ON ASYMPTOTICS FOR C₀-SEMIGROUPS

MARAT V. MARKIN

ABSTRACT. We stretch the spectral bound equal growth bound condition along with a generalized Lyapunov stability theorem, known to hold for C_0 -semigroups of normal operators on complex Hilbert spaces, to C_0 -semigroups of scalar type spectral operators on complex Banach spaces. For such semigroups, we obtain exponential estimates with the best stability constants. We also extend to a Banach space setting a celebrated characterization of uniform exponential stability for C_0 -semigroups on complex Hilbert spaces and thereby acquire a characterization of uniform exponential stability for scalar type spectral and eventually norm-continuous C_0 -semigroups.

1. INTRODUCTION

Based on the recently established fact that C_0 -semigroups of scalar type spectral operators on complex Banach spaces are subject to a precise weak spectral mapping theorem [18], we stretch to such semigroups the spectral bound equal growth bound condition along with a generalized Lyapunov stability theorem (see Preliminaries), known to hold for C_0 -semigroups of normal operators on complex Hilbert spaces, and further obtain exponential estimates with the best stability constants for them. We also extend to a Banach space setting the celebrated Gearhart-Prüss-Greiner characterization of uniform exponential stability for C_0 -semigroups on complex Hilbert spaces [9, Theorem V.3.8] (see [10, 11, 25]) and thereby acquire a characterization of uniform exponential stability for scalar type spectral and eventually norm-continuous C_0 -semigroups.

2. Preliminaries

Here, we outline certain essential preliminaries.

2.1. Resolvent Set and Spectrum.

For a closed linear operator A in a complex Banach space X, the set

$$p(A) := \left\{ \lambda \in \mathbb{C} \mid \exists R(\lambda, A) := (A - \lambda I)^{-1} \in L(X) \right\}$$

(*I* is the *identity operator* on *X*, L(X) is the space of bounded linear operators on *X*) and its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ are called its *resolvent set* and *spectrum*, respectively.

²⁰²⁰ Mathematics Subject Classification. Primary 47A10, 47B40, 47D03; Secondary 47B15, 47D06, 47D60.

Key words and phrases. Scalar type spectral operator, normal operator, C_0 -semigroup, spectral bound, growth bound, uniform exponential stability.

2.2. C_0 -Semigroups.

A C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space $(X, \|\cdot\|)$ with generator A subject to a *weak spectral mapping theorem*

(WSMT)
$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \setminus \{0\}, \ t \ge 0,$$

($\overline{\cdot}$ is the closure of a set) satisfies the following spectral bound equal growth bound condition:

(SBeGB)
$$s(A) = \omega_0,$$

where

$$s(A) := \sup \{ \operatorname{Re} \lambda \, | \, \lambda \in \sigma(A) \} \ (s(A)) := -\infty \text{ if } \sigma(A) = \emptyset \}$$

is the spectral bound of the generator A and

(2.1)
$$\omega_0 := \inf \left\{ \omega \in \mathbb{R} \mid \exists M = M(\omega) \ge 1 : \|T(t)\| \le M e^{\omega t}, \ t \ge 0 \right\}$$

(here and henceforth, we use the same notation for the *operator norm* as for the norm on X) is the *growth bound* of the semigroup [9, Proposition V.2.3] (see also [22]).

Generally,

$$-\infty \le s(A) \le \omega_0 = \inf_{t>0} \frac{1}{t} \ln ||T(t)|| = \lim_{t \to \infty} \frac{1}{t} \ln ||T(t)|| < \infty$$

(see, e.g., [9, Proposition V.1.22]).

An eventually norm-continuous C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space, i.e., such that, for some $t_0 > 0$, the operator function

$$[t_0,\infty) \ni t \mapsto T(t) \in L(X),$$

is *continuous* relative to the *operator norm*, is subject to the following stronger version of (WSMT):

(SMT)
$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \ t \ge 0,$$

called a spectral mapping theorem (see [9, Proposition V.2.3] and [9, Theorem V.2.8]). The class of eventually norm-continuous C_0 -semigroups encompasses C_0 -semigroups with certain regularity properties, such as eventually compact and eventually differentiable, in particular analytic and uniformly continuous (see [9, Section II.5]).

