$T\overline{T}$ deformation of the Ising model and its ultraviolet completion

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Pure $T\overline{T}$ deformations of conformal field theories are generally asymptotically incomplete in the ultra-violet (UV) due to square-root singularities in the ground state energy on a cylinder of circumference R, such that the theory is ill-defined for distances shorter than some critical R_*. In this article we show how a theory can be completed if one includes an infinite number of additional irrelevant perturbations. This is fully demonstrated in the case of the Ising model at $c_{IR} = 1/2$ in the infra-red (IR), where we find two completions with central charges $c_{UV} = 3/2$ and $c_{UV} = 7/10$, the latter being the tri-critical Ising model. Both of these UV completions have $\mathcal{N} = 1$ supersymmetry which is broken in the renormalization group flow to low energies. We also consider multiple $T\overline{T}$ deformations of a free massless boson, where we cannot find a UV completion that is consistent with the c-theorem. For negative coupling g, which violates the c-theorem, in both cases we find $c_{UV} = -c_{IR}$ as $g \to -\infty$. Finally we also study pure $T\overline{T}$ deformations of the off-critical Ising model.

Contents

I. INTRODUCTION

Our current understand of Renormalization Group (RG) flows and effective field theory makes an important distinction between relevant verses irrelevant operators. In the ideal scenario, at very high energies the theory is governed by an ultra-violet (UV) fixed point of the RG, which is a conformal field theory. At lower energies the behavior is determined by relevant perturbations of the UV fixed point. Relevant operators, which are usually finite in number, generally introduce a mass scale and the spectrum consists of massive particles. Under RG the theory flows to an infra-red (IR) fixed point described by a different conformal field theory (CFT). If all the particles are massive, at low energies the particles effectively have infinite mass and decouple leaving an empty theory. However, if there are massless particles they can survive the flow to the deep IR such that the theory flows to a non-trivial IR fixed point. In the vicinity of the IR fixed point the behavior is described by irrelevant perturbations of the IR CFT. Perturbation of the UV fixed point has a high level of predictability since there are usually only a finite number of relevant operators

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and one can in principle predict the low energy properties. However the reverse is not true: there are an infinite number of irrelevant operators which introduces an infinite number of couplings in the low energy theory. In general it is impossible to reconstruct the UV behavior from what is known at low energies, since some UV features are lost in the flow to the IR; for example the massive fields that decoupled. One says that the RG flow to the IR is irreversible, and this is captured for instance by the c-theorem [1]. This irreversibility is central to many of the difficulties currently faced in fundamental High Energy Physics and Quantum Gravity. The so-called hierarchy problem in the Standard Model is of this nature. Also, quantization of Einstein gravity involves a perturbation of free massless gravitons by an infinite number of irrelevant operators. In light of these remarks, an interesting and fundamental question arises: Is it possible to fully reconstruct the UV behavior from the properties of the IR fixed point CFT and its irrelevant perturbations? In other words: Is it possible to reverse the RG flow *without* violating the c-theorem and the like? Since based on our current understanding the answer is likely to be No, it would be very interesting to find concrete examples where the answer is Yes. For the latter, the theory is referred to as being asymptotically UV complete. In the context of quantum gravity this property of the RG flow is referred to as asymptotic safety [2]. In this article we will refer to this phenomenon as asymptotic UV reversibility, and will distinguish between such flows that violate the c-theorem and those that do not.

A concrete framework in which to study the above profound issues is in the context of so-called $T\overline{T}$ deformations, i.e. perturbations, of integrable quantum field theory in 2 space-time dimensions. The original works are are due to Smirnov and Zamolodchikov, and Cavaglià et. al. [3–5]. The models can be formulated with the action

$$\mathcal{S} = \mathcal{S}_{IR} + \delta\mathcal{S} \tag{1}$$

where $\mathcal{S}_{IR}$ is formally the action of the theory in the deep IR, and $\delta\mathcal{S}$ is a perturbation by irrelevant operators. The theory $\mathcal{S}_{IR}$ can be taken to be a massive integrable theory or a CFT, however as we will explain the case where $\mathcal{S}_{IR}$ is an IR fixed point CFT is the most interesting. Of special interest is the "$T\overline{T}$" perturbation, since it is typically the lowest dimension irrelevant operator. Every theory has a conserved stress-energy tensor,

$$T = -2\pi\, T_{zz}, \quad \overline{T} = -2\pi\, T_{\overline{z}\overline{z}}, \quad \Theta = 2\pi\, T_{z\overline{z}}, \tag{2}$$

where $z = x + iy$, $\overline{z} = x - iy$ are euclidean light-cone coordinates. If $\mathcal{S}_{IR}$ is a CFT then the trace of the stress-energy tensor $\Theta = 0$. It was shown in [3] that the dimension 4 irrelevant operator

$$T\overline{T} = 4\pi^2\left(T_{zz}T_{\overline{z}\overline{z}} - (T_{z\overline{z}})^2\right) \tag{3}$$

is well defined. It was shown in [4] that the theory defined by

$$\delta\mathcal{S} = \frac{\alpha}{\pi^2}\int d^2x\, T\overline{T} \tag{4}$$

continues to be integrable. Here α is a coupling with dimension $1/\text{mass}^2$. (The $1/\pi^2$ in the normalization is chosen for the convenience of simplifying the Burgers differential equation below.) It was also shown that the $T\overline{T}$ perturbation leads to an additional CDD factor $S_{\text{cdd}}(\theta)$ in the two particle S-matrix where θ is the usual rapidity parameterizing the energy of a particle $E = m\cosh\theta$. There has been a large amount of interesting subsequent work which focuses primarily on pure $T\overline{T}$ deformations. A partial list includes [5–7]. These models are also of interest in connection to 2-dimensional gravity and string theory, for instance in JT gravity; the $T\overline{T}$ perturbation may be viewed as a gravitational dressing of $\mathcal{S}_{IR}$ [8–15]. For a review see [16].

