# REMARKS ON ALGEBRAIC DYNAMICS IN POSITIVE CHARACTERISTIC 

JUNYI XIE


#### Abstract

In this paper, we study arithmetic dynamics in arbitrary characteristic, in particular in positive characteristic. We generalise some basic facts on arithmetic degree and canonical height in positive characteristic. As applications, we prove the dynamical Mordell-Lang conjecture for automorphisms of projective surfaces of positive entropy, the Zariski dense orbit conjecture for automorphisms of projective surfaces and for endomorphisms of projective varieties with large first dynamical degree. We also study ergodic theory for constructible topology. For example, we prove the equidistribution of backward orbits for finite flat endomorphisms with large topological degree. As applications, we give a simple proof for weak dynamical Mordell-Lang and prove a counting result for backward orbits without multiplicities. This gives some applications for equidistributions on Berkovich spaces.


## 1. Introduction

Let $\mathbf{k}$ be an algebraically closed field. In this paper, most of the time (from Section 2 to Section 4), we are mainly interested in the case char $\mathbf{k}>0$.

Many problems in arithmetic dynamics, such as Dynamical Mordell-Lang conjecture, Zariski dense orbit conjecture are proposed in characteristic 0. Indeed, their original statements do not hold in positive characteristic. But their known counter-examples often involve some Frobenius actions or some group structures. We suspect that the original statement of these conjecture still valid for "general" dynamical systems in positive characteristic.

The $p$-adic interpolation lemma ([47, Theorem 1] and [6, Theorem 3.3]) is a fundamental tool in arithmetic dynamics. It has important applications in Dynamical Mordell-Lang and Zariski dense orbit conjecture [9, 6, 3, 2, 55]. But this lemma does not work in positive characteristic. Because this, some very basic cases of Dynamical Mordell-Lang and Zariski dense orbit conjecture are still open in positive characteristic. We hope that some corollaries of the p-adic interpolation lemma still survive in positive characteristic. For this, I propose the following conjecture.
Conjecture 1.1. Set $K:=\overline{\mathbb{F}_{p}}((t))$ and $\left.K^{\circ}=\overline{\mathbb{F}_{p}}[t t]\right]$ its valuation ring. Let $f:\left(K^{\circ}\right)^{r} \rightarrow\left(K^{\circ}\right)^{r}$ be an analytic automorphism satisfying $f=\mathrm{id} \bmod t$. If there is no $n \geq 1$ such that $f^{n}=\mathrm{id}$, then the $f$-periodic points are not dense in $\left(K^{\circ}\right)^{r}$ w.r.t. $t$-adic topology.

Date: December 30, 2021.
The author is partially supported by project "Fatou" ANR-17-CE40-0002-01 and PEPS CNRS.

On the other hand, we observed that, under certain assumption on the complexity of $f$, a global argument using height can be used to replace the local argument using the p-adic interpolation lemma. We generalise the notion of arithmetic degree and prove some basic properties of it in positive characteristic. In particular, we generalise Kawaguchi-Silverman-Matsuzawa's upper bound for arithmetic degree [42, Theorem 1.4] in positive characteristic. With such notion, we apply our observation to dynamical system in positive characteristic. In particular, we prove the Dynamical Mordell-Lang and Zariski dense orbit conjecture in some cases (see Section 1.1 and 1.2).

Another aim of this paper is to study the ergodic theory on algebraic variety w.r.t constructible topology. Using this, we get some equidistribution reults and apply them to get some weak verisons of Dynamical Mordell-Lang, ManinMumford conjecture in arbitrary characteristic. This also gives some applications for equidistributions on Berkovich spaces.
1.1. Dynamical Mordell-Lang conjecture. Let $X$ be a variety over $\mathbf{k}$ and $f: X \rightarrow X$ be a rational self-map.

Definition 1.2. We say $(X, f)$ satisfies the $D M L$ property if for every $x \in X(\mathbf{k})$ whose $f$-orbit is well defined and every subvariety $V$ of $X$, the set $\left\{n \geq 0 \mid f^{n}(x) \in\right.$ $V\}$ is a finite union of arithmetic progressions.

Here an arithmetic progression is a set of the form $\{a n+b \mid n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$ possibly with $a=0$.

Dynamical Mordell-Lang Conjecture. If char $\mathbf{k}=0$, then $(X, f)$ satisfies the DML property.

It was proved when $f$ is unramified [6] and when $f$ is an endomorphism of $\mathbb{A}_{\mathbb{Q}}^{2}$ [52]. See [9, 30] for other known results. In general, this conjecture does not hold in positive characteristic. An example is [9, Example 3.4.5.1] as follows (see [25, 16] for more examples).

Example 1.3. Let $\mathbf{k}=\overline{\mathbb{F}_{p}(t)}, f: \mathbb{A}^{2} \rightarrow \mathbb{A}^{2}$ be the endomorphism defined by $(x, y) \mapsto(t x,(1-t) y)$. Set $V:=\{x-y=0\}$ and $e=(1,1)$. Then $\left\{n \geq 0 \mid f^{n}(e) \in\right.$ $V\}=\left\{p^{n} \mid n \geq 0\right\}$.

In [9, Conjecture 13.2.0.1], Ghioca and Scanlon proposed a variant of the Dynamical Mordell-Lang conjecture in positive characteristic ( $=p$-DML), which asked $\left\{n \geq 0 \mid f^{n}(x) \in V\right\}$ to be a finite union of arithmetic progressions along with finitely many sets taking form

$$
\left\{\sum_{i=1}^{m} c_{i} p^{l_{i} n_{i}} \mid n_{i} \in \mathbb{Z}_{\geq 0}, i=1, \ldots, m\right\}
$$

where $m \in \mathbb{Z}_{>1}, k_{i} \in \mathbb{Z}_{\geq 0}, c_{i} \in \mathbb{Q}$. See [25, 16] for known results of $p$-DML. However, we suspect that for a "general" dynamical system in positive characteristic still has the DML property.

Theorem 1.4. Let $X$ be a projective surface over $\mathbf{k}$. Let $f: X \rightarrow X$ be an automorphism. Assume that $\lambda_{1}(f)>1$. Then the pair $(X, f)$ satisfies the $D M L$ property.

Here $\lambda_{i}(f)$ is the $i$-th dynamical degree of $f$ (see Section 2.1). The following is a similar result for birational endomorphisms of $\mathbb{A}^{2}$. In 50, Theorem A], it is stated in characteristic 0 . But when $\lambda_{1}(f)>1$, its proof works in any characteristic.

Theorem 1.5. [50, Theorem A] Let $f: \mathbb{A}^{2} \rightarrow \mathbb{A}^{2}$ be a birational endomorphism over $\mathbf{k}$. If $\lambda_{1}(f)>1,\left(\mathbb{A}^{2}, f\right)$ satisfies the DML property.
1.2. Zariski dense orbit conjecture. Let $X$ be a variety over $\mathbf{k}$ and $f: X \rightarrow$ $X$ be a dominant rational self-map. Denote by $\mathbf{k}(X)^{f}$ the field of $f$-invariant rational functions on $X$. Let $X_{f}(\mathbf{k})$ is the set of $X(\mathbf{k})$ whose orbit is well-defined. For $x \in X_{f}(\mathbf{k}), O_{f}(x)$ is the orbit of $x$.

Definition 1.6. We say $(X, f)$ satisfies the $S Z D O$ property if there is $x \in X_{f}(\mathbf{k})$ such that $O_{f}(x)$ is Zariski dense in $X$.

We say $(X, f)$ satisfies the $Z D O$ property if either $\mathbf{k}(X)^{f} \neq \mathbf{k}$ or it satisfies SZDO property.

The Zariski dense orbit conjecture was proposed by Medvedev and Scanlon [44, Conjecture 5.10], by Amerik, Bogomolov and Rovinsky [2] and strengthens a conjecture of Zhang [56].

Zariski dense orbit Conjecture. If char $\mathbf{k}=0$, then $(X, f)$ satisfies the ZDO property.

This conjecture was proved for endomorphisms of projective surfaces [37, 55], endomorphisms of $\left(\mathbb{P}^{1}\right)^{N}$ [45, 55] and endomorphisms of $\mathbb{A}^{2}$ [53]. See [3, 1, 2, 23, 4. 5, 29, 26, 28, 7, 36] for other known results.

The original statement of Zariski dense orbit conjecture is not true in characteristic $p>0$. It is completely wrong over $\mathbf{k}=\overline{\mathbb{F}_{p}}$ and has counter-examples even when tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$ (see [55, Section 1.6] and [27, Remark 1.2]). Concerning the variants of the Zariski dense orbit conjecture in positive characteristic proposed in [55, Section 1.6] and [27, Conjecture 1.3], we get the following result.

Proposition 1.7. Let $K$ be an algebraically closed field extension of $\mathbf{k}$ with tr.d. $\mathbf{. k}^{\prime} K \geq \operatorname{dim} X$. Then $\left(f_{K}, X_{K}\right)$ satisfies the $Z D O$ property. Here $X_{K}$ and $f_{K}$ are the base change by $K$ of $X$ and $f$.

The following example shows that the assumption $\operatorname{tr} ._{._{\mathbf{k}}} K \geq \operatorname{dim} X$ is sharp.
Example 1.8. Let $X$ be a variety over $\mathbf{k}:=\overline{\mathbb{F}_{p}}$ of dimension $d \geq 1$. Assume that $X$ is defined over $\mathbb{F}_{p}$. Let $F: X \rightarrow X$ be the Frobenius endomorphism. It is clear that $\overline{\mathbb{F}_{p}}(X)^{F}=\overline{\mathbb{F}_{p}}$. For every algebraically closed field extension $K$ of $\mathbf{k}$ with tr.d.k $K \leq d-1$, and every $x \in X_{K}(K), O_{F_{K}}(x)$ is not Zariski dense in $X_{K}$.

On the other hand, the known counter-examples often involve some Frobenius actions. See [27, Theorem 1.5, Question 1.7] for this phenomenon. We suspect
that when $\operatorname{tr} . \mathrm{d} \cdot \overline{\mathbb{F}_{p}} \mathbf{k} \geq 1$, a "general" dynamical system in positive characteristic still have the ZDO property. Applying arguments using height, we get the following results.
Theorem 1.9. Assume that char $\mathbf{k}=p>0$ and tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$. Let $f: X \rightarrow X$ be a dominant endomorphism of a projective variety. If $\lambda_{1}(f)>1$, then for every nonempty Zariski open subset $U$ of $X$, there is $x \in U(\mathbf{k})$ with infinite orbit and $O_{f}(x) \subseteq U$.

Theorem 1.9 can be viewed as a weak version of [1, Corollary 9] in positive characteristic.

Theorem 1.10. Assume that char $\mathbf{k}=p>0$ and tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$. Let $f: X \rightarrow X$ be an automorphism of a projective surface. Then $(X, f)$ satisfies the $Z D O$ property.

The following result is a generalization of [36, Theorem 1.12 (iii)] in positive characteristic.

Theorem 1.11. Assume that char $\mathbf{k}=p>0$ and tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$. Let $f: X \rightarrow X$ be a dominant endomorphism of a projective variety. Assume that $X$ is smooth of dimension $d \geq 2$, and $\lambda_{1}(f)>\max _{i=2}^{d}\left\{\lambda_{i}(f)\right\}$. Then $(X, f)$ satisfies the $S Z D O$ property.
1.3. Ergodic theory. Let $X$ be a variety over $\mathbf{k}$. Denote by $|X|$ the underling set of $X$ with the constructible topology i.e. the topology on a $X$ generated by the constructible subsets (see [34, Section (1.9) and in particular (1.9.13)]). In particular every constructible subset is open and closed. This topology is finer than the Zariski topology on $X$. Moreover $|X|$ is (Hausdorff) compact.

Denote by $\mathcal{M}(|X|)$ the space of Radon measures on $X$ endowed with the weak-* topology.
Theorem 1.12. Every $\mu \in \mathcal{M}(|X|)$ takes form

$$
\mu=\sum_{i \geq 0} a_{i} \delta_{x_{i}}
$$

where $\delta_{x_{i}}$ is the Dirac measure at $x_{i} \in X, a_{i} \geq 0$.
Remark 1.13. Theorem 1.12 is inspired by [32, Theorem A]. In [32, Theorem A], Gignac worked on the Zariski topology, which is not Hausdorff. Here, we use the constructible topology systematically. We think that the constructible topology is the right topology for studying ergodic theory in algebraic dynamics. For example, using constructible topology, we may avoid the conception of finite signed Borel measure used in [32, Theorem A]. Instead of it, we use the more standard notion of Radon measure.

A sequence $x_{n} \in X, n \geq 0$ is said to be generic, if every subsequence $x_{n_{i}}, i \geq 0$ is Zariski dense in $X$.
Corollary 1.14. A sequence $x_{n} \in X, n \geq 0$ is generic if and only if

$$
\lim _{n \rightarrow \infty} \delta_{x_{n}}=\delta_{\eta}
$$

where $\eta$ is the generic point of $X$.

Let $f: X \rightarrow X$ be a dominant rational self-map. Set $|X|_{f}:=|X| \backslash\left(\cup_{i \geq 1} I\left(f^{i}\right)\right)$. Because every Zariski closed subset of $X$ is open and closed in the constructible topology, $|X|_{f}$ is a closed subset of $|X|$. The restriction of $f$ to $|X|_{f}$ is continuous. We still denote by $f$ this restriction.
1.3.1. DML problems. Applying Corolary 1.14, the dynamical Moredell-Lang conjecture can be interpreted as the following equidistribution statement:

Dynamical Mordell-Lang Conjecture (DML in form of equidistribution). For $x \in X_{f}(\mathbf{k})$, if $O_{f}(x)$ is Zariski dense in $X$, then

$$
\lim _{n \rightarrow \infty} \delta_{f^{n}(x)}=\delta_{\eta}
$$

Remark 1.15. Here the assumption that $O_{f}(x)$ is Zariski dense in $X$ does not cause any problem. Because after replacing $x$ by some $f^{m}(x)$ and $f$ by a suitable iterate, we may assume that $\overline{O_{f}(x)}$ is irreducible. Then after replacing $X$ by $\overline{O_{f}(x)}$, we may assume that $O_{f}(x)$ is Zariski dense in $X$.

Using Theorem 1.12, we give a fast proof of the weak dynamical Mordell-Lang. Same result was proved in [8, Corollary 1.5] (see also [24, Theorem 2.5.8], [32, Theorem D, Theorem E], [46, Theorem 2], [10, Theorem 1.10]).

Theorem 1.16 (Weak DML). Let $x$ be a points $\in X_{f}(\mathbf{k})$ with $\overline{O_{f}(x)}=X$. Let $V$ be a proper subvariety of $X$. Then $\left\{n \geq 0 \mid f^{n}(x) \in V\right\}$ is of Banach density zero in $\mathbb{Z}_{\geq 0}$ i.e. for every sequence of intervals $I_{n}, n \geq 0$ in $\mathbb{Z}_{\geq 0}$ with $\lim _{n \rightarrow \infty} \# I_{n}=+\infty$, we have

$$
\lim _{n \rightarrow \infty} \frac{\#\left(\left\{n \geq 0 \mid f^{n}(x) \in V\right\} \cap I_{n}\right)}{\# I_{n}}=0
$$

We also prove the weak dynamical Mordell-Lang for coherent backward orbits. A slightly weaker version was proved in [32, Theorem F]. This can be viewed as a weak version of [54, Conjecture 1.5].

Theorem 1.17 (Weak DML for coherent backward orbits). Let $x_{n} \in X_{f}(\mathbf{k}), n \leq$ 0 be a sequence of points such that $\overline{\left\{x_{n}, n \leq 0\right\}}=X$ and $f\left(x_{n}\right)=x_{n+1}$ for all $n \leq-1$. Let $V$ be a proper subvariety of $X$. Then $\left\{n \leq 0 \mid x_{n} \in V\right\}$ is of Banach density zero in $\mathbb{Z}_{\leq 0}$
1.3.2. Backward orbits. Now assume that $f: X \rightarrow X$ is a flat and finite endomorphism. Let $d_{f}:=\left[\mathbf{k}(X) / f^{*} \mathbf{k}(X)\right]$ be topological degree of $f$. It is just the $(\operatorname{dim} X)$-th dynamical degree of $f$.

Recall that for every $x \in X$, the multiplicity of $f$ at $x$ is

$$
m_{f}(x):=\operatorname{dim}_{\kappa(f(x))}\left(O_{X, x} / m_{f(x)} O_{X, x}\right) \in \mathbb{Z}_{\geq 1}
$$

where $O_{X, x}$ is viewed as an $O_{X, f(x)}$-module via $f$. For every $x \in X$, we have $\sum_{y \in f^{-1}(x)} m_{f}(y)=d_{f}$ (see [31, Theorem 2.4]).

In Section 5.2, we define a natural pullback $f^{*}: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ which is continuous and for every $x \in X$,

$$
f^{*} \delta_{x}=\sum_{y \in f^{-1}(x)} m_{f}(y) \delta_{y} .
$$

We get the following equidistribution result.
Theorem 1.18. Let $f: X \rightarrow X$ be a flat and finite endomorphism. Let $x \in X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$. Then for every sequence of intervals $I_{n}, n \geq 0$ in $\mathbb{Z}_{\geq 0}$ with $\lim _{n \rightarrow \infty} \# I_{n}=+\infty$, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{\# I_{n}}\left(\sum_{i \in I_{n}} d_{f}^{-i}\left(f^{i}\right)^{*} \delta_{x}\right)=\delta_{\eta}
$$

Remark 1.19. The assumption $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$ is necessary. Otherwise,

$$
\frac{1}{\# I_{n}}\left(\sum_{i \in I_{n}} d_{f}^{-i}\left(f^{i}\right)^{*} \delta_{x}\right), n \geq 0
$$

are supported on the proper closed subset $\overline{\bigcup_{i \geq 0} f^{-i}(x)}$ of $X$.
Applying Theorem 1.18, we count the preimages of a point without multiplicities.

