INVARIANT GRAPH AND RANDOM BONY ATTRACTORS

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ABSTRACT. In this paper, we deal with random attractors for dynamical systems forced by a deterministic noise. These kind of systems are modeled as skew products where the dynamics of the forcing process are described by the base transformation. Here, we consider skew products over the Bernoulli shift with the unit interval fiber. We study the geometric structure of maximal attractors, the orbit stability and stability of mixing of these skew products under random perturbations of the fiber maps. We show that there exists an open set \mathcal{U} in the space of such skew products so that any skew product belonging to this set admits an attractor which is either a continuous invariant graph or a bony graph attractor. These skew products have negative fiber Lyapunov exponents and their fiber maps are non-uniformly contracting, hence the non-uniform contraction rates are measured by Lyapunov exponents. Furthermore, each skew product of \mathcal{U} admits an invariant ergodic measure whose support is contained in that attractor. Additionally, we show that the invariant measure for the perturbed system is continuous in the Hutchinson metric.

1. INTRODUCTION

The qualitative study of dynamical systems is concerned with the study of attractors. Knowledge of the attractors may indicate the long time behavior of the orbits. In the most simple cases, an attractor of a dynamical system is a union of finite set of smooth manifolds. There are interesting examples of locally dynamical systems having more complicated attractors. For example in [20], Kudryashov introduced a new type of attractors so-called bony attractors, then he presented an open set in the space of step skew products over the Bernoulli shift such that any of them had a bony attractor. Following [20], an attractor *A* of a skew product is *bony* if *A* is the union of the graph of a continuous function on some subset of the base and an uncountable set of vertical closed intervals (bones) contained in the closure of the graph. This feature is similar to porcupine horseshoes discovered by Diaz and Gelfert in [10]. Indeed, from a topological point of view, a porcupine is a transitive set that looks like a horseshoe with infinitely many spines attached at various levels and in a dense way.

The objective of this article is to extend aforementioned result from [20] to the random case, where the skew products are general (not necessarily step). One novelty here is that, in our context, in contrast the Kudryashov' case, fiber maps are non-uniformly contracting, therefore the contraction rates are non-uniform and hence measured by Lyapunov exponents.

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We will discuss maximal attractors of this kind of skew products and show that they are either a continuous invariant graph or a bony attractor. Moreover, maximal attractors, carry an invariant ergodic measure that projects to the Bernoulli measure in the base.

Notice that, in general, dynamical systems under the external forcing are modeled, in discrete time, as skew products,

$$F: \Omega \times M \to \Omega \times M, \ F(\omega, x) = (\theta \omega, f_{\omega}(x)), \tag{1}$$

where the dynamics of the forcing process are described by the base transformation θ which is assumed to be a measure-preserving transformation of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (random forcing). An *invariant graph* of *F* is the graph of a measurable function $\gamma : \Omega \to M$ which satisfies $f_{\omega}(\gamma(\omega)) = \gamma(\theta(\omega))$, for \mathbb{P} -almost all $\omega \in \Omega$.

In the study of forced dynamical systems of the above form, invariant graphs play a central role since they are the natural substitutes of a stable fixed point to the case of forced systems. Furthermore, the existence of such invariant graphs considerably simplifies the dynamics of the forced systems. Moreover, Lyapunov exponents yield additional information about the stability and attractivity of invariant graphs. Attracting invariant graphs have a wide variety of applications in many branches of nonlinear dynamics (e.g. [8, 9, 16, 17, 21, 24, 25] etc.). A context in which the attractivity of invariant graphs plays a central role is generalised synchronisation, a phenomenon that has been widely studied in theoretical physics. In [22] Stark provides the conditions for the existence and regularity of invariant graphs and discusses a number of applications. His results include some generalizations to the case of non-uniform contraction. We mention that in skew product systems with uniformly contracting fiber maps, there exist continuous invariant attracting sets for the overall dynamics, see [14], Theorem 6.1a, [15]. Results in the non-uniform case, when the fiber map possesses negative Lyapunov exponents in the fibre [3, 11, 12, 29, 30], are very recent and invariant graphs are very sensitive to perturbations.

This work is organized as follows: In Subsections 1.1 and 1.2 we recall some standard definitions. Then we state our main result in Subsection 1.3. The proof of our main result, Theorem A below, is given in Section 2.

1.1. **Preliminaries.** Assume that *X* is a metric measure space. Denote by int(D) and Cl(D), respectively, the interior and the closure of any set *D*.

Let $(X; \mathcal{B}; \mu; f)$ be a measure preserving dynamical system. If f is invertible then, based on [6, 26], the system is *Bernoulli* if it is isomorphic to a Bernoulli shift. Clearly invertible systems cannot be isomorphic to non-invertible systems. But there is a construction to make a non-invertible system invertible, namely by passing to the natural extension. For noninvertible case, being Bernoulli means that the natural extension is isomorphic to a Bernoulli shift.

The map *f* is *mixing* (or *strong mixing*) if

 $\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B), \text{ as } n \to +\infty,$

for every $A, B \in \mathcal{B}$. Every mixing system [26] is necessarily ergodic.