An important probe is the ground state energy $E(R)$ on an infinite cylinder of circumference R, which was studied in [3–5]. In thermodynamic language the free energy density is $\mathcal{F}(T) = E(R)/R$, where $R = 1/T$ is the inverse temperature. It is standard to express these quantities in terms of a scaling function $c(mR)$

$$E(R) = -\frac{\pi}{6}\, c(mR)/R \tag{5}$$

where m can be identified with a physical energy scale, such as the mass of a particle, or the energy scale of massless particles. The UV limit is $r \equiv mR \to 0$, whereas the IR corresponds to $r \to \infty$. The normalization in (5) is such that $c = 1$ for a free massless boson. For a conformal theory, $c(mR)$ is scale invariant, i.e. independent of mR, and for unitary theories is equal to the Virasoro central charge. (For non-unitary theories it is shifted $c \to c - 12d_0$ where d_0 is the lowest scaling dimension of fields.) The quantity $c(mR)$ can be used to track the RG flow. In our recent work we studied the quantity $c(mR)$ with methods different from those in [4, 5], namely with the Thermodynamic Bethe Ansatz (TBA) [17]. The TBA was also studied recently in this context for more complicated supersymmetric models in [18]. Specifically we took $\mathcal{S}_{IR}$ to be a free massless boson or fermion and varied the IR central charge c_{IR} by varying the chemical potential. The S-matrix is then a pure CDD factor

$$S_{\text{cdd}}(\theta) = e^{ig\sinh\theta} \tag{6}$$

where g is a dimensionless coupling. From the TBA we found the general result

$$c(h) = \frac{2c_{IR}}{1 + \sqrt{1 - \frac{2\pi h}{3} c_{IR}}}, \qquad h \equiv \frac{g}{(mR)^2} \tag{7}$$

where c_{IR} is a constant $-\infty < c_{IR} < \infty$ identified as the IR central charge as $R \to \infty$, and m is the energy scale in the massless TBA. We can compare with the analogous result obtained in [4, 5] based on the inviscid Burgers equation

$$\partial_\alpha E + E \partial_R E = 0. \tag{8}$$

Indeed, one finds that (7) satisfies the above differential equation if one identifies

$$h = -\frac{\alpha}{R^2}, \qquad \implies \quad g = -\alpha \, m^2. \tag{9}$$

This successful comparison is an indication that the massless TBA equations proposed in [17], which involved factorizing the CDD factor, are indeed correct. We will review this proposal in Section V.

The issue of UV completeness is distinct from integrability. Based on the expression (7), in [17] we reached the following conclusions concerning the issue of UV completeness. First of all, the c-theorem states that $c(mR)$ increases toward the UV, and this requires $g > 0$, i.e. $\alpha < 0$. For $c_{IR} < 0$ the flow is well-defined for arbitrarily small R, i.e. in the limit $h \to \infty$. In this case the theory was interpreted as being UV complete with central charge $c_{UV} = 0$. However the fact that $c_{UV} = 0$ for *all* initial $c_{IR} < 0$ seems unsatisfactory; this issue will be resolved below. On the other hand, for the physically more interesting case of positive c_{IR}, $c(mR)$ develops a square-root singularity at $h = h_* \equiv 3/(2\pi c_{IR})$. At the singularity $c(h_*) \equiv c_{UV_*} = 2c_{IR}$ is finite. The flow cannot be extended to arbitrarily small R since $c(mR)$ becomes complex for $h > h_*$, and one should conclude that the theory is *not* UV complete. The above discussion is for $g > 0$ since that is what is consistent with the c-theorem, however $g < 0$ is still interesting as discussed in [17], and we will return to this below.

Having made these introductory remarks, we can now state the primary findings of our work presented below. By including an infinite number of additional irrelevant perturbations, and tuning the couplings appropriately, the theory can be asymptotically completed with a predictable UV fixed point with central charge c_{UV}. We demonstrate this in the following specific context. For concreteness, first consider $\mathcal{S}_{IR}$ as defining a massive integrable QFT. Integrability implies an infinite number of local conserved currents of spin $\pm(s+1)$ where s is a positive integer. From these currents one can construct left/right bilinear local fields X_s of dimension $(\text{mass})^{2(s+1)}$, where $X_1 = T\overline{T}/\pi^2$. It was shown by Smirnov and Zamolodchikov that perturbation by all of these operators

$$\delta\mathcal{S} = \sum_{s \geq 1} \alpha_s \int d^2x \, X_s(x) \tag{10}$$

continues to be formally integrable for any choice of couplings α_s (including $\alpha_s = 0$ for some s). Below we show that by choosing the couplings α_s appropriately, the theory is asymptotically complete in the UV, i.e. the RG flow extends to arbitrarily short distances where it becomes a different CFT with central charge c_{UV}. If the flow is consistent with the c-theorem, then $c_{UV} > c_{IR}$. Our analysis is based on the TBA which is particularly well suited to the study of these issues since we know the general form of the CDD factor. We do not demonstrate this in full generality, but rather in an example that is non-trivial enough for our purposes: $T\overline{T}$ perturbations of the critical or non-critical Ising model where $c_{IR} = 1/2$. We show how to choose the couplings α_s such that $c_{UV} = 3/2$ or $7/10$, the latter corresponding to the tri-critical Ising model. The fact that the UV completions are not unique is to be expected since there is some freedom in choosing the couplings α_s.

In the next section we first review the construction of the operators X_s [4]. In Section III we turn to the example of the Ising model, focusing on the massless critical case. The massive non-critical case with only $\alpha_1 \neq 0$ can also be dealt with in rather great detail as described in section VI.

