that

$$\pi_{\mathcal{X}_{Q(\sqrt{T}\mathbf{e}_d)}}(\Lambda, \mathbf{v}) = S_{T,\mathbf{v}} \cdot (\Lambda, \mathbf{v}) \underbrace{=}_{\text{recalling (4.1)}} (S_{T,\mathbf{v}}\Lambda, \theta(S_{T,\mathbf{v}})\mathbf{v}).$$

Next, using the matrices $S_{T,\mathbf{v}}$, $\mathbf{v} \in \mathbb{R}^d$, $T > 0$ which were defined above, we define $\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(L, P, \mathbf{v}) \stackrel{\text{def}}{=} S_{T,\mathbf{v}} \cdot (L, P, \mathbf{v}), \ (L, P, \mathbf{v}) \in \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}), \tag{4.17}$$

where $S_{T,\mathbf{v}} \cdot (L, P, \mathbf{v}) \underbrace{=}_{\text{recalling (4.6)}} (S_{T,\mathbf{v}}L, P, \theta(S_{T,\mathbf{v}})\mathbf{v}) = (S_{T,\mathbf{v}}L, P, \frac{1}{\sqrt{T}}\mathbf{v})$.

Remark. By identifying $\mathcal{Y}(\mathbb{R})$ as in Section 4.2.2, we observe that $\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}} : \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ takes the form

$$\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(\Lambda + \mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} \left(\frac{1}{T^{1/2(d-1)}}(\Lambda + \mathbf{u}), \frac{1}{T^{1/2}}\mathbf{v} \right). \tag{4.18}$$

It follows that $\pi_{\mathcal{X}_T}$ and $\pi_{\mathcal{Y}_T}$ are homeomorphisms for all $T > 0$, and we conclude the following commuting diagram

$$\begin{tikzcd} \mathcal{Y}_T(\mathbb{R}) \arrow[r, "\pi_\cap"] \arrow[d, "\pi_{\mathcal{Y}_T}"'] & \mathcal{X}_T(\mathbb{R}) \arrow[d, "\pi_{\mathcal{X}_T}"'] \\ \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \arrow[r, "\pi_\cap"] & \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R}) \end{tikzcd} \tag{4.19}$$

4.3.2. *Measures on moduli level sets.* As $\mathcal{Y}(\mathbb{R})$ and $\mathcal{X}(\mathbb{R})$ are fiber bundles over $\mathbb{R}^d \smallsetminus \mathbf{0}$, it follows (by (4.13)) that $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and $\mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$ are fiber bundles over the base space $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$. We will now define certain measures on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and $\mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by integrating the natural measures on the fibers of the maps $\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})$ and $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$, with respect to the measure on the base space $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$.

For $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ we denote by $g_\mathbf{v} \in \mathrm{SL}_d(\mathbb{R})$ a matrix satisfying

$$\tau(g_\mathbf{v}) \underbrace{=}_{\text{recalling (2.3)}} \theta(g_\mathbf{v})\mathbf{e}_d = \mathbf{v}.$$

Then, with the help of diagram (4.12), we observe that

$$\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v}) = g_{\mathbf{v}}\mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}),$$

and

$$\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v}) = g_{\mathbf{v}}\mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U,$$

which shows us explicitly the identification of the fibers of $\pi^{\mathcal{Y}}_{vec}$ with

$$Y_{d-1} \overset{\text{def}}{=} \mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}) = \left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{e}_d),$$

and the identification of the fibers of $\pi^{\mathcal{X}}_{vec}$ with

$$X_{d-1} \overset{\text{def}}{=} \mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U = \left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{e}_d). \tag{4.20}$$

We need the following technical definition which describes the normalization of the Haar measures we will be using.

Definition 4.5. Let G be a locally compact second countable group and let $\Gamma \leq G$ be a lattice. Let m_G be a left Haar measure on G and let $m_{G/\Gamma}$ be the unique left G-invariant probability measure on G/Γ. We say that m_G and $m_{G/\Gamma}$ are *Weil normalized* if for all $f \in C_c(G)$

$$\int_G f(x)dm_G(x) = \int_{G/\Gamma}\left(\sum_{\gamma\in\Gamma} f(x\gamma)\right)dm_{G/\Gamma}(x\Gamma).$$

To define a measure on $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$, we recall that $\mathrm{SO}_Q(\mathbb{R})$ acts transitively on $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ (by Witt's theorem, since we assume $Q(\mathbf{e}_d) \neq 0$) via the right action (2.1), which in turns implies the identification

$$\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R}) \cong \mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathrm{SO}_Q(\mathbb{R}),$$

where $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R}) \leq \mathrm{SO}_Q(\mathbb{R})$ denotes the stabilizer of $\mathbf{e}_d$. We let $m_{\mathrm{SO}_Q(\mathbb{R})}$ and $m_{\mathrm{SO}_Q(\mathbb{R})/\mathrm{SO}_Q(\mathbb{Z})}$ be Weil normalized, and we define the measure $\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}$ on $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})} \overset{\text{def}}{=} \left(\pi_{\mathbf{H}_{e_d}(\mathbb{R})}\right)_* m_{\mathrm{SO}_Q(\mathbb{R})},$$

where $\pi_{\mathbf{H}_{e_d}(\mathbb{R})} : \mathrm{SO}_Q(\mathbb{R}) \to \mathbf{H}_{e_d}(\mathbb{R})\backslash\mathrm{SO}_Q(\mathbb{R})$ is the natural quotient map ($\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}$ is well defined since we assume that $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ is compact).

We now proceed to define the measures on the fibers $\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})$ and $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$ for $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$. We let $m_{\mathrm{ASL}_{d-1}(\mathbb{R})}$ and $m_{Y_{d-1}}$ be Weil normalized, and we let $m_{X_{d-1}}$ the unique $\mathrm{ASL}_{d-1}(\mathbb{R})$ invariant measure on X_{d-1}. We define for $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ the measure $\mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})}$ on $\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})$ by

$$\mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})}(f) \overset{\text{def}}{=} \int f(g_{\mathbf{v}}x)dm_{Y_{d-1}}(x),\ \forall f \in C_c(\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v}))$$

and similarly, the measure $\mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}$ on $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$ by

$$\mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}(f) \overset{\text{def}}{=} \int f(g_{\mathbf{v}}x)dm_{X_{d-1}}(x),\ \forall f \in C_c(\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})).$$

We show now that $\mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})}$ and $\mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}$ are independent of the choice of $g_{\mathbf{v}}$. Indeed, if one chooses another $\tilde{g}_{\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ such that $\tau(\tilde{g}_{\mathbf{v}}) = \mathbf{v}$, then $\tau(g_{\mathbf{v}}^{-1}\tilde{g}_{\mathbf{v}}) = \mathbf{e}_d$, so that there exists $h \in \mathrm{ASL}_{d-1}(\mathbb{R})$, such that $\tilde{g}_{\mathbf{v}} = g_{\mathbf{v}}h$. Therefore we conclude for $\mathcal{M} \in \{X_{d-1}, Y_{d-1}\}$ that

$$\int f(\tilde{g}_{\mathbf{v}}x)dm_{\mathcal{M}}(x) = \int f(g_{\mathbf{v}}hx)dm_{\mathcal{M}}(x)$$

$$\underbrace{=}_{m_{\mathcal{M}} \text{ is } \mathrm{ASL}_{d-1}(\mathbb{R}) \text{ invariant}} \int f(g_{\mathbf{v}}x)dm_{\mathcal{M}}(x).$$

Finally, using the above, we define the following measures on the spaces $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and $\mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\mu_{\mathcal{Y}} \overset{\text{def}}{=} \int \mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})} \, d\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}(\mathbf{v}), \text{ and } \mu_{\mathcal{X}} \overset{\text{def}}{=} \int \mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})} \, d\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}(\mathbf{v}). \tag{4.21}$$

4.3.3. *Pushforwards.* We now turn to explain the relation between the measures $\mu_{\mathcal{Y}}$ and $\mu_{\mathcal{X}}$, as well as the connection between $\mu_{\mathcal{X}}$ and the natural measure on the space of shapes.

Recall that the map $\pi_\cap : \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$ defined in (4.7) has compact fibers, hence $(\pi_\cap)_* \mu_{\mathcal{Y}}$ is a well defined measure on $\mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R})$.

Lemma 4.6. *It holds that $(\pi_\cap)_* \mu_{\mathcal{Y}} = \mu_{\mathcal{X}}$.*

Proof. We notice that for all $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ it holds that $\pi_\cap(\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})) = \left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$, which shows that for all $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ the measure $(\pi_\cap)_*\mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})}$ is supported on $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$. Using (4.21), we conclude that it is sufficient to show

$$(\pi_\cap)_*\mu_{\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v})} = \mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}, \ \forall \mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R}). \tag{4.22}$$

in order to prove $(\pi_\cap)_* \mu_{\mathcal{Y}} = \mu_{\mathcal{X}}$.

We let $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$, and we observe that in terms of cosets, the restriction of $\pi_\cap$ to a fiber $\left(\pi^{\mathcal{Y}}_{vec}\right)^{-1}(\mathbf{v}) = g_{\mathbf{v}}\mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})$ takes the form

$$\pi_\cap(g_{\mathbf{v}}\eta\mathrm{ASL}_{d-1}(\mathbb{Z})) = g_{\mathbf{v}}\eta\mathrm{ASL}_{d-1}(\mathbb{Z})U, \quad \eta \in \mathrm{ASL}_{d-1}(\mathbb{R}),$$

(see (4.8)). Since the natural projection $\mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}) \to \mathrm{ASL}_{d-1}(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})U$ pushes $m_{Y_{d-1}}$ to $m_{X_{d-1}}$, we can deduce (4.22). □

Next, we recall the space of shapes $\mathcal{S}_{d-1} \overset{\text{def}}{=} K\backslash X_{d-1}$ (see Section 4.1.1), and we consider the product space

$$\mathcal{W} \overset{\text{def}}{=} \mathcal{S}_{d-1} \times \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R}).$$

We define the product measure $\mu_{\mathcal{W}} \overset{\text{def}}{=} \mu_{\mathcal{S}_{d-1}} \otimes \mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}$, where $\mu_{\mathcal{S}_{d-1}}$ is the push-forward of $m_{X_{d-1}}$ by a quotient from the left by K.

We define the map $(\text{shape} \times \pi^{\mathcal{X}}_{vec}) : \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{W}$ by

$$\left(\text{shape} \times \pi^{\mathcal{X}}_{vec}\right)(\Lambda, \mathbf{v}) \overset{\text{def}}{=} (\text{shape}(\Lambda, \mathbf{v}), \mathbf{v}), \tag{4.23}$$

where $\text{shape}(\Lambda, \mathbf{v})$ was defined by (4.4). As above, the map $\left(\text{shape} \times \pi^{\mathcal{X}}_{vec}\right)$ has compact fibers.

Lemma 4.7. *We have $\left(\text{shape} \times \pi^{\mathcal{X}}_{vec}\right)_* \mu_{\mathcal{X}} = \mu_{\mathcal{W}}$.*

Proof. Similarly to the proof of Lemma 4.6, we observe that it suffices to show that

$$(\text{shape})_*\mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})} = \mu_{\mathcal{S}_{d-1}}, \ \forall \mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R}). \tag{4.24}$$

We now describe $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v})$ in a more convenient way, which makes the description of $\text{shape}\,|_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}$ more transparent. Fix $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$. We recall the diagonal matrix (see Section 4.1.1)

$$d_{\|\mathbf{v}\|} \overset{\text{def}}{=} \begin{pmatrix} \|\mathbf{v}\|^{-1/(d-1)} I_{d-1} & \mathbf{0} \\ \mathbf{0} & \|\mathbf{v}\| \end{pmatrix},$$

and we let $\rho_{\mathbf{v}} \in \mathrm{SO}_d(\mathbb{R})$ such that $\rho_{\mathbf{v}}^{-1}\mathbf{e}_d = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. We denote

$$g_{\mathbf{v}} \overset{\text{def}}{=} \rho_{\mathbf{v}}^{-1} d_{\|\mathbf{v}\|}^{-1},$$

and we observe that

$$\tau(g_{\mathbf{v}}) = \rho_{\mathbf{v}}^{-1} d_{\|\mathbf{v}\|}\mathbf{e}_d = \rho_{\mathbf{v}}^{-1}\|\mathbf{v}\|\mathbf{e}_d = \mathbf{v}.$$

By using the action of $\mathrm{SL}_d(\mathbb{R})$ on $\mathcal{X}(\mathbb{R})$ (see (4.1)), we get $\left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{v}) = g_{\mathbf{v}} \cdot \left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{e}_d)$, and by recalling (4.4), we see that shape $|_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}$ takes the form

$$\text{shape}(g_{\mathbf{v}}\Lambda, \mathbf{v}) \underbrace{=}_{(4.4)} K g_{\mathbf{v}}^{-1} g_{\mathbf{v}} \Lambda = K\Lambda, \quad \forall \Lambda \perp \mathbf{e}_d \text{ such that } \mathrm{covol}(\Lambda) = 1. \tag{4.25}$$

Finally, by using (4.25), we see that the function $f_{\mathbf{v}} : X_{d-1} \to \mathbb{R}$, defined by

$$f_{\mathbf{v}}(x) \overset{\text{def}}{=} f \circ \text{shape}(g_{\mathbf{v}} x), \ x \in X_{d-1} = \left(\pi^{\mathcal{X}}_{vec}\right)^{-1}(\mathbf{e}_d),$$

is right K invariant, and by noting that

$$(\text{shape})_* \mu_{(\pi^{\mathcal{X}}_{vec})^{-1}(\mathbf{v})}(f) = \int f_{\mathbf{v}}(x) dm_{X_{d-1}}(x),$$

we obtain (4.24). □

4.4. **Statistics in moduli spaces.** We are now able to state our main results for the moduli spaces.

For $N \in \mathbb{N}$ and for $\mathcal{M} \in \{\mathcal{Y}, \mathcal{X}\}$, we define the following measures on $\mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\nu_N^{\mathcal{M}} \overset{\text{def}}{=} \frac{1}{|\mathcal{H}_{N,\text{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{x \in \mathcal{M}_N(\mathbb{Z})} \delta_{\pi_{\mathcal{M}_N}(x)},$$

(to recall $\pi_{\mathcal{M}_T}$ see (4.16) and (4.17)), and we define a measure on $\mathcal{W}$ by

$$\nu_N^{\mathcal{W}} \overset{\text{def}}{=} \frac{1}{|\mathcal{H}_{N,\text{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{\mathbf{v} \in \mathcal{H}_{N,\text{prim}}(\mathbb{Z})} \delta_{(\text{shape}(\Lambda_{\mathbf{v}}, \mathbf{v}), \frac{1}{\sqrt{N}}\mathbf{v})}.$$

Our first main theorem is as follows.

Theorem 4.8. *Assume that $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ such that $T_n \to \infty$ and such that for some fixed odd prime p_0, the (Q, p_0) co-isotropic property (to recall see Definition 3.6) holds. Then*

$$\lim_{n \to \infty} \nu_{T_n}^{\mathcal{M}}(f) = \mu_{\mathcal{M}}(f),$$

where $\mathcal{M} \in \{\mathcal{Y}, \mathcal{X}\}$, and $f \in C_c(\mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R}))$, or for $\mathcal{M} = \mathcal{W}$ and $f \in C_c(\mathcal{W})$.

Let $q \in \mathbb{N}$ and recall that ϑ_q denotes the natural reduction modulo q. For $N \in \mathbb{N}$ and for $\mathcal{M} \in \{\mathcal{Y}, \mathcal{X}\}$ we define measures on $\mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_{\vartheta_q(T)}(\mathbb{Z}/(q))$ by

$$\nu_N^{\mathcal{M},q} = \frac{1}{|\mathcal{H}_{N,\text{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{x \in \mathcal{M}_N(\mathbb{Z})} \delta_{(\pi_{\mathcal{M}_N}(x), \vartheta_q(\pi^{\mathcal{M}}_{vec}(x))},$$

and similarly a measure on $\mathcal{W} \times \mathcal{H}_{\vartheta_q(T)}(\mathbb{Z}/(q))$ by

$$\nu_N^{\mathcal{W},q} \overset{\text{def}}{=} \frac{1}{|\mathcal{H}_{N,\text{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{\mathbf{v} \in \mathcal{H}_{N,\text{prim}}(\mathbb{Z})} \delta_{(\text{shape}(\Lambda_{\mathbf{v}}, \mathbf{v}), \frac{1}{\sqrt{N}}\mathbf{v}, \vartheta_q(\mathbf{v}))}.$$

By adding some further assumptions on the sequence $\{T_n\}_{n=1}^{\infty}$ appearing in Theorem 4.8, we are able to obtain the following.

Theorem 4.9. *Let $q \in 2\mathbb{N} + 1$. In addition to our Standing Assumption on the form Q assume that Q is non-singular modulo q (see Definition 2.1). Let $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence of integers satisfying the (Q, p_0) for some odd prime p_0 and assume that there is a fixed $a \in (\mathbb{Z}/(q))^{\times}$ such that for all $n \in \mathbb{N}$ it holds $\vartheta_q(T_n) = a$. Then*

$$\lim_{n \to \infty} \nu_{T_n}^{\mathcal{M},q}(f) = \mu_{\mathcal{M}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q))}(f),$$

where $\mathcal{M} \in \{\mathcal{Y}, \mathcal{X}\}$, and $f \in C_c(\mathcal{M}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$, or for $\mathcal{M} = \mathcal{W}$ and $f \in C_c(\mathcal{W} \times \mathcal{H}_a(\mathbb{Z}/(q)))$.

4.5. **Proof of Theorems 1.1 and 1.2.** We now prove Theorems 1.1 and 1.2 by validating the assumptions of Theorems 4.8 and 4.9 for the form $Q_d(\mathbf{x}) \stackrel{\text{def}}{=} x_d^2 - \sum_{i=1}^{d-1} x_i^2$, for $d \geq 4$.

Fix $d \geq 4$. We observe that the form Q_d satisfies our Standing Assumption, since Q_d is clearly non-degenerate, since $Q_d(\mathbf{e}_d) = 1 \neq 0$ and since $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R}) \cong \mathrm{SO}_{d-1}(\mathbb{R})$ which is compact.

Since the determinant of Q_d's companion matrix is ± 1, the form Q_d is non-singular modulo p for any prime p.

We now claim that the sequence $\mathbb{N}$ has the $(Q_d, 5)$ co-isotropic property. Let $\mathbb{Q}_5$ be the field of 5-adic numbers. We note that $\sqrt{-1} \in \mathbb{Q}_5$ (by Hensel's lemma, since $2^2 = -1 \bmod 5$) and we observe that the plane

$$V \stackrel{\text{def}}{=} \mathrm{Span}_{\mathbb{Q}_5}\left\{\sqrt{-1}\mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_d\right\} \subseteq (\mathbb{Q}_5)^d,$$

consists of Q_d-isotropic vectors. For $N \in \mathbb{N}$ and for $\mathbf{v} \in \mathcal{H}_N(\mathbb{Q})$, we let $\mathbf{v}^{\perp(Q_d)}$ be the orthogonal space to $\mathbf{v}$ with respect to Q_d. Since $\mathbf{v}^{\perp(Q_d)} \otimes \mathbb{Q}_5$ is a $(d-1)$-dimensional subspace of $(\mathbb{Q}_5)^d$, we deduce that $V \cap \left(\mathbf{v}^{\perp(Q_d)} \otimes \mathbb{Q}_5\right) \neq \{\mathbf{0}\}$. By the remark below Definition 3.6 we deduce that the sequence $\mathbb{N}$ has the $(Q_d, 5)$ co-isotropic property.

We now verify that $\mathcal{H}_{N,\mathrm{prim}}(\mathbb{Z}) \neq \emptyset$ for all $N \in \mathbb{N}$. We recall that there exists $\mathbf{u} \in \mathbb{Z}^3_{\mathrm{prim}}$ such that

$$u_1^2 + u_2^2 + u_3^2 = m, \tag{4.26}$$

for all positive integers $m \neq 0, 4, 7$ modulo 8 (see e.g. [Gro85]). Since a square modulo 8 attains the residues $0, 1, 4$, for we deduce that all $N \in \mathbb{N}$ there exists $x_4 \in \mathbb{Z}$ such that $x_4^2 - N > 0$ and such that $x_4^2 - N \neq 0, 4, 7$, which implies by (4.26) that there exists $\mathbf{x} \in \mathbb{Z}^4_{\mathrm{prim}} \subseteq \mathbb{Z}^d_{\mathrm{prim}}$ such that

$$x_4^2 - x_1^2 - x_2^2 - x_3^2 = N.$$

5. The results for $\mathcal{Z}$ imply the results for $\mathcal{Y}$

Our goal in this section is to use Theorems 3.7 - 3.8 to deduce Theorems 4.8 - 4.9. We divide this section into two parts as follows.

- Section 5.1 proves Theorems 4.8 - 4.9 for $\mathcal{Y}$. This is the main difficulty in proving Theorems 4.8 - 4.9.
- Section 5.2 gives the proof for Theorems 4.8 - 4.9 for $\mathcal{X}$ and $\mathcal{W}$, which relies on Section 4.3.3 and Theorems 4.8 - 4.9 for $\mathcal{M} = \mathcal{Y}$.

5.1. **Proof of Theorems 4.8 - 4.9 for $\mathcal{Y}$.** We now outline our method for proving Theorems 4.8 - 4.9 for $\mathcal{M} = \mathcal{Y}$ which is based on the result of Theorems 3.7 - 3.8.

We claim that for all $T > 0$ it holds that

$$\mathcal{Y}_T(\mathbb{R}) \cong \mathcal{Z}_T(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}), \tag{5.1}$$

Indeed, we recall that $\mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z})$ identifies with $\mathcal{Y}(\mathbb{R})$ by the orbit map

$$\tau_{\mathcal{Y}}(g\mathrm{ASL}_{d-1}(\mathbb{Z})) \stackrel{\text{def}}{=} (g\mathbb{Z}^d, \mathrm{Span}_{\mathbb{R}}\{g\mathbf{e}_1, .., g\mathbf{e}_{d-1}\}, \tau(g)),\ g \in \mathrm{SL}_d(\mathbb{R}), \tag{5.2}$$

(see Section 4.2), and we observe that

$$\begin{aligned}\tau_{\mathcal{Y}}^{-1}(\mathcal{Y}_T(\mathbb{R})) &\underbrace{=}_{\text{recalling (4.13)}} \{g\mathrm{ASL}_{d-1}(\mathbb{Z}) \in \mathrm{SL}_d(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}) \mid \tau(g) \in \mathcal{H}_T(\mathbb{R})\} \\ &\underbrace{=}_{\text{recalling (4.13)}} \mathcal{Z}_T(\mathbb{R})/\mathrm{ASL}_{d-1}(\mathbb{Z}).\end{aligned} \tag{5.3}$$

Similarly, we obtain for all $N \in \mathbb{N}$ that

$$\mathcal{Y}_N(\mathbb{Z}) \cong \mathcal{Z}_N(\mathbb{Z})/\mathrm{ASL}_{d-1}(\mathbb{Z}). \tag{5.4}$$

Using (5.1), we can relate the measure $\mu_{\mathcal{Y}}$ on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ to the measure $\mu_{\mathcal{Z}}$ on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by using "unfolding", as we will now explain. For $f \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}))$ we obtain $\bar{f} \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}))$

by defining

$$\bar{f}(g\mathrm{ASL}_{d-1}(\mathbb{Z})) \overset{\mathrm{def}}{=} \sum_{\gamma\in\mathrm{ASL}_{d-1}(\mathbb{Z})} f(g\gamma), \tag{5.5}$$

and, as we show in Section 5.1.1, it holds that the map $f\mapsto\bar{f}$ is onto $C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}))$ and that $\mu_{\mathcal{Z}}(f)=\mu_{\mathcal{Y}}(\bar{f})$ for all $f\in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}))$.

Next, recall that $\pi_{\mathcal{Z}_T}:\mathcal{Z}_T(\mathbb{R})\to\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ (defined in (3.16)), is right $\mathrm{ASL}_{d-1}(\mathbb{R})$ equivariant, namely

$$\pi_{\mathcal{Z}_T}(g\eta)=\pi_{\mathcal{Z}_T}(g)\eta,\ \forall g\in\mathcal{Z}_T(\mathbb{R}),\ \eta\in\mathrm{ASL}_{d-1}(\mathbb{R}). \tag{5.6}$$

Using the equivariance of $\pi_{\mathcal{Z}_T}$ and using (5.1) we define $\pi^Q_{\mathcal{Y}_T}:\mathcal{Y}_T(\mathbb{R})\to\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\pi^Q_{\mathcal{Y}_T}(z\mathrm{ASL}_{d-1}(\mathbb{Z}))\overset{\mathrm{def}}{=}\pi_{\mathcal{Z}_T}(z)\mathrm{ASL}_{d-1}(\mathbb{Z}). \tag{5.7}$$

The main reason for introducing $\pi^Q_{\mathcal{Y}_T}$ is that by assuming the asymptotics of the form

$$\sum_{g\in\mathcal{Z}_N(\mathbb{Z})} f(\pi_{\mathcal{Z}_N}(g))\sim c(T)\mu_{\mathcal{Z}}(f),\ \text{as } N\to\infty,$$

we are able to obtain the asymptotics

$$\sum_{y\in\mathcal{Y}_N(\mathbb{Z})} \bar{f}(\pi^Q_{\mathcal{Y}_N}(y))\sim c(T)\mu_{\mathcal{Y}}(\bar{f}),\ \text{as } N\to\infty,$$

by observing that

$$\begin{aligned}\sum_{g\in\mathcal{Z}_N(\mathbb{Z})} f(\pi_{\mathcal{Z}_N}(g)) &= \sum_{g\mathrm{ASL}_{d-1}(\mathbb{Z})\in\mathcal{Z}_N(\mathbb{Z})/\mathrm{ASL}_{d-1}(\mathbb{Z})}\ \sum_{\gamma\in\mathrm{ASL}_{d-1}(\mathbb{Z})} f(\pi_{\mathcal{Z}_N}(g\gamma))\\ &\underbrace{=}_{(5.6)} \sum_{g\mathrm{ASL}_{d-1}(\mathbb{Z})\in\mathcal{Z}_N(\mathbb{Z})/\mathrm{ASL}_{d-1}(\mathbb{Z})}\ \sum_{\gamma\in\mathrm{ASL}_{d-1}(\mathbb{Z})} f(\pi_{\mathcal{Z}_N}(g)\gamma)\\ &\underbrace{=}_{(5.4)} \sum_{y\in\mathcal{Y}_N(\mathbb{Z})} \bar{f}(\pi^Q_{\mathcal{Y}_N}(y)),\end{aligned}$$

and by using that $\mu_{\mathcal{Z}}(f)=\mu_{\mathcal{Y}}(\bar{f})$.

