RECONSTRUCTING A MINIMAL TOPOLOGICAL DYNAMICAL SYSTEM FROM A SET OF RETURN TIMES

KAMIL BULINSKI AND ALEXANDER FISH

ABSTRACT. We investigate to what extent a minimal topological dynamical system is uniquely determined by a set of return times to some open set. We show that in many situations this is indeed the case as long as the closure of this open set has no non-trivial translational symmetries. For instance, we show that under this assumption two Kronecker systems with the same set of return times must be isomorphic. More generally, we show that if a minimal dynamical system has a set of return times that coincides with a set of return times to some open set in a Kronecker system with translationarily asymmetric closure, then that Kronecker system must be a factor. We also study similar problems involving Nilsystems and polynomial return times. We state a number of questions on whether these results extend to other homogeneous spaces and transitive group actions, some of which are already interesting for finite groups.

1. INTRODUCTION

We study the question of whether knowing the exact times a point in an unknown dynamical system enters an open set is enough to determine the system. In this paper, a topological dynamical system is a pair (X, T) where X is a compact metric space with metric d (all metrics will be refered to as d even for different spaces if no ambiguity arises) and $T : X \to X$ is a homeomorphism. Recall that (X, T) is *minimal* if for all $x_0 \in X$ the orbit $T^{\mathbb{Z}} x_0 = \{T^n x_0 \mid n \in \mathbb{Z}\}$ is dense and is *transitive* if some orbit is dense. If $x_0 \in X$ and $U \subset X$ is an open set then we can define the set of return times

$$\mathcal{R}_{(X,T)}(x_0, U) = \{n \in \mathbb{Z} \mid T^n x_0 \in U\}.$$

Thus our question may now be stated as follows: Given two minimal topological dynamical systems (X_1, T_1) and (X_2, T_2) with $x_1 \in X_1, x_2 \in X_2$ and $U_1 \subset X_1, U_2 \subset X_2$ open such that $\mathcal{R}_{(X_1,T_1)}(x_1, U_1) = \mathcal{R}_{(X_2,T_2)}(x_2, U_2)$, then is it necessarily true that (X_1, T_1) and (X_2, T_2) are isomorphic? If so, is there an isomorphism mapping x_1 to x_2?

Recall that an isomorphism $\phi : (X_1, T_1) \to (X_2, T_2)$ is a homeomorphism $\phi : X_1 \to X_2$ such that $\phi \circ T_1 = T_2 \circ \phi$. One can contrast this to the Taken's Reconstruction Theorem [6] as well as recent developments [2] where one instead is given a system (X, T) where the *dimension* of X is d and asks whether the delay observation mapping $X \to [0, 1]^{2d+1}$ given by $x \mapsto (h(x), h(Tx), \ldots, h(T^{2d+1}x))$ is injective for some *generic* continuous $h : X \to [0, 1]$. Thus our question can instead be posed as asking to what extent the mapping

$$(X, T, x_0, U) \mapsto (\mathbb{1}_U(T^n x_0))_{n \in \mathbb{Z}}$$

is injective up to isomorphism.

1.1. **Reconstructing Kronecker Systems.** Let us now start with some motivating examples where no such isomorphism exists.

Example 1.1. Let $X_1 = X_2 = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and suppose $\alpha \in \mathbb{T}$ is irrational. Let

$$U_1 = U_2 = (-\epsilon, \epsilon) \cup (-\epsilon + \tfrac{1}{2}, \epsilon + \tfrac{1}{2}) \subset \mathbb{T}.$$

Then for each $n \in \mathbb{Z}$ we have that $n\alpha \in U_1$ if and only if $n(\alpha + \tfrac{1}{2}) \in U_2$. However, the minimal systems $(\mathbb{T}, x \mapsto x + \alpha)$ and $(\mathbb{T}, x \mapsto x + \alpha + \tfrac{1}{2})$ are not isomorphic.

Example 1.2. Let $X_1 = \mathbb{T}$, $X_2 = \mathbb{T}^2$, $\alpha_1 = \sqrt{2} \in \mathbb{T}$ and $\alpha_2 = (\sqrt{2}, \sqrt{3}) \in \mathbb{T}^2$. Let $U_1 \subset \mathbb{T}$ be any non-empty proper open subset and let $U_2 = U_1 \times \mathbb{T}$. Then clearly $n\alpha_1 \in U_1$ if and only if $n\alpha_2 \in U_2$. Yet X_1 and X_2 are not even homeomorphic.

These two examples highlight that open sets with non-trivial translational symmetries are a source of issues. For a compact abelian metrizable group $(K, +)$ and $A \subset K$ we can define the stabilizer

$$\mathrm{Stab}_K(A) = \{k \in K \mid A + k = A\}.$$

Note that in our examples above the stabilizers are not trivial; the stabilizer of $(-\epsilon, \epsilon) \cup (-\epsilon + \tfrac{1}{2}, \epsilon + \tfrac{1}{2})$ is $\{0, \tfrac{1}{2}\}$ while the stabilizer of $U_1 \times \mathbb{T}$, for $U_1 \subset \mathbb{T}$ non-empty and proper, is the vertical subgroup $\{0\} \times \mathbb{T}$.

Our first main result demonstrates that the only way two non-isomorphic Kronecker systems can yield the same set of return times is if the closure of one of the defining open sets has a non-trivial stabilizer.

Theorem 1.3. Let $(K_1, +)$ and $(K_2, +)$ be two compact metrizable abelian groups and suppose that $\alpha_1 \in K_1$ and $\alpha_2 \in K_2$ are such that $\overline{\mathbb{Z}\alpha_i} = K_i$ for $i = 1, 2$. Suppose that $U_1 \subset K_1$ and $U_2 \subset K_2$ are open sets such that the stabilizers of their closures are trivial, i.e., $\mathrm{Stab}_{K_i}(\overline{U_i}) = \{0\}$. Then if

$$\{n \in \mathbb{Z} \mid n\alpha_1 \in U_1\} = \{n \in \mathbb{Z} \mid n\alpha_2 \in U_2\}$$

then there exists an isomorphism of topological groups (so continuous) $K_1 \to K_2$ mapping α_1 to α_2.

Note that if one removes a point not on the trajectory $\{n\alpha_i \mid n \in \mathbb{Z}\}$ from U_i then U_i remains open and the return times and the closures do not change, so it is indeed natural to study the stabilizer of the closure $\overline{U_i}$ rather than U_i.

1.2. Detecting Kronecker Factors of Minimal Systems.
We now turn our attention to minimal topological systems. As demonstrated in Example 1.2 if a minimal system has a non-trivial factor, then it shares a set of return times with it. Of course, this is also the case if one replaces in that example the rotation $(x, y) \mapsto (x + \sqrt{2}, y + \sqrt{3})$ with an ergodic skew product such as $\mathbb{T}^2 \to \mathbb{T}^2 : (x, y) \mapsto (x + \sqrt{2}, x + y)$. Our next result shows that essentially the only way a minimal system can share a set of return times with a Kronecker system is if the latter is a factor of the former.

Theorem 1.4. Let (X, T) be a minimal topological dynamical system and $U \subset X$ be a non-empty open set and $x_0 \in X$. Let $(K, +)$ be a compact metrizable abelian group and $\alpha \in K$ such that $\overline{\mathbb{Z}\alpha} = K$ and let $U' \subset K$ be a non-empty open set. Suppose that

$$(1) \qquad\qquad \{n \in \mathbb{Z} \mid T^n x_0 \in U\} = \{n \in \mathbb{Z} \mid n\alpha \in U'\}$$

and that $\mathrm{Stab}_K(\overline{U'}) = \{0\}$. Then the pointed Kronecker system $(K, 0, k \mapsto k + \alpha)$ is a factor of the pointed system (X, x_0, T), i.e., there is a continuous map $\phi : X \to K$ with $\phi(x_0) = 0$ satisfying

$$\phi(Tx) = \phi(x) + \alpha \text{ for all } x \in X.$$

Note that Theorem 1.4 immediately implies Theorem 1.3 (apply twice to both Kronecker systems to obtain two factor maps that are inverses) since an isomorphism of Kronecker systems that preserves the zero elements is necessarily a group isomorphism. The following example shows that we cannot replace the minimality assumption with transitivity.