II. PERTURBATIONS BY $T\overline{T}$ AND OTHER IRRELEVANT OPERATORS BASED ON HIGHER INTEGRALS OF MOTION

In this section we review the main results presented in the pioneering work [3–5]. Let us first consider the case where $\mathcal{S}_{IR}$ corresponds to a massive integrable model. Then there exists an infinite number of conserved local currents satisfying the continuity equations

$$\partial_{\overline{z}} T_{s+1} = \partial_z \Theta_{s-1}, \qquad \partial_z \overline{T}_{s+1} = \partial_{\overline{z}} \overline{\Theta}_{s-1}, \tag{11}$$

where s is a positive integer with $s+1$ and $-(s+1)$ the spins of T_{s+1} and $\overline{T}_{s+1}$ respectively. For $s=1$ these are the components of the stress-energy tensor and the conserved charges are left and right components of momentum. The spectrum of the integers $\{s\}$ depends on the model in question. For models based on su(2) such as the sine-Gordon or sinh-Gordon models, s is an odd integer. Zamolodchikov showed that from these one can construct well defined local operators X_s:

$$X_s = T_{s+1}\overline{T}_{s+1} - \Theta_{s-1}\overline{\Theta}_{s-1} \tag{12}$$

with scaling dimension (mass)$^{2(s+1)}$. More importantly, perturbation by such operators preserves the integrability. Thus we consider the theory defined by the action

$$\mathcal{S} = \mathcal{S}_{IR} + \sum_{s\geq 1} \alpha_s \int d^2x\, X_s(x) \tag{13}$$

where α_s are coupling constants of scaling dimension (mass)$^{-2s}$. Our convention for α_1 is that $X_1 = T\overline{T}/\pi^2$. We will refer to the above multi-parameter perturbations simply as $T\overline{T}$.

Since the theory is integrable, it can be described by a factorizable scattering theory for some fundamental particles. For simplicity we suppose the spectrum consists of a single particle of mass m with energy and momentum parameterized as usual by the rapidity θ:

$$E = m\cosh\theta, \qquad p = m\sinh\theta, \tag{14}$$

and the S-matrix for the unperturbed theory is $S_{IR}(\theta)$. Then the effect of the perturbation is to multiply the S-matrix by a CDD factor

$$S(\theta) = S_{IR}(\theta) \cdot S_{\mathrm{cdd}}(\theta), \qquad S_{\mathrm{cdd}}(\theta) = \exp\left(i\sum_{s\geq 0} g_s\,\sinh(s\theta)\right). \tag{15}$$

The normalization of the operators X_s can be chosen such that

$$g_s = -\alpha_s\, m^{2s}, \tag{16}$$

and the convention for $\alpha_1 = \alpha$ is the same as in [4].

We already encounter a serious difficulty: The infinite series in S_{cdd} has very little domain of convergence. As we will explain this can be resolved if one first takes a massless limit and then factorizes S_{cdd}, as was proposed for the pure case in [17].

III. ASYMPTOTIC REVERSIBILITY OF THE $T\overline{T}$ PERTURBED ISING MODEL

A. Generalities

Whether the unperturbed theory is massive or massless is clearly an IR property. At very high energies compared to the physical mass of particles, the theory is effectively massless. For this reason, as far as the issues raised in the Introduction that we wish to address, there is little to be gained by studying $T\overline{T}$ perturbations of massive theories. For instance, a pure $T\overline{T}$ perturbation where only $g_1 \neq 0$ of a massive theory is expected to have the same UV singularity as the perturbation of the massless case since the behavior in the UV is dominated by irrelevant operators. We will show this explicitly for an example in the next section. A precise statement is that a massive spectrum is associated with a relevant operator whereas in the ultra-violet the behavior is controlled by the irrelevant operators since the latter are less and less important at low energies, hence the terminology "irrelevant". If both relevant and irrelevant operators are present, they compete: the relevant operators win at low energies whereas the irrelevant ones win at high energies. However a massive theory plays an indirect role if one is interested in CFT's perturbed by irrelevant operators since a relevant perturbation selects a spectrum of particles that describes the IR CFT which in turn affects the analysis of the UV properties. Namely, a massive theory selects a spectrum of particles and a scattering description of the IR CFT in terms of Left-Left and Right-Right S-matrices S_{LL} and S_{RR}, and this in turn determines the operators X_s. The formulation of massless factorized scattering in [23, 24] thus plays a central role. It is important to recognize that the massless scattering description of a CFT is not unique and this implies that the UV properties of $T\overline{T}$ deformations are also not unique, since the operators X_s themselves depend on the IR description

of the IR CFT. On the other hand every CFT has a stress-energy tensor and a resulting X_1, thus there is simply not enough information in a pure $T\overline{T}$ deformation to determine a UV completion. For instance the critical Ising model has two integrable perturbations, either by the energy operator, which is just a mass term for the Majorana fermion, or by the spin field. The latter requires working with 8 fundamental massless particles related to the Lie algebra E_8 as discovered by A. Zamolodchikov. For a comprehensive review of integrable perturbations of CFT, the TBA, etc., we refer to the book by Mussardo [21].

In summary, we define a $T\overline{T}$ perturbation of a CFT_{IR} by the following steps:

(i)   Begin with a CFT_{IR} which will eventually be identified as the IR fixed point of the $T\overline{T}$ flow. Before turning on $T\overline{T}$, we first consider a relevant perturbation of CFT_{IR} which defines an integrable massive theory with a known spectrum of particles, their S-matrices, and the operators X_s.

(ii)   Second, we take the massless UV limit of the theory in (i) to define a scattering description of the original CFT_{IR} in terms of scattering matrices S_{LL} and S_{RR}. The value of c_{IR} is built in from the beginning and depends on the spectrum and S-matrices S_{LL}, S_{RR}.

(iii)   Third, we turn on X_s perturbations of the CFT_{IR}, where the latter is now the IR fixed point of the $T\overline{T}$ flow. Note that CFT_{IR} was the UV limit in step (ii), but now serves as the IR fixed point of the $T\overline{T}$ deformation. For this reason the order of UV limits matters.

(iv)   Finally we study whether the $T\overline{T}$ perturbation is completed in the UV by a different CFT_{UV}.

We itemized these steps in detail since the two UV limits involved in the procedure do not obviously commute. In this section we assume we first take the UV limit which leads to a massless scattering description of CFT_{IR} before turning on $T\overline{T}$.

The UV issues we wish to resolve are already present for non-interacting theories $\mathcal{S}_{IR}$ where in the massless limit $S_{LL} = S_{RR} = \sigma = \pm 1$, where $\sigma = 1, -1$ corresponds to bosonic verses fermionic quantum statistics. For massive interacting theories the statistics is always fermionic since the bosonic case is unstable [22]. However in the present context bosonic statistics cannot be disregarded since the $T\overline{T}$ interactions are too soft to modify the statistics. For instance, both the bosonic and fermionic cases were needed in [17] in order to cover the full range of c_{IR}. For this reason we will display σ in many TBA formulas even though our primary example is the fermionic Ising model, which we now turn to in detail.