Theorem 1.20. Let $f: X \rightarrow X$ be a flat and finite endomorphism. Assume that the field extension $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable. Let $x \in X(\mathbf{k})$ be a point with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$. For $c \in(0,1], n \geq 0$, define

$$
S_{c}^{n}:=\min \left\{\# S \mid S \subseteq f^{-n}(x), \sum_{y \in S} m_{f^{n}}(y) \geq c d_{f}^{n}\right\}
$$

Then for every $c \in(0,1]$, we have

$$
\lim _{n \rightarrow \infty}\left(S_{c}^{n}\right)^{1 / n}=d_{f}
$$

Taking $c=1$ in Theorem 1.20, we get the following corollary.
Corollary 1.21. Let $f: X \rightarrow X$ be a flat and finite endomorphism. If the field extension $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable, then for every $x \in X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=$ X,

$$
\lim _{n \rightarrow \infty}\left(\# f^{-n}(x)\right)^{1 / n}=d_{f}
$$

If the topological degree is large, we have the following stronger equidistribution result.
Theorem 1.22. Let $f: X \rightarrow X$ be a flat and finite endomorphism of a quasiprojective variety. Assume that

$$
\begin{equation*}
d_{f}:=\lambda_{\operatorname{dim} X}(f)>\max _{1 \leq i \leq \operatorname{dim} X-1} \lambda_{i} \tag{1.1}
\end{equation*}
$$

If the field extension $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable, then for every $x \in X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$,

$$
\lim _{n \rightarrow \infty} d_{f}^{-n}\left(f^{n}\right)^{*} \delta_{x}=\delta_{\eta}
$$

Moreover, for every irreducible subvariety $V$ of $X$ of dimension $d_{V} \leq \operatorname{dim} X-1$,

$$
\limsup _{n \rightarrow \infty} \#\left(f^{-n}(x) \cap V\right)^{1 / n} \leq \lambda_{d_{V}}<d_{f}
$$

Assumption 5.5 holds for polarized endomorphisms on projective varieties. A similar statement for polarized endomorphisms can be fund in [31, Theorem 5.1]. See [35, 20] for according result for complex topology.

Theorem 1.22 is not true without Assumption 5.5.
Example 1.23. Under the notation of Example 1.3, Set $g:=f^{-1}$. Then $\lambda_{i}(g)=$ $1, i=0,1,2$. Denote by $1_{V}$ the characteristic function of $V$. Since $V$ is open and closed in $\left|\mathbb{A}^{2}\right|, 1_{V}$ is continuous. We have

$$
\lim _{n \rightarrow \infty} \int 1_{V}\left(g^{-p^{n}}\right)^{*} \delta_{e}=\lim _{n \rightarrow \infty} 1_{V}\left(f^{p^{n}}(e)\right)=1 \neq 0=\int 1_{V} \delta_{\eta}
$$

1.4. Relation to Berkovich spaces. We will see in Section 5.4, $|X|$ can be viewed as a closed subset of the Berkovich analytification $X^{\text {an }}$ of $X$ w.r.t the trivial norm on $\mathbf{k}$. So the statements in ergodic theory on $|X|$ can be translated to statements on $X^{\text {an }}$. See the translation of Corollary 1.14 and Theorem 1.22 in Section 5.4 .

Using reduction map, we may also use ergodic theory w.r.t. the constructible topology to study endomorphisms on Berkovich spaces with good reduction. In Section 5.6, we apply Theorem 1.22 to get an equidistribution result for endomorphisms of large topological degree with good reduction.

### 1.5. Notation and Terminology.

- For a set $S$, denote by $\# S$ the cardinality of $S$.
- A variety is an irreducible separated scheme of finite type over a field. A subvariety of a variety $X$ is a closed subset of $X$.
- For a variety $X$ (resp. a rational self-map $f: X \rightarrow Y$ ) over a field $k$ and a subfield $K$ of $k$, we say that $X$ (resp. $f$ ) is defined over $K$ if there is a variety $X_{K}$ (resp. a rational map $f_{K}$ ) over $K$ such that $X$ (resp. $f$ ) is the base change by $k$ of $X$ (resp. $f$ ).
- For a rational map $f: X \rightarrow Y$ between varieties. Denote by $I(f)$ the indeterminacy locus of $f$.
- For a dominant rational self-map $f: X \rightarrow X$ between varieties, a subvariety $V$ of $X$ is said to be $f$-invariant if $I(f)$ does not contain any irreducible component of $V$ and $f(V) \subseteq V$.
- For a projective variety $X, N^{i}(X)$ is the the group of numerical $i$-cycles of $X$ and $N^{i}(X)_{\mathbb{R}}:=N^{i}(X) \otimes \mathbb{R}$.
- For two Cartier $\mathbb{R}$-divisors $D_{1}, D_{2}$, write $D_{1} \equiv D_{2}$ if $D_{1}, D_{2}$ are numerically equivalent.
- For a field extension $k / K$, tr.d. ${ }_{K} k$ is the transcendence degree of $k / K$.

Acknowledgement. I would like to thank Xinyi Yuan. Section 5 of this paper is motivated by some interesting discussion with him.

## 2. Dynamical Degree and arithmetic Degree

2.1. The dynamical degrees. In this section we recall the definition and some basic facts on the dynamical degree.

Let $X$ be a variety over $\mathbf{k}$ and $f: X \rightarrow X$ a dominant rational self-map. Let $X^{\prime}$ be a normal projective variety which is birational to $X$. Let $L$ be an ample (or just nef and big) divisor on $X^{\prime}$. Denote by $f^{\prime}$ the rational self-map of $X^{\prime}$ induced by $f$.

For $i=0,1, \ldots, \operatorname{dim} X$, and $n \geq 0,\left(f^{\prime n}\right)^{*}\left(L^{i}\right)$ is the $(\operatorname{dim} X-i)$-cycle on $X^{\prime}$ as follows: let $\Gamma$ be a normal projective variety with a birational morphism $\pi_{1}: \Gamma \rightarrow X^{\prime}$ and a morphism $\pi_{2}: \Gamma \rightarrow X^{\prime}$ such that $f^{\prime n}=\pi_{2} \circ \pi_{1}^{-1}$. Then $\left(f^{\prime n}\right)^{*}\left(L^{i}\right):=\left(\pi_{1}\right)_{*} \pi_{2}^{*}\left(L^{i}\right)$. The definition of $\left(f^{\prime n}\right)^{*}\left(L^{i}\right)$ does not depend on the choice of $\Gamma, \pi_{1}$ and $\pi_{2}$. The $i$-th dynamical degree of $f$ is

$$
\lambda_{i}(f):=\lim _{n \rightarrow \infty}\left(\left(f^{\prime n}\right)^{*}\left(L^{i}\right) \cdot L^{\operatorname{dim} X-i}\right)^{1 / n}
$$

The limit converges and does not depend on the choice of $X^{\prime}$ and $L$ [48, 21, 49, 18]. Moreover, if $\pi: X \rightarrow Y$ is a generically finite and dominant rational map between varieties and $g: Y \rightarrow Y$ is a rational self-map such that $g \circ \pi=\pi \circ f$, then $\lambda_{i}(f)=\lambda_{i}(g)$ for all $i$; for details, we refer to [18, Theorem 1] (and the projection formula), or Theorem 4 in its arXiv version [17].

The following result is easy when $\mathbf{k}$ is of characteristic 0 and $Z \nsubseteq \operatorname{Sing} X$.
Proposition 2.1. [36, Proposition 3.2] Let $X$ be a variety over $\mathbf{k}$ and $f: X \rightarrow X$ a dominant rational self-map. Let $Z$ be an irreducible subvariety in $X$ which is not contained in $I(f)$ such that $\left.f\right|_{Z}$ induces a dominant rational self-map of $Z$. Then $\lambda_{i}\left(\left.f\right|_{Z}\right) \leq \lambda_{i}(f)$ for $i=0,1, \ldots, \operatorname{dim} Z$.
2.2. Arithmetic degree. The arithmetic degree was defined in [38] over a number field or a function field of characteristic zero. In this section we extend this definition to the case over function field of positive characteristic and we prove some basic fact of it.

Let $\mathbf{k}=\overline{K(B)}$, where $K$ is an algebraically closed field and $B$ is a smooth projective curve.
2.2.1. Weil height. Let $X$ be a normal and projective variety over k. For every $L \in \operatorname{Pic}(X)$, we denote by $h_{L}: X(\mathbf{k}) \rightarrow \mathbb{R}$ a Weil height associated to $L$ and the function field $K(B)$. It is unique up to adding a bounded function.

Example 2.2. Assume that $X$ is defined over $K(B)$ i.e. there is a projective morphism $\pi: X_{B} \rightarrow B$ where $X_{B}$ is normal, projective and geometric generic fiber of $\pi$ is $X$. Assume that there is a line bundle $L_{B}$ on $X_{B}$ whose restriction on $X$ is $L$. In this case, for every $x \in X(\mathbf{k})$, we may take $h_{L}$ to be

$$
h_{\left(X_{B}, L_{B}\right)}(x)=[K(B)(x): K(B)]^{-1}(\bar{x} \cdot L),
$$

where $\bar{x}$ is the Zariski closure of $x$ in $X_{B}$.
Keep the notations in Example 2.2. Let $b$ be a point in $B(K)$. It induces a norm $|\cdot|_{b}$ on $K(B)$. Denote by $K(B)_{b}$ the completion of $K(B)$ w.r.t. $|\cdot|_{b}$. Denote by $\mathbb{C}_{b}$ the completion of $\overline{K(B)_{b}}$. Every field embedding $\tau: \mathbf{k}=\overline{K(B)} \hookrightarrow \mathbb{C}_{b}$ induces
an embedding $\phi_{\tau}: X(\mathbf{k}) \hookrightarrow X\left(\mathbb{C}_{b}\right)$. On $X\left(\mathbb{C}_{b}\right)$, we have a natural $b$-adic topology induced by $|\cdot|_{b}$.
Remark 2.3. Let $x_{b}$ be a point in $X_{b}$. Then $x_{b}$ defines a nonempty open subset $U_{x_{b}}$ consisting of all points in $X\left(\mathbb{C}_{b}\right)$ whose reduction is $x_{b} \in X_{b}(K)$. Then for every $x \in \phi_{\tau}^{-1}\left(U_{x_{b}}\right), x_{0}$ is contained in the Zariski closure of $x$ in $X_{B}$.

Lemma 2.4. There is $d \geq 1$ such that for every $b \in B(K)$, every nonempty $b$-adic open subset of $U \subseteq X\left(\mathbb{C}_{b}\right)$, and every $l \geq 1$, there is $x \in X(\mathbf{k})$ such that $\operatorname{deg}(x) \leq d$ and $h_{L}(x) \geq l$.

Proof. By Noether normalization lemma, we only need to prove the lemma when $X=\mathbb{P}^{N}$ and $L=O(1)$. After replace $K(B)$ by a finite extension, a changing of coordinates, we may assume that $0 \in U$. We may assume that $h_{L}$ is the naive height on $\mathbb{P}^{N}$ i.e. the height defined by the model $\left(\mathbb{P}_{B}^{N}, O_{\mathbb{P}^{N}(B)}(1)\right)$. Pick any rational function $g \in K(B) \backslash\{0\}$ with $g(b)=0$. Then for $n \geq 1, x_{n}:=$ $\left(g^{n}, \ldots, g^{n}\right) \in \mathbb{A}^{N}(K(B))$. We have $h_{L}\left(x_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$ and $\phi_{\tau}\left(x_{n}\right) \rightarrow 0$ in the $b$-adic topology. This concludes the proof.
2.2.2. Admissible triples. As in [36], we define an admissible triple to be $(X, f, x)$ where $X$ is a quasi-projective variety over $\mathbf{k}, f: X \rightarrow X$ is a dominant rational self-map and $x \in X_{f}(\mathbf{k})$.

We say that $(X, f, x)$ dominates (resp. generically finitely dominates) $(Y, g, y)$ if there is a dominant rational map (resp. generically finite and dominant rational map) $\pi: X \longrightarrow Y$ such $\pi \circ f=g \circ \pi, \pi$ is well defined along $O_{f}(x)$ and $\pi(x)=y$.

We say that $(X, f, x)$ is birational to $(Y, g, y)$ if there is a birational map $\pi: X \longrightarrow Y$ such $\pi \circ f=g \circ \pi$ and if there is a Zariski dense open subset $V$ of $Y$ containing $O_{g}(y)$ such that $\left.\pi\right|_{U}: U:=\pi^{-1}(V) \rightarrow V$ is a well-defined isomorphism and $\pi(x)=y$. In particular, if $(X, f, x)$ is birational to $(Y, g, y)$, then $(X, f, x)$ generically finitely dominates $(Y, g, y)$.

## Remark 2.5.

(1) If $(X, f, x)$ dominates $(Y, g, y)$ and if $O_{f}(x)$ is Zariski dense in $X$, then $O_{g}(y)$ is Zariski dense in $Y$. Moreover, if $(X, f, x)$ generically finitely dominates $(Y, g, y)$, then $O_{f}(x)$ is Zariski dense in $X$ if and only if $O_{g}(y)$ is Zariski dense in $Y$.
(2) Every admissible triple $(X, f, x)$ is birational to an admissible triple $\left(X^{\prime}, f^{\prime}, x^{\prime}\right)$ where $X^{\prime}$ is projective. Indeed, we may pick $X^{\prime}$ to be any projective compactification of $X, f^{\prime}$ the self-map of $X^{\prime}$ induced from $f$, and $x^{\prime}=x$.
2.2.3. The set $A_{f}(x)$. As in [36], we will associate to an admissible triple $(X, f, x)$ a subset

$$
A_{f}(x) \subseteq[1, \infty]
$$

Remark 2.6. We will show in Proposition 2.10 that $A_{f}(x) \subseteq\left[1, \lambda_{1}(f)\right]$.
We first define it when $X$ is projective. Let $L$ be an ample divisor on $X$, we define

$$
A_{f}(x) \subseteq[1, \infty]
$$

to be the limit set of the sequence $\left(h_{L}^{+}\left(f^{n}(x)\right)\right)^{1 / n}, n \geq 0$, where $h_{L}^{+}(\cdot):=$ $\max \left\{h_{L}(\cdot), 1\right\}$.

The following lemma was proved in [36, Lemma 3.8] when $\mathbf{k}=\overline{\mathbb{Q}}$, but its proof still works our case. It shows that the set $A_{f}(x)$ does not depend on the choice of $L$ and is invariant in the birational equivalence class of $(X, f, x)$.

Lemma 2.7. [36, Lemma 3.8] Let $\pi: X \rightarrow Y$ be a dominant rational map between projective varieties. Let $U$ be a Zariski dense open subset of $X$ such that $\left.\pi\right|_{U}: U \rightarrow Y$ is well-defined. Let $L$ be an ample divisor on $X$ and $M$ an ample divisor on $Y$. Then there are constants $C \geq 1$ and $D>0$ such that for every $x \in U$, we have

$$
\begin{equation*}
h_{M}(\pi(x)) \leq C h_{L}(x)+D . \tag{2.1}
\end{equation*}
$$

Moreover if $V:=\pi(U)$ is open in $Y$ and $\left.\pi\right|_{U}: U \rightarrow V$ is an isomorphism, then there are constants $C \geq 1$ and $D>0$ such that for every $x \in U$, we have

$$
\begin{equation*}
C^{-1} h_{L}(x)-D \leq h_{M}(\pi(x)) \leq C h_{L}(x)+D \tag{2.2}
\end{equation*}
$$

Now for every admissible triple $(X, f, x)$, we define $A_{f}(x)$ to be $A_{f^{\prime}}\left(x^{\prime}\right)$ where ( $X^{\prime}, f^{\prime}, x^{\prime}$ ) is an admissible triple which is birational to $(X, f, x)$ such that $X^{\prime}$ is projective. By Lemma [2.7, this definition does not depend on the choice of $\left(X^{\prime}, f^{\prime}, x^{\prime}\right)$.
2.2.4. The arithmetic degree. We define (see also [38]):

$$
\bar{\alpha}_{f}(x):=\sup A_{f}(x), \quad \underline{\alpha}_{f}(x):=\inf A_{f}(x) .
$$

We say that $\alpha_{f}(x)$ is well-defined and call it the arithmetic degree of $f$ at $x$, if $\bar{\alpha}_{f}(x)=\underline{\alpha}_{f}(x)$; and, in this case, we set

$$
\alpha_{f}(x):=\bar{\alpha}_{f}(x)=\underline{\alpha}_{f}(x) .
$$

By Lemma [2.7, if $(X, f, x)$ dominates $(Y, g, y)$, then $\bar{\alpha}_{f}(x) \geq \bar{\alpha}_{g}(y)$ and $\underline{\alpha}_{f}(x) \geq$ $\underline{\alpha}_{g}(y)$.

Applying Inequality (2.1) of Lemma 2.7 to the case where $Y=X$ and $M=L$, we get the following trivial upper bound: let $f: X \rightarrow X$ be a dominant rational self-map, $L$ any ample line bundle on $X$ and $h_{L}$ a Weil height function associated to $L$; then there is a constant $C \geq 1$ such that for every $x \in X \backslash I(f)$, we have

$$
\begin{equation*}
h_{L}^{+}(f(x)) \leq C h_{L}^{+}(x) \tag{2.3}
\end{equation*}
$$

For a subset $A \subseteq[1, \infty)$, define $A^{1 / \ell}:=\left\{a^{1 / \ell} \mid a \in A\right\}$.
We have the following simple properties, where the second half of 3 used Inequality (2.3).
Proposition 2.8. We have:
(1) $A_{f}(x) \subseteq[1, \infty)$.
(2) $A_{f}(x)=A_{f}\left(f^{\ell}(x)\right)$, for any $\ell \geq 0$.
(3) $A_{f}(x)=\bigcup_{i=0}^{\ell-1}\left(A_{f^{\ell}}\left(f^{i}(x)\right)\right)^{1 / \ell}$. In particular, $\bar{\alpha}_{f^{\ell}}(x)=\bar{\alpha}_{f}(x)^{\ell}, \underline{\alpha}_{f^{\ell}}(x)=$ $\underline{\alpha}_{f}(x)^{\ell}$.
The following lemma is easy.