Example 1.5. (Insufficient to assume transitivity instead of minimality) Let $X \subset [-1, 1]$ be the set $X = \{x_n \mid n \in \mathbb{Z} \cup \{-\infty, \infty\}\}$ where $x_{\pm\infty} = \pm 1$ and for $n \in \mathbb{Z}$,

$$
x_n = \begin{cases} -1 + \frac{1}{|n|} & n < 0 \\ 0 & n = 0 \\ 1 - \frac{1}{n} & n > 0 \end{cases}
$$

Thus X is closed and all x_n with $n \in \mathbb{Z}$ are isolated points. So we have a homeomorphism $T : X \to X$ mapping x_n to x_{n+1} and fixing $x_{\pm\infty}$. Hence it is transitive (all x_n with $n \in \mathbb{Z}$ have dense orbits) but not minimal. Now $U = \{x_n \mid n \in 2\mathbb{Z}\}$ is open and the return times set $2\mathbb{Z} = \{n \in \mathbb{Z} \mid T^n x_0 \in U\}$ is also equal to $\{n \in \mathbb{Z} \mid n\alpha \in U'\}$ where $U' = \{0\} \subset K = \mathbb{Z}/2\mathbb{Z}$ is open and $\alpha = 1 \in K$. But there is no factor π from (X, T) to the Kronecker system $(K, k \mapsto k + 1)$ as it would yield the contradiction $\pi(x_\infty) = \pi(T x_\infty) = \pi(x_\infty) + 1$.

1.3. **Return times along a polynomial sequence.** We recall the following result which follows immediately from the Polynomial Weyl Equidistribution [7] and the fact that connected compact abelian groups have no non-trivial characters with finite image.

Proposition 1.6. Let K be a **connected** compact abelian group and suppose that $P(x) \in \mathbb{Z}[x]$ is a non-constant polynomial. If $\alpha \in K$ is such that $\overline{\mathbb{Z}\alpha} = K$ then $P(\mathbb{Z})\alpha = \{P(n)\alpha \mid n \in \mathbb{Z}\}$ is dense in K.

Given this, it is natural to ask whether we can extend Theorem 1.3 to the case where we only know the set of return times inside a polynomial sequence, i.e., the set $\{n \mid P(n)\alpha \in U\}$. The next result shows that this is indeed the case.

Theorem 1.7. Let K_1 and K_2 be compact connected abelian groups and suppose that $\alpha_i \in K_i$ with $\overline{\mathbb{Z}\alpha_i} = K_i$. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with $P(0) = 0$. Let $U_i \subset K_i$ be non-empty open sets and let $\mathcal{R}_i = \{n \in \mathbb{Z} \mid P(n)\alpha_i \in U_i\}$. If $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathrm{Stab}_{K_1}(\overline{U_1}) = \{0\}$ and $\mathrm{Stab}_{K_2}(\overline{U_2}) = \{0\}$ then there exists an isomorphism $K_1 \to K_2$ (of topological groups, so continuous) mapping $P(n)\alpha_1$ to $P(n)\alpha_2$ for each $n \in \mathbb{Z}$. In particular, if $P(\mathbb{Z})$ is not contained in any proper subgroup of $\mathbb{Z}$ then this isomorphism maps α_1 to α_2.

Example 1.8. Consider the polynomial $P(n) = n^5 - n$ and let $\alpha_1 = \sqrt{2} \in \mathbb{T}$ and $\alpha_2 = \sqrt{2} + \frac{1}{5} \in \mathbb{T}$. Then for any open set $U \subset \mathbb{T}$ we have that $\{n \in \mathbb{Z} \mid P(n)\alpha_1 \in U\} = \{n \in \mathbb{Z} \mid P(n)\alpha_2 \in U\}$ and indeed there is an isomorphism (the identity map) mapping $P(n)\alpha_1$ to $P(n)\alpha_2$ as they are equal, but there is no isomorphism mapping α_1 to α_2.

We remark that it seems that there is no obvious way to extend Theorem 1.4 to polynomial times under the assumption that X is connected since an example of Pavlov [4] shows that there exists a minimal connected system (X, T) such that $\{T^{n^2} x_0 \mid n \in \mathbb{Z}\}$ is not dense for some $x_0 \in X$.

1.4. **Return times sets of nilmanifolds.** It is natural to try to extend Theorem 1.3 on the return times in Kronecker systems to other homogeneous spaces such as nilmanifolds. The following is a partial result on this which establishes that if a Kronecker system $(K, +)$ and a minimal nilsystem $(G/\Gamma, T)$ share the same return times, with the closures of the defining open sets having trivial stabilizers under translations of K and G respectively, then they must be isomorphic as dynamical systems.

Theorem 1.9. Let $X = G/\Gamma$ be a nilmanifold, where G is a nilpotent Lie group and $\Gamma \leq G$ is a cocompact discrete subgroup. Suppose that $\tau \in G$ is such that (X, T) is minimal, where $T : X \to X$ is given by $Tx = \tau x$ for $x \in X$. Suppose that $U \subset X$ is open and satisfies that $\mathrm{Stab}_G(\overline{U}) := \{g \in G \mid g\overline{U} = \overline{U}\}$ consists of only those $g \in G$ such that $gx = x$ for all $x \in X$. Now suppose that $(K, +)$ is a compact abelian group and $\alpha \in K$ is such that $\overline{\mathbb{Z}\alpha} = K$ and $V \subset K$ is open such that $\mathrm{Stab}_K(\overline{V}) = \{0\}$ and

$$\{n \in \mathbb{Z} \mid T^n x_0 \in U\} = \{n \in \mathbb{Z} \mid n\alpha \in V\}$$

for some $x_0 \in X$. Then the pointed system (X, x_0, T) is isomorphic to the pointed Kronecker system $(K, 0, k \mapsto k + \alpha)$, i.e., there is a homeomorphism $\phi : X \to K$ with $\phi(x_0) = 0$ satisfying

$$\phi(Tx) = \alpha + \phi(x) \text{ for all } x \in X.$$

Note that Theorem 1.4 shows that such a continuous map ϕ exists but does not show that it is a homeomorphism (and indeed it may not be without the assumption on $\mathrm{Stab}_G(\overline{U})$). Thus it suffices to show the existence of an inverse to ϕ, which will be established in Section 4.

It is interesting to ask whether the same result holds more generally for two nilsystems rather than a nilsystem and a Kronecker system. Furthermore, one can extend this question to more general homogeneous spaces.

1.5. **An explicit reconstruction for Jordan Measurable sets.** One way of stating Theorem 1.4 is that one can uniquely reconstruct a Kronecker system $(K, x \mapsto x + \alpha)$, where K is a compact group with $\overline{\mathbb{Z}\alpha} = K$, from the set $\mathcal{R} = \{n \mid n\alpha \in U\}$ provided that U is some (unknown) open set with $\mathrm{Stab}_K(\overline{U})$ is trivial. If we further assume that U is Jordan measurable (recall that this means that $m_K(\overline{U}) = m_K(U)$ where m_K is the Haar measure) then the following result shows that there is in fact a rather explicit reconstruction of (K, α) from $\mathcal{R}$. Let $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the unit complex numbers.

Theorem 1.10. Let K be a compact abelian group and let $\alpha \in K$ be such that $\overline{\mathbb{Z}\alpha} = K$ and suppose $U \subset K$ is a Jordan measurable open set such that $\mathrm{Stab}_K(\overline{U})$ is trivial. Let $\mathcal{R} = \{n \mid n\alpha \in U\}$ and let

$$\Lambda = \left\{\lambda \in \mathbb{U} \mid \frac{1}{N} \sum_{n=1}^{N} \lambda^n \mathbb{1}_{\mathcal{R}}(n) \text{ does not converge to 0 as } N \to \infty\right\}.$$

Then $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is countable and there is an injective continuous group homomorphism $K \to \mathbb{U}^{\mathbb{N}}$ mapping α to $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots)$. In particular, K is isomorphic to the closure of the subgroup generated by $\vec{\lambda}$.

