

MAXIMAL PRONILFACTORS AND A TOPOLOGICAL WIENER-WINTNER THEOREM

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To Benjamin Weiss with great respect.

ABSTRACT. For strictly ergodic systems, we introduce the class of CF- $\text{Nil}(k)$ systems: systems for which the maximal measurable and maximal topological k -step pronilfactors coincide as measure-preserving systems. Weiss' theorem implies that such systems are abundant in a precise sense. We show that the CF- $\text{Nil}(k)$ systems are precisely the class of minimal systems for which the k -step nilsequence version of the Wiener-Wintner average converges everywhere. As part of the proof we establish that pronilfactors are *coalescent* both in the measurable and topological categories. In addition, we characterize a CF- $\text{Nil}(k)$ system in terms of its $(k+1)$ -th dynamical cubespace. In particular, for $k = 1$, this provides for strictly ergodic systems a new condition equivalent to the property that every measurable eigenfunction has a continuous version.

CONTENTS

1. Introduction.	2
2. Preliminaries.	5
2.1. Dynamical systems.	5
2.2. Topological models.	7
2.3. Conditional expectation.	7
2.4. Pronilfactors and nilsequences.	8
2.5. Host-Kra structure theory machinery.	9
2.6. Maximal measurable pronilfactors.	10
2.7. Maximal topological pronilfactors.	11
2.8. CF- $\text{Nil}(k)$ systems.	12

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2.9. A CF- $\text{Nil}(k)$ topological model.	13
3. Coalescence and universality for maximal pronilfactors.	13
3.1. Coalescence	13
3.2. Universality	15
4. Cubespace characterization of CF- $\text{Nil}(k)$.	16
5. A topological Wiener-Wintner theorem.	21
References	24

1. INTRODUCTION.

In recent years there has been an increase in interest in pronilfactors both for measure-preserving systems (m.p.s.) and topological dynamical systems (t.d.s.). Pronilfactors of a given system are either measurable or topological (depending on the category) factors given by an inverse limit of nilsystems. A t.d.s. (m.p.s.) is called a topological (measurable) d -step pronilsystem if it is a topological (measurable) inverse limit of nilsystems of degree at most d .¹ In the theory of measure preserving systems (X, \mathcal{X}, μ, T) maximal measurable pronilfactors appear in connection with the L^2 -convergence of the nonconventional ergodic averages

$$(1) \quad \frac{1}{N} \sum f_1(T^n x) \dots f_k(T^{kn} x)$$

for $f_1, \dots, f_k \in L^\infty(X, \mu)$ ([HK05, Zie07]). In the theory of topological dynamical systems maximal topological pronilfactors appear in connection with the higher order regionally proximal relations ([HKM10, SY12, GGY18]).

When a system possesses both measurable and topological structure, it seems worthwhile to investigate pronilfactors both from a measurable and topological point of view. A natural meeting ground are strictly ergodic systems - minimal topological dynamical systems (X, T) possessing a unique invariant measure μ . For $k \in \mathbb{Z}$ let us denote by $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ respectively $(W_k(X), T)$ the maximal k -step measurable respectively topological pronilfactor² of (X, T) . Clearly $(W_k(X), T)$ has a unique invariant measure ω_k . We thus pose the question when is $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s.? We call a t.d.s. which is strictly ergodic and for which $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s., a *CF- $\text{Nil}(k)$* system³. Note that $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ is always a measurable factor of $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$. At first glance it may seem

¹It is a classical fact that every (measurable) ergodic d -step pronilsystem is isomorphic as m.p.s. to a (topological) minimal d -step pronilsystem.

²Both these objects exist and are unique in a precise sense. See Subsection 3.2.

³This terminology is explained in Subsection 2.8.

that CF- $\text{Nil}(k)$ systems are rare however a theorem by Benjamin Weiss regarding topological models for measurable extensions implies that every ergodic m.p.s. is measurably isomorphic to a CF- $\text{Nil}(k)$ system⁴.

We give two characterizations of CF- $\text{Nil}(k)$ systems. The first characterization is related to the Wiener-Wintner theorem while the second characterization is related to *k-cube uniquely ergodic* systems - a class of topological dynamical systems introduced in [GL19].

The Wiener-Wintner theorem ([WW41]) states that for an ergodic system (X, \mathcal{X}, μ, T) , for μ -a.e. $x \in X$, any $\lambda \in \mathbb{S}^1$ and any $f \in L^\infty(\mu)$, the following limit exists:

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda^n f(T^n x)$$

Denote by $M_T \subset \mathbb{S}^1$ the set of measurable eigenvalues⁵ of (X, \mathcal{X}, μ, T) . Let $P_\lambda f$ be the projection of f to the eigenspace corresponding to λ (in particular for $\lambda \notin M_T$, $P_\lambda f \equiv 0$). For fixed $\lambda \in \mathbb{S}^1$, one can show (2) converges a.s. to $P_\lambda f$.

In [Les96] Lesigne proved that a.s. convergence in (2) still holds when the term λ^n is replaced by a (continuous function) of a real-valued polynomial $P(n)$, $P \in \mathbb{R}[t]$. In [Fra06] Frantzikinakis established a *uniform version*⁶ of this theorem. In [HK09], Host and Kra showed that a.s. convergence in (2) still holds when the term λ^n is replaced by a *nilsequence*. In [EZK13] Eisner and Zorin-Kranich established a uniform version of this theorem.

For topological dynamical systems one may investigate the question of *everywhere* convergence in the Wiener-Wintner theorem. In [Rob94], Robinson proved that for an uniquely ergodic system (X, μ, T) , for any $f \in C(X)$, if every measurable eigenfunction of (X, \mathcal{X}, μ, T) has a continuous version then the limit (2) converges everywhere. He noted however that if $P_\lambda f \neq 0$ for some $\lambda \in M_T$, then the convergence of (2) is not uniform in (x, λ) , since the limit function $P_\lambda f(x)$ is not continuous on $X \times \mathbb{S}^1$.⁷ Moreover Robinson constructed a strictly ergodic system (X, T) such that (2) does not converge for some continuous function $f \in C(X)$, some $\lambda \in \mathbb{C}$ and some $x \in X$. Other topological versions of the Wiener-Wintner theorem may be found in [Ass92, Fan18]⁸.

The first main result of this article is the following theorem:

⁴See Subsection 2.9.

⁵Measurable and topological eigenvalues are defined in Subsection 2.1.

⁶In the context of the Wiener-Wintner theorem, *uniform versions* are a.s. convergence results involving a supremum over weights belonging to a given class. The first result of this type was obtained by Bourgain in [Bou90].

⁷Note M_T is countable.

⁸One should also note that topological Wiener-Winter theorems have been investigated in the generality of operator semigroups by Schreiber and Bartoszek and Śpiewak ([Sch14, BŚ17]).

Theorem A. *Let (X, T) be a minimal system. Then for $k \geq 0$ the following are equivalent:*

- (I). (X, T) is a CF- $\text{Nil}(k)$ system.
- (II). For any k -step nilsequence $\{a(n)\}_{n \in \mathbb{Z}}$, any continuous function $f \in C(X)$ and any $x \in X$,

$$(3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n) f(T^n x)$$

exists.

We remark that the direction (I) \Rightarrow (II) of Theorem A follows from [HK09] whereas the case $k = 1$ of Theorem A follows from [Rob94, Theorem 1.1].

As part of the proof of Theorem A we established a fundamental property for pronilsystems:

Theorem B. *Let (Y, ν, T) be a minimal (uniquely ergodic) k -step pronil-system. Then*

- (I). (Y, ν, T) is measurably coalescent, i.e. if $\pi : (Y, \nu, T) \rightarrow (Y, \nu, T)$ is a measurable factor map, then π is a measurable isomorphism.

and

- (II). (Y, T) is topologically coalescent, i.e. if $\Phi : (Y, T) \rightarrow (Y, T)$ is a topological factor map, then Φ is a topological isomorphism.

As part of the theory of higher order regionally proximal relations, Host, Kra and Maass introduced in [HKM10] the *dynamical cubespaces* $C_T^n(X) \subset X^{2^n}$, $n \in \mathbb{N} := \{1, 2, \dots\}$. These compact sets enjoy a natural action by the *Host-Kra cube groups* $\mathcal{HK}^n(T)$. According to the terminology introduced in [GL19], a t.d.s. (X, T) is called *k -cube uniquely ergodic* if $(C_T^k(X), \mathcal{HK}^k(T))$ is uniquely ergodic. The third main result of this article is the following theorem:

Theorem C. *Let (X, T) be a minimal t.d.s. Then the following are equivalent for any $k \geq 0$:*

- (I). (X, T) is a CF- $\text{Nil}(k)$ system.
- (II). (X, T) is $(k + 1)$ -cube uniquely ergodic.

We remark that the direction (I) \Rightarrow (II) follows from [HSY17].

In the context of various classes of strictly ergodic systems, several authors have investigated the question of whether every measurable eigenfunction has a continuous version. Famously in [Hos86] (see also [Que10, Page 170]), Host established this is the case for *admissible substitution dynamical systems*. In [BDM10, Theorem 27] an affirmative answer was given for strictly ergodic *Toeplitz type systems of finite*

rank. In [DFM19], the continuous and measurable eigenvalues of minimal Cantor systems were studied.

It is easy to see that for strictly ergodic systems (X, T) the condition that every measurable eigenfunction has a continuous version is equivalent to the fact that (X, T) is CF- $\text{Nil}(1)$. Thus Theorem C provides for strictly ergodic systems a new condition equivalent to the property that every measurable eigenfunction has a continuous version. Namely this holds iff $(C_T^2(X), \mathcal{HK}^2(T))$ is uniquely ergodic. As the last condition seems quite manageable one wonders if this new equivalence may turn out to be useful in future applications.

Structure of the paper. In Subsections 2.1–2.3 we review some definitions and classical facts; In Subsections 2.4–2.8, we introduce the topological and measurable maximal pronilfactors and define the CF- $\text{Nil}(k)$ systems; In Subsection 2.9, we use Weiss’s Theorem to show that the CF- $\text{Nil}(k)$ systems are abundant; In Section 3, we prove Theorem B and then establish *universality* for maximal pronilfactors; In Section 4, we prove Theorem C; In Section 5, we prove Theorem A.

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2. PRELIMINARIES.