However, we are interested in proving Theorems 4.8 - 4.9 for $\mathcal{M}=\mathcal{Y}$ which concern the asymptotics of averages of the form

$$\sum_{y\in\mathcal{Y}_N(\mathbb{Z})} \bar{f}(\pi_{\mathcal{Y}_N}(y)),\ \text{as } N\to\infty,$$

where $\pi_{\mathcal{Y}_T}:\mathcal{Y}_T(\mathbb{R})\to\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ was defined in (4.18). Fortunately, it turns out that $\pi^Q_{\mathcal{Y}_T}$ and $\pi_{\mathcal{Y}_T}$ differ asymptotically uniformly by a fixed map that preserves the measure $\mu_{\mathcal{Y}}$, allowing us to prove Theorems 4.8 - 4.9.

Remark. Observe that the right $\mathrm{SO}_Q(\mathbb{R})$-actions on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and on $\mathcal{Y}_T(\mathbb{R})$ given by

$$(L,P,\mathbf{v})\cdot\rho\overset{\mathrm{def}}{=}\left(\theta(\rho^{-1})L,\theta(\rho^{-1})P,\rho^{-1}\mathbf{v}\right),\ (L,P,\mathbf{v})\in\mathcal{Y}_s(\mathbb{R}),\ \rho\in\mathrm{SO}_Q(\mathbb{R}),$$

are equivariant with respect to the map π^Q_T. Yet, as we will see in Section 5.1.2, this statement is wrong in general for $\pi_{\mathcal{Y}_T}$.

The structure of the rest of the section is as follows:

- Section 5.1.1 relates the measure $\mu_{\mathcal{Y}}$ and $\mu_{\mathcal{Z}}$ by "unfolding".
- Section 5.1.2 compares $\pi^Q_{\mathcal{Y}_T}$ and $\pi_{\mathcal{Y}_T}$.
- Section 5.1.3 proves Theorems 4.8 - 4.9 for $\mathcal{M}=\mathcal{Y}$.

5.1.1. *Unfolding the measure on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$.* To relate the measure $\mu_{\mathcal{Z}}$ on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ (defined in Section 3.3) with $\mu_{\mathcal{Y}}$ (defined in Section 4.3.2), we now give $\mu_{\mathcal{Z}}$ a different description, which is conceptually similar to the definition of $\mu_{\mathcal{Y}}$. We observe that $\tau : \mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ endows $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ with a fiber bundle structure over $\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ with fibers being right $\mathrm{ASL}_{d-1}(\mathbb{R})$ cosets (to recall τ, see (2.3)). As for $\mu_{\mathcal{Y}}$, we define for each $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ a measure on the fiber $\tau^{-1}(\mathbf{v})$ by

$$\mu_{\tau^{-1}(\mathbf{v})}(f) \overset{\text{def}}{=} \int f(g_{\mathbf{v}}x)dm_{\mathrm{ASL}_{d-1}(\mathbb{R})}(x),\ \mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R}),\ f \in C_c(\tau^{-1}(\mathbf{v})),$$

where $g_{\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ is chosen such that $\tau(g_{\mathbf{v}}) = \mathbf{v}$. By integrating the measures on the fibers we define the measure $\nu_{\mathcal{Z}}$ on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ by

$$\nu_{\mathcal{Z}} \overset{\text{def}}{=} \int \mu_{\tau^{-1}(\mathbf{v})}\ d\mu_{\mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})}(\mathbf{v}). \tag{5.8}$$

We obtain the lemma below which we leave the reader to verify.

Lemma 5.1. *It holds that $\nu_{\mathcal{Z}} = \mu_{\mathcal{Z}}$, where $\mu_{\mathcal{Z}}$ was defined in* (3.17)*.*

The unfolding relation between $\mu_{\mathcal{Y}}$ and $\mu_{\mathcal{Z}}$ is given by the following lemma.

Lemma 5.2. *For all $f \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}))$ it holds that $\mu_{\mathcal{Z}}(f) = \mu_{\mathcal{Y}}(\bar{f})$, where $\bar{f}$ is given by* (5.5)*.*

Proof. Using Lemma 5.1 and by recalling the definition of $\mu_{\mathcal{Y}}$ in (4.21), we see that it is sufficient to prove that $\mu_{\tau^{-1}(\mathbf{v})}(f) = \mu_{(\pi^{\mathcal{Y}}_{vec})^{-1}(\mathbf{v})}(\bar{f})$ for all $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$. Let $g_{\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ such that $\tau(g_{\mathbf{v}}) = \mathbf{v}$, and recall that $m_{\mathrm{ASL}_{d-1}(\mathbb{R})}$ and $m_{Y_{d-1}}$ are Weil normalized (see Definition 4.5). Then,

$$\begin{aligned}\mu_{(\pi^{\mathcal{Y}}_{vec})^{-1}(\mathbf{v})}(\bar{f}) &= \int \left(\sum_{\gamma \in \mathrm{ASL}_{d-1}(\mathbb{Z})} f(g_{\mathbf{v}}x\gamma) \right) dm_{Y_{d-1}}(x\mathrm{ASL}_{d-1}(\mathbb{Z})) \\ &= \int f(g_{\mathbf{v}}x)dm_{\mathrm{ASL}_{d-1}(\mathbb{R})}(x) \\ &= \mu_{\tau^{-1}(\mathbf{v})}(f).\end{aligned}$$

□

We now turn to show that for all $T > 0$ the map $\bar{*} : C_c(\mathcal{Z}_T(\mathbb{R})) \to C_c(\mathcal{Y}_T(\mathbb{R}))$ defined by $f \mapsto \bar{f}$ is onto (to recall $\bar{f}$ see (5.5)). To prove the latter, we note the following general lemma.

Lemma 5.3. *Let G be a locally compact, second countable group, $K \leq G$ be compact, and $\Gamma \leq G$ be discrete. Then the map*

$$\bar{*} : C_c(K\backslash G) \to C_c(K\backslash G/\Gamma)$$

defined by $\bar{f}(Kg\Gamma) \overset{def}{=} \sum_{\gamma\in\Gamma} f(Kg\gamma)$ is onto.

Proof. We let $\pi_K : G/\Gamma \to K\backslash G/\Gamma$ be the natural map. Since K is compact, for $\varphi \in C_c(K\backslash G/\Gamma)$ it holds that $\varphi \circ \pi_K \in C_c(G/\Gamma)$. We recall that [Fol15, Proposition 2.50] tells us there exists $\tilde{f} \in C_c(G)$ such that

$$\varphi \circ \pi_K(g\Gamma) = \sum_{\gamma\in\Gamma} \tilde{f}(g\gamma).$$

We let m_K be the Haar probability measure on K and we observe that

$$\begin{aligned}\varphi \circ \pi_K(g\Gamma) &= \int \varphi \circ \pi_K(kg\Gamma) dm_K(k) \\ &= \int \left(\sum_{\gamma \in \Gamma} \tilde{f}(kg\gamma) \right) dm_K(k) \\ &= \sum_{\gamma \in \Gamma} \int \tilde{f}(kg\gamma) dm_K(k).\end{aligned}$$

where in the last line we used that for all $g \in G$, the sum $\sum_{\gamma \in \Gamma} \tilde{f}(kg\gamma)$ is a finite sum, where the number of summands is bounded uniformly in $k \in K$ (this follows by Lemma A.4). The proof is complete by denoting $f(Kg) \stackrel{\text{def}}{=} \int \tilde{f}(kg) dm_K(k)$ and by observing that $f \in C_c(K \backslash G)$. ☐

Let $G \stackrel{\text{def}}{=} (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$, $K \stackrel{\text{def}}{=} H$ which was defined in (3.14), and $\Gamma \stackrel{\text{def}}{=} \{\mathbf{e}\} \times \mathrm{ASL}_{d-1}(\mathbb{Z}) \leq (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$. Lemma 5.4 below shows that $\mathcal{Y}_T(\mathbb{R}) \cong K \backslash G / \Gamma$. Since $\mathcal{Z}_T(\mathbb{R}) \cong K \backslash G$, the proof that $\bar{*} : C_c(\mathcal{Z}_T(\mathbb{R})) \to C_c(\mathcal{Y}_T(\mathbb{R}))$ is onto will be done by Lemma 5.3 and Lemma 5.4.

Lemma 5.4. *For all* $T > 0$, $H \backslash (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R}) / \mathrm{ASL}_{d-1}(\mathbb{Z})$ *is homeomorphic to* $\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$, *by the map*

$$\tilde{\Phi}(H(\rho, \eta)\mathrm{ASL}_{d-1}(\mathbb{Z})) = \tau_{\mathcal{Y}} \left(\theta(\rho^{-1}) a_T \eta \mathrm{ASL}_{d-1}(\mathbb{Z}) \right),$$

where $\tau_{\mathcal{Y}}$ *is given by* (5.2) *and* $a_T \in \mathrm{SL}_d(\mathbb{R})$ *is given by Definition 3.4.*

Proof. We recall that $\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$ is identified with $H \backslash (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ by the map

$$\Phi(H(\rho, \eta)) = \theta(\rho^{-1}) a_T \eta,$$

(to recall, see (3.14) defining H, and see below (3.15)) which shows that

$$\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}) / \mathrm{ASL}_{d-1}(\mathbb{Z}) \cong H \backslash (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R}) / \mathrm{ASL}_{d-1}(\mathbb{Z})$$

by the map

$$\tilde{\Phi}(H(\rho, \eta)\mathrm{ASL}_{d-1}(\mathbb{Z})) = \theta(\rho^{-1}) a_T \eta \mathrm{ASL}_{d-1}(\mathbb{Z}).$$

Because $\mathcal{Y}_T(\mathbb{R})$ is identified with $\mathcal{Z}_T(\mathbb{R}) / \mathrm{ASL}_{d-1}(\mathbb{Z})$ for all $T > 0$ via $\tau_{\mathcal{Y}}$ (see below (5.1)), the proof is complete. ☐

5.1.2. *Comparing of* $\pi_{\mathcal{Y}_T}$ *and* $\pi^Q_{\mathcal{Y}_T}$. We will now discuss the difference between $\pi_{\mathcal{Y}_T}$ and $\pi^Q_{\mathcal{Y}_T}$ with the goal of showing that it converges as $T \to \infty$ in a certain uniform way to a fixed map that preserves the measure on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$.

We recall that for $T > 0$ and $(L, P, \mathbf{v}) \in \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$,

$$\pi_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(L, P, \mathbf{v}) = (S_{T,\mathbf{v}} L, P, \frac{1}{\sqrt{T}} \mathbf{v}), \tag{5.9}$$

where $S_{T,\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ acts by scalar multiplication of a factor $T^{-\frac{1}{2(d-1)}}$ on $P = \mathbf{v}^\perp$ and acts on the line $\mathbb{R}\mathbf{v}$ by scalar multiplication by a factor $T^{1/2}$ (see Section 4.3.1).

Next, we describe $\pi^Q_{\mathcal{Y}_T}$ in a manner similar to (5.9).

Definition 5.5. Recall the form Q^* defined in (3.11). For $\mathbf{v} \in \mathbb{R}^d \smallsetminus \mathbf{0}$ such that $Q(\mathbf{v}) > 0$, we denote by $\mathbf{v}_Q \in \mathbb{R}^d \smallsetminus \mathbf{0}$ the unique vector orthogonal with respect to the form Q^* to the hyperplane $\mathbf{v}^\perp$, having the normalization

$$\mathbf{v}_Q = \mathbf{v} + \hat{\mathbf{v}}_Q,$$

where $\hat{\mathbf{v}}_Q \in \mathbf{v}^\perp$. We define $S^Q_{T,\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ which acts by scalar multiplication of a factor $T^{-\frac{1}{2(d-1)}}$ on the hyperplane $\mathbf{v}^\perp$ and which acts on $\mathbb{R}\mathbf{v}_Q$ (the orthogonal line to the hyperplane $\mathbf{v}^\perp$ with respect to the form Q^*) by scalar multiplication of a factor $T^{1/2}$.

Remark. We observe that $\mathbf{v}_Q = \frac{1}{Q(\mathbf{v})} M\mathbf{v}$, where M is the companion matrix of the form Q. This implies that the map $\mathbf{v} \mapsto \mathbf{v}_Q$ is continuous.

Lemma 5.6. *For all $T > 0$ it holds that*

$$\pi^Q_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(L, P, \mathbf{v}) \overset{def}{=} (S^Q_{T,\mathbf{v}} L, P, \frac{1}{\sqrt{T}}\mathbf{v}),\ \forall (L, P, \mathbf{v}) \in \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R}). \tag{5.10}$$

Proof. Let $(L, \mathbf{v}^\perp, \mathbf{v}) \in \mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$. By using the identification (5.1), we take $g \in \mathcal{Z}_T(\mathbb{R})$ such that $(L, \mathbf{v}^\perp, \mathbf{v}) = \tau_{\mathcal{Y}}(g\mathrm{ASL}_{d-1}(\mathbb{Z}))$.

Using (3.15), we take $(\rho, \eta) \in (\mathrm{SO}_Q \times \mathrm{ASL}_{d-1})(\mathbb{R})$ such that

$$g = \theta(\rho^{-1}) a_T \eta,$$

and we observe that

$$\begin{aligned} \pi^Q_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(L, \mathbf{v}^\perp, \mathbf{v}) &\underbrace{=}_{\text{recalling } (5.7)} \tau_{\mathcal{Y}}(\pi_{\mathcal{Z}_{Q(\sqrt{T}\mathbf{e}_d)}}(g)\mathrm{ASL}_{d-1}(\mathbb{Z})) \\ &\underbrace{=}_{\text{recalling } (3.16)} \tau_{\mathcal{Y}}(\theta(\rho^{-1}) a_T^{-1} \theta(\rho) g \mathrm{ASL}_{d-1}(\mathbb{Z})). \end{aligned} \tag{5.11}$$

By recalling that $\theta(\rho^{-1}) \in \mathrm{SO}_{Q^*}(\mathbb{R})$ (see Lemma 3.3) and by recalling the definition of a_T (see Definition 3.4), we deduce that

$$\theta(\rho^{-1}) a_T^{-1} \theta(\rho) = S^Q_{T,\mathbf{v}},$$

where $S^Q_{T,\mathbf{v}}$ was given in Definition 5.5. Then by (5.11),

$$\begin{aligned} \pi^Q_{\mathcal{Y}_{Q(\sqrt{T}\mathbf{e}_d)}}(L, \mathbf{v}^\perp, \mathbf{v}) &\underbrace{=}_{\text{recalling } (5.2)} (S^Q_{T,\mathbf{v}} L, S^Q_{T,\mathbf{v}} \mathbf{v}^\perp, \theta(S^Q_{T,\mathbf{v}})\mathbf{v}) \\ &= (S^Q_{T,\mathbf{v}} L, \mathbf{v}^\perp, \frac{1}{\sqrt{T}}\mathbf{v}). \end{aligned}$$

□

Lemma 5.7. *Let $(L, P, \mathbf{v}) \in \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$, and consider the unipotent matrix $u^Q_{\mathbf{v}}$ which satisfies that $u^Q_{\mathbf{v}}\mathbf{v} = \mathbf{v}_Q$ and acts as identity on $\mathbf{v}^\perp$. Let*

$$u^Q_{T,\mathbf{v}} \overset{def}{=} \left(S^Q_{T,\mathbf{v}}\right) S^{-1}_{T,\mathbf{v}}, \tag{5.12}$$

then

$$\lim_{T\to\infty} \left(u^Q_{\mathbf{v}}\right)^{-1} u^Q_{T,\mathbf{v}} = I_d,$$

and the convergence is uniform when $\mathbf{v}$ is restricted to a compact subset of $\mathbb{R}^d \setminus \mathbf{0}$.

Proof. It is easy to verify that $\left(S^Q_{T,\mathbf{v}}\right) S^{-1}_{T,\mathbf{v}}$ acts as identity on $\mathbf{v}^\perp$, namely $\left(S^Q_{T,\mathbf{v}}\right) S^{-1}_{T,\mathbf{v}}$ and $u^Q_{\mathbf{v}}$ agree on $\mathbf{v}^\perp$. Next,

$$\begin{aligned} \left(S^Q_{T,\mathbf{v}}\right) S^{-1}_{T,\mathbf{v}}\mathbf{v} &= S^Q_{T,\mathbf{v}} \left(\frac{1}{\sqrt{T}}\mathbf{v}\right) \\ &= \frac{1}{\sqrt{T}} S^Q_{T,\mathbf{v}} \left(\mathbf{v}_Q - \hat{\mathbf{v}}_Q\right) \\ &= \mathbf{v}_Q - T^{-\frac{d}{2(d-1)}} \hat{\mathbf{v}}_Q, \end{aligned}$$

namely

$$\left(u^Q_{\mathbf{v}} - \left(S^Q_{T,\mathbf{v}}\right) S^{-1}_{T,\mathbf{v}}\right)\mathbf{v} = -T^{-\frac{d}{2(d-1)}} \hat{\mathbf{v}}_Q.$$

By multiplying both sides of the preceding equality by $\left(u^Q_{\mathbf{v}}\right)^{-1}$, and by recalling (3.4), we get

$$\left(I_d - \left(u^Q_{\mathbf{v}}\right)^{-1} u^Q_{T,\mathbf{v}}\right)\mathbf{v} = -T^{-\frac{d}{2(d-1)}} \hat{\mathbf{v}}_Q.$$

Since the map $\mathbf{v} \mapsto \mathbf{v}_Q$ is continuous (see remark below Definition 5.5), we deduce that $\lim_{T\to\infty} \left(u^Q_{\mathbf{v}}\right)^{-1} u^Q_{T,\mathbf{v}} = I_d$ converges uniformly when $\mathbf{v}$ varies in a compact set of $\mathbb{R}^d \smallsetminus \mathbf{0}$. $\square$

Now let $u^Q : \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ be defined by $u^Q(L, P, \mathbf{v}) \overset{\text{def}}{=} (u^Q_{\mathbf{v}} L, P, \mathbf{v})$.

Lemma 5.8. *The map u^Q preserves the measure $\mu_{\mathcal{Y}}$ on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$.*

Proof. Let $f \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}))$. By recalling the definition of $\mu_{\mathcal{Y}}$ in (4.21), it is sufficient to prove that $\mu_{(\pi^{\mathcal{Y}}_{vec})^{-1}(\mathbf{v})}(f \circ u^Q) = \mu_{(\pi^{\mathcal{Y}}_{vec})^{-1}(\mathbf{v})}(f)$ for all $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$. Let $\mathbf{v} \in \mathcal{H}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and let $g_{\mathbf{v}} \in \mathrm{SL}_d(\mathbb{R})$ such that $\tau(g_{\mathbf{v}}) = \mathbf{v}$. Then

$$\begin{aligned}\mu_{(\pi^{\mathcal{Y}}_{vec})^{-1}(\mathbf{v})}(f \circ u^Q) &= \int f(u^Q_{\mathbf{v}} g_{\mathbf{v}} x) dm_{Y_{d-1}}(x) \\ &= \int f(g_{\mathbf{v}}(g_{\mathbf{v}}^{-1} u^Q_{\mathbf{v}} g_{\mathbf{v}})x) dm_{Y_{d-1}}(x).\end{aligned}$$

As the reader may verify, it follows that $g_{\mathbf{v}}^{-1} u^Q_{\mathbf{v}} g_{\mathbf{v}} \in \mathrm{ASL}_{d-1}(\mathbb{R})$, and by recalling that $m_{Y_{d-1}}$ is left $\mathrm{ASL}_{d-1}(\mathbb{R})$ invariant, the proof is done. $\square$

Consider $\delta^Q_T : \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \to \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ defined by $\delta^Q_T \overset{\text{def}}{=} \left(u^Q\right)^{-1} \circ \pi^Q_{\mathcal{Y}_T} \circ \pi^{-1}_{\mathcal{Y}_T}$. Using Lemma 5.7, we obtain that δ^Q_T converges to the identity transformation on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ as $T \to \infty$ in the following uniform manner.

Corollary 5.9. *Assume that $y_n \to y_0$ in $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ and let $\{T_n\}_{n=1}^\infty \subseteq \mathbb{R}_{>0}$ such that $T_n \to \infty$. Then $\delta^Q_{T_n}(y_n) \to y_0$ and $\left(\delta^Q_{T_n}\right)^{-1}(y_n) \to y_0$.*

Proof. We write $y_n = (L_n, P_n, \mathbf{v}_n)$ and $y_0 = (L_0, P_0, \mathbf{v}_0)$, and we observe that $y_n \to y_0$ implies that $L_n \to L_0$ and $\mathbf{v}_n \to \mathbf{v}_0$ in the usual topology of X_d and $\mathbb{R}^d$ correspondingly.

Let for $T_n > 0$ and $\mathbf{v}_n \in \mathbb{R}^d \smallsetminus \mathbf{0}$, let $I_{T_n,\mathbf{v}_n} \in \mathrm{SL}_d(\mathbb{R})$ be defined by

$$I_{T_n,\mathbf{v}_n} \overset{\text{def}}{=} \left(u^Q_{\mathbf{v}_n}\right)^{-1} \left(S^Q_{T,\mathbf{v}_n}\right) S^{-1}_{T,\mathbf{v}_n},$$

and observe that

$$\delta^Q_{T_n}(L_n, P_n, \mathbf{v}_n) = (I_{T_n,\mathbf{v}_n} L_n, P_n, \mathbf{v}_n).$$

Since $L_n \to L_0$, since $\mathbf{v}_n \to \mathbf{v}_0$ and since $I_{T_n,\mathbf{v}_n} \to I_d$ uniformly when $\mathbf{v}$ is restricted to a compact subset of $\mathbb{R}^d \smallsetminus \mathbf{0}$ (by Lemma 5.7), we conclude that $I_{T_n,\mathbf{v}_n} L_n \to L_0$, which shows $\delta^Q_{T_n}(y_n) \to y_0$.

Similarly, we have that

$$\left(\delta^Q_{T_n}\right)^{-1}(L_n, P_n, \mathbf{v}_n) = (I^{-1}_{T_n,\mathbf{v}_n} L_n, P_n, \mathbf{v}_n),$$

and since $I^{-1}_{T_n,\mathbf{v}_n} \to I_d$ converges uniformly when $\mathbf{v}$ is restricted to a compact subset of $\mathbb{R}^d \smallsetminus \mathbf{0}$ (which follows by Lemma 5.7), we also obtain that $\left(\delta^Q_{T_n}\right)^{-1}(y_n) \to y_0$. $\square$

Lemma 5.10. *Let X be a manifold and assume that $\{\varphi_T\}_{T\in\mathbb{R}_{>0}}$ is a family of bijections $\varphi_T : X \to X$ such that for any sequence $\{x_n\} \subseteq X$ with $\lim_{n\to\infty} x_n = x_0$ and any $\{T_n\}_{n=1}^\infty \subseteq \mathbb{R}_{>0}$ such that $T_n \to \infty$ it holds that $\lim_{n\to\infty} \varphi_{T_n}(x_n) = x_0$ and $\lim_{n\to\infty} \varphi^{-1}_{T_n}(x_n) = x_0$. Then for all $f \in C_c(X)$, $f \circ \varphi_T$ converges to f uniformly. Namely, for all $f \in C_c(X)$ and all $\epsilon > 0$ there is $T_0 > 0$ such that*

$$|f \circ \varphi_T(x) - f(x)| < \epsilon, \ \forall T > T_0, \ \forall x \in X.$$

Proof. Let $f \in C_c(X)$ and assume for contradiction that $f \circ \varphi_T$ doesn't converge uniformly to f. Then there exists a $\delta > 0$, a sequence $T_n \to \infty$ and a sequence $\{x_n\} \subseteq X$ such that $|f \circ \varphi_{T_n}(x_n) - f(x_n)| > \delta$ for all $n \in \mathbb{N}$. Let $K \overset{\text{def}}{=} \mathrm{supp}(f)$ and observe by the preceding inequality that either $\varphi_{T_n}(x_n) \in K$ infinitely often or $x_n \in K$ infinitely often. Assume that $\varphi_{T_n}(x_n) \in K$ infinitely often. By sequential compactness we may assume that $\varphi_{T_n}(x_n) \to x_0$

which implies by assumption on φ_T^{-1} that $x_n = \varphi_{T_n}^{-1}(\varphi_{T_n}(x_n)) \to x_0$. We reach a contradiction since

$$|f \circ \varphi_{T_n}(x_n) - f(x_n)| \leq |f \circ \varphi_{T_n}(x_n) - f(x_0)| + |f(x_0) - f(x_n)|,$$

and since the continuity of f implies $|f \circ \varphi_{T_n}(x_n) - f(x_0)| \to 0$ and $|f(x_0) - f(x_n)| \to 0$.