B.   The Ising case

The Ising model at its critical temperature T_c is known to be described by a massless Majorana fermion with fields $\psi, \overline{\psi}$, which is a conformal field theory with $c = 1/2$. Perturbing the temperature away from T_c corresponds to a mass term with mass $m \propto (T - T_c)$. The integrals of motion are known to exist for s an odd integer. We thus consider a $T\overline{T}$ perturbation defined by the action

$$\mathcal{S} = \frac{1}{4\pi} \int d^2x \left(\psi \partial_{\overline{z}} \psi + \overline{\psi} \partial_z \overline{\psi} + m \, \overline{\psi} \psi \right) + \sum_{s \geq 1, \text{ odd}} \alpha_s \int d^2x \, X_s. \tag{17}$$

As discussed above, we first take the UV limit $m \to 0$, then consider the $T\overline{T}$ perturbations of the $c = 1/2$ free massless Majorana fermion.

The massless TBA requires distinguishing between Left (L) and Right (R) movers, where the energy and momentum of a single particle is

$$\begin{aligned} \text{right movers}: \quad E &= p = \tfrac{m}{2} e^{\theta} \\ \text{left movers}: \quad E &= -p = \tfrac{m}{2} e^{-\theta}. \end{aligned} \tag{18}$$

The parameter m is needed to give E, p units of energy. Here m is not the mass of a physical particle, but is a physical mass scale arising from the dimension-full parameters α_s as in (9). Since the unperturbed theory has S-matrices $S_{LL} = S_{RR} = \sigma = -1$, the TBA equations just couple the L,R pseudo-energies $\varepsilon_{L,R}$ and have the standard form

$$\varepsilon_R(\theta) = \tfrac{mR}{2} e^{\theta} + \sigma \, G_{RL} \star \log \left(1 - \sigma \, e^{-\varepsilon_L(\theta)} \right)$$

$$\varepsilon_L(\theta) \; = \; \tfrac{mR}{2}e^{-\theta} + \sigma \, G_{LR} \star \log\left(1 - \sigma \, e^{-\varepsilon_R(\theta)}\right), \tag{19}$$

where we have defined the convolution

$$(G \star f)(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \, G(\theta - \theta') f(\theta') \tag{20}$$

for an arbitrary function $f(\theta)$. Finally,

$$c(mR) = c_L(mR) + c_R(mR) \tag{21}$$

where

$$\begin{aligned}
c_R &= -\frac{3\sigma}{\pi^2} \int_{-\infty}^{\infty} d\theta \, \tfrac{mR}{2} \, e^{\theta} \, \log\left(1 - \sigma \, e^{-\varepsilon_R(\theta)}\right) \\
c_L &= -\frac{3\sigma}{\pi^2} \int_{-\infty}^{\infty} d\theta \, \tfrac{mR}{2} \, e^{-\theta} \, \log\left(1 - \sigma \, e^{-\varepsilon_L(\theta)}\right).
\end{aligned} \tag{22}$$

It remains to determine the kernels G, which must follow from the CDD factor S_{cdd}. For the pure case, we previously proposed that it is necessary to factorize the CDD factor [17]. The essential reason is that if one chooses $G_{RL} = G_{LR} = G(\theta) = -i\partial_\theta \log S_{\mathrm{cdd}}(\theta)$, then the convolution integrals $\star$ do not converge in an iterative solution, and the same is true for multiple $T\overline{T}$ perturbations. A justification for this factorization is that it leads to results that agree with the solutions that follow from the Burgers equation, as discussed in the Introduction. The necessary factorization is

$$S_{\mathrm{cdd}}(\theta) = S_{LR}(\theta)S_{RL}(\theta), \qquad S_{RL}(\theta) = \exp\left(i\sum_{s \geq 1} g_s e^{s\theta}/2\right), \quad S_{LR}(\theta) = \exp\left(-i\sum_{s \geq 1} g_s e^{-s\theta}/2\right). \tag{23}$$

The kernels which follow from these S-matrices, $G_{RL}(\theta) = -i\partial_\theta \log S_{RL}(\theta)$ and similarly for G_{LR}, are

$$G_{RL}(\theta) = G_{LR}(-\theta) = \sum_{s \geq 1} s \, g_s \, e^{s\theta}/2. \tag{24}$$

The $\theta \to -\theta$ symmetry of the TBA equations implies

$$\varepsilon_L(\theta) = \varepsilon_R(-\theta), \qquad \implies \quad c_L = c_R. \tag{25}$$

The above factorization can resolve the problem of the convergence of the multiple CDD factor itself. The kernel

$$G(\theta) = -i\partial_\theta \log S_{\mathrm{cdd}}(\theta) = \sum_{s \geq 1} s g_s \, \cosh(s\theta), \tag{26}$$

has a very limited domain of convergence. On the other hand, suppose that the couplings g_s are such that the kernels in (24) converge to a well-defined function $\widehat{G}^-$ for $\theta < 0$:

$$G_{RL}(\theta) = G_{LR}(-\theta) = \widehat{G}^-(\theta) \qquad \text{for} \;\; \theta < 0. \tag{27}$$

If the function $\widehat{G}^-$ extends to positive θ via $\widehat{G}^-(\theta) = \widehat{G}^-(-\theta) \equiv \widehat{G}(\theta)$, then for all θ we define the kernels from this function

$$G_{RL}(\theta) = G_{LR}(\theta) = \widehat{G}(\theta) = \widehat{G}(-\theta). \tag{28}$$

The above equation ensures (25). The precise meaning of this construction will be clear in the next subsection.