Note that this implies Theorem 1.4 in the case where the open sets are Jordan measurable but it is not difficult to construct examples with U not Jordan measurable where this reconstruction is invalid despite Theorem 1.4 still guaranteeing the uniqueness of the Kronecker system.

1.6. **Further questions.** We now gather some open questions. Our first question asks about a natural extension of Theorem 1.9 to two nilsystems and more general homogeneous spaces.

Question 1.11. Let $X_1 = G_1/\Gamma_1$ and $X_2 = G_2/\Gamma_2$ be homogeneous spaces, where G_i are Polish groups and $\Gamma_i \leq G_i$ are cocompact discrete subgroups. For all $i = 1, 2$, suppose that $U_i \subset X_i$ is an open set with closure having a trivial stabilizer in the sense that whenever $g \in G$ is such that $g\overline{U_i} = \overline{U_i}$ then $gx = x$ for all $x \in X_i$. Suppose that $\tau_i \in G_i$ are such that $(X_i, x \mapsto \tau_i x)$ are minimal systems and

$$\{n \in \mathbb{Z} \mid \tau_1^n x_1 \in U_1\} = \{n \in \mathbb{Z} \mid \tau_2^n x_2 \in U_2\}$$

for some $x_1 \in X_1$ and $x_2 \in X_2$. Then are the systems $(X_1, x \mapsto \tau_1 x)$ and $(X_2, x \mapsto \tau_2 x)$ isomorphic (via a map sending x_1 to x_2)? If not true in general, is it true if G_i are nilpotent Lie groups?

The next question asks whether one can extend our results on minimal dynamical $\mathbb{Z}$ systems to actions of other groups. It turns out that this seems difficult even without any topology, so we assume full orbits rather than just dense orbits as follows.

Question 1.12. Suppose that G is a group acting on sets X_1 and X_2 transitively. Thus there are $x_1 \in X_1$ and $x_2 \in X_2$ with $Gx_1 = X_1$ and $Gx_2 = X_2$. Suppose that $U_1 \subset X_1$ and $U_2 \subset X_2$ are subsets with trivial setwise stabilizer, i.e.,

$$\{g \in G \mid gU_1 = U_1\} = \{1\} = \{g \in G \mid gU_2 = U_2\}.$$

If $\{g \in G \mid gx_1 \in U_1\} = \{g \in G \mid gx_2 \in U_2\}$ then does it mean that the two actions are isomorphic? That is, does there exist a bijection $\phi : X_1 \to X_2$ such that $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_1$? Does there exist one mapping x_1 to x_2?

Using the language of [1] and [5] such group actions satisfying the trivial setwise stabilizer hypothesis are called 2-distinguishable. This question is already interesting for G finite. In fact, we can simplify this to the following interesting question, which again is already interesting for finite sets (we will show this equivalence in Section 6).

Definition 1.13. Let G be a group acting on a set X. We say that $U \subset X$ is *simple* (for this action) if whenever $\pi : X \to X'$ is a factor (i.e., X' is a G-set and $\pi : X \to X'$ is a surjective map such that $\pi(gx) = g\pi(x)$ for all $g \in G$, $x \in X$) such that $U = \pi^{-1}(U')$ for some $U' \subset X'$, then π is a bijection (isomorphism of G-sets). In other words, U is simple if it can not be realised as a preimage of a factor in a non-trivial way (i.e., the factor is not an isomorphism).

Question 1.14. Let G be a group acting transitively on a set X and suppose that $U \subset X$ has trivial setwise stabilizer, i.e., $\{g \in G \mid gU = U\}$ consists of only $1 \in G$. Then is it true that U is simple for this G-action? In other words, is it true that if $\mathcal{B} \subset 2^X$ is a G-invariant partition of X such that U is a union of elements in $\mathcal{B}$, then $\mathcal{B}$ consists of only singletons?

2. Bohr Set Return Times of Minimal Systems (Proof of Theorem 1.4)

Proof of Theorem 1.4. Consider the product system (Z, S) where $Z = X \times K$ and $S : Z \to Z$ is given by

$$S(x, k) = (Tx, k + \alpha).$$

Let $\Theta = \overline{S^{\mathbb{Z}} z_0}$ be the orbit closure of $z_0 = (x_0, 0) \in Z$. Let $\pi_X : \Theta \to X$ and $\pi_K : \Theta \to K$ denote the projection maps, which are surjective by the minimality of (X, T) and $(K, k \mapsto k + \alpha)$. Let

$$H = \{h \in K \mid (x_0, h) \in \Theta\}.$$

Claim: H is a closed subgroup of K.

Note that $(x_0, 0) = z_0 \in \Theta$ so $0 \in H$. Now if $h, h' \in H$ then that means $(x_0, h), (x_0, h') \in \Theta$ and thus there exists a sequence $n_1, n_2, \ldots \in \mathbb{Z}$ such that

$$S^{n_i} z_0 = (T^{n_i} x_0, n_i \alpha) \to (x_0, h)$$

as $i \to \infty$ and another sequence $n'_1, n'_2 \ldots \in \mathbb{Z}$ such that $(T^{n'_i} x_0, n'_i \alpha) \to (x_0, h')$. Now fix $\epsilon > 0$. We may choose $J = J(\epsilon) \in \mathbb{N}$ such that[1] $n_J \alpha \approx_\epsilon h$ and choose $\delta > 0$ such that

$$d(T^{n_J} x, x_0) < \epsilon \text{ whenever } d(x, x_0) < \delta.$$

Now take I large enough so that $d(T^{n'_I} x_0, x_0) < \delta$ and $n'_I \alpha \approx_\epsilon h'$. Thus $d(T^{n_J + n'_I} x_0, x_0) < \epsilon$ and $(n_J + n'_I)\alpha \approx_{2\epsilon} h + h'$. As $\epsilon > 0$ is arbitrary, this means $(x_0, h + h') \in \overline{S^{\mathbb{Z}} z_0} = \Theta$, thus $h + h' \in H$. Finally, notice that H is closed (as it is $\pi_K(\Theta \cap (\{x_0\} \times K))$) and so it is a closed non-empty sub-semigroup and thus a closed subgroup as K is compact.

Now for $x \in X$ let $H_x = \{k \in K \mid (x, k) \in \Theta\}$, thus $H = H_{x_0}$.

Claim: For each $x \in X$, we have that H_x is a coset of H, i.e., H_x and $H_{x'}$ are translates of each other for all $x, x' \in X$.

To see this note that by minimality of (X, T), there exists a sequence $n_1, n_2, \ldots \in \mathbb{Z}$ such that $T^{n_i} x \to x'$ and by compactness we suppose that $n_i \alpha \to \alpha_{x,x'} \in K$. It follows that $H_x + \alpha_{x,x'} \subset H_{x'}$. In particular this means that $\alpha_{0,x} + \alpha_{x,0} \in H$, so

$$H + \alpha_{0,x} = H - (\alpha_{0,x} + \alpha_{x,0}) + \alpha_{0,x} = H - \alpha_{x,0} \supset H_x$$

and thus $H_x = H + \alpha_{0,x}$ is a coset of H as desired.

Claim: $H \subset \text{Stab}_K(\overline{U'})$.