2.1. Dynamical systems. Throughout this article we assume every topological space to be metrizable. A **\mathbb{Z} -topological dynamical system (t.d.s.)** is a pair (X, T) , where X is a compact space and T is a homeomorphism on X . Denote by $C(X)$ the set of real-valued continuous functions on X . The **orbit** $\mathcal{O}(x)$ of $x \in X$ is the set $\mathcal{O}(x) = \{T^n x : n \in \mathbb{Z}\}$. Its closure is denoted by $\overline{\mathcal{O}}(x)$. A t.d.s. is **minimal** if $\overline{\mathcal{O}}(x) = X$ for all $x \in X$. A t.d.s. (X, T) is **distal** if for a compatible metric d_X of X , for any $x \neq y \in X$, $\inf_{n \in \mathbb{Z}} d_X(T^n x, T^n y) > 0$. We say $\pi : (Y, S) \rightarrow (X, T)$ is a **topological factor map** if π is a continuous and surjective map such that for any $x \in X$, $\pi(Sx) = T\pi(x)$. Given such a map, (X, T) is called a **topological factor** of (Y, S) and (X, T) is said to **factor continuously** on (Y, S) . If in addition π is injective then it is called a **topological isomorphism** and (Y, S) and (X, T) are said to be **isomorphic as t.d.s.** A factor map $\pi : (Y, S) \rightarrow (X, T)$ is called a **topological group extension** by a compact group K if there exists a continuous action $\alpha : K \times Y \rightarrow Y$ such that the actions S and K commute and for all $x, y \in Y$, $\pi(x) = \pi(y)$ iff there exists a unique $k \in K$ such that $kx = y$. A **(topological) eigenvalue** of a t.d.s. (X, T) is a complex number $\lambda \in \mathbb{S}^1$ such that an equation of the form $f(Tx) = \lambda f(x)$ holds for some $f \in C(X, \mathbb{C})$ and all $x \in X$. The function f is referred to as a **continuous** or **topological eigenfunction**.

Let $\{(X_m, T_m)\}_{m \in \mathbb{N}}$ be a sequence of t.d.s. and for any $m \geq n$, $\pi_{m,n} : (X_n, T_n) \rightarrow (X_m, T_m)$ factor maps such that $\pi_{i,l} = \pi_{i,j} \circ \pi_{j,l}$ for all $1 \leq i \leq j \leq l$. The **inverse limit** of $\{(X_m, T_m)\}_{m \in \mathbb{N}}$ is defined to be the system (X, T) , where

$$X = \{(x_m)_{m \in \mathbb{N}} \in \prod_{m \in \mathbb{N}} X_m : \pi_{m+1}(x_{m+1}) = x_m \text{ for } m \geq 1\}$$

equipped with the product topology and $T(x_m)_{m \in \mathbb{N}} \triangleq (T_m x_m)_{m \in \mathbb{N}}$. We write $(X, T) = \varprojlim (X_m, T_m)$.

A **measure preserving probability system (m.p.s.)** is a quadruple (X, \mathcal{X}, μ, T) , where (X, \mathcal{X}, μ) is a standard Borel probability space (in particular X is a Polish space and \mathcal{X} is its Borel σ -algebra) and T is an invertible Borel measure-preserving map ($\mu(TA) = \mu(A)$ for all $A \in \mathcal{X}$). An m.p.s. (X, \mathcal{X}, μ, T) is **ergodic** if for every set $A \in \mathcal{X}$ such that $T(A) = A$, one has $\mu(A) = 0$ or 1 . A **measurable factor map** is a Borel map $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ which is induced by a G -invariant sub- σ -algebra of \mathcal{X} ([Gla03, Chapter 2.2]). Given such a map, (Y, \mathcal{Y}, ν, S) is called a **measurable factor** of (X, \mathcal{X}, μ, T) . If π is in addition invertible on a set of full measure then π is called a **measurable isomorphism** and (X, \mathcal{X}, μ, T) and (Y, \mathcal{Y}, ν, S) are said to be **isomorphic as m.p.s.** Let (Y, \mathcal{Y}, ν, S) be an m.p.s. and A a compact group with Borel σ -algebra \mathcal{A} and Haar measure m . A **skew-product** $(Y \times A, \mathcal{Y} \otimes \mathcal{A}, \nu \times m, T)$ is given by the action $T(y, u) = (Sy, \beta(y)u)$, where $\beta : Y \rightarrow A$ is a Borel map, the so-called *cocycle* of the skew-product. The projection $(Y \times A, \mathcal{Y} \otimes \mathcal{A}, \nu \times m, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ given by $(y, a) \mapsto y$ is called a **measurable group extension** (cf. [Gla03, Theorem 3.29]).

A **(measurable) eigenvalue** of a m.p.s. (X, \mathcal{X}, μ, T) is a complex number $\lambda \in \mathbb{S}^1$ such that an equation of the form $f(Tx) = \lambda f(x)$ holds for μ -a.e. $x \in X$ for some Borel function $f : X \rightarrow \mathbb{C}$. The function f is referred to as a **measurable eigenfunction**.

Denote by $P_T(X)$ the set of T -invariant Borel probability measures of X . A t.d.s. (X, T) is called **uniquely ergodic** if $|P_T(X)| = 1$. If in addition it is minimal then it is called **strictly ergodic**. For a strictly ergodic system (X, T) with a (unique) invariant measure μ , we will use the notation (X, μ, T) . When considered as a m.p.s. it is with respect to its Borel σ -algebra.

Occasionally in this article we will consider more general group actions than \mathbb{Z} -actions. Thus a **G -topological dynamical system (t.d.s.)** is a pair (G, X) consisting of a (metrizable) topological group G acting on a (metrizable) compact space X . For $g \in G$ and $x \in X$ we denote the action both by gx and $g.x$. We will need the following proposition:

Proposition 2.1. *Let G be an amenable group. Let (G, X) be uniquely ergodic and let $(G, X) \rightarrow (G, Y)$ be a topological factor map. Then (G, Y) is uniquely ergodic.*

Proof. See proof of Proposition 8.1 of [AKL14]. □

2.2. Topological models.

Definition 2.2. Let (X, \mathcal{X}, μ, T) be a m.p.s. We say that a t.d.s. (\hat{X}, \hat{T}) is a **topological model** for (X, \mathcal{X}, μ, T) w.r.t. to a \hat{T} -invariant probability measure $\hat{\mu}$ on $\hat{\mathcal{X}}$, the Borel σ -algebra of \hat{X} , if the system (X, \mathcal{X}, μ, T) is isomorphic to $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ as m.p.s., that is, there exist a T -invariant Borel subset $C \subset X$ and a \hat{T} -invariant Borel subset $\hat{C} \subset \hat{X}$ of full measure and a (bi)measurable and equivariant measure preserving bijective Borel map $p : C \rightarrow \hat{C}$. Notice that oftentimes in this article (\hat{X}, \hat{T}) will be uniquely ergodic so that $\hat{\mu}$ will be the unique \hat{T} -invariant probability measure of \hat{X} .

Definition 2.3. Let (X, \mathcal{X}, μ, T) , (Y, \mathcal{Y}, ν, S) be m.p.s. Let (\hat{X}, \hat{T}) , (\hat{Y}, \hat{S}) be t.d.s. which are topological models of (X, \mathcal{X}, μ, T) and (Y, \mathcal{Y}, ν, S) w.r.t. measures $\hat{\mu}$ and $\hat{\nu}$ as witnessed by maps ϕ and ψ respectively. We say that $\hat{\pi} : (\hat{X}, \hat{T}) \rightarrow (\hat{Y}, \hat{S})$ is a **topological model** for a factor map $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ if $\hat{\pi}$ is a topological factor and the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \hat{X} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ Y & \xrightarrow{\psi} & \hat{Y} \end{array}$$

is commutative, i.e. $\hat{\pi}\phi = \psi\pi$

2.3. Conditional expectation. Let (X, \mathcal{X}, μ) be a probability space and let \mathcal{B} be a sub- σ -algebra of \mathcal{X} . For $f \in L^1(\mu)$, the **conditional expectation** of f w.r.t. \mathcal{B} is the unique function $\mathbb{E}(f|\mathcal{B}) \in L^1(X, \mathcal{B}, \mu)$ satisfying

$$(4) \quad \int_B f d\mu = \int_B \mathbb{E}(f|\mathcal{B}) d\mu$$

for every $B \in \mathcal{B}$. For $f \in L^1(\mu)$ and $g \in L^\infty(X, \mathcal{B}, \mu)$, it holds (see [HK18, Chapter 2, Section 2.4]):

$$(5) \quad \int_X fg d\mu = \int_X \mathbb{E}(f|\mathcal{B})g d\mu.$$

Let (X, \mathcal{X}, μ) and (Y, \mathcal{Y}, ν) be probability spaces and let $\pi : X \rightarrow Y$ be a measurable map such that $\pi_*\mu = \nu$. Denote by $\mathbb{E}(f|Y) \in L^1(Y, \nu)$ the function such that $\mathbb{E}(f|Y) = \mathbb{E}(f|\pi^{-1}(\mathcal{Y})) \circ \pi^{-1}$. Note this is well-defined. Thus the difference between $\mathbb{E}(f|Y)$ and $\mathbb{E}(f|\pi^{-1}(\mathcal{Y}))$ is that

the first function is considered as a function on Y and the second as a function on X .

2.4. Pronilsystems and nilsequences.

Definition 2.4. A (real) **Lie group** is a group that is also a finite dimensional real smooth manifold such that the group operations of multiplication and inversion are smooth. Let G be a Lie group. Let $G_1 = G$ and $G_k = [G_{k-1}, G]$ for $k \geq 2$, where $[G, H] = \{[g, h] : g \in G, h \in H\}$ and $[g, h] = g^{-1}h^{-1}gh$. If there exists some $d \geq 1$ such that $G_{d+1} = \{e\}$, G is called a **d -step nilpotent Lie group**. We say that a discrete subgroup Γ of a Lie group G is **cocompact** if G/Γ , endowed with the quotient topology, is compact. We say that quotient $X = G/\Gamma$ is a **d -step nilmanifold** if G is a d -step nilpotent Lie group and Γ is a discrete, cocompact subgroup. The nilmanifold X admits a natural action by G through **translations** $g.a\Gamma = ga\Gamma$, $g, a \in G$. The **Haar measure** of X is the unique Borel probability measure on X which is invariant under this action. A **nilsystem of degree at most d** , (X, T) , is given by an d -step nilmanifold $X = G/\Gamma$ and $T \in G$ with action $T.a\Gamma = Ta\Gamma$. When a nilsystem is considered as a m.p.s. it is always w.r.t. its Haar measure.

Definition 2.5. A t.d.s. (m.p.s) is called a topological (measurable) **d -step pronilsystem** if it is a topological (measurable) inverse limit of nilsystems of degree at most d . By convention a 0-step pronilsystem is the one-point trivial system.

Remark 2.6. By [HK18, p. 233] if an ergodic measurable d -step pronilsystem is presented as the inverse limit $(X, \mathcal{X}, \nu, T) = \varprojlim (X_m, \mathcal{X}_m, \nu_m, T_m)$ given by the measurable factor maps $\pi_m : (X_m, \mathcal{X}_m, \nu_m, T_m) \rightarrow (X_{m-1}, \mathcal{X}_{m-1}, \nu_{m-1}, T_{m-1})$ between nilsystems of degree at most d then there exist topological factor maps $\tilde{\pi}_m : (X_m, T_m) \rightarrow (X_{m-1}, T_{m-1})$ such that $\tilde{\pi} = \pi$ ν_m -a.e. and so effectively one can consider (X, \mathcal{X}, ν, T) as a (minimal) topological pronilsystem. Moreover any two d -step pronilsystem topological models of (X, \mathcal{X}, ν, T) are isomorphic as t.d.s. (Theorem 3.3).