C. Possible UV completions of Ising with $T\overline{T}$

Since our main purpose is to resolve the UV singularity found in the pure $T\overline{T}$ perturbation, we chose couplings such that they agree with previous conventions for X_1, namely $g_1 = g = -\alpha \, m^2$ where g is a continuous variable. (See the Introduction for definitions of g and α.) We know that for the Ising model X_s exist for $s = 2n + 1$ a positive odd

integer. As we will explain, an important criterion for the existence of a UV fixed point is that the following integral converges

$$\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \, \widehat{G}(\theta) \equiv k. \tag{29}$$

Note that for a pure $T\overline{T}$ perturbation with only g_1 the above integral does not converge, which essentially explains why the pure theory is not UV complete.

A very natural choice of couplings is the following:

$$g_{2n+1} = \frac{g \, (-)^n}{2n+1}, \qquad n \geq 0. \tag{30}$$

Then the kernels in (24) converge for $\theta < 0$ and $\widehat{G}^-$ can be identified with $1/\cosh\theta$. This leads to

$$\widehat{G}(\theta) = \frac{g}{4} \frac{1}{\cosh\theta}. \tag{31}$$

The integral in (29) now converges and one finds

$$k = g/8. \tag{32}$$

The above kernel is associated with an S-matrix $\widehat{S}$:

$$\widehat{G}(\theta) = -i\partial_\theta \log \widehat{S}(\theta), \qquad \widehat{S}(\theta) = (S_0(\theta))^{g/4}, \tag{33}$$

where

$$S_0(\theta) = -i \tanh\left(\frac{\theta}{2} - \frac{i\pi}{4}\right). \tag{34}$$

This suggests that g equal to 4 times an integer is special, and this will indeed turn out to be the case.

The computation of the UV central charge c_{UV} is standard and reviewed in the Appendix. For convenience let us repeat the main formulas here. We have

$$c_{UV} = \frac{6\sigma}{\pi^2} \left(2 \operatorname{Lr_2}\left(\sigma \, e^{-\varepsilon_0}\right) - \operatorname{Li_2}(\sigma)\right), \tag{35}$$

where ε_0 a solution to the transcendental equation

$$\varepsilon_0 = \sigma k \log\left(1 - \sigma e^{-\varepsilon_0}\right). \tag{36}$$

Above, Li is the dilogarithm and Lr the Rogers dilogarithm,

$$\operatorname{Lr_2}(z) = \operatorname{Li_2}(z) + \tfrac{1}{2} \log|z| \log(1-z). \tag{37}$$

For $k > 1$, i.e. $g > 8$, the equation (36) has no real solutions. As $g \to 8$, $\varepsilon_0 \to -\infty$ and

$$c_{UV} = 3/2 \qquad \text{when} \quad g = 8. \tag{38}$$

On the other hand, the equation has solutions for all $g < 8$, including negative. One finds

$$z_0 = e^{-\varepsilon_0} \approx -\frac{W(-k)}{k}, \qquad k = g/8 \to -\infty \tag{39}$$

where W is the Lambert W-function. In this limit $z_0 \to 0$ and $\operatorname{Lr}(0) = 0$. Thus

$$c_{UV} = -c_{IR} = -1/2 \qquad \text{as} \quad g \to -\infty \tag{40}$$

This is interesting since for the pure $T\overline{T}$ case, $c_{UV} = 0$ as $g \to -\infty$ for all c_{IR} [17], whereas the above result at least distinguishes between different c_{IR}.

In Figure 1 we plot c_{UV} as a function of k. One sees the feature anticipated in [17]: for $g > 0$, $c(r)$ increases toward the UV $r \to 0$, i.e. is consistent with the c-theorem, however for $g < 0$ the c-theorem is violated, i.e. c decreases toward the UV. We thus conclude that based on our proposed kernel $\widehat{G}$,

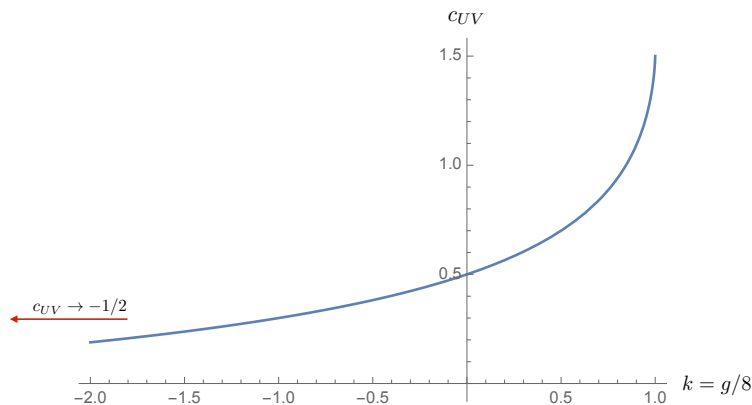$$-1/2 \leq c_{UV} \leq 3/2. \tag{41}$$

FIG. 1: c_{UV} as a function of $k = g/8$ for $T\overline{T}$ perturbations of the critical Ising model. For $k > 1$ there is no solution to the transcendental equation (36).

Our analysis thus far does not at all ensure that there is a complete formulation of these flows as relevant perturbations of a known UV CFT with the above value of c_{UV}, especially since the S-matrix $\widehat{S}$ is peculiar for $g/4$ not equal to an integer.

There is one more interesting case which serves as a check of our construction. As noted above, $g = 4$ is also special since $\widehat{S}$ is a single power of S_0. One finds from the above formulas that z_0 is the golden ratio:

$$z_0 = \frac{1 + \sqrt{5}}{2} \tag{42}$$

and

$$c_{UV} = \frac{7}{10} \qquad (g = 4), \tag{43}$$

which is the central charge of the tri-critical Ising model, the next model in the minimal unitary series.