To see this suppose that $h \in H$ and $u' \in U'$. Then by the surjectivity of $\pi_K : \Theta \to K$ there exists $x \in X$ such that $(x, u') \in \Theta$. Now we may find a sequence $n_1, n_2, \ldots \in \mathbb{Z}$ such that $n_i \alpha \to u'$ and $T^{n_i} x_0 \to x$. So for large enough i, we have $n_i \alpha \in U'$ and thus by the assumption of the equality of return times (1) we have $T^{n_i} x_0 \in U$. Since $(T^{n_i} x_0, n_i \alpha) \in \Theta$ we must have that $(T^{n_i} x_0, n_i \alpha + h) \in \Theta$ as $H_{T^{n_i} x_0}$ is a coset of H. This means that, for each fixed i, we may find a sequence $m_1, m_2, \ldots \in \mathbb{Z}$ such that $S^{m_j} z_0 \to (T^{n_i} x_0, n_i \alpha + h)$. So for large enough j we have that $T^{m_j} x_0$ is so close to $T^{n_i} x_0 \in U$ that $T^{m_j} x_0 \in U$ and so again by the equality of return times we have $m_j \alpha \in U'$. But as $m_j \alpha \to n_i \alpha + h$, we must have that $n_i \alpha + h \in \overline{U'}$. Finally, as $n_i \alpha \to u'$ we must have that $u' + h \in \overline{U'}$. Thus as $u' \in U'$ is arbitrary we must have that $h + U' \subset \overline{U'}$ and by continuity of translation $h + \overline{U'} \subset \overline{U'}$. Thus $h \in \text{Stab}_K(\overline{U'})$ as claimed.

Now by the assumption that this stabilizer is trivial, we get that H is trivial and thus $\pi_X : \Theta \to X$ is injective and hence a homeomorphism. Thus our desired factor is $\pi_K \circ \pi_X^{-1} : X \to K$. $\qquad\square$

---

[1] By this we mean that we have fixed an invariant metric d_K on K and we write $k \approx_\epsilon k'$ if $d_K(k, k') < \epsilon$ for $k, k' \in K$.

3. Polynomial return times (proof of Theorem 1.7)

3.1. The case of compact connected abelian Lie groups. We now embark on proving Theorem 1.7. We first prove this for the case where K_1 and K_2 are compact connected abelian **Lie** groups, i.e., subgroups of a finite dimensional torus. We leave the more subtle case of non-Lie groups to the next subsection.

Proposition 3.1. Let K be a compact abelian Lie group with $\alpha \in K$ such that $\mathbb{Z}\alpha$ is dense in K. Let K_0 be the connected component of the identity, thus $d = |K/K_0|$ is finite (as K is a compact Lie group). Let $P(x) \in \mathbb{Z}[x]$ be a non-constant polynomial with $P(0) = 0$. Then

$$\overline{P(d\mathbb{Z})\alpha} = K_0$$

and

$$\overline{P(\mathbb{Z})\alpha} = \bigcup_{a \in A} (K_0 + a)$$

for some finite set $A \subset K$.

Proof. Note that $dk \in K_0$ for all $k \in K$. In particular $\mathbb{Z}d\alpha \subset K_0$. Let $K'_0 = \overline{\mathbb{Z}d\alpha}$. Note that $K'_0 \subset K_0$ and it is of finite index in K since

$$K = \overline{\mathbb{Z}\alpha} = \bigcup_{m=0}^{d-1} \overline{(d\mathbb{Z} + m)\alpha} = \bigcup_{m=0}^{d-1} (K'_0 + m\alpha).$$

Thus K'_0 is of finite index in K_0 as well. But as K_0 is connected, we must have that $K'_0 = K_0$. Now let $Q(x) = \frac{1}{d}P(dx) \in \mathbb{Z}[x]$ (as $P(0) = 0$). From Weyl equidistribution and connectedness of K_0 we see that the sequence $Q(1)d\alpha, Q(2)d\alpha, Q(3)d\alpha \ldots$ equidistributes in K_0. As $Q(n)(d\alpha) = P(dn)\alpha$ this implies the first claim that

$$\overline{P(d\mathbb{Z})\alpha} = K_0.$$

Now for the second claim, we write

$$\overline{P(\mathbb{Z})\alpha} = \bigcup_{m=0}^{d-1} \overline{P(d\mathbb{Z} + m)\alpha} = \bigcup_{m=0}^{d-1} \overline{P_m(d\mathbb{Z})\alpha + a_m}$$

for some non-constant $P_m(x) \in \mathbb{Z}[x]$ with $P_m(0) = 0$ and $a_m \in K$. Thus applying the previous claim to the P_m we get that $\overline{P_m(d\mathbb{Z})\alpha + a_m}$ is a coset of K_0, as required. $\square$

Proposition 3.2. Let K_1 and K_2 be compact connected abelian Lie groups and suppose that $\alpha_i \in K_i$ with $\overline{\mathbb{Z}\alpha_i} = K_i$. Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with $P(0) = 0$ such that $P(\mathbb{Z})$ is not contained in any proper subgroup of $\mathbb{Z}$. Let $U_i \subset K_i$ be non-empty open sets and let $\mathcal{R}_i = \{n \in \mathbb{Z} \mid P(n)\alpha_i \in U_i\}$. If $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathrm{Stab}_{K_1}(\overline{U_1}) = \{0\}$ and $\mathrm{Stab}_{K_2}(\overline{U_2}) = \{0\}$ then there exists an isomorphism $K_1 \to K_2$ (of topological groups, so continuous) mapping α_1 to α_2.

Proof. Let $\alpha = (\alpha_1, \alpha_2) \in K_1 \times K_2$ and let $K = \overline{\mathbb{Z}\alpha} \subset K_1 \times K_2$. Let $\Theta = \overline{P(\mathbb{Z})\alpha}$. We know that

$$\Theta = \bigcup_{a \in A} (K_0 + a)$$

for some finite $A \subset K$ where K_0 is the identity connected component of K. Let $\pi_i : K \to K_i$ be the projection maps. We know that π_i is surjective by the assumption that $\overline{\mathbb{Z}\alpha_i} = K_i$. In fact, since K_0 is finite index in K, we have that $\pi_i(K_0)$ of finite index in K_i and thus $\pi_i(K_0) = K_i$ by the assumption that K_i is connected. We now wish to show that π_i is injective on Θ. To see this, suppose that $\theta, \theta' \in \Theta$ are such that $\pi_1(\theta) = \pi_1(\theta')$. We know that $\theta \in K_0 + a$ and $\theta' \in K_0 + a'$ for some $a, a' \in A$. We now wish

to show that $\pi_2(\theta - \theta') \in \mathrm{Stab}_{K_2}(\overline{U_2})$. To do this, suppose that $u_2 \in U_2$. From $\pi_2(K_0) = K_2$ we get that $\pi_2(K_0 + a') = K_2$, thus there exists $x \in K_1$ such that $(x, u_2) \in K_0 + a'$. Now

$$(x, u_2 + \pi_2(\theta - \theta')) = (x, u_2) + \theta - \theta' \in (K_0 + a') + (K_0 + a - a') = K_0 + a,$$

that is $(x, u_2 + \pi_2(\theta - \theta')) \in \Theta$. Now since $(x, u_2) \in \Theta$ this means there exists a sequence $n_1, n_2, \ldots$ of integers such that $P(n_j)\alpha \to (x, u_2)$. In particular, $P(n_j)\alpha_2 \to u_2$ so $P(n_j)\alpha_2 \in U_2$ for sufficiently large j. But since $\mathcal{R}_1 = \mathcal{R}_2$ this means that $P(n_j)\alpha_1 \in U_1$. Now since $(x, u_2) \in K_0 + a'$ and $K_0 + a'$ is open in K, we must have that $P(n_j)\alpha \in K_0 + a'$ for sufficiently large j. Thus $P(n_j)\alpha + (\theta - \theta') \in K_0 + a \subset \Theta$ for sufficiently large j. In particular, this means that we may choose an integer n'_j so that $P(n'_j)\alpha$ is so close to $P(n_j)\alpha + (\theta - \theta')$ so that $P(n'_j)\alpha_1$ is sufficiently close to $P(n_j)\alpha_1 \in U_1$ so that $P(n'_j)\alpha_1 \in U_1$ and so that $P(n'_j)\alpha \to (x, u_2 + \pi_2(\theta - \theta'))$. But as $\mathcal{R}_1 = \mathcal{R}_2$ this means that $P(n'_j)\alpha_2 \in U_2$. But $P(n'_j)\alpha_2 \to u_2 + \pi_2(\theta - \theta')$, thus showing that $u_2 + \pi_2(\theta - \theta') \in \overline{U_2}$. So we have shown that if $u_2 \in U_2$ then $u_2 + \pi_2(\theta - \theta') \in \overline{U_2}$, and by continuity of addition this holds more generally for $u_2 \in \overline{U_2}$, thus $\pi_2(\theta - \theta') \in \mathrm{Stab}_{K_2}(\overline{U_2})$. But by the assumption that $\mathrm{Stab}_{K_2}(\overline{U_2}) = \{0\}$ we get that $\pi_2(\theta) = \pi_2(\theta')$, which means that $\theta = \theta'$. Thus we have shown that $\pi_1 : \Theta \to K_1$ is injective, and thus a homeomorphism. As K_1 is connected this means that Θ is connected and so in fact $\Theta = K_0$ (as $P(0)\alpha = (0,0)$). So $\pi_1 : K_0 \to K_1$ is in fact an isomorphism of topological groups. By symmetry (using now the assumption that $\mathrm{Stab}_{K_1}(\overline{U_1}) = \{0\}$) we get that $\pi_2 : K_0 \to K_2$ is an isomorphism of topological groups. Thus we have an isomorphism $K_1 \to K_2$ mapping $P(n)\alpha_1$ to $P(n)\alpha_2$ for all $n \in \mathbb{Z}$. Finally, from the assumption that $P(\mathbb{Z})$ is not contained in any proper subgroup, we have that $1 = \sum_i a_i P(n_i)$ for some integers a_i and n_i. Thus $\alpha_1 = \sum_i a_i P(n_i)\alpha_1$ is mapped to $\sum_i a_i P(n_i)\alpha_2 = \alpha_2$ under this isomorphism. $\qquad\square$