For a metric space *X*, putting

$$\operatorname{Lip}_1(X) = \{ f : X \to \mathbb{R} : |f(x) - f(y)| \le d(x, y) \text{ for all } x, y \in X \},\$$

define the *Hutchinson metric* on the set $\mathcal{M}(X)$, the space of all Borel probability measures, by

$$d_H(\nu,\mu) = \sup\{|\int_X f d\nu - \int_X f d\mu : f \in \operatorname{Lip}_1(X)|\}.$$
(2)

In [19, Thm. 3.1], the author proved that for every metric space *X*, the topology \mathcal{T} on $\mathcal{M}(X)$ generated by $d_H(v, \mu)$ coincides with the topology \mathcal{W} of weak convergence if and only if diam(*X*) < ∞ . Moreover, the space $\mathcal{M}(X)$ is complete in the metric d_H if and only if *X* is complete (see [19, Thm. 4.2]).

The concept of a weak contraction map was introduced in 1997 by Alber and Guerre-Delabriere [2]. We say that a continuous map f is *weak contraction* (or *distance decreasing* [4]) whenever for each $x, y \in X$ with $x \neq y$, d(f(x), f(y)) < d(x, y).

It is a well-known fact [18, Coro. 3] (see also [4]) that if f is weak contraction and X is compact then there exists a unique fixed point $x \in X$ of the map f. Furthermore, for every $y \in X$, $\lim_{k\to\infty} f^k(y) = x$ uniformly. Then we say that x is a *weak attracting fixed point*. Clearly if f is a weak contraction map then

$$d(f^n(y), f^n(z)) \to 0$$
, as, $n \to \infty$,

for each $y, z \in X$.

1.2. **Random maps and skew products.** A *random map* with base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, in the sense of Arnold [1], is a skew product of the form (1) where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\theta : \Omega \to \Omega$ is a bi-measurable and ergodic measure-preserving bijection and *M* is a measurable space. If *M* is a smooth manifold and all fibre maps f_{ω} are C^r , we call *F* a random C^r -map.

Take $\Sigma_k^+ = \{0, \ldots, k-1\}^{\mathbb{N}}$ and $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ endowed with the product topology and equip them with the Bernoulli measures v^+ and v, respectively, corresponding to some distribution of probabilities p_0, \ldots, p_{k-1} , which gives us the probability with which we apply f_i . Here, assume that the probabilities p_i , $i = 0, \ldots, k-1$, are the same and equal to 1/k. Let $\sigma : \Sigma_k \to \Sigma_k$ and $\sigma^+ : \Sigma_k^+ \to \Sigma_k^+$ denote the one-sided and two-sided left shift. It is well known that [28] σ^+ and σ are ergodic transformations preserving the probabilities v^+ and v, respectively.

Let *M* be a compact smooth manifold. Here, we consider skew products of the form

$$F: \Sigma_k \times M \to \Sigma_k \times M; \ (\omega, x) \to (\sigma\omega, f_w(x))$$
(3)

which is called a *skew product over the Bernoulli shift*, where $\omega \in \Sigma_k$, $x \in M$ and the maps f_{ω} are C^r diffeomorphisms on M. The space Σ_k is called the *base*, the space M is called the *fiber*, and the maps f_{ω} are called the *fiber maps*. Thus each skew product of the form (3) is a random C^r -map.

A skew product over the Bernoulli shift is a *step skew product* if the fiber maps f_{ω} depend only on the digit ω_0 and not on the whole sequence ω . We emphasise, in contrast to step skew products, the fiber maps of (general) skew products of the form (3) depend on the whole sequence ω . When treating a step skew product for one sided time \mathbb{N} , this results in the skew product system F^+ on $\Sigma_k^+ \times M$:

$$F^{+}: \Sigma_{k}^{+} \times M \to \Sigma_{k}^{+} \times M; \ (\omega, x) \to (\sigma^{+}\omega, f_{w_{0}}(x)).$$

$$\tag{4}$$

We denote iterates of a skew product system *F* of the form (3) as $F^n(\omega, x) = (\sigma^n(\omega), f_{\omega}^n(x))$. Here, for $n \ge 1$

$$f_{\omega}^{n}(x) := f_{\sigma^{n-1}\omega} \circ \ldots \circ f_{\omega}(x)$$

For a step skew product system this becomes

$$f_{\omega}^{n}(x) := f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_{0}}(x)$$

where $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_n, \ldots) \in \Sigma_k$.

In the rest of this article we assume that the fiber *M* is always the unit interval *I*.

Take *C*(*I*) the space of all random *C*²-maps (general skew products) acting on $\Sigma_k \times I$ defined by *C*² interval diffeomorphisms. We equip *C*(*I*) with the following metric:

$$\operatorname{dist}_{C^2}(F,G) := \sup_{\omega \in \Sigma_k} (\operatorname{dist}_{C^2}(f_{\omega}^{\pm 1}, g_{\omega}^{\pm 1})), \text{ for each } F, G \in C(I),$$
(5)

where f_{ω} and g_{ω} are the fiber maps of *F* and *G*, respectively.

Let $F : \Sigma_k \times I \to \Sigma_k \times I$ be a homeomorphism onto its image, but suppose its image is contained strictly in $\Sigma_k \times I$. The (global) *maximal attractor* of *F* is defined as:

$$A_{max}(F) := \bigcap_{n=0}^{\infty} F^n(\Sigma_k \times I).$$
(6)

1.3. **Main results.** To state the main result precisely, the concept of a bony attractor may need to be introduced.

Definition 1.1. Following [20], an attractor Λ of a skew product F is a bony graph attractor if Λ is the union of the graph of a continuous function γ defined on some set of full measure of the base and a set of vertical closed intervals ("bones") contained in the closure of the graph.