It turns out that the $g = 4$ case was already known from the top down [25], however we "discovered" it purely from the IR data. In a larger context, let $\mathcal{M}_p$ denote the p-th unitary minimal model of CFT with $c = 1 - \frac{6}{(p+2)(p+3)}$ where $p = 1, 2, \ldots$. The Ising model is $\mathcal{M}_1$ and $\mathcal{M}_2$ is the tri-critical Ising model. Consider the integrable perturbation of $\mathcal{M}_p$ by the relevant operator $\delta\mathcal{S} = \lambda \int d^2x \, \Phi_{1,3}$ where "$\Phi_{1,3}$" has scaling dimension $\frac{2(p+1)}{(p+3)}$. For $\lambda < 0$ the theory is massive, however it is massless for $\lambda > 0$. The latter flows to $\mathcal{M}_{p-1}$ in the IR where it arrives there via the irrelevant operator $\Phi_{3,1}$ which has dimension $\frac{2(p+4)}{(p+2)}$ in $\mathcal{M}_p$ [26, 27]. For the Ising model $\mathcal{M}_1$, the operator $\Phi_{3,1}$ does not exist, and rather the flow from the tri-critical Ising model to the Ising model actually arrives via the operator $T\overline{T}$ [25].

We have shown that there are at least two different UV completions of Ising for the class of kernel $\widehat{G}$ we have considered where $g/4 = 1$ and 2. It is interesting to note that these two most interesting cases are both related to $\mathcal{N} = 1$ supersymmetry. The theory with $c_{UV} = 3/2$ can be considered as a free boson plus a free Majorana fermion, which is a supersymmetric theory. Apparently there exists a relevant perturbation of this theory that breaks the SUSY and in the flow to the IR the boson becomes decoupled leaving only the Majorana fermion. Also, $c_{UV} = 7/10$ is the lowest member of the $\mathcal{N} = 1$ SUSY minimal CFT's. What is conceptually interesting about these flows is that, whereas the single fermion IR theory shows no signature of SUSY, we were able to restore the broken SUSY in the flow to the UV, whereas one is normally interested in a top down approach. This implies there is a hidden non-linear SUSY in the Ising model perturbed by $T\overline{T}$. Al. Zamolodchikov interpreted the massless Majorana fermion as the goldstino of the spontaneously broken $\mathcal{N} = 1$ SUSY [25].

Recall that the instability of the pure $T\overline{T}$ perturbation occurs at $c_{UV_*} = 2c_{IR} = 1$, which is between $7/10$ and $3/2$. Thus the UV completion with $c_{UV} = 7/10$ occurs before the instability is reached, whereas the other completion occurs beyond at even shorter distances.
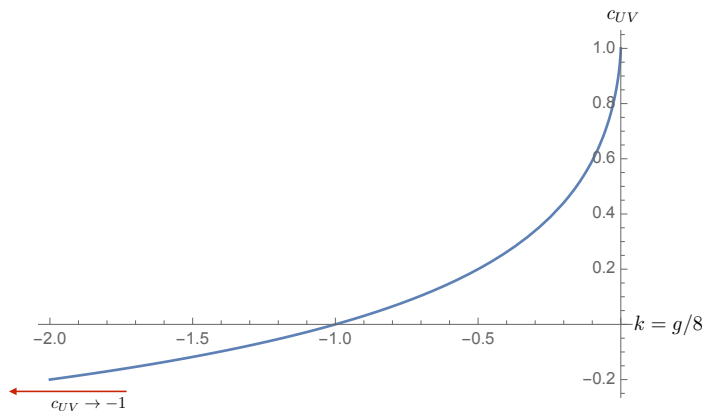
FIG. 2: c_{UV} as a function of $k = g/8$ for $T\overline{T}$ perturbations of the free massless boson. For $k > 0$ there is no solution to the transcendental equation (36).

IV.   REMARKS ON THE BOSONIC CASE

In this case the equation (36) has no solutions for $g > 0$, thus there is no resolution of the square-root singularity in the UV, and thus no evidence of a UV completion based on our kernel $\widehat{G}$. On the other hand there are solutions for all $g < 0$. As $g \to -\infty$ the formula (39) still applies for the bosonic case, and (40) still applies where now $c_{UV} = -1$. Thus

$$-1 \leq c_{UV} \leq 1 \tag{44}$$

and all these flows violate the c-theorem. See Figure 2.

V.   PURE $T\overline{T}$ FOR THE OFF-CRITICAL ISING MODEL AND FREE MASSIVE BOSON.

In this section we consider pure $T\overline{T}$ deformations of free *massive* theories where only $g_1 = g = -\alpha\, m^2$ is non-zero. As stated above, there is not a great deal to learn conceptually from these cases, however we present them here for a few reasons. First, this allows us to see explicitly that, as expected, these theories have the same UV singularity as the massless case since a non-zero mass is an IR property. Second, these examples clarify the discussion in Section IIIA: for multiple X_s perturbations one must first take a massless limit in the IR otherwise the kernel in (26) in general does not converge. Third, it turns out that the resulting TBA equations can be solved exactly in a way similar to the massless case presented in [17].

Suppose a theory consists of a single particle of mass m. The pseudo-energy $\varepsilon(\theta)$ is now a solution to the single integral equation:

$$\varepsilon(\theta) = mR\,\cosh\theta + \sigma G \star \log\left(1 - \sigma e^{-\varepsilon}\right). \tag{45}$$

The kernel G is $G(\theta) = -i\partial_\theta \log S(\theta)$, and the scale dependent central charge is

$$c(mR) = -\frac{3\sigma}{\pi^2} \int_{-\infty}^{\infty} d\theta\; mR\,\cosh\theta\,\log\left(1 - \sigma e^{-\varepsilon(\theta)}\right). \tag{46}$$

If the unperturbed theory is a free massive boson or fermion then the S-matrix is due entirely to the CDD factor, with kernel

$$G(\theta) = G_{\mathrm{cdd}}(\theta) = g\cosh\theta. \tag{47}$$

Due to the above simple form of the kernel, the solution to the TBA equation is greatly simplified. Let us first carry out the bosonic $\sigma = +1$ case. One can express

$$\varepsilon(\theta) = (1 + h\,B)\,r\cosh\theta \tag{48}$$

where

$$r \equiv mR, \qquad h \equiv \frac{g}{r^2}\,. \tag{49}$$

Plugging this expression into the TBA integral equation, B is a function of r, h and satisfies the integral equation

$$B = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\,r\cosh\theta\,\log\left[1 - e^{-(1+h\,B)\,r\cosh\theta}\right]. \tag{50}$$

The scale dependent c has the simple expression

$$c(r, h) = -\frac{6}{\pi}\,B(r, h). \tag{51}$$

The equation (50) is much easier to solve than a general TBA equation since B is just a constant independent of θ.