The asymptotic behavior of a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space with generator A satisfying spectral bound equal growth bound condition (SBeGB) is governed by the spectral bound of its generator and, in particular, is subject to the subsequent generalization of the classical Lyapunov Stability Theorem [9, Theorem V.3.6] (see also [15]).

Theorem 2.1 (Generalized Lyapunov Stability Theorem).

A C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space X with generator A, for which spectral bound equal growth bound condition (SBeGB) holds, is uniformly exponentially stable, i.e., $\omega_0 < 0$, or equivalently,

$$\exists \, \omega < 0, \ \exists \, M = M(\omega) \ge 1: \ \|T(t)\| \le M e^{\omega t}, \ t \ge 0,$$

iff

$$s(A) < 0.$$

Cf. [9, Definition V.3.1], [9, Proposition V.3.5], and [9, Theorem V.3.7].

The following statement [10, 11, 25] characterizes uniform exponential stability for C_0 -semigroups on complex Hilbert spaces.

Theorem 2.2 (Gearhart-Prüss-Greiner Characterization [9, Theorem V.3.8]). A C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Hilbert space space $(X, (\cdot, \cdot), \|\cdot\|)$ with generator A is uniformly exponentially stable iff

$$\{\lambda \in \mathbb{C} \, | \, \mathrm{Re} \, \lambda \geq 0\} \subseteq \rho(A) \quad and \quad \sup_{\mathrm{Re} \, \lambda \geq 0} \|R(\lambda, A)\| < \infty.$$

A C_0 -semigroup $\{T(t)\}_{t\geq 0}$ (of normal operators) on a complex Hilbert space generated by a normal operator A is subject to the following precise version of weak spectral mapping theorem (WSMT):

(PWSMT)
$$\sigma(T(t)) = e^{t\sigma(A)}, \ t \ge 0,$$

[9, Corollary V.2.12] without being a priori eventually norm continuous, and hence, to spectral bound equal growth bound condition (SBeGB) along with the Generalized Lyapunov Stability Theorem (Theorem 2.1).

2.3. Scalar Type Spectral Operators.

A scalar type spectral operator is a densely defined closed linear operator A in a complex Banach space with strongly σ -additive spectral measure (the resolution of the identity) $E_A(\cdot)$, which assigns to the Borel sets of the complex plane projection operators on X and has the operator's spectrum $\sigma(A)$ as its support [4,5,8].

Associated with such an operator is the *Borel operational calculus*, assigning to each Borel measurable function $F : \sigma(A) \to \overline{\mathbb{C}}$ ($\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) is the extended complex plane) with $E_A(\{\lambda \in \mathbb{C} \mid F(\lambda) = \infty\}) = 0$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)$$

in X, whose spectral measure is the image of $E_A(\cdot)$ under $F(\cdot)$, i.e.,

$$E_{F(A)}(\delta) = E_A(F^{-1}(\delta)), \ \delta \in \mathscr{B}(\mathbb{C}),$$

 $(\mathscr{B}(\mathbb{C})$ is the Borel σ -algebra on \mathbb{C}), with

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda)$$

[1, 4, 5, 8].

On a complex finite-dimensional Banach space, scalar type spectral operators are those linear operators, which furnish an *eigenbasis* for the space, i.e., allow a diagonal matrix representation (see, e.g., [4, 5, 8]).

In a complex Hilbert space, scalar type spectral operators are those that are similar to *normal operators* [26] (see also [14, 16]), the latter being the scalar type spectral

operators for which the corresponding spectral measure projections are *orthogonal* (see, e.g., [7, 24]).

Various examples of scalar type spectral operators, including differential operators arising in the study of linear systems of partial differential equations, in particular perturbed Laplacians, can be found in [8].

Due to its strong σ -additivity, the spectral measure is uniformly bounded, i.e.,

(2.2)
$$\exists M \ge 1 \ \forall \delta \in \mathscr{B}(\mathbb{C}) : \ \|E_A(\delta)\| \le M$$

(see, e.g., [6]).