In a manner similar to the preceding, we obtain a contradiction when assuming that $x_n \in K$ infinitely often. □

Corollary 5.11. *Let $f \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$, let $\epsilon > 0$ and let $\mathcal{K} \supsetneq \mathrm{Supp}(f)$ be an open precompact set. Then, there exists $T_0 > 0$ such that for all $T > T_0$ the following hold*

(1) $\left|f((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \mathbf{v}) - f(\pi_{\mathcal{Y}_T}(y), \mathbf{v})\right| < \epsilon,\ \forall\, (y, \mathbf{v}) \in \mathcal{Y}_T(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q))$.

(2) *if* $((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \mathbf{v}) \notin \mathcal{K}$, *then* $(\pi_{\mathcal{Y}_T}(y), \mathbf{v}) \notin \mathrm{Supp}(f)$.

Proof. Let $f \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$ and let $\epsilon \in (0, 1)$. Using Corollary 5.9 and Lemma 5.10 with the fact that $\mathcal{H}_a(\mathbb{Z}/(q))$ is a finite set, we obtain $T_1 > 0$ such that for all $T > T_1$ it holds

$$\left|f\left(\delta^Q_T(y'), \mathbf{v}\right) - f\left(y', \mathbf{v}\right)\right| < \epsilon,\ \forall \left(y', \mathbf{v}\right) \in \mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)).$$

Then, by substituting $y' = \pi_{\mathcal{Y}_T}(y)$, we obtain for all $T > T_1$ that

$$\left|f\left(((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \mathbf{v}\right) - f\left((\pi_{\mathcal{Y}_T}(y), \mathbf{v}\right)\right| < \epsilon,\ \forall\, (y, \mathbf{v}) \in \mathcal{Y}_T(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)).$$

Let $\mathcal{K} \supsetneq \mathrm{Supp}(f)$ be an open precompact set. By Urysohn's lemma there exists $\varphi : C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q))) \to [0, 1]$ such that

$$\varphi(y, \mathbf{v}) \overset{\mathrm{def}}{=} \begin{cases} 0 & (y, \mathbf{v}) \notin \mathcal{K} \\ 1 & (y, \mathbf{v}) \in \mathrm{Supp}(f). \end{cases}$$

As above, there exists $T_2 > 0$ such that for all $T > T_2$

$$\left|\varphi((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \mathbf{v}) - \varphi(\pi_{\mathcal{Y}_T}(y), \mathbf{v})\right| < \epsilon,\ \forall (y, \mathbf{v}) \in \mathcal{Y}_T(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)). \tag{5.13}$$

Assuming $((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \mathbf{v}) \notin \mathcal{K}$, we see by (5.13) and by the definition of φ that $\varphi(\pi_{\mathcal{Y}_T}(y), \mathbf{v}) = 0$, which implies that $(\pi_{\mathcal{Y}_T}(y), \mathbf{v}) \notin \mathrm{Supp}(f)$.

By defining $T_0 \overset{\mathrm{def}}{=} \max\{T_1, T_2\}$ the proof of the statements of Corollary 5.11 is done. □

Fix $q \in \mathbb{N}$ and let $\{T_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be an unbounded sequence such that $\vartheta_q(T_n) = a$, where $a \in \mathbb{Z}/(q)$ is fixed. We consider the following measure on $\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q))$ defined by

$$\nu^{\mathcal{Y},Q,q}_{T_n} \overset{\mathrm{def}}{=} \frac{1}{|\mathcal{H}_{T_n,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{y \in \mathcal{Y}_{T_n}(\mathbb{Z})} \delta_{(\pi^Q_{\mathcal{Y}_{T_n}}(y), \vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)))} \tag{5.14}$$

Corollary 5.12. *For all $f \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$ it holds that*

$$\lim_{n \to \infty} \nu^{\mathcal{Y},Q,q}_{T_n}(f \circ \left(u^Q\right)^{-1}) - \nu^{\mathcal{Y},q}_{T_n}(f) = 0, \tag{5.15}$$

where we recall that

$$\nu^{\mathcal{Y},q}_T = \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{y \in \mathcal{Y}_T(\mathbb{Z})} \delta_{(\pi_{\mathcal{Y}_T}(y), \vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)))}.$$

Proof. We let $f \in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q)))$ and we denote

$$\phi_T(y) \overset{\mathrm{def}}{=} f((u^Q)^{-1} \circ \pi^Q_{\mathcal{Y}_T}(y), \vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)) - f(\pi_{\mathcal{Y}_T}(y), \vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)),\ y \in \mathcal{Y}_T(\mathbb{Z}).$$

Then

$$\begin{aligned}\nu_{T_n}^{\mathcal{Y},Q,q}\left(f\circ\left(u^Q\right)^{-1}\right)-\nu_{T_n}^{\mathcal{Y},q}(f)&=\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{y\in\mathcal{Y}_T(\mathbb{Z})}\phi_T(y)\\ &=\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{y\mathrm{SO}_Q(\mathbb{Z})\in\mathcal{Y}_T(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})}\sum_{\gamma\in\mathrm{SO}_Q(\mathbb{Z})}\phi_T(y\cdot\gamma)\end{aligned}\tag{5.16}$$

Let $\epsilon>0$ and let $\mathcal{K}\supsetneq\mathrm{Supp}(f)$ be an open precompact set. We fix $T_0>0$ such that Corollary 5.11 holds. By Corollary 5.11,(1) it holds for all $T>T_0$

$$|\phi_T(y)|\le\epsilon,\ \forall y\in\mathcal{Y}_T(\mathbb{Z}).\tag{5.17}$$

We now claim that there exists a constant $c=c(f)>0$ such that for all $y\mathrm{SO}_Q(\mathbb{Z})\in\mathcal{Y}_T(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})$ it holds that

$$|\{\gamma\in\mathrm{SO}_Q(\mathbb{Z})\mid y\cdot\gamma\in\mathrm{Supp}(\phi_T)\}|\le c.\tag{5.18}$$

By Lemma A.4 (for $G=(\mathrm{SO}_Q\times\mathrm{ASL}_{d-1})(\mathbb{R})$, $K=H$, $\Gamma=(\mathrm{SO}_Q\times\mathrm{ASL}_{d-1})(\mathbb{Z})$ and $\tilde{\Gamma}=\{e\}\times\mathrm{ASL}_{d-1}(\mathbb{Z})$), we obtain that for any precompact set $\mathcal{C}\subseteq\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$ there exists a uniform constant $c>0$ such that for all $y_0\in\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$

$$|\{\gamma\in\mathrm{SO}_Q(\mathbb{Z})\mid y_0\cdot\gamma\in\mathcal{C}\}|\le c.\tag{5.19}$$

We recall that $\pi^Q_{\mathcal{Y}_T}$ is $\mathrm{SO}_Q(\mathbb{Z})$ equivariant, so that

$$(u^Q)^{-1}\circ\pi^Q_{\mathcal{Y}_T}(y\cdot\gamma)=(u^Q)^{-1}(\pi^Q_{\mathcal{Y}_T}(y)\cdot\gamma).$$

By Corollary 5.11,(2), for all $T>T_0$ and for $\gamma\in\mathrm{SO}_Q(\mathbb{Z})$ such that

$$(\pi^Q_{\mathcal{Y}_T}(y)\cdot\gamma,\vartheta_q(\pi^{\mathcal{Y}}_{vec}(y\cdot\gamma)))\notin u^Q(\mathcal{K}),$$

we have $|\phi_T(y\cdot\gamma)|=0$, namely $\mathrm{Supp}(\phi_T)\subseteq u^Q(\mathcal{K})$, which shows

$$|\{\gamma\in\mathrm{SO}_Q(\mathbb{Z})\mid y\cdot\gamma\in\mathrm{Supp}(\phi_T)\}|\le\left|\left\{\gamma\in\mathrm{SO}_Q(\mathbb{Z})\mid(\pi^Q_{\mathcal{Y}_T}(y)\cdot\gamma,\vartheta_q(\pi^{\mathcal{Y}}_{vec}(y\cdot\gamma)))\in u^Q(\mathcal{K})\right\}\right|$$

Consider the natural map $\pi_\infty:\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{H}_a(\mathbb{Z}/(q))\to\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})$. Since u^Q is a homeomorphism, and as $\mathcal{K}$ is precompact, by (5.19) there is a constant $c>0$ such that for all $y\in\mathcal{Y}_T(\mathbb{Z})$

$$\left|\left\{\gamma\in\mathrm{SO}_Q(\mathbb{Z})\mid\pi^Q_{\mathcal{Y}_T}(y)\cdot\gamma\in\pi_\infty(u^Q(\mathcal{K}))\right\}\right|\le c,$$

which shows (5.18). Finally, by (5.16), (5.17) and (5.18) we obtain for all $T>T_0$

$$\left|\nu_{T_n}^{\mathcal{Y},Q,q}\left(f\circ\left(u^Q\right)^{-1}\right)-\nu_{T_n}^{\mathcal{Y},q}(f)\right|\le\frac{|\mathcal{Y}_T(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\epsilon c.$$

Now the map $\pi^{\mathcal{Y}}_{vec}:\mathcal{Y}_T(\mathbb{Z})\to\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})$ is a bijection which is equivariant with respect to the right $\mathrm{SO}_Q(\mathbb{Z})$ action, which shows that

$$\frac{|\mathcal{Y}_T(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}=1,$$

and completes our proof. $\square$

5.1.3. *Concluding the proof that the results for $\mathcal{Z}$ imply the results for $\mathcal{Y}$.* We now give a detailed proof that Theorem 3.8 implies Theorem 4.9 for $\mathcal{M}=\mathcal{Y}$. The proof that Theorem 3.7 implies Theorem 4.8 follows along the same lines, and is left for the reader.

In the following we fix $q\in 2\mathbb{N}+1$ and we let $a\in(\mathbb{Z}/(q))^\times$.

Let $f\in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{H}_a(\mathbb{Z}/(q)))$, and consider $\bar{\varphi}_f\in C_c(\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{H}_a(\mathbb{Z}/(q)))$ given by

$$\bar{\varphi}_f\overset{\mathrm{def}}{=}f\circ(u^Q)^{-1},$$

where we abuse notations with $f\circ(u^Q)^{-1}(y,\mathbf{v})=f\left((u^Q)^{-1}(y),\mathbf{v}\right)$.

By Lemma 5.3, there exists $\varphi_f \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{H}_a(\mathbb{Z}/(q))$ such that

$$\bar{\varphi}_f(z\mathrm{ASL}_{d-1}(\mathbb{Z}), \mathbf{v}) = \sum_{\gamma \in \mathrm{ASL}_{d-1}(\mathbb{Z})} \varphi_f(z\gamma, \mathbf{v}).$$

Let $\varphi_f^\tau \in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_a(\mathbb{Z}/(q))$ be defined by $\varphi_f^\tau(z,g) \overset{\mathrm{def}}{=} \varphi_f(z,\tau(g))$ (where τ defined in (2.3)) We claim that

$$\nu_T^{\mathcal{Z},q}(\varphi_f^\tau) = \nu_T^{\mathcal{Y},Q,q}(\bar{\varphi}_f),$$

where $\nu_T^{\mathcal{Z},q}$ defined in (3.19) and $\nu_T^{\mathcal{Y},Q,q}$ defined in (5.14). We have

$$\begin{aligned}
\nu_T^{\mathcal{Z},q}(\varphi_f^\tau) =& \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{z \in \mathcal{Z}_T(\mathbb{Z})} \varphi_f^\tau(\pi_{\mathcal{Z}_T}(z), \vartheta_q(z)) \\
=& \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{z \in \mathcal{Z}_T(\mathbb{Z})} \varphi_f(\pi_{\mathcal{Z}_T}(z), \vartheta_q(\tau(z))) \\
=& \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{z\mathrm{ASL}_{d-1}(\mathbb{Z}) \in \mathcal{Z}_T(\mathbb{Z})/\mathrm{ASL}_{d-1}(\mathbb{Z})} \sum_{\gamma \in \mathrm{ASL}_{d-1}(\mathbb{Z})} \varphi_f(\pi_{\mathcal{Z}_T}(z\gamma), \vartheta_q(\tau(z\gamma))) \\
=& \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{z\mathrm{ASL}_{d-1}(\mathbb{Z}) \in \mathcal{Z}_T(\mathbb{Z})/\mathrm{ASL}_{d-1}(\mathbb{Z})} \sum_{\gamma \in \mathrm{ASL}_{d-1}(\mathbb{Z})} \varphi_f(\pi_{\mathcal{Z}_T}(z)\gamma, \vartheta_q(\tau(z))) \\
=& \frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|} \sum_{y \in \mathcal{Y}_T(\mathbb{Z})} \bar{\varphi}_f(\pi^Q_{\mathcal{Y}_T}(y), \vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)) \\
=& \nu_T^{\mathcal{Y},Q,q}(\bar{\varphi}_f).
\end{aligned}$$

Assume that Q is non-singular modulo $q \in 2\mathbb{N}+1$. Let $\{T_n\}_{n=1}^\infty \subseteq \mathbb{N}$ be an unbounded sequence of integers satisfying the (Q,p_0) co-isotropic property for some p_0 and assume that $\vartheta_q(T_n) = a$, $\forall n \in \mathbb{N}$. Then by assuming Theorem 3.8, we get

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},Q,q}(\bar{\varphi}_f) = \lim_{n\to\infty} \nu_{T_n}^{\mathcal{Z},q}(\varphi_f^\tau) = \mu_{\mathcal{Z}} \otimes \mu_{\mathcal{Z}_a(\mathbb{Z}/(q)}(\varphi_f^\tau).$$

We recall by the proof of Corollary 3.1 that $\tau(\mathcal{Z}_a(\mathbb{Z}/(q)) = \mathcal{H}_a(\mathbb{Z}/(q))$ and we observe that

$$\mu_{\mathcal{Z}} \otimes \mu_{\mathcal{Z}_a(\mathbb{Z}/(q))}(\varphi_f^\tau) = \mu_{\mathcal{Z}} \otimes \tau_* \mu_{\mathcal{Z}_a(\mathbb{Z}/(q))}(\varphi_f) = \mu_{\mathcal{Z}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q))}(\varphi_f).$$

By Lemma 5.2

$$\mu_{\mathcal{Z}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(\varphi_f) = \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(\bar{\varphi}_f),$$

which implies in turn that

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},Q,q}(\bar{\varphi}_f) = \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(\bar{\varphi}_f). \tag{5.20}$$

Our goal now is to show that (5.20) implies

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},q}(f) = \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(f), \tag{5.21}$$

which is the statement of Theorem 4.9.

We have by definition of $\bar{\varphi}_f$

$$\begin{aligned}
\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},Q,q}(f \circ (u^Q)^{-1}) &= \lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},Q,q}(\bar{\varphi}_f) \\
&= \lim_{n\to\infty} \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(\bar{\varphi}_f) \\
&= \lim_{n\to\infty} \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(f \circ (u^Q)^{-1}).
\end{aligned} \tag{5.22}$$

By Corollary 5.12 and by (5.22) we obtain that

$$\lim_{n\to\infty} \nu_{T_n}^{\mathcal{Y},q}(f) = \mu_{\mathcal{Y}} \otimes \mu_{\mathcal{H}_a(\mathbb{Z}/(q)}(f \circ (u^Q)^{-1}),$$

and finally, since u^Q preserves $\mu_{\mathcal{Y}}$ (see Lemma 5.8) we obtain (5.21).

5.2. **The results for $\mathcal{Y}$ imply the results for $\mathcal{X}$ and $\mathcal{W}$.** In the following we show that Theorems 4.8 - 4.9 for $\mathcal{M}=\mathcal{Y}$ imply Theorems 4.8 - 4.9 for $\mathcal{M}\in\{\mathcal{X},\mathcal{W}\}$. It may be helpful for the reader to recall Section 4.3.3.

We fix $T\in\mathbb{N}$ and we note the following commuting diagram (which follows from (4.19)),

$$\begin{tikzcd}
\mathcal{Y}_T(\mathbb{Z}) \arrow[r, leftrightarrow, "\pi_\cap"] \arrow[d, hook, "\pi_{\mathcal{Y}_T}"'] \arrow[rr, leftrightarrow, bend left, "\pi^{\mathcal{Y}}_{vec}"] & \mathcal{X}_T(\mathbb{Z}) \arrow[r, leftrightarrow, "\pi^{\mathcal{X}}_{vec}"] \arrow[d, hook, "\pi_{\mathcal{X}_T}"] & \mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z}) \\
\mathcal{Y}_{Q(\mathbf{e}_d)}(\mathbb{R}) \arrow[r, "\pi_\cap"] & \mathcal{X}_{Q(\mathbf{e}_d)}(\mathbb{R}) &
\end{tikzcd}$$

which shows that

$$\begin{aligned}
\nu_T^{\mathcal{X},q} =&\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{x\in\mathcal{X}_T(\mathbb{Z})}\delta_{(\pi_{\mathcal{X}_T}(x),\vartheta_q(\pi^{\mathcal{X}}_{vec}(x)))}\\
=&\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{y\in\mathcal{Y}_T(\mathbb{Z})}\delta_{(\pi_\cap\circ\pi_{\mathcal{Y}_T}(y),\vartheta_q(\pi^{\mathcal{Y}}_{vec}(y)))}\\
=&\left(\pi_\cap\times id\right)_*\nu_T^{\mathcal{Y},q}.
\end{aligned}$$

By Lemma 4.6 we have $(\pi_\cap)_*\,\mu_{\mathcal{Y}}=\mu_{\mathcal{X}}$, hence we obtain the limits for $\mathcal{X}$ from the limits of $\mathcal{Y}$.

Next, we observe that

$$\begin{aligned}
\nu_T^{\mathcal{W},q} =&\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{\mathbf{v}\in\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})}\delta_{\left(\mathrm{shape}(\Lambda_{\mathbf{v}}),\frac{1}{\sqrt{T}}\mathbf{v},\vartheta_q(\mathbf{v})\right)}\\
=&\frac{1}{|\mathcal{H}_{T,\mathrm{prim}}(\mathbb{Z})/\mathrm{SO}_Q(\mathbb{Z})|}\sum_{x\in\mathcal{X}_T(\mathbb{Z})}\delta_{\left((\mathrm{shape}\times\pi^{\mathcal{X}}_{vec})(\pi_{\mathcal{X}_T}(x)),\vartheta_q(\pi^{\mathcal{X}}_{vec}(x))\right)}\\
=&\left((\mathrm{shape}\times\pi^{\mathcal{X}}_{vec})\times id\right)_*\nu_T^{\mathcal{X},q},
\end{aligned}$$

and by Lemma 4.7, we have $\left(\pi^{\mathcal{X}}_{vec}\times\mathrm{shape}\right)_*\mu_{\mathcal{X}}=\mu_{\mathcal{W}}$, which shows that the limits for $\mathcal{W}$ follow from the limits of $\mathcal{X}$.

6. Some technicalities

This section discusses several technical facts about quadratic forms that will be used in the rest of the paper (mainly in Section 7).

For a prime p we denote by $\mathbb{Z}_p$ the ring of p-adic integers and by $\mathbb{Q}_p$ the field of p-adic numbers.

Lemma 6.1. *Let Q be an integral form which is non-singular modulo q (see Definition 2.1) for $q\in 2\mathbb{N}+1$ and let S_q be the set of primes appearing in the prime decomposition of q. Then the following hold:*

(1) The reduction map $\vartheta_{p^k}:\mathrm{SO}_Q(\mathbb{Z}_p)\to\mathrm{SO}_Q(\mathbb{Z}/(p^k))$ is onto for all $p\in S_q$ and $k\geq 1$.
(2) Q is isotropic over $\mathbb{Q}_p$ for all $p\in S_q$.

Proof. (1) Fix $p\in S_q$. To prove that $\vartheta_{p^k}:\mathrm{SO}_Q(\mathbb{Z}_p)\to\mathrm{SO}_Q(\mathbb{Z}/(p^k))$ is onto, we will prove that the natural projection

$$\pi_k:\mathrm{SO}_Q(\mathbb{Z}/(p^{k+1}))\to\mathrm{SO}_Q(\mathbb{Z}/(p^k))$$

is onto for all $k\geq 1$. We let $\bar{g}\in\mathrm{SO}_Q(\mathbb{Z}/(p^k))$ and we take $F\in M_d(\mathbb{Z}_p)$ such that $\vartheta_{p^k}(F)=\bar{g}$. Since $\det(\bar{g})=1$, it follows that $\det(F)\in\mathbb{Z}_p^\times$, which implies that $F\in\mathrm{GL}_d(\mathbb{Z}_p)$. Fix a symmetric matrix $M\in M_d(\mathbb{Z})$ such that

$$Q(\mathbf{x})=\mathbf{x}^tM\mathbf{x}$$

Since Q is non-singular modulo q it follows that $\det(M) \in \mathbb{Z}_p^\times$ for all $p \in S_q$, namely $M \in \mathrm{GL}_d(\mathbb{Z}_p)$ for all $p \in S_q$. We may now define $S \in M_d(\mathbb{Z}_p)$ by

$$S \overset{\text{def}}{=} \frac{1}{2}\left(M^{-1}\left(F^t\right)^{-1} M - F\right). \tag{6.1}$$

By noting that $\vartheta_{p^k}(F^t M F) = \vartheta_{p^k}(M)$ we obtain that $\vartheta_{p^k}(S) = 0$, so that in particular $\vartheta_{p^k}(F+S) = \bar{g}$. To finish the proof, it is sufficient to show that $\vartheta_{p^{k+1}}(F+S) \in \mathrm{SO}_Q(\mathbb{Z}/(p^{k+1}))$. We observe that

$$(F+S)^t M(F+S) = F^t M F + F^t M S + S^t M F + S^t M S. \tag{6.2}$$

We treat each of the terms appearing in (6.2) separately.

(a) The term $F^t M S$. By substituting (6.1) in S, we obtain that

$$F^t M S = \frac{1}{2} F^t M\left(M^{-1}\left(F^{-1}\right)^t M - F\right) = \frac{1}{2} M - \frac{1}{2} F^t M F.$$

(b) The term $S^t M F$. By substituting (6.1) in S^t, we obtain that

$$S^t M F = \frac{1}{2}\left(M^t F^{-1}\left(M^t\right)^{-1} - F^t\right) M F \underbrace{=}_{M^t = M} \frac{1}{2} M - \frac{1}{2} F^t M F.$$

Hence we deduce by the above that

$$(F+S)^t M(F+S) = M + S^t M S,$$

Since $\vartheta_{p^k}(S) = 0$, we obtain that $\vartheta_{p^{2k}}(S^t M S) = 0$. Namely $\vartheta_{p^{k+1}}\left((F+S)^t M(F+S)\right) = \vartheta_{p^{k+1}}(M)$, which completes the proof.

(2) Let M be the companion matrix of Q. By definition of non-singularity modulo q (see Definition 2.1) we have that $|\det(M)|_p = 1$ for all $p \in S_q$, where $|\cdot|_p$ denotes the p-adic valuation. Fix $p \in S_q$. By [Cas78, Chapter 8, Theorem 3.1] there exists $g \in \mathrm{GL}_d(\mathbb{Z}_p)$ such that

$$g^t M g = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix},$$

where $a_1, ..., a_d \in \mathbb{Z}_p$. Now

$$|a_1|_p \cdot ... \cdot |a_d|_p = \left|\det(g^t M g)\right|_p = |\det(M)|_p = 1.$$

Hence $|a_1|_p = ... = |a_d|_p = 1$, and by [Cas78, Chapter 3, Lemma 1.7] we get that $Q(g\mathbf{x}) = a_1 x_1^2 + ... + a_d x_d^2$ has an isotropic vector over $\mathbb{Q}_p$.

□

For $g \in \mathrm{SL}_d(\mathbb{Z})$ and $\gamma \in \mathrm{SL}_{d-1}(\mathbb{Q})$ we define a quadratic form $\varphi_g^\gamma : \mathbb{Q}^{d-1} \to \mathbb{Q}$, by

$$\varphi_g^\gamma(u) \overset{\text{def}}{=} Q^* \circ g \circ \gamma(u) \tag{6.3}$$

(see definition of Q^* in (3.11)), where we identify $\mathbb{Q}^{d-1}$ with $\mathbb{Q}^{d-1} \times \{0\}$. We will denote $\varphi_g \overset{\text{def}}{=} \varphi_g^{I_{d-1}}$.

Let $\hat{g} \in M_{d \times d-1}(\mathbb{R})$ be the matrix formed by the first $d-1$ columns of g. Then the matrix

$$M_{\varphi_g^\gamma} \overset{\text{def}}{=} \gamma^t \hat{g}^t M^{-1} \hat{g} \gamma \tag{6.4}$$

is a companion matrix for the form φ_g^γ.

Lemma 6.2. *It holds that* $\det\left(M_{\varphi_g^\gamma}\right) = \frac{1}{\det(M)} Q(\tau(g))$.