Lemma 2.9. Let $f: X \rightarrow X$ be a dominant rational self-map of a projective variety $X$ and $W \subseteq X$ an $f$-invariant subvariety. Then $X_{f}(\mathbf{k}) \cap W(\mathbf{k}) \subseteq W_{f \mid W}(\mathbf{k})$ and for every $x \in X_{f}(\mathbf{k}) \cap W(\mathbf{k}), \alpha_{f \mid W}(x)=\alpha_{f}(x)$.

When $\mathbf{k}=\overline{\mathbb{Q}}$, the next result was proved in [42, Theorem 1.4] in the smooth case and in [36, Proposition 3.11] in the singular case. The proof here in the function field case is much easier.

Proposition 2.10 (Kawaguchi-Silverman-Matsuzawa's upper bound). For every admissible triple $\left(X, f, x_{0}\right)$, we have $\bar{\alpha}_{f}\left(x_{0}\right) \leq \lambda_{1}(f)$.
Proof. We may assume that $X$ is projective. Set $d:=\operatorname{dim} X$. After replacing $f$ by a suitable iteration and $x_{0}$ by $f^{n}\left(x_{0}\right)$ for some $n \geq 0$ and noting that $\lambda_{1}\left(f^{n}\right)=\lambda_{1}(f)^{n}$ and by Proposition 2.8, we may assume that the Zariski closure $Z_{f}\left(x_{0}\right)$ of $O_{f}\left(x_{0}\right)$ is irreducible. By Proposition 2.1 and Lemma 2.9, we may replace $X$ by $Z_{f}\left(x_{0}\right)$ and assume that $O_{f}\left(x_{0}\right)$ is Zariski dense in $X$.

Assume that $X$ is defined over $K(B)$ i.e. there is a projective morphism $\pi$ : $\mathcal{X} \rightarrow B$ where $\mathcal{X}$ is projective, normal and geometric generic fiber of $\pi$ is $X$. Pick an ample line bundle $L_{B}$ on $\mathcal{X}$ and let $L$ be its restriction to $X$. We take the Weil height $h_{L}: X(\mathbf{k}) \rightarrow \mathbb{R}$ as follows: for every $x \in X(\mathbf{k})$,

$$
h_{L}(x):=h_{\left(\mathcal{X}, L_{B}\right)}(x)=[K(B)(x): K(B)]^{-1}(\bar{x} \cdot \mathcal{L}) .
$$

We may assume that $x_{0}$ is defined over $K(B)$.
Let $F: \mathcal{X} \rightarrow \mathcal{X}$ be the rational self-map over $B$ induced by $f$. The relative dynamical degree formula [17, Theorem 4], shows that

$$
\lambda_{1}(F)=\max \left\{1, \lambda_{1}(f)\right\}=\lambda_{1}(f)
$$

So for every $r>0$, there is $C_{r}>0$ such that for every $n \geq 0$,

$$
\begin{equation*}
\left(\left(F^{n}\right)^{*} L_{B} \cdot L_{B}^{d}\right) \leq C_{r}\left(\lambda_{1}(f)+r\right)^{n} \tag{2.4}
\end{equation*}
$$

Let $\mathcal{I}$ be the ideal sheaf of $\overline{x_{0}}$ on $\mathcal{X}$. After replacing $L_{B}$ be a suitable multiple, we may assume that $\mathcal{L} \otimes \mathcal{I}$ is globally generated. For every $n \geq 0$, there are divisors $H_{i}, i=0, \ldots, d$ in $\left|L_{B}\right|$ such that $\operatorname{dim} H_{1} \cap \cdots \cap H_{d}=1$ and containing $\overline{x_{0}}$ as an irreducible component.

Set $V_{n}:=H_{1} \cdots \cdots H_{d}$. Let $\Gamma$ be a normal projective variety with a birational morphism $\pi_{1}: \Gamma \rightarrow \mathcal{X}$ and a morphism $\pi_{2}: \Gamma \rightarrow \mathcal{X}$ such that $F^{n}=\pi_{2} \circ \pi_{1}^{-1}$. Write $\left(\pi_{1}\right)^{\#} \overline{x_{0}}$ the strict transform of $V^{n} \overline{x_{0}}$ by $\pi_{1}^{N}$. Then $\left(\pi_{1}\right)^{\#} \overline{x_{0}}$ is an irreducible component of $\cap_{i=1}^{d}\left(\pi_{1}^{*} H_{i}\right)$. In $N^{1}(\Gamma)$, we have $\pi_{1}^{*} V_{n}=\pi_{1}^{*} H_{1} \cdots \cdot \pi^{*} H_{d}$. By [36, Lemma 3.3], $\pi_{1}^{*} V_{n}-\left(\pi_{1}\right)^{\#} \overline{x_{0}}$ is pseudo-effective. Then we have

$$
\begin{gathered}
h_{L}\left(f^{n}\left(x_{0}\right)\right)=\left(\overline{f^{n}\left(x_{0}\right)} \cdot L_{B}\right)=\left(\left(\pi_{1}\right)^{\#} \overline{x_{0}} \cdot \pi_{2}^{*} L_{B}\right) \\
\leq\left(\pi_{1}^{*} H_{1} \cdots \cdots \pi_{1}^{*} H_{d} \cdot \pi_{2}^{*} L_{B}\right)=\left(\left(F^{n}\right)^{*} L_{B} \cdot L_{B}^{d}\right) . \\
\leq C_{r}\left(\lambda_{1}(f)+r\right)^{n} .
\end{gathered}
$$

It follows that

$$
\bar{\alpha}_{f}\left(x_{0}\right)=\limsup _{n \rightarrow \infty} h_{L}\left(f^{n}\left(x_{0}\right)\right)^{1 / n} \leq \lim _{n \rightarrow \infty}\left(C_{r}\left(\lambda_{1}(f)+r\right)^{n}\right)^{1 / n}=\lambda_{1}(f)+r .
$$

Letting $r \rightarrow \infty$, we conclude the proof.
2.3. Canonical height. Let $X$ be a normal projective variety and $f: X \rightarrow X$ a surjective endomorphism.

Let $A$ be an ample divisor of $X$, denote by $h_{A}$ a Weil height on $X(\mathbf{k})$ associated to $A$ with $h_{A} \geq 1$.

Proposition 2.11. Let $D$ be a nonzero Cartier $\mathbb{R}$-divisor such that $f^{*} D \equiv \beta D$ where $\beta>\lambda_{1}(f)^{1 / 2}$. Let $[D] \in N^{1}(X)_{\mathbb{R}}$ be the numerical class of $D$. Then for every $x \in X(\mathbf{k})$, the limit $h_{[D]}^{+}(x):=\lim _{n \rightarrow \infty} h_{D}\left(f^{n}(x)\right) / \beta^{n}$ exist, only depend on the numerical class $[D]$ and satisfies the following properties:
(i) $h_{[D]}^{+}=h_{D}+O\left(h_{A}^{1 / 2}\right)$;
(ii) $h_{[D]}^{+} \circ f=\beta h^{+}$.

Proof. This result was proved in [38, Theorem 5] in characteristic zero. The proof presented here is the same as [38, Theorem 5], but slightly shorter.

By [42, Proposition B.3], there is $C>0$ such that for every $x \in X(\mathbf{k})$,

$$
\left|h_{D}(f(x))-\beta h_{D}(x)\right| \leq C h_{A}(x)^{1 / 2} .
$$

Pick $\mu \in\left(\lambda_{1}(f)^{1 / 2}, \beta\right)$, by Proposition [2.10, for every $x \in X(\mathbf{k})$, there is $C_{x}>0$ such that,

$$
h_{A}\left(f^{n}(x)\right) \leq C_{x} \mu^{2 n} h_{A}(x) .
$$

Then we have

$$
\begin{aligned}
& \left|h_{D}\left(f^{n}(x)\right) / \beta^{n}-h_{D}\left(f^{n-1}(x)\right) / \beta^{n-1}\right|=\beta^{-n}\left|h_{D}\left(f^{n}(x)\right)-\beta h_{D}\left(f^{n-1}(x)\right)\right| \\
& \leq \beta^{-n} C h_{A}\left(f^{n-1}(x)\right)^{1 / 2} \leq \beta^{-n} C C_{x}^{1 / 2} \mu^{n} h_{A}(x)^{1 / 2}=C C_{x}^{1 / 2}(\mu / \beta)^{n} h_{A}(x)^{1 / 2}
\end{aligned}
$$

Since $0<\mu / \beta<1$,

$$
h_{[D]}^{+}(x)=h_{D}(x)+\sum_{n \geq 1}\left(h_{D}\left(f^{n}(x)\right) / \beta^{n}-h_{D}\left(f^{n-1}(x)\right) / \beta^{n-1}\right)
$$

converges and

$$
\begin{aligned}
\mid h_{[D]}^{+}(x) & -h_{D}(x)\left|\leq \sum_{n \geq 1}\right| h_{D}\left(f^{n}(x)\right) / \beta^{n}-h_{D}\left(f^{n-1}(x)\right) / \beta^{n-1} \mid \\
& \leq\left(\sum_{n \geq 1} C C_{x}^{1 / 2}(\mu / \beta)^{n}\right) h_{A}(x)^{1 / 2}=O\left(h_{A}(x)^{1 / 2}\right)
\end{aligned}
$$

Then we get (i). The statement (ii) follows from the definition.
For $D^{\prime} \equiv D$, by [42, Proposition B.3], there is $B>0$ such that for every $x \in X(\mathbf{k})$,

$$
\left|h_{D^{\prime}}(x)-h_{D}(x)\right| \leq B h_{A}(x)^{1 / 2} .
$$

Then

$$
\begin{gathered}
\left|h_{\left[D^{\prime}\right]}^{+}(x)-h_{[D]}^{+}(x)\right|:=\lim _{n \rightarrow \infty}\left|h_{D^{\prime}}\left(f^{n}(x)\right)-h_{D}\left(f^{n}(x)\right)\right| / \beta^{n} \\
\leq \limsup _{n \rightarrow \infty} B h_{A}\left(f^{n}(x)\right)^{1 / 2} / \beta^{n} \leq \limsup _{n \rightarrow \infty} B C_{x} h_{A}(x)^{1 / 2}(\mu / \beta)^{n}=0,
\end{gathered}
$$

which concludes the proof.

The following was proved in [43, Lemma 9.1] when $\mathbf{k}=\overline{\mathbb{Q}}$ and $X$ is smooth. After replacing [38, Theorem 5] by Proposition [2.11, [43, Lemma 9.1] is still valid when $\mathbf{k}=K(B)$ and $X$ is singular.

Proposition 2.12. Assume that $\lambda_{1}(f)>1$. Let $D \not \equiv 0$ be a nef $\mathbb{R}$-Cartier divisor on $X$ such that $f^{*} D \equiv \lambda_{1}(f) D$. Let $V \subseteq X$ be a subvariety of positive dimension such that $\left(D^{\operatorname{dim} V} \cdot V\right)>0$. Then there exists a nonempty open subset $U \subseteq V$ and a set $S \subseteq U(\mathbf{k})$ of bounded height such that for every $x \in U(\mathbf{k}) \backslash S$ we have $\alpha_{f}(x)=\lambda_{1}(f)$.

Corollary 2.13. Keep the notation in Proposition 2.12. For every Zariski dense open subset $U$ of $X$, there is $x \in U(\mathbf{k})$ such that $\alpha_{f}(x)=\lambda_{1}(f)$ and $O_{f}(x) \subseteq U$.

Proof of Corollary 2.13. We may assume that $X$ is normal and $X, f, D$ and $U$ are defined over $K(B)$. There is a normal and projective $B$-scheme $\pi: X_{B} \rightarrow B$ and a rational self-map $f_{B}: X_{B} \rightarrow X_{B}$ over $B$ such that the geometric generic fiber of $\left(X_{B}, f_{B}\right)$ is $(X, f)$. Let $b$ be a general point of $B(K)$ and denote by $\left(X_{b}, f_{b}\right)$ the fiber of $\left(X_{B}, f_{B}\right)$ above $b$. Then $f_{b}$ is an endomorphism of $X_{b}$. Set $Z:=X \backslash U$. Let $Z_{B}$ be the Zariski closure of $Z$ in $X_{B}$. Then $U_{b}:=X_{b} \backslash Z_{B}$. By Proposition 4.1 (see Section 4.1 for its proof), there is $x_{b} \in\left(U_{b}\right)_{f_{b} \mid U_{b}}(K)$. Let $M$ be a very ample line bundle on $X_{B}$. Taking $W_{B}$ to be the intersection of $\operatorname{dim} X-1$ general elements of $|10 M|$ of $X_{B}$ passing through $x_{b}$. By [11, Theorem $0.4], W_{B}$ is irreducible. Let $W \subseteq X$ be the generic fiber of $W_{B}$. It is of pure dimension 1. Then $(W \cap D)>0$. Because $W_{B}$ is irreducible, for every irreducible component $W^{\prime}$ of $W,\left(W^{\prime} \cdot D\right)>0$. By Lemma 2.4 and Remark [2.3, there are $x_{n} \in W^{\prime}(\mathbf{k}), n \geq 0$ such that $x_{b} \in \overline{\left\{x_{n}\right\}}$ and the height of $x_{n}$ tends to $+\infty$. Because $O_{f_{b}}\left(x_{b}\right) \subseteq U, O_{f}\left(x_{n}\right) \subseteq U$ for all $n \geq 0$. By Proposition 2.12, for $n \gg 0$, we have $x_{n} \in V(\mathbf{k}) \cap U$ and $\alpha_{f}\left(x_{n}\right)=\lambda_{1}(f)$.

## 3. Proof of Theorem 1.4

This proof mixes the ideas from 50 and 40 .
3.1. Reduce to the smooth case. By 41, there is a minimal desingularization $\pi: X^{\prime} \rightarrow X$. Then one may lift $f$ to an automorphism $f^{\prime}$ of $X^{\prime}$. The following lemma allows us to replace $(X, f)$ by $\left(X^{\prime}, f^{\prime}\right)$ and assume that $X$ is smooth.

Lemma 3.1. If $\left(X^{\prime}, f^{\prime}\right)$ satisfies the DML property, then $(X, f)$ satisfies the $D M L$ property.

Proof. Assume that $\left(X^{\prime}, f^{\prime}\right)$ satisfies the DML property. We only need to prove the following statement: for every $x \in X(\mathbf{k})$ and an irreducible curve $C \subseteq X(\mathbf{k})$, if $O_{f}(x) \cap C$ is infinite, then $C$ is $f$-periodic.

Pick $x^{\prime} \in \pi^{-1}(x)(\mathbf{k})$. There is an irreducible component $C^{\prime}$ of $\pi^{-1}(C)$ such that $O_{f^{\prime}}\left(x^{\prime}\right) \cap C^{\prime}$ is infinite. We have $\operatorname{dim} C^{\prime} \leq 1$. If $\pi\left(C^{\prime}\right) \neq C$, then $\pi\left(C^{\prime}\right)$ is a point. Then $x=\pi\left(x^{\prime}\right)$ is periodic. So $\pi\left(C^{\prime}\right)=C$ and $\operatorname{dim} C^{\prime}=1$. Since $\left(X^{\prime}, f^{\prime}\right)$ satisfies the DML property, $C^{\prime}$ is $f^{\prime}$-periodic. So $C=\pi\left(C^{\prime}\right)$ is $f^{\prime}$-periodic.
3.2. Numerical geometry. Set $\lambda:=\lambda_{1}(f)>1$. There is a nef class $\theta^{*} \in$ $N^{1}(X)_{\mathbb{R}} \backslash\{0\}$ such that $f^{*} \theta^{*}=\lambda \theta^{*}$. By projection formula $\lambda_{1}\left(f^{-1}\right)=\lambda$. So there is a nef class $\theta^{*} \in N^{1}(X)_{\mathbb{R}} \backslash\{0\}$ such that $\left(f^{-1}\right)^{*} \theta_{*}=\lambda \theta_{*}$. Then $f^{*} \theta_{*}=\lambda^{-1} \theta_{*}$. Since $\lambda^{2}\left(\theta^{* 2}\right)=\left(f^{*} \theta^{* 2}\right)=\left(\theta^{* 2}\right)$, we get $\left(\theta^{* 2}\right)=0$. Similarly, $\left(\theta_{*}{ }^{2}\right)=0$. By Hodge index theorem, $\left(\theta^{*} \cdot \theta_{*}\right)>0$. It follows that $\left(\theta^{*}+\theta_{*}\right)^{2}>0$. So $\theta^{*}+\theta_{*}$ is big and nef.

Set $H:=\left\{\alpha \in \mathbb{N}^{1}(X)_{\mathbb{R}} \mid\left(\theta^{*} \cdot \alpha\right)=\left(\theta_{*} \cdot \alpha\right)=0\right\}$. It is clear that $\mathbb{N}^{1}(X)_{\mathbb{R}}=$ $\mathbb{R} \theta^{*} \oplus \mathbb{R} \theta_{*} \oplus H$ and $f^{*} H=H$. By Hodge index theorem, the intersection form on $H$ is negative define. Since $f^{*}$ preserves the intersection form, all eigenvalues of $\left.f^{*}\right|_{H}$ are of norm 1 .

Since $f^{*}$ is an automorphism of the lattes $N^{1}(X) \subseteq N^{1}(X)_{\mathbb{R}}$, all eigenvalues of $f^{*}: N^{1}(X)_{\mathbb{R}} \rightarrow N^{1}(X)_{\mathbb{R}}$ are algebraic integers. In particular both $\lambda$ and $\lambda^{-1}$ are algebraic integers.

Lemma 3.2. There is $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ such that $\sigma(\lambda)=\lambda^{-1}$.
Proof of Lemma 3.2. Since $\lambda_{1}$ is an algebraic integer with $|\lambda|>1$, by product formula, there is $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ such that $\left|\sigma\left(\lambda_{1}\right)\right|<1$. Because $\sigma\left(\lambda_{1}\right)$ is an eigenvalue of $f^{*}$ and $\lambda_{1}^{-1}$ is the unique eigenvalue of $f^{*}$ with norm $<1$, we have $\sigma\left(\lambda_{1}\right)=\lambda_{1}^{-1}$.