Definition 2.7. ([HKM10, Definition 2.2]) A bounded sequence $\{a(n)\}_{n \in \mathbb{Z}}$ is called a **d -step nilsequence** if there exists a d -step pronilsystem (X, T) , $x_0 \in X$ and a continuous function $f \in C(X)$ such that $a(n) = f(T^n x_0)$ for $n \in \mathbb{Z}$.

Theorem 2.8. ([HK09, Theorem 3.1]) *Let (X, T) be a nilsystem. Then (X, T) is uniquely ergodic if and only if (X, T) is ergodic w.r.t. the Haar measure if and only if (X, T) is minimal.*

The following proposition is an immediate corollary of the previous theorem.

Proposition 2.9. *Let (X, T) be a pronilsystem. Then (X, T) is uniquely ergodic if and only if (X, T) is minimal.*

Definition 2.10. Let (X, μ, T) be a strictly ergodic t.d.s. We say that a t.d.s. (Y, T) is a **topological k -step pronilfactor** of (X, T) if it is a topological factor of (X, T) and if it is isomorphic to a k -step pronilsystem as a t.d.s. We say that a m.p.s. (Y, \mathcal{Y}, ν, T) is a **measurable k -step pronilfactor** of (X, T) if it is a measurable factor of (X, \mathcal{X}, μ, T) and if it is isomorphic to a k -step pronilsystem as a m.p.s.

2.5. Host-Kra structure theory machinery. By a **face** of the discrete cube $\{0, 1\}^k$ we mean a subcube obtained by fixing some subset of the coordinates. For $k \in \mathbb{N}$, let $[k] = \{0, 1\}^k$. Thus $X^{[k]} = X \times \cdots \times X$, 2^k times and similarly $T^{[k]} = T \times \cdots \times T$, 2^k times. For $x \in X$, $x^{[k]} = (x, \dots, x) \in X^{[k]}$. Let $[k]_* = \{0, 1\}^k \setminus \{\vec{0}\}$ and define $X_*^{[k]} = X^{[k]_*}$.

Definition 2.11. ([HK05]) Let (X, \mathcal{X}, μ, T) be an ergodic m.p.s. For $1 \leq j \leq k$, let $\bar{\alpha}_j = \{v \in \{0, 1\}^k : v_j = 1\}$ be the j -th **upper face** of $\{0, 1\}^k$. For any face $F \subset \{0, 1\}^k$, define

$$(T^F)_v = \begin{cases} T & v \in F \\ \text{Id} & v \notin F. \end{cases}$$

Define the **face group** $\mathcal{F}^k(T) \subset \text{Homeo}(X^{[k]})$ to be the group generated by the elements $\{T^{\bar{\alpha}_j} : 1 \leq j \leq k\}$. Define the the k -th **Host-Kra cube group** $\mathcal{HK}^k(T)$ to be the subgroup of $\text{Homeo}(X^{[k]})$ generated by $\mathcal{F}^k(T)$ and $T^{[k]}$.

Definition 2.12. ([HK05]) Let (X, \mathcal{B}, μ, T) be an ergodic m.p.s. Let $\mu^{[1]} = \mu \times \mu$. For $k \in \mathbb{N}$, let $\mathcal{I}_{T^{[k]}}$ be the $T^{[k]}$ -invariant σ -algebra of $(X^{[k]}, \mathcal{X}^{[k]}, \mu^{[k]})$. Define $\mu^{[k+1]}$ to be the relative independent joining of two copies of $\mu^{[k]}$ over $\mathcal{I}_{T^{[k]}}$. That is, for $f_v \in L^\infty(\mu)$, $v \in \{0, 1\}^{k+1}$:

$$\begin{aligned} \int_{X^{[k+1]}} \prod_{v \in \{0, 1\}^{k+1}} f_v(x_v) d\mu^{[k+1]}(x) &= \\ \int_{X^{[k]}} \mathbb{E}\left(\prod_{v \in \{0, 1\}^k} f_{v0} | \mathcal{I}_{T^{[k]}}\right)(x) \mathbb{E}\left(\prod_{v \in \{0, 1\}^k} f_{v1} | \mathcal{I}_{T^{[k]}}\right)(x) d\mu^{[k]}(x). \end{aligned}$$

In particular, from Equation (5), it follows that for all measurable functions $H_1, H_2 \in L^\infty(X^{[k]}, \mu^{[k]})$,

$$(6) \quad \int_{X^{[k]}} \mathbb{E}(H_1 | \mathcal{I}_{T^{[k]}})(c) \mathbb{E}(H_2 | \mathcal{I}_{T^{[k]}})(c) d\mu^{[k]}(c) = \int_{X^{[k]}} \mathbb{E}(H_1 | \mathcal{I}_{T^{[k]}})(c) H_2(c) d\mu^{[k]}(c).$$

Note $\mu^{[k]}$ is $\mathcal{HK}^k(T)$ -invariant ([HK18, Chapter 9, Proposition 2]).

Definition 2.13. [HK18, Chapter 9, Section 1] For $k \in \mathbb{N}$, let \mathcal{J}_*^k be the σ -algebras of sets invariant under $\mathcal{F}^k(T)$ on $X_*^{[k]}$.

Definition 2.14. [HK18, Subsection 9.1] Let (X, \mathcal{X}, μ, T) be an ergodic m.p.s. For $k \in \mathbb{N}$, define $\mathcal{Z}_k(X)$ to be the σ -algebra consisting of measurable sets B such that there exists a \mathcal{J}_*^{k+1} -measurable set $A \subset X_*^{[k+1]}$ so that up to $\mu^{[k+1]}$ -measure zero it holds:

$$X \times A = B \times X_*^{[k+1]}$$

Define the k -th **Host-Kra factor** $Z_k(X)$ as the measurable factor of X induced by $\mathcal{Z}_k(X)$ and denote by $\pi_k : X \rightarrow Z_k(X)$ the **(measurable) canonical k -th projection**. Let μ_k be the projection of μ w.r.t. π_k .

Definition 2.15. Let (X, \mathcal{X}, μ, T) be an m.p.s. and $k \in \mathbb{N}$. The **Host-Kra-Gowers seminorms** on $L^\infty(\mu)$ are defined as follows:

$$\|f\|_k = \left(\int \prod_{v \in \{0,1\}^k} \mathcal{C}^{|v|} f d\mu^{[k]} \right)^{1/2^k},$$

where $|(v_1, \dots, v_{k+1})| = \sum_{i=1}^{k+1} v_i$ and $\mathcal{C}^n z = z$ if n is even and $\mathcal{C}^n z = \bar{z}$ if n is odd. By [HK18, Subsection 8.3], $\|\cdot\|_k$ is a seminorm.

Lemma 2.16. [HK18, Chapter 9, Theorem 7] *Let (X, \mathcal{X}, μ, T) be an ergodic m.p.s. and $k \in \mathbb{N}$. Then for $f \in L^\infty(\mu)$, $\|f\|_{k+1} = 0$ if and only if $\mathbb{E}(f | \mathcal{Z}_k(X)) = 0$.*

2.6. Maximal measurable pronilfactors.

Definition 2.17. Let $k \in \mathbb{N}$. A m.p.s. (X, \mathcal{X}, μ, T) is called a **(measurable) system of order k** if it is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$.

Theorem 2.18. ([HK05, Theorem 10.1], [HK18, Chapter 16, Theorem 1], *for an alternative proof see [GL19, Theorem 5.3]*) *An ergodic m.p.s. is a system of order k iff it is isomorphic to a minimal k -step pronilsystem as m.p.s.*

Remark 2.19. Let (X, \mathcal{X}, μ, T) be an ergodic m.p.s. In the literature $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is referred to as the **maximal measurable k -step pronilfactor** or as the *maximal factor which is a system of order k* (see [HK18, Chapter 9, Theorem 18]). By this it is meant that any measurable factor map $\phi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ where (Y, \mathcal{Y}, ν, S) is a minimal k -step pronilsystem, factors through the canonical k -th projection $\pi_k : (X, \mathcal{X}, \mu, T) \rightarrow (Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$, i.e., there exists a unique (up to measure zero) $\psi : (Z_k(X), \mathcal{Z}_k(X), \mu_k, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ such that $\phi = \psi \circ \pi_k$ a.s. In section 3 we establish the complementary property of *universality* for $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$.

Remark 2.20. In [HKM14, Corollary 2.2] a criterion for an ergodic m.p.s. (X, \mathcal{X}, μ, T) to have $Z_k(X) = Z_1(X)$ for all $k \geq 1$ is given. Indeed this is the case for ergodic systems whose spectrum does not

admit a Lebesgue component with infinite multiplicity. In particular this holds true for weakly mixing systems, systems with singular maximal spectral type and systems with finite spectral multiplicity.

2.7. Maximal topological pronilfactors. Recall the Definition of $\mathcal{HK}^k(T)$ and $\mathcal{F}^k(T)$ (Definition 2.11).

Definition 2.21. Let (X, T) be a minimal t.d.s. Define the **induced $(k + 1)$ -th dynamical cubespace** by:

$$C_T^{k+1}(X) = \overline{\{gx^{[k+1]} \mid g \in \mathcal{HK}^{k+1}(T)\}}.$$

Definition 2.22. ([HKM10, Definition 3.2]) Let (X, T) be a topological dynamical system and $k \geq 1$. The points $x, y \in X$ are said to be **regionally proximal of order k** , denoted $(x, y) \in \text{RP}^{[k]}(X)$, if there are sequences of elements $f_i \in \mathcal{F}^k(T)$, $x_i, y_i \in X$, $a_* \in X_*^{[k]}$ such that

$$\lim_{i \rightarrow \infty} (f_i x_i^{[k]}, f_i y_i^{[k]}) = (x, a_*, y, a_*).$$

Theorem 2.23. ([SY12, Theorem 3.5]⁹) *Let (X, T) be a minimal t.d.s. and $k \geq 1$. Then $\text{RP}^{[k]}(X)$ is a closed T -invariant equivalence relation.*

Definition 2.24. A t.d.s. (X, T) is called a **(topological) system of order k** if $\text{RP}^{[k]}(X) = \{(x, x) \mid x \in X\}$.

Theorem 2.25. ([HKM10, Theorem 1.2], *for an alternative proof see [GMV20, Theorem 1.30]*) *A minimal t.d.s. is a topological system of order k iff it is isomorphic to a minimal k -step pronilsystem as t.d.s.*

Theorem 2.23 allows us to give the following definition.

Definition 2.26. Let (X, T) be a minimal t.d.s. Define the **maximal k -step nilfactor** by $W_k(X) = X / \text{RP}^{[k]}(X)$. Denote the associated map $\pi_k^{\text{top}} : X \rightarrow W_k(X)$ as the **(topological) canonical k -th projection**.

Remark 2.27. The terminology of Definition 2.26 is justified by the following property: Any topological factor map $\phi : (X, T) \rightarrow (Y, T)$ where (Y, T) is a system of order k , factors through the canonical k -th projection $\pi_k^{\text{top}} : (X, T) \rightarrow (W_k(X), T)$, i.e., there exists a unique $\psi : (W_k(X), T) \rightarrow (Y, T)$ such that $\phi = \psi \circ \pi_k^{\text{top}}$ ([HKM10, Proposition 4.5]). In section 3 we establish the complementary property of *universality* for $(W_k(X), T)$.