Proof. First, by the multiplicativity of the determinant, we get that $\det\left(M_{\varphi_g^\gamma}\right) = \det\left(M_{\varphi_g}\right)$. Next, we observe that

$$\begin{aligned} Q(\tau(g)) &= \left\langle \left(g^t\right)^{-1}\mathbf{e}_d, M\left(g^t\right)^{-1}\mathbf{e}_d\right\rangle \\ &= \left\langle \mathbf{e}_d, \left(g^t M^{-1} g\right)^{-1}\mathbf{e}_d\right\rangle, \end{aligned}$$

which is the d,d entry of the matrix $\left(g^t M^{-1} g\right)^{-1}$. Now

$$\begin{aligned} \left(g^t M^{-1} g\right)^{-1} &= \frac{1}{\det(g^t M^{-1} g)}\operatorname{adj}\left(g^t M^{-1} g\right) \\ &= \det(M)\cdot \operatorname{adj}\left(g^t M^{-1} g\right). \end{aligned}$$

We note that the d,d entry of adj $\left(g^t M^{-1} g\right)$ is given by the minor $\det\left(\hat{g}^t M^{-1}\hat{g}\right) = \det\left(M_{\varphi_g}\right)$, which proves our claim. □

Consider the natural map

$$\pi_{\mathrm{SL}_{d-1}} : \mathrm{ASL}_{d-1} \to \mathrm{SL}_{d-1},$$

given by

$$\begin{pmatrix} m & * \\ & 1 \end{pmatrix} \mapsto m.$$

Lemma 6.3. *We have*

$$\gamma^{-1}\pi_{\mathrm{SL}_{d-1}}\left(g^{-1}\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right)g\right)\gamma = \mathrm{SO}_{\varphi_g^{\gamma}}(\mathbb{R}).$$

Proof. We recall by Lemma 3.3 that $\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right)$ is the subgroup of $\mathrm{SO}_{Q^*}(\mathbb{R})$ that preserves the hyperplane $\mathrm{Span}_{\mathbb{R}}\{\mathbf{g}_1,...,\mathbf{g}_{d-1}\}$, where $\mathbf{g}_i$ denotes the i'th column of g. Therefore group $g^{-1}\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right)g$ is the subgroup of $\mathrm{SO}_{Q^*\circ g}(\mathbb{R})$ which preserves the hyperplane $\mathbb{R}^{d-1}\times\{0\}$ by the left linear action. Hence $\pi_{\mathrm{SL}_{d-1}}\left(g^{-1}\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right)g\right)$ is the restriction of $\mathrm{SO}_{Q^*\circ g}(\mathbb{R})$ to the hyperplane $\mathbb{R}^{d-1}\times\{0\}$, which shows $\pi_{\mathrm{SL}_{d-1}}\left(g^{-1}\theta\left(\mathbf{H}_{\tau(g)}(\mathbb{R})\right)g\right) = \mathrm{SO}_{\varphi_g}(\mathbb{R})$. Finally we note that

$$\gamma^{-1}\mathrm{SO}_{\varphi_g}(\mathbb{R})\gamma = \mathrm{SO}_{\varphi_g^{\gamma}}(\mathbb{R}).$$

□

Lemma 6.4. *Let $A\in M_d(\mathbb{Z})\cap\mathrm{GL}_d(\mathbb{Q})$. Then for any $g\in\mathrm{SL}_d(\mathbb{Z})$, the g.c.d of the entries of the integral matrix*

$$A_g \overset{\mathrm{def}}{=} \hat{g}^t A\hat{g}$$

where $\hat{g}$ is the matrix formed by the first $d-1$ columns of g, is at-most $\det A$.

Proof. To prove our claim it is sufficient to show that there exist two integral vectors $\mathbf{b},\mathbf{a}\in\mathbb{Z}^{d-1}$ such that

$$\mathbf{b}^t A_g\mathbf{a} = \alpha,$$

for $\alpha\in\mathbb{Z}$ satisfying that $\alpha\mid\det(A)$. This will be done by a variation on the geometric argument given in the proof of [AES16a, Lemma 3.2]. Let $\mathbf{u}_1\in\mathbb{Z}^d_{prim}\bigcap(A\mathbf{g}_d)^{\perp}\bigcap\hat{g}\mathbb{Q}^{d-1}$ where $\mathbf{g}_d\overset{\mathrm{def}}{=}g\mathbf{e}_d$ (such a vector exists since $(A\mathbf{g}_d)^{\perp}\bigcap\hat{g}\mathbb{Q}^{d-1}$ is the intersection of two rational hyperplanes). Namely, we choose $\mathbf{u}_1\in\mathbb{Z}^d_{prim}$ such that

$$\mathbf{u}_1 = \hat{g}\mathbf{a},\ \mathbf{a}\in\mathbb{Z}^{d-1},$$

(the entries of $\mathbf{a}$ are integral since the columns of g form a $\mathbb{Z}$ basis for $\mathbb{Z}^d$) and

$$0 = (A\mathbf{g}_d)^t\mathbf{u}_1 = \mathbf{g}_d^t A\mathbf{u}_1. \tag{6.5}$$

Let $\alpha\in\mathbb{N}$ be the g.c.d. of the entries of $A\mathbf{u}_1$. Since $\mathbf{u}_1\in\mathbb{Z}^d_{\mathrm{prim}}$, we may use [Cas97, Chapter 1, Theorem 1.B] to deduce that $\alpha\mid\det(A)$. Let $\tilde{\mathbf{u}}_2\in\mathbb{Z}^d$ such that

$$\tilde{\mathbf{u}}_2^t(A\mathbf{u}_1) = \alpha. \tag{6.6}$$

Since $\mathbf{g}_1,...,\mathbf{g}_d$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$, we may write

$$\tilde{\mathbf{u}}_2 = \hat{g}\mathbf{b} + b_d\mathbf{g}_d,\ \mathbf{b}\in\mathbb{Z}^{d-1},\ b_d\in\mathbb{Z},$$

then by (6.5) and (6.6) we obtain

$$\alpha = (\hat{g}\mathbf{b})^t (A\hat{g}\mathbf{a}) = \mathbf{b}^t A_g \mathbf{a},$$

which completes the proof. $\square$

7. A revisit to the S-arithemetic theorem of [AES16a]

The purpose of this section is to prove Theorem 7.1 below, which concerns the equidistribution of a sequence of compact orbits in an S-arithemetic space. We note that Theorem 7.1 generalizes Theorem 3.1 of [AES16a] by taking into account more general quadratic forms and by also taking into account more than one prime. Yet, we note that our proof of Theorem 7.1 strongly relies on the ideas and methods which already appear in the proof of Theorem 3.1 of [AES16a].

In the following we consider algebraic groups defined over $\mathbb{Q}$, and we follow the notations and conventions as in [PR94], Chapter 2.

To ease the notation, we introduce

$$\mathbb{G}_1 \overset{\text{def}}{=} \mathrm{SO}_Q,\ \mathbb{G}_2 \overset{\text{def}}{=} \mathrm{ASL}_{d-1},\ \mathbb{G} \overset{\text{def}}{=} \mathbb{G}_1 \times \mathbb{G}_2,$$

where we recall that Q is as in our Standing Assumption. For a finite set of primes S, we denote by $\mathbb{Q}_S \overset{\text{def}}{=} \prod_{p\in S}\mathbb{Q}_p$, where $\mathbb{Q}_p$ is the field of p-adic numbers, by $\mathbb{Z}_S \overset{\text{def}}{=} \prod_{p\in S}\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, and by $\mathbb{Z}\left[S^{-1}\right] \overset{\text{def}}{=} \mathbb{Z}\left[\frac{1}{p}; p\in S\right]$.

We consider

$$\mathbb{G}(\mathbb{R}\times\mathbb{Q}_S) \overset{\text{def}}{=} \mathbb{G}(\mathbb{R})\times\mathbb{G}(\mathbb{Q}_S),$$

we define $\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right) \leq \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)$ by

$$\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right) \overset{\text{def}}{=} \left\{(\gamma_1,\gamma_2,\gamma_1,\gamma_2) \mid \gamma_i \in \mathbb{G}_i(\mathbb{Z}\left[S^{-1}\right])\right\},$$

we recall that $\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ is a lattice in $\mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)$ (see [PR94], Chapter 5), and we define

$$\mathcal{Y}_S \overset{\text{def}}{=} \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)/\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right)$$

(we use the above notation in this section only. Note not to be confused with the notation $\mathcal{Y}$ for the space of oriented grids). Let $g\in \mathrm{SL}_d(\mathbb{Z})$ such that $Q(\tau(g))>0$. Using the transitivity of the $\mathbb{G}(\mathbb{R})$-action on $\mathcal{Z}_{Q(\tau(g))}(\mathbb{R})$ (see Corollary 3.1), we choose $t_g = \left((t_g)_1, (t_g)_2\right) \in \mathbb{G}(\mathbb{R})$ such that

$$g = a_{Q(\tau(g))}\cdot t_g \underbrace{=}_{\text{definition of the } \mathbb{G} \text{ action on } \mathrm{SL}_d} \theta\left((t_g)_1^{-1}\right) a_{Q(\tau(g))}\,(t_g)_2\,, \tag{7.1}$$

where $a_{Q(\tau(g))} \in \mathcal{Z}_{Q(\tau(g))}(\mathbb{R})$ was defined in Definition 3.4.

We define the twisted orbit

$$O_{g,S} \overset{\text{def}}{=} (t_g, e_S)\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]), \tag{7.2}$$

where $\mathbf{L}_g \leq \mathbb{G}$ is the stabilizer of g (see Lemma 3.2).

We observe that $\mathbf{L}_g(\mathbb{R})$ is a compact group since by assuming that $Q(\tau(g))>0$, it follows that $Q(\tau(g)) = TQ(\mathbf{e}_d)$ for $T>0$ implying that $\mathbf{H}_{\tau(g)}(\mathbb{R})$ (which is isomorphic to $\mathbf{L}_g(\mathbb{R})$) is conjugate to $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ (the action of $\mathrm{SO}_Q(\mathbb{R})$ is transitive on $\mathcal{H}_{Q(\sqrt{T}\mathbf{e}_d)}(\mathbb{R})$), which is compact by our Standing Assumption. Then by [PR94, Theorem 5.7] we obtain that $\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right) \subseteq \mathcal{Y}_S$ is a compact orbit, and we define

$$\mu_{g,S} \overset{\text{def}}{=} (t_g, e_S)_* \mu_{\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)\mathbb{G}(\mathbb{Z}[S^{-1}])} \tag{7.3}$$

where $\mu_{\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)\mathbb{G}(\mathbb{Z}[S^{-1}])}$ is the $\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)$-invariant probability measure supported on $\mathbf{L}_g(\mathbb{R}\times\mathbb{Q}_S)\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right)$.

Theorem 7.1. *Assume that S is a finite set of odd primes such that Q is isotropic over $\mathbb{Q}_p$ for all $p \in S$. Let $\{g_n\}_{n=1}^{\infty} \subseteq \mathrm{SL}_d(\mathbb{Z})$ such that $Q(\tau(g_n)) > 0$ for all $n \in \mathbb{N}$, such that $Q(\tau(g_n)) \to \infty$, and such that there exists $p_0 \in S$ for which $\tau(g_n)$ is (Q, p_0) co-isotropic for all $n \in \mathbb{N}$ (see Definition 3.6). Then,*

$$\mu_{g_n,S} \overset{weak^*}{\longrightarrow} \mu_{\mathcal{Y}_S},$$

where $\mu_{\mathcal{Y}_S}$ is the $\mathbb{G}(\mathbb{R} \times \mathbb{Q}_S)$-invariant probability measure on $\mathcal{Y}_S$.

7.1. **Proof of Theorem 7.1.** The key input for the proof of Theorem 7.1 is [GO11, Theorem 4.6], which we state in a simplified form in Theorem 7.2 below.

For the rest of the section, we will denote the simply connected covering of a semi-simple algebraic group $\mathbb{L}$ defined over $\mathbb{Q}$ by $\tilde{\mathbb{L}}$ and the universal covering map by $\pi : \tilde{\mathbb{L}} \to \mathbb{L}$ (for more details see e.g. [PR94, Section 2.1.13]).

Theorem 7.2. *Let $\mathbf{G}$ be a connected semi-simple algebraic group defined over $\mathbb{Q}$, let S be a finite set of primes and let $\mathbb{L}_n$, $n \in \mathbb{N}$, be a sequence of connected semi-simple $\mathbb{Q}$-subgroups of $\mathbf{G}$. Consider a sequence $\{t_n\}_{n=1}^{\infty} \subseteq \mathbf{G}(\mathbb{R}\times\mathbb{Q}_S)$ and let $\nu_n \overset{def}{=} (t_n)_* \,\mu_{\pi(\tilde{\mathbb{L}}_n(\mathbb{R}\times\mathbb{Q}_S))\mathbf{G}(\mathbb{Z}[S^{-1}])}$, where $\mu_{\pi(\tilde{\mathbb{L}}_n(\mathbb{R}\times\mathbb{Q}_S))\mathbf{G}(\mathbb{Z}[S^{-1}])}$ is the unique $\pi\left(\tilde{\mathbb{L}}_n\left(\mathbb{R}\times\mathbb{Q}_S\right)\right)$-invariant probability measure supported on $\pi\left(\tilde{\mathbb{L}}_n\left(\mathbb{R}\times\mathbb{Q}_S\right)\right)\mathbf{G}\left(\mathbb{Z}\left[S^{-1}\right]\right)$.*

$1: *Assume that there exists $p \in S$ such that for all $n \in \mathbb{N}$ and all connected non-trivial normal $\mathbb{Q}_p$-subgroups $\mathbf{N} \trianglelefteq \mathbb{L}_n$ it holds that $\mathbf{N}(\mathbb{Q}_p)$ is non-compact (in terms of [GO11], S is strongly isotropic).*

Let ν be a probability measure on $\mathbf{G}(\mathbb{R}\times\mathbb{Q}_S)/\mathbf{G}(\mathbb{Z}\left[S^{-1}\right])$ which is a weak-star limit of $\{\nu_n\}_{n=1}^{\infty}$. Then:

1. *There exists a connected $\mathbb{Q}$-algebraic subgroup $\mathbb{M} \leq \mathbb{G}$ such that $\nu = (t_0)_*\mu_{M\mathbb{G}(\mathbb{Z}[S^{-1}])}$ where M is a closed finite index subgroup of $\mathbb{M}(\mathbb{R}\times\mathbb{Q}_S)$, $t_0 \in \mathbf{G}(\mathbb{R}\times\mathbb{Q}_S)$ and $\mu_{M\mathbb{G}(\mathbb{Z}[S^{-1}])}$ is the left M-invariant probability measure supported on $M\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$.*
2. *There exists a sequence $\{\gamma_n\}_{n=1}^{\infty} \subseteq \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$, such that for all large enough n it holds that*
$$\gamma_n^{-1}\mathbb{L}_n\gamma_n \subseteq \mathbb{M}.$$
3. *There exists a sequence $\{l_n\}_{n=1}^{\infty} \subseteq \pi(\tilde{\mathbb{L}}_n(\mathbb{R}\times\mathbb{Q}_S))$ such that*
$$\lim_{n\to\infty} t_n l_n \gamma_n = t_0.$$

In addition,

$2: *assume that for all $n \in \mathbb{N}$ the centralizer of $\mathbf{L}_n$ in $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.*

Then the sequence of measures $\{\nu_n\}_{n=1}^{\infty}$ is relatively compact in the space of probability measures on $\mathbf{G}(\mathbb{R}\times\mathbb{Q}_S)/\mathbf{G}(\mathbb{Z}\left[S^{-1}\right])$, and the group $\mathbb{M}$ above is semi-simple.

For the rest of this section, we fix a finite set of odd primes S and a sequence $\{g_n\}_{n=1}^{\infty} \subseteq \mathrm{SL}_d(\mathbb{Z})$ which meets the assumptions of Theorem 7.1.

Recall that our goal is to find the limit of the measures $\mu_{g_n,S}$ (defined in (7.3)), but note that Theorem 7.2 applies for a sequence of measures of the form $(x_n)_*\,\mu_{\pi(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S))\mathbb{G}(\mathbb{Z}[S^{-1}])}$. As we will see in Section 7.1.1, the subgroup $\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right) \leq \mathbf{L}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)$ has a fixed finite index for all $n \in \mathbb{N}$. Using this fact, we will partition the orbits $O_{g_n,S}$ (defined in (7.2)) into finitely many pieces $O_{g_n,S,\mathbf{i}}$ (defined in (7.8) below), and we will be able to apply Theorem 7.2 to the sequence of natural measures $\mu_{g_n,S,\mathbf{i}}$ (see (7.10)) supported on $O_{g_n,S,\mathbf{i}}$. By finding the limiting measure of the sequence of $\mu_{g_n,S,\mathbf{i}}$ for each choice of $\mathbf{i}$, we will obtain limit of the measures $\mu_{g_n,S}$ (see Section 7.1.2).

7.1.1. *The universal covering of* $\mathbf{L}_g$ *and of* $\mathbb{G}_1$. In the following we will recall some facts concerning the Spin group which is the universal covering of an orthogonal group of a quadratic form, and then we will be able to describe the subgroup $\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)$ in a useful way.

Assume that $m \geq 3$ and let φ be a non-degenerate rational quadratic form in m variables. We denote by SO_φ its special orthogonal group, and for a field $\mathbb{F} \supseteq \mathbb{Q}$ we consider the spinor norm $\phi : \mathrm{SO}_\varphi(\mathbb{F}) \to \mathbb{F}^\times/(\mathbb{F}^\times)^2$ (see e.g. [Cas78, Chapter 10] for more details). We recall that the spin group Spin_φ is the simply connected covering of SO_φ (see [Cas78, Chapter 10], or [PR94, Section 2.3.2] for more details) and we note the following exact sequences (see [EV08, Lemma 1]). For an odd prime p it holds

$$\mathrm{Spin}_\varphi(\mathbb{Q}_p) \xrightarrow{\pi} \mathrm{SO}_\varphi(\mathbb{Q}_p) \xrightarrow{\phi} \mathbb{Q}_p^\times/\left(\mathbb{Q}_p^\times\right)^2 \to 0, \tag{7.4}$$

for a positive definite φ we have

$$\mathrm{Spin}_\varphi(\mathbb{R}) \xrightarrow{\pi} \mathrm{SO}_\varphi(\mathbb{R}) \xrightarrow{\phi} 0, \tag{7.5}$$

and for an indefinite φ it holds

$$\mathrm{Spin}_\varphi(\mathbb{R}) \xrightarrow{\pi} \mathrm{SO}_\varphi(\mathbb{R}) \xrightarrow{\phi} \{\pm 1\} \to 0. \tag{7.6}$$

We also note that $\pi(\mathrm{Spin}_\varphi(\mathbb{R}))$ equals to the connected component of $\mathrm{SO}_\varphi(\mathbb{R})$.

Returning to our case, we let $\tau(g_n)^{\perp(Q)}$ be the hyperplane orthogonal to $\tau(g_n)$ with respect to the form Q. We observe that $\mathrm{Spin}_{Q|_{\tau(g_n)^{\perp(Q)}}}(\mathbb{F})$ naturally identifies with $\tilde{\mathbf{H}}_{\tau(g_n)}(\mathbb{F})$ (see [EV08, Section 2.4, footnote 6]). Since we assume that $\mathbf{H}_{\tau(g_n)}(\mathbb{R})$ is compact, it follows that $Q\mid_{\tau(g_n)^{\perp(Q)}}$ is positive definite. Therefore we may conclude by (7.5) that

$$\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right) = \mathbf{L}_{g_n}(\mathbb{R}) \times \pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{Q}_S)\right). \tag{7.7}$$

For $\mathbf{i} \in \prod_{p\in S}\mathbb{Q}_p^\times/\left(\mathbb{Q}_p^\times\right)^2$ we pick $h_{g_n}^{(\mathbf{i})} \in \mathbf{H}_{\tau(g_n)}(\mathbb{Q}_S)$ such that $\phi(h_{g_n}^{(\mathbf{i})}) = \mathbf{i}$. By (7.4) and (7.7) we deduce that $\left(e_{1,\infty}, e_{2,\infty}, h_{g_n}^{(\mathbf{i})}, g_n^{-1}\theta\left(h_{g_n}^{(\mathbf{i})}\right)g_n\right)$, $\mathbf{i} \in \prod_{p\in S}\mathbb{Q}_p^\times/\left(\mathbb{Q}_p^\times\right)^2$ is a complete set of representatives of $\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)$ cosets in $\mathbf{L}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)$. We define

$$O_{g_n,S,\mathbf{i}} \stackrel{\text{def}}{=} l_{g_n}^{(\mathbf{i})}\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]), \tag{7.8}$$

where

$$l_{g_n}^{(\mathbf{i})} \stackrel{\text{def}}{=} \left(t_{g_n}, h_{g_n}^{(\mathbf{i})}, g_n^{-1}\theta\left(h_{g_n}^{(\mathbf{i})}\right)g_n\right), \tag{7.9}$$

(to recall t_g, see (7.1)) and we let

$$\mu_{g_n,S,\mathbf{i}} \stackrel{\text{def}}{=} \left(l_{g_n}^{(\mathbf{i})}\right)_* \mu_{\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}[S^{-1}])}, \tag{7.10}$$

where $\mu_{\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}[S^{-1}])}$ is the left $\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)$-invariant probability measure on the orbit $\pi\left(\tilde{\mathbf{L}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$.

7.1.2. *A reduction - The limit of* $\mu_{g_n,S,\mathbf{i}}$ *implies Theorem 7.1.* We recall the following lemma from [AES16a].

Lemma 7.3. *Let* $N \trianglelefteq K \leq G$ *be locally compact groups such that* N *is of index* $k \in \mathbb{N}$ *in* K. *Assume that* $\Gamma \leq G$ *is a lattice, let* $Kx\Gamma$ *be a finite volume orbit and denote its* K*-invariant probability measure by* $\mu_{Kx\Gamma}$. *Then*

$$\mu_{Kx\Gamma} = \frac{1}{k}\sum_{i=1}^{k}\mu_{k_iNx\Gamma},$$

where $k_1, ..., k_N$ *is a complete list of representatives for* N *cosets in* K *and* $\mu_{k_iNx\Gamma}$ *is the* N*-invariant probability measure on* $k_iNx\Gamma = Nk_ix\Gamma$.

An immediate corollary from Lemma 7.3 is that

$$\mu_{g_n,S} = \frac{1}{k_S}\sum_{\mathbf{i}} \mu_{g_n,S,\mathbf{i}}, \tag{7.11}$$

where $k_S = \left|\prod_{p\in S}\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2\right|$.

Next, for each $\mathbf{i}\in\prod_{p\in S}\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ we choose $\rho_S^{(\mathbf{i})}\in \mathrm{SO}_Q(\mathbb{Q}_S)$ such that $\phi(\rho_S^{(\mathbf{i})})=\mathbf{i}$, and we denote

$$\mathcal{Y}_{S,\mathbf{i}} \overset{\mathrm{def}}{=} (e_{1,\infty}, e_{2,\infty}, \rho_S^{(\mathbf{i})}, e_{2,S})\pi\left(\tilde{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]).$$

We claim that

$$\pi\left(\tilde{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) = \left(\mathbb{G}(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]). \tag{7.12}$$

Indeed, recall that $\mathbb{G}_2$ is simply connected, so that

$$\pi\left(\tilde{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) = \left(\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))\times\mathbb{G}_2(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]), \tag{7.13}$$

hence to prove (7.12) it is sufficient by (7.13) to show that

$$\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))\mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right]) = \mathbb{G}_1(\mathbb{R})\mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right]).$$

If Q is positive definite then we deduce by (7.5) that $\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))=\mathbb{G}_1(\mathbb{R})$. If on the other-hand Q is indefinite, we note that there exists $\gamma\in\mathbb{G}_1(\mathbb{Z})$ with $\phi(\gamma) = -\mathbb{R}^\times/\left(\mathbb{R}^\times\right)^2$ (there are integral vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ such that $Q(\mathbf{v}_\pm)\in\pm\mathbb{R}^\times$. The orthogonal transformation γ obtained by the composition of the associated reflections $\gamma=\tau_{\mathbf{v}_+}\circ\tau_{\mathbf{v}_-}$ has $\phi(\gamma) = -\mathbb{R}^\times/\left(\mathbb{R}^\times\right)^2$), which shows that

$$\begin{aligned}\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))\mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right]) &= \left(\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))\bigcup\pi(\tilde{\mathbb{G}}_1(\mathbb{R}))\gamma\right)\mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right])\\ &\underbrace{=}_{(7.6)}\mathbb{G}_1(\mathbb{R})\mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right]).\end{aligned}$$

We let

$$\mu_{\mathcal{Y}_{S,\mathbf{i}}} \overset{\mathrm{def}}{=} (e_{1,\infty}, e_{2,\infty}, \rho_S^{(\mathbf{i})}, e_{2,S})_*\mu_{\mathbb{G}(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))\mathbb{G}(\mathbb{Z}[S^{-1}])},$$

where $\mu_{\mathbb{G}(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))\mathbb{G}(\mathbb{Z}[S^{-1}])}$ is the $\mathbb{G}(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))$ invariant probability measure supported on $\pi\left(\tilde{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)\right)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$.

Running over $\mathbf{i}\in\prod_{p\in S}\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ we obtain by (7.4) that $(e_{1,\infty}, e_{2,\infty}, \rho_S^{(\mathbf{i})}, e_{2,S})$ is a complete list of representatives of $\mathbb{G}(\mathbb{R})\times\pi(\tilde{\mathbb{G}}(\mathbb{Q}_S))$ cosets in $\mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)$, and we conclude by Lemma 7.3 that

$$\mu_{\mathcal{Y}_S} = \frac{1}{k_S}\sum_{\mathbf{i}}\mu_{\mathcal{Y}_{S,\mathbf{i}}}. \tag{7.14}$$

We note that for each $\mathbf{i}\in\prod_{p\in S}\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ we have $O_{g_n,S,\mathbf{i}}\subseteq\mathcal{Y}_{S,\mathbf{i}}$, and our goal in the following will be to prove that

$$\mu_{g_n,S,\mathbf{i}}\to\mathcal{Y}_{S,\mathbf{i}}, \tag{7.15}$$

which by (7.11) and (7.14) will imply Theorem 7.1.

For the rest of the proof we fix $\mathbf{i}\in\prod_{p\in S}\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$. We now proceed to prove (7.15), which will be done in two steps.

7.1.3. *First step - the limit of* $\underline{\mu}_{g_n,S,\mathbf{i}}$. We cannot apply Theorem 7.2 as is because $\mathcal{Y}_S$ is not a quotient of a semi-simple group. Therefore in our first step below we project to a smaller space in which we can apply Theorem 7.2.