In this article, we will show that maximal attractors of a certain class of general skew products (random maps) are either a continuous invariant graph or a bony attractor. Our novelty here is that the fiber maps of such systems depend on the whole sequence ω and hence they are not necessarily step skew products. Moreover, the fiber contraction rates are non-uniform and hence measured by Lyapunov exponents, in addition, the attractors carry an ergodic measure. Our result thus extends work by Kudryashov in [20] who treated step skew products over the Bernoulli shift having bony attractors.

Theorem A. There exists an open nonempty set \mathcal{U} in the space C^2 random maps C(I) given by (5) such that any system G belonging to this set has a maximal attractor $A_{max}(G)$ satisfies the following properties:

(1) the maximal attractor $A_{max}(G)$ is either a continuous invariant graph or a bony graph attractor;

- (2) there exists an invariant ergodic measure μ_G whose support is the closure of the graph Γ_G , in particular, (G, Γ_G, μ_G) is Bernoulli and therefore it is mixing, additionally, the invariant measure for the perturbed system is continuous in the Hutchinson metric;
- (3) the fiber Lyapunov exponent of G is negative;

Moreover, the set of random maps of \mathcal{U} which admit a bony graph attractor is nonempty.

2. Proof of Theorem A

To prove Theorem A, we provide a single step skew product *F*, and show that every skew product (random map) $G \in C(I)$ (not necessarily step) which is close enough to *F* (with respect to dist_{C²} given by (5)) satisfies the conclusion of the theorem.

Consider the interval I = [0, 1] and let $\{f_0, ..., f_{k-1}\}$ a finite family of orientation preserving (strictly increasing) C^2 -diffeomorphisms defined on I enjoying the following conditions:

- (a) The mappings f_i , i = 0, ..., k 1, bring the unit interval *I* strictly into itself and they are C^2 close to the identity.
- (b) f_0 is a weak contraction with a unique weak attracting fixed point p_0 , i.e. $Df_0(p_0) = 1$.
- (c) f_i , i = 1, ..., k 1, are uniformly contracting maps such that any of them has a unique attracting fixed point p_i .
- (d) We have "contraction on average", i.e. for each $x \in I$, $\prod_{i=0}^{k-1} Df_i(x) < 1$.
- (e) The fixed points p_i , i = 0, ..., k 1, are pairwise disjoint, $p_i \neq 0, 1$ and satisfy the no-cycle condition, i.e. $f_i(p_i) \neq p_k$ for each distinct indices i, j and k.
- (f) Let $p_0 < p_1 < ... < p_{k-1}$ and $J = [p_0, p_1]$. Assume we have "covering property", i.e. there exists the points x_0 and x_1 such that $p_0 < x_0 < x_1 < p_1$ and the interval $B = (x_0, x_1) \subset int(J)$ for which the following holds: $\forall x \in [x_0, x_1]$, $||Df_i(x)|| < 1$ and $Cl(B) \subset f_0(B) \cup f_1(B)$.

Write the step skew product

$$F: \Sigma_k \times I \to \Sigma_k \times I; \ (\omega, x) \to (\sigma\omega, f_{w_0}(x))$$

$$\tag{7}$$

whose fiber maps are the mapping f_i , i = 0, ..., k - 1, which satisfy the properties listed above. Fix the skew product *F* and take a small open ball \mathcal{U} around *F* in the space *C*(*I*). To prove Theorem A, we show that any system *G* belonging to this set satisfies the conclusion of the theorem.

We define the *Transfer Operator* $T : \mathcal{M}(I) \to \mathcal{M}(I)$ by the formula,

$$T(\mu)(B) := \frac{1}{k} \sum_{i=1}^{k} \mu(f_i^{-1}(B)),$$

for any Borel subset *B* and for each measure $\mu \in \mathcal{M}(I)$, where $\mathcal{M}(I)$ is the space of all Borel probability measures on *I*. If a measure $\mu \in \mathcal{M}(I)$ is a fixed point of the transfer operator we say that μ is a *stationary measure*.

Remark 2.1. The following two facts hold:

- (1) The contraction on average condition given by (d) ensuring [23] the existence of a unique attractive stationary probability measure m in the sense that $T^n\mu$ converges weakly to m, for any probability measure $\mu \in \mathcal{M}(I)$.
- (2) For the skew product F^+ of the form (4) with the fiber maps f_i , i = 0, ..., k 1, the product measure $v^+ \times m$ is an ergodic invariant measure. The skew product F given by (7) is the natural extension of F^+ . Invariant measures for F^+ with marginal v^+ and invariant measures for F with marginal v are in one to one relationship, as detailed in [1]. A stationary measure m thus, through the invariant measure $v^+ \times m$ for F^+ , gives rise to an invariant measure μ for F, with marginal v.