Definition 2.28. ([GL19, Definition 3.1]) A t.d.s. (X, T) is called **k -cube uniquely ergodic** if $(C_T^k(X), \mathcal{HK}^k(T))$ is uniquely ergodic.

⁹This theorem was generalized to arbitrary minimal group actions in [GGY18, Theorem 3.8].

2.8. CF- $\text{Nil}(k)$ systems.

Definition 2.29. For $k \geq 0$, we say (X, T) is a **CF- $\text{Nil}(k)$** system if (X, T) is strictly ergodic and $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is isomorphic to $(W_k(X), \omega_k, T)$ as m.p.s. where μ_k and ω_k are the images of the unique invariant measure of (X, T) under the measurable, respectively topological canonical k -th projections.

Remark 2.30. By convention $Z_0(X) = W_0(X) = \{\bullet\}$. Thus every strictly ergodic (X, T) is CF- $\text{Nil}(0)$.

The term " (X, μ, T) is CF- $\text{Nil}(k)$ " is an abbreviation of

" (X, μ, T) Continuously **F**actors on a **k**-step pro**N**ilsystem which is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s."

Indeed if $(W_k(X), \omega_k, T)$ is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s. then obviously this condition holds. The reverse implication is given by the following proposition which has been (implicitly) used several times in the literature ([HK09, HKM14, HSY19]). Its proof is given at the end of Subsection 3.2.

Proposition 2.31. *Let (X, T) be a strictly ergodic t.d.s. which topologically factors on a (minimal) k -step pronilsystem (\hat{Z}_k, T) with the unique ergodic measure γ_k . If $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is isomorphic to (\hat{Z}_k, γ_k, T) as m.p.s., then (\hat{Z}_k, T) and $(W_k(X), T)$ are isomorphic as t.d.s. In particular (X, μ, T) is CF- $\text{Nil}(k)$.*

Theorem C allows us to give a remarkable simple proof of the following Theorem.

Theorem 2.32. *Let (X, T) be a CF- $\text{Nil}(k)$ system. The following holds:*

- (1) *If $\pi : (X, T) \rightarrow (Y, T)$ is a topological factor map, then (Y, T) is a CF- $\text{Nil}(k)$ system.*
- (2) *(X, T) is a CF- $\text{Nil}(i)$ system for $0 \leq i \leq k$.*

Proof. To prove (1) we note (Y, T) is minimal being a factor of a minimal system and $(C_T^{k+1}(Y), \mathcal{HK}^{k+1}(T))$ is uniquely ergodic being a factor of $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ under the natural topological factor map induced from $\pi : (X, T) \rightarrow (Y, T)$ (see Proposition 2.1). By Theorem C this implies (Y, T) is a CF- $\text{Nil}(k)$ system.

Similarly, to prove (2), we consider $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T)) \rightarrow (C_T^{i+1}(X), \mathcal{HK}^{i+1}(T))$ given by

$$(c_{v_1, \dots, v_{k+1}})_{(v_1, \dots, v_{k+1}) \in \{0,1\}^{k+1}} \mapsto (c_{v_1, \dots, v_{i+1}, 0, \dots, 0})_{(v_1, \dots, v_{i+1}) \in \{0,1\}^{i+1}}$$

□

2.9. A CF- $\text{Nil}(k)$ topological model. Recall the definitions of Subsection 2.2. In [Wei85, Theorem 2] Benjamin Weiss proved the following theorem:

Theorem 2.33. *(Weiss) Let (Z, ν, S) be a strictly ergodic t.d.s. and (X, \mathcal{X}, μ, T) an ergodic m.p.s. such that there exists a measurable factor $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Z, \mathcal{Z}, \nu, S)$. Then π has a topological model $\hat{\pi} : (\hat{X}, \hat{T}) \rightarrow (Z, S)$ where (\hat{X}, \hat{T}) is strictly ergodic.*

The following theorem is already implicit in [HSY19].

Theorem 2.34. *Let $k \in \mathbb{Z}$. Every ergodic system (X, \mathcal{X}, μ, T) has a topological model (\hat{X}, \hat{T}) such that (\hat{X}, \hat{T}) is CF- $\text{Nil}(k)$.*

Proof. By Theorem 2.18, $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is measurably isomorphic to a strictly ergodic inverse limit of k -step nilsystems (\hat{Z}_k, \hat{T}) . By Theorem 2.33, (X, \mathcal{X}, μ, T) admits a strictly ergodic topological model (\hat{X}, \hat{T}) such that there exists a topological factor map $(\hat{X}, \hat{T}) \rightarrow (\hat{Z}_k, \hat{T})$ which is a topological model of $(X, \mathcal{X}, \mu, T) \rightarrow (Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$. By Proposition 2.31, (\hat{X}, \hat{T}) is CF- $\text{Nil}(k)$. \square

Remark 2.35. One can easily construct a strictly ergodic system which is not CF- $\text{Nil}(k)$. Let (X, \mathcal{X}, μ, T) be an irrational rotation on the circle. By [Leh87], there exists a topologically mixing and strictly ergodic model $(\hat{X}, \hat{\mu}, T)$ of (X, μ, T) . As X is an irrational rotation, $Z_1(\hat{X}) = \hat{X}$ and therefore for all $k \geq 1$, $Z_k(\hat{X}) = \hat{X}$. As \hat{X} is topologically mixing, it is topologically weakly mixing and therefore for all $k \geq 1$, $W_k(\hat{X}) = \{\bullet\}$ ([SY12, Theorem 3.13(1)]). It follows for all $k \geq 1$ one has that $(W_k(\hat{X}), T)$ is not isomorphic to $(Z_k(\hat{X}), \hat{\mu}_1, T)$ as m.p.s.

3. COALESCENCE AND UNIVERSALITY FOR MAXIMAL PRONILFACTORS.

3.1. Coalescence. In this section we establish Theorem B, i.e., both *topological coalescence* (introduced in [Aus63]) and *measurable coalescence* (introduced in [HP68]) for minimal pronilsystems¹⁰. There is a vast literature dedicated to coalescence (see [LLT92] and references within). Coalescence plays an important role in the next subsection.

Theorem 3.1. *(Topological coalescence for minimal pronilsystems) Let (Y, T) be a minimal k -step pronilsystem. Then (Y, T) is topologically coalescent, i.e. if $\Phi : (Y, T) \rightarrow (Y, T)$ is a topological factor map, then Φ is a topological isomorphism.*

Proof. Recall that the Ellis semigroup is defined as $E = E(Y, T) = \overline{\{T^n : n \in \mathbb{Z}\}}$, where the closure is w.r.t. the product topology on Y^Y

¹⁰The definitions of these concepts appear as part of the statements of Theorems 3.1 and 3.3 respectively.

(see [Ell58] for more details). By a theorem of Donoso [Don14, Theorem 1.1], $E(Y, T)$ is a k -step nilpotent group, i.e. for $E_1 = E$, $E_{i+1} = [E_i, E]$, $i \geq 1$, one has that $E_{k+1} = \{\text{Id}\}$. As Φ is continuous, one has that E and Φ commute, i.e. for any $g \in E$, $\Phi \circ g = g \circ \Phi$. For any $z \in Y$, we define the group $\mathcal{G}(Y, z) = \{\alpha \in E(Y, T), \alpha z = z\}$. Let $x, y \in Y$ such that $\Phi(x) = y$. If $u \in \mathcal{G}(Y, x)$, one always has that $uy = u(\Phi(x)) = \Phi(ux) = \Phi(x) = y$, i.e. $u \in \mathcal{G}(Y, y)$. Thus $\mathcal{G}(Y, x) \subset \mathcal{G}(Y, y)$.

Assume that Φ is not one-to-one, then there exists $x_1 \neq x_2 \in Y$ such that $\Phi(x_1) = \Phi(x_2)$. As (Y, T) is minimal, there exists $p_1, p_2 \in E(Y, T)$ such that $x_1 = p_1x$, $x_2 = p_2x$. Then $p_1y = \Phi(p_1x) = \Phi(x_1) = \Phi(x_2) = \Phi(p_2x) = p_2y$. Thus $p_1^{-1}p_2 \in \mathcal{G}(Y, y)$. As $p_2x = x_2 \neq x_1 = p_1x$, we have

$$p_1^{-1}p_2x \neq x,$$

which implies that $p_1^{-1}p_2 \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$.

Let $\beta_0 = p_1^{-1}p_2$. As (Y, T) is minimal, there exists $u \in E(Y, T)$ such that $ux = y$. Then $\mathcal{G}(Y, x) = u^{-1}\mathcal{G}(Y, y)u$. Let $\beta_1 = (u^{-1}\beta_0^{-1}u)\beta_0$. As $\beta_0 \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$, one has that

$$(7) \quad \beta_0 \notin \mathcal{G}(Y, x), \beta_0 \in \mathcal{G}(Y, y) \text{ and } (u^{-1}\beta_0^{-1}u) \in u^{-1}\mathcal{G}(Y, y)u = \mathcal{G}(Y, x) \subset \mathcal{G}(Y, y).$$

Thus we can show that $\beta_1 \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$. Indeed, by (7) we know that $\beta_1 = (u^{-1}\beta_0^{-1}u)\beta_0 \in \mathcal{G}(Y, y)$ as $\mathcal{G}(Y, y)$ is a group. If $\beta_1 \in \mathcal{G}(Y, x)$, then $\beta_0 = (u^{-1}\beta_0^{-1}u)^{-1}\beta_1 \in \mathcal{G}(Y, x)$, which constitutes a contradiction. Therefore $\beta_1 \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$ and $(u^{-1}\beta_1^{-1}u) \in u^{-1}\mathcal{G}(Y, y)u = \mathcal{G}(Y, x)$.

Similarly, we define $\beta_{i+1} = (u^{-1}\beta_i^{-1}u)\beta_i$ for $i \geq 1$. By the same argument, one has that $\beta_{i+1} \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$. But notice that $\beta_i \in E_{i+1}$ and $E_{k+1} = \{\text{Id}\}$, therefore $\text{Id} = \beta_k \in \mathcal{G}(Y, y) \setminus \mathcal{G}(Y, x)$. Contradiction.

Thus Φ is a one-to-one topological factor map, which implies it is a topological isomorphism. \square

Proposition 3.2. [HK18, Chapter 13, Proposition 15] *Let (Y, ν, T) , (Y', ν', T) be minimal (uniquely ergodic) k -step pronilsystems. Let $\pi : (Y, \nu, T) \rightarrow (Y', \nu', T)$ be a measurable factor map. Then there exists a topological factor map $\hat{\pi} : (Y, T) \rightarrow (Y', T)$ such that $\pi(y) = \hat{\pi}(y)$ for ν -a.e. y .*

Combining Theorem 3.1 and Proposition 3.2 we immediately have the following theorem.