Denote $\underline{\mathbb{G}}_2 \overset{\text{def}}{=} \mathrm{SL}_{d-1}$, and define $\underline{\mathbb{G}} \overset{\text{def}}{=} \mathbb{G}_1 \times \underline{\mathbb{G}}_2$. Consider the natural map

$$\pi_{\underline{\mathbb{G}}_2} : \mathbb{G}_2 \to \underline{\mathbb{G}}_2,$$

given by

$$\begin{pmatrix} m & * \\ \mathbf{0} & 1 \end{pmatrix} \mapsto m,$$

and let $\pi_{\underline{\mathbb{G}}} : \mathbb{G} \to \underline{\mathbb{G}}$ be defined by $\pi_{\underline{\mathbb{G}}}(\rho,\eta) \overset{\text{def}}{=} (\rho, \pi_{\underline{\mathbb{G}}_2}(\eta)),\ \forall\,(\rho,\eta) \in \mathbb{G}$.

We define

$$\underline{\mathbf{L}}_g \overset{\text{def}}{=} \pi_{\underline{\mathbb{G}}}(\mathbf{L}_g) = \left\{ \left(h, \pi_{\underline{\mathbb{G}}_2}\left(g^{-1}\theta\left(h\right)g\right)\right) \mid\ h \in \mathbf{H}_{\tau(g)} \right\}.$$

The following lemma has essentially the same content as [AES16a, Lemma 3.4] (the proof is also essentially the same).

Lemma 7.4. *Let* $g \in \mathrm{SL}_d(\mathbb{Z})$ *such that* $Q(\tau(g)) > 0$. *Then:*

1. $\mathbf{H}_{\tau(g)}(\mathbb{R})$ *(resp.* $\pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(\mathbf{H}_{\tau(g)}(\mathbb{R}))g))$ *is maximal among connected algebraic subgroups of* $\mathbb{G}_1(\mathbb{R})$ *(resp.* $\underline{\mathbb{G}}_2(\mathbb{R})$*).*
2. *Assumption* \$2 *of Theorem 7.2 holds for* $\underline{\mathbf{L}}_g$.
3. *Let* p *be an odd prime, and assume that there exists* $\mathbf{u} \in \mathbb{Q}_p \otimes \tau(g)^{\perp(Q)}$ *such that* $Q(\mathbf{u}) = 0$. *Then assumption* \$1 *of Theorem 7.2 is valid for* $\mathbf{L}_g(\mathbb{Q}_p)$ *and for* $\underline{\mathbf{L}}_g(\mathbb{Q}_p) \overset{def}{=} \{(h, \pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(h)g)) \mid h \in \mathbf{H}_{\tau(g)}\}$.

Proof. To obtain (1) we recall by [Dyn52] that the stabilizer of a non-isotropic vector in a special orthogonal group of a non-degenerate quadratic form is a maximal connected Lie subgroup of $\mathbb{G}_1$, hence it follows that $\mathbf{H}_{\tau(g)}(\mathbb{R})$ is maximal among connected algebraic subgroups of $\mathbb{G}_1(\mathbb{R})$. Next, by Lemma 6.3, $\pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(\mathbf{H}_{\tau(g)}(\mathbb{R}))g))$ is the stabilizer of a non-degenerate quadratic form in $d-1$ variables, and since $\mathbf{H}_{\tau(g)}(\mathbb{R})$ is compact, we have $\pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(\mathbf{H}_{\tau(g)}(\mathbb{R}))g)) \cong \mathrm{SO}_{d-1}(\mathbb{R})$, which is a well known maximal Lie subgroup of $\underline{\mathbb{G}}_2(\mathbb{R})$. Next, to prove (2), it is sufficient to prove that the centralizer of $\mathbf{H}_{\tau(g)}(\mathbb{R})$ (resp. $\pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(\mathbf{H}_{\tau(g)}(\mathbb{R}))g))$ in $\mathbb{G}_1(\mathbb{R})$ (resp. $\underline{\mathbb{G}}_2(\mathbb{R})$) is finite. In fact, if not, we would obtain a proper connected algebraic subgroup containing $\mathbf{H}_{\tau(g)}(\mathbb{R})$ (resp. $\pi_{\underline{\mathbb{G}}_2}(g^{-1}\theta(\mathbf{H}_{\tau(g)}(\mathbb{R}))g)$, which is a contradiction to (1). Finally, if we assume that there exists $\mathbf{u} \in \mathbb{Q}_p \otimes \tau(g)^{\perp(Q)}$ for an odd prime p such that $Q(\mathbf{u}) = 0$, then by the proof of [AES16a, Lemma 3.4] we get that assumption \$1 of Theorem 7.2 is valid for $\mathbf{H}_{\tau(g)}(\mathbb{Q}_p)$. Since $\mathbf{H}_{\tau(g)}(\mathbb{Q}_p) \cong \mathbf{L}_g(\mathbb{Q}_p) \cong \underline{\mathbf{L}}_g(\mathbb{Q}_p)$, assumption \$1 of Theorem 7.2 is valid for $\mathbf{L}_g(\mathbb{Q}_p)$ and $\underline{\mathbf{L}}_g(\mathbb{Q}_p)$. □

Consider $\vartheta_{\underline{\mathbb{G}}} : \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)/\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) \to \underline{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)/\underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right])$ be the map induced by $\pi_{\underline{\mathbb{G}}}$, and note that $\vartheta_{\underline{\mathbb{G}}}$ has compact fibers. We define $\mathcal{X}_{S,\mathbf{i}} \overset{\text{def}}{=} \vartheta_{\underline{\mathbb{G}}}(\mathcal{Y}_{S,\mathbf{i}})$, and $\underline{O}_{g,S,\mathbf{i}} \overset{\text{def}}{=} \vartheta_{\underline{\mathbb{G}}}\left(O_{g,S,\mathbf{i}}\right)$, which are equivalently described by

$$\mathcal{X}_{S,\mathbf{i}} \overset{\text{def}}{=} \pi\left(\tilde{\underline{\mathbb{G}}}(\mathbb{R}\times\mathbb{Q}_S)\right)(e_{1,\infty}, e_{2,\infty}, \rho_S^{(\mathbf{i})}, e_{2,S})\underline{\mathbb{G}}\left(\mathbb{Z}\left[S^{-1}\right]\right),$$

$$\underline{O}_{g,S,\mathbf{i}} \overset{\text{def}}{=} \underline{l}_{g_n}^{(\mathbf{i})}\pi\left(\tilde{\underline{\mathbf{L}}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right]),$$

where $\underline{l}_{g_n}^{(\mathbf{i})} \overset{\text{def}}{=} \pi_{\underline{\mathbb{G}}}(l_{g_n}^{(\mathbf{i})})$ (see (7.9) for the definition of $l_{g_n}^{(\mathbf{i})}$).

Let $\underline{\mu}_{g_n,S,\mathbf{i}} \overset{\text{def}}{=} \left(\vartheta_{\underline{\mathbb{G}}}\right)_* \mu_{g_n,S,\mathbf{i}}$, and we note that

$$\underline{\mu}_{g_n,S,\mathbf{i}} = (\underline{l}_{g_n}^{(\mathbf{i})})_* \mu_{\pi(\tilde{\underline{\mathbf{L}}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S))\underline{\mathbb{G}}(\mathbb{Z}[S^{-1}])},$$

where $\mu_{\pi(\tilde{\underline{\mathbf{L}}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S))\underline{\mathbb{G}}(\mathbb{Z}[S^{-1}])}$ is the $\pi\left(\tilde{\underline{\mathbf{L}}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)$-invariant probability measure supported on $\pi\left(\tilde{\underline{\mathbf{L}}}_{g_n}(\mathbb{R}\times\mathbb{Q}_S)\right)\underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right])$.

Let ν be a weak star limit of a subsequence $\underline{\mu}_{g_n,S,\mathbf{i}}$, $n \in C_1 \subseteq \mathbb{N}$. Then by Lemma 7.4,(2) and Theorem 7.2, $\underline{\nu}$ is a probability measure and there exists a semi-simple connected $\mathbb{Q}$-algebraic subgroup $\mathbb{M} \leq \mathbb{G}$ such that

$$\underline{\nu} = (t_0)_* \, \mu_{M\mathbb{G}(\mathbb{Z}[S^{-1}])}, \tag{7.16}$$

where M is a closed finite index subgroup of $\mathbb{M}(\mathbb{R} \times \mathbb{Q}_S)$ and $t_0 \in \mathbb{G}(\mathbb{R} \times \mathbb{Q}_S)$.

For the rest of this section, our goal will be to prove that $\mathbb{M} = \underline{\mathbb{G}}$, and as we now show, this will prove that

$$\underline{\nu} = \mu_{\mathcal{X}_{S,\mathbf{i}}} \stackrel{\text{def}}{=} \left(\vartheta_{\underline{\mathbb{G}}}\right)_* \mu_{\mathcal{Y}_{S,\mathbf{i}}}, \tag{7.17}$$

which is the unique $\pi\left(\underline{\tilde{\mathbb{G}}}(\mathbb{R} \times \mathbb{Q}_S)\right)$-invariant probability measure on $\mathcal{X}_{S,\mathbf{i}}$.

So assume that $M \leq \underline{\mathbb{G}}$ is of finite index. We now show that

$$\pi(\underline{\tilde{\mathbb{G}}}(\mathbb{R} \times \mathbb{Q}_S)) \subseteq M. \tag{7.18}$$

Since $\pi(\underline{\tilde{\mathbb{G}}}(\mathbb{R})) \times \{e_S\}$ is the connected component of $\underline{\mathbb{G}}(\mathbb{R}) \times \{e_S\}$, we get

$$\pi(\underline{\tilde{\mathbb{G}}}(\mathbb{R})) \times \{e_S\} \subseteq M \cap (\underline{\mathbb{G}}(\mathbb{R}) \times \{e_S\})\,. \tag{7.19}$$

Let $\underline{\mathbb{G}}^+(\mathbb{Q}_S)$ the group generated by unipotent elements of $\underline{\mathbb{G}}(\mathbb{Q}_S)$. By Corollary 6.7 of [BT73], any subgroup of finite index contains the group $\underline{\mathbb{G}}^+(\mathbb{Q}_S)$. Since $M \cap (\{e_\infty\} \times \underline{\mathbb{G}}(\mathbb{Q}_S)) \leq \{e_\infty\} \times \underline{\mathbb{G}}(\mathbb{Q}_S)$ is of finite index, we deduce

$$\{e_\infty\} \times \underline{\mathbb{G}}^+(\mathbb{Q}_S) \subseteq M \cap (\{e_\infty\} \times \underline{\mathbb{G}}(\mathbb{Q}_S))\,. \tag{7.20}$$

Since we assume that Q is isotropic for all $p \in S$, by Lemma 1 of [EV08] we have that $\mathbb{G}_1^+(\mathbb{Q}_S) = \pi(\tilde{\mathbb{G}}_1(\mathbb{Q}_S))$, and it is well known that $\underline{\mathbb{G}}_2^+(\mathbb{Q}_S) = \underline{\mathbb{G}}_2(\mathbb{Q}_S) = \pi\left(\underline{\tilde{\mathbb{G}}}_2(\mathbb{Q}_S)\right)$. Thus we conclude that

$$\underline{\mathbb{G}}^+(\mathbb{Q}_S) = \pi(\underline{\tilde{\mathbb{G}}}(\mathbb{Q}_S)), \tag{7.21}$$

and by (7.19), (7.20), and (7.21) we deduce that (7.18) holds. Since for all n, the measure $\underline{\mu}_{g_n,S,\mathbf{i}}$ is supported on $\underline{O}_{g,S,\mathbf{i}} \subseteq \mathcal{X}_{S,\mathbf{i}}$, we deduce that $t_0 M\underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right]) \subseteq \mathcal{X}_{S,\mathbf{i}}$, and by (7.18) we conclude that $t_0 M\underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right]) \supseteq \mathcal{X}_{S,\mathbf{i}}$, which shows the implication $\mathbb{M} = \underline{\mathbb{G}} \implies$ (7.17).

Now assume for contradiction that $\mathbb{M} \lneq \underline{\mathbb{G}}$. Let

$$\pi_1 : \underline{\mathbb{G}} \to \mathbb{G}_1,\ \pi_2 : \underline{\mathbb{G}} \to \underline{\mathbb{G}}_2,$$

be the natural maps. Since $\mathbb{M}$ is semi-simple and since $\mathbb{G}_1$ and $\underline{\mathbb{G}}_2$ have no isomorphic simple Lie factors (due to ambient dimensions, accidental isomorphisms play no role), it follows that $\pi_1(\mathbb{M}) \lneq \mathbb{G}_1$ or $\pi_2(\mathbb{M}) \lneq \underline{\mathbb{G}}_2$.

By Theorem 7.2, we let $\{\gamma_{g_n}\}_{n=1}^\infty \subseteq \underline{\mathbb{G}}(\mathbb{Z}\left[S^{-1}\right])$ and a further subsequence $C_2 \subseteq C_1$ such that $|C_1 \setminus C_2| < \infty$, which satisfies

$$\gamma_{g_n}^{-1}\underline{\mathbf{L}}_{g_n}\gamma_{g_n} \subseteq \mathbb{M},\ \forall n \in C_2, \tag{7.22}$$

and we let $\{l_n\}_{n=1}^\infty \subseteq \pi(\underline{\tilde{\mathbf{L}}}_{g_n}(\mathbb{R} \times \mathbb{Q}_S))$ such that

$$(\underline{l}_{g_n}^{(\mathbf{i})})l_{g_n}\gamma_{g_n} \to t_0. \tag{7.23}$$

In case $\pi_1(\mathbb{M}) \lneq \mathbb{G}_1$. Let $\delta_{g_n} \stackrel{\text{def}}{=} \pi_1(\gamma_{g_n}) \in \mathbb{G}_1(\mathbb{Z}\left[S^{-1}\right])$. Since $\pi_1(\mathbb{M})$ is a strict, connected, semi-simple $\mathbb{Q}$ subgroup of $\mathbb{G}_1$, we obtain that $\pi_1(\mathbb{M}(\mathbb{R})) \lneq \mathbb{G}_1(\mathbb{R})$, and by maximality of the subgroups $\mathbf{H}_{\tau(g_n)}(\mathbb{R})$, (see Lemma 7.4,(1)) we obtain that for all $i, j \in C_2$

$$\delta_{g_i}^{-1}\mathbf{H}_{\tau(g_i)}(\mathbb{R})\delta_{g_i} = \delta_{g_j}^{-1}\mathbf{H}_{\tau(g_j)}(\mathbb{R})\delta_{g_j} = \pi_1(\mathbb{M}(\mathbb{R})),$$

which implies that

$$\mathbf{H}_{\delta_{g_i}^{-1}\tau(g_i)}(\mathbb{R}) = \mathbf{H}_{\delta_{g_j}^{-1}\tau(g_j)}(\mathbb{R}). \tag{7.24}$$

We fix $i \in C_2$, and by (7.24) we may deduce that for each $j \in C_2$ there exists $0 \neq \alpha_j \in \mathbb{Z}\left[S^{-1}\right]$ such that

$$\delta_{g_i}^{-1}\tau(g_i) = \alpha_j\delta_{g_j}^{-1}\tau(g_j). \tag{7.25}$$

We will now show that $\{\alpha_j\}_{j\in C_2}$ is bounded and bounded away from 0, which will be a contradiction since $Q(\tau(g_j))\to\infty$, since i is fixed, and since by (7.25) we have

$$Q\left(\tau(g_i)\right)=Q\left(\delta_{g_i}^{-1}\tau(g_i)\right)=\alpha_j^2 Q\left(\tau(g_j)\right).$$

By recalling that $\{\tau(g_j)\}_{j\in C_2}$ is a sequence of primitive integral vectors, we deduce that $\left\{\delta_{g_j}^{-1}\tau(g_j)\right\}_{j\in C_2}$ are primitive vectors in $\mathbb{Z}\left[S^{-1}\right]^d$ considered as a $\mathbb{Z}\left[S^{-1}\right]$ module. This implies that $\alpha_j\in\mathbb{Z}\left[S^{-1}\right]^\times$ where

$$\mathbb{Z}\left[S^{-1}\right]^\times=\left\{\prod_{p\in S}p^{n_p}\mid n_p\in\mathbb{Z}\right\}.$$

By (7.23) we obtain a sequence $\left\{h_{g_j}\right\}_{j\in C_2}$ with $h_{g_j}\in\pi\left(\tilde{\mathbf{H}}_{\tau(g_j)}(\mathbb{Q}_S)\right)$ such that

$$h_{g_j}^{(\mathbf{i})}h_{g_j}\delta_{g_j}\to\pi_{1,S}(t_0),$$

where $\pi_{1,S}:\underline{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S)\to\mathbb{G}_1(\mathbb{Q}_S)$ is the natural map and t_0 is given in (7.16). By multiplying both sides of (7.25) with $h_{g_j}^{(\mathbf{i})}h_{g_j}\delta_{g_j}$, we obtain that

$$\lim_{C_2\ni j\to\infty}\alpha_j\tau(g_j)=\pi_{1,S}(t_0)\delta_{g_i}^{-1}\tau(g_i). \tag{7.26}$$

Since $\tau(g_j)$ is a primitive integral vector, $\|\tau(g_j)\|_p$ (the maximum of the p-adic valuations of the entries) is constant in j for all $p\in S$. Thus, by (7.26), the p-adic valuation of α_j is bounded, and since $\alpha_j\in\mathbb{Z}\left[S^{-1}\right]^\times$, we conclude that $\{\alpha_j\}_{j\in C_2}$ is bounded and bounded away from 0.

In case $\pi_2(\mathbb{M})\lneq\underline{\mathbb{G}}_2$. We will obtain a contradiction in a similar way as we had in the case that $\pi_1(\mathbb{M})\lneq\underline{\mathbb{G}}_2$. We denote $\eta_{g_n}\stackrel{\text{def}}{=}\pi_2(\gamma_{g_n})\in\underline{\mathbb{G}}_2(\mathbb{Z}\left[S^{-1}\right])$. Since $\pi_2(\mathbb{M})$ is a strict, connected semi-simple $\mathbb{Q}$ subgroup of $\underline{\mathbb{G}}_2$, we obtain by maximality (see Lemma 7.4,(1)) and by recalling (7.22), that for all $i,j\in C_2$

$$\eta_{g_i}^{-1}\pi_{\underline{\mathbb{G}}_2}\left(g_i^{-1}\theta\left(\mathbf{H}_{\tau(g_i)}(\mathbb{R})\right)g_i\right)\eta_{g_i}=\eta_{g_j}^{-1}\pi_{\underline{\mathbb{G}}_2}\left(g_j^{-1}\theta\left(\mathbf{H}_{\tau(g_j)}(\mathbb{R})\right)g_j\right)\eta_{g_j}. \tag{7.27}$$

By Lemma 6.3, we find that (7.27) can be rewritten by

$$\mathrm{SO}_{\varphi_{g_j}^{\eta_{g_j}}}(\mathbb{R})=\mathrm{SO}_{\varphi_{g_i}^{\eta_{g_i}}}(\mathbb{R}),$$

where the quadratic form φ_g^γ is a given by (6.3). By recalling Lemma 3.3 of [AES16a], we find that there exists $\alpha_j\in\mathbb{Q}^\times$ such that

$$\alpha_j\varphi_{g_j}^{\eta_{g_j}}=\varphi_{g_i}^{\eta_{g_i}}, \tag{7.28}$$

where we fix i and let $j\in C_2$ vary. Our plan now is to show that $\{\alpha_j\}_{j\in C_2}$ is bounded and bounded away from 0. This will be a contradiction since we assume that $Q(\tau(g_j))\to\infty$ and since by Lemma 6.2 we have that $\mathrm{disc}(\varphi_{g_j}^{\eta_{g_j}})=\frac{1}{\mathrm{disc}(Q)}Q(\tau(g_j))$, where $\mathrm{disc}(\varphi)$ denotes the determinant of the companion matrix of a quadratic form φ.

We recall that (see (6.4))

$$\varphi_g^\eta(\mathbf{u})=\mathbf{u}^t\left(\eta^t\hat{g}^tM^{-1}\hat{g}\eta\right)\mathbf{u},$$

where $\hat{g}$ is the matrix formed by the first $d-1$ columns of g and where M is the companion matrix of Q. Therefore, by (7.28) we deduce

$$\alpha_j(\eta_j^t\hat{g}_j{}^tM^{-1}\hat{g}_j\eta_j)=\eta_i^t\hat{g}_i{}^tM^{-1}\hat{g}_i\eta_i,$$

which in turn implies that

$$\alpha_j(\eta_j^t\hat{g}_j{}^t\mathrm{adj}(M)\hat{g}_j\eta_j)=\eta_i^t\hat{g}_i{}^t\mathrm{adj}(M)\hat{g}_i\eta_i, \tag{7.29}$$

where $\mathrm{adj}(M)$ is the matrix adjugate of M, which has integral entries as M is integral. We denote

$$\bar{M}_{\varphi_g^\eta}\stackrel{\text{def}}{=}\eta^t\hat{g}^t\mathrm{adj}(M)\hat{g}\eta,$$

and for $l \in C_2$ we let $q_l \in \mathbb{N}$ be defined by

$$q_l \overset{\mathrm{def}}{=} g.c.d(\hat{g_l}^t \mathrm{adj}(M)\hat{g_l}).$$

We rewrite (7.29) to

$$\frac{\alpha_j q_j}{q_i}\left(\frac{1}{q_j}\bar{M}_{\varphi_{g_j}^{\eta_j}}\right) = \frac{1}{q_i}\bar{M}_{\varphi_{g_i}^{\eta_i}}, \tag{7.30}$$

and by noting that $\frac{1}{q_l}\hat{g_l}^t \mathrm{adj}(M)\hat{g_l}$ has co-primes entries, we may deduce that $\frac{\alpha_j q_j}{q_i} \in \left(\mathbb{Z}\left[S^{-1}\right]\right)^{\times}$.

By (7.23) there exists a sequence $\left\{k_{g_j}\right\}_{j\in C_2}$ with $k_{g_j} \in \mathrm{SO}_{\varphi_{g_j}}(\mathbb{Q}_S)$ such that

$$\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\eta_{g_j} \to \pi_{2,S}\left(t_0\right), \tag{7.31}$$

where $\underline{k}_{g_j}^{(\mathbf{i})} \overset{\mathrm{def}}{=} \pi_{\underline{\mathbb{G}}_2}\left(g_j^{-1}\theta\left(h_{g_j}^{(\mathbf{i})}\right)g_j\right) \in \mathrm{SO}_{\varphi_{g_j}}(\mathbb{Q}_S)$, $\pi_{2,S} : \underline{\mathbb{G}}(\mathbb{R}\times\mathbb{Q}_S) \to \underline{\mathbb{G}}_2(\mathbb{Q}_S)$ is the natural map and t_0 is given in (7.16). We conclude by denoting $\bar{M}_{\varphi_{g_j}} = \bar{M}_{\varphi_{g_j}^e}$, and by noting that $\bar{M}_{\varphi_{g_j}}$ is a multiple of the companion matrix of the quadratic form φ_{g_j}, that

$$\begin{aligned}\left(\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\eta_{g_j}\right)^t\right)^{-1} \bar{M}_{\varphi_{g_j}^{\eta_{g_j}}}\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\eta_{g_j}\right)^{-1} &= \left(\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\right)^{-1}\right)^t \bar{M}_{\varphi_{g_j}}\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\right)^{-1} \\ &= \bar{M}_{\varphi_{g_j}}.\end{aligned} \tag{7.32}$$

To simplify notation, we denote the fixed matrix $\frac{1}{q_i}\bar{M}_{\varphi_{g_i}^{\eta_{g_i}}}$ by B and we deduce by (7.30) and (7.32) that

$$\frac{\alpha_j q_j}{q_i}\left(\frac{1}{q_j}\bar{M}_{g_j}\right) = \left(\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\eta_{g_j}\right)^t\right)^{-1} B\left(\left(\underline{k}_{g_j}^{(\mathbf{i})} k_{g_j}\eta_{g_j}\right)\right)^{-1}. \tag{7.33}$$

We conclude by (7.31) and (7.33) that the p-adic norm of $\frac{\alpha_j q_j}{q_i}\left(\frac{1}{q_j}\bar{M}_{g_j}\right)$ is bounded for all $p \in S$, and since $\frac{1}{q_j}\bar{M}_{g_j}$ is a primitive integral matrix, the p-adic norm of $\frac{\alpha_j q_j}{q_i}\left(\frac{1}{q_j}\bar{M}_{g_j}\right)$ equals to the p-adic valuation $\left|\frac{\alpha_j q_j}{q_i}\right|_p$ for all $p \in S$. Since $\left\{\frac{\alpha_j q_j}{q_i}\right\}_{j\in C_2} \subseteq \left(\mathbb{Z}\left[S^{-1}\right]\right)^{\times}$, we conclude that $\left\{\frac{\alpha_j q_j}{q_i}\right\}_{j\in C_2}$ is bounded in absolute value from above and away from 0.

Finally, using Lemma 6.4 we deduce that q_j is uniformly bounded in $j \in C_2$ from above and below, which implies in turn that α_j is bounded in $j \in C_2$ from above and away from 0.

7.1.4. *Second step - Upgrading to $\mathbb{G}$.* In a summary of the first step, it holds that

$$\left(\vartheta_{\underline{\mathbb{G}}}\right)_* \mu_{g_n,S,\mathbf{i}} = \underline{\mu}_{g_n,S,\mathbf{i}}, \tag{7.34}$$

and it holds that $\underline{\mu}_{g_n,S,\mathbf{i}} \to \mu_{\mathcal{X}_{S,\mathbf{i}}}$, where

$$\left(\vartheta_{\underline{\mathbb{G}}}\right)_* \mu_{\mathcal{Y}_{S,\mathbf{i}}} = \mu_{\mathcal{X}_{S,\mathbf{i}}}. \tag{7.35}$$

Let ν be a weak-star limit of a subsequence $\{\mu_{g_n,S,\mathbf{i}}\}_{n\in C_1}$, for $C_1 \subseteq \mathbb{N}$. Using (7.34) and (7.35), we deduce that ν is a probability measure.