For non-zero mass one must still solve (50) numerically, unlike the massless case presented in [17]. Let us show how to obtain the latter result from the above formulas. The massless limit is taken as follows. Make a change of variables $\theta \to \theta + a$ and perform the limits

$$\lim_{m\to 0, a\to\pm\infty} = \begin{cases} \frac{\widetilde{m}}{2}\,e^{\theta}, & \widetilde{m} = me^{a} \text{ held fixed} & (\mathrm{Right-movers}) \\ \frac{\widetilde{m}}{2}\,e^{-\theta}, & \widetilde{m} = me^{-a} \text{ held fixed} & (\mathrm{Left-movers}). \end{cases} \tag{52}$$

We will continue to label $\widetilde{m}$ as m. This leads to Left and Right pseudo-energies $\varepsilon_{L,R}$. As explained in the last section, and proposed in [17], the TBA equations for ε_L and ε_R are coupled with kernels that require factorizing $S_{\mathrm{cdd}} = S_{LR}S_{RL}$, which leads to

$$G_{RL}(\theta) = G_{LR}(-\theta) = g\,e^{\theta}/2. \tag{53}$$

The coupled TBA equations are as in (19).

Due to the simple form of the kernels in (53) the pseudo-energies can now be expressed as

$$\varepsilon_R(\theta) = (1 + 2hB_R)\,r\,e^{\theta}/2, \qquad \varepsilon_L(\theta) = (1 + 2hB_L)\,r\,e^{-\theta}/2, \tag{54}$$

where B_R only depends on h and satisfies

$$B_R = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\,\left(re^{\theta}/2\right)\log\left[1 - e^{-(1+2h\,B_R)\,re^{\theta}/2}\right], \tag{55}$$

with $B_L = B_R$ and $\varepsilon_L(\theta) = \varepsilon_R(-\theta)$. The central charge is

$$c(h) = -\frac{6}{\pi}\,(B_R(h) + B_L(h)) = -\frac{12}{\pi}B_R(h). \tag{56}$$

The integral (55) can be done using

$$\int_{-\infty}^{\infty} d\theta\,e^{\theta}\,\log\left(1 - ze^{-y\,e^{\theta}}\right) = -\frac{\mathrm{Li}_2(z)}{y}, \qquad \Re(y) > 0, \tag{57}$$

where $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$. For complex z, $\mathrm{Li}_2(z)$ is the dilogarithm. We only need $z = \pm 1$ for bosons verses fermions and $\mathrm{Li}_2(1) = -2\mathrm{Li}_2(-1) = \zeta(1) = \pi^2/6$. For $\sigma = +1$, we obtain the simple algebraic equation

$$B_R = -\frac{\pi}{12(1 + 2hB_R)}. \tag{58}$$

In order to see that this is equivalent to the result in [17], which was obtained in a different manner directly from the massless TBA, define $B_R = -\frac{\pi}{12}\frac{1}{A}$. Then A satisfies the quadratic equation $A = 1 - \pi h/(6A)$ as in [17], which is easily solved. The result for c is the simple expression (7) with $c_{IR} = 1$.

Repeating the above analysis for fermions, i.e. the non-critical Ising model, one finds nearly identical formulas apart from a few signs. One has

$$\varepsilon(\theta) = (1 - hB)r\cosh\theta \tag{59}$$

where now $B(r, h)$ satisfies the integral equation

$$B = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, r\cosh\theta\, \log\left[1 + e^{-(1-h\,B)\,r\cosh\theta}\right]. \tag{60}$$

The scale dependent c is now given by

$$c(r, h) = \frac{6}{\pi}\, B(r, h). \tag{61}$$

The massless limit is also straightforward, namely, $B = B_R + B_L$ where $B_R(h)$ satisfies

$$B_R = \frac{\pi}{24(1 - 2hB_R)}. \tag{62}$$

Solving this quadratic equation one finds

$$B_R(h) = \frac{\pi}{12}\left(1 + \sqrt{1 - \frac{\pi h}{3}}\right)^{-1}, \tag{63}$$

and $c(h)$ is as in (7) with $c_{IR} = 1/2$.

VI. CONCLUSIONS AND OUTLOOK

Whereas pure $T\overline{T}$ deformations are generally UV incomplete, we have shown that such theories can be completed by including an infinite number of perturbations by more irrelevant operators. Our study was based on the thermodynamic Bethe ansatz. Consistent UV completions are expected to be both rare and not necessarily unique. For instance, for the Ising model with $c_{IR} = 1/2$, i.e. a free Majorana fermion, we found two such completions with $c_{UV} = 3/2$ and $7/10$, both of which are $\mathcal{N} = 1$ supersymmetric. The SUSY is broken in the IR and not at all anticipated; rather this SUSY becomes visible only after reconstruction and completion of the UV.

Ultra-violet incompleteness appears to be unavoidable for theories with $c_{IR} > 0$ and flows that are consistent with the c-theorem. Nevertheless, the pure $T\overline{T}$ deformations can still be interesting, possibly with applications as theories with a minimal, shortest possible distance.

We only considered in detail free conformal field theories in the IR since the main issues we attempted to understand are already present there. However the ideas presented in this paper can be extended to more complicated IR CFT's. This requires dealing with non-trivial Left-Left and Right-Right scattering matrices S_{LL} and S_{RR} which are not simply equal to ± 1, and are appropriate to the massless IR CFT.

It is commonly thought that irrelevant perturbations of an IR theory are intractable due to an infinite number of couplings and the irreversibility of RG flows from the UV to the IR. Perhaps the models considered in this paper provide some new lessons concerning the possibilities for RG flows since they provide counter examples to the commonly accepted properties of such flows. For instance, although in hindsight the flow of the tri-critical Ising model to the critical one was already known from the top down, i.e. starting from the UV, we were able to "rediscover" this flow in the reversed direction, i.e. starting purely in the IR. We also found a different completion with $c_{UV} = 3/2$ which was unanticipated. It is likely that symmetries play a large role in the possibility of such RG flows.