2.1. **Fiber Lyapunov exponents.** For each Lipschitz map $f : I \rightarrow I$ we define the norm $\|.\|$ by

$$||f|| := \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}.$$
(8)

It is easily seen that, whenever f is C^1 , by Mean value theorem for real-valued functions,

$$||f|| = \sup\{||Df(x)|| : x \in I\}.$$

For the skew product $F(\omega, x) = (\sigma\omega, f(\omega, x)) = (\sigma\omega, f_{\omega_0}(x))$ given by (7), consider a sequence of functions φ^n defined by $\varphi^n(\omega) = ||f^n(\omega, .)||$. It is simply verified that the family of functions $\{a_n\}$ defined by $a_n(\omega) = \log(\varphi^n(\omega))$ is subadditive. By definition of the mappings f_i , i = 0, ..., k - 1, and the functions φ^n one has that $\log^+(\varphi^1) \in L^1(\nu)$, hence by Kingman's Subadditive Theorem [26, Thm. 3.3.3], the limit

$$\lambda(\omega) := \lim_{n \to \infty} \frac{1}{n} \log \|f^n(\omega, .)\|$$
(9)

exists at *v*-almost every point. Moreover, the function $\lambda \in L^1(v)$ and

$$\lim_{n \to \infty} \frac{1}{n} \log \|f^n(\omega, .)\| = \inf_n \frac{1}{n} \log \|f^n(\omega, .)\|.$$
(10)

By ergodicity of v, the limit (9) is constant, denoted by λ . The contraction on average property given by condition (*d*) ensures that λ is negative. The constant λ is called the *m*-fiber Lyapunov exponent with respect to v.

Lemma 2.2. There exists an open subset $\mathcal{U} \subset C(I)$ containing F such that any skew product G belonging to this set admits a negative *m*-fiber Lyapunov exponents with respect to v.

Proof. Take small neighborhoods $U_i \subset \text{Diff}^2(I)$ of the fiber maps f_i , i = 0, ..., k - 1, of F and let $\mathcal{U} \subset C(I)$ a small open neighborhood of F enjoying the following property: there exists a constant C > 0 such that for any $G \in \mathcal{U}$ with $G(\omega, x) = (\sigma \omega, g(\omega, x)) = (\sigma \omega, g_\omega(x))$ one has that

$$\forall \omega \in \Sigma_k$$
, the map $g_\omega \in U_{\omega_0}$, and $\operatorname{dist}_{C^2}(g_\omega, g_{\omega'}) < Cd(\omega, \omega')$, for any $\omega, \omega' \in \Sigma_k$. (11)

Then, by this fact and (10), for given a sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that if diam(U_i) < δ then

$$\lim_{n \to \infty} \frac{1}{n} \log \|g^n(\omega, .)\| = \lambda + \varepsilon < 0, \text{ for } \nu \text{ a.e. } \omega \in \Sigma_k.$$
(12)

In particular, *G* possesses a negative *m*-fiber Lyapunov exponent.

2.2. **Maximal attractors and invariant graphs.** For the step skew product *F* given by (7) and any general skew product $G \in \mathcal{U}$, consider the maximal attractors $A_{max}(F)$ and $A_{max}(G)$, respectively, defined by

$$A_{max}(F) := \bigcap_{n \ge 0} F^n(\Sigma_k \times I), \quad A_{max}(G) := \bigcap_{n \ge 0} G^n(\Sigma_k \times I).$$
(13)

A first main step in the proof of Theorem A is to show that the attractor is an invariant graph. For that, we get the next proposition which is an analogue of [7, Thm. 5], [22, Thm. 1.4] and [5, Pro. 2.3] to our setting.

Proposition 2.3. Consider the skew product F given by (7). For each general skew product $G \in \mathcal{U}$, given by Lemma 2.2, there exists a measurable function $\gamma_G : \Omega \subseteq \Sigma_k \to I$, with $\nu(\Omega) = 1$ such that Γ_G the graph of γ_G is invariant under G. The closure of the graph Γ_G is the support of an invariant ergodic measure μ_G , in particular, (G, Γ_G, μ_G) is Bernoulli and hence it is mixing. Furthermore, Γ_G is attracting in the sense that for $\omega \in \Omega$, $\lim_{n\to\infty} |\pi_x(G^n(\omega, x)) - \gamma_G(\sigma^n \omega)| = 0$ for every $x \in I$, where π_x is the natural projection from $\Sigma_k \times I$ to I.

Proof. By Lemma 2.2, each $G \in \mathcal{U}$ has negative *m*-fiber Lyapunov exponent. By (12), given $\varepsilon > 0$ there exists a measurable function $C : \Sigma_k \to \mathbb{R}^+$ such that for ν a.e. $\omega \in \Sigma_k$, we have

$$\|g^n(\omega,.)\| < C(\omega)e^{(\lambda+\varepsilon)n}, \text{ for all } n > 0.$$
(14)

Since the Bernoulli shift σ is ergodic and invertible hence σ^{-1} is ergodic with respect to v and has the same spectrum of Lyapunov exponents by Furstenberg-Kesten Theorem [13]. Thus if we define

$$h_n(\omega, x) := g^n(\sigma^{-n}\omega, x)$$

then by (12)

$$\lim_{n\to\infty}\frac{1}{n}\log||h_n(\omega,.)|| = \lambda + \varepsilon < 0, \text{ for } \nu \text{ a.e. } \omega \in \Sigma_k.$$

Hence there exists $\ell(\omega)$ such that

$$||h_n(\omega, .)|| < e^{n(\lambda + \varepsilon)} \quad \forall n \ge \ell(\omega).$$

Thus given $\varepsilon > 0$ there exists a measurable function $C : \Sigma_k \to \mathbb{R}^+$ such that for ν a.e. $\omega \in \Sigma_k$, we have

$$\|h_n(\omega, .)\| < C(\omega)e^{(\lambda+\varepsilon)n}, \text{ for all } n > 0.$$
(15)