In order to prove that $\nu = \mu_{\mathcal{Y}_{S,\mathbf{i}}}$, we will apply Theorem 7.2 in the ambient space

$$\mathbb{G}'(\mathbb{R}\times\mathbb{Q}_S)/\mathbb{G}'(\mathbb{Z}\left[S^{-1}\right]),$$

where $\mathbb{G}' \overset{\mathrm{def}}{=} \mathbb{G}_1 \times \mathrm{SL}_d$.

By Theorem 7.2 and Lemma 7.4,(3) there exists a connected $\mathbb{Q}$-algebraic subgroup $\mathbb{M} \leq \mathbb{G}'$ such that

$$\nu = \left(t_0\right)_* \mu_{M\mathbb{G}'(\mathbb{Z}[S^{-1}])}, \tag{7.36}$$

where $M \leq \mathbb{M}(\mathbb{R}\times\mathbb{Q}_S)$ is a closed finite index subgroup and $t_0 \in \mathbb{G}'(\mathbb{R}\times\mathbb{Q}_S)$.

As explained in [AES16a] (see below equation (4.5) in [AES16a]), it follows that $\mathbb{M} \leq \mathbb{G}$, that $t_0 \in \mathbb{G}(\mathbb{R} \times \mathbb{Q}_S)$, that there exists a sequence $\{\gamma_{g_n}\}_{n \in C_2} \subseteq \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$, where $C_2 \subseteq C_1$, $|C_1 \smallsetminus C_2| < \infty$ such that

$$\gamma_{g_n}^{-1} \mathbf{L}_{g_n} \gamma_{g_n} \subseteq \mathbb{M}, \tag{7.37}$$

and that either $\mathbb{M} = \mathbb{G}$ or $\mathbb{M} = \mathbb{G}_1 \times \mathrm{SL}_{d-1}^{\mathbf{t}}$ where

$$\mathrm{SL}_{d-1}^{\mathbf{t}} \overset{\text{def}}{=} c_{\mathbf{t}} \iota \left(\mathrm{SL}_{d-1}\right) c_{\mathbf{t}}^{-1},\ c_{\mathbf{t}} \overset{\text{def}}{=} \begin{pmatrix} I_{d-1} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix},\ \mathbf{t} \in \mathbb{Q}^{d-1},$$

and $\iota : \mathrm{SL}_{d-1} \to \mathrm{ASL}_{d-1}$ is the natural embedding which maps $m \mapsto \begin{pmatrix} m & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$. As in Section 7.1.3 (by the same argument), the proof will be done once we show that $\mathbb{M} = \mathbb{G}$.

Assume by contradiction that $\mathbb{M} = \mathbb{G}_1 \times \mathrm{SL}_{d-1}^{\mathbf{t}}$. We let $\pi_2 : \mathbb{G} \to \mathbb{G}_2$ be the coordinate map, and we denote $\eta_g \overset{\text{def}}{=} \pi_2(\gamma_g)$. By definition of $\mathrm{SL}_{d-1}^{\mathbf{t}}$ and by (7.37) we obtain $\forall n \in C_2$ that

$$c_{\mathbf{t}}^{-1} \eta_{g_n}^{-1} g_n^{-1} \theta\left(\mathbf{H}_{\tau(g_n)}\right) g_n \eta_{g_n} c_{\mathbf{t}} \subseteq \iota\left(\mathrm{SL}_{d-1}\right). \tag{7.38}$$

We fix $N \in \mathbb{N}$ such that $N\mathbf{t} \in \mathbb{Z}_{\text{prim}}^{d-1}$. By using that $\iota\left(\mathrm{SL}_{d-1}\right)$ fixes $\mathbf{e}_d$ and by using (7.38) we conclude that

$$\tilde{\mathbf{v}}_n \overset{\text{def}}{=} \left(g_n \eta_{g_n} c_{\mathbf{t}}\right)\left(N e_d\right), \tag{7.39}$$

is fixed by the left linear action of $\theta\left(\mathbf{H}_{\tau(g_n)}\right)$. By Lemma 3.3, the group $\theta(\mathbf{H}_{\tau(g_n)})$ is the stabilizer subgroup of the non-isotropic vector $M\tau(g_n)$ under the left linear action of SO_{Q^*}. The space of fixed vectors for such groups is one-dimensional, hence there exists $\alpha_{g_n} \in \mathbb{Q}$ such that

$$\alpha_{g_n}\left(M\tau(g_n)\right) = \tilde{\mathbf{v}}_n. \tag{7.40}$$

Again as above, we will show that $\{\alpha_{g_n}\}_{n \in C_2}$ is bounded and bounded away from 0.

Before continuing, we will now explain why the boundedness of $\{\alpha_{g_n}\}_{n \in C_2}$ yields a contradiction. By definition of $\tilde{\mathbf{v}}_n$ in (7.39), we may express $\tilde{\mathbf{v}}_n$ by

$$\tilde{\mathbf{v}}_n = \sum_{i=1}^{d-1} a_i\left(g_n \mathbf{e}_i\right) + N\left(g_n \mathbf{e}_d\right),$$

where $a_1, ..., a_{d-1} \in \mathbb{Q}$. We now observe that

$$\begin{aligned} \alpha_{g_n} Q(\tau(g_n)) =& \alpha_{g_n} \tau(g_n)^t M \tau(g_n) \underbrace{=}_{(7.40)} \tau(g_n)^t \tilde{\mathbf{v}}_n \\ =& \sum_{i=1}^{d-1} a_i \underbrace{\langle \tau(g_n), g_n \mathbf{e}_i \rangle}_{=0} + N \underbrace{\langle \tau(g_n), g_n \mathbf{e}_d \rangle}_{=1} \\ =& N, \end{aligned}$$

and since $Q(\tau(g_n)) \to \infty$ and N is fixed, this will be a contradiction.

We now proceed to show the boundedness of $\{\alpha_{g_n}\}_{n \in C_2}$. We denote for $n \in \mathbb{N}$ by q_{g_n} the g.c.d of $M\tau(g_n)$, and we rewrite (7.40) by

$$\left(\alpha_{g_n} q_{g_n}\right)\left(\frac{1}{q_{g_n}} M\tau(g_n)\right) = g_n \eta_{g_n}(N c_{\mathbf{t}} \mathbf{e}_d)$$

Using that $\frac{1}{q_{g_n}} M\tau(g_n)$ and $N c_{\mathbf{t}} \mathbf{e}_d$ are primitive integral vectors, we deduce by the preceding equality that $\alpha_{g_n} q_{g_n} \in \mathbb{Z}\left[S^{-1}\right]^{\times}$. By Theorem 7.2,(3) there exists a sequence $\{h_{g_n}\}_{n \in C_2}$ with $h_{g_n} \in \pi\left(\tilde{\mathbf{H}}_{\tau(g)}(\mathbb{Q}_S)\right)$ such that

$$h_{g_n}^{(\mathbf{i})} h_{g_n} \delta_{g_n} \to \pi_{1,S}\left(t_0\right),$$

where $\delta_{g_n} \overset{\text{def}}{=} \pi_1(\gamma_{g_n})$, $\pi_{1,S} : \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S) \to \mathbb{G}_1(\mathbb{Q}_S)$ is the natural map, t_0 is given in (7.36), and

$$g_n^{-1}\theta\left(h^{(\mathbf{i})}_{g_n} h_{g_n}\right) g_n\eta_{g_n} \to \pi_{2,S}(t_0)\,, \tag{7.41}$$

where $\eta_{g_n} \overset{\text{def}}{=} \pi_2(\gamma_{g_n})$, $\pi_{2,S} : \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S) \to \mathbb{G}_2(\mathbb{Q}_S)$ is the natural map. We obtain by (7.40) and by recalling that $\theta(h^{(\mathbf{i})}_{g_n} h_{g_n})$ stabilizes $M\tau(g_n)$ (see Lemma 3.3), that

$$\begin{aligned}\alpha_{g_n} M\tau(g_n) &= \theta(h^{(\mathbf{i})}_{g_n} h_{g_n})\tilde{\mathbf{v}}_n \\ &\underbrace{=}_{\text{recalling (7.39)}} g_n\left(g_n^{-1}\theta(h^{(\mathbf{i})}_{g_n} h_{g_n}) g_n\eta_{g_n}\right) c_{\mathbf{t}}(Ne_d)\,.\end{aligned} \tag{7.42}$$

Since $g_n \in \mathrm{SL}_d(\mathbb{Z})$, we get for any $p \in S$ that

$$\left\| g_n\left(g_n^{-1}\theta(h^{(\mathbf{i})}_{g_n} h_{g_n})_p g_n\eta_{g_n}\right) c_{\mathbf{t}}(Ne_d)\right\|_p = \left\|\left(g_n^{-1}\theta(h^{(\mathbf{i})}_{g_n} h_{g_n})_p g_n\eta_{g_n}\right) c_{\mathbf{t}}(Ne_d)\right\|_p,$$

where $\|\cdot\|_p$ is the maximum of p-adic valuations of the entries, and $\theta(h^{(\mathbf{i})}_{g_n} h_{g_n})_p$ is the p'th component of $\theta(h^{(\mathbf{i})}_{g_n} h_{g_n}) \in \mathbb{G}(\mathbb{Q}_S)$. By (7.41) and (7.42) we deduce for all $p \in S$ that the p-adic valuation of $\|\alpha_{g_n} M\tau(g_n)\|_p$ is bounded. Since $\frac{1}{q_{g_n}} M\tau(g_n)$ is a primitive integral vector, we get that

$$\left\|\alpha_{g_n} q_{g_n}\left(\frac{1}{q_{g_n}} M\tau(g_n)\right)\right\|_p = |\alpha_{g_n} q_{g_n}|_p\,,$$

which implies in turn that $|\alpha_{g_n} q_{g_n}|_p$ is bounded in $n \in \mathbb{N}$, for all $p \in S$. By recalling that $\alpha_{g_n} q_{g_n} \in (\mathbb{Z}\left[S^{-1}\right])^\times$, we conclude that $\{\alpha_{g_n} q_{g_n}\}_{n\in C_2}$ is bounded and bounded away from 0. Finally, since $M\mathbb{Z}^d \subseteq \mathbb{Z}^d$ and since $M\tau(g_n) \in M\mathbb{Z}^d$ is a primitive vector in the lattice $M\mathbb{Z}^d$, we get by [Cas97], Chapter 1, Theorem 1,B. that $q_{g_n} \le \det(M)$, which completes the proof.

8. Equivalence classes of integral points and their relation to the S-arithmetic orbits

In this section we define for each $T > 0$ an equivalence relation on $\mathcal{Z}_T(\mathbb{Z})$ for which there are finitely many equivalence classes $E_{g_1} \bigsqcup ... \bigsqcup E_{g_N} = \mathcal{Z}_T(\mathbb{Z})$, see Section 8.1. The motivation for this equivalence relation is a connection established in Section 8.3 between each equivalence class E_g and the orbit $O_{g,S}$ (the main result is Corollary 8.4).

Outline for the rest of the paper. The current section may be viewed as a prelude to Section 9 in which we use the aforementioned connection (Corollary 8.4) and Theorem 7.1 to prove Theorem 9.1, which gives the limiting distribution of the normalized counting measures on the subsets $\left\{(\pi_{\mathcal{Z}_{Q(\tau(g))}}(x), \vartheta_q(x)) \mid x \in E_g\right\} \subseteq \mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_a(\mathbb{Z}/(q))$, as $Q(\tau(g)) \to \infty$.

In Section 10 we achieve our main goal of proving Theorems 3.7 - 3.8 concerning the limit of the normalized counting measures supported on $\{(\pi_{\mathcal{Z}_T}(x), \vartheta_q(x)) \mid x \in \mathcal{Z}_T(\mathbb{Z})\}$, $T \in \mathbb{N}$, by rewriting the counting measures on $\{(\pi_{\mathcal{Z}_T}(x), \vartheta_q(x)) \mid x \in \mathcal{Z}_T(\mathbb{Z})\}$ as an average of the counting measures on $\{(\pi_{\mathcal{Z}_T}(x), \vartheta_q(x)) \mid x \in E_g\}$ and by employing Theorem 8.1.

8.1. **The equivalence relation.** A natural way to "generate" integral points on $\mathcal{Z}_T(\mathbb{Z})$ from a given $g \in \mathcal{Z}_T(\mathbb{Z})$ is to view g as a point in $\mathcal{Z}_T(\mathbb{Q}_S)$ and to consider the intersection of orbits

$$E_g \overset{\text{def}}{=} g\cdot\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) \bigcap g\cdot\mathbb{G}(\mathbb{Z}_S) \tag{8.1}$$

(to recall the definition of the right action of $\mathbb{G}$ on $\mathcal{Z}_T$ see (3.6)). We define our equivalence relation on $\mathcal{Z}_T(\mathbb{Z})$ by $g \sim g' \iff E_g = E_{g'}$. Clearly, the equivalence class of each $g \in \mathcal{Z}_T(\mathbb{Z})$ is given by E_g.

Lemma 8.1. *For each $T > 0$, it holds that each equivalence class E_g is composed of finitely many $\mathbb{G}(\mathbb{Z})$ orbits, and it holds that there are finitely many equivalence classes.*

Proof. Note that each equivalence class is $\mathbb{G}(\mathbb{Z})$ invariant, hence each equivalence class is composed of $\mathbb{G}(\mathbb{Z})$ orbits. There are finitely many $\mathbb{G}(\mathbb{Z})$ orbits in $\mathcal{Z}_T(\mathbb{Z})$ by Corollary 3.1, (3), which proves our claim. $\square$

8.2. **A decomposition of the orbits $O_{g,S}$.** For the rest of this section we fix a finite set of primes S, and we take $g \in \mathrm{SL}_d(\mathbb{Z})$ such that $Q(\tau(g)) > 0$.

The goal of this section is to deduce the decomposition (8.8), which is a technical fact that we will need in Section 8.3 to relate the orbit $O_{g,S}$ with E_g.

We recall the definition of $O_{g,S}$ and we rewrite it as follows

$$\begin{aligned} O_{g,S} =& (t_g, e_S)\mathbf{L}_g(\mathbb{R} \times \mathbb{Q}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) \\ =& \left(t_g\mathbf{L}_g(\mathbb{R})t_g^{-1} \times \mathbf{L}_g(\mathbb{Q}_S)\right)(t_g, e_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]), \end{aligned} \tag{8.2}$$

where t_g is defined in (7.1). By Lemma 3.5 we deduce that $t_g\mathbf{L}_g(\mathbb{R})t_g^{-1} = H$ (where $H = \mathbf{L}_{I_d}(\mathbb{R})$) and by (8.2) we deduce that

$$O_{g,S} = H \times \mathbf{L}_g(\mathbb{Q}_S)(t_g, e_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]). \tag{8.3}$$

We have that $\mathbf{L}_g$ is a $\mathbb{Q}$-group, hence we obtain

$$\mathbf{L}_g(\mathbb{Q}_S) = \bigsqcup_{h \in M} \mathbf{L}_g(\mathbb{Z}_S)h\mathbf{L}_g(\mathbb{Z}\left[S^{-1}\right]), \tag{8.4}$$

where $M = M(g)$ is a finite set of representatives of the double coset space (see [PR94, Chapter 5]). Using (8.3) and (8.4) we obtain the decomposition

$$O_{g,S} = \bigsqcup_{h \in M} O_{g,S,h}, \tag{8.5}$$

where

$$O_{g,S,h} \overset{\text{def}}{=} \left(H \times \mathbf{L}_g(\mathbb{Z}_S)\right)(t_g, h)\,\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right). \tag{8.6}$$

8.2.1. *Intersection with the principle genus.* We will be actually interested in the intersection $O_{g,S} \cap \mathcal{U}_S$, where $\mathcal{U}_S \subseteq \mathbb{G}(\mathbb{R} \times \mathbb{Q}_S)/\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ is the clopen orbit of the clopen subgroup $\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$ passing through the identity coset $\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$, namely

$$\mathcal{U}_S \overset{\text{def}}{=} \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]). \tag{8.7}$$

Since $\mathbb{G}(\mathbb{Z}) \leq \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$, where $\mathbb{G}(\mathbb{Z}) \subseteq \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$ is diagonally embedded, is the stabilizer subgroup stabilizing $\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ by the natural left action, we conclude that $\mathcal{U}_S$ is naturally identified with $\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$, where each element $(g_\infty, g_S)\,\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) \in \mathcal{U}_S$ viewed as a point in $\mathbb{G}(\mathbb{R} \times \mathbb{Q}_S)/\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ identifies with $\left(g_\infty\gamma^{-1}, g_S\gamma^{-1}\right)\mathbb{G}(\mathbb{Z}) \in \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$, where $\gamma \in \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ is an arbitrary element which gives that $g_S\gamma^{-1} \in \mathbb{G}(\mathbb{Z}_S)$.

We observe that $O_{g,S,h} \cap \mathcal{U}_S \neq \emptyset$ if and only if $O_{g,S,h} \subseteq \mathcal{U}_S$, which shows that

$$O_{g,S} \cap \mathcal{U}_S = \bigsqcup_{h \in M_0} O_{g,S,h}, \tag{8.8}$$

where $M_0 = M_0(g) \subseteq M(g)$ is a finite subset.

For all $h \in M_0$, since $O_{g,S,h} \subseteq \mathcal{U}_S$, we obtain that $h \in \mathbf{L}_g(\mathbb{Q}_S) \cap \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$. Namely there are $c \in \mathbb{G}(\mathbb{Z}_S)$ and $\gamma \in \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ such that

$$h = c\gamma^{-1}. \tag{8.9}$$

Then, for $h \in M_0$, we get that the orbit $O_{g,S,h}$ (defined in (8.13)) is identified by

$$O_{g,S,h} = (H \times \mathbf{L}_g(\mathbb{Z}_S))(t_g\gamma, c)\mathbb{G}(\mathbb{Z}). \tag{8.10}$$

8.3. **A duality principle relating E_g with $O_{g,S} \cap \mathcal{U}_S$.** The main idea that stands behind the relation of E_g with $O_{g,S}$ can be roughly described as a "duality" principle by which we transfer a left $(H \times \mathbf{L}_g(\mathbb{Z}_S))$-orbit (8.10) in $\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$ to a right $\mathbb{G}(\mathbb{Z})$-orbit in $(H \times \mathbf{L}_g(\mathbb{Z}_S))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$ via the following diagram of natural maps

$$\begin{tikzcd} & \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S) \arrow[dl] \arrow[dr] & \\ (H \times \mathbf{L}_g(\mathbb{Z}_S))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S) & & \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z}) \end{tikzcd} \tag{8.11}$$

Namely, by (8.11), we transfer an orbit (8.10) to a $\mathbb{G}(\mathbb{Z})$-orbit $\mathcal{Q}_{g,S,h} \subseteq (H \times \mathbf{L}_g(\mathbb{Z}_S))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$ passing through the base point $(H \times \mathbf{L}_g(\mathbb{Z}_S))(t_g\gamma, c)$. By using the right action of $\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$ on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_{Q(\tau(g))}(\mathbb{Z}_S)$, and by recalling that $H \times \mathbf{L}_g(\mathbb{Z}_S)$ is the stabilizer of (I_d, g), we may identify $\mathcal{Q}_{g,S,h}$ with (where we abuse notations)

$$\mathcal{Q}_{g,S,h} = (I_d \cdot (t_g\gamma), g \cdot c) \cdot \mathbb{G}(\mathbb{Z}) \subseteq \mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}) \times \mathcal{Z}_{Q(\tau(g))}(\mathbb{Z}_S). \tag{8.12}$$

To relax notations, we denote the homeomorphism $\pi_{\mathcal{Z}_T} : \mathcal{Z}_T(\mathbb{R}) \to \mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})$ (defined in (3.16)) by $\pi_{\mathcal{Z}}$. The lemma below gives the key correspondence between $O_{g,S} \cap \mathcal{U}_S$ and E_g.

Lemma 8.2. *It holds that* $\bigsqcup_{h \in M_0} \mathcal{Q}_{g,S,h} = \{(\pi_{\mathcal{Z}}(x), x) \mid x \in E_g\}$, *and that* $|M_0| = |E_g/\mathbb{G}(\mathbb{Z})|$.

Proof. Let us first show that for each $h \in M_0$ it holds that $\mathcal{Q}_{g,S,h} \subseteq \{(\pi_{\mathcal{Z}}(x), x) \mid x \in E_g\}$. By writing $h = c\gamma^{-1}$ for $c \in \mathbb{G}(\mathbb{Z}_S)$ and $\gamma \in \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ and by noting that $g = g \cdot h = g \cdot (c\gamma^{-1})$, we obtain that $g \cdot \gamma = g \cdot c$. Then, g' defined by

$$g' \overset{\text{def}}{=} g \cdot \gamma = g \cdot c.$$

is in E_g. By definition of t_g (see (7.1)) we have

$$g' = g \cdot \gamma = a_{Q(\tau(g))} \cdot (t_g\gamma),$$

and by using the equivariance of the map $\pi_{\mathcal{Z}}$, we get

$$\pi_{\mathcal{Z}}(g') = \pi_{\mathcal{Z}}(g \cdot \gamma) = \pi_{\mathcal{Z}}(a_{Q(\tau(g))} \cdot (t_g\gamma)) \underbrace{=}_{\text{recalling (3.16)}} I_d \cdot (t_g\gamma).$$

We may now conclude that

$$(\pi_{\mathcal{Z}}(g'), g') \cdot \mathbb{G}(\mathbb{Z}) = \mathcal{Q}_{g,S,h},$$

and by using equivariance of $\pi_{\mathcal{Z}}$, we deduce that

$$(\pi_{\mathcal{Z}}(g'), g') \cdot \mathbb{G}(\mathbb{Z}) = \left\{(\pi_{\mathcal{Z}}(g' \cdot \gamma), g' \cdot \gamma) \mid \gamma \in \mathbb{G}(\mathbb{Z})\right\} \subseteq \{(\pi_{\mathcal{Z}}(x), x) \mid x \in E_g\}.$$

We will now prove the inclusion in the opposite direction. We let $g' \in E_g$ and we note that, according to the definition of E_g (see (8.1)), there are $c \in \mathbb{G}(\mathbb{Z}_S)$ and $\gamma \in \mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ such that

$$g' = g \cdot \gamma = g \cdot c.$$

We can deduce from the preceding equality that $h \overset{\text{def}}{=} c\gamma^{-1}$ is an element of $\mathbf{L}_g(\mathbb{Q}_S) \cap \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$, and we conclude by the preceding paragraph that $(\pi_{\mathcal{Z}}(g'), g') \in \mathcal{Q}_{g,S,h}$.

Finally, since $O_{g,S,h}$, $h \in M_0$, are disjoint $(H \times \mathbf{L}_g(\mathbb{Z}_S))$-orbits in $\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$, it follows that $\mathcal{Q}_{g,S,h}$, $h \in M_0$, are disjoint $\mathbb{G}(\mathbb{Z})$-orbits in $(H \times \mathbf{L}_g(\mathbb{Z}_S))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)$. As $\{(\pi_{\mathcal{Z}}(x), x) \mid x \in E_g\}/\mathbb{G}(\mathbb{Z})$ is in bijection with $E_g/\mathbb{G}(\mathbb{Z})$, it follows that $M_0 = |E_g/\mathbb{G}(\mathbb{Z})|$. □

We are actually interested in the set $\{(\pi_{\mathcal{Z}}(x), \vartheta_q(x)) \mid x \in E_g\}$, and in order to relate it to the orbits $O_{g,S,h}$ we will consider the projection modulo q in the following subsection.

8.3.1. *Taking the residue modulo q.* We note that the natural ring homomorphism

$$\vartheta_{p^k} : \mathbb{Z}_p \to \mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/(p^k),$$

induces a homomorphism $\vartheta_{p^k} : \mathbb{G}(\mathbb{Z}_p) \to \mathbb{G}(\mathbb{Z}/(p^k))$. Let $q \in \mathbb{N}$ and assume that S includes the primes S_q appearing in the prime decomposition of q. The Chinese remainder theorem yields the identification

$$\prod_{p_i \in S',} \mathbb{G}(\mathbb{Z}/p_i^{k_i}\mathbb{Z}) \cong \mathbb{G}(\mathbb{Z}/(q)),$$

and so we obtain the map $\vartheta_q : \mathbb{G}(\mathbb{Z}_S) \to \mathbb{G}(\mathbb{Z}/(q))$ in the obvious way. We also note that $\vartheta_q(\mathbf{L}_g(\mathbb{Z}_S)) \subseteq \mathbf{L}_{\vartheta_q(g)}(\mathbb{Z}/(q))$.

We consider the map $(id_\infty \times \vartheta_q) : \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S) \to \mathbb{G}(\mathbb{R} \times \mathbb{Z}/(q))$ given by

$$(id_\infty \times \vartheta_q)(g_\infty, g_S) \stackrel{\text{def}}{=} (g_\infty, \vartheta_q(g_S)),$$

and we upgrade Diagram (8.11) to the following diagram

$$\begin{tikzcd}
 & \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S) \arrow[dl] \arrow[dr] & \\
(H \times \mathbf{L}_g(\mathbb{Z}_S))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}_S) \arrow[d, "(id_\infty \times \vartheta_q)"'] & & \mathbb{G}(\mathbb{R} \times \mathbb{Z}_S)/\mathbb{G}(\mathbb{Z}) \arrow[d, "(id_\infty \times \vartheta_q)"'] \\
(H \times \mathbf{L}_{\vartheta_q(g)}(\mathbb{Z}/(q)))\backslash\mathbb{G}(\mathbb{R} \times \mathbb{Z}/(q)) & & \mathbb{G}(\mathbb{R} \times \mathbb{Z}/(q))/\mathbb{G}_{(q)}(\mathbb{Z})
\end{tikzcd}$$

where

$$\mathbb{G}_{(q)}(\mathbb{Z}) \stackrel{\text{def}}{=} \{(u, \vartheta_q(u)) \mid u \in \mathbb{G}(\mathbb{Z})\} \leq \mathbb{G}(\mathbb{R} \times \mathbb{Z}/(q)).$$

We let

$$\begin{aligned} O_{g,q,h} &\stackrel{\text{def}}{=} (id_\infty \times \vartheta_q) \circ O_{g,S,h} \\ &= (H \times \vartheta_q(\mathbf{L}_g(\mathbb{Z}_S)))(t_g\gamma, \vartheta_q(c))\mathbb{G}_{(q)}(\mathbb{Z}) \end{aligned} \tag{8.13}$$

and we let

$$\mathcal{Q}_{g,q,h} \stackrel{\text{def}}{=} (id_\infty \times \vartheta_q)(\mathcal{Q}_{g,S,h}) \tag{8.14}$$

where $\mathcal{Q}_{g,q,h}$ is the right $\mathbb{G}_{(q)}(\mathbb{Z})$-orbit passing through $(H \times \mathbf{L}_{\vartheta_q(g)}(\mathbb{Z}/(q)))(t_g\gamma, \vartheta_q(c))$.