Then $f^{*} \sigma\left(\theta^{*}\right)=\sigma\left(f^{*} \theta^{*}\right)=\sigma(\lambda) \sigma\left(\theta^{*}\right)=\lambda^{-1} \sigma\left(\theta^{*}\right)$. So there is $c>0$ such that $\theta_{*}=c \sigma\left(\theta^{*}\right)$. After replacing $\theta_{*}$ by $c^{-1} \theta_{*}$, we may assume that $\sigma\left(\theta^{*}\right)=\theta_{*}$.

Corollary 3.3. For every curve $C$ of $X,\left(\theta^{*} \cdot C\right)=0$ if and only if $\left(\theta_{*} \cdot C\right)=0$.
Proof of Corollary 3.3. The subspace $P:=\left\{\alpha \in N^{1}(X)_{\mathbb{C}} \mid(\alpha \cdot C)=0\right\}$ is a hyperplane of $N^{1}(X)_{\mathbb{C}}$ defined over $\mathbb{Q}$. We have $\sigma(P)=P$. Embed $N^{1}(X)_{\mathbb{R}}$ in $N^{1}(X)_{\mathbb{C}}$. Then $\theta^{*} \in P$ if and only if $\theta_{*}=\sigma\left(\theta^{*}\right) \in \sigma(P)=P$.
3.3. Canonical height. In this section, we assume
(i) either $\mathbf{k}=\overline{\mathbb{Q}}$;
(ii) or there is an algebraically closed subfield $K \subseteq \mathbf{k}$, a curve $B$ over $K$, such that $X$ and $f$ are defined over $K(B)$ and $\mathbf{k}=\overline{K(B)}$.
Let $A$ be an ample divisor of $X$, denote by $h_{A}$ a Weil height on $X(\mathbf{k})$ associated to $A$ with $h_{A} \geq 1$. Pick $\mathbb{R}$-divisors $D^{*}$ and $D_{*}$ with numerical classes $\theta^{*}, \theta_{*}$. By [38, Theorem 5] and [39] in characteristic zero and Proposition 2.11 in positive characteristic, for every $y \in X(\mathbf{k})$, the limits

$$
h^{+}(y):=\lim _{n \rightarrow \infty} h_{D^{*}}\left(f^{n}(y)\right) / \lambda^{n}
$$

and

$$
h^{-}(y):=\lim _{n \rightarrow \infty} h_{D_{*}}\left(f^{-n}(y)\right) / \lambda^{n}
$$

exist, do not depend on the choice of $D^{*}, D_{*}, h_{D^{*}}$ and $h_{D_{*}}$, and satisfies the following properties:
(i) $h^{+}=h_{D^{*}}+O\left(h_{A}^{1 / 2}\right), h^{-}=h_{D_{*}}+O\left(h_{A}^{1 / 2}\right)$;
(ii) $h^{+} \circ f=\lambda h^{+}$and $h^{-} \circ f=\lambda^{-1} h^{-}$.

Lemma 3.4. Let $C$ be an irreducible curve of $X$ such that $\left(C \cdot \theta_{*}\right)>0$. Then for every $M \geq 0$, there is $M^{\prime} \geq 0$, such that

$$
\left\{y \in C(\mathbf{k}) \mid h^{-}(y) \leq M\right\} \subseteq\left\{y \in C(\mathbf{k}) \mid h_{A}(y) \leq M^{\prime}\right\}
$$

Proof of Lemma 3.4. There is $d>0$, such that

$$
h^{-} \geq h_{D_{*}}-d h_{A}^{1 / 2}
$$

Pick $a>0$ such that $a\left(D_{*} \cdot C\right)>(A \cdot C)$. Then there is $b>0$ such that for every $y \in C$,

$$
a h_{D^{*}}(y)+b \geq h_{A}(y)
$$

So for every $y \in C$,

$$
h^{-}(y) \geq a^{-1}\left(h_{A}(y)-b\right)-d h_{A}^{1 / 2}(y)
$$

If $h^{-}(y) \leq M$, we get

$$
M \geq a^{-1}\left(h_{A}(y)-b\right)-d h_{A}^{1 / 2}(y)=\left(a^{-1} h_{A}^{1 / 2}(y)-d\right) h_{A}^{1 / 2}(y)-a^{-1} b
$$

This implies that

$$
h_{A}^{1 / 2}(y) \leq \max \{a d, a M+b+a d\}=a M+b+a d
$$

Then we get $h_{A}(y) \leq(a M+b+a d)^{2}$.

### 3.4. The case $\left(C \cdot \theta_{*}\right)>0$.

Lemma 3.5. Let $C$ be an irreducible curve of $X$ such that $\left(C \cdot \theta_{*}\right)>0$. For every $x \in X(\mathbf{k}), O_{f}(x) \cap C$ is finite.
Proof of Lemma 3.5. Let $\mathbb{F}$ be the minimal algebraically closed subfield of $\mathbf{k}$. So $\mathbb{F}=\overline{\mathbb{Q}}$ if char $\mathbf{k}=0$ and $\mathbb{F}=\overline{\mathbb{F}_{p}}$ when char $\mathbf{k}=p>0$. There is an algebraically closed subfield $\mathbf{k}^{\prime}$ of $\mathbf{k}$ with $\operatorname{tr} . \mathrm{d} \cdot{ }_{\mathbb{F}} \mathbf{k}^{\prime}<\infty$ such that $X, f, C$ and $x$ are defined over $\mathbf{k}^{\prime}$. After replacing $\mathbf{k}$ by $\mathbf{k}^{\prime}$, we may assume $\operatorname{tr}$.d. ${ }_{\mathbb{F}} \mathbf{k}<\infty$. Now we prove Lemma 3.5 by induction on $\operatorname{tr}$.d. ${ }_{F} \mathbf{k}$.

When $\mathbf{k}=\overline{\mathbb{F}_{p}}$ for some prime $p>0, O_{f}(x)$ is finite. Then Lemma 3.5 holds.
Assume $\mathbf{k}=\overline{\mathbb{Q}}$. Set $I:=\left\{i \geq 0 \mid f^{i}(x) \in C\right\}$. For every $i \geq I, h^{-}\left(f^{i}(x)\right)=$ $\lambda^{-i} h^{-}(x) \leq h^{-}(x)$. By Lemma 3.4, there is $M>0$ such that $h_{A}\left(f^{i}(x)\right)<M$ for every $i \in I$. We conclude the proof by the Northcott property.

Now we may assume that tr.d. ${ }_{\mathbb{F}} \mathbf{k} \geq 1$. There is an algebraically closed subfield $K \subseteq \mathbf{k}$, a smooth irreducible projective curve $B$ over $K$, such that $X, f, C$ and $x$ are defined over $K(B)$ and $\mathbf{k}=\overline{K(B)}$.

There is a projective morphism $\pi: \mathcal{X} \rightarrow B$ whose geometric generic fiber is $X$. The automorphism $f$ extends to a birational self-map $f_{B}: \mathcal{X} \rightarrow \mathcal{X}$ over $B$. Let $A_{B}$ be an ample divisor on $\mathcal{B}$. Let $C_{B}$ be the Zariski closure of $C$ in $\mathcal{X}$. Let $A$ be the restriction of $A_{B}$ on the generic fiber $X$. There is a nonempty open subset $U$ of $B$, such that $\pi$ is smooth above $U$ and $\left.f_{B}\right|_{\pi^{-1}(U)}$ is an automorphism. Assume that $\left(A \cdot \theta_{*}\right)=1$.

For every $b \in B$, let $X_{b}:=\pi^{-1}(b), C_{b}:=C \cap X_{b}, f_{b}$ be the restriction of $f$ to $X_{b}$ and $A_{b}$ be the restriction of $L_{B}$ to $X_{b}$. After shrinking $U$, we may assume
that $C_{b}$ is irreducible for every $b \in U$. For every $n \geq 0$ and $b \in U$, we have $\left(\left(f^{n}\right)^{*} A \cdot A\right)=\left(\left(f_{b}^{n}\right)^{*} A_{b} \cdot A_{b}\right)$. So $\lambda_{1}\left(f_{b}\right)=\lambda_{1}(f)=\lambda>1$. For $b \in U$, set

$$
\theta_{*, b}^{\prime}:=\lim _{n \rightarrow \infty}\left(\left(f_{b}^{-n}\right)^{*} A_{b} \cdot A_{b}\right) / \lambda^{n} .
$$

The discussion in Section 3.2 shows that $\mathbb{R} \theta_{*, b}^{\prime}$ the eigenspace of $\left(f_{b}^{-1}\right)^{*}$ in $N^{1}\left(X_{b}\right)$ for eigenvalue $\lambda$. Set $\theta_{*, b}:=\theta_{*, b}^{\prime} /\left(\theta_{*, b}^{\prime} \cdot A\right)$. We have

$$
\left(\theta_{*, b} \cdot C_{b}\right)=\left(\theta_{*} \cdot C\right)>0
$$

Set $I:=\left\{i \geq 0 \mid f^{i}(x) \in C\right\}$. For every $i \geq I, h^{-}\left(f^{i}(x)\right)=\lambda^{-i} h^{-}(x) \leq h^{-}(x)$. By Lemma 3.4, there is $M>0$ such that $h_{A}\left(f^{i}(x)\right)<M$ for every $i \in I$.

For every point $y \in X$ defined over $K(B)$, its closure $s_{y}$ in $\mathcal{X}$ is a section of $\pi$. We may assume that for every $y \in X(K(B)), h_{A}(y)=\left(A_{B} \cdot s_{y}\right)$. Also, for every section $s$ of $\pi$, its generic fiber defines a point $y_{s} \in X(K(B))$. For every $y \in X(K(B)), \pi$ induces an isomorphism from $s_{y}$ to the curve B. Consider the Hilbert polynomial

$$
\chi\left(s_{y}^{*} \mathcal{O}\left(n A_{B}\right)\right)=1-g(B)+n\left(s_{y} \cdot A_{B}\right)=1-g(B)+n h_{A}(y)
$$

So there is a quasi-projective $K$-variety $\mathcal{M}_{M}$ that parameterizes the sections $s$ of $\pi$ with $h_{A}\left(y_{s}\right) \leq M$ (see [19]). For every $b \in U$, denote by $e_{b}: \mathcal{M}_{M} \rightarrow X_{b}$ the morphism $s \mapsto s(b)$. Pick a sequence $b_{i}, i \geq 1$ of distinct points in $U(K)$. For $s_{1}, s_{2} \in \mathcal{M}_{M}, s_{1}=s_{2}$ if and only if $e_{b_{i}}\left(s_{1}\right)=e_{b_{i}}\left(s_{2}\right)$ for every $i \geq 1$. For $l \geq 1$, set

$$
e_{l}:=\prod_{i=1}^{l} e_{b_{i}}: \mathcal{M}_{M} \rightarrow \prod_{i=1}^{l} X_{b_{i}}
$$

By [50, Lemma 8.1], there is $L \geq 1$ such that $e_{L}$ is quasi-finite. For $j \in I$, $f^{j}(x)$ defines a point $s_{f^{j}(x)} \in \mathcal{M}_{M}$. The induction hypothesis shows that, for $i=1, \ldots, L$,

$$
e_{b_{i}}\left(\left\{f^{j}(x) \mid j \in I\right\}\right)=\left\{f_{b_{i}}^{j}\left(x_{b_{i}}\right) \mid j \in I\right\} \subseteq O_{f_{b_{i}}}\left(x_{b_{i}}\right) \cap C_{b_{i}}
$$

is finite. So $e_{L}\left(\left\{f^{j}(x) \mid j \in I\right\}\right)$ is finite. Since $e_{L}$ is quasi-finite, $O_{f}(x) \cap C=$ $\left\{f^{j}(x) \mid j \in I\right\}$ is finite.
3.5. Conclusion. Let $x \in X(\mathbf{k})$ and $C$ be an irreducible curve of $X$. If $\left(C \cdot \theta_{*}\right)>$ 0 , we conclude the proof by Lemma 3.5.

Now assume that $\left(C \cdot \theta_{*}\right)=0$. Let $B(f)$ be the set of curves $C^{\prime}$ with $\left(C^{\prime}\right.$. $\left.\theta_{*}\right)=0$. By Corollary 3.3, $C^{\prime} \in B(f)$ if and only if $\left(C^{\prime} \cdot \theta^{*}\right)=0$, if and only if $\left(C^{\prime} \cdot\left(\theta^{*}+\theta_{*}\right)\right)=0$. Since $\theta^{*}+\theta_{*}$ is big and nef, $B(f)$ is finite. Since $f^{*} \theta^{*}=\lambda_{1} \theta^{*}$, $C^{\prime} \in B(f)$ if and only if $f\left(C^{\prime}\right) \in B(f)$. So every curve in $B(f)$ is periodic. Since $C \in B(f), C$ is periodic.

## 4. ZARISKI DENSE ORBIT CONJECTURE

Let $X$ be a variety over $\mathbf{k}$ of dimension $d_{X}$. Let $f: X \rightarrow X$ be a dominant rational self-map.
4.1. Existence of well-defined orbits. In characteristic 0 , the following result is well know. In positive characteristic, the proof is similar.

Proposition 4.1. For every Zariski dense open subset $U$ of $X$, there is $x \in U(\mathbf{k})$ whose $f$-orbit is well defined and contained in $U$.

Proof of Proposition 4.1. After replacing $X, f$ by $U,\left.f\right|_{U}$, we may assume that $X=U$. So we only need to show that $X_{f}(\mathbf{k}) \neq \emptyset$.

Let $\mathbb{F}$ be the smallest algebraically closed subfield of $\mathbf{k}$. So $\mathbb{F}=\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_{p}}$. We may replace $\mathbf{k}$ by an algebraically closed subfield $\mathbf{k}^{\prime}$ of $\mathbf{k}$ with $\operatorname{tr} .{ }^{\text {d }} \cdot \mathbf{F} \mathbf{k}^{\prime}<\infty$ such that $X, f$ are defined over $\mathbf{k}^{\prime}$. Now assume that tr.d. $\mathbf{F} \mathbf{k}<\infty$. If char $\mathbf{k}=0$, we conclude the proof by [55, Proposition 3.22]. Now assume that char $\mathbf{k}=p>0$.

The case $\mathbf{k}=\overline{\mathbb{F}_{p}}$ is essentially proved in [22, Proposition 5.5]. On may also see [51, Proposition 6.2]. In [51, Proposition 6.2], $f$ is assumed to be birational, but its proof works for arbitrary dominant rational self-map.

Now assume that tr.d. ${ }_{\mathbb{F}} \mathbf{k} \geq 1$. There is a subfield $L$ of $K$ which is finitely generated over $\mathbf{k}$ such that $X, f$ are defined over $L$. Let $B$ be a projective and normal variety over $\mathbb{F}$ such that $L=\mathbf{k}$. There is a $B$-scheme $\pi: X_{B} \rightarrow B$ and a rational self-map $f_{B}: X_{B} \rightarrow X_{B}$ over $B$ such that the geometric generic fiber of $\left(X_{B}, f_{B}\right)$ is $(X, f)$. Let $b$ be a general point of $B(\mathbb{F})$ and denote by $\left(X_{b}, f_{b}\right)$ the fiber of $\left(X_{B}, f_{B}\right)$ above $b$. Then $V_{b}:=X_{b} \backslash I\left(f_{B}\right)$ and $f_{b}$ is dominant. Applying the case over $\overline{\mathbb{F}_{p}}$ to $\left(V_{b},\left.f_{b}\right|_{V_{b}}\right)$, there is $x_{b} \in\left(V_{b}\right)_{f_{b} \mid V_{b}}(\mathbb{F})$. Cutting by general hyperplanes of $X_{B}$, there is an irreducible subvariety $S$ of $X_{B}$ of dimension $\operatorname{dim} S=\operatorname{dim} B$ passing through $b$ with $\pi(S)=B$. Then the generic point of $S$ defines a point $x \in X_{f}(\mathbf{k})$, which concludes the proof.
4.2. Tautological upper bound. The following lemmas was proved in characteristic zero, but their proof works in any characteristic.

Lemma 4.2. [36, Lemma 2.15] Let $K$ be an algebraically closed field extension of $\mathbf{k}$. Then $\mathbf{k}(X)^{f}=\mathbf{k}$ if and only if, $K\left(X_{K}\right)^{f_{K}}=K$.

Lemma 4.3. [55, Lemma 2.1] Let $X^{\prime}$ be an irreducible variety over $\mathbf{k}, f^{\prime}: X^{\prime} \rightarrow$ $X^{\prime}$ be a rational self-map and $\pi: X^{\prime} \rightarrow X$ be a generically finite dominant rational map satisfying $f \circ \pi=\pi \circ f^{\prime}$, then we have the following properties.
(i) If there exists $m \geq 1$, and $H \in \mathbf{k}(X)^{f^{m}} \backslash \mathbf{k}$, then there exists $G \in \mathbf{k}(X)^{f} \backslash \mathbf{k}$.
(ii) There exists $H^{\prime} \in \mathbf{k}\left(X^{\prime}\right)^{f^{\prime}} \backslash \mathbf{k}$, if and only if there exists $H \in \mathbf{k}(X)^{f} \backslash \mathbf{k}$.

They show that the assumption $\mathbf{k}(X)^{f}=\mathbf{k}$ is stable under base change, under positive iterate and under semiconjugacy by generaically finite dominant morphism. As an example of realization problems, the author asked the following question in [55, Section 1.6].

Question 4.4. What is the minimal transcendence degree $R(\mathbf{k}, X, f)$ of an algebraically closed field extension $K$ of $\mathbf{k}$ such that ( $X_{K}, f_{K}$ ) satisfies the ZDO property?

Proposition 1.7 gives a tautological upper bound of $R(\mathbf{k}, X, f)$.

Proof of Proposition 1.7. We may assume that $\mathbf{k}(X)^{f}=\mathbf{k}$. By Lemma 4.2, $K\left(X_{K}\right)^{f_{K}}=K$.