Applying the approach used in the proof of [5, Pro. 2.3], we conclude that the sequence $\{h_{\ell}(\omega, x)\}$ is a Cauchy sequence for every $x \in I$ and a.e. $\omega \in \Sigma_k$. Indeed, let

$$\alpha(x) := \sup_{\omega \in \Sigma_k} |x - g(\omega, x)|$$

and note that for *x* fixed $\alpha(x)$ is finite as Σ_k is compact and *g* is continuous. Given any $\varepsilon' > 0$, choose $\ell^*(\omega)$ sufficiently large that

$$\alpha(x)C(\omega)\sum_{j=\ell^*(\omega)}^{\infty}e^{j(\lambda+\varepsilon)}<\varepsilon'.$$

Then if $m > \ell > \ell^*(\omega)$

$$|h_m(\omega, x) - h_\ell(\omega, x)| \le \alpha(x) \sum_{j=\ell}^{\infty} ||h_j(\omega, .)|| \le \alpha(x) c(\omega) \sum_{j=\ell}^{\infty} e^{j(\lambda + \varepsilon)} < \varepsilon'.$$

To see this note that

$$|h_m(\omega, x) - h_\ell(\omega, x)| = |h_m(\omega, x) - h_{m-1}(\omega, x) + \ldots + h_{\ell+1}(\omega, x) - h_\ell(\omega, x)|.$$

Note that applying *G* once to $(\sigma^{-k}(\omega), x)$, gives $G(\sigma^{-k}(\omega), x) = (\sigma^{-(k-1)}(\omega), g(\sigma^{-k}(\omega), x))$. Thus, $h_k(\omega, x) - h_{k-1}(\omega, x) = h_{k-1}(\omega, g(\sigma^{-k}(\omega), x))) - h_{k-1}(\omega, x)$. As a result,

$$|h_{k-1}(\omega, x) - h_k(\omega, x)| \le ||h_{k-1}(\omega, .)|||x - g(\sigma^{-k}(\omega), x))|$$

Hence

$$\begin{split} |h_{m}(\omega, x) - h_{\ell}(\omega, x)| &\leq |h_{m}(\omega, x) - h_{m-1}(\omega, x)| + \ldots + |h_{\ell+1}(\omega, x) - h_{\ell}(\omega, x)| \\ &= \sum_{j=\ell+1}^{m} |h_{j}(\omega, x) - h_{j-1}(\omega, x)| \\ &\leq \sum_{j=\ell+1}^{m} ||h_{j-1}(\omega, .)|| |x - g(\sigma^{-j}(\omega), x))| \\ &\leq \sum_{j=\ell+1}^{\infty} ||h_{j-1}(\omega, .)|| \alpha(x). \end{split}$$

Thus

$$\begin{aligned} |h_m(\omega, x) - h_\ell(\omega, x)| &\leq \alpha(x) \sum_{j=\ell+1}^{\infty} ||h_{j-1}(\omega, .)|| \\ &= \alpha(x) C(\omega) \sum_{i=\ell+1}^{\infty} e^{(j-1)(\lambda+\varepsilon)} < \varepsilon' \end{aligned}$$

as $\ell > \ell^*(\omega)$. Thus there exists a subset $\Omega \subseteq \Sigma_k$, with $\nu(\Omega) = 1$, so that for each $\omega \in \Omega$ the sequence $\{h_m(\omega, x)\}$ is a Cauchy sequence for every $x \in I$. Define

$$\gamma_G: \Omega \to I, \ \gamma_G(\omega) := \lim_{n \to +\infty} h_n(\omega, 0). \tag{16}$$

Since

$$G(\omega, h_{\ell}(\omega, 0)) = (\sigma \omega, h_{\ell+1}(\sigma \omega, 0)),$$

we see that

$$G(\omega, \gamma_G(\omega)) = (\sigma\omega, \gamma_G(\sigma\omega))$$

and hence Γ_G , the graph of γ_G is invariant under *G*. Furthermore, by construction, for every $\omega \in \Omega$, one has

$$\lim_{n \to +\infty} |g^n(\omega, x) - g^n(\omega, \gamma_G(\omega))| = \lim_{n \to +\infty} |g^n(\omega, x) - g^n(\omega, 0)| = 0.$$
(17)

This is because

$$|g^n(\omega, x) - g^n(\omega, 0)| \le ||g^n(\omega, .)|||x$$

and $||g^n(\omega, .)|| \to 0$ as $n \to +\infty$.

Therefore, for every $\omega \in \Omega$,

$$\lim_{n \to +\infty} g_{\sigma^{-1}\omega} \circ \ldots \circ g_{\sigma^{-n}\omega}(I) = \lim_{n \to +\infty} g_{\sigma^{-1}\omega} \circ \ldots \circ g_{\sigma^{-n}\omega}(0) = \gamma_G(\omega).$$
(18)

Hence, γ_G induces an invariant graph for *G* which is an attracting set by (18).

Consider the projection $p_G : \Gamma_G \to \Sigma_k, p_G(\omega, \gamma_G(\omega)) = \omega$, which is an isomorphism onto its image and the measure

$$\mu_G = (p_G)_* \nu = \nu \circ (id \times \gamma_G)^{-1}.$$
⁽¹⁹⁾

Then the mixing properties of the base transformation (σ, Σ_k, ν) lift to the transformation $(G, \Gamma_G \Gamma, \mu_G)$. In particular, μ_G is Bernoulli which implies that it is mixing. Every system that is mixing is also ergodic. Hence μ_G is an ergodic measure.

We now point out that the previous proposition with together the next two results establish assertions (1) and (2) of the main result of this article, Theorem A.

Proposition 2.4. For each skew product $G \in \mathcal{U}$ the maximal attractor $A_{max}(G)$ is either a continuous invariant graph or a bony attractor.