Lemma 8.3. *Let $h, h' \in M_0$ be two different elements and let $\gamma, \gamma' \in \mathbb{G}(\mathbb{Z}[S^{-1}])$ which appear in a decomposition* (8.9) *of h, h' correspondingly. Then $Ht_g\gamma\mathbb{G}(\mathbb{Z}) \cap Ht_g\gamma'\mathbb{G}(\mathbb{Z}) = \emptyset$.*

Proof. Assume for contradiction that $Ht_g\gamma\mathbb{G}(\mathbb{Z}) \cap Ht_g\gamma'\mathbb{G}(\mathbb{Z}) \neq \emptyset$. Then there exists $\kappa \in H$ and $u \in \mathbb{G}(\mathbb{Z})$ such that

$$t_g^{-1}\kappa t_g\gamma u = \gamma'. \tag{8.15}$$

This gives that

$$\left(t_g^{-1}\kappa t_g\right) h^{-1} \left(cuc'^{-1}\right) = h'^{-1}, \tag{8.16}$$

where $c, c' \in \mathbb{G}(\mathbb{Z}_S)$ appear in the decomposition (8.9) of h, h' correspondingly. By the definition of t_g and by (8.15) we conclude that $t_g^{-1}\kappa t_g \in \mathbf{L}_g(\mathbb{R}) \cap \mathbb{G}(\mathbb{Z}[S^{-1}]) = \mathbf{L}_g(\mathbb{Z}[S^{-1}])$, and by (8.16) we get $\left(cuc'^{-1}\right) \in \mathbf{L}_g(\mathbb{Q}_S) \cap \mathbb{G}(\mathbb{Z}_S) = \mathbf{L}_g(\mathbb{Z}_S)$. Hence (8.16) shows that h and h' are equivalent, which is a contradiction since h, h' are representatives for two different cosets in the space $\mathbf{L}_g(\mathbb{Z}_S)\backslash\mathbf{L}_g(\mathbb{Q}_S)/\mathbf{L}_g(\mathbb{Z}[S^{-1}])$. □

From Lemma 8.2, we obtain the following corollary, which is the main conclusion of our discussion in this section.

Corollary 8.4. *It holds that $\bigsqcup_{h \in M_0} \mathcal{Q}_{g,q,h} = \{(\pi_{\mathcal{Z}}(x), \vartheta_q(x)) \mid x \in E_g\}$, and that $M_0 = |E_g/\mathbb{G}(\mathbb{Z})|$.*

Proof. By Lemma 8.3 it follows that $\bigsqcup_{h\in M_0}\mathcal{Q}_{g,q,h}$ is indeed a disjoint union, and by using Lemma 8.2 we obtain that

$$\begin{aligned}\bigsqcup_{h\in M_0}\mathcal{Q}_{g,q,h}&=(id_\infty\times\vartheta_q)\left(\bigsqcup_{h\in M_0}\mathcal{Q}_{g,S,h}\right)\\&=(id_\infty\times\vartheta_q)\left(\{(\pi_{\mathcal{Z}}(x),x)\mid x\in E_g\}\right)\\&=\{(\pi_{\mathcal{Z}}(x),\vartheta_q(x))\mid x\in E_g\}.\end{aligned}$$

□

9. Statistics of the equivalence classes E_g

We are now ready to study the statistics of E_g as $Q(\tau(g))\to\infty$ by using the limiting distribution of the orbits $O_{g,S}$ (Theorem 7.1), and by exploiting the connection between the equivalence classes E_g and the orbits $O_{g,S}$ (Corollary 8.4).

We now list the assumptions that will hold throughout this section, which will allow us to employ Theorem 7.1.

- Q is a form as in our Standing Assumption and $q\in 2\mathbb{N}+1$ is such that Q is non-singular modulo q.
- $\{g_n\}_{n=1}^\infty\subseteq \mathrm{SL}_d(\mathbb{Z})$ satisfy that $Q(\tau(g_n))\to\infty$ and for all $n\in\mathbb{N}$
  - $Q(\tau(g_n))>0$
  - there is a prime p_0 for which $\tau(g_n)$ is (Q,p_0) co-isotropic (see Definition 3.6),
  - The reduction mod q is fixed in n, namely $\vartheta_q(g_n)=\bar{g}$, for all $n\in\mathbb{N}$.
- S_q denotes the set of primes decomposing q and $S\overset{\text{def}}{=}S_q\cup\{p_0\}$.

By Lemma 6.1(2), we deduce that the assumptions of Theorem 7.1 indeed hold for S and the sequence $\{g_n\}_{n=1}^\infty$.

We denote $a\overset{\text{def}}{=}Q(\tau(\bar{g}))\in\mathbb{Z}/(q)$ and consider the measures on $\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{Z}_a(\mathbb{Z}/(q))$ given by

$$\nu_{g_n}^q\overset{\text{def}}{=}\frac{1}{|E_{g_n}/\mathbb{G}(\mathbb{Z})|}\sum_{x\in E_{g_n}}\delta_{(\pi_{\mathcal{Z}}(x),\vartheta_q(x))},\ n\in\mathbb{N}.$$

Our main goal in this section is to prove the following theorem.

Theorem 9.1. *Consider $O_{\bar{g}}\subseteq\mathcal{Z}_a(\mathbb{Z}/(q))$ defined by*

$$O_{\bar{g}}\overset{def}{=}\bar{g}\cdot\mathbb{G}(\mathbb{Z}/(q)),$$

and let $\mu_{O_{\bar{g}}}$ be the normalized counting measure on $O_{\bar{g}}$. Then for all $f\in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{Z}_a(\mathbb{Z}/(q)))$ it holds that

$$\lim_{n\to\infty}\nu_{g_n}^q(f)=\mu_{\mathcal{Z}}\otimes\mu_{O_{\bar{g}}}(f).$$

9.1. **Outline of proof for Theorem 9.1.** We now outline the method we will use in the proof of Theorem 9.1, building on Theorem 7.1 and the link between the equivalence classes E_{g_n} and the orbits $O_{g_n,S}$.

We denote

$$G\overset{\text{def}}{=}\mathbb{G}(\mathbb{R}\times\mathbb{Z}/(q)),\ K\overset{\text{def}}{=}(H\times\mathbf{L}_{\bar{g}}(\mathbb{Z}/(q))),\ \Gamma=\mathbb{G}_{(q)}(\mathbb{Z}),\tag{9.1}$$

and we consider the following diagram of natural maps

$$\begin{tikzcd} K\backslash G \arrow[dr, "\pi_\Gamma"] & & G/\Gamma \arrow[dl, "\pi_K"'] \\ & K\backslash G/\Gamma & \end{tikzcd}$$

where π_K and π_Γ denote the natural quotient map.

We recall that $\bigsqcup_{h\in M_0}O_{g,q,h}$ is a disjoint union of finitely many $(H\times\vartheta_q(\mathbf{L}_{g_n}(\mathbb{Z}_S)))$-orbits and we recall that $(H\times\vartheta_q(\mathbf{L}_{g_n}(\mathbb{Z}_S)))\subseteq K$. Hence $\mathcal{R}_{g_n,q}\subseteq K\backslash G/\Gamma$ defined by

$$\mathcal{R}_{g_n,q} \overset{\text{def}}{=} \pi_K(\bigsqcup_{h\in M_0} O_{g,q,h}), \tag{9.2}$$

is a finite set, and by Lemma 8.3 we obtain that $|\mathcal{R}_{g_n,q}| = |M_0| = |E_{g_n}/\mathbb{G}(\mathbb{Z})|$. In Section 9.2, we will prove, by relying on Theorem 7.1, that the uniform probability counting measures $\lambda_{g_n,q}$ on $\mathcal{R}_{g_n,q}$ equidistribute towards the natural probability measure $\mu_{K\backslash G/\Gamma}$ on $K\backslash G/\Gamma$.

To deduce the limit of our counting measures that are supported on $\{(\pi_{\mathcal{Z}}(x), \vartheta_q(x)) \mid x \in E_{g_n}\}$ by the equidistribution of $\{\lambda_{g_n,q}\}_{n=1}^{\infty}$ we observe that $\mathcal{R}_{g_n,q}$ can be also described by

$$\mathcal{R}_{g_n,q} \overset{\text{def}}{=} \pi_\Gamma(\bigsqcup_{h\in M_0} \mathcal{Q}_{g,q,h}) = \pi_\Gamma(\{(\pi_{\mathcal{Z}}(x), \vartheta_q(x)) \mid x \in E_g\}), \tag{9.3}$$

and we use "unfolding" technique (similarly to Section 5.1.1) to lift the measure $\lambda_{g_n,q}$ for $n \in \mathbb{N}$ to the counting measure on $K\backslash G$ supported on $\{(\pi_{\mathcal{Z}}(x), \vartheta_q(x)) \mid x \in E_g\}$.

9.1.1. *Unfolding.* We now discuss the "unfolding" process mentioned above which lifts an equidistribution result in $K\backslash G/\Gamma$ to an equidistribution result in $K\backslash G$.

Let m_G, $m_{G/\Gamma}$, be G-invariant measures on G, G/Γ respectively, such that $m_{G/\Gamma}$ is a probability measure and all the measures are Weil normalized (a notion introduced in Section 4.3.2), namely such that for all $\varphi \in C_c(G)$

$$\int_G \varphi(g)dm_G(g) = \int_{G/\Gamma}\left(\sum_{\gamma\in\Gamma}\varphi(g\gamma)\right)dm_{G/\Gamma}(g\Gamma). \tag{9.4}$$

We define a measure on $K\backslash G$ by $\mu_{K\backslash G} \overset{\text{def}}{=} (\pi_K)_* m_G$, and a measure on $K\backslash G/\Gamma$ by $\mu_{K\backslash G/\Gamma} \overset{\text{def}}{=} (\pi_K)_* m_{G/\Gamma}$ (which is well defined, since we assume that K is compact).

Assume that $S_n \subseteq K\backslash G/\Gamma$ is a finite set, and consider the measures $\bar{\nu}_n$ supported on $K\backslash G$ defined by

$$\bar{\nu}_n \overset{\text{def}}{=} \frac{1}{|S_n|}\sum_{x\in(\pi_\Gamma)^{-1}(S_n)}\delta_x.$$

Let

$$\begin{aligned}\mathcal{F} &\overset{\text{def}}{=} \{Kg\Gamma \mid |\mathrm{Stab}_\Gamma(Kg)| > 1\} \\ &= \left\{Kg\Gamma \mid \left|g^{-1}Kg\cap\Gamma\right| > 1\right\}.\end{aligned} \tag{9.5}$$

Lemma 9.2. *Assume that $S_n \subseteq K\backslash G/\Gamma$, $n \in \mathbb{N}$, are finite sets such that the probability counting measures supported on S_n converge weakly to $\mu_{K\backslash G/\Gamma}$, and that*

$$\frac{|\mathcal{F}\cap S_n|}{|S_n|} \to 0. \tag{9.6}$$

Then for every $f \in C_c(K\backslash G)$, it holds that $\bar{\nu}_n(f) \to \mu_{K\backslash G}(f)$.

The proof of Lemma 9.2 involves elementary tools, hence we decided to include the complete details in the appendix.

Our goal in the following section is to verify the assumptions of Lemma 9.2 for $S_n = \mathcal{R}_{g_n,q}$, which will prove Theorem 9.1.

9.2. **Equidistribution in $K\backslash G/\Gamma$.** Let $\tilde{\eta}_{g_n,S}$ be the measure supported on $O_{g_n,S}\cap\mathcal{U}_S$, given by

$$\tilde{\eta}_{g_n,S} \overset{\text{def}}{=} \mu_{g_n,S}\mid_{\mathcal{U}_S}, \tag{9.7}$$

where $\mathcal{U}_S = \mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)\mathbb{G}(\mathbb{Z}\left[S^{-1}\right]) \cong \mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$ and $\mu_{g_n,S}$ defined in (7.3) is the natural probability measure supported on $O_{g_n,S}$. We consider the following probability measure $\eta_{g_n,q}$ on $K\backslash G/\Gamma$ supported on $\mathcal{R}_{g_n,q}$ (by Corollary 8.4) defined by

$$\eta_{g_n,q} \overset{\text{def}}{=} (\pi_K\circ(id_\infty\times\vartheta_q))_*\tilde{\eta}_{g_n,S}.$$

We obtain the following corollary which follows by Theorem 7.1.

Corollary 9.3. *It holds that*

$$\eta_{g_n,q} \to \mu_{K\backslash G/\Gamma}, \tag{9.8}$$

where $\mu_{K\backslash G/\Gamma}$ is the push-forward by the natural quotient map π_K of the unique G-invariant probability measure on G/Γ.

Proof. Since $\mathcal{U}_S \subseteq \mathbb{G}(\mathbb{R}\times\mathbb{Q}_S)/\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ is a clopen set, we get by Theorem 7.1 that

$$\tilde{\eta}_{g_n,S} \overset{\text{weak}^*}{\longrightarrow} \mu_{\mathcal{U}_S}, \tag{9.9}$$

where $\mu_{\mathcal{U}_S}$ is the unique $\mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)$ invariant probability on

$$\mathcal{U}_S \cong \mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)/\mathbb{G}(\mathbb{Z}).$$

By Lemma 6.1,(1), by the Chinese remainder theorem, and by noting that $\vartheta_q\left(\mathrm{ASL}_{d-1}(\mathbb{Z}_S)\right) = \mathrm{ASL}_{d-1}(\mathbb{Z}/(q))$, we conclude that

$$\vartheta_q\left(\mathbb{G}(\mathbb{Z}_S)\right) = \mathbb{G}(\mathbb{Z}/(q)).$$

Hence $(id_\infty\times\vartheta_q) : \mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)/\mathbb{G}(\mathbb{Z}) \to G/\Gamma$ is onto. It now follows that

$$\begin{aligned}\eta_{g_n,q} = (\pi_K\circ(id_\infty\times\vartheta_q))_*\tilde{\eta}_{g_n,S} \to &(\pi_K)_*(id_\infty\times\vartheta_q)_*\mu_{\mathcal{U}_S}\\ =&(\pi_K)_*\mu_{G/\Gamma}\\ =&\mu_{K\backslash G/\Gamma}.\end{aligned}$$

$\square$

9.2.1. *Weights of the measures $\eta_{g_n,q}$.* In the following we study the weights of the atoms of the measures $\eta_{g_n,q}$ which are supported on the finite sets $\mathcal{R}_{g_n,q}$.

We express $\eta_{g_n,q}$ by

$$\eta_{g_n,q} = \sum_{h\in M_0}\alpha_h^{(n)}\delta_{\pi_K(O_{g_n,q,h})}, \tag{9.10}$$

and by recalling (9.7) and the decomposition (8.8) of $O_{g_n,S}\cap\mathcal{U}_S$, we conclude that

$$\alpha_h^{(n)} = \tilde{\eta}_{g_n,S}\left((\pi_K\circ(id_\infty\times\vartheta_q))^{-1}\left(O_{g_n,q,h}\right)\right) \underbrace{=}_{\text{Lemma } 8.3} \tilde{\eta}_{g_n,S}(O_{g_n,S,h}).$$

It follows that

$$\alpha_h^{(n)} = \tilde{\eta}_{g_n,S}(O_{g_n,S,h}) = \frac{\alpha^{(n)}}{\left|\mathrm{stab}_{H\times\mathbf{L}_{g_n}(\mathbb{Z}_S)}\left((t_{g_n}\gamma,c)\mathbb{G}(\mathbb{Z})\right)\right|}, \tag{9.11}$$

where $c\in\mathbb{G}(\mathbb{Z}_S)$, $\gamma\in\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$ decompose h as in (8.9), where $\mathrm{stab}_{H\times\mathbf{L}_{g_n}(\mathbb{Z}_S)}(x)$ for $x\in\mathbb{G}(\mathbb{R}\times\mathbb{Z}_S)/\mathbb{G}(\mathbb{Z})$ is the stabilizer of x under the natural left action of $H\times\mathbf{L}_{g_n}(\mathbb{Z}_S)$, and where $\alpha^{(n)}\in\mathbb{R}_{>0}$ is a normalizing factor which turns $\eta_{g_n,q}$ to a probability measure.

Lemma 9.4. *Let $g\in\{g_n\}_{n=1}^\infty$, and let $h\in M_0$ be such that $h=c\gamma^{-1}$, for $\gamma\in\mathbb{G}\left(\mathbb{Z}\left[S^{-1}\right]\right)$ and $c\in\mathbb{G}(\mathbb{Z}_S)$. Then*

$$\left|stab_{H\times\mathbf{L}_{g_n}(\mathbb{Z}_S)}\left((t_g\gamma,c)\mathbb{G}(\mathbb{Z})\right)\right| \le \left|\mathbf{H}_{\tau(I_d\cdot(t_g\gamma))}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|.$$

Proof. We have that

$$\mathrm{stab}_{H\times\mathbf{L}_g(\mathbb{Z}_S)}\left((t_g\gamma,c)\mathbb{G}(\mathbb{Z})\right) = (H\times\mathbf{L}_g(\mathbb{Z}_S))\cap x_{g,h}\mathbb{G}(\mathbb{Z})x_{g,h}^{-1}.$$

where $x_{g,h} = (t_g\gamma,c)$. We recall that $H\times\mathbf{L}_g(\mathbb{Z}_S)$ is a graph of a function $f:\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\times\mathbf{H}_{\tau(g)}(\mathbb{Z}_S)\to\mathbb{G}(\mathbb{R})\times\mathbb{G}(\mathbb{Z}\left[S^{-1}\right])$, (see Lemma 3.2), which gives

$$\left|(H\times\mathbf{L}_g(\mathbb{Z}_S))\cap x_{g,h}\mathbb{G}(\mathbb{Z})x_{g,h}^{-1}\right| \le \left|\left(\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\times\mathbf{H}_{\tau(g)}(\mathbb{Z}_S)\right)\cap\pi_1(x_{g,h})\mathbb{G}_1(\mathbb{Z})\pi_1(x_{g,h})^{-1}\right|,$$

where $\pi_1:\mathbb{G}\to\mathbb{G}_1$ is the natural projection, and $\pi_1(x_{g,h}) = (\pi_1(t_g\gamma),\pi_1(c))$. We observe that

$$\begin{aligned}\Big|\big(\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\times\mathbf{H}_{\tau(g)}(\mathbb{Z}_S)\big)\bigcap\pi_1(x_{g,h})\mathbb{G}_1(\mathbb{Z})\pi_1(x_{g,h})^{-1}\Big|\\ =\Big|\pi_1(x_{g,h})^{-1}\left(\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\times\mathbf{H}_{\tau(g)}(\mathbb{Z}_S)\right)\ \pi_1(x_{g,h})\bigcap\mathbb{G}_1(\mathbb{Z})\Big|\\ \underbrace{\leq}_{\mathbb{G}_1(\mathbb{Z})\text{ is diagonally embedded}}\Big|\pi_1(t_g\gamma)^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\pi_1(t_g\gamma)\bigcap\mathbb{G}_1(\mathbb{Z})\Big|.\end{aligned}$$

We conclude that

$$\left|\mathrm{stab}_{H\times\mathbf{L}_g(\mathbb{Z}_S)}\left((t_g\gamma,c)\mathbb{G}(\mathbb{Z})\right)\right|\leq\Big|\pi_1(t_g\gamma)^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\pi_1(t_g\gamma)\bigcap\mathbb{G}_1(\mathbb{Z})\Big|,$$

and we note that we may finish the proof by verifying that

$$\pi_1(t_g\gamma)^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\pi_1(t_g\gamma)=\mathbf{H}_{\tau(I_d\cdot(t_g\gamma))}(\mathbb{R}). \tag{9.12}$$

To prove the latter equality we recall that the right $\mathrm{SO}_Q(\mathbb{R})$ actions on $\mathrm{SL}_d(\mathbb{R})$ and on $\mathbb{R}^d\smallsetminus\{\mathbf{0}\}$ are equivariant with respect to $\tau:\mathrm{SL}_d(\mathbb{R})\to\mathbb{R}^d\smallsetminus\{\mathbf{0}\}$ (to recall, see (3.5)), which shows that

$$\mathbf{e}_d\cdot\pi_1(t_g\gamma)=\tau(I_d\cdot(t_g\gamma)),$$

and which in turn implies (9.12). □

For $g_\infty\in\mathbb{G}(\mathbb{R})$, $\eta\in H$ and $u\in\mathbb{G}(\mathbb{Z})$ we note that

$$\left|\mathbf{H}_{\tau(I_d\cdot g_\infty)}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|=\left|\mathbf{H}_{\tau(I_d\cdot(\eta g_\infty u))}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|,$$

and we define $\mathcal{E}\subseteq K\backslash G/\Gamma$ by

$$\mathcal{E}\overset{\text{def}}{=}\left\{\left(H\times\mathbf{L}_{\bar{g}}(\mathbb{Z}/(q))\right)(g_\infty,g_{(q)})\mathbb{G}_{(q)}(\mathbb{Z})\mid\left|\mathbf{H}_{\tau(I_d\cdot g_\infty)}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|>1\right\}. \tag{9.13}$$

Lemma 9.5. *We denote $\alpha^{(n)}_{max}=\max_{h\in M_0}\{\alpha^{(n)}_h\}$, where $\alpha^{(n)}_h$ are the weights of the atoms of $\eta_{g_n,q}$ (see (9.10)). Then there exists $m>0$ such that $\frac{\alpha^{(n)}_{max}}{m}\leq\alpha^{(n)}_h\leq\alpha^{(n)}_{max}$, $\forall n\in\mathbb{N}$. Moreover, for all $h\in M_0$ such that $\pi_K\left(O_{g_n,q,h}\right)\notin\mathcal{E}$, it holds that $\alpha^{(n)}_h=\alpha^{(n)}_{max}$.*

Proof. It follows by Lemma 9.4 and by (9.11) that

$$\frac{\alpha^{(n)}}{\left|\mathbf{H}_{\tau(I_d\cdot(t_{g_n}\gamma))}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|}\leq\alpha^{(n)}_h\leq\alpha^{(n)}. \tag{9.14}$$

We recall that

$$\pi_K\left(O_{g_n,q,h}\right)=K\left(t_{g_n}\gamma,\vartheta_q\left(c\right)\right)\Gamma,$$

and we conclude by (9.13) and (9.14) that

$$\alpha^{(n)}_h=\alpha^{(n)}_{\max}=\alpha^{(n)}\Longleftarrow\pi_K\left(O_{g_n,q,h}\right)\notin\mathcal{E}.$$

Finally, we show that $\left|\mathbf{H}_{\tau(I_d\cdot(t_{g_n}\gamma))}(\mathbb{R})\cap\mathbb{G}_1(\mathbb{Z})\right|$ is uniformly bounded from above. Indeed, since $\mathbf{H}_{\tau(I_d\cdot(t_{g_n}\gamma))}(\mathbb{R})$ is compact (being a conjugate of $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$, which is compact by our Standing Assumption), we obtain that the subgroup $\mathbf{H}_{\tau(I_d\cdot(t_{g_n}\gamma))}(\mathbb{R})\cap\mathbb{G}(\mathbb{Z})\leq\mathrm{GL}_d(\mathbb{Z})$ is finite. For a fixed $d\in\mathbb{N}$, the size of finite subgroups of $\mathrm{GL}_d(\mathbb{Z})$ is uniformly bounded (see for example [Fri97]), which implies that there exists $m>0$ such that $\frac{\alpha^{(n)}_{\max}}{m}\leq\alpha^{(n)}_h$. □

Lemma 9.6. *It holds that $\frac{|\mathcal{R}_{g_n,q}\cap\mathcal{E}|}{|\mathcal{R}_{g_n,q}|}\to0$ as $n\to\infty$.*

Proof. We claim that in order to prove $\lim_{n\to\infty}\frac{|\mathcal{R}_{g_n,q}\cap\mathcal{E}|}{|\mathcal{R}_{g_n,q}|}=0$, it is sufficient to show that $\mathcal{E}\subseteq K\backslash G/\Gamma$ is closed and that

$$\mu_{K\backslash G/\Gamma}(\mathcal{E})=0. \tag{9.15}$$

Indeed, by assuming the preceding limit, the proof will be complete since

$$\begin{aligned}0 =&\mu_{K\backslash G/\Gamma}(\mathcal{E}) \underbrace{\geq}_{\text{Corollary 9.3}} \limsup_{n\to\infty} \eta_{g_n,q}(\mathcal{E})\\ =&\limsup_{n\to\infty} \frac{\eta_{g_n,q}(\mathcal{E}\cap R_{g_n,q})}{\eta_{g_n,q}(\mathcal{R}_{g_n,q})} \underbrace{\geq}_{\text{Lemma 9.5}} \limsup_{n\to\infty} \frac{\frac{\alpha^{(n)}_{\max}}{m}|\mathcal{R}_{g_n,q}\cap\mathcal{E}|}{\alpha^{(n)}_{\max}|\mathcal{R}_{g_n,q}|}.\end{aligned}$$