3.2. **Infinite dimensional polynomial orbit.** We now remove the assumption that our groups K_1 and K_2 are Lie groups (i.e., embedded in a finite dimensional torus). Thus let K_1 and K_2 be compact metrizable connected abelian groups. Note that this means they are closed subgroups of the countable dimensional torus $\mathbb{T}^{\mathbb{N}}$ and thus we may approximate them by compact Lie groups as follows.

Definition 3.3. Let (X, d_X) and (Y, d_Y) be metric spaces. For $\delta > 0$, we say that a map $\phi : X \to Y$ is a δ-almost isometry if for all $x_1, x_2 \in X$ we have that

$$|d_X(x_1, x_2) - d_Y(\phi(x_1), \phi(x_2))| < \delta.$$

Thus we have a decreasing sequence $\delta_1 > \delta_2 > \ldots$ of positive real numbers converging to 0 and, for each positive integer D, surjective continuous group homomorphisms $\phi_D^1 : K_1 \to F_D^1$ and $\phi_D^2 : K_2 \to F_D^2$ where F_D^1 and F_D^2 are closed subgroups of $\mathbb{T}^D$ and ϕ_D^1 and ϕ_D^2 are δ_D almost isometries (we have fixed an invariant metric on each compact metrizable abelian group K, which we shall call d_K or d if clear from the context). For concreteness, we construct these maps conveniently as follows. Equip $\mathbb{T}^D$ with the metric

$$d(x, y) = \sum_{i=1}^{D} 2^{-i} \|x_i - y_i\|,$$

where $\|v\|$ is the shortest distance from $v \in \mathbb{T}$ to $0 \in \mathbb{T}$, and $\mathbb{T}^{\mathbb{N}}$ with the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \|x_i - y_i\|,$$

that way ϕ_D^1 and ϕ_D^2 are (restrictions to K_1 and K_2 respectively) the projections onto the first D co-ordinates, which are clearly $O(2^{-D})$-almost isometries. The F_D^1 and F_D^2 are then defined to be the images of these, thus $\phi_D^1 : K_1 \to F_D^1$ and $\phi_D^2 : K_2 \to F_D^2$ are surjective.

Now let $P(x) \in \mathbb{Z}[x]$ be a non-constant polynomial such that $P(0) = 0$. We now let

$$\Theta = \overline{P(\mathbb{Z})\alpha} \subset K_1 \times K_2$$

where $\alpha = (\alpha_1, \alpha_2) \in K_1 \times K_2$ are such that $\overline{\mathbb{Z}\alpha_i} = K_i$ for $i = 1, 2$. We let $\phi_D : K_1 \times K_2 \to F_D^1 \times F_D^2$ be the map $\phi_D = \phi_D^1 \times \phi_D^2$. We also let $\Theta_D = \phi_D(\Theta) \subset F_D^1 \times F_D^2$. We equip $K_1 \times K_2$ and $F_D^1 \times F_D^2$ with the metrics $d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$, that way ϕ_D is also a $O(2^{-D})$-almost isometry. Observe that

$$\Theta_D = \overline{P(\mathbb{Z})\phi_D(\alpha)}$$

and thus we may apply Proposition 3.1 to obtain that

$$\tag{2} \Theta_D = \bigcup_{a \in A_D} (\Gamma_D + a)$$

for some finite $A_D \subset F_D^1 \times F_D^2$ and closed connected subgroup $\Gamma_D \leq F_D^1 \times F_D^2$. Note that $\Gamma_D \subset \Theta_D$ as $0 \in \Theta_D$ since $P(0) = 0$.

Let

$$\Gamma = \bigcap_{D=1}^{\infty} \phi_D^{-1}(\Gamma_D)$$

and notice that this is a closed subgroup.

Lemma 3.4. We have $\Gamma \subset \Theta$.

Proof. Since ϕ_D is a $O(2^{-D})$-almost isometry, we have that if $p \in \phi_D^{-1}(\Gamma_D)$ then $\phi_D(p) \in \Gamma_D \subset \phi_D(\Theta)$ and so $d(p, \Theta) < O(2^{-D})$. But Θ is closed, so the proof is complete by letting $D \to \infty$. $\square$

Lemma 3.5. We have that $\phi_D^{-1}(\Gamma_D) \supset \phi_{D+1}^{-1}(\Gamma_{D+1})$.

Proof. The natural projection $F_{D+1}^1 \times F_{D+1}^2 \to F_D^1 \times F_D^2$ maps Θ_{D+1} to Θ_D, so in particular maps the connected component Γ_{D+1} to the connected component Γ_D. $\square$

We let $\pi_i : K_1 \times K_2 \to K_i$ and $\pi_{i,D} : F_D^1 \times F_D^2 \to F_D^i$ denote the projections.

Lemma 3.6. For $i = 1, 2$ we have that $\pi_i(\Gamma) = K_i$.

Proof. Fix $i \in \{1, 2\}$. We first show that $\pi_{i,D}(\Gamma_D) = F_D^i$. First notice that

$$\pi_{i,D}(\Theta_D) = \overline{P(\mathbb{Z})\phi_D(\alpha_i)} = F_D^i$$

where the last equality follows from Weyl equdisitribution, connectendess of F_D^i and $\overline{\mathbb{Z}\phi_D(\alpha_i)} = F_D^i$. But

$$\pi_{i,D}(\Theta_D) = \bigcup_{a \in A_D} (\pi_{i,D}(\Gamma_D) + \pi_{i,D}(a))$$

is a disjoint union of closed sets, thus by connectedness they must all be the same and equal to $\pi_{i,D}(\Gamma_D)$, so indeed $\pi_{i,D}(\Gamma_D) = F_D^i$. Now let $k \in K_i$. Then by our first claim we have that $\phi_D^i(k) = \pi_{i,D}(\gamma_D)$ for some $\gamma_D \in \Gamma_D$. Now by surjectivity of ϕ_D^i, we may take $\beta_D \in \phi_D^{-1}(\Gamma_D)$ such that $\phi_D(\beta_D) = \gamma_D$. By compactness, we may pass to a subsequence such that $\beta_{D_j} \to \beta$ for some $\beta \in K_1 \times K_2$. But it now follows from Lemma 3.5 that $\beta \in \Gamma$. The proof will be complete if we can show that $\pi_i(\beta) = k$. To see this, note that $\phi_D^i(\pi_i(\beta)) = \pi_{i,D}(\phi_D(\beta))$ but $\phi_D(\beta)$ can be made arbitrarily close to $\phi_D(\beta_D) = \gamma_D$ for large enough D. So $\phi_D^i(\pi_i(\beta))$ can be made arbitrarily close to $\pi_{i,D}(\gamma_D) = \phi_D^i(k)$. It now follows that $\pi_i(\beta)$ is arbitrarily close to k for large enough D (as ϕ_D^i is a $O(2^{-D})$-almost isometry), thus they are equal. $\square$