Proof. Take a skew product $G \in \mathcal{U}$ with $G(\omega, x) = (\sigma \omega, g(\omega, x)) = (\sigma \omega, g_{\omega}(x))$. By the previous proposition there exists a measurable function $\gamma_G : \Omega \subseteq \Sigma_k \to I$, with $\nu(\Omega) = 1$, such that Γ_G the graph of γ_G is invariant under G. We claim that $\Gamma_G \subset A_{max}(G)$.

Indeed, since $A_{\omega} := A_{max}(G) \cap I_{\omega} = \bigcap_{n \ge 0} I(\omega, n)$, where $I(\omega, n) := g_{\sigma^{-1}\omega} \circ \ldots \circ g_{\sigma^{-n}\omega}(I)$ and $I_{\omega} := \{\omega\} \times I$, and by using (18), one has

$$\lim_{n \to +\infty} g_{\sigma^{-1}(\omega)} \circ \cdots \circ g_{\sigma^{-n}(\omega)}(I) = \lim_{n \to +\infty} g_{\sigma^{-1}(\omega)} \circ \cdots \circ g_{\sigma^{-n}(\omega)}(0) = \gamma_G(\omega),$$

for each $\omega \in \Omega$, hence we observe that $\Gamma_G \subset A_{max}(G)$, as claimed.

Note that $I(\omega, n)$ is a sequence of nested intervals, and thus $A_{\omega} = A_{max}(G) \cap I_{\omega}$ is either an interval or a single point. Also note that if some sequences ω and ω' are close enough to each other, say,

$$\omega'_{-n} = \omega_{-n}, \ldots, \omega'_{-1} = \omega_{-1}$$

then, using $I(\omega', n) \supset A_{\omega'}$, we deduce $I(\omega, n) \supset A_{\omega'}$. This implies the upper-semicontinuity of A_{ω} . This semicontinuity, will immediately imply the continuity of its graph part.

Now there are two possibilities: either $\Omega = \Sigma_k$ and hence $A_{max}(G)$ is a continuous invariant graph, or the bones exist. In the later case, to verify that $A_{max}(G)$ is actually a bony attractor, it is enough to show that the set of bones contained in the closure of the graph. This will be done in the following lemma which completes the proof of the proposition.

Lemma 2.5. Let $G \in \mathcal{U}$ be a small perturbation of the skew product F given by (7) such that its maximal attractor $A_{max}(G)$ contains the bones with a graph function γ_G defined on a full measure subset $\Omega \subset \Sigma_k$. Then the bones are contained in the closure of the graph Γ_G .

Proof. To prove the lemma it is enough to show that the maximal attractor $A_{max}(G)$ coincides with the closure of the intersection $A_{max}(G) \cap (\Omega \times I)$.

First, we notice that the fiber maps f_i , i = 1, ..., k - 1, of F are uniformly contracting maps, by condition (*c*), and the skew product G is C^2 -close to F, hence, by (11), every sequence $\omega \in \Sigma_k$ without a tail of 0's to the left belong to Ω . Assume $(\omega, x) \in A_{max}(G)$ with $\omega \in \Sigma_k \setminus \Omega$. Then the sequence ω has a tail of 0's to the left (i.e. there exists $n_0 \in \mathbb{N}$ so that for each $n > n_0$, one has $\omega_{-n} = 0$). We denote the set of sequences ω' such that $\omega_i = \omega'_i$ for $i \in [-N, N]$ by $U_N(\omega)$ and the ε -neighborhood of the point x by $V_{\varepsilon}(x)$. Take $n > n_0 > N$ with $n = n_0 + 2m$ for large enough m and $h = g_{\sigma^{-(n_0+m)}\omega}^{-1} \circ \ldots \circ g_{\sigma^{-1}\omega}^{-1}$. Note that $\sigma^{-(n_0+m)}\omega$ has the following form

$$\sigma^{-(n_0+m)}\omega = (\dots, 0, \dots, 0; \underbrace{0, \dots, 0}_{m-times}, \omega_{-n_0}, \dots, \omega_{-1}, \omega_0, \omega_1, \omega_0, \dots)$$

Then the point $(\omega', x') = (\sigma^{-(n_0+m)}(\omega), h(x)) \in A_{max}(G)$. Now we take the sequence

$$\widetilde{\omega} = (\dots, 1, 1, \underbrace{0, \dots, 0}_{2m-times}, \omega_{-n_0}, \dots, \omega_{-1}; \omega_0, \omega_1, \dots)$$

which has a tail of 1's to the left. Since f_1 is a uniformly contracting map, G is C^2 -close to F and by (11), we conclude that $\operatorname{diam}(g_{\widetilde{\omega}_{-n}}, \ldots, g_{\widetilde{\omega}_{-1}}(I)) \to 0$ whenever $n \to +\infty$. Thus $\widetilde{\omega} \in \Omega$. Moreover, for $0 < \delta < \varepsilon < 1$ small and large enough k, one has $\operatorname{diam}(g_{\sigma^{-n_0-2m-k_{\widetilde{\omega}}}} \circ \ldots \circ g_{\sigma^{-n_0-2m-2k_{\widetilde{\omega}}}}(I)) < \delta < \varepsilon$. Let us take $I_{\delta} = g_{\sigma^{-n_0-2m-k_{\widetilde{\omega}}}} \circ \ldots \circ g_{\sigma^{-n_0-2m-2k_{\widetilde{\omega}}}}(I)$. Then for large enough m, we have

$$g_{\sigma^{-n_0-m-1}\widetilde{\omega}} \circ \ldots \circ g_{\sigma^{-n_0-2m}\widetilde{\omega}}(I_{\delta}) \subset h(V_{\varepsilon}(x))$$

Thus, the pair $(\tilde{\omega}, \gamma_G(\tilde{\omega}))$ belongs to the intersection $A_{max}(G) \cap (\Omega \times I) \cap (U_N(\omega) \times V_{\varepsilon}(x))$ and hence the conclusion of the lemma holds.