An irreducible $f_{K}$-invariant variety $V$ is said to be maximal, if the only irreducible $f_{K}$-invariant variety $W$ containing $V$ is $X_{K}$. We note that $I\left(f_{K}\right)=$ $I(f) \otimes_{\mathbf{k}} K$ is defined over $\mathbf{k}$.

Lemma 4.5. Let $V$ be an irreducible $f_{K}$-invariant variety. Then $V$ is over defined over $\mathbf{k}$.

Proof of Lemma4.5. Set $r:=\operatorname{dim} V<d_{X}$. There is a subfield $L$ of $K$ which is finitely generated over $\mathbf{k}$ such that $V$ is defined over $L$. Let $B$ be a projective and normal variety over $\mathbf{k}$ such that $L=\mathbf{k}(B)$.

Then there is a subvariety $V_{B}$ of $X \times B$ such that $\pi_{2}\left(V_{B}\right)=B$ where $\pi_{2}$ : $X \times B \rightarrow B$ is the projection to the second coordinate and $V=V_{\eta} \times_{L} K$ where $\eta$ is the generic point of $B$ and $V_{\eta}$ is the generic fiber of $\left.\pi_{2}\right|_{V_{B}}$. We have $\operatorname{dim} V_{B}=\operatorname{dim} B+r$. Since $V$ is $f_{K}$-invariant, $V_{B} \subseteq X \times B$ is $f_{B}:=f \times \mathrm{id}$ invariant.

Consider $\pi_{1}: X \times B \rightarrow B$ the projection to the first coordinate. It is clear that $\pi_{1}(V)$ is irreducible and $f$-invariant. Since $V \subseteq \pi_{1}(V)_{K}$ and $V$ is maximal, we get either $V_{B}=\pi_{1}^{-1}\left(\pi_{1}\left(V_{B}\right)\right)$ or $\pi_{2}\left(V_{B}\right)=X$. In the former case $V=\pi_{1}\left(V_{B}\right)_{K}$ is defined over $\mathbf{k}$. Now we assume that $\pi_{2}\left(V_{B}\right)=X$. Then $\operatorname{dim} B=\operatorname{dim} V_{B}-r \geq$ $d_{X}-r \geq 1$ and $\mathbf{k} \subsetneq \pi_{2}^{*}(\mathbf{k}(B)) \subseteq \mathbf{k}\left(V_{B}\right)^{f_{B} \mid V_{B}}$.

If $\operatorname{dim} V_{B}=d_{X}$, we conclude the proof by Lemma 4.3. Now assume that $\operatorname{dim} V_{B} \geq d_{X}+1$. So a general fiber of $\left.\pi_{1}\right|_{V_{B}}$ has dimension $s \geq 1$. We have $\operatorname{dim} B=d_{X}+s-r>s$. Let $H_{1}, \ldots, H_{2}$ be very ample divisors on $B$ which are general in their linear system. Then the intersection of $\pi_{2}^{-1}\left(H_{i}\right), i=1, \ldots, s$ and a general fiber of $\left.\pi_{1}\right|_{V_{B}}$ is of dimension 0 and $W^{\prime}:=V_{B} \cap H_{1} \cdots \cap H_{s}$ is $f_{B^{-}}$ invariant. Because $\pi_{1}\left(W^{\prime}\right)=X$, there is an irreducible component $W$ of $W^{\prime}$ with $\pi_{1}(W)=X$ and there is $l \geq 1$ such that $W$ is $f^{l}$-invariant. Because $d_{X}=\operatorname{dim} W$ and $\operatorname{dim} \pi_{2}(W)=d_{X}-r>0$. So $\mathbf{k} \subsetneq \mathbf{k}(W)^{\left(\left.f_{B}\right|_{W}\right)^{l}}$, which is a contradiction by Lemma 4.3.

We only need to treat the case tr.d. ${ }_{\mathbf{k}} K=d$. So we may assume that $K=\overline{\mathbf{k}(X)}$. The diagonal $\Delta$ of $X \times X$ defines a point $o$ in $X_{K}(K)$. Here we view $X_{K}$ as the geometric generic fiber of the second projection $\pi_{2}: X \times X \rightarrow X$. Because $\pi_{1}(\Delta)=X$ where $\pi_{1}: X \times X \rightarrow X$ is the first projection, $O_{f_{K}}(o)$ is well defined and for every $n \geq 0, f_{K}^{n}(o)$ is not contained in any proper subvariety of $X_{K}$ defined over $\mathbf{k}$. An irreducible component $W$ of $\overline{O_{f_{K}}(o)}$ of maximal dimension is $f_{K}$-periodic and does not contained in any proper subvariety of $X_{K}$ defined over k. By Lemma 4.5, $W=X_{K}$ which concludes the proof.

In fact, with a slight modification, we prove a stronger result related to the strong form of the Zariski dense orbit conjecture [55, Conjecture 1.4].

Proposition 4.6. Assume that $\mathbf{k}(X)^{f}=\mathbf{k}$. Let $K$ be an algebraically closed field extension of $\mathbf{k}$ with tr. $d_{\cdot \mathbf{k}} K \geq \operatorname{dim} X$. Then for every nonempty Zariski open subset $U$ of $X_{K}$, there is a point $x \in U(K)$ whose $f_{K}$-orbit is well defined and contained in $U$.

Proof of Proposition 4.6. Keep the notation in the proof of Proposition 1.7. Pick a general point $b \in X(\mathbf{k})$. Then $U_{b}:=X \backslash \overline{\left(X_{K} \cap U\right)}$ is not empty. By Proposition 4.1. there is $x_{b} \in U_{b}$, whose $f$ orbit is well defined and contained in $U_{b}$. Cutting by general hyperplanes of $X \times X$, there is an irreducible subvariety $S$ of $X \times X$ of dimension $\operatorname{dim} S=\operatorname{dim} X$ passing through $\left(x_{b}, b\right)$ such that $\pi_{1}(S)=X$ and $\pi_{2}(S)=X$. The generic point of $S$ defines a point in $x \in X_{K}(K)$. Then the $f_{K^{-}}$ orbit of $x$ is well defined and contained in $U$. After replacing $o$ by $x$, the argument in the last paragraph of the proof of Proposition 1.7 shows that $O_{f_{K}}(x)$ is Zariski dense in $X_{K}$.
4.3. Height argument. The aim of this section is to prove Theorem 1.9, 4.9 and 1.11.

Assume that char $\mathbf{k}=p>0$ and tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$. Let $f: X \rightarrow X$ be a dominant endomorphism of a projective variety. There is a algebraically closed subfield $K$ of $\mathbf{k}$ such that $\operatorname{tr} . \mathrm{d}_{K} \mathbf{k}=1$. So there is smooth projective curve $B$ over $K$, such that $f, X$ are defined over $K(B)$. The Weil heights appeared in the section are associated to the function field $K(B)$.

Proof of Theorem 1.9. By Corollary 2.13, there exists a point $x \in U(\mathbf{k})$ with $\alpha_{f}(x)=\lambda_{1}(f)>1$ and $O_{f}(x) \subseteq U$. So $x$ has infinite orbit.

Proof of Theorem 1.11. The proof of [36, Proposition 8.6] shows that for every $f$ periodic proper subvariety $V$ of period $m \geq 1, \lambda_{1}\left(\left.f^{m}\right|_{V}\right)<\lambda_{1}\left(f^{m}\right)$. By Propositon 2.12, there exists a point $x \in X(\mathbf{k})$ with $\alpha_{f}(x)=\lambda_{1}(f)>1$. Let $W$ be an irreducible component of $\overline{O_{f}(x)}$ of maximal dimension. There is $m \geq 1$ with $f^{m}(W)=W$. There is $l \geq 0$ such that $f^{l}(x) \in W$.

If $W \neq X$, by Proposition 2.10 and Lemma 2.9, we get

$$
\lambda_{1}(f)^{m}=\bar{\alpha}_{f}(x)^{m}=\bar{\alpha}_{f^{m}}\left(f^{l}(x)\right) \leq \lambda_{1}\left(\left.f^{m}\right|_{W}\right)<\lambda_{1}(f)^{m}
$$

We get a contradiction. So $W=X$, which concludes the proof.
The following theorem was proved in [14, Theorem 1], but when $f$ is an automorphism, its proof work in arbitrary characteristic.

Theorem 4.7. If $f$ is an automorphism and it preserves infinitely many (not necessarily irreducible) hyperplanes, then $\mathbf{k}(X)^{f} \neq \mathbf{k}$.

Proposition 4.8. Let $X$ be a projective variety over $\mathbf{k}$ of dimension $d_{X}$. Let $L$ be an ample line bundle on $X$. Let $f: X \rightarrow X$ be an automorphism such that $\left(\left(f^{n}\right)^{*} L \cdot L^{d_{X}-1}\right), n \geq 0$ is bounded. Then $(X, f)$ satisfies the ZDO property.

Proof of Proposition 4.8. Let $\operatorname{Aut}(X)$ be the scheme of automorphisms of $X$. Every connected component of $\operatorname{Aut}(X)$ is a variety over $\mathbf{k}$, but $\operatorname{Aut}(X)$ may have infinite connected component.

Because $\left(\left(f^{n}\right)^{*} L \cdot L^{d_{X}-1}\right), n \geq 0$ is bounded, the Zariski closure $G$ of $f^{n}, n \geq 0$ in $\operatorname{Aut}(X)$ is a commutative algebraic group. After replacing $f$ by a suitable iterate, we may assume that $G$ is irreducible. We may assume that $f$ is of infinite order. So $\operatorname{dim} G \geq 1$.

For every $x \in X(\mathbf{k}), \overline{O_{f}(x)}=\overline{G . x}$. Consider the morphism $\Phi: G \times X \rightarrow X \times X$ sending $(g, x)$ to $(g(x), x)$. Denote by $\pi_{i}: X \times X \rightarrow X$ the $i$-th projection. Consider the $G$-action on $X \times X$ by $g .(x, y)=(g(x), y)$. Set $F:=f \times$ id : $X \times X \rightarrow X \times X$.

The image $W$ of $\Phi$ is a constructible subset of $X \times X$. Let $Y$ be the Zariski closure of $W$ in $X \times X$. It is irreducible and $F$-invariant. Let $\Delta$ be the diagonal of $X \times X$. Then $\Delta \subseteq W \subseteq Y$. So $\pi_{1}(Y)=\pi_{2}(Y)=X$. Because $\operatorname{dim} G \geq 1$ and the action of $G$ on $X$ is faithful, $Y \neq \Delta$. So the general fiber of $\left.\pi_{2}\right|_{Y}$ has dimension $r \geq 1$. If $r=\operatorname{dim} X$, then for a general $x \in X(\mathbf{k}), \overline{O_{f}(x)}=\overline{G \cdot x}=X$ which concludes the proof. Now assume that $r<\operatorname{dim} X$.

We have $\operatorname{dim} Y=\operatorname{dim} X+r$. The general fiber of $\left.\pi_{1}\right|_{Y}$ also has dimension $r \geq 1$. Let $H_{1}, \ldots, H_{r}$ be very ample hyperplanes of $X$ which are general in their linear system. The intersection of $\pi_{2}^{*} H_{1}, \ldots, \pi_{2}^{*} H_{r}$ and a general fiber of $\left.\pi_{1}\right|_{Y}$ is proper. Set $Z:=\pi_{2}^{-1}\left(\cap_{i=1}^{r} H_{i}\right)$. We have $\pi_{1}(Z)=X, \operatorname{dim} Z=\operatorname{dim} X$ and $\operatorname{dim} \pi_{2}(Z)=\operatorname{dim}\left(H_{1} \cap \cdots \cap H_{r}\right)=\operatorname{dim} X-r \geq 1$. Because $G$ is connected, every irreducible component of $Z$ is $G$-invariant. In particular, let $T$ be an irreducible component of $Z$ with $\pi_{1}(T)=X$, then $T$ is $F$ invariant and we have $\operatorname{dim} T=$ $\operatorname{dim} X, \operatorname{dim} \pi_{2}(T)=\operatorname{dim} X-r \geq 1$. Because $\mathbf{k} \subsetneq \mathbf{k}(T)^{\left.F\right|_{T}}$ and $\left.\pi_{1} \circ F\right|_{T}=f \circ \pi_{2}$, we conclude the proof by Lemma 4.3.

Theorem 4.9. Assume that char $\mathbf{k}=p>0$ and tr.d. $\overline{F_{p}} \mathbf{k} \geq 1$. Let $f: X \rightarrow X$ be an automorphism of a projective surface. Then $(X, f)$ satisfies the $Z D O$ property.

Proof of Theorem 4.9. By [41], there is a minimal desingularization $\pi: X^{\prime} \rightarrow$ $X$. Then one may lift $f$ to an automorphism $f^{\prime}$ of $X^{\prime}$. Easy to see that $(X, f)$ satisfies the ZDO property if and only if $\left(X^{\prime}, f^{\prime}\right)$ satisfies the ZDO property. After replacing $(X, f)$ by $\left(X^{\prime}, f^{\prime}\right)$, we may assume that $X$ is smooth. By Theorem 1.11, we may assume that $\lambda_{1}(f)=1$. Let $L$ be an ample line bundle on $X$.

If $\left(\left(f^{n}\right)^{*} L \cdot L\right), n \geq 0$ is unbounded, by Gizatullin [33], there is a surjective morphism $\pi: X \rightarrow C$ to a smooth projective curve $C$ and an automorphism $f_{C}: C \rightarrow C$ such that $f_{C} \circ \pi=\pi \circ f$. ${ }^{1}$ After replacing $\pi: X \rightarrow C$ by a minimal resolution of $\pi$, we may assume that $\pi$ is a morphism. There is $m \geq 1$ such that $f_{C}^{m}=\mathrm{id}$, we have $\mathbf{k} \subsetneq \pi^{*}\left(\mathbf{k}(C)^{f_{C}}\right) \subseteq \mathbf{k}(X)^{f}$.

Now we may assume that $\left(\left(f^{n}\right)^{*} L \cdot L\right), n \geq 0$ is bounded. We conclude the proof by Proposition 4.8.

## 5. Ergodic theory

Let $X$ be a variety over $\mathbf{k}$. Denote by $|X|$ the underling set of $X$ with the constructible topology i.e. the topology on a $X$ generated by the constructible subsets. This topology is finer than the Zariski topology on $X$. Moreover $|X|$ is (Hausdorff) compact. Denote by $\eta$ the generic point of $X$.

Using the Zariski topology, on may define a partial ordering on $|X|$ by $x \geq y$ if and only if $y \in \bar{x}$. The noetherianity of $X$ implies that this partial ordering

[^0]satisfies the descending chain condition: for every chain in $|X|$,
$$
x_{1} \geq x_{2} \geq \ldots
$$
there is $N \geq 1$ such that $x_{n}=x_{N}$ for every $n \geq N$. For every $x \in|X|$, the Zariski closure of $x$ in $X$ is $U_{x}:=\overline{\{x\}}=\{y \in|X| \mid y \leq x\}$ which is open and closed in $|X|$.

Let $\mathcal{M}(X)$ be the space of Radon measure on $X$ endowed with the weak-* topology and $\mathcal{M}^{1}(|X|)$ be the space of probability Radon measure on $|X|$. Note that $\mathcal{M}^{1}(|X|)$ is compact.

Proof of Theorem 1.12. We claim that for every Radon measure $\mu$ on $|X|$ with $\mu(|X|)>0$, there exists $x \in X$ such that $\mu(x)>0$.

Then for every Radon measure $\mu$ on $|X|$, set $S(\mu):=\{x \in|X| \mid \mu(x)>0\}$. Then $S(\mu)$ is at most countable and we have $c:=\sum_{x \in S(\mu)} \mu(x) \in(0, \mu(|X|)]$. If $c=\mu(|X|)$, then we have $\mu=\sum_{x \in S(\mu)} \mu(x) \delta_{x}$, which concludes the proof. Assume that $c<\mu(|X|)$, set

$$
\alpha:=\mu-\sum_{x \in S(\mu)} \mu(x) \delta_{x}
$$

Then $\alpha$ is a Radon measure with $\alpha(|X|)=\mu(|X|)-c>0$ and $S(\alpha)=\emptyset$. This contradicts our claim.

Now we only need to prove the claim.
Lemma 5.1. For $x \in|X|$, if $\mu\left(U_{x}\right)>0$ and $\mu(x)=0$, then there exists $y \in$ $U_{x} \backslash\{x\}$ such that $\mu\left(U_{y}\right)>0$.

Now assume that for every $x \in|X|, \mu(x)=0$. Since $|X|=\cup_{x \in X} U_{x}$ and $|X|$ is compact, there exists a finite subset $F$ of $|X|$ such that $|X|=\cup_{x \in F} U_{x}$. Then there exists $x_{0} \in F$ such that $\mu\left(U_{x_{0}}\right)>0$. Since $\mu\left(x_{0}\right)=0$ by the assumption, by Lemma 5.1, we get a sequence of points $x_{i}, i \geq 0, x_{i}>x_{i+1}$ such that $\mu\left(U_{x_{i}}\right)>$ $0, \mu\left(x_{i}\right)=0$. This contradicts the descending chain condition.

Proof of Lemma 5.1. Observe that $U_{x} \backslash\{x\}$ is open and $\mu\left(U_{x} \backslash\{x\}\right)>0$. Since $\mu$ is Radon, there exists a compact subset $K \subseteq U_{x} \backslash\{x\}$ such that $\mu(K)>0$. Since $K \subseteq \cup_{z \in K} U_{z}$, there exists a finite set $x_{1}, \ldots, x_{m}$ in $K$ such that $K \subseteq \cup_{i=1}^{m} U_{x_{i}}$. Since $\sum_{i=1}^{m} \mu\left(U_{x_{i}}\right) \geq \mu(K)>0$, there exists some $1 \leq i \leq m$ such that $\mu\left(U_{x_{i}}\right)>0$. Set $y:=x_{i}$, we concludes the proof.