VII. APPENDIX: COMPUTING UV CENTRAL CHARGES FROM THE TBA

The calculation of c_{UV} from the TBA is a standard computation [19–21] which we briefly review.

A. Massive case

Given a massive integrable model with a single bosonic or fermionic particle of mass m with S-matrix $S(\theta)$, kernel $G(\theta) = -i\partial_\theta \log S(\theta)$, and pseudo-energy $\varepsilon(\theta)$, let us define

$$L \equiv -\sigma \log\left(1 - \sigma\, e^{-\varepsilon}\right). \tag{64}$$

Then the TBA equation reads

$$\varepsilon(\theta) = r\,\cosh\theta - (G \star L)(\theta). \tag{65}$$

Since ε and L are even in θ, we can write

$$c(r) = \frac{6}{\pi^2} \int_0^\infty d\theta\, r\cosh\theta\, L(\theta) \tag{66}$$

where $r = mR$. The UV central charge is $c_{UV} = \lim_{r\to 0} c(r)$. The calculation of c_{UV} is relevant to steps (i) and (ii) in Section IIIA.

As $r \to 0$, $d\varepsilon/d\theta \approx 0$ for $|\theta| \ll \log(2/r)$. For this region of θ, L is approximately constant and ε is a constant ε_0 which satisfies the transcendental equation

$$\varepsilon_0 = \sigma k \log\left(1 - \sigma e^{-\varepsilon_0}\right), \tag{67}$$

where

$$k \equiv \int_{-\infty}^\infty \frac{d\theta}{2\pi}\, G(\theta). \tag{68}$$

For most of the θ region of integration for positive θ, $m\cosh\theta \approx m\, e^\theta/2$, thus the UV limit essentially corresponds to a massless theory with decoupled pseudo-energies $\varepsilon_{L,R}$ which satisfy TBA equations with $S_{LL} = S_{RR} = S$. One then has

$$c_{UV} = \lim_{r\to 0} \frac{6}{\pi^2} \int_0^\infty d\theta\, \tfrac{r}{2} e^\theta\, L(\theta) \tag{69}$$

where ε now satisfies the massless TBA equation

$$\varepsilon(\theta) = \tfrac{r}{2} e^\theta - (G \star L)(\theta). \tag{70}$$

Taking the derivative of the above equation and substituting,

$$c_{UV} = \lim_{r\to 0} \frac{6}{\pi^2} \int_0^\infty d\theta\, \left[\partial_\theta \varepsilon + \partial_\theta(G \star L)\right] L(\theta). \tag{71}$$

For the first term the integral over θ can be traded for an integral over ε:

$$c_1 = -\frac{6\sigma}{\pi^2} \int_{\varepsilon_0}^\infty d\varepsilon\, \log\left(1 - \sigma e^{-\varepsilon}\right) = \frac{6\sigma}{\pi^2}\, \mathrm{Li}_2(\sigma e^{-\varepsilon_0}) \tag{72}$$

where $\mathrm{Li}_2(z) = \sum_{n=1}^\infty z^n/n^2$ is the dilogarithm. Integration by parts of the second term, assuming $G(\infty) = 0$, gives a contribution that depends only on logs. The final result is

$$c_{UV} = \frac{6\sigma}{\pi^2}\, \mathrm{Lr}_2(\sigma e^{-\varepsilon_0}) \tag{73}$$

where Lr_2 is the Rogers dilogarithm:

$$\mathrm{Lr}_2(z) = \mathrm{Li}_2(z) + \tfrac{1}{2} \log|z| \log(1-z). \tag{74}$$

B. Massless case

This case is similar to the massive case but not identical since the L,R pseudo-energies are now coupled. Let us write the TBA equations (19) as follows

$$\varepsilon_R = \tfrac{r}{2}e^{\theta} - G \star L_L, \qquad \varepsilon_L = \tfrac{r}{2}e^{-\theta} - G \star L_R. \tag{75}$$

where

$$G_{RL}(\theta) = G_{LR}(\theta) = G(\theta) = G(-\theta) \tag{76}$$

as in (28). The symmetry of the kernel implies $\varepsilon_L(\theta) = \varepsilon_R(-\theta)$ and $c_L = c_R$. We can use this to express $c(r)$ as an integral over positive θ only:

$$c(r) = 2\left(c_R^{(+)} + c_L^{(+)}\right) \tag{77}$$

where $c^{(+)} = \int_0^{\infty} \cdots$. As in the massive case, for $\theta \ll \log(2/r)$, $\varepsilon_L = \varepsilon_R = \varepsilon_0$ where ε_0 again satisfies the equation (67). For $\theta \gg \log(2/r)$, $\varepsilon_{R,L} \approx r e^{\pm\theta}/2$.

Repeating the analysis of the massive case, one finds

$$2c_R^{(+)} = \frac{6\sigma}{\pi^2} \,\mathrm{Lr}_2\left(\sigma e^{-\varepsilon_0}\right). \tag{78}$$

For $c_L^{(+)}$ the limits of integration are different:

$$2c_L^{(+)} = -\frac{6}{\pi^2} \int_0^{\varepsilon_0} d\varepsilon_L \, L(\varepsilon_L) + \delta c_L^{(+)} \tag{79}$$

where $\delta c_L^{(+)}$ arises from the integration by parts which again produces the $\log \cdot \log$ term in Lr_2. Replacing $\int_0^{\varepsilon_0} = \int_0^{\infty} - \int_{\varepsilon_0}^{\infty}$ one obtains

$$2c_L^{(+)} = \frac{6\sigma}{\pi^2} \left(\mathrm{Lr}_2\left(\sigma e^{-\varepsilon_0}\right) - \mathrm{Li}_2(\sigma)\right). \tag{80}$$

Putting this all together one gets

$$c_{UV} = \frac{6\sigma}{\pi^2}\left(2\,\mathrm{Lr}_2\left(\sigma\, e^{-\varepsilon_0}\right) - \mathrm{Li}_2(\sigma)\right). \tag{81}$$

Note that $6\sigma\mathrm{Li}_2(\sigma)/\pi^2$ is the central charge of the free theory which is subtracted in the final expression.

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