Theorem 3.3. *(Measurable coalescence for minimal pronilsystems) Let (Y, ν, T) be a minimal (uniquely ergodic) k -step pronilsystem. Then (Y, ν, T) is measurably coalescent, i.e. if $\pi : (Y, \nu, T) \rightarrow (Y, \nu, T)$ is a measurable factor map, then π is a measurable isomorphism (which equals a.s. a topological isomorphism).*

Proof. By Proposition 3.2, there exists a topological factor map $\hat{\pi} : (Y, \nu, T) \rightarrow (Y, \nu, T)$ such that $\pi(y) = \hat{\pi}(y)$ for ν -a.e. $y \in Y$. By

Theorem 3.1, $\hat{\pi}$ is a topological isomorphism. As π equals a.s. $\hat{\pi}$, one may find a T -invariant Borel set $Y_0 \subset Y$ with $\nu(Y_0) = 1$, $\pi|_{Y_0} = \hat{\pi}|_{Y_0}$. As $\hat{\pi}$ is one-to-one, $\pi|_{Y_0}^{-1}(\pi|_{Y_0}(Y_0)) = Y_0$ and therefore $\nu(\pi|_{Y_0}(Y_0)) = 1$. Thus $\pi|_{Y_0} : Y_0 \rightarrow \hat{\pi}(Y_0)$ is a Borel measurable one-to-one map between two T -invariant sets of full measure, which implies that π is a measurable isomorphism. \square

Corollary 3.4. *Let (X, \mathcal{X}, μ, T) be an ergodic m.p.s. and $k \in \mathbb{N}$. Let (Y, \mathcal{Y}, ν, S) be a minimal k -step pronilfactor isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$. Let $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ be a factor map. The following holds:*

- (1) *There is a (topological) isomorphism $p : (Z_k(X), \mathcal{Z}_k(X), \mu_k, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ such that $\pi = p \circ \pi_k$ a.s.*
- (2) *For every measurable factor map $\phi : (X, \mathcal{X}, \mu, T) \rightarrow (Y', \mathcal{Y}', \nu', S')$ where $(Y', \mathcal{Y}', \nu', S')$ is a minimal k -step pronilfactor, factors through π , i.e., there exists a unique (up to measure zero) $\psi : (Y, \mathcal{Y}, \nu, S) \rightarrow (Y', \mathcal{Y}', \nu', S')$ such that $\phi = \psi \circ \pi$ a.s.*

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow \pi_k & \downarrow \pi & \searrow \phi & \\
 Z_k & \xleftarrow{p} & Y & \xrightarrow{\psi} & Y'
 \end{array}$$

Proof. By the maximality of π_k (see Subsection 2.6) there is a measurable factor map $p : (Z_k(X), \mathcal{Z}_k(X), \mu_k, T) \rightarrow (Y, \mathcal{Y}, \nu, S)$ such that $\pi = p \circ \pi_k$ a.s. By assumption there is a measurable isomorphism $i : (Y, \mathcal{Y}, \nu, S) \rightarrow (Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ (which equals a.s. a topological isomorphism). By Theorem 3.3, $i \circ p$ is a measurable isomorphism and therefore p is a measurable isomorphism. This establishes (1). Thus π inherits the maximality property of π_k . This establishes (2). \square

Remark 3.5. Bernard Host has pointed out to us that it is possible to prove Theorem B using results from [HK18, Chapter 13].

3.2. Universality.

Definition 3.6. Let (X, μ, T) be a strictly ergodic t.d.s. Denote by C_k^{top} the collection of (topological) isomorphism equivalence classes of topological k -step pronilfactors of (X, T) . Denote by C_k^{meas} the collection of (measurable) isomorphism equivalence classes of measurable k -step pronilfactors of (X, T) . An (equivalence class of) t.d.s. $(M, T) \in C_k^{\text{top}}$ is called C_k^{top} -**universal**¹¹ if every $(N, S) \in C_k^{\text{top}}$ is a topological factor of (M, T) . An (equivalence class of) m.p.s. $(M, \mathcal{M}, \mu, T) \in C_k^{\text{meas}}$

¹¹This terminology is frequently used in the literature, see [dV93, GL13].

is called C_k^{meas} -**universal** if every $(N, \mathcal{N}, \nu, S) \in C_k^{\text{meas}}$ is a measurable factor of (M, \mathcal{M}, μ, T) .

The following theorem establishes a complementary property to maximality as described in Remark 2.19 and Remark 2.27.

Theorem 3.7. *Let (X, μ, T) be a strictly ergodic t.d.s., then $(W_k(X), T)$ is the unique C_k^{top} -universal topological k -step pronilfactor of (X, T) and $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is the unique C_k^{meas} -universal measurable k -step pronilfactor of (X, T) .*

Proof. By Remark 2.19 $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ is a C_k^{meas} -universal measurable k -step pronilfactor of (X, T) . Assume $(Z'_k(X), \mathcal{Z}'_k(X), \mu'_k, T)$ is another C_k^{meas} -universal measurable k -step pronilfactor of (X, T) . By universality one has measurable factor maps $Z'_k(X) \rightarrow \mathcal{Z}'_k(X)$ and $Z_k(X) \rightarrow \mathcal{Z}'_k(X)$. By Theorem 3.3, $Z_k(X)$ and $\mathcal{Z}'_k(X)$ are isomorphic.

By Remark 2.27 $(W_k(X), T)$ is a C_k^{top} -universal topological k -step pronilfactor of (X, T) . By Theorem 3.1 it is unique. \square

Proof of Proposition 2.31. By Remark 2.27, one can find a topological factor map $q : (W_k(X), T) \rightarrow (\hat{Z}_k, T)$. Let ω_k be the unique ergodic measure of $(W_k(X), T)$. By Remark 2.19, one can find a measurable factor map $\psi : (\hat{Z}_k, \gamma_k, T) \rightarrow (W_k(X), \omega_k, T)$.

$$\hat{Z}_k \begin{array}{c} \xrightarrow{\psi} \\ \xleftarrow{q} \end{array} W_k$$

By Proposition 3.2, there exists a topological factor map $\hat{\psi} : (\hat{Z}_k, \gamma_k, T) \rightarrow (W_k(X), \omega_k, T)$ such that $\hat{\psi} = \psi$ a.s. In particular, $\hat{\psi} \circ q : (W_k(X), \omega_k, T) \rightarrow (W_k(X), \omega_k, T)$ is a topological factor map. By Theorem 3.1, $\hat{\psi} \circ q$ is a topological isomorphism. Thus q is a topological isomorphism. As (\hat{Z}_k, T) and (W_k, T) are uniquely ergodic, q is also a measurable isomorphism. In particular $(W_k(X), \mathcal{W}_k(X), \omega_k, T)$ and $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ are isomorphic as m.p.s. and (X, μ, T) is CF-Nil(k). \square

4. CUBESPACE CHARACTERIZATION OF CF-NIL(k).

In this section, we prove Theorem C. We need some lemmas.

Lemma 4.1. [HKM10, Lemma 5.6] *Let (X, T) be a minimal topological dynamical system and μ be an invariant ergodic measure on X . Then the measure $\mu^{[k]}$ is supported on $C_T^k(X)$ for any $k \geq 1$.*

Proof. The Lemma is proven in [HKM10, Lemma 5.6] with the help of the so called L^2 -convergence of cubical averages theorem [HK05, Theorem 1.2]. This is a deep theorem with a highly non-trivial proof. We note that we are able to give a direct proof of this Lemma which we hope to publish elsewhere. \square

Definition 4.2. Let G be a countable amenable group. A **Følner sequence** $\{F_N\}_{N \in \mathbb{N}}$ is a sequence of finite subsets of G such that for any $g \in G$, $\lim_{N \rightarrow \infty} |gF_N \cap F_N|/|F_N| = 1$.

Theorem 4.3. (*Lindenstrauss*) Let G be an amenable group acting on a measure space (X, \mathcal{X}, μ) by measure preserving transformations. Let \mathcal{I}_G be the G -invariant σ -algebra of (X, \mathcal{X}, μ) . There is a Følner sequence $\{F_N\}_{N \in \mathbb{N}}$ such that for any $f \in L^\infty(\mu)$, for μ -a.e. $x \in X$,

$$\lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{g \in F_N} f(gx) = \mathbb{E}(f|\mathcal{I}_G)(x),$$

In particular, if the G action is ergodic, for μ -a.e. $x \in X$,

$$\lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{g \in F_N} f(gx) = \int f(x) d\mu \text{ a.e.}$$

Proof. The theorem follows from [Lin01, Theorem 1.2] and [Lin01, Proposition 1.4]. In [Lin01, Theorem 1.2] the statement reads

$$(8) \quad \lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{g \in F_N} f(gx) = \bar{f}(x) \text{ a.e.}$$

for some G -invariant $\bar{f} \in L^\infty(\mu)$.

Note that if we replace f by $\mathbb{E}(f|\mathcal{I}_G)$ in (8), we have trivially as $\mathbb{E}(f|\mathcal{I}_G)$ is G -invariant:

$$\mathbb{E}(f|\mathcal{I}_G)(x) = \lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \mathbb{E}(f|\mathcal{I}_G)(gx)$$

Using the Lebesgue dominated convergence theorem for conditional expectation¹² one has:

$$\mathbb{E}(f|\mathcal{I}_G)(x) = \lim_{N \rightarrow \infty} \mathbb{E}\left(\frac{1}{|F_N|} \sum_{g \in F_N} f(g \cdot) | \mathcal{I}_G\right)(x) = \mathbb{E}(\bar{f} | \mathcal{I}_G)(x) = \bar{f}(x) \text{ a.e.}$$

Thus $\bar{f}(x) = \mathbb{E}(f|\mathcal{I}_G)(x)$, which gives the statement above. \square

Proof of Theorem C. (I) \Rightarrow (II): This follows from the proof in [HSY17, Section 4.4.3], where it is shown that if one has a commutative diagram of the following form:

$$\begin{array}{ccc} (X, \mathcal{X}, \mu, T) & \xrightarrow{\phi} & (\hat{X}, T) \\ \pi_k \downarrow & & \downarrow \hat{\pi}_k \\ (Z_k(X), \mathcal{Z}_k(X), \mu_k, T) & \xrightarrow{\text{Id}} & (Z_k(X), T), \end{array}$$

¹²It follows easily from applying the Lebesgue dominated convergence theorem in Equation (4).

then $(C_T^{k+1}(\hat{X}), \mathcal{HK}^{k+1}(T))$ is uniquely ergodic. Here (X, \mathcal{X}, μ, T) is an ergodic system, (\hat{X}, T) is strictly ergodic, ϕ is a measurable isomorphism w.r.t. the uniquely ergodic measure of (\hat{X}, T) and $\hat{\pi}_k$ is a topological factor map. Indeed, it is easy to obtain such a diagram for a CF- $\text{Nil}(k)$ system using Proposition 2.31.

(II) \Rightarrow (I): We assume that $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ is uniquely ergodic. By Lemma 4.1, the unique invariant measure is $\mu^{[k+1]}$. As (X, T) is a topological factor of $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ w.r.t. the projection to the first coordinate, (X, T) is uniquely ergodic.