Proposition 3.7. There exists $A \subset K_1 \times K_2$, with $0 \in A$, such that

$$\Theta = \bigcup_{a \in A} (\Gamma + a).$$

Proof. We already know that $\Gamma \subset \Theta$ by Lemma 3.4. Thus it remains to show that if $\gamma_0 + a \in \Theta$ for some $\gamma_0 \in \Gamma$ and $a \in K_1 \times K_2$ then for every $\gamma \in \Gamma$ we have that $\gamma + a \in \Theta$. Let D be a positive integer, we thus have that $\phi_D(\gamma_0 + a) \in \Theta_D$ and so (2) and the fact that $\Gamma \subset \phi_D^{-1}(\Gamma_D)$ (by definition) implies that $\Gamma_D + \phi_D(a) \subset \Theta_D$. In particular, this means that $\phi_D(\gamma + a) \in \Theta_D$. Thus there exists a $\beta_D \in \Theta$ such that $\phi_D(\gamma + a) = \phi_D(\beta_D)$. But since ϕ_D is a $O(2^{-D})$-almost isometry, we have that $d(\gamma + a, \beta_D) < O(2^{-D})$. Thus $\beta_D \to \gamma + a$ and since Θ is closed we must have $\gamma + a \in \Theta$. $\square$

Now let $U_1 \subset K_1$ and $U_2 \subset K_2$ be open sets. Let

$$\mathcal{R}_i = \{n \in \mathbb{Z} \mid P(n)\alpha_i \in U_i\}.$$

Proposition 3.8. If $\mathcal{R}_1 = \mathcal{R}_2$ and $\theta, \theta' \in \Theta$ are such that $\pi_1(\theta_1) = \pi_1(\theta_2)$, then $\pi_2(\theta - \theta') \in \mathrm{Stab}_{K_2}(\overline{U_2})$.

Proof. We have $\theta' - \theta = (0, v)$ for some $v \in K_2$, so we wish to show $v \in \mathrm{Stab}_{K_2}(\overline{U_2})$. By Lemma 3.7 we have $a, a' \in A$ such that $\theta \in \Gamma + a$ and $\theta' \in \Gamma + a'$. Now let $u_2 \in U_2$. Since $\pi_2(\Gamma) = K_2$ there exists $x \in K_1$ such that $(x, u_2) \in \Gamma + a$. Now fix $\epsilon > 0$ such that the ball of radius ϵ centred at u_2 is contained in U_2. Since $\pi_1|_\Gamma : \Gamma \to K_1$ is a surjective homomorphism between compact groups, it is an open map and thus there exists a $\delta > 0$ such that if $g_1 \in K_1$ with $d(0, g_1) < \delta$ then there exists a $\gamma \in \Gamma$ with $d(0, \gamma) < \epsilon$ and $\pi_1(\gamma) = g_1$. Now since $(x, u_2) \in \Gamma + a \subset \Theta$ there exists a positive integer n_1 such that $d(P(n_1)\alpha, (x, u_2)) < \max\{\delta, \epsilon\}$. Writing $P(n_1)\alpha = (x_1, y_1)$ with $x_1 \in K_1, y_1 \in K_2$ we see that $d(y_1, u_2) < \epsilon$ and thus $y_1 \in U_2$ and thus since $n_1 \in \mathcal{R}_2 = \mathcal{R}_1$ we have $x_1 \in U_1$. We set $g_1 = x_1 - x$. Notice that $d(0, g_1) < \delta$ and so there exists a $\gamma \in \Gamma$ with $d(0, \gamma) < \epsilon$ such that $\pi_1(\gamma) = g_1$. Writing $\gamma = (g_1, g_2)$ for some $g_2 \in K_2$ we get that

$$(x_1, u_2 + g_2) = (x, u_2) + \gamma \in \Gamma + a + \gamma = \Gamma + a.$$

Thus $\theta'' := (x_1, u_2 + g_2 + v) \in \Gamma + a' \subset \Theta$. This means that there exists $n_2 \in \mathbb{Z}$ such that $P(n_2)\alpha$ is so close to θ'' that $\pi_1(P(n_2)\alpha)$ is so close to x_1 that $\pi_1(P(n_2)\alpha) \in U_1$ and $d(\pi_2(P(n_2)\alpha), u_2 + g_2 + v) < \epsilon$. Thus $n_2 \in \mathcal{R}_1 = \mathcal{R}_2$ and so $\pi_2(P(n_2)\alpha) \in U_2$. This means that

$$d(u_2 + v, U_2) \leq d(u_2 + v + g_2, U_2) + \epsilon \leq d(u_2 + v + g_2, \pi_2(P(n_2)\alpha)) + \epsilon < 2\epsilon.$$

As $\epsilon > 0$ was arbitrary and independent of v, we may take $\epsilon \to 0$ to get that $d(u_2 + v, U_2) = 0$, and thus $u_2 + v \in \overline{U_2}$. Thus $U_2 + v \subset \overline{U_2}$ and by continuity of addition we have that $\overline{U_2} + v \subset \overline{U_2}$. The reverse inclusion holds by swapping the roles of θ and θ', thus $v \in \mathrm{Stab}_{K_2}(\overline{U_2})$. $\square$

Proposition 3.9. If $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathrm{Stab}_{K_1}(\overline{U_1})$ and $\mathrm{Stab}_{K_2}(\overline{U_2})$ are trivial, then there exists an isomorphism of topological groups $K_1 \to K_2$ mapping $P(n)\alpha_1$ to $P(n)\alpha_2$ for all $n \in \mathbb{Z}$.

Proof. By Proposition 3.8 and the assumption $\mathrm{Stab}_{K_2}(\overline{U_2}) = \{0\}$ we have that the mapping $\pi_1|_\Theta : \Theta \to K_1$ is injective and thus a homeomorphism. But since $\pi_1(\Gamma) = K_1$ by Lemma 3.6, we must have that $\Theta = \Gamma$, so $\pi_1|_\Theta : \Theta \to K_1$ is also a group homomorpishm. By symmetry (this time using $\mathrm{Stab}_{K_1}(\overline{U_1}) = \{0\}$) we get that $\pi_2|_\Theta : \Theta \to K_2$ is also an isomorphism of topological groups. Thus $\pi_2|_\Theta \circ (\pi_1|_\Theta)^{-1} : K_1 \to K_2$ is the desired isomorphism. $\square$

4. Return times sets for nilsystems

We now prove Theorem 1.9. As noted, Theorem 1.4 already defines the map $\phi : X \to K$ and so it remains to provide an inverse $\psi : K \to X$, which is established by the following result.

Proposition 4.1. Let $X = G/\Gamma$ be a nilmanifold, where G is a nilpotent Lie group and $\Gamma \leq G$ is a cocompact discrete subgroup. Suppose that $\tau \in G$ is such that (X,T) is minimal, where $T : X \to X$ is given by $Tx = \tau x$ for $x \in X$. Suppose that $U \subset X$ is open and satisfies that $\text{Stab}_G(\overline{U}) := \{g \in G \mid g\overline{U} = \overline{U}\}$ consists of only those $g \in G$ such that $gx = x$ for all $x \in X$. Now suppose that $(K, +)$ is a compact Lie group and $\alpha \in K$ is such that $\overline{\mathbb{Z}\alpha} = K$ and $V \subset K$ is open such that

$$\{n \in \mathbb{Z} \mid T^n x_0 \in U\} = \{n \in \mathbb{Z} \mid n\alpha \in V\}$$

for some $x_0 \in X$. Then the pointed system (X, x_0, T) is a factor of the pointed Kronecker system $(K, 0, k \mapsto k + \alpha)$, i.e., there is a continuous map $\psi : K \to X$ with $\psi(0) = x_0$ satisfying