By [8, Theorem XVIII.2.11 (c)], for a Borel measurable function $F : \sigma(A) \to \overline{\mathbb{C}}$, the operator F(A) is bounded iff $F(\cdot)$ is E_A -essentially bounded, i.e.,

$$E_A \operatorname{-ess\,sup}_{\lambda \in \sigma(A)} |F(\lambda)| < \infty_{\underline{\gamma}}$$

in which case

(2.3)
$$E_{A}\operatorname{-ess\,sup}|F(\lambda)| \le \|F(A)\| \le 4M E_{A}\operatorname{-ess\,sup}|F(\lambda)|,$$
$$\underset{\lambda \in \sigma(A)}{\overset{\lambda \in \sigma($$

where $M \ge 1$ is from (2.2).

A scalar type spectral C_0 -semigroup $\{T(t)\}_{t\geq 0}$ (i.e., a C_0 -semigroup of scalar type spectral operators) on a complex Banach space X is generated by a scalar type spectral operator [3,23], which is the case *iff*

$$s(A) < \infty$$

with

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \ge 0,$$

[20, Proposition 3.1], the orbit maps of the semigroup

$$\Gamma(t)f = e^{tA}f, \ t \ge 0, f \in X,$$

being the weak solutions (also called the $mild \ solutions$) of the associated abstract evolution equation

 $y'(t) = Ay(t), \ t \ge 0,$

[21] (see also [2,9]).

3. SBEGB CONDITION AND EXPONENTIAL ESTIMATES

By the Precise Weak Spectral Mapping Theorem [18, Theorem 5.1], a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator A is subject to precise weak spectral mapping theorem (PWSMT). Thus, by [9, Proposition V.2.3] (see Preliminaries), we arrive at the following

Corollary 3.1 (SBeGB Condition).

For a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator A, spectral bound equal growth bound condition (SBeGB) holds.

4

Remarks 3.1.

• Considering that, for a scalar type spectral operator A in a complex Banach space, $\sigma(A) \neq \emptyset$, when such an operator generates a C_0 -semigroup $\{T(t)\}_{t>0}$,

$$-\infty < s(A) = \omega_0 < \infty$$

(see Preliminaries).

• For a C_0 -semigroup $\{T(t)\}_{t>0}$, the exponential estimates

$$\exists \, \omega \in \mathbb{R}, \ \exists \, M = M(\omega) \geq 1: \ \|T(t)\| \leq M e^{\omega t}, \ t \geq 0,$$

hold for all $\omega > \omega_0$ with some $M = M(\omega) \ge 1$ (see Preliminaries).

However, the *best stability constants*, i.e., the smallest numbers ω and M for which (EE) is valid, (cf. [13]) need not exist, i.e., the infimum in (2.1) may not be attained even when $\omega_0 \in \mathbb{R}$ (see [9, Section I.1] and Example 3.1).

Proposition 3.1 (Exponential Estimate for Scalar Type Spectral C_0 -Semigroups). Let $\{T(t)\}_{t\geq 0}$ be a C_0 -semigroup (of scalar type spectral operators) on a complex Banach space $(X, \|\cdot\|)$ generated by a scalar type spectral operator A with spectral measure $E_A(\cdot)$. Then

(3.1)
$$||T(t)|| \le 4M_0 e^{\omega_0 t}, t \ge 0,$$

where $\omega_0 = s(A)$ is the best stability constant in the exponent and

(3.2)
$$M_0 := \sup_{\delta \in \mathscr{B}(\mathbb{C})} \| E_A(\delta) \| \ge 1.$$

Proof. Since

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \ge 0,$$

(see Preliminaries), by (2.3), for any $t \ge 0$,

$$\begin{aligned} \|T(t)\| &= \|e^{tA}\| = \left\| \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda) \right\| \le 4M E_A \operatorname{-ess\,sup} |e^{t\lambda}| \le 4M \sup_{\lambda \in \sigma(A)} |e^{t\lambda}| \\ &= 4M \sup_{\lambda \in \sigma(A)} e^{t\operatorname{Re}\lambda} \le 4M e^{s(A)t} \\ & \text{ in view of } s(A) = \omega_0; \end{aligned}$$

 $=4Me^{\omega_0 t},$

where $M \ge 1$ is from (2.2).