Let $p_k : (X, T) \rightarrow (W_k(X), T)$ be the topological canonical k -th projection. By Proposition 2.1, as (X, T) is uniquely ergodic so is $(W_k(X), T)$. Let us denote by ω_k the unique invariant measure of $(W_k(X), T)$. Obviously $(p_k)_*\mu = \omega_k$. Thus $p_k : (X, \mu, T) \rightarrow (W_k(X), \omega_k, T)$ is a measurable factor map. Let \mathcal{W}_k be the σ -algebra corresponding to the map p_k . Let \mathcal{Z}_k be the σ -algebra corresponding to the measurable canonical k -th projection $\pi_k : (X, \mu, T) \rightarrow (Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$. We will show that $\mathcal{W}_k = \mathcal{Z}_k$, which implies that $(W_k(X), \omega_k, T)$ is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s. The map $p_k : (X, T) \rightarrow (W_k(X), T)$ induces a factor map

$$(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T)) \rightarrow (C_T^{k+1}(W_k(X)), \mathcal{HK}^{k+1}(T)).$$

By Proposition 2.1, as $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ is uniquely ergodic so is $(C_T^{k+1}(W_k(X)), \mathcal{HK}^{k+1}(T))$. By Lemma 4.1 the unique invariant measure on $(C_T^{k+1}(W_k(X)), \mathcal{HK}^{k+1}(T))$ is $\omega_k^{[k+1]}$. Let γ_{k+1} be the *conditional product measure relative to $(W_k(X)^{[k+1]}, \omega_k^{[k+1]})$* on $X^{[k+1]}$ ([Fur77, Definition 9.1]). This is the unique measure on $X^{[k+1]}$ such that for all $f_v \in L^\infty(X, \mu)$, $v \in \{0, 1\}^{k+1}$ ([Fur77, Lemma 9.1]):

$$(9) \quad \int_{X^{[k+1]}} \prod_{v \in \{0, 1\}^{k+1}} f_v(c_v) d\gamma_{k+1}(c) = \int_{W_k(X)^{[k+1]}} \prod_{v \in \{0, 1\}^{k+1}} \mathbb{E}(f_v | W_k(X))(c_v) d\omega_k^{[k+1]}(c).$$

As $\mathbb{E}(\cdot | W_k(X))$ commutes with T and $\omega_k^{[k+1]}$ is $\mathcal{HK}^{k+1}(T)$ -invariant, one has that γ_{k+1} is $\mathcal{HK}^{k+1}(T)$ -invariant. It is natural to introduce the measure γ_{k+1} as by [HK18, Chapter 9, Theorem 14], $\mu^{[k+1]}$ is the conditional product measure relative to $\mu_k^{[k+1]}$. Thus if $\mu_k = \omega_k$ then $\gamma_{k+1} = \mu^{[k+1]}$. It turns out one can reverse the direction of implications. Indeed we claim that $\gamma_{k+1}(C_T^{k+1}(X)) = 1$. Assuming this claim and recalling the assumption that $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ is uniquely ergodic, one has by Lemma 4.1 that $\gamma_{k+1} = \mu^{[k+1]}$. With the end goal of showing $\mathcal{Z}_k = \mathcal{W}_k$ we start by showing $\mathcal{Z}_k \subset \mathcal{W}_k$. It is enough to show $L^\infty(\mu) \cap L^2(\mathcal{W}_k)^\perp \subset L^\infty(\mu) \cap L^2(\mathcal{Z}_k)^\perp$. To this end we will show

that for any function $f \in L^\infty(\mu)$ such that $\mathbb{E}(f|\mathcal{W}_k) = 0$, it holds that $\mathbb{E}(f|\mathcal{Z}_k) = 0$. By Definition 2.15, as $\gamma_{k+1} = \mu^{[k+1]}$,

$$\begin{aligned} \|f\|_{k+1}^{2^{k+1}} &= \int \prod_{v \in \{0,1\}^{k+1}} \mathcal{C}^{|v|} f(c_v) d\gamma_{k+1}(c) = \\ &= \int \prod_{v \in \{0,1\}^{k+1}} \mathbb{E}(\mathcal{C}^{|v|} f | W_k(X))(c_v) d\omega_k^{[k+1]}(c). \end{aligned}$$

As $\mathbb{E}(f|\mathcal{W}_k) \equiv 0$, it holds that $\mathbb{E}(\mathcal{C}^{|v|} f | W_k(X)) \equiv 0$ for any $v \in \{0,1\}^{k+1}$. Therefore $\|f\|_{k+1} = 0$. This implies by Lemma 2.16 that $\mathbb{E}(f|\mathcal{Z}_k) = 0$ as desired. By Remark 2.19, $Z_k(X)$ is the maximal measurable k -step pronilfactor of (X, μ, T) . As $(W_k(X), \omega_k, T)$ is a k -step pronilfactor of (X, T) , one has that $\mathcal{W}_k \subset \mathcal{Z}_k$. Thus $\mathcal{W}_k = \mathcal{Z}_k$, which implies that $(W_k(X), \omega_k, T)$ is isomorphic to $(Z_k(X), \mathcal{Z}_k(X), \mu_k, T)$ as m.p.s.

As a final step, we will now show that $\gamma_{k+1}(C_T^{k+1}(X)) = 1$. Let $f_v \in L^\infty(X, \mu)$, $v \in \{0,1\}^{k+1}$ and set $H_0 = \prod_{v \in \{0\} \times \{0,1\}^k} f_v$ and $H_1 = \prod_{v \in \{1\} \times \{0,1\}^k} f_v$ as well as $\hat{H}_0 = \prod_{v \in \{0\} \times \{0,1\}^k} \mathbb{E}(f_v | W_k(X))$, $\hat{H}_1 = \prod_{v \in \{1\} \times \{0,1\}^k} \mathbb{E}(f_v | W_k(X))$. By Equation (9), we have

$$(10) \quad \int_{X^{[k+1]}} H_0(c) H_1(c') d\gamma_{k+1}(c, c') = \int_{W_k(X)^{[k+1]}} \hat{H}_0(c) \hat{H}_1(c') d\omega_k^{[k+1]}(c, c').$$

By Equation (6) in Definition 2.12,

$$(11) \quad \int_{W_k(X)^{[k+1]}} \hat{H}_0(c) \hat{H}_1(c') d\omega_k^{[k+1]}(c, c') = \int_{W_k(X)^{[k]}} \mathbb{E}(\hat{H}_0 | \mathcal{I}_{T^{[k]}})(c) \hat{H}_1(c) d\omega_k^{[k]}(c).$$

By Birkhoff's ergodic theorem (see also Theorem 4.3), one has that

$$(12) \quad \begin{aligned} &\int_{W_k(X)^{[k]}} \mathbb{E}(\hat{H}_0 | \mathcal{I}_{T^{[k]}})(c) \hat{H}_1(c) d\omega_k^{[k]}(c) \\ &= \int \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \hat{H}_0((T^{[k]})^n c) \hat{H}_1(c) d\omega_k^{[k]}(c) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \hat{H}_0((T^{[k]})^n c) \hat{H}_1(c) d\omega_k^{[k]}(c), \end{aligned}$$

here we used the Lebesgue dominated convergence theorem.

Abusing notation one may consider \hat{H}_0 and \hat{H}_1 as defined on $X^{[k]}$ (see Subsection 2.3). As $p_k : (X, \mu, T) \rightarrow (W_k(X), \omega_k, T)$ is a measurable factor map, one has

$$\int \hat{H}_0((T^{[k]})^n c) \hat{H}_1(c) d\omega_k^{[k]}(c) = \int \hat{H}_0((T^{[k]})^n c) \hat{H}_1(c) d\mu^{[k]}(c).$$

As $(C_T^k(X), \mathcal{HK}^k(T))$ is a topological factor of $(C_T^{k+1}(X), \mathcal{HK}^{k+1}(T))$ w.r.t. the "lower" 2^k coordinates, $(C_T^k(X), \mathcal{HK}^k(T))$ is uniquely ergodic.

By Lemma 4.1, the unique ergodic measure is $\mu^{[k]}$. By Theorem 4.3 applied to $(C_T^k(X), \mu^{[k]}, \mathcal{HK}^k(T))$, there is a Følner sequence $\{F_M \subset \mathcal{HK}^k(T)\}_{M \in \mathbb{N}}$ such that

$$(13) \quad \int \hat{H}_0((T^{[k]})^n c) \hat{H}_1(c) d\mu^{[k]}(c) = \lim_{M \rightarrow \infty} \frac{1}{|F_M|} \sum_{h \in F_M} \hat{H}_0((T^{[k]})^n h s) \hat{H}_1(h s)$$

for $\mu^{[k]}$ -a.e. $s \in C_T^k(X)$. Thus from Equations (10), (11), (12) and (13), it holds for arbitrary $f_v \in L^\infty(X, \mu)$, $v \in \{0, 1\}^{k+1}$, $H_0 = \prod_{v \in \{0\} \times \{0, 1\}^k} f_v$ and $H_1 = \prod_{v \in \{1\} \times \{0, 1\}^k} f_v$, for $\mu^{[k]}$ -a.e. $s \in C_T^k(X)$,

$$(14) \quad \int_{X^{[k+1]}} H_0(c) H_1(c') d\gamma_{k+1}(c, c') = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \lim_{M \rightarrow \infty} \frac{1}{|F_M|} \sum_{h \in F_M} \hat{H}_0((T^{[k]})^n h s) \hat{H}_1(h s)$$

Let $R \in C(X^{[k+1]}, \mathbb{R})$ be a continuous function. We claim for $\mu^{[k]}$ -a.e. $s \in C_T^k(X)$,

$$(15) \quad \int R(c) d\gamma_{k+1}(c) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \lim_{M \rightarrow \infty} \frac{1}{|F_M|} \sum_{h \in F_M} R((T^{[k]})^n h s, h s)$$

Notice that it follows from Definitions 2.11 and 2.21 that if $s \in C_T^k(X)$, then $((T^{[k]})^n h s, h s) \in C_T^{k+1}(X)$ for arbitrary $h \in \mathcal{HK}^k(T)$ and $n \in \mathbb{Z}^+$ (see also [GGY18, Subsection A.2]). Thus using Equation (15) with functions $R_\delta \in C(X^{[k+1]}, [0, 1])$ such that $R_\delta|_{C_T^{k+1}(X)} \equiv 1$ and $R_\delta|_{X^{[k+1]} \setminus B_\delta(C_T^{k+1}(X))} \equiv 0$, (taking δ to zero) one obtains:

$$\gamma_{k+1}(C_T^{k+1}(X)) = 1.$$

We now prove (15). For $d \in \mathbb{N}$, let $H_d^{(i)}$ be functions of the form $\prod_{v \in \{0, 1\}^{k+1}} h_v^{(i)}$, $i \in I_d$ for some finite set I_d , such that $|R(z) - \sum_{i \in I_d} H_d^{(i)}(z)| < \frac{1}{2d}$ for all $z \in C_T^{k+1}(X)$. Denote by $C(R) = \int R(c) d\gamma_{k+1}(c)$ the (LHS) of (15). Denote by $D(R)(z)$ be the (RHS) of Equation (15). By Equation (14), $C(H_d^{(i)}) = D(H_d^{(i)})(z)$ for $\mu^{[k]}$ -a.e. $z \in C_T^k(X)$. Note that $|C(R) - \sum_{i \in I_d} C(H_d^{(i)})| < \frac{1}{2d}$ and $|D(R)(y) - \sum_{i \in I_d} D(H_d^{(i)})(y)| < \frac{1}{2d}$ for all $y \in C_T^k(X)$. Thus for any d , $E_d := \{y \in C_T^k(X) : |C(R)(y) - D(R)(y)| < \frac{1}{d}\}$ has full $\mu^{[k]}$ measure. Let $E = \bigcap_{d \in \mathbb{N}} E_d$, then $\mu^{[k]}(E) = 1$ and for any $y \in E$, Equation (15) holds. \square

The following remark may be of interest:

Remark 4.4. In [GHSY20, Section 6] an example is given showing there exists a strictly ergodic *distal* system which is *not* CF-Nil(1).