$$\psi(\alpha + k) = T\psi(x) \text{ for all } k \in K.$$

Proof. Let $Z = K \times X$ and $z_0 = (0, x_0) \in Z$. Then $Z = (K \oplus G)/(\{0\} \oplus \Gamma)$ is a nilmanifold and thus (Z, S) is a nilsystem where $S : Z \to Z$ is given by $T(k, x) = (k + \alpha, Tx)$. Now let $\Theta = \overline{S^{\mathbb{Z}} z_0}$ be the closure of the S-orbit of z_0. Then (see Theorem 9 in Chapter 11 of [3]) we have that

$$\Theta = H z_0$$

for some closed subgroup H of $K \oplus G$. Now let $\pi_K : \Theta \to K$ and $\pi_X : \Theta \to X$ denote the projection maps. We wish to show that π_K is injective as it would then be a homeomorphism and our desired factor map would be $\pi_X \circ \pi_K^{-1} : K \to X$. Thus suppose that $\theta_1, \theta_2 \in \Theta$ are such that $\pi_K(\theta_1) = \pi_K(\theta_2)$. Thus we can write $\theta_1 = h_1 z_0$ and $\theta_2 = h_2 z_0$ for some $h_1, h_2 \in H$. For $i = 1, 2$, we may write $h_i = (k_i, g_i)$ where $k_i \in K$ and $g_i \in G$, thus $\theta_i = (k_i, g_i x_0)$. But $k_1 = k_2$ and so we have $h_0 := h_2 h_1^{-1} = (0, g_2 g_1^{-1}) \in H$ and we have the relation $h_0 \theta_1 = \theta_2$. Now let $u \in U$ be arbitrary. By surjectivity of $\pi_X : \Theta \to X$ we may find a $h' \in H$ such that $h' z_0 = (k', u) \in \Theta$ for some $k' \in K$. Thus we may find a sequence of integers $n_1, n_2, \ldots$ such that $S^{n_i} z_0 \to h' z_0$. In particular $T^{n_i} x_0 \to u$ and thus $T^{n_i} x_0 \in U$ for large enough n_i. This must mean that $n_i \alpha \in V$. Now $h_0 S^{n_i} z_0 = (n_i \alpha, g_2 g_1^{-1} T^{n_i} z_0) \in \Theta$ and so, for each fixed i, we may find a sequence of integers $m_1, m_2, \ldots$ such that $\lim_{j \to \infty} S^{m_j} z_0 = (n_i \alpha, g_2 g_1^{-1} T^{n_i} x_0)$. In particular this means that $m_j \alpha \to n_i \alpha \in V$ and so for large enough j we have $m_j \alpha \in V$ and so $T^{m_j} x_0 \in U$. But $T^{m_j} x_0 \to g_2 g_1^{-1} T^{n_i} x_0$ and so $g_2 g_1^{-1} T^{n_i} x_0 \in \overline{U}$. Finally taking the limit as $i \to \infty$ and using $T^{n_i} x_0 \to u$ we have that $g_2 g_1^{-1} u \in \overline{U}$. As $u \in U$ was arbitrary, this shows that $g_2 g_1^{-1} U \subset \overline{U}$ and thus $g_2 g_1^{-1} \overline{U} \subset \overline{U}$. The reverse inclusion follows from swapping the roles of θ_1 and θ_2, thus $g_2 g_1^{-1} \in \text{Stab}_G(\overline{U})$. Thus $g_2 g_1^{-1}$ stabilizes every $x \in X$ thus $g_2 x_0 = g_2 g_1^{-1} g_1 x_0 = g_1 x_0$, so in fact $\theta_2 = \theta_1$. Thus π_K is injective as desired. $\square$

5. Spectral construction for Jordan Measurable sets (Proof of Theorem 1.10)

Let $U \subset K$ be a Jordan measurable open subset of a compact abelian group K. Let $\alpha \in K$ be an element such that $\mathbb{Z}\alpha$ is dense in K. Let Λ denote the set of unit complex numbers λ such that

$$\frac{1}{N} \sum_{n=1}^{N} \lambda^n \mathbb{1}_U(n\alpha)$$

does not converge to 0 as $N \to \infty$.

Lemma 5.1. We have $\Lambda \subset \{\chi(\alpha) \mid \chi \in \widehat{K}\}$. In fact, if

$$\mathbb{1}_U = \sum_\chi c_\chi \chi$$

is the Fourier decomposition in $L^2(K)$, then Λ is the set of those $\chi(\alpha)$ for which $c_\chi \neq 0$.

Proof. Let $\mathbb{U}$ be the unit complex numbers. Suppose that $\lambda \in \mathbb{U}$ but $\lambda \notin \{\chi(\alpha) \mid \chi \in \widehat{K}\}$. Note that for any character χ on K we have that

$$\frac{1}{N} \sum_{n=1}^N \lambda^n \chi(n\alpha) = \frac{1}{N} \sum_{n=1}^N (\lambda\chi(\alpha))^n \to 0.$$

As any continuous function $f : K \to \mathbb{C}$ can be uniformly approximated by a linear combination of characters, we have that

$$\frac{1}{N} \sum_{n=1}^N \lambda^n f(n\alpha) \to 0.$$

Now observe that there exist continuous functions $f_\epsilon^+ : K \to [0, 1]$ such that

$$f_\epsilon^+ \searrow \mathbb{1}_{\overline{U}}, \quad \text{as } \epsilon \to 0.$$

In particular we choose these functions so that

$$m_K(\operatorname{supp} f_\epsilon^+ \setminus U) = m_K(\operatorname{supp} f_\epsilon^+ \setminus \overline{U}) < \epsilon$$

where m_K denotes the Haar measure on K and in the first equality we used that $\overline{U}$ and U are m_K almost the same (as U is Jordan measurable). Now as $\operatorname{supp} f_\epsilon^+ \setminus U$ is a closed set its indicator function can be written as a pointwise decreasing limit of continuous functions, thus we have that

$$\limsup_{N \to \infty} \frac{1}{N} |\{n \in [1, N] \mid n\alpha \in \operatorname{supp} f_\epsilon^+ \setminus U\}| \leq m_K(\operatorname{supp} f_\epsilon^+ \setminus U) < \epsilon.$$

This means that

$$\limsup_{N \to \infty} |\frac{1}{N} \sum_{n=1}^N \lambda^n f_\epsilon^+(n\alpha) - \frac{1}{N} \sum_{n=1}^N \lambda^n \mathbb{1}_U(n\alpha)| < \epsilon.$$

Thus

$$\frac{1}{N} \sum_{n=1}^N \lambda^n \mathbb{1}_U(n\alpha) \to 0$$

and hence we have shown the first claim that

$$\Lambda \subset \{\chi(\alpha) \mid \chi \in \widehat{K}\}.$$

Now suppose that $\lambda \in \{\chi(\alpha) \mid \chi \in \widehat{K}\}$. Thus $\lambda = \overline{\chi(\alpha)}$ for some unique (by density of $\mathbb{Z}\alpha$) character χ on K. But $\overline{\chi}\mathbb{1}_U$ is Riemann-integrable thus we have that

$$\frac{1}{N} \sum_{n=1}^N \lambda^n \mathbb{1}_U(n\alpha) = \frac{1}{N} \sum_{n=1}^N \overline{\chi}\mathbb{1}_U(n\alpha) \to \int \overline{\chi}\mathbb{1}_U dm_K = c_\chi.$$

$\square$

Now let $E = \{\chi(\alpha) \mid \chi \in \widehat{K}\}$. Note that the mapping $\widehat{K} \to E$ mapping χ to $\chi(\alpha)$ is injective (by density of $\mathbb{Z}\alpha$) and thus bijective. Consequently, let $\Gamma \subset \widehat{K}$ be the set corresponding to Λ, i.e., $\Lambda = \{\chi(\alpha) \mid \chi \in \Gamma\}$.

Now observe that the mapping $\iota : K \to \mathbb{U}^{\widehat{K}}$ given by $\iota(k) = (\chi(k))_{\chi \in \widehat{K}}$ is injective and continuous, thus a homeomorphism onto its image. Now let

$$\pi : K \to \mathbb{U}^\Gamma$$

be the projection given by

$$k \mapsto (\chi(k))_{\chi \in \Gamma}.$$

Proposition 5.2. $\ker(\pi) \subset \mathrm{Stab}_K(\overline{U})$.