The next result ensures that the subset of all skew products $G \in \mathcal{U}$ having a bony attractor is nonempty.

Lemma 2.6. There exists a small perturbation $G \in \mathcal{U}$ of F which admits a bony graph attractor in the sense of Definition 1.1. In particular, the subset of bones has the cardinality of the continuum and is dense in the attractor.

Proof. Consider the fiber map f_0 satisfies conditions (*a*) and (*b*) in the beginning of Section 2 with a weak attracting fixed point p_0 , and take a map g, C^2 -close to f_0 , such that g = id on a small neighborhood U of the point p_0 . Now take a small perturbation G of F so that for the sequence $\omega = (..., 0, 0, 0, ...) \in \Sigma_k$ one has $g_\omega = g$. As you have seen before, for $I_\omega = \{\omega\} \times I$, and $I(\omega, n) = g_{\sigma^{-1}(\omega)} \circ \cdots \circ g_{\sigma^{-n}(\omega)}(I)$, one has $A_{max}(G) \cap I_\omega = \bigcap_{n \ge 0} I(\omega, n)$. Thus, we get

$$I(\omega, n) = g_{\sigma^{-1}(\omega)} \circ \cdots \circ g_{\sigma^{-n}(\omega)}(I) = \underbrace{g \circ \ldots \circ g}_{(n)-times}(I) = g^{n}(I)$$

which ensures that $A_{max}(G) \cap I_{\omega} = \bigcap_{n \ge 0} I(\omega, n)$ is an interval, hence $A_{max}(G)$ is a bony attractor. Moreover, for each sequence $\omega' \in \Sigma_k$ of the form $\omega' = (\dots, 0, 0; \omega'_1, \omega'_2, \dots)$, it is not hard to see that $A_{max}(G) \cap I_{\omega'}$ is an interval.

Furthermore, by construction, for any finite word α of the alphabets $\{0, 1, ..., k-1\}$ and a sequence ρ of the form $\rho = (..., 0, 0, \alpha, 0, 0, ...)$ with α standing at the zero position, $A_{max}(G) \cap I_{\rho}$ contains an interval. Thus the subset of bones has the cardinality of the continuum and is dense in the attractor $A_{max}(G)$.

In what follows, we show that the maximal attractor $A_{max}(F)$ is thick, this means that the projection of $A_{max}(F)$ on the fiber has positive Lebesgue measure.

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Lemma 2.7. Consider the skew product F given by (7). Then the maximal attractor $A_{max}(F)$ is thick.

Proof. First, we recall conditions (a) - (f) in the beginning of this section. By Proposition 2.3 and conditions (b) and (c), the maximal attractor $A_{max}(F)$ is a continuous invariant graph. Consider the graph function $\gamma_F : \Sigma_k \to I$ and let $K = \gamma_F(\Sigma_k)$. We apply the covering property from condition (f) and show that the interval B with $B \subset J = [p_0, p_1]$ introduced by (f) is contained in K. By this fact, Remark 2.1, the definition of measure μ_F given by (19) and by construction, the maximal attractor $A_{max}(F)$ is thick. For that, we show that for each $x \in B$, there exists a sequence $(\omega_{-n})_{n\geq 1}$ of $\{0, 1\}$ so that

$$x = \lim_{n \to +\infty} f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(I).$$
⁽²⁰⁾

First, we define, inductively, a sequence $(\omega_{-n})_{n\geq 1}$ of $\{0, 1\}$ so that

$$x = \lim_{n \to +\infty} f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(B).$$
(21)

Assume that we have found $\omega_{-1}, \ldots, \omega_{-n} \in \{0, 1\}$ so that $x \in f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(B)$. Then the covering property implies that

$$x \in f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(B) \subset \bigcup_{i=0}^{1} f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}} \circ f_i(B),$$

hence we can find $\omega_{-(n+1)}$ such that $x \in f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}} \circ f_{\omega_{-(n+1)}}(B)$. Take any sequence $\omega' \in \Sigma_k$ so that for each $n \ge 1$, we have $\omega'_{-n} = \omega_{-n}$. Then it is easily seen that

$$x = \lim_{n \to \infty} f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(B) = \lim_{n \to \infty} f_{\omega_{-1}} \circ \ldots \circ f_{\omega_{-n}}(I) = \lim_{n \to \infty} f_{\omega'_{-1}} \circ \ldots \circ f_{\omega'_{-n}}(I),$$

as we claimed.

The next proposition is an analogue of [5, Thm. 3.1] to our setting. It asserts that the invariant measure for the perturbed system is continuous in the Hutchinson metric.

Proposition 2.8. Suppose $G \in \mathcal{U}$ with $G(\omega, x) = (\sigma \omega, g(\omega, x))$. Then for given $\varepsilon > 0$, by shrinking \mathcal{U} , for v almost every $\omega \in \Sigma_k$ and all $x \in I$, one has that

$$d(F^n(\omega, x), G^n(\omega, x)) < \varepsilon,$$

except for at most a fraction ε of times n, where the distance d between two points of $\Sigma_k \times I$ is the sum of the distances between their projections onto the base and onto the fiber.