Therefore,

$$||T(t)|| = ||e^{tA}|| \le 4M_0 e^{\omega_0 t}, \ t \ge 0,$$

where $\omega_0 = s(A)$ (see Corollary 3.1) is the best stability constant in the exponent and M_0 , defined by (3.2), is the smallest $M \ge 1$ for which (2.2) holds.

MARAT V. MARKIN

Remarks 3.2.

• Thus, for a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator Awith spectral measure $E_A(\cdot)$, $\omega_0 = s(A)$ is the best stability constant in the exponent and the other best stability constant satisfies the estimate

$$1 \le \min\left\{M \ge 1 \mid \|T(t)\| \le M e^{\omega_0 t}, t \ge 0\right\} \le 4 \sup_{\delta \in \mathscr{B}(\mathbb{C})} \|E_A(\delta)\|.$$

• As the next example demonstrates, for a non-scalar-type-spectral C_0 -semigroup on a complex Banach space, even under *spectral bound equal growth bound condition* (SBeGB), the best stability constants need not exist.

Example 3.1. On the complex Banach space $l_p^{(2)}$ $(1 \le p \le \infty)$, the bounded linear operator A of multiplication by the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which is nonzero *nilpotent*, and hence, not scalar type spectral (see, e.g., [8]), generates the *uniformly continuous* (not scalar type spectral) semigroup of its exponentials

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \ t \ge 0,$$

where e^{tA} , $t \ge 0$, is the bounded linear operator on $l_p^{(2)}$ of multiplication by the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

(see, e.g., [9, 19]).

Although, in the considered case,

$$s(A) = \omega_0 = 0,$$

the exponential estimate

$$||T(t)|| \le M e^{\omega_0 t} = M, \ t \ge 0,$$

holds for no $M \ge 1$ (cf. [9]).

For C_0 -semigroups of normal operators on complex Hilbert spaces exponential estimate (3.1) can be refined as follows.

Proposition 3.2 (Exponential Estimate for Normal C_0 -Semigroups).

Let $\{T(t)\}_{t\geq 0}$ be a C_0 -semigroup (of normal operators) on a complex Hilbert space $(X, (\cdot, \cdot), \|\cdot\|)$ generated by a normal operator A with spectral measure $E_A(\cdot)$. Then

(3.3)
$$||T(t)|| \le e^{\omega_0 t}, t \ge 0$$

with $\omega_0 = s(A)$ and $M_0 := 1$ being the best stability constants.

6

Proof. Since

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \ge 0,$$

(see, e.g., [9,24]), for arbitrary $t \ge 0$ and $f \in X$, by the properties of the Borel operational calsulus,

$$\|T(t)f\| = \|e^{tA}f\| = \left\| \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda)f \right\| = \left[\int_{\sigma(A)} |e^{t\lambda}|^2 d(E_A(\lambda)f, f) \right]^{1/2}$$
$$= \left[\int_{\sigma(A)} e^{2t\operatorname{Re}\lambda} d(E_A(\lambda)f, f) \right]^{1/2} \le \left[e^{2s(A)t} \|E_A(\sigma(A))f\|^2 \right]^{1/2}$$

since $E_A(\sigma(A)) = I;$

$$= \left[e^{2s(A)t} \|f\|^2\right]^{1/2} = e^{s(A)t} \|f\|$$

in view of $s(A) = \omega_0$;

 $= e^{\omega_0 t} \|f\|.$

Whence, we obtain exponential estimate (3.3), in which $\omega_0 = s(A)$ (see Corollary 3.1) and $M_0 := 1$ are the best stability constants.

4. Generalized Lyapunov Stability Theorem

From the SBeGB Condition Corollary (Corollary 3.1) and the Exponential Estimate for Scalar Type Spectral C_0 -Semigroups (Proposition 3.1), we arrive at the following version of the Generalized Lyapunov Stability Theorem (Theorem 2.1) for scalar type spectral C_0 -semigroups.