We will now proceed to prove (9.15). Consider the natural projection

$$\mathrm{p} : (H \times \mathbf{L}_{\bar{g}}(\mathbb{Z}/(q)))\,\backslash\mathbb{G}(\mathbb{R}\times\mathbb{Z}/(q))/\mathbb{G}_{(q)}(\mathbb{Z}) \to \mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathbb{G}_1(\mathbb{R})/\mathbb{G}_1(\mathbb{Z}),$$

and note that

$$\mathrm{p}(\mathcal{E}) = \left\{\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\rho\mathbb{G}_1(\mathbb{Z}) \mid \rho^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\rho \cap \mathbb{G}_1(\mathbb{Z}) \neq \{e\}\right\}.$$

We now recall some basic facts concerning orbifolds (we follow [Bor92]). Since $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ is compact, it follows that $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathbb{G}_1(\mathbb{R})/\mathbb{G}_1(\mathbb{Z})$ is an orbifold, and the set $\mathrm{p}(\mathcal{E})$ is known as its singular set (see [Bor92, Definition 25]). The singular set is closed and has empty interior, see [Bor92, Proposition 26], hence in particular $\mathcal{E}$ is closed (as a preimage of a closed set). Now since $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$ is compact, it is known that there exists a $\mathbb{G}_1(\mathbb{R})$ right invariant Riemannian metric on $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathbb{G}_1(\mathbb{R})$. Hence by [Bor92, Proposition 34], the singular set is locally the image of a union of finitely many sub-manifolds of $\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathbb{G}_1(\mathbb{R})$ under the natural quotient map. Therefore

$$\mu_{\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\backslash\mathbb{G}_1(\mathbb{R})/\mathbb{G}_1(\mathbb{Z})}(\mathrm{p}(\mathcal{E})) = 0,$$

which implies (9.15). □

Lemma 9.7. *It holds that $\mathcal{F}\subseteq\mathcal{E}$, where $\mathcal{F}\subseteq K\backslash G/\Gamma$ is given by* (9.5)*.*

Proof. We recall that $\mathcal{F}$ is given by

$$\mathcal{F} = \left\{K(g_\infty, g_{(q)})\Gamma \mid \left|(g_\infty, g_{(q)})^{-1}K(g_\infty, g_{(q)})\cap\Gamma\right| > 1\right\}.$$

We let $K(g_\infty, g_{(q)})\Gamma\in\mathcal{F}$, and upon recalling the notations of K, G and Γ in (9.1), we deduce that there exists $u\in\mathbb{G}(\mathbb{Z})\smallsetminus\{e\}$ and $h_\infty\in H$ such that

$$g_\infty^{-1}h_\infty g_\infty = u.$$

By recalling the definition of H (see (3.14)) we obtain that

$$\pi_1(g_\infty^{-1}h_\infty g_\infty) = \pi_1(u)\in\mathbb{G}_1(\mathbb{Z})\smallsetminus\{e\}, \tag{9.16}$$

where $\pi_1:\mathbb{G}\to\mathbb{G}_1$ is the natural projection. We have that

$$\pi_1(g_\infty^{-1}h_\infty g_\infty) = \pi_1(g_\infty)^{-1}\pi_1(h_\infty)\pi_1(g_\infty),$$

and that $\pi_1(h_\infty)\in\pi_1(H) = \mathbf{H}_{\mathbf{e}_d}(\mathbb{R})$, which implies by (9.16) that

$$\left|\pi_1(g_\infty)^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\pi_1(g_\infty)\cap\mathbb{G}_1(\mathbb{Z})\right| > 1. \tag{9.17}$$

By (9.17), by observing that

$$\pi_1(g_\infty)^{-1}\mathbf{H}_{\mathbf{e}_d}(\mathbb{R})\pi_1(g_\infty) = \mathbf{H}_{\mathbf{e}_d\cdot\pi_1(g_\infty)}(\mathbb{R}) \underbrace{=}_{(3.5)} \mathbf{H}_{\tau(I_d\cdot g_\infty)}(\mathbb{R}),$$

and by recalling (9.13) which defines $\mathcal{E}$, we obtain that $K(g_\infty, g_{(q)})\Gamma\in\mathcal{E}$. □

We now state the key corollary of this section, which verifies the assumptions of Lemma 9.2 and finishes our proof of Theorem 9.1.

Corollary 9.8. *It holds that*

$$\lim_{n\to\infty}\frac{|\mathcal{F}\cap\mathcal{R}_{g_n,q}|}{|\mathcal{R}_{g_n,q}|} = 0,$$

and it holds that the sequence probability counting measures $\lambda_{g_n,q}$ supported on $\mathcal{R}_{g_n,q}$ for $n\in\mathbb{N}$ converges to $\mu_{K\backslash G/\Gamma}$.

Proof. By Lemma 9.6 and Lemma 9.7, we deduce that $\lim_{n\to\infty}\frac{|\mathcal{F}\cap\mathcal{R}_{g_n,q}|}{|\mathcal{R}_{g_n,q}|}=0$. By Corollary 9.5, Lemma 9.6, we obtain

$$\eta_{g_n,q}-\lambda_{g_n,q}\to 0, \tag{9.18}$$

and by (9.8), we deduce that $\lambda_{g_n,q}\to\mu_{K\backslash G/\Gamma}$. □

10. Proof of theorems 4.8 and 4.9 for $\mathcal{Z}$

We let Q be as in our Standing Assumption. We consider a sequence $\{T_n\}_{n=1}^{\infty}\subseteq\mathbb{N}$ such that $T_n\to\infty$, and assume that there is an odd prime p_0 for which it holds that T_n has the (Q,p_0) co-isotropic property for all $n\in\mathbb{N}$ (see Definition 3.6).

For each $n\in\mathbb{N}$, let $g_{1,n},..,g_{m(n),n}\in\mathcal{Z}_{T_n}(\mathbb{Z})$ be a complete set of representatives for the equivalence relation defined in Section 8, namely

$$E_{g_{1,n}}\bigsqcup...\bigsqcup E_{g_{m(n),n}}=\mathcal{Z}_{T_n}(\mathbb{Z}).$$

We claim that each of vector of the list $\tau(g_{1,n}),..,\tau(g_{m(n),n})$ is also (Q,p_0) co-isotropic (see Definition 3.6). Indeed, by Witt's theorem, the action of $\mathrm{SO}_Q(\mathbb{Q})$ is transitive on $\mathcal{H}_{T_n}(\mathbb{Q})$, and if $\mathbf{v}\in\mathcal{H}_{T_n}(\mathbb{Q})$ is (Q,p) co-isotropic, then it follows that $\rho\mathbf{v}$ is (Q,p) co-isotropic, for $\rho\in\mathrm{SO}_Q(\mathbb{Q})$.

We now fix an arbitrary sequence $\{g_{j_n,n}\}_{n=1}^{\infty}$ for $1\le j_n\le m(n)$, we fix $q\in 2\mathbb{N}+1$ such that Q is non-singular modulo q and we let $S=S_q\cup\{p_0\}$ where S_q is the set of primes appearing in the prime decomposition of q.

10.1. **Proof of Theorem 3.7.** We partition the sequence $\{g_{j_n,n}\}_{n=1}^{\infty}$ into finitely many subsequences $\{g_{j_n,n}\}_{n\in C}$, $C\subseteq\mathbb{N}$ such that for all $n\in C$ the reduction mod q is fixed, say $\bar{g}\overset{\text{def}}{=}\vartheta_q(g_{j_n,n})$, $\forall n\in C$. Then, we may apply Theorem 9.1 to any of those unbounded subsequences.

We let $f\in C_c(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R}))$ and we consider $\tilde{f}\in C_c\left(\mathcal{Z}_{Q(\mathbf{e}_d)}(\mathbb{R})\times\mathcal{Z}_a(\mathbb{Z}/(q))\right)$ defined by $\tilde{f}(x,y)\overset{\text{def}}{=}f(x)$, where $a\overset{\text{def}}{=}Q(\bar{g})\in\mathbb{Z}/(q)$. Then, in the notations of Theorem 9.1, we have

$$\lim_{C\ni n\to\infty}\nu^q_{g_{j_n,n}}(\tilde{f})=\mu_{\mathcal{Z}}(f),$$

which implies in turn that for the full sequence (namely, without the assumption that $\vartheta_q(g_{j_n,n})$ is fixed in n) it holds that

$$\lim_{n\to\infty}\nu^q_{g_{j_n,n}}(\tilde{f})=\mu_{\mathcal{Z}}(f). \tag{10.1}$$

We recall that

$$\nu^q_{g_{j,n}}=\frac{1}{\left|E_{g_{j,n}}/\mathbb{G}(\mathbb{Z})\right|}\sum_{x\in E_{g_{j,n}}}\delta_{\left(\pi_{\mathcal{Z}_{T_n}}(x),\vartheta_q(x)\right)},$$

and that (see (3.19))

$$\nu^{\mathcal{Z},q}_{T_n}=\frac{1}{|\mathcal{Z}_{T_n}(\mathbb{Z})/\mathbb{G}(\mathbb{Z})|}\sum_{x\in\mathcal{Z}_{T_n}(\mathbb{Z})}\delta_{\left(\pi_{\mathcal{Z}_{T_n}}(x),\vartheta_q(x)\right)}.$$

It follows that

$$\sum_{j=1}^{m(n)}\left(\sum_{x\in E_{g_{j,n}}}\delta_{\left(\pi_{\mathcal{Z}_{T_n}}(x),\vartheta_q(x)\right)}\right)=\sum_{x\in\mathcal{Z}_{T_n}(\mathbb{Z})}\delta_{\left(\pi_{\mathcal{Z}_{T_n}}(x),\vartheta_q(x)\right)}, \tag{10.2}$$

and that

$$\sum_{j=1}^{m_n}\left|E_{g_{j_n,n}}/\mathbb{G}(\mathbb{Z})\right|=|\mathcal{Z}_{T_n}(\mathbb{Z})/\mathbb{G}(\mathbb{Z})|=|\mathcal{H}_{T_n,\mathrm{prim}}(\mathbb{Z})/\mathbb{G}_1(\mathbb{Z})|. \tag{10.3}$$

We now note the following elementary lemma (which we give without a proof).

Lemma 10.1. *Let* $\{a_{i,n}\}_{i=1,n=1}^{m_n,\infty}$ $\{b_{i,n}\}_{i=1,n=1}^{m_n,\infty}$ *be positive real sequences. Assume* $a_{i_n,n}/b_{i_n,n} \to L$*, for any sequence* $\{i_n\}_{n=1}^{\infty}$ *such that* $i_n \in \{1,..,m_n\}$*. Then* $\frac{\sum_{i=1}^{m_n} a_{i,n}}{\sum_{i=1}^{m_n} b_{i,n}} \to L$*.*

We may now deduce by Lemma 10.1 and (10.1), (10.2), (10.3) that

$$\nu_{T_n}^{\mathcal{Z}}(f) \underbrace{=}_{\text{recalling (3.18)}} \frac{1}{|\mathcal{H}_{T_n,\text{prim}}(\mathbb{Z})/\mathbb{G}_1(\mathbb{Z})|} \sum_{x\in\mathcal{Z}_{T_n}(\mathbb{Z})} f(\pi_{\mathcal{Z}_{T_n}}(x)) = \nu_{T_n}^{\mathcal{Z},q}(\tilde{f}) \to \mu_{\mathcal{Z}}(f),$$

which proves Theorem 3.7.

10.2. **Proof of Theorem 3.8.** We assume further that there is a fixed $a \in (\mathbb{Z}/(q))^{\times}$ such that $\vartheta_q(T_n) = a,\ \forall n \in \mathbb{N}$.

By Corollary 3.1, (2)

$$\mathcal{Z}_a(\mathbb{Z}/(q)) = \vartheta_q(g_{j,n}) \cdot \mathbb{G}(\mathbb{Z}/(q)),\ \forall n \in \mathbb{N},\ \forall j \leq m(n)$$

Then, by using Theorem 9.1, and following the same arguments as above, we obtain Theorem 3.8.

Appendix A. Unfolding

In the following we let G be locally compact second countable group, $\Gamma \leq G$ be a lattice, $\tilde{\Gamma} \leq \Gamma$, and $K \leq G$ be a compact subgroup. We will discuss in this section a mechanism which lifts an equidistribution result in $K\backslash G/\Gamma$ to an equidistribution result in $K\backslash G/\tilde{\Gamma}$ (see Corollary A.3).

Let m_G, $m_{G/\Gamma}$, $m_{G/\tilde{\Gamma}}$ be G-invariant measures on G, G/Γ, $G/\tilde{\Gamma}$ respectively, such that $m_{G/\Gamma}$ is a probability measure and such that all the measures are Weil normalized (a notion introduced in Section 4.3.2), namely such that for all $\varphi \in C_c(G)$

$$\int_G \varphi(g)dm_G(g) = \int_{G/\Gamma} \left(\sum_{\gamma\in\Gamma} \varphi(g\gamma)\right) dm_{G/\Gamma}(g\Gamma) = \int_{G/\tilde{\Gamma}} \left(\sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(g\tilde{\gamma})\right) dm_{G/\tilde{\Gamma}}(g\tilde{\Gamma}) \tag{A.1}$$

(such a normalization exists by Theorem 2.51 in [Fol15]). Let $f \in C_c(G/\tilde{\Gamma})$ and consider

$$\bar{f}(x\Gamma) \stackrel{\text{def}}{=} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(x\gamma\tilde{\Gamma}). \tag{A.2}$$

We claim that $\bar{f} \in C_c(G/\Gamma)$. Indeed, by [Fol15, Proposition 2.50] there exists $\varphi \in C_c(G)$ such that

$$f(x\tilde{\Gamma}) = \sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(x\tilde{\gamma}),$$

which shows that

$$\begin{aligned} \bar{f}(x\Gamma) &= \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(x\gamma\tilde{\Gamma}) \\ &= \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} \sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(x\gamma\tilde{\gamma}) = \sum_{\gamma\in\Gamma} \varphi(x\gamma), \end{aligned} \tag{A.3}$$

and we note that $\sum_{\gamma\in\Gamma} \varphi(x\gamma) \in C_c(G/\Gamma)$.

Lemma A.1. *It holds that*

$$\int_{G/\tilde{\Gamma}} f(x\tilde{\Gamma})dm_{G/\tilde{\Gamma}}(g\tilde{\Gamma}) = \int_{G/\Gamma} \bar{f}(x\Gamma)dm_{G/\Gamma}(g\Gamma), \tag{A.4}$$

for all $f \in C_c(G/\tilde{\Gamma})$

Proof. Let $\varphi \in C_c(G)$, and assume that $f(x\tilde{\Gamma}) = \sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(x\tilde{\gamma})$. Then

$$\begin{aligned}\int_{G/\tilde{\Gamma}} f(x\tilde{\Gamma}) dm_{G/\tilde{\Gamma}}(g\tilde{\Gamma}) &= \int_{G/\tilde{\Gamma}} \left(\sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(g\tilde{\gamma})\right) dm_{G/\tilde{\Gamma}}(g\tilde{\Gamma}) \\ &\underbrace{=}_{(A.1)} \int_G \varphi(g) dm_G(g) \\ &\underbrace{=}_{(A.1)} \int_{G/\Gamma} \left(\sum_{\gamma\in\Gamma} \varphi(g\gamma)\right) dm_{G/\Gamma}(g\Gamma) \\ &= \int_{G/\Gamma} \left(\sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} \left(\sum_{\tilde{\gamma}\in\tilde{\Gamma}} \varphi(g\gamma\tilde{\gamma})\right)\right) dm_{G/\Gamma}(g\Gamma) \\ &= \int_{G/\Gamma} \left(\sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(x\gamma\tilde{\Gamma})\right) dm_{G/\Gamma}(g\Gamma) \underbrace{=}_{(A.2)} \int_{G/\Gamma} \bar{f}(x\Gamma) dm_{G/\Gamma}(g\Gamma).\end{aligned}$$

□

We denote by π_K the natural quotient map $\pi_K : G \to K\backslash G$. We define a measure on $K\backslash G/\tilde{\Gamma}$ by $\mu_{K\backslash G/\tilde{\Gamma}} \overset{\text{def}}{=} (\pi_K)_* m_{G/\tilde{\Gamma}}$, and on $K\backslash G/\Gamma$ by $\mu_{K\backslash G/\Gamma} \overset{\text{def}}{=} (\pi_K)_* m_{G/\Gamma}$ (which is well defined, since we assume that K is compact).

Lemma A.2. *Assume that $S_n \subseteq K\backslash G/\Gamma$, $n \in \mathbb{N}$, are finite sets such that the uniform probability measures supported on S_n converge weakly to $\mu_{K\backslash G/\Gamma}$. Assume that $\left\{Kg_{i,n}\tilde{\Gamma}\right\} \subseteq K\backslash G/\tilde{\Gamma}$ are representatives for S_n (namely a choice of one point in the preimage of $Kg_{i,n}\Gamma$ under the natural projection for each $1 \le i \le |S_n|$). Then for all $f \in C_c(K\backslash G/\tilde{\Gamma})$ it holds that*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{i=1}^{|S_n|} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(Kg_{i,n}\gamma\tilde{\Gamma}) = \mu_{K\backslash G/\tilde{\Gamma}}(f).$$

Proof. Let $f \in C_c(K\backslash G/\tilde{\Gamma})$, consider

$$\bar{f}(Kx\Gamma) \overset{\text{def}}{=} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(Kx\gamma\tilde{\Gamma}),$$

and note that $\bar{f}(Kx\Gamma) \in C_c(K\backslash G/\Gamma)$ (indeed, since $f \circ \pi_K \in C_c(G/\tilde{\Gamma})$, it follows that $\bar{f}(Kx\Gamma) = \overline{f \circ \pi_K}(x\Gamma) \in C_c(G/\Gamma)$ by the discussion above Lemma A.1). By the assumption of the lemma, we have that

$$\frac{1}{|S_n|} \sum_{i=1}^{|S_n|} \bar{f}(Kg_{i,n}\Gamma) \to \int_{K\backslash G/\Gamma} \overline{f}(Kg\Gamma) d\mu_{K\backslash G/\Gamma}. \tag{A.5}$$

The proof is complete by observing that the left hand side of (A.5) may be rewritten by

$$\frac{1}{|S_n|} \sum_{i=1}^{|S_n|} \bar{f}(Kg_{i,n}\Gamma) = \frac{1}{|S_n|} \sum_{i=1}^{|S_n|} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f(Kg_{i,n}\gamma\tilde{\Gamma}),$$

and the right hand side of (A.5) may be rewritten by

$$\int_{K\backslash G/\Gamma} \overline{f}(Kg\Gamma) dm_{K\backslash G/\Gamma} = \int_{G/\Gamma} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} f \circ \pi_K(x\gamma\tilde{\Gamma}) dm_{G/\Gamma} =$$

$$\underbrace{=}_{(A.4)} \int_{G/\tilde{\Gamma}} f \circ \pi_K(x\tilde{\Gamma}) dm_{G/\tilde{\Gamma}} = \mu_{K\backslash G/\tilde{\Gamma}}(f).$$

□

Phrased differently, Lemma A.2 states that for the locally finite atomic measures

$$\nu_n \stackrel{\text{def}}{=} \frac{1}{|S_n|} \sum_{i=1}^{|S_n|} \sum_{\gamma\tilde{\Gamma}\in\Gamma/\tilde{\Gamma}} \delta_{Kg_{i,n}\gamma\tilde{\Gamma}},$$

it holds that $\nu_n(f) \to \mu_{K\backslash G/\tilde{\Gamma}}(f)$ for all $f \in C_c(K\backslash G/\tilde{\Gamma})$. We observe that ν_n are not uniform measures, namely, some atoms can have different weights. We let $\pi^{\tilde{\Gamma}} : G/\tilde{\Gamma} \to G/\Gamma$ be the natural map, and we note that the support of ν_n can be expressed by

$$\mathrm{supp}(\nu_n) \stackrel{\text{def}}{=} \left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n).$$

We define $\bar{\nu}_n$ to be the uniform measures supported on $\mathrm{supp}(\nu_n)$, namely

$$\bar{\nu}_n \stackrel{\text{def}}{=} \frac{1}{|S_n|} \sum_{x\in\left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n)} \delta_x.$$

Similarly to Lemma A.2, we would like to show $\bar{\nu}_n(f) \to \mu_{K\backslash G/\tilde{\Gamma}}(f)$, for all $f \in C_c(K\backslash G/\tilde{\Gamma})$. This requires an additional assumption that the points which are counted more then once are negligible. We define $\mathcal{F} \subseteq K\backslash G/\Gamma$ by

$$\begin{aligned}\mathcal{F} &\stackrel{\text{def}}{=} \{Kg\Gamma \mid |\mathrm{Stab}_\Gamma(Kg)| > 1\} \\ &= \left\{Kg\Gamma \mid \left|g^{-1}Kg \cap \Gamma\right| > 1\right\}.\end{aligned} \tag{A.6}$$

Corollary A.3. *Assume that $S_n \subseteq K\backslash G/\Gamma$, $n \in \mathbb{N}$, are finite sets such that the uniform probability measures supported on S_n converge weakly to $\mu_{K\backslash G/\Gamma}$, and assume that*

$$\frac{|\mathcal{F} \cap S_n|}{|S_n|} \to 0. \tag{A.7}$$

Then it holds that $\bar{\nu}_n(f) \to \mu_{K\backslash G/\tilde{\Gamma}}(f)$, for all $f \in C_c(K\backslash G/\tilde{\Gamma})$.

We require the following basic lemma for the proof of Corollary (A.3).

Lemma A.4. *Let $U \subseteq K\backslash G/\tilde{\Gamma}$ be a set with compact closure. Then there exists $m_U > 0$ such that for all $g \in G$ it holds that*

$$\left|\left\{\gamma\tilde{\Gamma} \in \Gamma/\tilde{\Gamma} \mid Kg\gamma\tilde{\Gamma} \in U\right\}\right| \le m_U. \tag{A.8}$$

Proof. Let $U \subseteq K\backslash G/\tilde{\Gamma}$ be a set with compact closure. We let $\tilde{U} \subseteq G$ be a compact set such that $\overline{U} = K\tilde{U}\tilde{\Gamma}$ (where $\overline{U}$ denotes the closure of U), and we observe that

$$\left|\left\{\gamma\tilde{\Gamma} \in \Gamma/\tilde{\Gamma} \mid Kg\gamma\tilde{\Gamma} \in U\right\}\right| = \left|\left\{\gamma\tilde{\Gamma} \in \Gamma/\tilde{\Gamma} \mid Kg\gamma\tilde{\Gamma} \in K\tilde{U}\tilde{\Gamma}\right\}\right| \le \left|\Gamma \cap g^{-1}K\tilde{U}\right|,$$

for all $g \in G$. We recall that a lattice subgroup is uniformly discrete, namely, there exists an open neighborhood of identity $\mathcal{N}$ such that $|u\mathcal{N} \cap \Gamma| \le 1$, $\forall u \in G$. Since $K\tilde{U}$ is compact, there exist $u_1, ..., u_{m_U} \in G$ such that $u_1\mathcal{N} \cup .. \cup u_{m_U}\mathcal{N} \supseteq K\tilde{U}$. This implies that $g^{-1}u_1\mathcal{N} \cup .. \cup g^{-1}u_{m_U}\mathcal{N} \supseteq g^{-1}K\tilde{U}$. Since there is at most one point of Γ in each set $g^{-1}u_i\mathcal{N}$, it follows that $\left|\Gamma \cap g^{-1}K\tilde{U}\right| \le m_U$, which implies (A.8). □

Proof of Corollary A.3. We denote by $\left\{Kg_{i,n}\tilde{\Gamma}\right\}_{i=1}^{|S_n|} \subseteq K\backslash G/\tilde{\Gamma}$ a set of representatives for $\left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n)$ (a choice of a unique point in each fiber) and we fix a *positive* function $f \in C_c(K\backslash G/\tilde{\Gamma})$.

By noting that the weights of the atoms of ν_n are larger then the weights of the atoms of $\bar{\nu}_n$, we find that

$$\bar{\nu}_n(f) \le \nu_n(f).$$

We consider the uniform counting measure ν_n^- supported on $\left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n \smallsetminus \mathcal{F})$ where each atom has mass $\frac{1}{|S_n|}$. We note that for all $Kg_{i,n}\tilde{\Gamma} \in \left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n \smallsetminus \mathcal{F})$ and for any two distinct $\gamma_1\tilde{\Gamma}, \gamma_2\tilde{\Gamma} \in \Gamma/\tilde{\Gamma}$ it holds that

$$Kg_{i,n}\gamma_1\tilde{\Gamma} \neq Kg_{i,n}\gamma_2\tilde{\Gamma}.$$

Namely, the weights of the atoms of ν_n^- and of $\bar{\nu}_n$ are the same on $\left(\pi^{\tilde{\Gamma}}\right)^{-1}(S_n \smallsetminus \mathcal{F})$, which implies that

$$\nu_n^-(f) \leq \bar{\nu}_n(f).$$

We observe that

$$\nu_n(f) - \nu_n^-(f) = \frac{1}{|S_n|} \sum_{Kg_{i,n}\tilde{\Gamma} \in \left(\pi^{\tilde{\Gamma}}\right)^{-1}(\mathcal{F}\cap S_n)} \sum_{\gamma\tilde{\Gamma} \in \Gamma/\tilde{\Gamma}} f(Kg_{i,n}\gamma\tilde{\Gamma}).$$

We denote by U the support of f, and we obtain by the triangle inequality and by Lemma A.4 that

$$\nu_n(f) - \nu_n^-(f) \leq \|f\|_\infty \frac{m_U}{|S_n|} |S_N \cap \mathcal{F}| \to 0.$$

Finally, since Lemma A.2 gives

$$\lim_{n\to\infty} \nu_n(f) = \mu_{K\backslash G/\tilde{\Gamma}}(f),$$

then we also get

$$\lim_{n\to\infty} \bar{\nu}_n(f) = \mu_{K\backslash G/\tilde{\Gamma}}(f).$$

□

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