Proof of Corollary 1.14. Let $x_{n} \in X, n \geq 0$ be a sequence of points.
We first assume that $x_{n} \in X, n \geq 0$ is generic. Because $\mathcal{M}^{1}(|X|)$ is compact, we only need to show that for every subsequence with $\lim _{i \rightarrow \infty} \delta_{x_{n_{i}}}=\mu$, we have $\mu=\delta_{\eta}$. By Theorem 1.12, we may write

$$
\mu=\sum_{i \geq 0}^{m} a_{i} \delta_{x_{i}}
$$

where $m \in \mathbb{Z}_{\geq 0} \cup\{\infty\}, x_{i}$ are distinct points, $a_{i}>0$ and $\sum_{i \geq 0} a_{i}=1$. If $\mu \neq \delta_{\eta}$, we may assume that $x_{0} \neq \eta$. Then $V:=\overline{\left\{x_{0}\right\}}$ is a closed proper subvariety of $X$. Then we have

$$
1_{V}\left(x_{n_{i}}\right)=\int 1_{V} \delta_{x_{n_{i}}} \rightarrow \int 1_{V} \mu>a_{0}
$$

as $n \rightarrow \infty$. So $x_{n_{i}} \in V$ for all but finitely many $i$, which is a contradiction.
Now assume that $\lim _{n \rightarrow \infty} \delta_{x_{n}}=\delta_{\eta}$. For every subsequence $x_{n_{i}}, i \geq 0$ and every closed proper subvariety $V$ of $X$,

$$
\lim _{i \rightarrow \infty} 1_{V}\left(x_{n_{i}}\right)=\lim _{i \rightarrow \infty} \int 1_{V} \delta_{x_{n_{i}}}=\int 1_{V} \delta_{\eta}=0
$$

So $x_{n_{i}} \notin V$ for all but finitely many $i$. So $x_{n_{i}}$ is Zariski dense in $X$.
5.1. DML problems. Let $f: X \rightarrow X$ be a dominant rational self-map. Set $|X|_{f}:=|X| \backslash\left(\cup_{i \geq 1} I\left(f^{i}\right)\right)$. Because every Zariski closed subset of $X$ is open and closed in the constructible topology, $|X|_{f}$ is a closed subset of $|X|$. The restriction of $f$ to $|X|_{f}$ is continuous. We still denote by $f$ this restriction.

Denote by $\mathcal{P}(X, f)$ the set of $f$-periodic points in $|X|_{f}$. Theorem 1.12 implies directly the following lemma.

Lemma 5.2. If $\mu \in \mathcal{M}^{1}\left(|X|_{f}\right)$ with $f_{*} \mu=\mu$, then there are $x_{i} \in \mathcal{P}(X, f), i \geq 0$ and $a_{i} \geq 0, i \geq 0$ with $\sum_{i=0} a_{i}=1$ such that

$$
\mu=\sum_{i \geq 0} \frac{a_{i}}{\# O_{f}(y)}\left(\sum_{y \in O_{f}\left(x_{i}\right)} \delta_{y}\right)
$$

Now we prove Theorem 1.16 and Theorem 1.17 ,
Proof of Theorem 1.16. Let $x$ be a points $\in X_{f}(\mathbf{k})$ with $\overline{O_{f}(x)}=X$. Let $V$ be a proper subvariety of $X$. Consider a sequence of intervals $I_{n}, n \geq 0$ in $\mathbb{Z}_{\geq 0}$ with $\lim _{n \rightarrow \infty} \# I_{n}=+\infty$. For every $n \geq 0$, set $\mu_{n}:=\left(\# I_{n}\right)^{-1}\left(\sum_{i \in I_{n}} \delta_{f^{i}(x)}\right) \in \mathcal{M}^{1}\left(|X|_{f}\right)$. Because

$$
\frac{\#\left(\left\{n \geq 0 \mid f^{n}(x) \in V\right\} \cap I_{n}\right)}{\# I_{n}}=\int 1_{V} \mu_{n}
$$

we only need to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mu_{n}=\delta_{\eta} \tag{5.1}
\end{equation*}
$$

Because $\mathcal{M}^{1}(|X|)$ is compact, we only need to show that for every convergence subsequence $\mu_{n_{i}}, i \geq 0, \mu_{n_{i}} \rightarrow \delta_{\eta}$ as $i \rightarrow \infty$. Set $\mu:=\lim _{n \rightarrow \infty} \mu_{n_{i}}$. We have

$$
\begin{gathered}
f_{*} \mu=\lim _{n \rightarrow \infty} f_{*} \mu_{n_{i}}=\lim _{i \rightarrow \infty} \mu_{n_{i}}+\lim _{i \rightarrow \infty}\left(\# I_{n_{i}}\right)^{-1}\left(\delta_{f^{\max I_{n_{i}}+1}(x)}-\delta_{f^{\min I_{n_{i}}(x)}}\right) \\
=\lim _{i \rightarrow \infty} \mu_{n_{i}}=\mu .
\end{gathered}
$$

For every $y \in \mathcal{P}(X, f) \backslash\{\eta\}, U_{y}$ is open and closed in $|X|_{f}$. Then

$$
Y:=|X|_{f} \backslash\left(\cup_{y \in \mathcal{P}(X, f)} U_{y}\right)
$$

is an $f$-invariant closed proper subset of $|X|_{f}$. Because $\overline{O_{f}(x)}=X, x \in Y$. So for every $n \geq 0$, Supp $\mu_{n} \subseteq Y$. Because $Y \cap \mathcal{P}(X, f)=\{\eta\}$, Lemma 5.2 shows that $\mu=\delta_{\eta}$.

Proof of Theorem 1.17. Let $x_{n} \in X_{f}(\mathbf{k}), n \leq 0$ be a sequence of points such that $\overline{\left\{x_{n}, n \leq 0\right\}}=X$ and $f\left(x_{n}\right)=x_{n+1}$ for all $n \leq-1$. Consider a sequence of intervals $I_{n}, n \geq 0$ in $\mathbb{Z}_{\leq 0}$ with $\lim _{n \rightarrow \infty} \# I_{n}=+\infty$. For $n \geq 1$, define $x_{n}:=f^{n}\left(x_{0}\right)$.

For every $n \geq 0$, set $\mu_{n}:=\left(\# I_{n}\right)^{-1}\left(\sum_{i \in I_{n}} \delta_{x_{i}}\right) \in \mathcal{M}^{1}\left(|X|_{f}\right)$. As the proof of Theorem 1.16, we only need to show

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mu_{n}=\delta_{\eta} . \tag{5.2}
\end{equation*}
$$

Because $\mathcal{M}^{1}(|X|)$ is compact, we only need to show that for every convergence subsequence $\mu_{n_{i}}, i \geq 0, \mu_{n_{i}} \rightarrow \delta_{\eta}$ as $i \rightarrow \infty$. Set $\mu:=\lim _{n \rightarrow \infty} \mu_{n_{i}}$. We have

$$
\begin{gathered}
f_{*} \mu=\lim _{n \rightarrow \infty} f_{*} \mu_{n_{i}}=\lim _{i \rightarrow \infty} \mu_{n_{i}}+\lim _{i \rightarrow \infty}\left(\# I_{n_{i}}\right)^{-1}\left(\delta_{x_{\max I_{n+1}+1}}-\delta_{x_{\min I_{n+1}+1}}\right) \\
=\lim _{i \rightarrow \infty} \mu_{n_{i}}=\mu .
\end{gathered}
$$

For every $y \in \mathcal{P}(X, f) \backslash\{\eta\}, U_{y} \cap\left\{x_{i}, i \leq 0\right\}$ is finite. Otherwise $\left\{x_{i}, i \leq 0\right\} \subseteq$ $\cup_{z \in O_{f}(y)} U_{z}$ is not Zariski dense in $X$. This implies that $\mu\left(U_{y}\right)=\lim _{i \rightarrow \infty} \mu_{n_{i}}\left(U_{y}\right)=$ 0 . So Supp $\mu \subseteq Y:=|X|_{f} \backslash\left(\cup_{y \in \mathcal{P}(X, f)} U_{y}\right)$. Because $Y \cap \mathcal{P}(X, f)=\{\eta\}$, Lemma 5.2 shows that $\mu=\delta_{\eta}$.
5.2. Functoriality. Assume that $f: X \rightarrow X$ is a flat and finite endomorphism. Because the image by $f$ of every constructible subset is constructible, $f$ is open w.r.t the constructible topology. Moreover, for every $x \in X, f\left(U_{x}\right)=U_{f(x)}$.

Denote by $C(|X|)$ the space of continuous $\mathbb{R}$-valued functions on $|X|$ with the $L_{\infty}$ norm $\|\cdot\|$. For every $\phi \in C(|X|)$, define $f_{*} \phi$ to be the function

$$
x \in|X| \mapsto f_{*} \phi:=\sum_{y \in f^{-1}(x)} m_{f}(y) \phi(y)
$$

The following Lemma shows that $f_{*}$ is a bounded linear operator on $C(|X|)$.
Lemma 5.3. For every $\phi \in C(|X|), f_{*} \phi$ is continuous and $\left\|f_{*} \phi\right\| \leq d_{f}\|\phi\|$.
Proof. By [31, Proposition 2.8], for every $x \in|X|$, there is an open subset $V_{x} \subseteq U_{x}$ containing $x$ such that $V_{x}=f^{-1}\left(f\left(V_{x}\right)\right) \cap U_{x}$ and for every $y \in f\left(V_{x}\right)$,

$$
m_{f}(x)=\sum_{z \in f^{-1}(y) \cap V_{x}} m_{f}(z) .
$$

Because $\{x\}=f^{-1}(f(x)) \cap U_{x}$, such $V_{x}$ can be taken arbritarily small.
Because $\phi \in C(|X|)$, for every $x \in|X|$ and $r>0$, there is an open subset $V_{x}^{r}$ containing $x$ such that for every $y \in V_{x}^{r},|\phi(y)-\phi(x)|<r$.

Let $w$ be a point in $|X|$. There are open neighborhoods $O_{y}$ of $y \in f^{-1}(w)$, such that for distinct $y_{1}, y_{2} \in f^{-1}(w), O_{y_{1}} \cap O_{y_{2}}=\emptyset$. For every $r>0$, and $y \in f^{-1}(w)$, we may take $V_{y}$ as in the first paragraph such that $V_{y} \subseteq O_{y} \cap V_{y}^{r / d_{f}}$. Then $W_{w}^{r}:=\cap_{y \in f^{-1}(w)} f\left(V_{y}\right)$ is an open set containing $w$. For every $x \in W_{w}^{r}$ and distinct $y_{1}, y_{2} \in f^{-1}(w)$, we have

$$
\left(f^{-1}(x) \cap V_{y_{1}}\right) \cap\left(f^{-1}(x) \cap V_{y_{2}}\right)=\emptyset .
$$

Since

$$
d_{f}=\sum_{z \in f^{-1}(x)} m_{f}(z) \geq \sum_{y \in f^{-1}(w)} \sum_{z \in f^{-1}(x) \cap V_{y}} m_{f}(x)=\sum_{y \in f^{-1}(w)} m_{f}(y)=d_{f},
$$

we have

$$
f^{-1}(x)=\sqcup_{y \in f^{-1}(w)}\left(f^{-1}(x) \cap V_{y}\right) .
$$

Then we get

$$
\begin{gathered}
\left|f_{*} \phi(x)-f_{*} \phi(w)\right| \leq \sum_{y \in f^{-1}(w)}\left|m_{f}(y) \phi(y)-\sum_{z \in V_{y} \cap f^{-1}(x)} m_{f}(z) \phi(z)\right| \\
\leq \sum_{y \in f^{-1}(w)} \sum_{z \in V_{y} \cap f^{-1}(x)} m_{f}(z)|\phi(y)-\phi(z)|<\sum_{y \in f^{-1}(w)} \sum_{z \in V_{y} \cap f^{-1}(x)} m_{f}(z) r / d_{f}=r .
\end{gathered}
$$

So $f_{*} \phi$ is continuous. Moreover for every $x \in|X|$

$$
f_{*} \phi(x)=\left|\sum_{y \in f^{-1}(x)} m_{f}(x) \phi(y)\right| \leq \sum_{y \in f^{-1}(x)} m_{f}(x)\|\phi\|=d_{f}\|\phi\|,
$$

which concludes the proof.
Now one may define the pullback $f^{*}: \mathcal{M}(|X|) \rightarrow \mathcal{M}(|X|)$ by the duality: for every $\mu \in \mathcal{M}(|X|)$ and $\phi \in C(|X|)$,

$$
\int \phi\left(f^{*} \mu\right)=\int\left(f_{*} \phi\right) \mu
$$

In particular, $f^{*} \mu(|X|)=d_{f} \mu(|X|)$. The pullback $f^{*}: \mathcal{M}(|X|) \rightarrow \mathcal{M}(|X|)$ is continuous w.r.t. the weak-* topology on $\mathcal{M}(|X|)$ and one may check that for every $x \in|X|$,

$$
f^{*} \delta_{x}=\sum_{y \in f^{-1}(x)} m_{f}(y) \delta(y)
$$

5.3. Backward orbits. Assume that $f: X \rightarrow X$ is a flat and finite endomorphism. In particular, $f$ is surjective. The aim of this section is to prove Theorem $1.18,1.20$ and 1.22 .

Let $T P(X, f)$ be the point $x \in|X|$ such that $\cup_{n \geq 0} f^{-n}(x)$ is finite. It is clear that $f^{*} T P(X, f) \subseteq T P(X, f)$. For $x \in T P(X, f)$, since $f: \cup_{n \geq 1} f^{-n}(x) \rightarrow$ $\cup_{n \geq 0} f^{-n}(x)$ is surjective, it is bijective. So $x$ is periodic. Then $f^{-1}(\bar{T} P(X, f))=$ $T P(X, f)$ and for every $x \in T P(X, f), f^{-1}(x)$ is a single point. For the simplicity, we still denote by $f^{-1}(x)$ the unique points in it.

For every $x \in T P(X, f), f^{-1}\left(U_{x}\right)=\cup_{y \in f^{-1}(x)} U_{f^{-1}(y)}$. Then

$$
Y:=X \backslash \cup_{x \in T P(X, f) \backslash\{\eta\}} U_{x}
$$

is a closed subset of $|X|$ such that $f^{-1}(Y)=f(Y)=Y$. It is clear that $Y$ is exactly the subset of $x \in|X|$ such that $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$.

Lemma 5.4. For $\mu \in \mathcal{M}(|X|)$ supported in $Y$, if $d_{f}^{-1} f^{*} \mu=\mu$, then $\mu=\delta_{\eta}$.

Proof. Assume that $\mu \neq \delta_{\eta}$. We may assume that $\mu(\eta)=0$. Otherwise, we may replace $\mu$ by $\mu-\mu(\eta) \delta_{\eta}$. By Theorem 1.12, one may write

$$
\mu=\sum_{i=0}^{m} a_{i} \delta_{x_{i}}
$$

where $m \in \mathbb{Z}_{\geq 0} \cup\{\infty\}, x_{i}$ are distinct points in $Y \backslash\{\eta\}, a_{i}>0$ and $\sum_{i \geq 0} a_{i}=1$. We have

$$
\mu=d_{f}^{-1} f^{*} \mu=\sum_{i=0}^{m} \sum_{y \in f^{-1}(x)} \frac{a_{i} m_{f}(y)}{d_{f}} \delta_{y} .
$$

Terms in the right hand side have distinct supports.
Assume that $a_{i}$ is decreasing. We claim that for every $i, f^{-1}\left(x_{i}\right)$ is a single point. Otherwise, pick $l$ minimal such that $f^{-1}\left(x_{l}\right)$ is not a single point. Assume that $s \geq 0$ is maximal such that $a_{l+s}=a_{l}$. Think $\mu$ as a function $\mu:|X| \rightarrow[0,1]$ sending $x$ to $\mu(x)$. We have $\mu^{-1}\left(a_{l}\right)=s+1$. On the other hand

$$
\left(d_{f}^{-1} f^{*} \mu\right)^{-1}\left(a_{l}\right)=\left\{i=l, \ldots, l+s \mid f^{-1}\left(x_{i}\right) \text { is a single point }\right\} \leq s
$$

which is a contradiction. Then we get $\mu=\sum_{i=0}^{m} a_{i} \delta_{f^{-1}\left(x_{i}\right)}$. Because for every $r>0,\left\{i=0, \ldots, m \mid a_{i} \geq r\right\}$ is finite, all $x_{i}, i=0, \ldots, m$ are contained in $T P(X, f) \cap(Y \backslash\{\eta\})=\emptyset$. We get a contradiction.

Proof of Theorem 1.18. Let $x$ be a point in $X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$. Let $I_{n}, n \geq 0$ be a sequence of intervals in $\mathbb{Z}_{\geq 0}$ with $\lim _{n \rightarrow \infty} \# I_{n}=+\infty$. Set

$$
\mu_{n}:=\frac{1}{\# I_{n}}\left(\sum_{i \in I_{n}} d_{f}^{-i}\left(f^{i}\right)^{*} \delta_{x}\right) \in \mathcal{M}^{1}(|X|)
$$

Because $\mathcal{M}^{1}(|X|)$ is compact, only need to show that for every convergence subsequence $\mu_{n_{i}}, i \geq 0, \mu_{n_{i}} \rightarrow \delta_{\eta}$ as $i \rightarrow \infty$. Set $\mu:=\lim _{n \rightarrow \infty} \mu_{n_{i}}$.