Theorem 4.1 (GLST for Scalar Type Spectral C_0 -Semigroups).

A C_0 -Semigroup $\{T(t)\}_{t\geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator A is uniformly exponentially stable iff

$$s(A) < 0,$$

in which case

$$||T(t)|| \le 4M_0 e^{\omega_0 t}, \ t \ge 0,$$

where

$$\omega_0 = s(A)$$
 and $M_0 := \sup_{\delta \in \mathscr{B}(\mathbb{C})} \|E_A(\delta)\| \ge 1.$

For the C_0 -semigroups of normal operators on complex Hilbert spaces, in view of the *Exponential Estimate for Normal* C_0 -Semigroups (Proposition 3.2), the exponential estimate in the prior statement can be refined as follows.

Corollary 4.1 (GLST for Normal C_0 -Semigroups). A C_0 -Semigroup $\{T(t)\}_{t>0}$ (of normal operators) on a complex Hilbert space gen-

erated by a normal operator A is uniformly exponentially stable iff

s(A) < 0,

in which case

$$||T(t)|| \le e^{\omega_0 t}, \ t \ge 0,$$

where $\omega_0 = s(A)$.

5. UNIFORM EXPONENTIAL STABILITY

The following statement extends the *Gearhart-Prüss-Greiner Characterization* (Theorem 2.2) [10, 11, 25] to a Banach space setting.

Theorem 5.1 (Characterization of Uniform Exponential Stability).

For a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space $(X, \|\cdot\|)$ with generator A to be uniformly exponentially stable it is necessary and, provided spectral bound equal growth bound condition (SBeGB) holds, sufficient that

(5.1)
$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \ge 0\} \subseteq \rho(A) \quad and \quad \sup_{\operatorname{Re} \lambda \ge 0} \|R(\lambda, A)\| < \infty.$$

Proof.

Necessity. Suppose that a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space $(X, \|\cdot\|)$ with generator A is uniformly exponentially stable. Then

(5.2)
$$\exists \omega < 0, \ \exists M = M(\omega) \ge 1: \ \|T(t)\| \le M e^{\omega t}, \ t \ge 0.$$

Therefore,

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \ge 0\} \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\} \subseteq \rho(A).$$

Also, for arbitrary $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$ and $f \in X$,

$$R(\lambda, A)f = -\int_0^\infty e^{-\lambda t} T(t)f \, dt$$

(see, e.g., [9, 12]) and

$$\begin{aligned} \|R(\lambda,A)f\| &= \left\| -\int_0^\infty e^{-\lambda t} T(t)f\,dt \right\| \le \int_0^\infty \left\| e^{-\lambda t} T(t)f \right\|\,dt \\ &\le \int_0^\infty e^{-\operatorname{Re}\lambda t} \|T(t)\| \|f\|\,dt \end{aligned}$$

by (5.2);

$$\leq M \int_0^\infty e^{-(\operatorname{Re}\lambda - \omega)t} \, dt \|f\| = \frac{M}{\operatorname{Re}\lambda - \omega} \|f\|,$$

which implies that

$$\sup_{\operatorname{Re}\lambda\geq 0} \|R(\lambda,A)\| \leq \sup_{\operatorname{Re}\lambda\geq 0} \frac{M}{\operatorname{Re}\lambda - \omega} = -\frac{M}{\omega} < \infty,$$

completing the proof of the *necessity*.

Sufficiency. Let us prove this part by contradiction.

Suppose that a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space $(X, \|\cdot\|)$ with generator A is subject to spectral bound equal growth bound condition (SBeGB) and conditions (5.1) but is not uniformly exponentially stable, which, by the Generalized Lyapunov Stability Theorem (Theorem 2.1), is equivalent to the fact that

(5.3)
$$s(A) := \sup \{ \operatorname{Re} \lambda \, | \, \lambda \in \sigma(A) \} \ge 0$$

Observe that this implies, in particular, that $\sigma(A) \neq \emptyset$ (see Preliminaries).