Proof. Suppose that $k \in \ker \pi$. Thus $\chi(k) = 1$ for all $\chi \in \Gamma$. But since

$$\mathbb{1}_{\overline{U}} = \sum_{\chi \in \Gamma} c_\chi \chi$$

we have that $\mathbb{1}_{\overline{U}}(x + k) = \mathbb{1}_{\overline{U}}(x)$ for almost all $x \in K$. As U and $\overline{U}$ are m_K almost equal, this means that the open set $(U - k) \setminus \overline{U}$ has zero measure and is thus empty. This means that $u - k \in \overline{U}$ for all $u \in U$ and $k \in \ker(\pi)$. Thus $\ker(\pi) \subset \mathrm{Stab}_K(\overline{U})$. $\qquad\square$

Proof of Theorem 1.10. The assumption that $\mathrm{Stab}_K(\overline{U})$ is trivial together with Proposition 5.2 implies that π is injective and thus an isomorphism onto its image. But the image of π is the closed subgroup of $\mathbb{U}^\Gamma$ generated by

$$\pi(\alpha) = (\chi(\alpha))_{\chi \in \Gamma}.$$

Finally, by applying the natural isomorphism $\mathbb{U}^\Gamma \cong \mathbb{U}^\Lambda$ (i.e., the one induced by the bijection $\chi \mapsto \chi(\alpha)$ from Γ to Λ) we get that K is isomorphic to the closed subgroup of $\mathbb{U}^\Lambda$ generated by $(\lambda)_{\lambda \in \Lambda}$. $\qquad\square$

6. Equivalence of Question 1.12 and Question 1.14

We now show that Question 1.12 and Question 1.14 are equivalent. Recall that a *block system* of a group action $G \curvearrowright X$ is a partition $\mathcal{B} \subset 2^X$ of X (collection of non-empty disjoint sets whose union is X) that is G-invariant (i.e., if $g \in G$ and $B \in \mathcal{B}$ then $gB \in \mathcal{B}$). Let us define the *block system generated by $U \subset X$* (for this action) to be the smallest (with respect to inclusion) block system such that U is a union of elements of $\mathcal{B}$. More concretely, the the block system generated by U consists of those minimal non-empty sets that can be written as intersections of elements in $\{gU \mid g \in G\} \cup \{X \setminus gU \mid g \in G\}$.

Proposition 6.1. Let G be a group acting transitively on a set X_1 and also acting transitively on a set X_2 and suppose that $U_1 \subset X_1$ and $U_2 \subset X_2$ are simple for the respective actions. Suppose that $x_1 \in X_1$ and $x_2 \in X_2$ are such that

$$\{g \in G \mid gx_1 \in U_1\} = \{g \in G \mid gx_2 \in U_2\}.$$

Then there is an isomorphism $\phi : X_1 \to X_2$ of G-sets mapping x_1 to x_2, i.e., $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_1$.

Proof. Let $\mathcal{B}$ denote the block system generated by U_1. Thus there is a well defined map $\pi : X_1 \to \mathcal{B}$ where for $x \in X_1$ we define $\pi(x)$ to be the unique element of $\mathcal{B}$ containing $x \in X_1$. Note that π is a morphism of G-actions, i.e., $g\pi(x) = \pi(gx)$ for all $g \in G$ and $x \in X_1$. Note that

$$U_1 = \pi^{-1}\left(\bigcup_{\{B \in \mathcal{B} \ \mid \ B \subset U_1\}} B \right)$$

since by definition U_1 is a union of elements of $\mathcal{B}$. Thus since U_1 is simple we must have that π is an isomorphism. In particular this means $\{x\} \in \mathcal{B}$ for all $x \in X_1$. Now let

$$R(V) = \{g \in G \mid gx_1 \in V\}$$

for any $V \subset X_1$. Notice the properties $R(gV) = gR(V)$ and $R(\bigcap_{i \in \mathcal{I}} V_i) = \bigcap_{i \in \mathcal{I}} R(V_i)$ and $R(X_1 \setminus V) = G \setminus R(V)$. Now let $\mathcal{B}_G$ denote the block system generated by $R(U_1)$ for the action of $G \curvearrowright G$ by left multiplication. By the aforementioned properties we have that $\mathcal{B}_G$ consists of those sets of the form $R(B)$ for $B \in \mathcal{B}$ and so we have that $\mathcal{B}_G$ consists of exactly the sets of the form $R(\{x\})$ for $x \in X_1$. Note that $R(\{x_1\}) = \{g \in G \mid gx_1 = x_1\} = \operatorname{Stab}(x_1)$ and any other $R(\{x\})$ must be a coset of $\operatorname{Stab}(x_1)$ (by transitivity $x = gx_1$ for some $g \in G$ and so $R(\{x\}) = gR(\{x_1\})$). So in fact $\operatorname{Stab}(x_1)$ may be desribed as member of $\mathcal{B}_G$ that contains $1 \in G$. Since $X_1 \cong G/\operatorname{Stab}(x_1)$, this means that we have an isomorphism of G-actions $X_1 \to \mathcal{B}_G$ that maps x_1 to the unique element of $\mathcal{B}_G$ containing $1 \in G$ (it maps x to $R(\{x\})$). Notice that $\mathcal{B}_G$ is the same if we replace X_1, x_1, U_1 with X_2, x_2, U_2 respectively by the assumption that

$$\{g \in G \mid gx_1 \in U_1\} = \{g \in G \mid gx_2 \in U_2\}.$$

Thus indeed we have an isomorphism $X_1 \cong X_2$ of G-actions (they are both isomorphic to G acting on $\mathcal{B}_G$). $\qquad\square$

This demonstrates the equivalence of the two questions as follows. If the answer to Question 1.14 is affirmative, then the two sets in Question 1.12 are simple and thus the Proposition 6.1 provides an affirmative answer. Conversely, if $U \subset X$ is a subset in a transitive G-set X with a trivial setwise stabilizer, then we have a factor $\pi : X \to \mathcal{B}$, where $\mathcal{B}$ is the block system generated by U, mapping $x \in X$ to the unique element of $\mathcal{B}$ containing x (as constructed in the proof of the Propostion 6.1). But $\pi(U)$ also has a trivial setwise stabilizer and $\{g \in G \mid gx_0 \in U\} = \{g \in G \mid g\pi(x_0) \in \pi(U)\}$ so if the answer to Question 1.12 is affirmative, then π must be an isomorphism and so $\mathcal{B}$ consists of singletons.

References

[1] Albertson, M. O.; Collins, K. L. *Symmetry breaking in graphs.* Electron. J. Combin. 3 (1996), no. 1, Research Paper 18, approx. 17 pp.

[2] Gutman, Y. *Taken's embedding theorem with a continuous observable.* Ergodic theory, 134–141, De Gruyter, Berlin, 2016.

[3] Host, B.; Kra, B. *Nilpotent structures in ergodic theory.* Mathematical Surveys and Monographs, 236. American Mathematical Society, Providence, RI, 2018. X+427 pp. ISBN: 978-1-4704-4780-9

[4] Pavlov, R. *Some counterexamples in topological dynamics.* Ergodic Theory Dynam. Systems 28 (2008), no. 4, 1291–1322.

[5] Smith, S. M.; Tucker, T. W.; Watkins, M. E. *Distinguishability of infinite groups and graphs.* Electron. J. Combin. 19 (2012), no. 2, Paper 27, 10 pp.

[6] Takens, F. *Detecting strange attractors in turbulence.* Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980), pp. 366–381, Lecture Notes in Math., 898, Springer, Berlin-New York, 1981.

[7] Weyl, H. *Über die Gleichverteilung von Zahlen mod. Eins.* (German) Math. Ann. 77 (1916), no. 3, 313–352.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, AUSTRALIA

Email address: `kamil.bulinski@sydney.edu.au`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, AUSTRALIA

Email address: `alexander.fish@sydney.edu.au`