5. A TOPOLOGICAL WIENER-WINTNER THEOREM.

In this section, we prove Theorem A.

Definition 5.1. Let (X, T) be a t.d.s. and $\mu \in P_T(X)$. A point $x \in X$ is **generic** (for μ) if for all $f \in C(X)$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N f(T^n x) = \int f d\mu$$

Lemma 5.2. Let (X, T) be a t.d.s. and $x_0 \in X$. Assume that for all $f \in C(X)$, there exists $c_f \in \mathbb{R}$, a constant depending on f , so that :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N f(T^n x_0) = c_f$$

Then x_0 is generic for some $\mu \in P_T(X)$.

Proof. Define the functional $\phi : C(X) \rightarrow \mathbb{R}$ by $\phi(f) = c_f$. It is easy to see that ϕ is a bounded linear positive functional of supremum norm 1. By the Riesz representation theorem $c_f = \int f d\mu$ for some Borel probability measure μ on X ([Rud06, Theorem 2.14]). As $c_f = c_{Tf}$ for all $f \in C(X)$, it follows that $\mu \in P_T(X)$. Thus x_0 is generic by Definition 5.1. \square

Theorem 5.3. ([Gla03, Theorem 4.10]) Let (X, T) be a minimal t.d.s., then (X, T) is uniquely ergodic iff every $x \in X$ is generic for some $\mu \in P_T(X)$ (depending on x).

Lemma 5.4. Let (X, T) be a t.d.s. and $\mu \in P_T(X)$. If a point $x \in X$ is generic for μ , then μ is supported on $\overline{\mathcal{O}}(x)$.

Proof. Let f be a non-negative function supported outside $\overline{\mathcal{O}}(x)$. Then $\int f d\mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) = 0$. Q.E.D. \square

Proof of Theorem A. (I) \Rightarrow (II). It follows from [HK09, Theorem 2.19 and Proposition 7.1].

We will show (II) \Rightarrow (I) inductively. For $k = 0$ note that Condition (II) with the constant nilsequence $a(n) \equiv 1$ implies that for a fixed arbitrary $x \in X$ and every $f \in C(X)$, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n) f(T^n x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x)$ exists. From Lemma 5.2, $x \in X$ is generic for some $\mu_x \in P_T(X)$. By Theorem 5.3, (X, T) is uniquely ergodic. By assumption (X, T) is minimal and thus (X, T) is a CF- $\text{Nil}(k)$ system.

Assume the (II) \Rightarrow (I) holds for $k - 1$. We will now show \sim (I) $\Rightarrow \sim$ (II) for k . Thus we assume that (X, T) is not CF- $\text{Nil}(k)$. If (X, T) is not CF- $\text{Nil}(k-1)$ then the result follows from the inductive assumption. Thus we may assume (X, T) is CF- $\text{Nil}(k-1)$ and in particular uniquely ergodic. Denote the unique probability measure of (X, T) by μ . By definition one has that $(Z_{k-1}(X), \mathcal{Z}_{k-1}(X), \mu_{k-1}, T)$ is isomorphic as

an m.p.s. to $(W_{k-1}(X), \omega_{k-1}, T)$, where ω_{k-1} is the unique ergodic measure of $(W_{k-1}(X), T)$.

An important result of the *Host-Kra structure theory* is that $\pi : Z_k(X) \rightarrow Z_{k-1}(X)$, determined by $\pi_{k-1} = \pi \circ \pi_k$ (as defined in Definition 2.14), is a measurable group extension w.r.t. some abelian group A (See [HK05, Section 6.2], [HK18, Chapter 9, Section 2.3]). By [GL19, Theorem 1.1, proof of Theorem 5.3], we can find a topological model $\hat{\pi} : (\hat{Z}_k, T) \rightarrow (\hat{Z}_{k-1}, T)$ of π which is an abelian topological group extension w.r.t. the abelian group A such that (\hat{Z}_k, T) is a minimal k -step pronilsystem and (\hat{Z}_{k-1}, T) is a minimal $(k-1)$ -step pronilsystem. Denote by ϕ and ψ the measurable isomorphisms between $Z_k(X)$ and $\hat{Z}_k(X)$ and $Z_{k-1}(X)$ and $\hat{Z}_{k-1}(X)$ respectively.

$$\begin{array}{ccc} Z_k(X) & \xrightarrow{\phi} & \hat{Z}_k(X) \\ \pi \downarrow & & \downarrow \hat{\pi} \\ Z_{k-1}(X) & \xrightarrow{\psi} & \hat{Z}_{k-1}(X) \end{array}$$

For clarity denote $\pi_{Z_k} := \pi_k$ from the previous paragraph.

Define $\pi_{\hat{Z}_k} = \phi \circ \pi_{Z_k}$. Let $p_{k-1} : X \rightarrow W_{k-1}(X)$ be the topological canonical $(k-1)$ -th projection. Let $\pi_{\hat{Z}_{k-1}} = \hat{\pi} \circ \pi_{\hat{Z}_k}$. By Corollary 3.4(2), $\hat{\pi} \circ \pi_{\hat{Z}_k}$ inherits the maximality property of $\pi_{k-1} = \pi \circ \pi_{Z_k}$. By Corollary 3.4(1), there exists a measurable factor map $p : \hat{Z}_{k-1}(X) \rightarrow W_{k-1}(X)$ such that $p_{k-1} = p \circ \hat{\pi} \circ \pi_{\hat{Z}_k(X)}$ a.s. As $\hat{Z}_{k-1}(X)$ is isomorphic to both $Z_{k-1}(X)$ and $W_{k-1}(X)$ as m.p.s.¹³, by Theorem 3.3, p may be chosen to be a topological isomorphism. W.l.o.g. we will assume $p = \text{Id}$. Thus we have:

$$(16) \quad p_{k-1}(x) = \hat{\pi} \circ \pi_{\hat{Z}_k(X)}(x) \text{ for } \mu\text{-a.e. } x \in X.$$

$$\begin{array}{ccccc} X & \xleftrightarrow{\text{Id}} & X & \xleftrightarrow{\text{Id}} & X \\ \pi_{Z_k} \downarrow & & \downarrow \pi_{\hat{Z}_k} & & \downarrow p_{k-1} \\ Z_k(X) & \xleftrightarrow{\phi} & \hat{Z}_k(X) & & \\ \pi \downarrow & & \downarrow \hat{\pi} & & \\ Z_{k-1}(X) & \xleftrightarrow{\psi} & \hat{Z}_{k-1}(X) & \xleftrightarrow{\text{Id}} & W_{k-1}(X) \end{array}$$

We claim that there exists a minimal subsystem $(Y, T \times T) \subset (X \times \hat{Z}_k, T \times T)$ such that $(Y, T \times T)$ is not uniquely ergodic. Assuming this,

¹³Here we use that (X, T) is CF- $\text{Nil}(k-1)$.

as by Theorem 5.3 a minimal system is uniquely ergodic if and only if every point is generic, there exists $(x_3, u_3) \in Y$ such that (x_3, u_3) is not a generic point for any measure. By Lemma 5.2, there exist continuous functions $h \in C(\hat{Z}_k)$, $f \in C(X)$ such that

$$(17) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N h(T^n u_3) f(T^n x_3)$$

does not exist. As (\hat{Z}_k, T) is a k -step pronilsystem, $h(T^n u_3)$ is a k -step nilsequence (Definition 2.7). Thus (II) does not hold as required.

Our strategy in proving the claim is finding a minimal subsystem $(Y, T \times T)$ of $(X \times \hat{Z}_k, T \times T)$ which supports an invariant measure ν , w.r.t which $(Y, T \times T)$ is isomorphic to (X, μ, T) as an m.p.s. We then assume for a contradiction that $(Y, T \times T)$ is uniquely ergodic. Next we notice that the strictly ergodic system $(Y, T \times T)$, being measurably isomorphic to (X, μ, T) , has $Z_k(Y) \simeq Z_k(X)$. Moreover as $(Y, T \times T)$ is a minimal subsystem of a product of the two minimal systems, (X, T) and (\hat{Z}_k, T) , it maps onto each of them through the first, respectively second coordinate projections. From the projection on (\hat{Z}_k, T) , we conclude that (Y, T) has a topological k -step pronilfactor \hat{Z}_k which is measurably isomorphic to $Z_k(Y)$. By Proposition 2.31, one has that (Y, T) is CF- $\text{Nil}(k)$. From the projection on (X, T) , we conclude by Proposition 2.32, that (X, T) is CF- $\text{Nil}(k)$. This constitutes a contradiction implying that (Y, T) is not uniquely ergodic as desired.

A natural copy of (X, μ, T) inside $(X \times \hat{Z}_k, T \times T)$ is given by the *graph joining* of $\pi_{\hat{Z}_k(X)}$, defined by the measure $\mu^{(k)} = (\text{Id} \times \pi_{\hat{Z}_k(X)})_* \mu := \int \delta_x \times \delta_{\pi_{\hat{Z}_k(X)}(x)} d\mu(x)$ on $(X \times \hat{Z}_k, T)$ (see [Gla03, Chapter 6, Example 6.3]). Clearly

$$(18) \quad \text{Id} \times \pi_{\hat{Z}_k(X)} : (X, \mathcal{X}, \mu, T) \rightarrow (X \times \hat{Z}_k, \mathcal{X} \times \hat{\mathcal{Z}}_k, \mu^{(k)}, T \times T), \quad x \mapsto (x, \pi_{\hat{Z}_k(X)}(x)).$$

is a measurable isomorphism and in particular $\mu^{(k)}$ is an ergodic measure of $(X \times \hat{Z}_k, T \times T)$. However $(X \times \hat{Z}_k, \mathcal{X} \times \hat{\mathcal{Z}}_k, \mu^{(k)}, T \times T)$ is a m.p.s. and not a (minimal) t.d.s. We consider an orbit closure of a $\mu^{(k)}$ -generic point $(x_1, \pi_{\hat{Z}_k(X)}(x_1))$ to be determined later. By Lemma 5.4, $\mu^{(k)}$ is supported on $\overline{\mathcal{O}}(x_1, \pi_{\hat{Z}_k(X)}(x_1))$. However $(\overline{\mathcal{O}}(x_1, \pi_{\hat{Z}_k(X)}(x_1)), T \times T)$ is not necessarily minimal. We thus pass to an (arbitrary) minimal subsystem $(Y, T \times T) \subset (\overline{\mathcal{O}}(x_1, \pi_{\hat{Z}_k(X)}(x_1)), T \times T)$. However $\mu^{(k)}$ is not necessarily supported on Y . As explained in the previous paragraph, our final aim will be to find (a possibly different) invariant measure $\nu \in \mathbb{P}_{T \times T}(Y)$ which is isomorphic to μ .