In view of the inclusion

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \ge 0\} \subseteq \rho(A),\$$

(5.3) implies that

$$(5.4) s(A) = 0.$$

The operator A being *closed*, for an arbitrary $\lambda \in \rho(A)$,

$$||R(\lambda, A)|| \ge \frac{1}{\operatorname{dist}(\lambda, \sigma(A))},$$

where

$$\operatorname{dist}(\lambda, \sigma(A)) := \inf_{\mu \in \sigma(A)} |\mu - \lambda|,$$

(see, e.g., [6, 19]), which, in view of (5.4), implies that

$$\sup_{\operatorname{Re}\lambda\geq 0} \|R(\lambda,A)\| \geq \sup_{\operatorname{Re}\lambda\geq 0} \frac{1}{\operatorname{dist}(\lambda,\sigma(A))} = \frac{1}{\operatorname{inf}_{\operatorname{Re}\lambda\geq 0}\operatorname{dist}(\lambda,\sigma(A))} = \infty,$$

contradicting (5.1).

The obtained contradiction completes the proof of the *sufficiency*, and hence, of the entire statement. $\hfill \Box$

Remarks 5.1.

- Thus, the necessity of the Gearhart-Prüss-Greiner characterization of the uniform exponential stability [9, Theorem V.3.8] holds in a Banach space setting.
- As the next example shows, the requirement that the semigroup be subject to *spectral bound equal growth bound condition* (SBeGB) in the *sufficiency* of the prior characterization is essential and cannot be dropped.

Example 5.1. As discussed in [9, Counterexample V.1.26], the left translation C_0 -semigroup

$$[T(t)f](x) := f(x+t), \ t, x \ge 0,$$

on the complex Banach space

$$X := \left\{ f \in C(\mathbb{R}_+) \, \middle| \, \lim_{x \to \infty} f(x) = 0 \text{ and } \int_0^\infty |f(x)| e^x \, dx < \infty \right\}$$

 $(\mathbb{R}_+ := [0, \infty))$ with the norm

$$X \ni f \mapsto ||f|| := ||f||_{\infty} + ||f||_{1} = \sup_{x \ge 0} |f(x)| + \int_{0}^{\infty} |f(x)|e^{x} dx$$

is generated by the differentiation operator

$$Af := f'$$

with the *domain*

$$D(A) := \{ f \in X \mid f \in C^1(\mathbb{R}_+), f' \in X \},\$$

for which

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} \, | \, \operatorname{Re} \lambda \le -1 \right\},\,$$

and hence s(A) = -1.

The semigroup is not uniformly exponentially stable since

$$||T(t)|| = 1, \ t \ge 0,$$

and hence, $\omega_0 = 0$.

Thus,

$$s(A) \neq \omega_0.$$

However, the resolvent

$$R(\lambda, A)f = -\int_0^\infty e^{-\lambda t} T(t)f \, dt, \ f \in X,$$

of the generator A exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > -1$ and

$$\sup_{\operatorname{Re}\lambda\geq 0}\|R(\lambda,A)\|<\infty$$

(see [9, Comments V.3.9]).

By the *SBeGB Condition Corollary* (Corollary 3.1) and [9, Corollary V.2.9] (see also [9, Corollary V.2.10]), we arrive at

Corollary 5.1 (Characterization of Uniform Exponential Stability).

For a C_0 -semigroup $\{T(t)\}_{t\geq 0}$ on a complex Banach space $(X, \|\cdot\|)$ with generator A to be uniformly exponentially stable it is necessary and, provided the semigroup is scalar type spectral or eventually norm-continuous, in particular eventually compact, eventually differentiable, analytic, or uniformly continuous, sufficient that

$$\{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\, \lambda \ge 0\} \subseteq \rho(A) \quad and \quad \sup_{\mathrm{Re}\, \lambda \ge 0} \|R(\lambda,A)\| < \infty.$$

On the regularity of scalar type spectral C_0 -semigroups, see [17].