Furthermore, $d_H(\mu_F, \mu_G) < \varepsilon$ *, where* d_H *is Hutchinson metric given by* (2) *and* μ_F *and* μ_G *are the measures obtained from Proposition 2.3 for F and G.*

References

- [1] L. Arnold, Random Dynamical Systems, Springer Verlag, 1998.
- [2] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New results in operator theory, *Advances and Appl.* 98 (ed. by I. Gohberg and Yu Lyubich), *Birkhauser Verlag, Basel*, 1997.
- [3] L. Arnold and H. Crauel, Iterated function systems and multiplicative ergodic theory, in Diffusion processes and related problems in analysis, Vol. II, *Progr. Probab.*, 27, *Birkhauser Boston* (1992), 283-305.

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- [4] A. Bielecki, Iterated function systems analogues on compact metric spaces and their attractors, Univ. Iagel. Acta Math. 32 (1995) 187-192.
- [5] D. Broomhead, D. Hadjiloucas and M. Nicol, Random and deterministic perturbation of a class of skewproduct systems, *Dynamics and Stability of Systems*, 14 (1999), 115-128.
- [6] H. Bruin, Notes on Ergodic Theory, Preprint, November 5, 2014.
- [7] K.M. Campbell, Observational noise in skew product systems, *Physica D*, 107 (1997), 43-56.
- [8] K.M. Campbell and M.E. Davies, The existence of inertial functions in skew product systems, *Nonlinearity*, 9 (1996), 801-817.
- [9] M.E. Davies and K.M. Campbell, Linear recursive filters and nonlinear dynamics, *Nonlinearity*, 9 (1996), 487-499.
- [10] L. J. Diaz and K. Gelfert, Porcupine-like horseshoes: transitivity, Lyapunov spectrum, and phase transitions, *Fund. Math.*, 216(1) (2012), 55-100.
- [11] J. H. Elton, An ergodic theorem for iterated maps, Ergodic Th. and Dynam. sys., 7 (1987), 481-488.
- [12] J. H. Elton, A multiplicative ergodic theorem for Lipschitz maps, Stochastic Processes and their Applications, 34 (1990), 39-47.
- [13] H. Furstenberg and H. Kesten, Products of random matrices, Ann. Math. Stat., 31 (1960), 457-469.
- [14] Hirsch, M. and C. Pugh, Stable manifolds and hyperbolic sets, Bull. Amer. Math. Soc., 75 (1969), 149-152.
- [15] Hirsch, M., C. Pugh and M. Shub, Invariant Manifolds, Lecture Notes in Mathematics, 583, Springer, 1977.
- [16] B.R. Hunt, E. Ott and J.A. Yorke, Fractal dimensions of chaotic saddles of dynamical systems, *Phys. Rev. E*, 54 (1996), 4819-4823.
- [17] B.R. Hunt, E. Ott and J.A. Yorke, Differentiable generalized synchronization of chaos, *Phys. Rev. E.*, 55 (1997), 4029-4034.
- [18] J. R. Jachymski, An fixed point criterion for continuous self mappings on a complete metric space, Aequations Math., 48 (1994), 163-170.
- [19] A. S. Kravchenko, Completeness of the spaces of separable measures in the Kantrovich-Rubinshtein metric. (Russian summary) Sibrisk. Math. Zh. 47 (1) (2006), 85-96; translation in Siberian Math. J. 47 (1) (2006), 68-76.
- [20] Yu. G. Kudryashov, Bony attractors, Funkts. Anal. Prilozhen., 44 (3) (2010), 73-76; English transl., Functional Anal. Appl., 44 (3) (2010), 219-222.
- [21] L.M. Pecora and T.L. Carroll, Discontinuous and nondifferentiable functions and dimension increase induced by filtering chaotic data, *chaos*, 6 (1996), 432-439.
- [22] J. Stark, Regularity of invariant graphs for forced systems, Ergod. Th. Dyn. Systems, 19 (1999), 155-199.
- [23] O. Stenflo, A survey of average contractive iterated function systems, *Journal of Difference Equations and Appl.*, 18 (8) (2012) 1355-1380.
- [24] J. Stark and M.E. Davies, Recursive filters driven by chaotic signals, in *IEE Colloquium on Exploiting Chaos in Signal Processing*, 1 (1994), 431-516.
- [25] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematics Series, Vol. 68, Springer, Berlin, 1988.
- [26] M. Viana and K. Oliveira, Foundations of Ergodic Theory, Cambridge Studies in Advanced Mathematics, 2016.
- [27] D. Volk, Persistent massive attractors of smooth maps, *Ergodic Theory and Dynamical Systems*. 34(2) (2012), 693-704.
- [28] P. Walters, An Introduction to Ergodic Theorem, Springer-Verlag, 1982.
- [29] M. Zaj, A. Fakhari, F. H. Ghane, A. Ehsani, physical measures for certain class of non-uniformly hyperbolic endomorphisms on the solid torus, *Discrete Continuous Dynamical Systems-Series A*, 38 (4) (2018), 1777–1807.
- [30] M. Zaj, F. H. Ghane, Non Hyperbolic Solenoidal Thick Bony Attractors, Qual. Theory Dyn. Syst. 17 (3) (2018).

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