As $\hat{\pi}$ is a topological group extension w.r.t. the abelian group A ,

$$(19) \quad \text{Id} \times \hat{\pi} : (X \times \hat{Z}_k, T \times T) \rightarrow (X \times W_{k-1}(X), T \times T) : (x, z) \mapsto (x, \hat{\pi}(z))$$

is also a topological group extension w.r.t. the abelian group A . Thus A acts on the fibers of $\text{Id} \times \hat{\pi}$ transitively and continuously by homeomorphisms. Moreover for all $a \in A$, $(\text{Id} \times a)_* \mu^{(k)}$ is an invariant measure on $(X \times \hat{Z}_k, T \times T)$ isomorphic to $\mu^{(k)}$ and thus isomorphic to μ . We will find $\nu \in \text{P}_{T \times T}(Y)$ of the form $\nu = (\text{Id} \times a)_* \mu^{(k)}$. Indeed for μ -a.e. $x \in X$, $(x, \pi_{\hat{Z}_k(X)}(x))$ is a generic point of $\mu^{(k)}$. Using (16), one may choose $x_1 \in X$ such that

- $(x_1, \pi_{\hat{Z}_k(X)}(x_1))$ is a generic point of $\mu^{(k)}$;
- $\hat{\pi}(\pi_{\hat{Z}_k(X)}(x_1)) = p_{k-1}(x_1)$.

From the second point it follows that:

$$\text{Id} \times \hat{\pi} : (\overline{\mathcal{O}}(x_1, \pi_{\hat{Z}_k(X)}(x_1)), T \times T) \rightarrow (\overline{\mathcal{O}}(x_1, p_{k-1}(x_1)), T \times T)$$

is a topological factor map. As p_{k-1} is a topological factor map,

$$(20) \quad \text{Id} \times p_{k-1} : (X, T) \rightarrow (\overline{\mathcal{O}}(x_1, p_{k-1}(x_1)), T \times T), \quad x \rightarrow (x, p_{k-1}(x))$$

is a topological isomorphism. Therefore $(\overline{\mathcal{O}}(x_1, p_{k-1}(x_1)), T \times T)$ is minimal. Thus $(\text{Id} \times \hat{\pi})|_Y : (Y, T) \rightarrow (\overline{\mathcal{O}}(x_1, p_{k-1}(x_1)), T)$ factors onto. In particular there exists $z_1 \in \hat{Z}_k(X)$, such that $(x_1, z_1) \in Y$ and $\hat{\pi}(z_1) = p_{k-1}(x_1)$. As by assumption $\hat{\pi}(\pi_{\hat{Z}_k(X)}(x_1)) = p_{k-1}(x_1)$, we can find $a \in A$ such that $a \cdot \pi_{\hat{Z}_k(X)}(x_1) = z_1$. As $(x_1, \pi_{\hat{Z}_k(X)}(x_1))$ is a generic point of $\mu^{(k)}$, it follows that $(x_1, a \cdot \pi_{\hat{Z}_k(X)}(x_1)) = (x_1, z_1)$ is a generic point of $\nu := (\text{Id} \times a)_* \mu^{(k)}$. Therefore by Lemma 5.4, the invariant measure $\nu \simeq \mu$ is supported on the minimal subsystem $\overline{\mathcal{O}}(x_1, z_1) = Y$. This ends the proof. \square

REFERENCES

- [AKL14] Omer Angel, Alexander S. Kechris, and Russell Lyons. Random orderings and unique ergodicity of automorphism groups. *Journal of the European Mathematical Society*, 16(10):2059–2095, 2014.
- [Ass92] I. Assani. Uniform Wiener-Wintner theorems for weakly mixing dynamical systems. *Unpublished preprint*, 1992.
- [Aus63] Joseph Auslander. Endomorphisms of minimal sets. *Duke Mathematical Journal*, 30(4):605–614, 1963.
- [BDM10] Xavier Bressaud, Fabien Durand, and Alejandro Maass. On the eigenvalues of finite rank Bratteli-Vershik dynamical systems. *Ergodic Theory Dynam. Systems*, 30:639–664, 2010.
- [Bou90] J. Bourgain. Double recurrence and almost sure convergence. *J. Reine Angew. Math.*, 404:140–161, 1990.
- [BS17] Wojciech Bartoszek and Adam Śpiewak. A note on a Wiener-Wintner theorem for mean ergodic Markov amenable semigroups. *Proceedings of the American Mathematical Society*, 145(7):2997–3003, 2017.
- [DFM19] Fabien Durand, Alexander Frank, and Alejandro Maass. Eigenvalues of minimal cantor systems. *J. Eur. Math. Soc.*, 21:727–775, 2019.
- [Don14] Sebastián Donoso. Enveloping semigroups of systems of order d . *Discrete Contin. Dyn. Syst.*, 34(7):2729–2740, 2014.

- [dV93] J. de Vries. *Elements of topological dynamics*, volume 257 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993.
- [Ell58] Robert Ellis. Distal transformation groups. *Pacific J. Math.*, 8:401–405, 1958.
- [EZK13] Tanja Eisner and Pavel Zorin-Kranich. Uniformity in the Wiener-Wintner theorem for nilsequences. *Discrete Contin. Dyn. Syst.*, 33(8):497–516, 2013.
- [Fan18] Ai-Hua Fan. Topological Wiener-Wintner ergodic theorem with polynomial weights. *Chaos, Solitons and Fractals*, 117:105–116, 2018.
- [Fra06] Nikos Frantzikinakis. Uniformity in the polynomial Wiener-Wintner theorem. *Ergodic Theory Dynam. Systems*, 26(4):1061–1071, 2006.
- [Fur77] Harry Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Analyse Math.*, 31:204–256, 1977.
- [GGY18] Eli Glasner, Yonatan Gutman, and XiangDong Ye. Higher order regionally proximal equivalence relations for general minimal group actions. *Advances in Mathematics*, 333(6), 2018.
- [GHSY20] Eli Glasner, Wen Huang, Song Shao, and Xiangdong Ye. Regionally proximal relation of order d along arithmetic progressions and nilsystems. *Sci. China Math.*, 63(9):1757–1776, 2020.
- [GL13] Yonatan Gutman and Hanfeng Li. A new short proof for the uniqueness of the universal minimal space. *Proceedings of the American Mathematical Society*, 141(1):265–267, 2013.
- [GL19] Yonatan Gutman and Zhengxing Lian. Strictly ergodic distal models and a new approach to the Host-Kra factors. *Preprint*. <https://arxiv.org/abs/1909.11349>, 2019.
- [Gla03] Eli Glasner. *Ergodic theory via joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
- [GMV20] Yonatan Gutman, Freddie Manners, and Péter P. Varjú. The structure theory of nilspaces III: Inverse limit representations and topological dynamics. *Advances in Mathematics*, 365(13), 2020.
- [HK05] Bernard Host and Bryna Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. of Math. (2)*, 161(1):397–488, 2005.
- [HK09] Bernard Host and Bryna Kra. Uniformity seminorms on ℓ^∞ and applications. *J. Anal. Math.*, 108:219–276, 2009.
- [HK18] Bernard Host and Bryna Kra. *Nilpotent Structures in Ergodic Theory*. American Mathematical Society, 2018.
- [HKM10] Bernard Host, Bryna Kra, and Alejandro Maass. Nilsequences and a structure theorem for topological dynamical systems. *Adv. Math.*, 224(1):103–129, 2010.
- [HKM14] Bernard Host, Bryna Kra, and Alejandro Maass. Complexity of nilsystems and systems lacking nilfactors. *Journal d'Analyse Mathématique*, 124(1):261–295, 2014.
- [Hos86] Bernard Host. Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable. *Erg. Theory Dyn. Systems*, 6:529–540, 1986.
- [HP68] Frank Hahn and William Parry. Some characteristic properties of dynamical systems with quasi-discrete spectra. *Mathematical systems theory*, 2(2):179–190, 1968.

- [HSY17] Wen Huang, Song Shao, and Xiangdong Ye. Strictly ergodic models under face and parallelepiped group actions. *Commun. Math. Stat*, 5:93–122, 2017.
- [HSY19] W. Huang, S. Shao, and X. Ye. Pointwise convergence of multiple ergodic averages and strictly ergodic models. *Journal d Analyse Mathématique*, 139(2), 2019.
- [Leh87] Ehud Lehrer. Topological mixing and uniquely ergodic systems. *Israel Journal of Mathematics*, 57(2):239–255, 1987.
- [Les96] Emmanuel Lesigne. Un théorème de disjonction de systèmes dynamiques et une généralisation du théorème ergodique de Wiener-Wintner. *Ergodic Theory Dynam. Systems*, 10:513–521, 1996.
- [Lin01] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.
- [LLT92] Mariusz Lemańczyk, Pierre Liardet, and Jean-Paul Thouvenot. Coalescence of circle extensions of measure-preserving transformations. *Ergodic Theory and Dynamical Systems*, 12(4):769–789, 1992.
- [Que10] Martine Queffélec. *Substitution dynamical systems - spectral analysis*. Lecture Notes in Mathematics, Springer, 2010.
- [Rob94] E. Arthur Robinson. On uniform convergence in the Wiener-Wintner theorem. *Journal of the London Mathematical Society*, 49, 1994.
- [Rud06] Walter Rudin. *Real and complex analysis*. Tata McGraw-hill education, 2006.
- [Sch14] Marco Schreiber. Topological Wiener–Wintner theorems for amenable operator semigroups. *Ergodic Theory and Dynamical Systems*, 34(5):1674–1698, 2014.
- [SY12] Song Shao and Xiangdong Ye. Regionally proximal relation of order d is an equivalence one for minimal systems and a combinatorial consequence. *Adv. Math.*, 231(3-4):1786–1817, 2012.
- [Wei85] Benjamin Weiss. Strictly ergodic models for dynamical systems. *Bull. Amer. Math. Soc. (N.S.)*, 13(2):143–146, 1985.
- [WW41] Norbert Wiener and Aurel Wintner. Harmonic analysis and ergodic theory. *Amer J Math*, 63:415–426, 1941.
- [Zie07] Tamar Ziegler. Universal characteristic factors and Furstenberg averages. *J. Amer. Math. Soc.*, 20(1):53–97 (electronic), 2007.

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