Then

$$
\begin{gathered}
f^{*} \mu=\lim _{i \rightarrow \infty} f^{*} \mu_{n_{i}}=\lim _{i \rightarrow \infty} \frac{1}{\# I_{n}}\left(\sum_{j \in I_{n_{i}}} d_{f}^{-j}\left(f^{j+1}\right)^{*} \delta_{x}\right) \\
\lim _{i \rightarrow \infty} d_{f} \mu_{n_{i}}+\lim _{i \rightarrow \infty} \frac{d_{f}}{\# I_{n}}\left(d_{f}^{-\max I_{n_{i}}-1}\left(f^{\max I_{n_{i}}+1}\right)^{*} \delta_{x}-d_{f}^{-\min I_{n_{i}}}\left(f^{\min I_{n_{i}}}\right)^{*} \delta_{x}\right)
\end{gathered}
$$

Because $d_{f}^{-\max I_{n_{i}}-1}\left(f^{\max I_{n_{i}}+1}\right)^{*} \delta_{x}(|X|)=d_{f}^{-\min I_{n_{i}}}\left(f^{\min I_{n_{i}}}\right)^{*} \delta_{x}(|X|)=1$, we get

$$
f^{*} \mu=\lim _{i \rightarrow \infty} d_{f} \mu_{n_{i}}=d_{f} \mu
$$

Because $x \in Y$, for every $n \geq 0$, $\operatorname{Supp} \mu_{n} \subseteq Y$. So $\mu \subseteq Y$. We conclude the proof by Lemma 5.4 ,

Proof of Theorem 1.20. Assume that $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable. Let $x \in X(\mathbf{k})$ be a point with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$. Pick $c \in(0,1]$ Because

$$
\# f^{n}(x) \leq \sum_{y \in f^{-n}(x)} m_{f^{n}}(y)=d_{f}^{n}
$$

we have

$$
\limsup _{n \rightarrow \infty}\left(S_{c}^{n}\right)^{1 / n} \leq \limsup _{n \rightarrow \infty} \# f^{n}(x)^{1 / n} \leq d_{f} .
$$

We now prove the inequality in the other direction.
By [31, Theorem 2.1] and [31, Proposition 2.3], there is a proper Zariski closed subset $R$ of $X$, such that for every $y \in X(\mathbf{k}) \backslash R, m_{f}(y)=1$. Set

$$
\mu_{n}:=\frac{1}{n}\left(\sum_{i=1}^{n} d_{f}^{-i}\left(f^{i}\right)^{*} \delta_{x}\right) \in \mathcal{M}^{1}(|X|)
$$

By Theorem 1.18,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mu_{n}=\delta_{\eta} . \tag{5.3}
\end{equation*}
$$

Set $D:=\left\{1, \ldots, d_{f}\right\}$. Let $\Omega:=\sqcup_{n \geq 0} D^{n}$ be the set of words in $D$ of finite length. In particular $D^{0}=\{\emptyset\}$. By induction, one may define a map

$$
\phi: \Omega \rightarrow \sqcup_{n \geq 0} f^{-n}(x) \subseteq \sqcup_{n \geq 0} X
$$

such that
(i) $\theta\left(D^{n}\right)=f^{-n}(x)$, in particular $\phi(\emptyset)=x$.
(ii) for every word $w_{1} \ldots w_{n} \in D^{n}, n \geq 1$,

$$
\theta\left(w_{1} \ldots w_{n-1}\right)=f\left(\theta\left(w_{1} \ldots w_{n}\right)\right)
$$

(iii) for every $y \in f^{-n-1}(x)$ and $w_{1} \ldots w_{n} \in D^{n}$ satisfying $\theta\left(w_{1} \ldots w_{n}\right)=f(y)$,

$$
\#\left\{w \in D \mid \theta\left(w_{1} \ldots w_{n} w\right)=y\right\}=m_{f}(y) .
$$

By [31, Proposition 2.5], for every $y \in f^{-n-1}(x), m_{f^{n+1}}(y)=m_{f^{n}}(f(y)) m_{f}(y)$. This implies that for every $y \in f^{-n}(x)$,

$$
\#\left\{\omega \in D^{n} \mid \theta(\omega)=y\right\}=m_{f^{n}}(y)
$$

Define a function $A: \Omega \rightarrow(0,1]$ by

$$
A: \omega \in D^{n} \mapsto m_{f^{n}}(\theta(\omega))^{-1}
$$

We have
(i) $\sum_{\omega \in D^{n}} A(\omega)=\# f^{-n}(x)$;
(ii) for every $w_{1} \ldots w_{n+1} \in D^{n+1}$,

$$
A\left(w_{1} \ldots w_{n+1}\right)=m_{f}\left(\theta\left(w_{1} \ldots w_{n+1}\right)\right)^{-1} A\left(w_{1} \ldots w_{n}\right) .
$$

We have $A(\emptyset)=1$ and

$$
A\left(w_{1} \ldots w_{n+1}\right) \geq d_{f}^{-1_{R}\left(\theta\left(w_{1} \ldots w_{n+1}\right)\right)} A\left(w_{1} \ldots w_{n}\right)
$$

Then we have

$$
\begin{aligned}
& \prod_{\omega \in D^{n+1}} A(\omega)=\prod_{\omega \in D^{n}} \prod_{w \in D} A(\omega w) \geq \prod_{\omega \in D^{n}} \prod_{w \in D} d_{f}^{-1_{R}\left(\theta\left(w_{1} \ldots w_{n+1}\right)\right)} A(\omega) \\
= & \left(\prod_{\omega \in D^{n+1}} d_{f}^{-1_{R}(\theta(\omega))}\right)\left(\prod_{\omega \in D^{n}} A(\omega)\right)^{d_{f}}=d_{f}^{-\int 1_{R}\left(f^{n+1}\right)^{*} \delta_{x}}\left(\prod_{\omega \in D^{n}} A(\omega)\right)^{d_{f}} .
\end{aligned}
$$

Set $B_{n}:=\log _{d_{f}} \prod_{\omega \in D^{n}} A(\omega)$. We get

$$
B_{n+1} / d_{f}^{n+1} \geq-d_{f}^{-n-1} \int 1_{R}\left(f^{n+1}\right)^{*} \delta_{x}+B_{n} / d_{f}^{n}
$$

Then we get

$$
B_{n} / d_{f}^{n} \geq \sum_{i=1}^{n}-d_{f}^{-i} \int 1_{R}\left(f^{i}\right)^{*} \delta_{x}=-n \int 1_{R} \mu_{n}
$$

For every $n \geq 0$, pick $E_{n} \subseteq f^{-n}(x)$, such that

$$
\sum_{y \in E_{n}} m_{f^{n}}(y) \geq c d_{f}^{n}
$$

and $\# E_{n}=S_{c}^{n}$. So

$$
\# \theta^{-1}\left(E_{n}\right)=\sum_{y \in E_{n}} m_{f^{n}}(y) \geq c d_{f}^{n}
$$

By Inequality of arithmetic and geometric means, we have

$$
\begin{gathered}
S_{c}^{n}=\sum_{\omega \in \theta^{-1}\left(E_{n}\right)} A(\omega) \geq \# \theta^{-1}\left(E_{n}\right)\left(\prod_{\omega \in \theta^{-1}\left(E_{n}\right)} A(\omega)\right)^{\frac{1}{\# \theta^{-1}\left(E_{n}\right)}} \\
\geq c d_{f}^{n}\left(\prod_{\omega \in \theta^{-1}\left(E_{n}\right)} A(\omega)\right)^{\frac{1}{c d_{f}^{n}}} \geq c d_{f}^{n}\left(\prod_{\omega \in D^{n}} A(\omega)\right)^{\frac{1}{c d_{f}^{n}}} \\
\quad=c d_{f}^{n+B_{n} / c d_{f}^{n}} \geq c d_{f}^{n\left(1-c^{-1} \int 1_{R} \mu_{n}\right)} .
\end{gathered}
$$

So $\left(S_{c}^{n}\right)^{1 / n} \geq c^{1 / n} d_{f}^{1-\int 1_{R} \mu_{n}}$. By Equality 5.3.

$$
\liminf _{n \geq 0}\left(S_{c}^{n}\right)^{1 / n} \geq d_{f}
$$

whcih concludes the proof.

Proof of Theorem 1.2.2. Set $d_{X}:=\operatorname{dim} X$. Assume that $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable and

$$
\lambda_{\operatorname{dim} X}(f)>\max _{1 \leq i \leq \operatorname{dim} X-1} \lambda_{i} .
$$

Let $x$ be a point in $X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$.
We first show that for every irreducible subvariety $V$ of $X$ of $\operatorname{dim} V=d_{V}<d_{X}$,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \#\left(f^{-n}(x) \cap V\right)^{1 / n} \leq \lambda_{d_{V}} \tag{5.4}
\end{equation*}
$$

Let $Y$ be a normal and projective variety containing $X$ as an Zariski dense open subset. Let $Z$ be the Zariski closure of $V$ in $W$. Let $\mathcal{I}_{Z}$ be the ideal sheaf associated to $Z$. Let $H$ be a very ample divisor on $Y$ such that $\mathcal{O}(H) \otimes \mathcal{I}_{Z}$ is generated by global sections.

For every $n \geq 0$, consider the following commutative diagram

where $\pi_{1}^{n}$ is birational and it is an isomorphism above $X$. There are $H_{1}, \ldots, H_{d_{X}-d_{V}} \in$ $|H|$ such that the intersection of $H_{1}, \ldots, H_{d_{X}-d_{V}}$ is proper, $V$ is an irreducible component of $\cap_{i=1}^{d_{X}-d_{V}} H_{i}$ and $V$ is the unique irreducible component meeting $f^{-n}(x) \cap V$. Take $H_{1}^{\prime}, \ldots, H_{d_{V}}^{\prime}$ general in those elements of $|H|$ containing $x$. Then the intersection of $H_{1}^{\prime}, \ldots, H_{d_{V}}^{\prime}$ and $f(V)$ at $x$ is proper. Since $f$ is finite, the intersection of $f^{*}\left(H_{1}^{\prime}\right), \ldots, f^{*} H_{d_{V}}^{\prime}$ and $V$ is proper at every $y \in f^{-n}(x) \cap V$.

We have

$$
\left(\pi_{1}^{n}\right)^{-1}\left(f^{-n}(x) \cap V\right) \subseteq\left(\cap_{i=1}^{d_{X}-d_{V}}\left(\pi_{1}^{n}\right)^{*} H_{i}\right) \cap\left(\cap_{i=1}^{d_{V}}\left(\pi_{2}^{n}\right)^{*} H_{i}^{\prime}\right),
$$

and every point $y \in\left(\pi_{1}^{n}\right)^{-1}\left(f^{-n}(x) \cap V\right)$ is isolated in $\left(\cap_{i=1}^{d_{X}-d_{V}}\left(\pi_{1}^{n}\right)^{*} H_{i}\right) \cap\left(\cap_{i=1}^{d_{V}}\left(\pi_{2}^{n}\right)^{*} H_{i}^{\prime}\right)$. By [36, Lemma 3.3],

$$
\begin{gathered}
\left(H^{d_{X}-d_{V}} \cdot\left(f^{n}\right)^{*} H^{d_{V}}\right)=\left(\left(\pi_{1}^{n}\right)^{*} H_{1} \cdots \cdot\left(\pi_{1}^{n}\right)^{*} H_{d_{X}-d_{V}} \cdot\left(\pi_{2}^{n}\right)^{*} H_{1}^{\prime} \cdots \cdots\left(\pi_{2}^{n}\right)^{*} H_{d_{V}}^{\prime}\right) \\
\geq \#\left(\pi_{1}^{n}\right)^{-1}\left(f^{-n}(x) \cap V\right)=\#\left(f^{-n}(x) \cap V\right) .
\end{gathered}
$$

Then we get

$$
\limsup _{n \rightarrow \infty} \#\left(f^{-n}(x) \cap V\right)^{1 / n} \leq \lim _{n \rightarrow \infty}\left(H^{d_{X}-d_{V}} \cdot\left(f^{n}\right)^{*} H^{d_{V}}\right)^{1 / n}=\lambda_{d_{V}}
$$

Now we only need to show

$$
\lim _{n \rightarrow \infty} d_{f}^{-n}\left(f^{n}\right)^{*} \delta_{x}=\delta_{\eta}
$$

Because $\mathcal{M}^{1}(|X|)$ is compact, only need to show that for every convergence subsequence $d_{f}^{-n_{i}}\left(f^{n_{i}}\right)^{*} \delta_{x}, i \geq 0, \lim _{i \rightarrow \infty} d_{f}^{-n_{i}}\left(f^{n_{i}}\right)^{*} \delta_{x}=\delta_{\eta}$. Set $\mu:=\lim _{i \rightarrow \infty} d_{f}^{-n_{i}}\left(f^{n_{i}}\right)^{*} \delta_{x}$. By Theorem 1.12, we may write

$$
\mu=\sum_{i \geq 0}^{m} a_{i} \delta_{x_{i}}
$$

where $m \in \mathbb{Z}_{\geq 0} \cup\{\infty\}, x_{i}$ are distinct points, $a_{i}>0$ and $\sum_{i \geq 0} a_{i}=1$. Assume that $\mu \neq \delta_{\eta}$. Then we may assume that $a_{0}>0$ and $x_{0} \neq \eta$. Set $r:=\overline{\left\{x_{0}\right\}}<d_{X}$.

Then

$$
\int 1_{U_{x_{0}}} \mu \geq \int 1_{U_{x_{0}}} a_{0} \delta_{x_{0}}=a_{0} .
$$

Pick $c \in\left(0, a_{0}\right)$. Then there is $N \geq 0$ such that for every $i \geq N$,

$$
\frac{\sum_{y \in f^{-n_{i}}(x) \cap \overline{\left\{x_{0}\right\}}} m_{f^{n_{i}}}(y)}{d_{f}^{n_{i}}}=\int 1_{U_{x_{0}}} d_{f}^{-n_{i}}\left(f^{n_{i}}\right)^{*} \delta_{x} \geq c
$$

So $\sum_{y \in f^{-n_{i}}(x) \cap \overline{\left\{x_{0}\right\}}} m_{f^{n_{i}}}(y) \geq c d_{f}^{n_{i}}$, then $\#\left(f^{-n_{i}}(x) \cap \overline{\left\{x_{0}\right\}}\right) \geq S_{c}^{n_{i}}$. By Theorem 1.20 and Inequality 5.4, we get

$$
d_{f}>\lambda_{r} \geq \limsup _{i \rightarrow \infty}\left(\#\left(f^{-n_{i}}(x) \cap \overline{\left\{x_{0}\right\}}\right)\right)^{1 / n_{i}} \geq \liminf _{i \rightarrow \infty}\left(S_{c}^{n_{i}}\right)^{1 / n_{i}}=d_{f}
$$

which is a contradiction.
5.4. Berkovich spaces. In this section, $\mathbf{k}$ is a complete nonarchimedean valued field with norm $|\cdot|$. See [12] and [13] for basic theory of Berkovich spaces.

Let $X$ be a variety over $\mathbf{k}$. Recall that, as a topological space, Berkovich's analytification of $X$ is

$$
X^{\mathrm{an}}:=\left\{\left(x,|\cdot|_{x}\right)\left|x \in X,|\cdot|_{x} \text { is a norm on } \kappa(x) \text { which extends }\right| \cdot \mid \text { on } \mathbf{k}\right\}
$$

endowed with the weakest topology such that
(i) $\tau: X^{\text {an }} \rightarrow X$ by $\left(x,|\cdot|{ }_{x}\right) \mapsto x$ is continuous;
(ii) for every Zariski open $U \subseteq X$ and $\phi \in O(U)$, the map $|\phi|: \tau^{-1}(U) \rightarrow$ $[0+\infty)$ sending $\left(x,|\cdot|_{x}\right)$ to $|\phi|_{x}$ is continuous.
Let $\mathcal{M}\left(X^{\mathrm{an}}\right)$ be the space of Radon measures on $X^{\mathrm{an}}$ and let $\mathcal{M}^{1}\left(X^{\mathrm{an}}\right)$ be the space of probability Radon measures on $X^{\text {an }}$.
5.5. Trivial norm case. Assume that $|\cdot|$ is the trivial norm.

For every $x \in X$, let $|\cdot|_{x, 0}$ be the trivial norm on $\kappa(x)$. Then we have an embedding $\sigma: X \rightarrow X^{\text {an }}$ sending $x \in X$ to $\left(x,|\cdot|_{x, 0}\right)$. We have $\tau \circ \sigma=\mathrm{id}$. One may check that the constructible topology on $X$ is exact the topology induced by the topology on $X^{\text {an }}$ and the embedding $\sigma$. Because $|X|$ is compact, $\sigma(X)$ is closed in $X^{\text {an }}$ and $\sigma:|X| \rightarrow \sigma(|X|)$ is a homeomorphism.
Remark 5.5. We note that, if $X$ is endowed with the constructible topology, $\tau: X^{\text {an }} \rightarrow|X|$ is no longer continuous.

Using the embedding $\sigma$, Corollary 1.14 can be translated to a statement on $X^{\text {an }}$.

Corollary 5.6 (=Corollary (1.14). A sequence $x_{n} \in X, n \geq 0$ is generic if and only if in $\mathcal{M}\left(X^{\mathrm{an}}\right)$

$$
\lim _{n \rightarrow \infty} \delta_{\sigma\left(x_{n}\right)}=\delta_{\sigma(\eta)}
$$

Let $f: X \rightarrow X$ be a finite flat morphism. It induces a morphism $f^{\text {an }}: X^{\text {an }} \rightarrow$ $X^{\text {an }}$. We have

$$
f^{\mathrm{an}} \circ \sigma=\sigma \circ f \text { and } \tau \circ f^{\mathrm{an}}=f \circ \tau
$$

According to [31, Lemma 6.7], there is a natural pullback $f^{\text {an* }}: \mathcal{M}\left(X^{\mathrm{an}}\right) \rightarrow$ $\mathcal{M}\left(X^{\mathrm{an}}\right)$. One may check that the following diagram is commutative.


Then we may translate Theorem 1.22 to a statement on $X^{\text {an }}$.

Theorem 5.7 (=Theorem 1.22). Let $f: X \rightarrow X$ be a flat and finite endomorphism of a quasi-projective variety. Assume that

$$
\begin{equation*}
d_{f}:=\lambda_{\operatorname{dim} X}(f)>\max _{1 \leq i \leq \operatorname{dim} X-1} \lambda_{i} . \tag{5.5}
\end{equation*}
$$

If the field extension $\mathbf{k}(X) / f^{*} \mathbf{k}(X)$ is separable, then for every $x \in X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f^{-i}(x)}=X$,

$$
\lim _{n \rightarrow \infty} d_{f}^{-n}\left(f^{n}\right)^{*} \delta_{\sigma(x)}=\delta_{\sigma(\eta)} .
$$

5.6. Reduction. Let $\mathbf{k}^{\circ}$ be the valuation ring of $\mathbf{k}$ and $\mathbf{k}^{\circ \circ}$ the maximal ideal of $\mathbf{k}^{\circ}$. Set $\widetilde{\mathbf{k}}:=\mathbf{k}^{\circ} / \mathbf{k}^{\circ \circ}$ the residue field of $\mathbf{k}$. Let $\mathcal{X}$ be a flat projective scheme over $\mathbf{k}^{\circ}$. Denote by $X_{0}$ its special fiber, it is a (maybe reducible) variety over $\widetilde{\mathbf{k}}$. Let $X$ be the generic fiber of $\mathcal{X}$. Let $Y_{1}, \ldots, Y_{m}$ be the irreducible components of $X_{0}$ and $\eta_{i}, i=1, \ldots, m$ the generic points of $Y_{i}$. Set $\xi_{i}$ the unique point in $\operatorname{red}^{-1}\left(\eta_{i}\right)$.