6. Acknowledgments

My sincere gratitude is to Dr. Yuri Latushkin for his reference to the *Gearhart-Prüss-Greiner characterization* during our conversation at the Joint Mathematics Meetings 2019, thoughtful suggestions concerning the best stability constants, and insightful remarks.

10

References

- [1] W.G. Bade, Unbounded spectral operators, Pacific J. Math. 4 (1954), 373-392.
- [2] J.M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, Proc. Amer. Math. Soc. 63 (1977), no. 2, 101–107.
- [3] E. Berkson, Semi-groups of scalar type operators and a theorem of Stone, Illinois J. Math. 10 (1966), no. 2, 345—352.
- [4] N. Dunford, Spectral operators, Pacific J. Math. 4 (1954), 321–354.
- [5] _____, A survey of the theory of spectral operators, Bull. Amer. Math. Soc. 64 (1958), 217–274.
- [6] N. Dunford and J.T. Schwartz with the assistance of W.G. Bade and R.G. Bartle, *Linear Operators. Part I: General Theory*, Interscience Publishers, New York, 1958.
- [7] _____, Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space, Interscience Publishers, New York, 1963.
- [8] _____, Linear Operators. Part III: Spectral Operators, Interscience Publishers, New York, 1971.
- [9] K.-J. Engel and R. Nagel, A Short Course on Operator Semigroups, Universitext, Springer, New York, 2006.
- [10] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, Trans. Amer. Math. Math. Soc. 236 (1978), 385–394.
- [11] G. Greiner, Some applications of Fejer's theorem to one-parameter semigroups, Semesterbericht Funktionalanalysis Tübingen 7 (Wintersemester 1984/85), 33–50.
- [12] E. Hille and R.S. Phillips, Functional Analysis and Semi-groups, American Mathematical Society Colloquium Publications, vol. 31, Amer. Math. Soc., Providence, RI, 1957.
- [13] Yu. Latushkin and V. Yurov, Stability estimates for semigroups on Banach spaces, Discrete Contin. Dyn. Syst. 33 (2013), no. 11–12, 5203–5216.
- [14] E. R. Lorch, Bicontinuous linear transformations in certain vector spaces, Bull. Amer. Math. Soc. 45 (1939), 564–569.
- [15] A.M. Lyapunov, Stability of Motion, Ph.D. Thesis, Kharkov, 1892, English Translation, Academic Press, New York-London, 1966.
- [16] G.W. Mackey, Commutative Banach Algebras (ed. by A. Blair), Harvard Lecture Notes, 1952.
- [17] M.V. Markin, On the regularity of scalar type spectral C_0 -semigroups, arXiv:2103.05260.
- [18] _____, On spectral inclusion and mapping theorems for scalar type spectral operators and semigroups, arXiv:2002.09087.
- [19] _____, *Elementary Operator Theory*, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin/Boston, 2020.
- [20] _____, A note on the spectral operators of scalar type and semigroups of bounded linear operators, Int. J. Math. Math. Sci. 32 (2002), no. 10, 635–640.
- [21] _____, On an abstract evolution equation with a spectral operator of scalar type, Ibid. 32 (2002), no. 9, 555–563.
- [22] J.M.A.A. van Neerven, The Asymptotic Behaviour of Semigroups of Linear Operators, Oper. Theory Adv. Appl., vol. 88, Birkhäuser Verlag, Basel, 1996.
- [23] T.V. Panchapagesan, Semi-groups of scalar type operators in Banach spaces, Pacific J. Math. 30 (1969), no. 2, 489–517.
- [24] A.I. Plesner, Spectral Theory of Linear Operators, Nauka, Moscow, 1965 (Russian).
- [25] J. Prüss, On the spectrum of C₀-semigroups, Trans. Amer. Math. Math. Soc. 284 (1984), 847–857.
- [26] J. Wermer, Commuting spectral measures on Hilbert space, Pacific J. Math. 4 (1954), 355– 361.

DEPARTMENT OF MATHEMATICS CALIFORNIA STATE UNIVERSITY, FRESNO 5245 N. BACKER AVENUE, M/S PB 108 FRESNO, CA 93740-8001, USA

 $Email \ address: \tt mmarkin@csufresno.edu$