Denote by red : $X^{\text {an }} \rightarrow X_{0}$ the reduction map. It is anti-continuous i.e. for every Zariski open subset $U$ of $X_{0}, \operatorname{red}^{-1}(U)$ is closed. In particular, for constructible topology on $X_{0}$, red : $X^{\text {an }} \rightarrow\left|X_{0}\right|$ Borel measurable.

For every $\mu \in \mathcal{M}\left(X^{\text {an }}\right)$, we may define its push forward $\operatorname{red}_{*} \mu \in \mathcal{M}\left(\left|X_{0}\right|\right)$ as follows: For every $\phi \in C\left(\left|X_{0}\right|\right)$,

$$
\int \phi \operatorname{red}_{*} \mu:=\int\left(\operatorname{red}^{*} \phi\right) \mu
$$

Because $\operatorname{red}^{*} \phi$ is Borel measurable and bounded, $\int\left(\operatorname{red}^{*} \phi\right) \mu$ is well defined and we have $\left|\int\left(\operatorname{red}^{*} \phi\right) \mu\right| \leq\|\phi\|_{\infty} \mu\left(X^{\mathrm{an}}\right)$. We note that, in general, red ${ }_{*}: \mathcal{M}\left(X^{\mathrm{an}}\right) \rightarrow$ $\mathcal{M}\left(\left|X_{0}\right|\right)$ is not continuous.

Example 5.8. Let $\mathcal{X}=\mathbb{P}_{\mathbf{k}^{\circ}}^{N}$. Let $x_{n}, n \geq 0$ be the Gauss point of the polydisc $\left\{\left|T_{i}\right| \leq 1-1 /(n+2), i=1, \ldots, N\right\} \subseteq\left(\mathbb{A}^{N}\right)^{\text {an }} \subseteq\left(\mathbb{P}^{N}\right)^{\text {an }}$. We have $\delta_{x_{n}} \rightarrow \xi_{1}$ as $n \rightarrow \infty$, but for every $n \geq 0$,

$$
\operatorname{red}_{*} \delta_{x_{n}}=\delta_{\operatorname{red}\left(x_{n}\right)}=\delta_{[1: 0 ; \cdots: 0]} \neq \delta_{\eta_{1}}=\operatorname{red}_{*} \delta_{\xi_{1}} .
$$

Proposition 5.9. Let $\mu_{n} \in \mathcal{M}^{1}\left(X^{\mathrm{an}}\right), n \geq 0$ be a sequence of probability Radon measures on $X^{\mathrm{an}}$. Assume that there are $a_{i} \geq 0, i=1, \ldots, m$ with $\sum_{i=1}^{m} a_{i}=1$ such that

$$
\operatorname{red}_{*}\left(\mu_{n}\right) \rightarrow \sum_{i=1}^{m} a_{i} \delta_{\eta_{i}}
$$

as $n \rightarrow \infty$. Then we have

$$
\mu_{n} \rightarrow \sum_{i=1}^{m} a_{i} \delta_{\xi_{i}}
$$

as $n \rightarrow \infty$.
Proof. Because $X^{\text {an }}$ is compact, $\mathcal{M}^{1}\left(X^{\text {an }}\right)$ is weak-* compact. So we may assume that

$$
\lim _{n \rightarrow \infty} \mu_{n}=\mu
$$

for some $\mu \in \mathcal{M}^{1}\left(X^{\text {an }}\right)$. We first show that $\operatorname{Supp} \mu \subseteq\left\{\xi_{1} \ldots, \xi_{m}\right\}$. Otherwise $\mu\left(X^{\text {an }} \backslash\left\{\xi_{1} \ldots, \xi_{m}\right\}\right)=1$. Then there is a compact subset $K$ of $X^{\text {an }} \backslash\left\{\xi_{1} \ldots, \xi_{m}\right\}$
such that $\mu(K)>0$. For every $x \in K$, set $V_{x}:=\operatorname{red}^{-1}(\overline{\operatorname{red}(x)})$. It is an open neighborhood of $x$ in $X^{\text {an }} \backslash\left\{\xi_{1} \ldots, \xi_{m}\right\}$. Because $K$ is compact, there is one $x \in K$ such that $\mu\left(V_{x}\right)>0$. Set $Z:=\overline{\operatorname{red}(x)}$. There is a compact subset $S \subseteq V_{x}$ such that $\mu(S)>0$. By Urysohn's Lemma, there is a continuous function $\chi: X^{\mathrm{an}} \rightarrow[0,1]$ such that $\left.\chi\right|_{S}=1$ and $\left.\chi\right|_{X^{\text {an }} \backslash V_{x}}=0$. Then we have

$$
\begin{gathered}
0=\lim _{n \rightarrow \infty} \int 1_{Z} \operatorname{red}_{*} \mu_{n}=\lim _{n \rightarrow \infty} \int\left(\operatorname{red}^{*} 1_{Z}\right) \mu_{n}=\lim _{n \rightarrow \infty} \int 1_{V_{x}} \mu_{n} \\
\geq \lim _{n \rightarrow \infty} \int \chi \mu_{n}=\int \chi \mu \geq \mu(S)>0
\end{gathered}
$$

which is a contradiction.
Now we may write $\mu=\sum_{i=1}^{m} b_{i} \delta_{\xi_{i}}$ with $b_{i} \geq 0$ and $\sum_{i=1}^{m} b_{i}=1$. For each $i=1, \ldots, m$, set $U_{i}:=Z_{i} \backslash\left(\cup_{j \neq i} Z_{j}\right)$. Then $\operatorname{red}^{-1}\left(U_{i}\right)$ is a closed subset contained in the open subset red $^{-1}\left(Z_{i}\right)$. By Urysohn's Lemma, there is a continuous function $\chi_{i}: X^{\text {an }} \rightarrow[0,1]$ such that $\left.\chi\right|_{\mathrm{red}^{-1}\left(U_{i}\right)}=1$ and $\left.\chi\right|_{X^{\text {an }} \backslash \operatorname{red}^{-1}\left(Z_{i}\right)}=0$. Then we have

$$
\begin{gathered}
b_{i}=\int \chi_{i} \mu=\lim _{n \rightarrow \infty} \int \chi_{i} \mu_{n} \\
\geq \lim _{n \rightarrow \infty} \mu_{n}\left(\operatorname{red}^{-1}\left(U_{i}\right)\right)=\lim _{n \rightarrow \infty} \int 1_{U_{i}} \operatorname{red}_{*} \mu_{n} \\
=\int 1_{U_{i}}\left(\sum_{j=1}^{m} a_{j} \delta_{\eta_{j}}\right)=a_{i}
\end{gathered}
$$

Because $\sum_{i=1}^{m} b_{i}=\sum_{i=1}^{m} a_{i}=1$, we get $b_{i}=a_{i}$ for every $i=1, \ldots, m$. This concludes the proof.

Now assume that $X_{0}$ is irreducible and smooth. Denote by $\eta$ the generic point of $X_{0}$ and $\xi$ the unique point in $\operatorname{red}^{-1}(\eta)$. Let $F: \mathcal{X} \rightarrow \mathcal{X}$ be a finite endomorphism. Denote by $f, f_{0}$ the restriction of $F$ to $X, X_{0}$. We note that for $i=0, \ldots, \operatorname{dim} X$, one has $\lambda_{i}(f)=\lambda_{i}\left(f_{0}\right)$.

By Theorem 1.22 and Proposition 5.9, we get the following equidistribution result for endomorphisms of good reductions.

Corollary 5.10. Assume that

$$
d_{f}:=\lambda_{\operatorname{dim} X}(f)>\max _{1 \leq i \leq \operatorname{dim} X-1} \lambda_{i}
$$

If the field extension $\widetilde{\mathbf{k}}\left(X_{0}\right) / f_{0}^{*} \widetilde{\mathbf{k}}\left(X_{0}\right)$ is separable, then for every $x \in X(\mathbf{k})$ with $\overline{\cup_{i \geq 0} f_{0}^{-i}(\operatorname{red} x)}=X_{0}$,

$$
\lim _{n \rightarrow \infty} d_{f}^{-n}\left(f^{n}\right)^{*} \delta_{x}=\delta_{\xi}
$$

One may compare Corollary 5.10 with [31, Theorem A] for polarized endomorphism. See [35, 20] for according result for complex topology.

## References

[1] Ekaterina Amerik. Existence of non-preperiodic algebraic points for a rational self-map of infinite order. Math. Res. Lett., 18(2):251-256, 2011.
[2] Ekaterina Amerik, Fedor Bogomolov, and Marat Rovinsky. Remarks on endomorphisms and rational points. Compositio Math., 147:1819-1842, 2011.
[3] Ekaterina Amerik and Frédéric Campana. Fibrations méromorphes sur certaines variétés à fibré canonique trivial. Pure Appl. Math. Q., 4(2, part 1):509-545, 2008.
[4] Jason P. Bell, Dragos Ghioca, and Zinovy Reichstein. On a dynamical version of a theorem of Rosenlicht. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 17(1):187-204, 2017.
[5] Jason P. Bell, Dragos Ghioca, Zinovy Reichstein, and Matthew Satriano. On the MedvedevScanlon conjecture for minimal threefolds of nonnegative Kodaira dimension. New York J. Math., 23:1185-1203, 2017.
[6] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang problem for étale maps. Amer. J. Math., 132(6):1655-1675, 2010.
[7] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. Applications of p-adic analysis for bounding periods for subvarieties under étale maps. Int. Math. Res. Not. IMRN, (11):35763597, 2015.
[8] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang problem for Noetherian spaces. Funct. Approx. Comment. Math., 53(2):313-328, 2015.
[9] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang conjecture, volume 210 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2016.
[10] Jason P. Bell, Fei Hu, and Matthew Satriano. Height gap conjectures, $D$-finiteness, and a weak dynamical Mordell-Lang conjecture. Math. Ann., 378(3-4):971-992, 2020.
[11] Olivier Benoist. Le théorème de Bertini en famille. Bull. Soc. Math. France, 139(4):555-569, 2011.
[12] Vladimir G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. Number 33 in Mathematical Surveys and Monographs. American Mathematical Society, 1990.
[13] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. Inst. Hautes Études Sci. Publ. Math., (78):5-161 (1994), 1993.
[14] Serge Cantat. Invariant hypersurfaces in holomorphic dynamics. Math. Research Letters, 17(5):833-841, 2010.
[15] Serge Cantat, Andriy Regeta, and Junyi Xie. Families of commuting automorphisms, and a characterization of the affine space. arXiv:1912.01567, 2019.
[16] Pietro Corvaja, Dragos Ghioca, Thomas Scanlon, and Umberto Zannier. The dynamical Mordell-Lang conjecture for endomorphisms of semiabelian varieties defined over fields of positive characteristic. J. Inst. Math. Jussieu, 20(2):669-698, 2021.
[17] Nguyen-Bac Dang. Degrees of Iterates of Rational Maps on Normal Projective Varieties.
[18] Nguyen-Bac Dang. Degrees of iterates of rational maps on normal projective varieties. Proc. Lond. Math. Soc. (3), 121(5):1268-1310, 2020.
[19] Olivier Debarre. Higher-dimensional algebraic geometry. Universitext. Springer-Verlag, New York, 2001.
[20] Tien-Cuong Dinh, Viêt-Anh Nguyên, and Tuyen Trung Truong. Equidistribution for meromorphic maps with dominant topological degree. Indiana Univ. Math. J., 64(6):1805-1828, 2015.
[21] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. Ann. of Math. (2), 161(3):1637-1644, 2005.
[22] Najmuddin Fakhruddin. Questions on self maps of algebraic varieties. J. Ramanujan Math. Soc., 18(2):109-122, 2003.
[23] Najmuddin Fakhruddin. The algebraic dynamics of generic endomorphisms of $\mathbb{P}^{n}$. Algebra Number Theory, 8(3):587-608, 2014.
[24] Charles Favre. Dynamique des applications rationnelles. PhD thesis, 2000.
[25] Dragos Ghioca. The dynamical Mordell-Lang conjecture in positive characteristic. Trans. Amer. Math. Soc., 371(2):1151-1167, 2019.
[26] Dragos Ghioca and Fei Hu. Density of orbits of endomorphisms of commutative linear algebraic groups. New York J. Math., 24:375-388, 2018.
[27] Dragos Ghioca and Sina Saleh. Zariski dense orbits for regular self-maps of tori in positive characteristic. (with sina saleh).
[28] Dragos Ghioca and Matthew Satriano. Density of orbits of dominant regular self-maps of semiabelian varieties. Trans. Amer. Math. Soc., 371(9):6341-6358, 2019.
[29] Dragos Ghioca and Thomas Scanlon. Density of orbits of endomorphisms of abelian varieties. Trans. Amer. Math. Soc., 369(1):447-466, 2017.
[30] Dragos Ghioca and Junyi Xie. The dynamical mordellclang conjecture for skew-linear selfmaps. appendix by michael wibmer. International Mathematics Research Notices, page rny211, 2018.
[31] William Gignac. Equidistribution of preimages over nonarchimedean fields for maps of good reduction. Ann. Inst. Fourier (Grenoble), 64(4):1737-1779, 2014.
[32] William Gignac. Measures and dynamics on Noetherian spaces. J. Geom. Anal., 24(4):1770-1793, 2014.
[33] M. H. Gizatullin. Rational G-surfaces. Izv. Akad. Nauk SSSR Ser. Mat., 44(1):110-144, 239, 1980.
[34] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. Inst. Hautes Études Sci. Publ. Math., (20):259, 1964.
[35] Vincent Guedj. Ergodic properties of rational mappings with large topological degree. Ann. of Math. (2), 161(3):1589-1607, 2005.
[36] Jia Jia, Takahiro Shibata, Junyi Xie, and De-Qi Zhang. Endomorphisms of quasiprojective varieties - towards Zariski dense orbit and Kawaguchi-Silverman conjectures. arXiv:2104.05339, 2021.
[37] Jia Jia, Junyi Xie, and De-Qi Zhang. Surjective endomorphisms of projective surfaces the existence of infinitely many dense orbits. arXiv:2005.03628, 2020.
[38] Shu Kawaguchi and Joseph H. Silverman. On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. J. Reine Angew. Math., 713:21-48, 2016.
[39] Shu Kawaguchi and Joseph H. Silverman. Erratum to: "On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties" (J. Reine Angew. Math. 713 (2016), 21-48). J. Reine Angew. Math., 761:291-292, 2020.
[40] John Lesieutre and Matthew Satriano. Canonical Heights on Hyper-Kähler Varieties and the Kawaguchi-Silverman Conjecture. Int. Math. Res. Not. IMRN, (10):7677-7714, 2021.
[41] Joseph Lipman. Desingularization of two-dimensional schemes. Ann. of Math. (2), 107(1):151-207, 1978.
[42] Yohsuke Matsuzawa. On upper bounds of arithmetic degrees. Amer. J. Math., 142(6):17971820, 2020.
[43] Yohsuke Matsuzawa, Kaoru Sano, and Takahiro Shibata. Arithmetic degrees and dynamical degrees of endomorphisms on surfaces. Algebra Number Theory, 12(7):1635-1657, 2018.
[44] Alice Medvedev and Thomas Scanlon. Polynomial dynamics. arXiv:0901.2352v1.
[45] Alice Medvedev and Thomas Scanlon. Invariant varieties for polynomial dynamical systems. Ann. of Math. (2), 179(1):81-177, 2014.
[46] Clayton Petsche. On the distribution of orbits in affine varieties. Ergodic Theory Dynam. Systems, 35(7):2231-2241, 2015.
[47] Bjorn Poonen. p-adic interpolation of iterates. Bull. Lond. Math. Soc., 46(3):525-527, 2014.
[48] Alexander Russakovskii and Bernard Shiffman. Value distribution for sequences of rational mappings and complex dynamics. Indiana Univ. Math. J., 46(3):897-932, 1997.
[49] Tuyen Trung Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. J. Reine Angew. Math., 758:139-182, 2020.
[50] Junyi Xie. Dynamical Mordell-Lang conjecture for birational polynomial morphisms on $\mathbb{A}^{2}$. Math. Ann., 360(1-2):457-480, 2014.
[51] Junyi Xie. Periodic points of birational transformations on projective surfaces. Duke Math. J., 164(5):903-932, 2015.
[52] Junyi Xie. The dynamical Mordell-Lang conjecture for polynomial endomorphisms of the affine plane. Astérisque, (394):vi+110, 2017.
[53] Junyi Xie. The existence of Zariski dense orbits for polynomial endomorphisms of the affine plane. Compos. Math., 153(8):1658-1672, 2017.
[54] Junyi Xie. Algebraic dynamics of the lifts of Frobenius. Algebra Number Theory, 12(7):1715-1748, 2018.
[55] Junyi Xie. The existence of Zariski dense orbits for endomorphisms of projective surfaces (with an appendix in collaboration with Thomas Tucker). arXiv:1905.07021, 2019.
[56] Shou-Wu Zhang. Distributions in algebraic dynamics. In Surveys in differential geometry. Vol. X, volume 10 of Surv. Differ. Geom., pages 381-430. Int. Press, Somerville, MA, 2006.

Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
Email address: junyi.xie@univ-rennes1.fr


[^0]:    ${ }^{1}$ In [33, there is an assumption that char $\mathbf{k} \neq 2,3$. But, it is checked in 15 that such assumption in 33 can be removed.

